On the nonlinear electromagnetic coupling between a coil and an oscillating magnet

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2010 J. Phys. D: Appl. Phys. 43 295005

(<http://iopscience.iop.org/0022-3727/43/29/295005>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 152.3.237.20

The article was downloaded on 26/01/2012 at 17:57

Please note that [terms and conditions apply](#).

On the nonlinear electromagnetic coupling between a coil and an oscillating magnet

Adam J Sneller and Brian P Mann

Duke University, Department of Mechanical Engineering and Materials Science, Durham, NC 27708, USA

E-mail: adam.sneller@duke.edu

Received 23 February 2010, in final form 1 June 2010

Published 8 July 2010

Online at stacks.iop.org/JPhysD/43/295005

Abstract

The electromagnetic induction of voltage across a coil due to the motion of a magnet is among the fundamental problems of physics, and it has a broad range of practical applications. While Maxwell's equations exactly describe this phenomenon, the physical complexity inherent in most realistic situations often prevents the generation of closed-form expressions for the electromagnetic coupling. This paper uses basic principles to develop an approximate analytical expression for the induced voltage in terms of a set of physical parameters, and experimental results demonstrate a high level of validity in the model over the parameter values tested. For oscillatory magnet motion about a point on a coil's axis, it is shown that the induced voltage is an infinite sum of harmonics at integer multiples of the oscillation frequency; the relative amplitudes of these harmonics vary as the magnet's equilibrium position migrates along the coil's axis, causing the odd and even harmonics to vanish, reappear and reach peak values at predictable locations. Several simplifications to the model are considered, and their validity is investigated analytically over a range of parameters.

1. Introduction

Electromagnetic induction results from relative motion between a conductor and a magnetic field, and it is a process by which mechanical motion can be converted to and from electrical energy. The study of induction as a phenomenon began in the 1830s with the work of such luminaries as Michael Faraday and Joseph Henry, and its application has had a significant impact on the discourse of contemporary science and technology. Induction underlies the physics of a multitude of modern devices, including electric generators, motors, actuators and transducers [1, 2]; audio, video and telecommunications equipment [3] and sensors and controllers [4–6].

Some of the more recent applications of induction have been related to friction reduction through such means as magnetic bearings [7, 8]. Furthermore, the dissipative nature of induction has led to developments in such areas as magnetic brakes [9–11] and structural vibration suppression [12–16], effectively replacing friction as a means of energy dissipation. The friction reduction offered by these applications lends itself

to decreased wear and damage in various physical components, thus prolonging their useful service life.

Recent work has been directed towards vibrational energy harvesting, which utilizes electromagnetic induction to convert ambient motion into electricity [17–22]. This is typically achieved using a permanent magnet and a coil, which are mounted to the vibration source in such a way as to allow relative motion between them. Much of the current literature has assumed both linear electromagnetic coupling [23–27] and a linear suspension between the coil and magnet [26–30]. The nature of a harvester's suspension will ultimately depend on how it has been fabricated, and thus a harvester could be designed such that the suspension is linear over the anticipated operating regime. However, several studies have demonstrated that the response of the harvester can be broadened through the introduction of various types of mechanical nonlinearities [31–38]. By analogy, it could be argued that a truly optimized energy harvester would somehow make use of the nonlinearities inherent to the electromagnetic coupling, even though it may be possible to force the coupling into a linear regime.

This work uses basic principles to develop an analytical model for the electromagnetic coupling between a coil and a magnet, which is shown to be nonlinearly dependent on the position of the magnet in reference to the coil. For oscillatory magnet motion, the model predicts that the induced voltage will be an infinite sum of harmonic sinusoids, the amplitudes of which depend on the frequency, amplitude and equilibrium point of the motion. Experimental results demonstrate qualitative and quantitative agreement with theory over the range of parameters tested.

The model predicts that there exist two points on a coil's axis that will maximize the electromagnetic coupling term. The experimental results demonstrate that these points not only exist, but that the model accurately predicts their locations on the coil's axis. It is also shown that small oscillations about either point will induce a voltage with primarily odd harmonics of the oscillation frequency, and that increasingly large oscillation amplitudes will cause the even harmonic amplitudes to appear and begin increasing in magnitude. The electromagnetic coupling term is shown to be zero when the magnet is at the centre of the coil; for oscillations about this point it is shown that all of the odd harmonics of the voltage are zero, while the even harmonics are maximized. The extent of agreement between experiment and theory along with the relative simplicity of the model may make it useful in the development of advanced electromagnetic devices, such as large-stroke actuators and macroscale position and motion sensors.

This paper is organized as follows: the next section outlines the application of the dipole model to describe the magnetic field generated by a moving magnet. Faraday's Law of induction is then applied to each turn of the coil, and several assumptions are imposed in order to generate an approximate analytical expression for the total induced voltage. Subsequently, experimental data are compared against analytical predictions, thus allowing conclusions to be drawn regarding the model's validity. Finally, several simplifying assumptions are investigated numerically, including the often cited 'tightly wound' coil, for which all turns of the coil are assumed to be collocated [4].

