Essays

Notes on financial econometrics

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Abstract

The first part of the discussion reviews recent successes in modeling of discrete time financial data and argues that a direct approach is better suited than stochastic volatility. The second part reviews recent work on estimating continuous time models with emphasis on simulation-based techniques and joint estimation of the risk neutral and objective probability distributions. © 2001 Elsevier Science S.A. All rights reserved.

1. Discrete time data

1.1. Direct specification

Perhaps the most stunning empirical success is the extent to which we now essentially understand the statistical dynamics of a scalar financial price series. Suppose P_t is the price of a financial asset at time t , and we let $y_t = \log(P_t) - \log(P_{t-1})$ or more generally $y_t = \log(P_t + D_t) - \log(P_{t-1})$ if dividends are taken into account. Let $\mathcal{Y}_{t-1} = \{y_{t-k}\}_{k \geq 1}$ denote the lag history of $\{y_t\}$, and put

$$\begin{aligned}\mu_t &= \mathcal{E}(y_t | \mathcal{Y}_{t-1}), \\ \sigma_t^2 &= \text{Var}(y_t | \mathcal{Y}_{t-1}), \\ z_t &= \frac{y_t - \mu_t}{\sigma_t},\end{aligned}\tag{1}$$

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where for now, we suppose z_t is iid with density $f(z)$. This setup underlies much discrete-time financial modeling such as ARCH/GARCH and its various non-parametric relatives. We know from over 15 years of work with various series that the conditional mean is nearly constant, $\mu_t \approx \text{AR}(1)$ with autoregressive coefficient near zero, σ_t^2 is extremely persistent series that might be best captured by either long memory (Ding et al., 1993), (Baillie et al., 1996) or multiple components GARCH models (Engle and Lee, 1999), and that $f(z)$ is highly non-Gaussian with more mass at the origin and in the tails than the Gaussian distribution. If y_t is an interest rate, the basic facts are more complicated. The conditional mean is nearly that of a random walk and is perhaps nonlinear (Aït-Sahalia, 1996a). The conditional variance displays both GARCH-like behavior and a level effect (Andersen and Lund, 1997), while $f(z)$ is highly non-Gaussian with a shape much like that of an equity return.

These basic facts arise from what I will call a direct approach to specifying models for y_t . If we specify functional forms for $\mu_t = m(\mathcal{Y}_{t-1}, \theta)$, $\sigma_t^2 = \mathcal{V}(\mathcal{Y}_{t-1}, \theta)$, $f(z) = f(z|\theta)$, then transition density for y_t is

$$y_t \sim f \left[\frac{y_t - m(\mathcal{Y}_{t-1}, \theta)}{\sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta)}} \right] / \sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta)}, \quad (2)$$

which provides the basis for maximum-likelihood type estimation of the model from which much can be learned regarding the dynamics of y_t .

There are somewhat less parametric approaches to the same modeling task. In the SNP approach of Gallant and Tauchen (1989, 2000a), $f(z)$ is approximated by a modified Hermite series with $f_K(z, \theta)$ denoting the K th term in the series. In this case, the transition density of y_t

$$y_t \sim f_K \left[\frac{y_t - m(\mathcal{Y}_{t-1}, \theta_0)}{\sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta_0)}} \right] / \sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta_0)}, \quad (3)$$

where $\theta_K \in \Theta_K$ contains all of the parameters of the expansion, $\Theta_K \subset \Theta_{K+1}$, and the functions m and $\mathcal{V}$ retain their same parametric specifications for each K , as given in base specifications $K = 0$, which is usually a Gaussian model. In the semiparametric GARCH formulation of Engle and Gonzales–Rivera (1991), the transition density is

$$y_t \sim \hat{f} \left[\frac{y_t - m(\mathcal{Y}_{t-1}, \theta)}{\sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta)}} \right] / \sqrt{\mathcal{V}(\mathcal{Y}_{t-1}, \theta)}, \quad (4)$$

where $\hat{f}(z)$ is a kernel-based estimate of the transition density of z_t . For very long time series, the presumption of time homogeneity in the error density $f(z)$ becomes untenable (Gallant et al., 1992). The conditional skewness, kurtosis, and other higher order properties become state dependent. The SNP approach

accommodates such dependence by making the Hermite coefficients depend upon $\mathcal{Y}_{t-1}$ so the error density is $f(z|\mathcal{Y}_{t-1}, \theta_K)$, and the transition density takes a form similar to (3). It also has a natural multivariate generalization. Gallant and Tauchen (2000a) discuss computational details and provide computer code and worked examples.

