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Perfect Foresight, Expectational Consistency, and Macroeconomic Equilibrium

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This paper begins by introducing three alternative properties of expectations: weak consistency, strong consistency, perfect foresight. These concepts are then used to consider the relationship between beginning-of-period (stock) equilibrium and end-of-period (flow) equilibrium for both discrete and continuous time. We show that in the former case the consistency between them requires not only that there be perfect foresight in predicting certain relevant variables but also that there be no accumulation of assets. In the latter case the relationship between the two equilibria rests on much weaker conditions. They are equivalent provided expectations satisfy our assumption of weak consistency.

For nearly 40 years now, economists have been concerned with the distinction between stock equilibrium and flow equilibrium. Most recently Foley (1975) has analyzed this question and has shown how the relationship between these two equilibrium concepts depends crucially upon whether or not there is perfect foresight in the formation of certain relevant expectations. Foley's paper is an extremely interesting contribution and significantly advances our understanding of these important issues.

In this paper we wish to extend and make more explicit some of the

This paper was written while Burmeister was visiting the Faculty of Economics, Australian National University. He wishes to express his thanks to the ANU for making an enjoyable visit possible. The study is an outgrowth of previous work by Burmeister and Graham (1974), Burmeister and Turnovsky (1976), and Turnovsky (1977). We wish to acknowledge the helpful comments of two referees.

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issues raised by Foley. Specifically, we consider the relationship between beginning-of-period (stock) equilibrium and end-of-period (flow) equilibrium for strictly positive periods and also as the period becomes infinitesimally small. We show that in the latter case the relationship between the two equilibria rests on much weaker conditions than perfect foresight. They turn out to be equivalent, provided expectations are weakly consistent in the sense we define below.

Part of the confusion in the literature on stocks and flows, expectations, and plans stems from the fact that very often these quantities must be modeled as functions of two time variables— s , t , say—to specify precise meanings. For example, to completely describe expectations one should indicate (1) the time when they are formed (t), and (2) the time for which they are made (s), $s \geq t$. The same applies to the specification of plans. Similarly, flow variables should be written as functions of two time variables, indicating the beginning and the end of the period over which they are measured. This point has very rarely been observed by previous authors. The result is that when one moves to continuous time, what are really partial derivatives often are misinterpreted as total derivatives, and this fact has important analytical consequences. Accordingly, in our presentation below, we are careful to avoid this confusion by defining functions in the way just indicated.

Our analysis has three parts. In Section I we define three alternative properties of expectations: weak consistency, strong consistency, and perfect foresight. All of these notions arise in different contexts (including in particular the present), and the distinction between them is not always made clear. Section II uses these concepts in relating the two equilibrium concepts using both discrete and continuous time. In Section III we consider the dynamics of the equilibrium as these expectations evolve over time. Most attention is devoted to continuous time. We show that when one takes the limit, both equilibrium conditions involve partial differential equations in expectations. Combining this with an appropriate expectation assumption, which provides another partial differential equation, yields a total differential equation and completely determines the evolution of the system over time.¹

I. Three Related Properties of Expectations

Let $q(t)$ = actual value of some economic variable (say a price) at time t , and $q^*(t + h, t)$ = expectation formed at time t for the variable at time $t + h$, $h \geq 0$. Both $q(t)$ and $q^*(t + h, t)$ are assumed to be twice

¹ This approach was first used by Burmeister and Graham (1974), where price expectations equations, together with portfolio equilibrium conditions, determine the evolution of prices (see esp. pp. 326–27). The idea is significantly generalized by Burmeister and Turnovsky (1976).

continuously differentiable for all h and t . We are therefore ruling out the possibility of any discontinuous jumps in these variables.

We shall assume throughout the paper that expectations satisfy the following condition, "weak consistency":

$$W \quad q^*(t, t) = q(t). \quad (1)$$

This condition asserts that the expectation formed at time t for that same instant must equal the actual value prevailing at that time. Underlying it is the assumption that forecasters have instantaneous access to information so that at time t they know $q(t)$. Under these conditions any rational forecasting mechanism must satisfy W . If such information is not available instantaneously, then W need not hold. Further properties and implications of W are discussed at length in Burmeister and Turnovsky (1976).

