Price Expectations, Disequilibrium Adjustments, and Macroeconomic Price Stability

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1. INTRODUCTION

In two previous papers we have discussed the formulation of a logically consistent adaptive-type price expectations mechanism in continuous time and have applied our results to derive sufficient conditions for dynamic price stability in a variety of economic models (Burmeister and Turnovsky [5, 20]). In all cases the sufficient conditions for stability entail a "slow" rate of adaptation which implies a consistent error in the *direction* of the forecasted versus actual price movements. This is a highly undesirable feature since one cannot realistically expect economic agents to persistently predict in the wrong direction!

As will become clear in due course, the main reason behind this unsatisfactory aspect is the fact that our previous analysis was based on the conventional assumption of continuous market clearance. In effect, with markets adjusting infinitely fast, expectations were required to adjust sufficiently slowly in order for stability to be maintained. In this paper we allow for the possibility that markets may not clear instantaneously, and thereby we are able to derive sufficient dynamic stability conditions that admit "faster" adaptation rates for which expected and actual prices do always move in the same direction.

The basic underlying stability issue is nothing but the "saddlepoint instability" or "Hahn-Samuelson" problem that is familiar from the literature on descriptive heterogeneous capital good models; see, for example,

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Burmeister and Dobell [2], Burmeister and Graham [3], Hahn [8], Samuelson [15], and Shell and Stiglitz [17]. But this instability problem is not limited to heterogeneous capital good models; rather, as is clear from our own previous work and from this paper, the issue potentially arises in any model in which economic agents may hold their wealth in alternative assets, for then the dynamic price equations associated with portfolio equilibrium in competitive asset markets are generally unstable if all markets clear instantaneously and if agents possess "perfect myopic foresight," i.e., if expected and actual rates of price changes always coincide.¹

It has been argued that this saddlepoint instability problem can be circumvented if there exist perfect futures markets to infinity; then, one can show that all nonconvergent paths are inconsistent with the model (e.g., if the equations of motion are such that every nonconvergent path has at least one price that becomes negative in finite time), it follows that an initial equilibrium in all futures markets will assure that the economy follows a convergent path. There are three difficulties with this "solution." First of all, it is unrealistic to assume the existence of futures markets for every commodity, and in any event most future markets are limited to a horizon of about one year. Secondly, there is the technical difficulty of justifying the crucial assumption that all nonconvergent paths will reveal themselves to be inconsistent. And finally, if for some reason, perhaps because of an exogenous shock, an economy were to find itself on an errant, nonconvergent price path, then (unless some additional dynamic adjustment mechanisms are specified) the only way for the economy to reach a convergent path is to have a discontinuous jump in prices, as in Sargent and Wallace [16]. But the information necessary to ensure that the jump in prices will be precisely that required to move the system onto the stable manifold is extreme, rendering this possibility rather remote in a world of continuing stochastic disturbances. Furthermore, such jumps become infeasible if one drops the assumption of instantaneous market clearance and instead assumes (as we shall) a Walrasian price adjustment where prices move sluggishly in response to excess demand conditions.

For these reasons this paper introduces an alternative approach to the problem of saddlepoint instability, one which we feel contains substantial intuitive appeal. The new key element we introduce is the familiar postulate that prices adjust upwards (downwards) in response to excess demand (supply). Thus the dynamic price behavior for the i -th asset contains two parameters: α_i , the speed of market adjustment, and β_i , the rate of price expectation adaptation. When $\alpha_i \rightarrow \infty$, the i -th market clears instan-

¹ It has been recognized for some time that these conditions may give rise to "speculative fevers" or "tulip manias," although the problem seems to be ignored in the finance literature where one seldom finds dynamics formulated in a general equilibrium framework; for early insights to this problem see Samuelson [14].

taneously and demand is always equal to supply; when $\beta_i \rightarrow \infty$, the expected rate of change for the i -th price always equals the actual, i.e., there is "perfect myopic foresight." The previously cited literature on saddlepoint instability entails the double extreme case of $\alpha_i = \beta_i = \infty$ (all i), while our earlier work (Burmeister and Turnovsky, [5], [20]) assumed the single extreme $\alpha_i = \infty$ and $0 < \beta_i < 1$ (all i). Realistically one should expect the intermediate case $0 < \alpha_i < \infty$ and $0 < \beta_i < \infty$ (all i), and this case is precisely the one we analyze below. It then turns out that the sufficient conditions for dynamic stability involve a trade-off between the α_i and the β_i , and stability prevails provided the product $\alpha_i\beta_i$ is "not too large" in a sense that we should make precise. This feature allows us to eliminate the stated objectionable feature in our previous work; since the i -th rate of expected and actual price change moves in the same (opposite) direction when $\beta_i > 1$ ($\beta_i < 1$), we can now establish sufficient conditions for stability with $\beta_i > 1$ and $0 < \alpha_i < \infty$ (all i).

Our model, therefore, provides a unified general treatment of market and price expectations adjustments parameterized by the α_i and β_i coefficients, respectively. The limiting behavior as the α_i and/or β_i approach infinity corresponds to the polar cases previously studied by ourselves and other authors. Our formulation also points out an interesting identification problem. If expected rates of change are unobservable, and almost certainly they are not observable, one cannot obtain an estimate of the β_i price adaptation coefficients. In fact, the observable rates of price changes equations always involve the products $\alpha_i\beta_i$ which therefore can be estimated. Suppose, for example, we were to find that $\alpha_i\beta_i = 100$. It could be that expectations adjust "slowly" and markets clear "fast" with, say, $\beta_i = 0.1$ and $\alpha_i = 1,000$; or it could be that expectations adjust "rapidly" and market clearing is "sluggish" with $\beta_i = 100$ and $\alpha_i = 1.0$. We suspect that this observation has profound implications for the proper interpretation of existing econometric estimates, but this issue is a digression from our present objectives and will be pursued in our subsequent research work.

There is another issue we mention briefly. Because demand and supply are not equal when $\alpha_i < \infty$, "trade out of equilibrium" must continually be taking place.² We shall be concerned with analyzing asset markets, with the stocks of assets assumed to be fixed in supply, and therefore at each instant of time the stocks are held by someone. Accordingly, it seems most natural to assume that trade in fact must be occurring along a vertical supply curve.

The remainder of the paper proceeds as follows. In Section 2 we elaborate upon some of the difficulties associated with the adaptive expectations hypothesis, while in Section 3 we outline the nature of the financial markets we shall consider. In the next three sections we consider the stability of the

² For an extensive discussion of this issue see Barro and Grossman [1].

market adjustment under three alternative sets of assumptions: (i) three assets with fixed income, (ii) an arbitrary number of assets with fixed income, and (iii) three assets with endogenous income. The concluding section contains a brief summary and appraisal of our main conclusions.