1.2. Stochastic volatility

The general approach above is *direct* in that the investigator directly specifies the three key pieces: the conditional mean function $m(\mathcal{Y}_{t-1}, \theta)$, the conditional variance function $\mathcal{V}(\mathcal{Y}_{t-1}|\theta)$, and the error density $f(z|\mathcal{Y}_{t-1}, \theta)$. This specification can be done in either a fairly tightly parameterized manner or a more flexibly parameterized manner with a non-parametric interpretation.

The direct approach stands in contrast to stochastic volatility, for which the basic model is

$$y_t = z_t h_t, \quad (5)$$

where

$$\begin{aligned} z_t &\sim q(z) \\ h_t &\sim g(h|\mathcal{H}_{t-1}), \quad \mathcal{H}_{t-1} = \{h_{t-k}\}_{k \geq 1} \end{aligned} \quad (6)$$

and for simplicity the conditional mean is ignored here. In the above, h_t is unobserved stochastic volatility and z_t is a return shock. The basic model takes $q(z)$ as the standard Gaussian density and $\log(h_t)$ as Gaussian AR(1) process with possible correlation between volatility innovations and returns shocks; see Ghysels et al. (1995) for a survey. The appeal of the stochastic volatility model is its simplicity and ease of interpretation. A drawback, however, is that given (6) the conditional density of the observed process given its own past, $f(y_t|\mathcal{Y}_{t-1})$, is not available in a convenient closed form. This has led to a large number of method-of-moments based approaches and Bayesian-based approaches to estimation of (6); see Andersen and Sorensen (1996), Jacquier et al. (1994), and Kim et al. (1998), among others. These additional complications in estimation seem a small price to pay for the elegant simplicity of the basic specification. However, Gallant et al. (1997) find that a realistic stochastic volatility model has to be far more complicated if it is to actually fit the data. The error density $q(z)$ has to be made strongly thick-tailed and left skewed, while the dynamics of h_t have to be very rich with both short-term (Markov) and long-memory components. The entire apparatus becomes so complicated and so difficult to estimate that the appeal of stochastic volatility on grounds of simplicity is lost. The direct approach is better suited to the task than is stochastic volatility.

2. Continuous time estimation

2.1. Estimation of price dynamics

Continuous time estimation has attracted a huge amount of attention in the past five years. Lo (1988) points out what was considered the major obstacle: given a specification of the continuous time dynamics the conditional density of the discretely sampled price process is not available in closed form. This either precludes, or greatly complicates maximum likelihood estimation. Hansen and Scheinkman (1995) and Aït-Sahalia (1996a, b) are among the first works in this area. For reasons space, I will confine my discussion to simulation-based moments estimators, though there is progress on implementing maximum likelihood (Aït-Sahalia, 1999; Elerian et al., 1999) for scalar observed data. Some advantages of simulation-based procedures are that they can more readily handle multivariate situations with partially observed state vectors and path-dependent observed variables.