In addition we define the following two concepts: "strong consistency,"

$$S \quad \lim_{h \rightarrow 0} \left[\frac{q^*(t + h, t) - q(t)}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{q(t + h) - q(t)}{h} \right] \equiv \dot{q}(t), \quad (2)$$

and "perfect foresight,"

$$P \quad q^*(t + h, t) = q(t + h) \quad \text{for all } h \geq 0. \quad (3)$$

Strong consistency refers to forecasts of instantaneous rates of change, which given W are as defined in (2). If S holds, the expected instantaneous rate of change at time t must equal the actual instantaneous rate of change at that time. Perfect foresight concerns forecasting the future, and under P all future prices are predicted perfectly.

There is a simple ordering between these three concepts: $P \Rightarrow S \Rightarrow W$ but not the reverse. This can be seen as follows. From P , $q^*(t + h, t) = q(t + h)$ for all h . Inserting the latter into the left-hand side of (2), the condition S is obviously satisfied. Furthermore, S can be written as

$$\lim_{h \rightarrow 0} \left[\frac{q^*(t + h, t) - q(t + h)}{h} \right] = 0. \quad (2')$$

Since $h \rightarrow 0$, it follows that the numerator of (2') must also tend to zero, implying $q^*(t, t) = q(t)$; that is W . Thus the ordering $P \Rightarrow S \Rightarrow W$ is established.²

It is clear that in general the reverse ordering does not hold. For example, the fact that W holds does not imply S ; unless $q^*(t + h, t) \rightarrow q(t)$ at a faster-than-linear rate, the limit on the left-hand side of (2) does

² Note that perfect foresight with respect to future rates of change,

$$\frac{q^*(t + h, t) - q(t)}{h}, \quad h \geq 0,$$

is equivalent to P .

not exist. Likewise S and W , which describe instantaneous expectations, obviously have no implications for expectations pertaining to finite points of time in the future.

The quantity $[q^*(t + h, t) - q(t)]/h$ measures the expected rate of change of price over the finite interval h .³ Taking the limit as $h \rightarrow 0$ of this expression and using W , which as we have just seen is implied by S , yields the partial derivative $q_1^*(t, t)$. This expression is therefore the correct limiting measure for the expected instantaneous rate of change of price at a given time t . Accordingly S can be rewritten as

$$q_1^*(t, t) = \dot{q}(t). \quad (4)$$

Differentiating (1) with respect to q yields

$$\dot{q}^*(t, t) = q_1^*(t, t) + q_2^*(t, t) = \dot{q}(t). \quad (5)$$

Given the above interpretation of q_1^* , it follows that q_2^* equals the difference between the actual and anticipated rate of change of prices at time t ; that is, it measures the *unanticipated* rate of change of prices. Using (4), we see that the assumption of strong consistency is equivalent to

$$q_2^*(t, t) = 0. \quad (6)$$

To interpret this derivative further and to relate it to P , we apply the first mean value theorem to $q^*(t + h, t)$. Invoking the weak consistency condition W , we may write

$$q(t + h) - q^*(t + h, t) = q_2^*(t + h, \xi)h, \quad t < \xi < t + h. \quad (7)$$

Hence $q_2^*(t + h, \xi)$, which measures how the prediction for the fixed time $t + h$ is revised with the variable planning date ξ , is proportional to the forecast error and reflects the extent to which there is imperfect foresight. With perfect foresight it follows from (3) that

$$q_2^*(t + h, \xi) \equiv 0. \quad (8)$$

Letting $h \rightarrow 0$, in (8), we obtain $q_2^*(t, t) = 0$, from which we deduce that our assumption of strong consistency is identical with what is frequently referred to as "perfect myopic foresight."

However, there seems to be some confusion in the way perfect myopic foresight is defined in the literature. Many authors define it by the following relationship involving total derivatives:

$$\frac{\dot{q}^*(t, t)}{q(t)} = \frac{\dot{q}(t)}{q(t)}. \quad (5')$$

³ If W did not hold, the appropriate definition of the expected rate of change of price would be $[q^*(t + h, t) - q^*(t, t)]/h$.