2. ADAPTIVE EXPECTATIONS, RATIONALITY, AND PERFECT MYOPIC FORESIGHT

We begin by considering some further aspects of the continuous time version of the adaptive expectations hypothesis not discussed in our previous work. To do so consider the following discrete time version of the hypothesis:

$$\pi(t+h, t) - \pi(t, t-h) = \beta[p(t) - \pi(t, t-h)], \beta > 0, \quad (1)$$

where

- $p(t)$ = actual price level at time t ;
- $\pi(t+h, t)$ = the price expectations formed at time t for the actual price at time $t+h$, $h \geq 0$;
- $\beta = \text{constant}$ = rate of adaptation;
- h = length of time horizon or forecast interval.

We assume that forecasts satisfy the following *weak consistency axiom*:

$$\pi(t, t) = p(t). \quad (2)$$

That is, our weak consistency axiom asserts that the expectation formed now for the price now is equal to the actual prevailing price. Since the model contains no information costs and actual prices are known at every instant, (2) is necessary for efficient forecasting in continuous time.

Substituting (2) into (1), dividing both sides by h , and letting $h \rightarrow 0$ yields the following continuous time limit of (1):

$$\dot{\pi}(t) = \beta\pi_2(t, t) \quad (3)$$

where

$$\begin{aligned} \pi(t) &\equiv \pi(t, t);^3 \\ \pi_1(t, t) &\equiv \left. \frac{\partial \pi(s, t)}{\partial s} \right|_{s=t}; \\ \pi_2(t, t) &\equiv \left. \frac{\partial \pi(s, t)}{\partial t} \right|_{t=s}. \end{aligned}$$

³ Note that $\dot{\pi}(t) \equiv (d/dt)\pi(t, t) = \pi_1(t, t) + \pi_2(t, t)$.

This is the correct limiting version of adaptive expectations, on the assumption that the weak consistency axiom holds. It is a mixed total-partial differential equation and forms the basis for our previous analysis.

It is crucial for a proper understanding of our work to interpret $\pi_1(t, t)$ correctly. This is most easily done by writing

$$\pi_1(t, t) = \lim_{h \rightarrow 0} \left[\frac{\pi(t + h, t) - \pi(t, t)}{h} \right], \quad (4)$$

from which it becomes clear that $\pi_1(t, t)$ measures the *expected instantaneous rate of change of price at time t* .⁴

We shall assume that all variables are left- and/or right-hand continuously differentiable, as is appropriate from our definitions. Accordingly from the weak consistency axiom (2) we obtain

$$\dot{p}(t) = \pi_1(t, t) + \pi_2(t, t) = \dot{p}(t). \quad (5)$$

Since the right hand side of (5) measures the *actual* rate of price change, while $\pi_1(t, t)$ measures the corresponding *anticipated* rate of change, it follows from (5) that $\pi_2(t, t)$ measures the *unanticipated rate of price change*, or, in other words, the forecast error. The economic interpretation of Eq. (3) is thus precisely the same as that of the conventional discrete-time version of the hypothesis. It also follows from (5) that perfect myopic foresight, when $\pi_1(t, t) \equiv \dot{p}(t)$, is equivalent to $\pi_2(t, t) \equiv 0$. Substituting (5) into (3) we deduce that

$$\pi_1(t, t) = \left(\frac{\beta - 1}{\beta} \right) \dot{p}(t), \quad (6)$$

from which it is seen that perfect myopic foresight corresponds to the limiting case where $\beta \rightarrow \infty$. On the other hand, the razor's edge case of $\beta = 1$ corresponds to *static expectations*, when the anticipated rate of price change is zero.

In our previous applications of our adaptive type price expectations mechanism in continuous time (Burmeister and Turnovsky [5], [20]), it was postulated that the asset markets were in continuous equilibrium, with zero excess demand. Under this assumption, the sufficient conditions for short-run dynamic stability entailed $0 < \beta < 1$ for all asset prices. But from (6) it can be seen that this implies a very objectionable result, namely actual and expected rates of change of prices must always move in *opposite* directions. Surely such behavior cannot persist forever if economic agents are "rational" in any reasonable sense. In this paper we will drop the previously made

⁴ The relationship between perfect myopic foresight and the weak consistency axiom is discussed at length by Turnovsky and Burmeister [20].

assumption of asset market equilibrium, thereby enabling us to derive new sufficient conditions for dynamic price stability which allow for the possibility of $\beta > 1$. Thus loosely speaking we may think of β as parameterizing the "degree of rationality"; $\beta < 1$ is definitely "irrational," $\beta = 1$ is "static," while cases of $\beta > 1$ can be viewed as being at least "weakly rational" with the limiting case of $\beta \rightarrow \infty$ corresponding to "perfect myopic foresight" or "completely rational" expectations in the sense of Muth [13].

On the other hand, solving (1) recursively yields

$$\pi(t+h, t) = \beta \sum_{i=0}^{\infty} (1-\beta)^i p(t-ih), \quad (7)$$

from which it is seen that expectations will be explosive provided $\beta > 1$ and the forecast interval h is strictly positive. Accordingly, if we believe that actual prices are bounded, the condition $\beta > 1$ is most unsatisfactory.⁵

Thus there is a dilemma associated with the adaptive expectations assumption: either $\beta < 1$, in which case the *direction* of the instantaneous rate of change is always incorrectly predicted, or $\beta > 1$, in which case expectations formed over any finite forecast interval must ultimately diverge. But since our present concern is with introducing a parameter to describe the degree of rationality rather than with the adaptive hypothesis per se, the above problems need not present us with any difficulties. We can still parameterize the relationship between $\pi_1(t, t)$ and $\dot{p}(t)$ by

$$\pi_1(t, t) = \gamma \dot{p}(t), \quad \gamma \equiv \frac{\beta - 1}{\beta}, \quad (8)$$

with γ reflecting the degree of rationality. In accordance with the above notions, if $\gamma < 0$ expectations are "irrational," if $\gamma = 0$ they are "static," if $\gamma > 0$ they are "weakly rational," while the limiting case $\gamma = 1$ corresponds to "perfect myopic foresight" or "completely rational" expectations.

Our work does *not* necessitate an adaptive-type expectations mechanism since the key relationship, Eq. (8), can be derived from a completely different

⁵ At first glance it might appear that the seemingly odd behavior implied by (6) and (7) can be resolved by adopting the Sargent-Wallace procedure of integrating (or summing) the system forward in time rather than backwards and imposing a transversality condition rather than an initial condition, thereby leading to a reversal of the relevant stability conditions. While this procedure is feasible in the Sargent-Wallace context, it does not make any economic sense here. The reason is simply that although $\beta > 1$ would now ensure the stability of (7) together with the weak rationality of (6), it would make the predictions of prices formed at time t all depend upon the unknown *future* values of prices $p(\tau)$, $\tau > t$. But this contradicts the essential feature of the adaptive hypothesis (1) which underlies (6), (7), according to which price expectations are autoregressively determined; that is, they depend upon the known *past* values of prices.

underlying discrete-time framework. Thus suppose that instead of (1), we postulate the Hicksian extrapolative mechanism

$$\pi(t + h, t) = p(t) + \gamma[p(t) - p(t - h)]. \quad (9)$$

Dividing both sides by h and taking the limit $h \rightarrow 0$, along with the weak consistency axiom, yields precisely (8) as the limiting behavior; see Turnovsky and Burmeister [20]. The cases $\gamma < 0$ and $\gamma > 0$ correspond to regressive and extrapolative expectations, respectively, while $\gamma = 1$ is again the perfect foresight case.