Suppose the underlying state vector u_t of the economy evolves as

$$du_t = a(u_t)dt + B(u_t)dw_t, \quad (7)$$

where w_t is a vector of Brownian motions. Assume a vector of logged financial prices p_t evolve according to

$$dp_t = a_p(u_t)dt + B_p(u_t)dw_t. \quad (8)$$

Clearly, if one specifies the functional forms

$$a(u) = a(u, \rho) \quad B(u) = B(u, \rho) \quad a_p(u) = a_p(u, \rho) \quad B_p(u) = B_p(u, \rho), \quad (9)$$

where ρ is a parameter vector, then the price data generation process is determined in continuous time. The econometrician observes functions of the path of the price process at discrete time points:

$$y_t = \mathcal{O}[\{p_s\}_{s=t-1}^t], \quad (10)$$

where $\{p_s\}_{s=t-1}^t$ means the within-period continuous price path, y_t is the observed process for integer t , and $\mathcal{O}$ is the observation function. The form of $\mathcal{O}$ depends upon the application. For interest rate data, $\mathcal{O}$ just selects out the yields implied by bond prices; for equities data, which have a unit root, $\mathcal{O}$ selects out first differences of log prices; more generally, $\mathcal{O}$ also selects out path-dependent quantities such as the high/low range as in Gallant et al. (1999) or the quadratic variation as in Bollerslev and Zhou (2000).

Given the observed data set $Y_T = \{y_t\}_{t=1,2,\dots,T}$, the task is to estimate ρ and test the specification. Although $pdf(y_t|y_{t-1}, \dots, \rho)$ is not readily available, it is

clear that one can easily simulate from the system (7)–(10). For each candidate value of ρ one generates simulated data sets $Y_{\mathcal{T}}(\rho) = \{y_{\mathcal{T}}(\rho)\}_{\mathcal{T}=1}^{\mathcal{T}}$. Simulated method of moments (SMM) of Duffie and Singleton (1993) is feasible and there are some good ways to implement SMM. One approach is the Indirect Inference approach of Gourieroux et al. (1993). Suppose we consider an auxiliary model $f(y_t|y_{t-1}, \dots, \theta)$ for the observed data. Let $\hat{\theta} = B_T(Y_T)$ denote the QML estimator of θ based on f as a function $B_T(\cdot)$ of the observed data set Y_T . The Indirect Inference estimator minimizes $[\hat{\theta} - \bar{\theta}(\rho)]' W [\hat{\theta} - \bar{\theta}(\rho)]$ where W is a weight matrix and $\bar{\theta}(\rho)$ is given by the binding function $\bar{\theta}(\rho) = \lim_{\mathcal{T} \rightarrow \infty} B_{\mathcal{T}}[Y_{\mathcal{T}}(\rho)]$, which is approximated by $B_{\mathcal{T}}[Y_{\mathcal{T}}(\rho)]$ for large $\mathcal{T}$. Unless $f(y_t|y_{t-1}, \dots, \theta)$ is linear, or only mildly nonlinear, this approach is very computationally demanding as one needs to evaluate the binding function $\bar{\theta}(\rho)$ for any permissible value of ρ . The estimator of Gallant and Tauchen (1996, 2000b) circumvents the need to evaluate the binding function by using the score vector $(\partial/\partial\theta)\log[f(y_t|y_{t-1}, \dots, \theta)]$ to define the moment conditions. If the auxiliary model $f(y_t|y_{t-1}, \dots, \theta)$ is chosen flexibly with a suitable nonparametric interpretation, then the estimator achieves the asymptotic efficiency of maximum likelihood and has good power properties for detecting misspecification (Gallant and Long, 1997; Tauchen, 1997), hence the term efficient, method of moments (EMM). Some applications of EMM are Andersen and Lund (1997), Dai and Singleton (1999), and Gallant et al. (1999).

2.2. Joint estimation of objective and risk neutral distributions

One of the most interesting and exciting challenges in continuous time analysis is the prospect of joint estimation of the so-called objective and risk-neutral probability distributions. As before, assume the state vector u_t evolves according to

$$du_t = a(u_t)dt + B(u_t)dw_t, \quad (11)$$

Suppose we have traded security prices p_{jt} with cash flows $c_{jt} = \mathcal{C}_j(u_t)$. Internal consistency (no arbitrage) requires that each price be the present value of the expected cash flow

$$p_{jt} = \int_{s=0}^{\infty} \exp\left(-\int_{v=0}^s r_v dv\right) \hat{\mathcal{E}}(c_{j,t+s}|u_t) ds, \quad (12)$$

where r_t is instantaneous short rate of interest and $\hat{\mathcal{E}}$ denotes the expectation under risk neutral dynamics:

$$du_t = a^*(u_t)dt + B(u_t)dw_t. \quad (13)$$

Observe that the objective (i.e. actual) dynamics of the state vector u_t in (11) and the risk neutral dynamics (13) in general have different local drift functions $a(u_t)$ and $a^*(u_t)$ but they have the same local volatility structure $B(u_t)$.