But except for a constant of integration this is equivalent to W and is certainly implied by it. The correct relationship involves only the partial derivative of q^* and is given by (4), or alternatively,

$$\frac{q_1^*(t, t)}{q(t)} = \frac{\dot{q}(t)}{q(t)}. \quad (4')$$

As we have just seen this is a stronger condition than W . The difference between (4') and (5') is a subtle one and is not readily apparent until one introduces two time arguments in the expectational function to identify precisely the date and time horizon of the forecasts.⁴

II. Application to Macroeconomic Equilibrium

We shall apply these expectation notions to analyze the equilibrium of the simple macroeconomic model considered by Foley, paying particular attention to the specification of asset market equilibrium. We consider just two assets, money and homogeneous physical capital, the prices of which in terms of consumption goods are $p(t)$ and $q(t)$, respectively. Likewise the corresponding expected prices are denoted by $p^*(t + h, t)$ and $q^*(t + h, t)$. Capital pays a dividend $\bar{d}$ (in capital) at time t , which for simplicity is fixed and known; money pays a zero dividend. Thus the real (expected) rates of return on holding money and capital over the period $(t, t + h)$ are defined, respectively, by

$$r_1^*(t + h, t) = \frac{p^*(t + h, t) - p(t)}{hp(t)} \quad (9a)$$

and

$$r_2^*(t + h, t) = \frac{\bar{d}}{q(t)} + \frac{q^*(t + h, t) - q(t)}{hq(t)}. \quad (9b)$$

We define wealth at time t , $W(t)$, by

$$W(t) = p(t)M(t) + q(t)K(t), \quad (10)$$

where $M(t)$ and $K(t)$ denote the stocks of money and capital at time t . Similarly, planned holding of wealth for time $t + h$, made at time t , $W^d(t + h, t)$, is defined by

$$W^d(t + h, t) = p^*(t + h, t)M^d(t + h, t) + q^*(t + h, t)K^d(t + h, t) \quad (11)$$

⁴ Foley is not very explicit on his definition of perfect myopic foresight. It is possible to interpret his expectations as all being taken at the fixed time zero, in which case his total derivatives are consistent with our partial derivatives. That is, $q_2^*(0, 0) = 0$ because the time at which the expectation is being taken is being held constant.

where $M^d(t + h, t)$ = planned holding of money for time $t + h$, formed at time t , and $K^d(t + h, t)$ = planned holding of capital for time $t + h$, formed at time t .

Provided disposable income is defined appropriately one can derive⁵

$$S(t + h, t) = W^d(t + h, t) - W(t) \quad (12)$$

where

$$S(t + h, t) = \int_t^{t+h} s(\tau, t) d\tau$$

denotes the planned accumulated savings flow over the period $(t, t + h)$, and $s(\tau, t)$ is the planned rate of savings for time τ made at time t . Letting $h \rightarrow 0$, it follows that $S(t, t) = 0$. Since $s(t, t)$, the savings rate, is presumably nonzero and finite, saving is still occurring at time t . The condition $S(t, t) = 0$ simply means that the *accumulation* of this flow over a zero time period must be zero. Thus as $h \rightarrow 0$ in (12), we deduce

$$W^d(t, t) = W(t), \quad (13)$$

that is, as the planning horizon tends to zero desired and actual wealth must coincide.

Given these savings plans $S(t + h, t)$, $W^d(t + h, t)$ also equals the *expected* wealth for time $t + h$ formed at time t . Writing this as $W^*(t + h, t)$, we also require⁶

$$\begin{aligned} W^*(t + h, t) &= p^*(t + h, t)M^*(t + h, t) + q^*(t + h, t)K^*(t + h, t) \\ &= p^*(t + h, t)M^d(t + h, t) + q^*(t + h, t)K^d(t + h, t) \\ &= W^d(t + h, t) \end{aligned} \quad (14)$$