2. Mathematical model

2.1. The dipole model

A dipole is a hypothetical entity that possesses a magnetic moment $\mathbf{m}$ and occupies an infinitesimal volume. By definition [4], such a dipole will generate a magnetic field $\mathbf{B}$ of the form

$$\mathbf{B} = \frac{\mu_0}{4\pi} \nabla \frac{\mathbf{m} \cdot \mathbf{s}}{|\mathbf{s}|^3}, \quad (1)$$

where $\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$ is the magnetic constant, ∇ is the vector gradient and $\mathbf{s}$ is the position of a point of interest relative to the dipole's location.

Consider that there exists a magnet on the axis of the coil of figure 1(a), located at $x = y(t)$. It is assumed to feature a magnetic moment $\mathbf{m} = v B_r / \mu_0 \hat{\mathbf{e}}_x$, where v and B_r are the magnet's volume and residual magnetic flux density,

Figure 1. (a) Diagram showing the region occupied by a three-dimensional coil within the (r, x) coordinate system, along with a magnet located at $x = y(t)$; (b) schematic of the circuit that is powered by electromagnetic induction.

respectively, and $\hat{\mathbf{e}}_x$ is a unit vector pointing in the positive x -direction. By utilizing equation (1), the magnetic field generated by the magnet can be approximated as

$$\mathbf{B} = \frac{\lambda}{6\pi} \left(\frac{3r(x-y)\hat{\mathbf{e}}_r - [r^2 - 2(x-y)^2]\hat{\mathbf{e}}_x}{[r^2 + (x-y)^2]^{5/2}} \right), \quad (2)$$

where $\lambda = \frac{3}{2} v B_r$ is a conveniently defined parameter and $\hat{\mathbf{e}}_r$ is a unit vector pointing in the positive r -direction.

2.2. Faraday's law of induction

If the magnet is allowed to move along the coil's axis, an electric field will develop in the coil such as to oppose the magnet's motion. This field generates an electric potential V across the coil's leads, which Faraday's law of induction predicts to be proportional to the time rate of change in magnetic flux through the coil [3, 4]. Neglecting eddy current losses, the induced voltage can be expressed as

$$V = -\frac{d}{dt} \oint_S \mathbf{B} \cdot d\mathbf{a} = -\dot{\Phi}_{\text{tot}}, \quad (3)$$

where S is a surface that encloses the conductor, $d\mathbf{a}$ is an area element normal to S , Φ_{tot} is the magnetic flux through the coil and an overdot indicates a time derivative. Consider a coil located within the cylindrical coordinate system (r, x) shown in figure 1(a), where x_1 and x_2 are the x -coordinates occupied by coil's ends and r_1 and r_2 are the inner and outer radii of the region, respectively. By isolating the n th turn of the coil and treating it as a single-turn coil, equation (3) will reveal that

$$V_n^{(1)} = -\frac{d}{dt} \oint_{S_n} \mathbf{B} \cdot d\mathbf{a} = -\dot{\Phi}_n^{(1)}, \quad (4)$$

where the superscript (1) indicates a single-turn coil, and $V_n^{(1)}$, S_n and $\Phi_n^{(1)}$ are the induced voltage, enclosing surface, and magnetic flux associated with the single-turn coil. Consider that any point on the n th turn of the coil lies within an arbitrarily shaped cross-sectional area ΔA_n located at $(\hat{r}_n, \hat{x}_n)$. For convenience, it will be assumed that the volume created by revolving ΔA_n about the coil's axis encloses exactly one

loop of insulated wire along with some empty air space. Furthermore, the sum of all ΔA_n fills the entire planar region $r_1 \leq r \leq r_2$, $x_1 \leq x \leq x_2$, or equivalently

$$\sum_{n=1}^N \Delta A_n = (r_2 - r_1)(x_2 - x_1) \equiv A_c, \quad (5)$$

where A_c will be termed the coil's total cross-sectional area. The surface S_n will thus be approximated as the surface of a thin torus with radius $\hat{r}_n$ located at $\hat{x}_n$ and the magnetic flux through the S_n will be approximated by averaging the integral over the wire area A_{wn} to reveal

$$\Phi_n^{(1)} = \frac{1}{A_{wn}} \int_{A_{wn}} \left(\int_0^{2\pi} \int_0^r \mathbf{B}(r', x, y) \cdot \hat{\mathbf{e}}_x r' dr' d\varphi \right) dA, \quad (6)$$

$$\Phi_n^{(1)} = \frac{1}{A_{wn}} \int_{A_{wn}} -\frac{1}{3} \lambda r^2 [r^2 + (x - y)^2]^{-3/2} dA, \quad (7)$$

$$\Phi_n^{(1)} \approx -\frac{1}{3} \lambda \hat{r}_n^2 [\hat{r}_n^2 + (\hat{x}_n - y)^2]^{-3/2}, \quad (8)$$