At a fixed point in time t , one can actually estimate the risk-neutral distribution from a *cross section* of derivative prices $\{p_{jt}\}_{j=1}^J$. Finance economists are quite familiar with the calculation, which is undertaken routinely in industry. A stylized overview follows. One specifies functional forms $a^*(u) = a^*(u, \rho^*)$ and $B(u) = B(u, \rho^*)$ such that the expectation in (13) is relatively easy to compute and ρ^* means the parameterization under the risk neutral distribution. The expectation determines functional forms for the prices $p_j(u_t, \rho^*)$. Estimation proceeds via minimization of the pricing errors

$$(\hat{\rho}_t^*, \hat{u}_t) = \underset{\rho^*, u_t}{\operatorname{argmin}} \sum_{j=1}^J [p_{jt} - p_j(u_t, \rho^*)]^2, \quad (14)$$

where the unobserved (or partially observed) state u_t is estimated along with ρ^* . Given $(\hat{\rho}_t^*, \hat{u}_t)$, the dynamics (13), and the pricing equation (12), one can price any contingent claim (derivative security) as of date t . Of course, one can assume that ρ^* is constant across time and add the objective function (14) across days to produce a common estimate of ρ^* and a time series of estimates of the state vector $\{\hat{u}_t\}$. This approach is sensible but there are immediate econometric questions to raise. The first is the lack of a theory of the pricing error. Why should the error be expected to be serially uncorrelated of constant variance? More to the point, why does the model not fit the cross section exactly, as deviations entail possible arbitrage opportunities? This point at least should be pondered. A related issue is the appropriate econometric theory to apply in the face as many incidental parameters (u_t) as there are data points. Finally, the approach only delivers an estimate of the risk neutral dynamics (13).

A potentially very progressive approach, and one of the most exciting frontiers on financial econometrics, is to exploit the common local volatility structure of (11) and (13) and estimate jointly the objective and risk neutral distributions. Specifically, parameterize

$$\begin{aligned} du_t &= a(u_t, \rho) dt + B(u_t, \rho) dw_t, \\ du_t &= a(u_t, \rho^*) dt + B(u_t, \rho) dw_t^*, \end{aligned} \quad (15)$$

where w_t and w_t^* are independent Brownian motions and the restrictions is that the functional form of $B(u, \rho)$ must be the same across the two sets of dynamics. Financial prices are generated via

$$dp_t = a_p(u_t, \rho, \rho^*) dt + B_p(u_t, \rho, \rho^*) dw_t, \quad (16)$$

where the functional forms of a_p and B_p are determined by computing the expectations in (12) under the risk-neutral dynamics of (15). Given (16), then

joint estimations via SMM can proceed exactly as outlined in Section 3.1. There is research along these lines. For example, Chernov and Ghysels (2000) have recently undertaken exactly this approach for a multifactor stochastic volatility model, while Pan (1999) undertakes similar estimation via a GMM procedure for an affine jump diffusion model of interest rates.

3. Conclusion

The preceding remarks all pertain to estimation situations with long time series observations – often thousands – on a relatively modest number of series – often three or four at most. Another huge challenge is dealing with extremely dense data sets comprised of ultra high frequency data on many – possibly hundreds – of series. Two other issues are development of a sound theory of the pricing errors for derivatives models and development of practical ways to force fully articulated economic models to confront the rich dynamic structure of observed financial time series.

Acknowledgements

I would like to thank the editors for this opportunity and to apologize, in advance, for the selectivity of the topics and citations due to the necessary tight page constraints.

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