⁵ The household sector budget constraint over the time period $(t, t + h)$ is given by

$$\begin{aligned} p(t)M(t) + q(t)K(t) + Y(t + h, t) \\ + [p^*(t + h, t) - p(t)]M(t) + [q^*(t + h, t) - q(t) + dh]K(t) \\ = p^*(t + h, t)M^d(t + h, t) + q^*(t + h, t)K^d(t + h, t) + C(t + h, t) + U(t + h, t) \end{aligned}$$

where $Y(t + h, t)$ = accumulated income flow over the period $(t, t + h)$, $C(t + h, t)$ = accumulated consumption flow over the period $(t, t + h)$, $U(t + h, t)$ = accumulated flow of tax payments over the period $(t, t + h)$. This equation asserts that the household sector's consumption plans and planned holding of assets for time $t + h$ are constrained by (1) initial wealth, (2) exogenous income, (3) expected returns on holding money and capital, (4) tax commitments. From the identity $Y(t + h, t) \equiv C(t + h, t) + S(t + h, t) + U(t + h, t)$, equation (12) is obtained if and only if disposable income $Y^D(t + h, t)$ is defined by

$$\begin{aligned} Y^D(t + h, t) &= Y(t + h, t) - U(t + h, t) + [p^*(t + h, t) - p(t)]M(t) \\ &\quad + [q^*(t + h, t) - q(t) + dh]K(t). \end{aligned}$$

For further discussion of this and the implications for the consistent formulation of continuous time macroeconomic models see Turnovsky (1977).

⁶ Note that these convenient properties on the expectations quantities apply only because the model is nonstochastic so that we are dealing with point expectations.

where $M^*(t + h, t)$ and $K^*(t + h, t)$ are defined analogously to $W^*(t + h, t)$. This relationship is what Foley calls the "balance sheet constraint," and it asserts that individuals must plan to hold their expected wealth for any point of time in the form of money or capital. We differ from Foley slightly in that we start by specifying $S(t + h, t)$ and use the resulting $W^d(t + h, t)$ implied by (12) to define $W^*(t + h, t)$. We could have adopted Foley's procedure of beginning with (14) and using (12) to define an implicit savings plan. There is no substantive difference between these two approaches; they are equivalent.

Before proceeding, we must interpret the partial derivatives of $W^d(u, t)$ for arbitrary u, t . The first partial derivative $W_1^d(u, t)$ measures the changes in the desired holding of wealth as the planning horizon increases, at a given planning date t . On the other hand, $W_2^d(u, t)$ describes the change in the amount of wealth investors wish to have at a given specified date in the future as the planning date changes. Equations (13) and (14) enable us to write

$$\begin{aligned} W^d(t + h, t + h) - W^d(t + h, t) \\ = W^*(t + h, t + h) - W^*(t + h, t), \end{aligned} \quad (15)$$

and application of the mean value theorem to both sides of (15) yields

$$\begin{aligned} W_2^d(t + h, \xi_1)h = W_2^*(t + h, \xi_2)h \quad t < \xi_1 < t + h, \\ t < \xi_2 < t + h. \end{aligned} \quad (16)$$

Cancelling h and taking the partial derivative of W^* implies

$$\begin{aligned} W_2^d(t + h, \xi_1) = p_2^*(t + h, \xi_2)M^*(t + h, \xi_2) \\ + p^*(t + h, \xi_2)M_2^*(t + h, \xi_2) \\ + q_2^*(t + h, \xi_2)K^*(t + h, \xi_2) \\ + q^*(t + h, \xi_2)K_2^*(t + h, \xi_2). \end{aligned} \quad (17)$$

Thus $W_2^d(t + h, \xi_1)$ reflects the extent to which there is imperfect foresight in predicting stocks of assets, as well as in predicting asset prices. Hence if there is perfect foresight in predicting all these quantities $W_2^d(t + h, \xi_1) \equiv 0$. As we move to continuous time and let $h \rightarrow 0$, both ξ_1 and $\xi_2 \rightarrow t$ and (17) becomes

$$\begin{aligned} W_2^d(t, t) = p_2^*(t, t)M^*(t, t) + p^*(t, t)M_2^*(t, t) \\ + q_2^*(t, t)K^*(t, t) + q^*(t, t)K_2^*(t, t). \end{aligned} \quad (17')$$

In this case the weaker condition of strong consistency in all predictions will ensure $W_2^d(t, t) = 0$.

With these notions we now define end-of-period equilibrium and beginning-of-period equilibrium. Following Foley, we say that an asset market will be in end-of-period equilibrium if the quantity of that asset people plan at time t to hold at time $t + h$ equals the supply at time $t + h$. Moreover, these planned quantities are functions of expected wealth at time $t + h$ and expected returns over the following period $(t + h, t + 2h)$.