In summary, then, we shall use Eq. (8) where the parameter $\gamma > 0$ measures the “degree of rationality.” This equation is to be interpreted as the limiting behavior as the time horizon h approaches zero, and it is consistent with both the adaptive and extrapolative hypotheses in discrete time.

3. ASSET MARKET SPECIFICATIONS

We define *short-run dynamic equilibrium* as a state in which:

- (i) The stocks of all assets are instantaneously fixed,
- (ii) asset markets are in equilibrium in the sense that demand is equal to supply, and
- (iii) the relative prices of all assets are constant.

Our problem is to consider a *short run* situation in which only (i) obtains, and then to ask whether or not the economy asymptotically approaches (starting from arbitrarily given initial relative prices) a short-run dynamic equilibrium featured by (i), (ii) and (iii) above.

This approach differs from most conventional models in which short-run equilibrium is *defined* to be a state with constant relative prices. Such an assumption may be justified as a legitimate modeling device (in the sense of being a “sufficiently close approximation” to reality) if in fact a short-run *dynamic* equilibrium, as we have defined it, is always unique and stable with a “sufficiently rapid” speed of convergence. However, as we shall see, such uniqueness and stability require sufficient conditions that impose economically significant restrictions; in particular, restrictions are imposed on both the adjustment coefficients and the asset demand functions. When such sufficient conditions are violated, a short-run dynamic equilibrium may be *unstable*, in which case microeconomic models predicated upon the postulate that a short-run dynamic equilibrium prevails at every instant are no longer tenable.

Suppose now that we have $n + 1$ assets, with the $(n + 1)$ -st asset (money)

serving as *numéraire*. The actual prices of any asset at any times t and $t + h$ must equal the present discounted value of expected future earnings from times t and $t + h$, respectively. This means that given our weak consistency axiom, the instantaneous expected net money rate of return on the i -th asset, $r_i(t)$, must satisfy

$$r_i(t) = \frac{w_i(t)}{p_i(t)} + \frac{\pi_{i,1}(t, t)}{p_i(t)}, \quad i = 1, \dots, n, \quad (10)$$

where $w_i(t)$ is the net money “dividend” or “coupon rate” paid on the i -th asset at time t , $p_i(t)$ is the price of the i -th asset at time t , and $\pi_{i,1}(t, t)$ is the instantaneous expected rate of change of the price of the i -th asset at time t . By convention, $w_{n+1}(t) = 0$, and thus $r_{n+1}(t) = 0$ (“money” yields no return). Equation (10) simply states that the expected rate of return equals the sum of the known coupon rate of return $w_i(t)/p_i(t)$ and the expected rate of capital gains $\pi_{i,1}(t, t)/p_i(t)$; for a more detailed derivation see Burmeister and Turnovsky [5] or Burmeister and Graham [3].⁶ The instantaneous rate of change of price is assumed to be generated by a set of equations analogous to (8),

$$\pi_{i,2}(t, t) = \gamma_i \dot{p}_i(t), \quad i = 1, \dots, n, \quad (8')$$

where we shall assume $\gamma_i > 0$.

We now relax our previous assumption of continuous asset market clearance. Letting $F^i(\cdot)$ denote the value of demand for the i -th asset (in terms of the *numéraire*, money), excess demand is the difference between demand and supply, or

$$F^i(\cdot) - p_i k_i \quad (11)$$

where k_i is the fixed number of physical units of the i -th asset. We postulate that asset markets do not clear instantaneously and that the rate of changes in actual asset prices are proportional to excess demand:⁷

$$\frac{\dot{p}_i}{p_i} = \alpha_i [F^i(\cdot) - p_i k_i]$$

or

$$\dot{p}_i = p_i \alpha_i [F^i(\cdot) - p_i k_i], \quad i = 1, \dots, n, \quad (12)$$

⁶ When expected capital gains are ignored or assumed zero, (10) reduces to $r_i(t) = w_i(t)/p_i(t)$, the relationship often assumed in macroeconomics; see, for example, Tobin [18]. The restrictive conditions under which this expression is valid are discussed by Turnovsky [19].

⁷ It is clear that only n of the $n + 1$ excess demand functions are independent because of the wealth constraint.

where $\alpha_i > 0$ is the speed of market adjustment. The limiting case of $\alpha_i = \infty$ implies that markets clear instantaneously with excess demand identically zero.

The interaction between the market speed of adjustment coefficient, α_i , and the corresponding rate of price-expectation adaptation coefficient, γ_i , will become clear as we develop a specific model in the following section.

In order to facilitate the exposition, in Sections 4 and 5 below we take money income as fixed; this simplifying assumption will be dropped in Section 6. With money income fixed, the asset demand functions depend only upon the expected net money rates of return and wealth:

$$F^i(r_1, \dots, r_n; W) = (p_i k_i)^D, \quad i = 1, \dots, n+1, \quad (13)$$

where:

- $(p_i k_i)^D$ = the value demanded of the i -th asset;
- p_i = the actual price of the i -th asset;
- k_i = the fixed physical stock of the i -th asset;
- r_i = the expected net money rate of return for the i -th asset;
- W = the money value of wealth
- $= \sum_{i=1}^{n+1} p_i k_i$;
- $p_{n+1} \equiv 1$ = numéraire;
- $r_{n+1} \equiv 0$ = net rate of return on money in terms of money.

The index t for time has been dropped from every variable to simplify notation.

From Eqs. (8') and (10) we may derive

$$r_i = \frac{w_i + \gamma_i \dot{p}_i}{p_i} \quad (14)$$

(where as indicated we shall assume $\gamma_i > 0$).

Thus in general our dynamic price equations are obtained from (12), (13), and (14) and are of the form

$$\dot{p}_i = p_i \alpha_i \left[F^i \left(\frac{w_1 + \gamma_1 \dot{p}_1}{p_1}, \dots, \frac{w_n + \gamma_n \dot{p}_n}{p_n}; p_i k_i + \dots + p_n k_{n+1} \right) - p_i k_i \right],$$

$i = 1, \dots, n. \quad (15)$

We shall require the following assumption on the asset demand functions.