where dA is an infinitesimal area element within the wire cross-section A_{wn} , r' is a dummy radial coordinate, φ is a polar angle coordinate and equation (8) arises by assuming the integrand of equation (7) to be approximately constant within a given A_{wn} . Consider another coil that is identical to the original N -turn coil in every respect, except that the n th turn of the coil entirely occupies the region ΔA_n . Such a coil will be referred to as continuous, since conducting material is assumed to occupy the entire region of the coil except for infinitesimal gaps between individual turns. Suppose also that the external magnetic field $\mathbf{B}$ only acts on regions that correspond to A_{wn} of the original coil. The flux Φ_n through the n th turn of the continuous coil can be found by substituting ΔA_n for A_{wn} in equation (7). Since $\mathbf{B}$ is defined to be $\mathbf{0}$ outside of A_{wn} , it can be found that $\Phi_n = \Phi_n^{(1)} A_{wn} / \Delta A_n$. The total magnetic flux through the coil is thus $\Phi_{\text{tot}} = \sum_{n=1}^N \Phi_n$, and applying the chain rule to equation (3) will reveal

$$V = \lambda \dot{y} \sum_{n=1}^N \frac{\hat{r}_n^2 (\hat{x}_n - y)}{(\hat{r}_n^2 + (\hat{x}_n - y)^2)^{5/2}} \frac{A_{wn}}{\Delta A_n}, \quad (9)$$

where $\dot{y} = dy/dt$ is the velocity of the magnet. Equation (9) could be simplified further for a uniformly wound coil, in which case $\Delta A_n = \Delta A$ and $A_{wn} = A_w$ are both constant for all n . Previous works [39] have defined a coil's fill factor ξ as the proportion of the coil's cross-section that contains conducting material, or $\xi = N A_w / A_c$. Noting from equation (5) that $A_c = N \Delta A$ for a uniform coil, the fill factor can be expressed as

$$\xi = \frac{A_w}{\Delta A}, \quad (10)$$

and by utilizing equations (9) and (10) the total induced voltage can be expressed as

$$V = -\frac{\xi \lambda \dot{y}}{\Delta A} \sum_{n=1}^N \frac{\hat{r}_n^2 (\hat{x}_n - y)}{(\hat{r}_n^2 + (\hat{x}_n - y)^2)^{5/2}} \Delta A. \quad (11)$$

Assuming that the quantity within the summation varies semi-continuously between adjacent turns of the coil, the summation

Figure 2. Plot of the theoretical flux linkage $\Theta(y)$ for coil 1 ($\cdots$) and coil 2 (—), using magnet 1 and the parameters given in tables 1 and 2.

of equation (11) can be approximated as an integral over the coil's cross-sectional area A_c :

$$V = -\frac{\xi \lambda \dot{y}}{\Delta A} \int_{x_1}^{x_2} \int_{r_1}^{r_2} \frac{r^2 (x - y)}{(r^2 + (x - y)^2)^{5/2}} dr dx. \quad (12)$$

The above equation can be integrated directly, and simplifying the result will yield a closed-form expression for the voltage induced across the coil:

$$V = \dot{y} \Theta(y), \quad (13)$$

$$\Theta(y) = \frac{N \xi \lambda}{(r_2 - r_1)(x_2 - x_1)} \sum_{i,j=1}^2 (-1)^{i+j} \left(\ln(r_i + z_{ij}) - \frac{r_i}{z_{ij}} \right), \quad (14)$$

$$z_{ij}^2(y) = r_i^2 + (x_j - y)^2, \quad (15)$$

where the indices i and j arise from the integration limits of equation (12). The quantity $\Theta(y)$ of the above equations, which will be referred to as the coil's flux linkage, accounts for the nonlinear electromechanical coupling of the system. It should be noted that the assumptions made in the derivation of equations (13)–(15) may not be valid for all situations, and that more careful consideration may be necessary in those cases.

Figure 2 shows a plot of equation (14) for two different sets of parameters. It can be seen that $\Theta(y)$ is odd about $y = 0$, and that there are two positions $y = \pm y_p$ that cause $|\Theta|$ to reach a peak value. The value of y_p will depend on the coil's dimensions, as can be seen in figure 3. It can further be seen that the only zero of Θ occurs at the coil's centre. For many situations the magnet might be assumed to oscillate harmonically, in which case its velocity $\dot{y}$ will be maximum at the equilibrium position. The model given above suggests that in order to maximize the voltage output for any amplitude of oscillation, the ideal equilibrium position is either of the points y_p that maximizes $|\Theta|$. Additionally, harmonic magnet motion about the centre of a coil will induce a minimal amount of voltage, since the coupling term Θ is zero at $y = 0$.

Figure 3. Plot showing the influence of coil dimensions on the location of peak flux linkage, y_p , where $2h$ is taken to be the length of the coil.