Denoting the demand functions

$$p^*(t+h, t)M^d(t+h, t) \quad \text{and} \quad q^*(t+h, t)K^d(t+h, t)$$

by L and J , respectively, end-of-period equilibrium is defined by

$$E \left\{ \begin{array}{l} J \left[W^d(t+h, t), \frac{p^*(t+2h, t) - p^*(t+h, t)}{hp^*(t+h, t)}, \right. \\ \quad \left. \frac{\bar{d}}{q^*(t+h, t)} + \frac{q^*(t+2h, t) - q^*(t+h, t)}{hq^*(t+h, t)} \right] \quad (18a) \\ = q^*(t+h, t)K(t+h) \\ = q^*(t+h, t)\{K(t) + I[q(t), K(t)]h\} \\ L \left[W^d(t+h, t), \frac{p^*(t+2h, t) - p^*(t+h, t)}{hp^*(t+h, t)}, \right. \\ \quad \left. \frac{\bar{d}}{q^*(t+h, t)} + \frac{q^*(t+2h, t) - q^*(t+h, t)}{hq^*(t+h, t)} \right] \quad (18b) \\ = p^*(t+h, t)M(t+h) \\ = p^*(t+h, t)[M(t) + mh]. \end{array} \right.$$

Note that since we are concerned with planning decisions made at time t , the relevant expectations for time $(t+2h)$ are those formed at time t . Moreover, the relevant expected rates of capital gain over the period $(t+h, t+2h)$ are the expected gains deflated by $p^*(t+h, t)$, $q^*(t+h, t)$, respectively.

The right-hand side equality of (18a) defines the supply of capital at time t . This is specified to be a function of investment flow I , which itself depends upon the price of capital at time t and the existing stock of capital. The quantity of money at time $(t+h)$ is assumed to be an exogenous government policy decision with m denoting the rate at which money is being issued. We assume that asset demands vary positively with their own rates of return and negatively with the rate of return on the other asset, reflecting an assumption that they are gross substitutes. Thus we specify

$$\begin{aligned} 0 < J_1 < 1, J_2 < 0, J_3 > 0 \\ 0 < L_1 < 1, L_2 > 0, L_3 < 0 \end{aligned} \quad (19a)$$

together with the “adding up” conditions,

$$L_1 + J_1 = 1, J_2 + K_2 = 0, J_3 + L_3 = 0. \quad (19b)$$

In addition there is a consumption flow equation which, like Foley, we can write as

$$Q[K(t), q(t)]h = Ch \quad (20)$$

where Q denotes the flow of consumption goods dependent upon $K(t)$ and $q(t)$, and C is consumption demand, the arguments of which need not be specified precisely.⁷ As Foley shows, because of Walras's law only two of the three equations (18a), (18b), and (20) are independent, and these two independent equations determine $p(t)$ and $q(t)$, given expectations.

Beginning-of-period equilibrium, on the other hand, requires that wealth holders be content to hold the existing stock of assets available at the beginning of the period. In this case the asset demand functions depend upon current wealth at time t , together with the expected rates of return over the immediate period $(t, t + h)$. Thus beginning-of-period equilibrium is defined by

$$B \left\{ \begin{array}{l} J \left[W(t), \frac{p^*(t+h, t) - p(t)}{hp(t)}, \frac{\bar{d}}{q(t)} + \frac{q^*(t+h, t) - q(t)}{hq(t)} \right] \\ \quad = q(t)K(t) \\ L \left[W(t), \frac{p^*(t+h, t) - p(t)}{hp(t)}, \frac{\bar{d}}{q(t)} + \frac{q^*(t+h, t) - q(t)}{hq(t)} \right] \\ \quad = p(t)L(t). \end{array} \right. \quad \begin{array}{l} (21a) \\ (21b) \end{array}$$

Because of the wealth constraint only one of these two equations is independent, the other independent equation in this case being the flow relationship (20).