Assumption GS (Gross Substitutes).

$$\frac{\partial F^i}{\partial r_i} > 0 \quad \text{for all } i = 1, \dots, n$$

and

$$\frac{\partial F^i}{\partial r_j} \leq 0 \quad \text{for } i \neq j (i = 1, \dots, n+1; j = 1, \dots, n).$$

Assumption W.

$$1 > \frac{\partial F^{n+1}}{\partial W} > 0 \quad \text{for all } i = 1, \dots, n$$

and

$$1 > \frac{\partial F^i}{\partial W} \geq 0 \quad \text{for all } i = 1, \dots, n,$$

i.e., wealth has a positive effect on the demand for money and a nonnegative effect on all other assets.

We also have the usual “adding up” conditions imposed by the wealth constraint:

$$\sum_{i=1}^{n+1} \frac{\partial F^i}{\partial r_j} = 0, \quad j = 1, \dots, n+1, \quad (16a)$$

and

$$\sum_{i=1}^{n+1} \frac{\partial F^i}{\partial W} = 1. \quad (16b)$$

We now turn to the special, but economically important case, $n = 2$, when there is a total of $n + 1 = 3$ assets.

4. THE $n = 2$ CASE WITH CONSTANT INCOME

Conventional macroeconomic models with “bonds,” “capital,” and “money” correspond to our $n = 2$ case with $n + 1 = 3$ assets. This special case, obviously of special interest, is the one for which we are able to prove our strongest results.

With money income fixed,⁸ the dynamics are simply (15) for $n = 2$:

$$\dot{p}_i = p_i \alpha_i [F^i(r_1, r_2; p_1 k_1 + p_2 k_2 + k_3) - p_i k_i] \quad (17)$$

⁸ As we discuss in Section 6 below, continuous time models also include the flow constraint that the planned savings rate equal the planned rate of wealth accumulation. With stocks of assets held constant, it follows that wealth accumulation consists only of capital gains on existing assets. By treating income as constant, we are implicitly assuming that all current income is consumed and all capital gains are saved. This assumption is relaxed in Section 6.

or, in implicit form,

$$\begin{aligned} \phi^i(\dot{p}_1, \dot{p}_2, p_1, p_2) \\ = \dot{p}_i - p_i \alpha_i \left[F^i \left(\frac{w_1 + \gamma_1 \dot{p}_1}{p_1}, \frac{w_2 + \gamma_2 \dot{p}_2}{p_2}; p_1 k_1 + p_2 k_2 + k_3 \right) - p_i k_i \right] \\ = 0, \quad i = 1, 2. \end{aligned} \quad (18)$$

Provided the matrix

$$A \equiv \left[\frac{\partial \phi^i}{\partial p_j} \right] = \begin{bmatrix} 1 - \alpha_1 \gamma_1 F_1^1 & -\alpha_1 \gamma_2 F_2^1 p_1 / p_2 \\ -\alpha_2 p_2 F_1^2 \gamma_1 / p_1 & 1 - \alpha_2 \gamma_2 F_2^2 \end{bmatrix} \quad (19)$$

is a P -matrix, (18) can be solved globally for

$$\dot{p}_i = f^i(p_1, p_2).^9 \quad (20)$$

Thus Eqs. (20) exist *if, and only if*,

$$1 - \alpha_i \gamma_i F_i^i > 0, \quad i = 1, 2, \quad (21a)$$

and

$$(1 - \alpha_1 \gamma_1 F_1^1)(1 - \alpha_2 \gamma_2 F_2^2) - \alpha_1 \alpha_2 \gamma_1 \gamma_2 F_1^2 F_1^2 > 0. \quad (21b)$$

A sufficient condition for these two inequalities to be met is:

Condition (S).

$$1 - \alpha_1 \gamma_1 F_1^1 - \alpha_2 \gamma_2 F_2^2 > 0.$$

For $\gamma_i > 0$, this condition in effect imposes an upper bound on the weighted sum of the products of the rate of market adjustment and expectations adjustment in the two independent asset markets. The allowable rates of adjustment vary inversely with the responsiveness of the asset demands with respect to their own real rates of return. Note that if $\gamma_i < 0$ (as in our previous work), then (S) is satisfied trivially.

An alternative sufficient condition is derived from the fact that if a matrix A has a positive dominant diagonal, then it is a P -matrix; see Gale and

⁹ A matrix A is a P -matrix if every principle minor of A is positive. The Gale-Nikaido [7] univalence theorem asserts that the system $F^i(x, y) = 0$, $i = 1, \dots, m$, $x \in R^n$, $y \in R^m$, has a unique solution of the form $y = \phi^i(x)$ for all $x \in R^n$ if the matrix $[\partial F^i / \partial y_j]$ is a P -matrix.

Nikaido [7].¹⁰ Let us suppose that price expectations adapt sufficiently fast with $\beta_i > 1$, so that the γ_i defined by (8) are positive. Then, given assumptions (GS) and (W), from (19) we deduce the sign pattern

$$A \equiv \left[\frac{\partial \phi^i}{\partial p_j} \right] = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad (22)$$

and A has a (column) positive dominant diagonal provided we can find weights $\xi_i > 0$ such that

$$\xi_i(1 - \alpha_1 \gamma_1 F_1^1) + \xi_2 \alpha_2 \gamma_1 p_2 F_1^2 / p_1 > 0 \quad (23a)$$

and

$$\xi_2(1 - \alpha_2 \gamma_2 F_2^2) + \xi_1 \alpha_1 \gamma_2 p_1 F_2^1 / p_2 > 0. \quad (23b)$$

Taking $\xi_i = 1/\alpha_i p_i$ and using (W), we can show that the following condition suffices for (23a) and (23b):

Condition (S')

$$\alpha_i \gamma_i < 1/2F_i^i, \quad i = 1, 2.$$

For $\gamma_i > 0$ this condition imposes an upper bound on the product of the rate of market adjustment and expectations adjustment in the i -th independent market alone, with the upper bound varying inversely with the responsiveness of the asset demand function with respect to its own real rate of return. This condition is more stringent than (S) in that it is sufficient but not necessary for (S). But like (S) it holds trivially if $\gamma_i < 0$.

Accordingly, we have proved that the dynamic price equations may be written in the causal form (20) under either of the sufficient conditions (S) or (S'), given assumptions (GS) and (W) and postulating "rapid price expectation adaptation" with $\gamma_i > 0$ ($\beta_i > 1$).

A short-run dynamic equilibrium is a point $(p^*_1, p^*_2) > 0$ satisfying

$$0 = \dot{p}_i = f^i(p^*_1, p^*_2) \quad (24)$$

or

$$\phi^i(0, 0, p^*_1, p^*_2) = 0. \quad (25)$$

¹⁰ An $n \times n$ matrix X is said to have a (column) dominant diagonal if there exist positive numbers $d_1, \dots, d_n$ such that $d_j |x_{jj}| > \sum_{i \neq j} d_i |x_{ij}|$, $j = 1, \dots, n$. It is said to have a (row) dominant diagonal if $d_i |x_{ii}| > \sum_{j \neq i} d_j |x_{ij}|$, $i = 1, \dots, n$. In either case if $x_{jj} > 0$ (< 0) for all j , we say that the dominant diagonal is positive (negative).