2.3. Response of the inductive circuit to harmonic magnet motion

The electrical circuit features a coil with inductance L attached to a load resistor R_L as shown in figure 1(b). The equation describing its behaviour can be found using Kirchoff's voltage law to be

$$L\dot{I} + (R_i + R_L)I = V(t), \quad (16)$$

where R_i is the resistance internal to the coil, I is the electrical current through the circuit and V is the induced voltage that has been derived in section 2.2. In order to generate an expression for $V(t)$, it will be assumed that the magnet's motion takes the form $y = \bar{y} + a \sin \omega t$. Recalling from equation (13) that $V = \dot{y} \Theta$, a Taylor series expansion of $\Theta(y)$ about $\bar{y}$ can be used to express the voltage as

$$\begin{aligned} V &= \omega a \cos \omega t \sum_{k=0}^{\infty} \frac{(a \sin \omega t)^k}{k!} \left. \frac{\partial^k \Theta}{\partial y^k} \right|_{y=\bar{y}} \\ &= \sum_{k=1}^{\infty} V_k \sin(k\omega t - \psi_k), \end{aligned} \quad (17)$$

where the amplitudes V_k and phases ψ_k can be found using trigonometric identities and Θ is as defined in equation (14). By superimposing the solutions from each term of equation (17), equation (16) can be solved to reveal an expression for the induced current:

$$I = \sum_{k=1}^{\infty} \frac{V_k \sin(k\omega t - \psi_k - \vartheta_k)}{\sqrt{(R_i + R_L)^2 + (k\omega L)^2}}, \quad (18)$$

where ϑ_k is the phase lag of the inductive circuit at the k th harmonic of ω . From equations (17) and (18) it can be seen that the amplitudes of V_k will vary depending on the amplitude a , frequency ω and equilibrium position $\bar{y}$ of the magnet's motion. In section 4 the harmonic amplitudes predicted by equation (18) will be compared against experimental data.

3. The effect of geometry on model simplifications

The analytical model for the induced voltage presented in this work accounts for fluctuations in the B -field along both the radial and axial dimensions of the coil. In order to provide insight regarding the relative importance of these variations, several simplifications to the model will be considered. They will be constructed by assuming that one or both of $\hat{r}_n$ and $\hat{x}_n$ from equation (11) are constant for all n . The assumption that $\hat{r}_n = r_0$ for all n would propagate through to a modified equation (12), in which the quantity $\int_{r_1}^{r_2} dr = r_2 - r_1$ would factor out of the integral. An analogous statement could be made regarding collocation of turns at $x_n = x_0$, and the reduced integrals can be evaluated to find approximate expressions for Θ :

$$\hat{\Theta}_r(y) = \frac{N\xi\lambda r_0^2}{x_2 - x_1} \sum_{j=1}^2 (-1)^j \left(r_0^2 + (x_j - y)^2 \right)^{-3/2}, \quad (19)$$

$$\hat{\Theta}_x(y) = \frac{N\xi\lambda}{r_2 - r_1} \sum_{i=1}^2 (-1)^i \frac{r_i^3}{y} \left(r_i^2 + (x_0 - y)^2 \right)^{-3/2}, \quad (20)$$

$$\hat{\Theta}_{r,x}(y) = N\xi\lambda r_0^2 y \left(r_0^2 + (x_0 - y)^2 \right)^{-5/2}, \quad (21)$$

where the subscripts of $\hat{\Theta}$ denote the coil dimensions over which the B -field is being approximated as constant. In order to compare the analytical forms, the coordinates r_0 and x_0 must be defined in terms of real coil parameters. The most obvious choice for x_0 is at the coil's centre, or $x_0 = \frac{1}{2}(x_1 + x_2)$. In order to take into account radial fluctuations in the B -field, a more complex averaging scheme will be used to define r_0 . Defining $\phi(r)$ as the integrand of equation (7) evaluated at $x = x_0$, the coil's characteristic radius r_0 will be taken to be the solution to $\phi(r_0) = \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \phi dr$, or

$$r_0 = \frac{r_2 - r_1}{\ln(r_2/r_1)}, \quad x_0 = \frac{x_1 + x_2}{2}, \quad (22)$$

where the definition of x_0 is repeated for convenience. For this study it is convenient to define the coordinate system such that $x_{1,2} = \mp h$, and thus from equations (22) it is implied that $x_0 = 0$. Additionally, it will be useful to define the aspect ratios ρ_0 , ρ_1 and ρ_2 as the ratios r_0/h , r_1/h and r_2/h , respectively. As discussed in section 2.3, there exist two points $y = \pm y_p$ along the coil's axis that cause the flux linkage of equation (14) to reach an extreme value, or $\Theta(\pm y_p) = \pm \Theta_p$ (see figure 2). Given a particular set of coil aspect ratios, it can be understood that each approximation presented in equations (19)–(21) will feature an error in both in predicted location y_p and magnitude Θ_p of the peak flux linkage. In order to provide insight regarding to the validity of the approximate expressions, they will be compared against equation (14) for coils of various aspect ratios.