In both cases we shall treat the capital equilibrium condition together with the flow equilibrium relation as being the two independent equations determining the equilibrium of the system. We therefore need not consider the money market explicitly. Furthermore, we can simplify matters, without any significant loss of generality, if we assume a simple consumption function of the form

$$C = f\{Q[K(t), q(t)] + I[K(t), q(t)]q(t)\}. \quad (22)$$

Thus $q(t)$ is determined by the flow relationship and for given $K(t)$ can be solved in the form

$$q(t) = \psi[K(t)]. \quad (23)$$

Other cases in which $q(t)$ is determined solely by the flow relationship are discussed by Foley.

To compare the two equilibria, let us first consider a strictly positive time horizon $h > 0$. End-of-period equilibrium (E) in period $(t, t + h)$

⁷ As a referee has suggested, a more general formulation would be to postulate

$$C(t+h, t) = \int_t^{t+h} Q[K^*(\tau, t), q^*(\tau, t)] d\tau.$$

Dividing by h , letting $h \rightarrow 0$, and invoking W yields $C_1(t, t) = Q[K(t), q(t)]$, which in the limiting case with minor notational changes is identical to (20).

is determined by (18a) and (22); beginning-of-period equilibrium (B) in that period is determined by (21a) and (22). Consider now B in period $(t + h, t + 2h)$. This is obtained by replacing t by $t + h$ in (21a) and (22). A sufficient condition for the two sets of equilibrium conditions to be identical is (i) perfect foresight in predicting all quantities, namely, $p^*(t + h, t) = p(t + h)$, $q^*(t + h, t) = q(t + h)$, $M^*(t + h, t) = M(t + h)$, $K^*(t + h, t) = K(t + h)$; (ii) $K(t + h) = K(t)$.

While perfect foresight suffices to make the two asset market conditions identical, (ii) is required to make the flow conditions, and hence the overall equilibrium of the system, consistent in the two cases. Moreover, even if expectational variables are included in the consumption function, reflecting expected capital gains, etc., perfect foresight alone would not suffice to ensure that the two equilibria hold simultaneously unless there is no capital accumulation.

When we move to continuous time, the conditions for consistency of the two equilibria weaken drastically. Letting $h \rightarrow 0$ in (18a) and (21a), and recalling equation (13), namely, $W^d(t, t) = W(t)$, their wealth arguments coincide. Moreover, provided that weak consistency holds with respect to both $p^*(t, t)$ and $q^*(t, t)$, the respective rates of return also converge to a common value. Thus in the limit, the two capital stock equilibrium conditions both become

$$J \left[W(t), \frac{p_1^*(t, t)}{p(t)}, \frac{\bar{d}}{q(t)} + \frac{q_1^*(t, t)}{q(t)} \right] = q(t)K(t)$$

while the flow equation (20) is obviously identical in the two cases. Thus in the limit the two equilibria are consistent and we summarize this with the following proposition: provided price expectations satisfy the *weak consistency condition* W , in a continuous-time model, beginning-of-period equilibrium and end-of-period equilibrium coincide.

In this connection Foley has argued that E is not well defined as one moves to the limit and converts to continuous time unless perfect myopic foresight (our assumption, S) holds. However, as the above argument shows, the limit is in fact perfectly well defined under the much milder condition of weak consistency. And this result seems perfectly reasonable.⁸

Moreover, under these same weak conditions, the partial derivatives of the E condition are also well defined. We shall show this for the capital equilibrium condition, but the same argument applies to the money market as well. Recalling our definition of J , (18a) can be written as

$$K^d(t + h, t) = K(t) + I(t)h \equiv K(t + h). \quad (18a')$$

Letting $h \rightarrow 0$ yields

$$K^d(t, t) = K(t). \quad (24)$$

⁸ The economic intuition is clear: in continuous time "end" and "beginning" merge and hence are equivalent.

Substituting for $K(t)$ in (18a'), dividing by h , and letting $h \rightarrow 0$ we obtain

$$K_1^d(t, t) = I(t). \quad (25)$$

This is just the corresponding flow equilibrium relationship for investment. Now consider equation (24) at time $t + h$. This, together with (18a'), implies $K^d(t + h, t) = K^d(t + h, t + h)$, from which we deduce

$$K_2^d(t, t) = 0. \quad (26)$$

Thus recalling that $K^d \equiv K^*$, it follows that end-of-period equilibrium implies perfect myopic foresight in predicting stocks of assets. Finally, (25) and (26) together imply the total derivative

$$\dot{K}^d(t, t) = I(t). \quad (27)$$

Beginning-of-period equilibrium can be written as

$$K^d(t, t) = K(t), \quad (24)$$

which upon differentiation with respect to t yields (27), consistent with B .