We assume existence of such an equilibrium point (the interested reader can easily derive conditions on the asset demand functions $F^i(\cdot)$ which are sufficient for existence) and ask whether or not such a dynamic equilibrium is always *unique*. In fact any point $(p^*_1, p^*_2) > 0$ satisfying (25) is unique under the assumptions we have already made; to see this fact, simply note that the relevant Jacobian matrix of (25) is

$$J = \left[\frac{\partial \phi^i}{\partial p^{*j}} \right] = -A \quad (26)$$

and therefore J has a dominant negative diagonal. This is sufficient for uniqueness of (p^*_1, p^*_2) ; see Gale and Nikaido [7].

Local stability of the unique short-run dynamic equilibrium (p^*_1, p^*_2) is examined by differentiation of (18):

$$\begin{aligned} & \begin{bmatrix} 1 - \alpha_1 \gamma_1 F_1^1 & -\alpha_1 \gamma_2 F_2^1 p_1 / p_2 \\ -\alpha_2 \gamma_1 F_1^2 p_2 / p_1 & 1 - \alpha_2 \gamma_2 F_2^2 \end{bmatrix} \begin{bmatrix} \partial \dot{p}_1 / \partial p_1 & \partial \dot{p}_1 / \partial p_2 \\ \partial \dot{p}_2 / \partial p_1 & \partial \dot{p}_2 / \partial p_2 \end{bmatrix} \\ &= \begin{bmatrix} p_1 \alpha_1 \{ (F_w^1 - 1) k_1 - F_1^1 w_1 / p_1^2 \} & p_1 \alpha_1 \{ F_w^1 k_2 - F_2^1 w_2 / p_2^2 \} \\ p_2 \alpha_2 \{ F_w^2 k_1 - F_1^2 w_1 / p_1^2 \} & p_2 \alpha_2 \{ (F_w^2 - 1) k_2 - F_2^2 w_2 / p_2^2 \} \end{bmatrix} \quad (27) \end{aligned}$$

which we write as

$$AR = C \quad (28)$$

with A defined previously by (19). Stability hinges upon the properties of the matrix

$$R = A^{-1}C. \quad (29)$$

Given our assumptions, and either condition (S) or (S'), the sign pattern of

$$R = A^{-1}C$$

is identical to that of

$$E \equiv (\text{adj } A) C = [e_{ij}]$$

because A is a P -matrix with $\det A > 0$. But

$$\begin{aligned} e_{11} &= (1 - \alpha_2 \gamma_2 F_2^2) p_1 \alpha_1 \{ (F_w^1 - 1) k_1 - F_1^1 w_1 / p_1^2 \} \\ &\quad + (\alpha_1 \gamma_2 F_2^1 p_1 / p_2) p_2 \alpha_2 \{ F_w^2 k_1 - F_1^2 w_1 / p_1^2 \} \end{aligned}$$

which is negative under conditions (S) or (S') since then $1 - \alpha_2 \gamma_2 F_2^2 > 0$,

while from assumption (W) we have $F_w^1 - 1 < 0$ and from assumption (GS) we have $F_1^1 > 0, F_1^2 > 0, F_2^1 < 0$. Similarly

$$e_{21} = (\alpha_2 \gamma_1 F_1^2 p_2 / p_1) p_1 \alpha_1 \{(F_w^1 - 1) k_1 - F_1^1 w_1 / p_1^2\} \\ + (1 - \alpha_1 \gamma_1 F_1^1) p_2 \alpha_2 \{F_w^2 k_1 - F_1^2 w_1 / p_1^2\} > 0;$$

likewise $e_{22} < 0, e_{12} > 0$. Thus we conclude that S has the sign pattern

$$\begin{bmatrix} - & + \\ + & - \end{bmatrix}, \quad (30)$$

while from (29)

$$\det R = \frac{\det C}{\det A}. \quad (31)$$

Since $\text{tr}(R) < 0$ and $\det(R) > 0$ are necessary and sufficient for stability, and since $\text{tr}(R) < 0$ from (30) while $\det(A) > 0$, in view of (31) it remains to show that $\det C > 0$. We will prove that $[-C]$ has a dominant positive diagonal, implying that

$$\det C = (-1)^2 \det[-C] > 0.$$

Consider the first column of $[-C]$ with weights $\xi_i = 1/p_i \alpha_i > 0$; we require

$$\xi_1 p_1 \alpha_1 \{(1 - F_w^1) k_1 + F_1^1 w_1 / p_1^2\} - \xi_2 p_2 \alpha_2 \{F_w^2 k_1 - F_1^2 w_1 / p_1^2\} > 0$$

or

$$(1 - F_w^1 - F_w^2) k_1 + (F_1^1 + F_1^2) w_1 / p_1^2 > 0. \quad (32)$$

In view of assumption (W) and the “adding up” constraint (16), obviously the required inequality (32) holds.

We summarize our result for this $n = 2$, three asset case in the following theorem.

THEOREM 1. *Consider the dynamic price equations (18) under assumptions (GS), (W), and $\gamma_i > 0$ ($\beta_i > 1$), and conditions (S) or (S'). Then:*

(i) *The equations*

$$\dot{p}_i = f^i(p_1, p_2), \quad i = 1, 2,$$

exist globally.

(ii) Any dynamic equilibrium point $(p^*_1, p^*_2) > 0$ satisfying

$$0 = f^i(p^*_1, p^*_2), \quad i = 1, 2$$

is unique.

(iii) Such a dynamic equilibrium point $(p^*_1, p^*_2) > 0$ is locally asymptotically stable.^{11,12}

5. THE GENERAL CASE WITH CONSTANT INCOME

For the case of general n the dynamic price equations (15) are written as

$$\begin{aligned} & \phi^i(\dot{p}_1, \dots, \dot{p}_n, p_1, \dots, p_n) \\ & \equiv \dot{p}_1 - p_i \alpha_i \left[F^i \left(\frac{w_1 + \gamma_1 \dot{p}_1}{p_1}, \dots, \frac{w_n + \gamma_n \dot{p}_n}{p_n}; p_1 k_1 + \dots + p_n k_n + k_{n+1} \right) \right] \\ & = 0. \end{aligned} \quad (33)$$

The arguments are straightforward generalizations of those made in the previous section, and we will be brief. Differentiation of (33) yields

$$P \Delta A^* \Gamma P^{-1} R - P \Delta C^* = 0 \quad (34)$$

where:

$$P = \text{diag } [p_i]; \quad (35a)$$

$$\Delta = \text{diag } [\alpha_i]; \quad (35b)$$

$$\Gamma = \text{diag } [\gamma_i]; \quad (35c)$$

$$A^* = \begin{bmatrix} \frac{1}{\alpha_1 \gamma_1} - F_1^1 & -F_2^1 & \dots & -F_n^1 \\ \vdots & \vdots & & \vdots \\ -F_1^n & -F_2^n & \dots & \frac{1}{\alpha_n \gamma_n} - F_n^n \end{bmatrix}; \quad (35d)$$

$$R = \begin{bmatrix} \partial \dot{p}_1 / \partial p_1 & \dots & \partial \dot{p}_1 / \partial p_n \\ \vdots & & \vdots \\ \partial \dot{p}_n / \partial p_1 & \dots & \partial \dot{p}_n / \partial p_n \end{bmatrix}; \quad (35e)$$

¹¹ Note that the familiar conditions $\text{tr}(R) < 0$ and $\det(R) > 0$ which we have established are necessary and sufficient for the matrix S , evaluated at (p_1^*, p_2^*) , to have characteristic roots with negative real parts.