By normalizing each of equations (19)–(21) with respect to Θ_p , a set of expressions will be generated that depend only on y/y_p and the aspect ratios ρ_1 and ρ_2 . The approximate expressions can thus be compared graphically,

Figure 4. Plots showing the normalized flux linkage Θ/Θ_p (—) in comparison with the approximations $\hat{\Theta}_r/\Theta_p$ ($\times$), $\hat{\Theta}_x/\Theta_p$ ($\circ$) and $\hat{\Theta}_{r,x}/\Theta_p$ (∇) for a variety of coil aspect ratios: (a) $\rho_1 = 1$, $\rho_2 = 2$; (b) $\rho_1 = 5$, $\rho_2 = 50$; (c) $\rho_1 = 5$, $\rho_2 = 8$ and (d) $\rho_1 = 1$, $\rho_2 = 10$.

and this is achieved in figure 4 for representative aspect ratios. Figures 4(a) and (b) show scenarios where the closest matches are $\hat{\Theta}_r$ and $\hat{\Theta}_x$, respectively. In figure 4(c), both $\hat{\Theta}_r$ and $\hat{\Theta}_x$ appear to be good approximations and in figure 4(d) all three approximations are poor. It can be seen from figure 4 that $\hat{\Theta}_r$ is quite accurate for the realistic coil sizes of plots (a) and (c), while it loses validity for the extremely wide coils of plots (b) and (d). Based on the findings presented here, it appears as if axial and, to a lesser extent, radial fluctuations of the B -field within the coil play a significant role in the electromagnetic coupling between the coil and the magnet.

4. Experimental study

In this section, experiments will be used to test the validity of the analytical model for the induced voltage. The experimental apparatus consists of a horizontal shaker (APS Dynamics)¹ that displaces a neodymium magnet (K&J Magnetics) along the axis of a stationary wire coil, as shown in figure 5. The coil's leads are connected to a load resistor, across which the voltage is measured via data acquisition (National Instruments). Two coils were fabricated for use in this study, the properties of which are outlined in table 1. The coils were produced by winding copper magnet wire (Fay Electric Wire, MW35C/73C) about spools, which were printed via rapid prototyping (Dimension). The winding was carried out using a coil-winding

Figure 5. Image of the apparatus used to displace a magnet through a coil.

machine (Schatten Design), which allows for relatively high-quality coils to be wound in a short time frame. Industrial applications typically employ high-precision coil fabrication techniques [39], but the corresponding monetary and time investments were judged to be excessive for the purposes of study. For all experiments the load resistance was held at $R_L = 10 \text{ k}\Omega$, which was found to be sufficiently large as to effectively negate transience in the circuit response. It should be noted that practical applications would likely have a more rigorous criterion for selecting R_L , such as the optimization of power delivery to the load [20, 23].

The properties of the coils and magnets used in this study are provided in tables 1 and 2, respectively, and the numbering convention defined therein will be used for the rest of the text. For each set of experiments, the amplitude a and frequency ω of the magnet's displacement were held constant while the

¹ Manufacturer's names are included for completeness; no endorsement is implied.

Table 1. Summary of the physical parameters of the coils used in the experimental study: d_w and d_{ins} derive from the manufacturer's data, σ is the reported value for copper at 20 °C [40], ξ_{theo} is calculated from equation (10), and ξ_{expt} is described in section 4.

Quantity	Symbol	Coil 1	Coil 2	Units
Number of turns	N	1500	3000	—
Length	$2h$	12.70	25.40	mm
Inner radius	r_1	17.46	17.46	mm
Outer radius	r_2	24.70	24.29	mm
Peak flux point	y_p	11.43	14.84	mm
Inductance	L	99.44	290.2	mH
Internal resistance	R_i	160.5	437.8	Ω
Bare wire diameter	d_w	160.1	160.1	μm
Insulated wire diameter	d_{ins}	191.8	191.8	μm
Wire conductivity	σ	59	59	$\Omega^{-1}\mu\text{m}^{-1}$
Theoretical fill factor	ξ_{theo}	0.3285	0.3481	—
Experimental fill factor	ξ_{expt}	0.3319	0.3302	—

Table 2. Summary of the physical parameters of the cylindrical neodymium magnets used in the study; all values derive from the manufacturer's data.

Quantity	Symbol	Magnet 1	Magnet 2	Units
Length	—	3.175	12.70	mm
Inner diameter	—	6.350	6.350	mm
Outer diameter	—	12.70	19.05	mm
Volume	v	3218	301.6	mm^3
Residual flux density	B_r	1.31	1.31	T

oscillation point $\bar{y}$ was varied discretely along the coil's length. For each oscillation point, the steady-state voltage across the load resistor was measured, thus allowing the amplitudes of the first four harmonics of the forcing frequency to be determined and compared against the theoretical predictions outlined in section 2.3.