Foley also has a second proposition which asserts that given W , perfect myopic foresight in predicting prices is implied by E and B taken together. He establishes this result by differentiating the stock equilibrium conditions for the two assets with respect to t and showing by means of the "adding-up" conditions that the total derivatives of expected and actual prices are the same. That is, in our notation he shows

$$\frac{\dot{q}^*(t, t)}{q(t)} = \frac{\dot{q}(t)}{q(t)} \quad (5')$$

and similarly for the price of money. However, this equation does not imply perfect myopic foresight. Apart from a constant of integration, equation (5') is simply a restatement of W . As we discussed in Section I, the true relationship involves the partial derivative q_1^* and requires

$$\frac{q_1^*(t, t)}{q(t)} = \frac{\dot{q}(t)}{q(t)}. \quad (4')$$

Hence his proposition 2 does not appear to hold in general in our framework.⁹

⁹ The logic in this exercise is treacherously difficult and there may be circular results which, while logically correct, we regard as economically uninteresting. The fundamental observation is that $\dot{q}^*(t, t) \neq q_1^*(t, t)$ unless $q_2^*(t, t) = 0$, and for general t the assumption that $q_2^*(t, t) = 0$ is equivalent to our condition (4) or perfect myopic foresight. By writing q^* as a single function of t , one is implicitly assuming $q_2^*(t, t) = 0$, i.e., perfect myopic foresight and hence in essence begging the question. Note, however, that if one views all expectations as being evaluated at a fixed time (say zero), then $q_2^*(0, 0) = 0$, so that at this fixed point only Foley's proposition 2 will hold.

III. Evolution of the System over Time

To close the system and to trace out its path over time, we must hypothesize some expectations mechanism describing how $p^*(t + h, t)$ and $q^*(t + h, t)$ are generated, as well as describing how stocks of assets and their expectations are evolving. To keep things simple we shall focus on the dynamics associated with price expectations. We shall therefore assume that all stocks and their expectations remain constant, writing

$$K^*(t + h, t) \equiv K(t + h) \equiv \bar{K} \quad (28a)$$

$$M^*(t + h, t) \equiv M(t + h) \equiv \bar{M}. \quad (28b)$$

With these assumptions and simple consumption function (22), $q(t)$ must remain constant for all t (see [23]). Under these conditions it is obviously reasonable for their corresponding expectations to be perfect, and by appropriate choice of units we can let

$$q^*(t + h, t) = q(t + h) = 1. \quad (29)$$

Let us first briefly consider the discrete-time version of the model. Inserting (28) and (29) into (18a) and (21a), E and B become

$$E \quad J \left[p^*(t + h, t) \bar{M} + \bar{K}, \frac{p^*(t + 2h, t) - p^*(t + h, t)}{hp^*(t + h, t)}, d \right] = \bar{K} \quad (18a'')$$

$$B \quad J \left[p(t) \bar{M} + \bar{K}, \frac{p^*(t + h, t) - p(t)}{hp(t)}, d \right] = \bar{K}; \quad (21a'')$$

then postulating some expectations mechanism for $p^*(t + h, t)$ the evolution of $p(t)$ is determined. For example, taking $h = 1$ and postulating the simple extrapolative mechanism,

$$p^*(t + 1, t) = p(t) + \beta[p(t) - p(t - 1)] \quad (30)$$

yields the following nonlinear first-order difference equation for B :

$$J \left[p(t) \bar{M} + \bar{K}, \frac{\beta[p(t) - p(t - 1)]}{p(t)}, d \right] = \bar{K}.$$

This determines the evolution of the price of money under the assumption of beginning-of-period equilibrium. The stability properties of this equation can be investigated, at least locally.

The dynamics of end-of-period equilibrium can be analyzed similarly, although in this case a higher-order difference equation will be obtained. In general the time profile and stability properties of the two equilibria will differ, although with assets fixed under conditions of perfect foresight they will coincide.