¹² It is worth noting that if one assumes that the set of prices is bounded (say by a simplex in Euclidean space) and if the functions f^i satisfy Dierker's "inward pointing" condition near the boundary of the price set, then (iii) implies (ii); see Dierker [6, pp. 9, 114]. Essentially this condition means that if the price vector is near the boundary, prices will tend to move towards the interior of the feasible set of prices. But although the notion of "inward looking" is an intuitively appealing one, in practice it is hard to establish. These comments also apply to the corresponding statements in Theorems 2 and 3 below, and we are grateful to the referee for making this observation.

and

$$C^* = \begin{bmatrix} (F_w^1 - 1) k_1 - F_1^1 w_1 / p_1^2 & \cdots & F_w^1 k_n - F_n^1 w_n / p_n^2 \\ \vdots & & \vdots \\ F_w^n k_1 - F_1^n w_1 / p_1^2 & \cdots & (F_w^n - 1) k_n - F_n^n w_n / p_n^2 \end{bmatrix}. \quad (35f)$$

It is easily seen that the matrix

$$\left[\frac{\partial \phi^i}{\partial p_j} \right] = P \Delta A^* \Gamma P^{-1}, \quad (36)$$

and to prove that the latter is a P -matrix, note that it has a dominant positive diagonal by considering columns with weights $\xi_i = \alpha_i \gamma_i > 0$ under the following condition:

Condition (S)*

$$1 > \sum_{i=1}^n \alpha_i \gamma_i F_i^i.$$

Likewise the row sums are

$$\frac{1}{\alpha_i \gamma_i} - \sum_{j=1}^n F_i^j = [\text{using (16a)}]$$

which using (16a) equals

$$\frac{1}{\alpha_i \gamma_i} - F_1^1 - (F_1^1 + F_1^{n+1}). \quad (37)$$

Thus with assumption (W) all row sums are positive provided the following alternative condition is satisfied:

Condition (S)'*

$$\alpha_i \gamma_i < \frac{1}{2F_i^i}, \quad i = 1, \dots, n.$$

Either (S*) or (S*') imply the global existence of the causal dynamic equations

$$\dot{p}_i = f^i(p_1, \dots, p_n), \quad i = 1, \dots, n. \quad (38)$$

Uniqueness of a dynamic equilibrium $p^* = (p_1^*, \dots, p_n^*) > 0$, satisfying

$$0 = f^i(p_1^*, \dots, p_n^*), \quad i = 1, \dots, n, \quad (39)$$

and

$$\phi^i(0, \dots, 0, p_1^*, \dots, p_n^*), \quad i = 1, \dots, n, \quad (40)$$

hinges on the Jacobian matrix

$$J^* \equiv \left[\frac{\partial \phi^i}{\partial p_j^*} \right] = -P \Delta A^* \Gamma P^{-1}. \quad (41)$$

As in the previous section, the assumptions already made suffice for the uniqueness of p^* , since $-J^*$ is a P -matrix.

Local stability is analyzed by solving (34) for

$$\left[\frac{\partial \dot{p}_i}{\partial p_j} \right] \equiv R = P \Gamma^{-1} (A^*)^{-1} C^* \quad (42)$$

and considering the properties of R . The corresponding result obtained for the $n = 2$ case rested critically upon establishing the sign pattern of

$$\begin{bmatrix} - & + \\ + & - \end{bmatrix}$$

for R . Unfortunately this sign pattern does not generalize to arbitrary n without imposing additional restrictions. The problem is analogous to that of establishing generalizations of the Stolper-Samuelson and Rybczynski theorems beyond the two-factor-two-good case. Such generalizations depend upon establishing sign patterns similar to the above or relevant matrices; see, e.g., Inada [9], Uekawa *et al.* [21].

Our procedure involves first establishing conditions that will ensure the desired sign pattern for $D = [A^*]^{-1} C^*$ and then showing that D has a negative dominant diagonal. Then since both P and Γ are diagonal, the negative dominant diagonal property extends to R and stability follows.

Consider $D = [A^*]^{-1} C^*$. It can be shown from (35f) that $[-C^*]$ has the sign pattern

$$\begin{bmatrix} + & & & - \\ & \cdot & & \\ & & \cdot & \\ - & & & + \end{bmatrix}$$

and that it has a (column) positive dominant diagonal.¹³ It therefore follows that $(-C^*)^T$ is a Leontief matrix and that $[-C^*]^{-1} \geq 0$.¹⁴ Moreover, given

¹³ This can be established by summing the columns of C^* and using the adding up conditions (21).

¹⁴ A square matrix $X = [x_{ij}]$ is said to be a Leontief matrix if $x_{ij} \leq 0$, $i \neq j$, and if $Xu > 0$ for some $u > 0$, in which case it has a (row) positive dominant diagonal. It then can be established that $X^{-1} \geq 0$; see McKenzie [12].

condition (S'), $A^* \geq 0$, so that $-D^{-1} = [-C^*]^{-1} A^* \geq 0$. We now introduce the following:

DEFINITION. A square matrix $X \geq 0$ is said to have the Minkowski property if and only if its inverse X^{-1} has nonnegative diagonal elements and nonpositive off-diagonal elements.

With this definition we are in a position to apply the following theorem due to Uekawa *et al.* [21].

THEOREM. Let X be a nonnegative square matrix and Y be a Leontief matrix of order n . Then the matrix $A = Y^{-1}X$ has the Minkowski property if and only if X, Y satisfy the following condition.

Condition (UKW). For any non-empty proper subset J of the set $\{1, \dots, n\}$ and any given positive $\bar{u}_J$, the equation

$$X_J u_J - X_J \bar{u}_J = Y_J v_J - Y_J \bar{v}_J$$

has a positive solution $u_J > 0, v = (v_J, v_J) > 0$, where X_J is a submatrix of the matrix X which includes the j -th column vector of $X, j \in J, x_j$ is a sub-vector of the vector x which consists of the j -th component of $x, j \in J$.