The first set of experiments was conducted using coil 1 and magnet 2 with 12.7 mm motion at 4 Hz. The oscillation point $\bar{y}$ was varied discretely between $-2y_p$ and $2y_p$ at increments of $\frac{1}{10}y_p$, where $\bar{y} = 0$ corresponds to the centre of the coil and y_p is the peak flux point of the coil. This procedure was repeated using coil 2, and the experimental and theoretical results are compared for both coils in figure 6. The physical parameters used in obtaining the theoretical data were chosen directly from experimental parameters, with the exception of the fill factor ξ . It was judged that the uncertainty surrounding the theoretical value of ξ , as given in equation (10), was likely to be significantly larger than that of directly measurable parameters. In order to obtain the best possible match between experiment and theory, each coil's ξ was chosen from a least-squares fit of theoretical to experimental data for the first harmonic amplitude; these experimentally obtained values for ξ were used in generating all theoretical data presented hereafter, and they are reported in table 1. In order to test for differences in model validity between magnets of different sizes, the previously described experimental procedure was repeated using magnet 1, and the results can be seen in figure 7. In order to maintain clarity in the figures, only results obtained for oscillation points ranging from 0 to $2y_p$ are presented. The effect of differences in the magnet's motion was then investigated using magnet 2 and coil 2 for 6.35 mm motion

at 2, 4 and 6 Hz, and the results are plotted in figure 8 over $0 \leq \bar{y} \leq 2y_p$.

5. Results and discussion

The experimental results presented in section 4 provide significant insight regarding the validity of the analytical model. Figures 6–8 compare the amplitudes of the first four harmonics of the experimental voltage signal with those predicted by theory for several different sets of parameters. Figure 6 presents the data obtained for magnet 2 being displaced at 1.27 mm and 4 Hz, which was used to find experimental fill factors ξ_{expt} for both coils using a least-squares fit. For both coils, these data indicate a high degree of similarity between experiment and theory for the first harmonic amplitude of the load voltage. The higher harmonics demonstrate successively larger deviations between experiment and theory, but the overall shape of the curves remains quite similar. In order to account for some of the differences between experiment and theory, the true nature of the magnet displacement was investigated using data from an accelerometer (PCB Piezotronics) that was mounted to the shaker table. The theoretical data were generated using a pure sinusoid for $y(t)$, and the presence of any other frequencies in the experimental displacement signal would account for some of the deviation from theory. Accelerometer data from this experiment indicated the displacement amplitude at 8 Hz to be approximately 0.54% of the primary (4 Hz) amplitude, thus indicating that the displacement $y(t)$ is not a pure sinusoid. While the presence of higher-order harmonics in the displacement signal does not appear to affect the first harmonic of the voltage significantly, it can be seen from figure 6 that the theoretical model begins to lose predictive value for the higher harmonics of the voltage.

The plots of figure 7 present experimental and theoretical data collected using magnet 1 being displaced at 1.27 mm and 4 Hz. As with the first set of results, it can be seen that the best agreement between experiment and theory is obtained at the first harmonic of the load voltage, and that successive harmonics deviate further from the theoretical values. From table 2, the volume ratio of magnet 2 to magnet 1 can be found to be approximately 10.7 : 1. Interestingly, a very similar ratio can be obtained by comparing any data point of figure 7 with any corresponding point of figure 6. Since the induced voltage was found to scale directly with magnet volume, it can be concluded that magnet size does not impact the validity of the model. Since the dipole model is inherently most accurate for a small magnet [3], it can be further concluded that the use of the dipole model in describing the magnets did not significantly contribute to experimental error. The robustness of the model is made more apparent by noting the similarity in size between magnet 2 and coil 1 (table 1).

Figure 8 shows the data obtained for magnet 2 and coil 2 with a displacement amplitude of 6.35 mm at frequencies of 2, 4 and 6 Hz. As with the previous results, the experimental results are in close agreement with theory at the first harmonic of the forcing frequency, and successively higher harmonics reveal increasingly large deviations from theory. It can further

Figure 6. Plots showing how the amplitudes of the first four harmonics of the load voltage $R_L I$ (in V) vary as the oscillation point $\bar{y}$ of magnet 2 moves along the axis of coil 1 (a)–(d) and coil 2 (e)–(h) for 12.7 mm amplitude motion at 4 Hz for both experimental (●) and theoretical (—) data.

Figure 7. Plots showing how the amplitudes of the first four harmonics of the load voltage $R_L I$ (in V) vary as the oscillation point $\bar{y}$ of magnet 1 moves along the axis of coil 1 (a)–(d) and coil 2 (e)–(h) for 12.7 mm amplitude motion at 4 Hz for both experimental (●) and theoretical (—) data.

Figure 8. Plots showing how the amplitudes of the first four harmonics of the load voltage $R_L I$ (in V) vary as the oscillation point $\bar{y}$ of magnet 2 moves along the axis of coil 2 for 6.35 mm amplitude motion at 2 Hz (a)–(d), 4 Hz (e)–(h) and 6 Hz (i)–(l) for both experimental (●) and analytical (—) data.

be seen that these deviations become more pronounced as the forcing frequency increases. As previously, the loss of model validity at higher harmonics may have occurred due to an impure forcing signal. By analysing the accelerometer signal for each experiment, it was found that the amplitude of the second harmonic of the magnet displacement was approximately 0.18%, 0.26% and 0.72% of the primary harmonic for 2 Hz, 4 Hz and 6 Hz, respectively. It is therefore reasonable to conclude that the extent of deviation from theory is contingent upon how precisely the magnet displacement $y(t)$ is known.