The analysis becomes more interesting when we let $h \rightarrow 0$ and pass to continuous time. In this case, as we have seen, weak consistency ensures

that B and E coincide and under our simplifying assumptions reduce to

$$J \left[p(t)\bar{M} + \bar{K}, \frac{p_1^*(t, t)}{p(t)}, d \right] = \bar{K}. \quad (31)$$

Note that this is only a partial differential equation in p . To determine the dynamics completely, we must transform this to a total differential equation. One simple way this can be done is if one assumes strong consistency of expectations. In this case $p_2^*(t, t) = 0$, so that from (4), $p_1^*(t, t) = \dot{p}(t)$. Substituting into (31) yields the total differential equation in p ,

$$J \left[p(t)\bar{M} + \bar{K}, \frac{\dot{p}(t)}{p(t)}, d \right] = \bar{K}. \quad (32)$$

Under the assumption that $J_2 < 0$, this can be solved for

$$\dot{p}(t) = \phi[p(t), \bar{M}, \bar{K}]. \quad (33)$$

Differentiating (32) with respect to p , in the neighborhood of equilibrium we calculate

$$\frac{\partial \dot{p}}{\partial p} = \frac{-J_1 \bar{M} p}{J_2} > 0. \quad (34)$$

Hence we conclude that with strong consistency of expectations (perfect myopic foresight), $p(t)$ will be locally unstable. This result is not surprising and indeed is quite consistent with analogous results obtained in other contexts (see, e.g., Hahn 1966; Samuelson 1967; Shell and Stiglitz 1967; Burmeister et al. 1973; Burmeister and Graham 1974).

A second way of closing the model is to assume the extrapolative mechanism (30). For arbitrary h , this can be written as

$$\begin{aligned} p^*(t + h, t) &= p(t) + \beta[p(t) - p(t - h)] \\ &= (1 + \beta)p(t) - \beta p(t - h), \end{aligned} \quad (30')$$

where β is constant. Dividing by h , letting $h \rightarrow 0$, and invoking W yields the continuous analogue

$$p_1^*(t, t) = \beta \dot{p}(t). \quad (35)$$

From the calculations given above it is clear that the system will be locally stable as long as $\beta < 0$.

The final expectations mechanism we consider is the adaptive hypothesis. Assuming that at time t investors are concerned with forecasting over the period $(t, t + h)$, the hypothesis can be described by

$$p^*(t + h, t) - p^*(t, t - h) = \gamma[p(t) - p^*(t, t - h)]$$

or

$$p^*(t + h, t) = \gamma p(t) + (1 - \gamma)p^*(t, t - h)$$

where γ is a constant and is assumed to lie in the range $0 < \gamma < 1$. Dividing by h , letting $h \rightarrow 0$, and using W yields the limiting, continuous-time relationship¹⁰

$$\dot{p}(t) = p_1^*(t, t) + p_2^*(t, t) = \gamma p_2^*(t, t). \quad (37)$$

The derivation and properties of this equation, which is the logically consistent version of the adaptive hypothesis in continuous time, is discussed at length by Burmeister and Turnovsky (1976). The crucial point is that it involves the partial derivative $p_2^*(t, t)$, which as we have seen measures the rate of forecast error committed at time t . The two equations in (37) can be combined with (31) to yield a total differential equation in p or equivalently $p^*(t, t)$. This equation represents the simultaneous solution to both a price expectations equation and a portfolio balance equation.

From (37), it follows that

$$p_1^*(t, t) = \left(\frac{\gamma - 1}{\gamma} \right) \dot{p}(t). \quad (38)$$

Inserting (38) into (31) yields the differential equation

$$J \left[p(t) \bar{M} + \bar{K}, \frac{(\gamma - 1) \dot{p}}{\gamma p}, d \right] = \bar{K}. \quad (39)$$

Differentiating (39) with respect to p yields $\partial \dot{p} / \partial p = (-J_1 \bar{M} p / J_2) \times [\gamma / (\gamma - 1)]$ so that for $0 < \gamma < 1$ adaptive expectations yield a locally stable adjustment in p .

In Burmeister and Turnovsky (1976) we have extended this stability analysis to a multiasset system under adaptive expectations and have established sufficient conditions to ensure that it is locally asymptotically stable.

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¹⁰ This equation is implied by (7) in Burmeister and Graham (1974, p. 326).

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