Setting $X = A^*, Y = [-C^*]$ in the theorem, it follows that $[-C^*]^{-1} A^*$ has the Minkowski property, i.e. $[A^*]^{-1} [-C^*]$ has nonnegative diagonal elements and nonpositive off-diagonal elements if and only if $A^*, [-C^*]$ satisfy condition (UKW). That is, $[A^*]^{-1} C^*$ has the sign pattern

$$\begin{bmatrix} - & & & + \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ + & & & - \end{bmatrix}$$

if and only if $A^*, [-C^*]$ satisfy condition (UKW).¹⁵

Unfortunately, the critical condition (UKW) does not appear to lend itself to intuitive interpretation in the present context. However, it indicates clearly that the desired sign pattern on the matrix $[A^*]^{-1} C^*$ does not in general extend beyond the case $n = 2$ without the imposition of more stringent conditions.¹⁶

Assuming that $D = [A^*]^{-1} C^*$ does have the above sign pattern, we now

¹⁵ Some of the elements of $[A^*]^{-1} C^*$ may be zero, but this does not affect our argument.

¹⁶ The matrix $D^{-1} = [-C^*]^{-1} A^* \geq 0$. Markham [10] has established conditions under which the inverse of a nonnegative matrix has the Minkowski property. Unfortunately, the required conditions appear too complex to be useful in our context.

proceed to show that D has a negative dominant diagonal. Consider the equation

$$A^*D = C^*. \quad (43)$$

Equating the first column yields

$$\begin{array}{rclcl} (1/\alpha_1 - F_1^1) d_{11} - F_2^1 d_{21} & \cdots & -F_n^1 d_{n1} & = & c_{11}^* \\ -F_2^1 d_{11} + (1/\alpha_2 - F_2^2) d_{21} & \cdots & -F_n^2 d_{n1} & = & c_{21}^* \\ \vdots & & \vdots & & \\ -F_1^n d_{11} - F_2^n d_{21} & \cdots & + (1/\alpha_n - F_n^n) d_{n1} & = & c_{n1}^* \end{array}$$

where d_{ij} , c_{ij}^* denote the individual elements of D , C^* , respectively. Summing these equations and using the adding up condition (16), we obtain

$$(1/\alpha_1 + F_1^{n+1}) d_{11} + \cdots + (1/\alpha_n + F_n^{n+1}) d_{n1} = \sum_j c_{j1}^* < 0. \quad (44)$$

Now let $\theta_i = (1/\alpha_i + F_i^{n+1})$. It can be shown that $\theta_i > 0$ provided $(S^{*'})$ holds, and hence (44) can be rewritten as

$$\theta_1 |d_{11}| > \theta_2 |d_{21}| + \cdots + \theta_n |d_{n1}|. \quad (45)$$

When this procedure is repeated for any arbitrary column in (43), the same positive weights θ_i are obtained. Thus we deduce that the diagonal elements of D are *strictly* negative and that D has a negative dominant diagonal. It therefore follows that since P , Γ , are diagonal, R also must have a strictly negative dominant diagonal and therefore that it is stable. We may summarize our results for this general case in the following theorem.

THEOREM 2. *Consider the dynamic price equations (33) under assumptions (GS), (W). Then*

- (i) *If in addition either (S^*) or $(S^{*'})$ hold, the equations*

$$\dot{p}_i = f^i(p_1, \dots, p_n), \quad i = 1, \dots, n,$$

exist globally.

- (ii) *If in addition either (S^*) or $(S^{*'})$ hold, any dynamic equilibrium point $(p_1^*, \dots, p_n^*)$ satisfying*

$$0 = f^i(p_1^*, \dots, p_n^*), \quad i = 1, \dots, n,$$

is unique.

(iii) *If in addition (S*) and (UKW) hold, such a dynamic equilibrium point $(p^*_1, \dots, p^*_n)$ is locally asymptotically stable.*

6. THE $n = 2$ CASE WITH ENDOGENOUS INCOME

The important case of three assets ("bonds," "capital," and "money") analyzed in Section 4 will now be generalized to allow for an endogenous determination of income, $y(t)$. This involves the introduction of an additional relationship, namely the *flow* constraint equating the planned rate of wealth accumulation with the planned rate of savings, $s(t)$. Unless very special assumptions are made (see footnote 8 above), this flow constraint properly forms part of a continuous time model; see e.g. May [11] and Turnovsky [19]. With the stocks of assets fixed in the short run, this constraint is given by

$$s(t) = \left. \frac{\partial W^d(s, t)}{\partial s} \right|_{s=t} = \sum_{i=1}^{n+1} \pi_{i,1}(t, t) k_i \quad (46)$$

where $W^d(s, t)$ denotes the planned stock of wealth formed at time t for time $s \geq t$. As shown elsewhere, the flow constraint (46) holds if and only if disposable income is defined to be¹⁷

$$y^D(t) \equiv y(t) + \sum_{i=1}^{n+1} p_i(t) r_i(t) k_i. \quad (47)$$

To complete the model we must specify the savings function, which we postulate to depend upon disposable income and wealth:

$$s(t) = s[y^D(t), W(t)], \quad 1 > \frac{\partial s}{\partial y^D} \equiv s_1 > 0, \quad \frac{\partial s}{\partial W} \equiv s_2 < 0 \quad (48)$$

with y^D defined by (47). This relationship is in nominal terms. Thus the level of income is one of the endogenous variables in our model. Because the stocks of all assets, including productive capital goods, are fixed in our short run, y is proportional to money wage income. Hence if the money wage rate is also fixed, y and employment are proportional and our model determines the short-run employment in the usual Keynesian fashion.

Dropping the t for the time notation, we find that the flow constraint may be written as

$$\begin{aligned} & s[y + (w_1 + \gamma_1 \dot{p}_1) k_1 + (w_2 + \gamma_2 \dot{p}_2) k_2, p_1 k_1 + p_2 k_2 + k_3] \\ & = \gamma_1 \dot{p}_1 k_1 + \gamma_2 \dot{p}_2 k_2. \end{aligned} \quad (49)$$

¹⁷ See Burmeister and Turnovsky [5] and Turnovsky [20].