The plots discussed above can also be used to draw conclusions regarding the effect of displacement amplitude on the model. The parameters used in collecting the data of figures 7(e)–(h) and 8(e)–(h) are identical, with the exception being that the former's magnet displacement amplitude is twice that of the latter. The most obvious qualitative difference to be noted is that maximum value of the first harmonic of the voltage occurs very near to $\bar{y} = y_p$ for the smaller displacement amplitude, and that the larger displacement amplitude causes this peak point to shift to the right. Similar differences can be found that relate to the locations of maximum and minimum values of the other harmonics.

As described in section 4, the values used for ξ in generating the theoretical data were found using a least-squares fit against experimental data. From table 1, the

percentage difference between theoretical and experimental values for ξ can be found to be approximately 1.03% for coil 1 and 5.42% for coil 2. Possible sources for error include measurement error (for both voltage and displacement) and inaccuracies in the manufacturers' data relating to the coils and magnets. Additionally, some assumptions made in developing the model may not be wholly true, such as the coil being uniformly-wound, which would undoubtedly contribute to the error.

While the model seems to match experimental results for the conditions tested, there likely exist practical operating conditions that will invalidate one or more of the assumptions inherent to developing the theoretical model. In particular, high-frequency oscillations may lead to the formation of eddy currents in the coil, thus effectively wasting a portion of the power generated. It has been shown that the effect of eddy currents will be small as long as [40] $\mathcal{R}_m = \frac{1}{2}\mu_0\sigma\omega d_w^2 \ll 1$, where $\mathcal{R}_m$ is the magnetic Reynolds number, σ is the wire conductivity, ω is the frequency and d_w is the wire diameter. Parameter values from table 1 will reveal $\mathcal{R}_m = 3.6 \times 10^{-5}$ at 6 Hz, thus suggesting that eddy currents are unlikely to have contributed significantly to experimental error. Situations may arise that cause $\mathcal{R}_m \ll 1$, and in these cases it may be necessary to model additional losses in the circuit model developed in section 2.3. It should be noted that magnetic dampers operate by dissipating energy through eddy currents [15, 41],

and therefore a more comprehensive model than is presented in this work would be required to accurately describe these systems.

Another frequency-dependent phenomenon that may be encountered is the skin effect, which refers to the tendency of high-frequency current to concentrate on the outermost portion of the wire cross-section [3]. This reduces the coil's effective area, which would presumably reduce the fill factor ξ . In the literature a conductor's skin depth δ is often approximated by the relation [40] $\delta^2 = 2/\omega\mu_r\mu_0\sigma$, where σ and $\mu_r \approx 1$ [1] are the conductivity and relative magnetic permeability of copper, respectively. Values from table 1 indicate the experimental system's skin depth at 6 Hz to be $\delta = 26.8$ mm, which is far greater than the wire diameter $d_w = 160.1$ μ m. It will therefore be concluded that the skin effect did not contribute significantly to experimental error. It should be noted, however, that certain high-frequency applications may necessitate consideration of the skin effect in modelling the induced voltage.

Having established some confidence in the experimentally obtained values for ξ from a mathematical standpoint, it is necessary to discuss the fill factor in more general terms. Theoretical limits on ξ relate to both the presence of air gaps in the coil and the thickness of the wire insulation, and these concepts must be explored in order to draw meaningful conclusions regarding the model. Conceptually, it can be understood that the amount of empty air present in the coil relates to the care and precision afforded to the coil-winding process. Therefore it will be instructive to consider different types of coil winds, and typical fill factors that may be attributed to them. Beeby and Donnel [39] define three such types as scramble-, orthogonal- and orthocyclic-wound coils, listed in order of increasingly ordered cross-sections. The insulated fill factor ξ_{ins} is the ratio of wire and insulation to total coil cross-section, which relates to the true fill factor ξ by the following relation:

$$\xi = \frac{A_w}{A_{\text{ins}}} \xi_{\text{ins}} = \left(\frac{d_w}{d_{\text{ins}}} \right)^2 \xi_{\text{ins}}, \quad (23)$$

where A_w and A_{ins} are the bare wire and insulated wire cross-sectional areas, respectively, and d_w and d_{ins} are the corresponding diameters. The quantity ξ_{ins} can be understood to depend solely on the quality of the wind, and hence it is independent of wire insulation thickness. For a typical scramble-wound coil it may lie within $0.5 < \xi_{\text{ins}} < 0.6$ [1], which table 1 and equation (23) will convert to $0.35 < \xi < 0.42$. This range is reasonably close to the experimentally obtained values ($\xi_{\text{expt}}^{(1)} = 0.3319$, $\xi_{\text{expt}}^{(2)} = 0.3302$), and therefore these coils might be classified as scramble-wound. While requiring a great deal more effort to produce, orthogonal- and orthocyclic-wound coils have been reported to feature fill factors as high as $\xi_{\text{ins}} = 0.78$ and 0.9 , respectively [39]. Equation (23) estimates the corresponding true fill factors to be $\xi = 0.54$ and 0.63 , the latter of which might be considered to be a theoretical upper bound on ξ for coils wound with this particular wire.