As $1 > s_1 > 0$ by assumption, (49) may be solved for

$$y = y(\dot{p}_1, \dot{p}_2, p_1, p_2) \quad (50)$$

with

$$\frac{\partial y}{\partial \dot{p}_i} = \gamma_i \frac{(1 - s_1)}{s_1} > 0 \quad \text{for } \gamma_i > 0, \quad i = 1, 2, \quad (51a)$$

and

$$\frac{\partial y}{\partial p_i} = -\frac{s_2 k_i}{s_1} > 0, \quad i = 1, 2. \quad (51b)$$

The asset demand functions (13) now depend upon income and are generalized to

$$F^i(r_1, r_2; y, W) = (p_i k_i)^D, \quad i = 1, 2, 3, \quad (13')$$

and we have an additional "adding up" constraint, namely

$$F_y^1 + F_y^2 + F_y^3 = 0. \quad (52)$$

We shall assume a positive transactions demand for money:

Assumption (M).

$$F_y^3 > 0, F_y^1 \leq 0, F_y^2 \leq 0.$$

As before, we consider the implicit functions

$$\begin{aligned} \phi^i(\dot{p}_1, \dot{p}_2, p_1, p_2) \\ = \dot{p}_i - p_i \alpha_i \left[F^i \left(\frac{w_1 + \gamma_i p_i}{p_1}, \frac{w_2 + \gamma_i \dot{p}_2}{p_2}, y, p_1 k_1 + p_2 k_2 + k_3 - p_i k_i \right) \right] \\ = 0, \quad i = 1, 2, \end{aligned} \quad (53)$$

with y given by (50). We now must show that the matrix

$$\bar{A} = \begin{bmatrix} 1 - \alpha_1 \gamma_1 F_1^1 - p_1 \alpha_1 F_y^1 \frac{\partial y}{\partial \dot{p}_1} & -\gamma_2 \alpha_1 p_1 F_2^1 / p_2 - p_1 \alpha_1 F_y^1 \frac{\partial y}{\partial \dot{p}_2} \\ -\gamma_1 \alpha_2 p_2 F_1^2 / p_1 - p_2 \alpha_2 F_y^2 \frac{\partial y}{\partial \dot{p}_1} & 1 - \alpha_2 \gamma_2 F_2^2 - p_2 \alpha_2 F_y^2 \frac{\partial y}{\partial \dot{p}_2} \end{bmatrix} \quad (54)$$

is a P -matrix, which is sufficient for the global existence of the causal dynamic equations

$$\dot{p}_i = f^i(p_1, p_2), \quad i = 1, 2. \quad (55)$$

To do this, in addition to our previous assumptions (GS), (W), and (M), we also need the following symmetry condition which does not seem empirically too unrealistic:

Assumption (T).

The transactions demand for money is financed by equal reductions in the values of the holdings for the other two assets, i.e.

$$\theta \equiv F_y^1 = F_y^2 \leq 0, F_y^3 = -2\theta = -2F_y^1 = -2F_y^2 \geq 0.$$

Now, using our previous condition (S') that

$$0 < \alpha_i \gamma_i F_i^i < 2\alpha_i \gamma_i F_i^i < 1,$$

obviously the diagonal elements of $\bar{A}$ are positive in view of assumption (M) which implies $F_y^i \leq 0, i = 1, 2$. Also by direct calculation we can show that our assumptions suffice to ensure $\det(\bar{A}) > 0$, as is required for $\bar{A}$ to be a P -matrix.

Uniqueness of a dynamic equilibrium point $(p^*_1, p^*_2) > 0$ follows as in Section 4 since the Jacobian matrix

$$\left[\frac{\partial \phi^i(0, 0, p_1^*, p_2^*)}{\partial p_j^*} \right] = -\bar{A}. \quad (56)$$

Stability depends upon the matrix

$$\bar{S} = \bar{A}^{-1} \bar{C} \quad (57)$$

where:

$$\bar{S} = \begin{bmatrix} \frac{\partial \dot{p}_1}{\partial p_1} & \frac{\partial \dot{p}_1}{\partial p_2} \\ \frac{\partial \dot{p}_2}{\partial p_1} & \frac{\partial \dot{p}_2}{\partial p_2} \end{bmatrix}$$

and

$$\bar{C} = \begin{bmatrix} p_1 \alpha_1 [(F_W^1 - 1) k_1 - F_1^1 w_1 / p_1^2] + p_1 \alpha_1 \theta \frac{\partial y}{\partial p_1} \\ p_2 \alpha_2 [F_W^2 k_1 - F_1^2 w_1 / p_1^2] + p_2 \alpha_2 \theta \frac{\partial y}{\partial p_1} \\ p_1 \alpha_1 [F_W^1 k_2 - F_2^1 w_2 / p_2^2] + p_1 \alpha_1 \theta \frac{\partial y}{\partial p_2} \\ p_2 \alpha_2 [(F_W^2 - 1) k_2 - F_2^2 w_2 / p_2^2] + p_2 \alpha_2 \theta \frac{\partial y}{\partial p_2} \end{bmatrix}.$$

By direct calculation using the previous assumptions and the methods employed above, it is easily verified that $\det(\bar{S}) > 0$ and that the diagonal elements of $\bar{S}$ are negative, thereby establishing local asymptotic stability of a unique dynamic equilibrium point $(p^*_1, p^*_2) > 0$.

We summarize the results proved in this section as:

THEOREM 3. *Consider the dynamic prices equation (53) under assumptions (GS), (W), (M), (T), and condition (S'). Then*

(i) *The equations*

$$\dot{p}_i = f^i(p_1, p_2), \quad i = 1, 2,$$

exist globally.

(ii) *Any dynamic equilibrium point $(p^*_1, p^*_2) > 0$ satisfying*

$$0 = f^i(p^*_1, p^*_2), \quad i = 1, 2,$$

is unique.

(iii) *Such a dynamic equilibrium point $(p^*_1, p^*_2) > 0$ is locally asymptotically stable.*

7. CONCLUSIONS

The assumptions we have required to prove stability seem plausible, although they are obviously stringent. Of course, they are only sufficient conditions, and as an empirical matter our model may be stable even if some of our assumptions are violated. A problem deserving econometric analysis is the question of whether or not our sufficient conditions are approximately valid. In any event, short-run price dynamics are sufficiently complex that the conventional assumption that short-run dynamic equilibria prevail at every instant is at best dubious.

The novel feature of our approach is that we have parameterized both the "degree of rationality" of price expectations and the "speed of market adjustment." Since these parameters enter crucial conditions as a product, they cannot be identified from the structural equations of the model. This result stems from the fact that price expectations are unobservable, while the evolution of actual prices is determined by the simultaneous solution of expectations equations and portfolio equilibrium equations in a general equilibrium framework. In our opinion this feature itself is a notable advancement over much previous work since it seems unreasonable to us that price expectations are generated independently of other considerations such as asset market equilibrium.

Most importantly, we now have established that provided asset markets do not clear instantaneously, "rational" price expectations in which actual and expected prices always move in the same direction are consistent with stability. This result is to be contrasted with conventional adaptive-type expectations mechanisms in which the sufficient stability conditions entail expected and actual prices *always* moving in opposite directions, a situation which is obviously untenable and "irrational"!

Finally, we note that our formulation holds promise for providing a cornerstone to the construction of a dynamic macroeconomic model in which the supplies of assets are determined endogenously and move over time. We have taken a first albeit crucial step toward this ultimate objective.

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