6. Conclusions

This paper derived an approximate analytical expression for the voltage induced across a coil due to the motion of a magnet in terms of a set of physical parameters. The model predicted that the electromagnetic coupling term features nonlinear dependence on the magnet position, a notion that was confirmed through experimental study. Theory and experiment agreed that the electromagnetic coupling term reaches a peak value at either of two symmetric points on the coil's axis; magnet oscillations about either point were found to induce the maximum attainable voltage for a given amplitude and frequency of motion. Additionally, the model accurately predicted that the coupling term would be minimized at the coil's centre. Oscillations about the coil's centre would thus be predicted to induce a minimal amount of voltage, and this has been supported through experiment.

The results demonstrated that harmonic magnet motion along the coil's axis will generally induce a voltage with an infinite sum of harmonics of the oscillation frequency, and that the relative amplitudes of these harmonics depend on the amplitude, frequency and equilibrium position of the magnet's motion. The frequency content of a voltage signal could thus be used to locate the magnet's oscillation point in terms of coil parameters, a notion that has potential applications for electromagnetic sensors. This type of nonlinear behaviour is dependent on axial and radial variations of the magnetic field within the coil, which it was shown that the often cited 'tightly wound' coil approximation fails to capture accurately. Additionally, the results suggest that modelling the magnets as dipoles did not contribute to experimental error. Phenomena such as the formation of eddy currents and the skin effect were neglected in developing the model, and they were subsequently shown to be insignificant for the experimental system. However, these may gain importance in high-frequency applications, and improved modelling may be necessary in order to account for their effects.

Acknowledgments

The authors would like to acknowledge support from an ONR Young Investigator Award through program manager Dr Ronald Joslin. Additional thanks is given to Ben Owens for assistance in developing the experiments.

References

- [1] McLyman W T 2004 *Transformer and Inductor Design Handbook* 3rd edn (New York: Marcel Dekker)
- [2] Thibault J P and Rossi L 2003 Electromagnetic flow control: characteristic numbers and flow regimes of a wall-normal actuator *J. Phys. D: Appl. Phys.* **36** 2559–68
- [3] Furlani E P 2001 *Permanent Magnet and Electromechanical Devices* 1st edn (San Diego, CA: Academic)
- [4] Ulaby F T 2007 *Fundamentals of Applied Electromagnetics* 5th edn (Upper Saddle River, NJ: Prentice Hall)
- [5] Eyabi P and Washington G 2006 Modeling and sensorless control of an electromagnetic valve actuator *Mechatronics* **16** 159–75

- [6] Chen X, Zhu Q and Howe D 2009 Modeling and analysis of a tubular oscillating permanent-magnet actuator *IEEE Trans. Indust. Appl.* **45** 1961–70
- [7] Boehm J, Gerber R and Kiley N R C 1993 Sensors for magnetic bearings *IEEE Trans. Magn.* **29** 2962–4
- [8] Cheah S K and Sodano H A 2008 Novel eddy current damping mechanism for passive magnetic bearings *J. Vib. Control* **14** 1749–66
- [9] Nagaya K, Kojima H, Karube Y and Kibayashi H 1984 Braking forces and damping coefficients of eddy current brakes consisting of cylindrical magnets and plate conductors of arbitrary shape *IEEE Trans. Magn.* **20** 2136–45
- [10] Wiederick H D, Gauthier N, Campbell D A and Rochon P 1987 Magnetic braking: Simple theory and experiment *Am. J. Phys.* **55** 500–3
- [11] Cadwell L H 1996 Magnetic damping: analysis of an eddy current brake using an airtrack *Am. J. Phys.* **64** 917–23
- [12] Okada Y, Matsuda K and Hashitani H 1995 Self-sensing active vibration control using the moving-coil-type actuator *J. Vib. Acoust.* **117** 411–15
- [13] Paulitsch C, Gardonio P and Elliot S J 2006 Active vibration damping using self-sensing, electrodynamic actuators *Smart Mater. Struct.* **15** 499–508
- [14] Agashe J S and Arnold D P 2008 A study of scaling and geometry effects on the forces between cuboidal and cylindrical magnets using analytical force solutions *J. Phys. D: Appl. Phys.* **41** 1050
- [15] Cheng T H and Oh I K 2009 Coil-based electromagnetic damper and actuator for vibration suppression of cantilever beams *J. Intell. Mater. Syst. Struct.* **20** 2237–47
- [16] Ebrahimi B, Khamesee M B and Golnaraghi F 2009 A novel eddy current damper: theory and experiment *J. Phys. D: Appl. Phys.* **42** 075001
- [17] Sodano H A and Inman D J 2005 Comparison of piezoelectric energy harvesting devices for recharging batteries *J. Intell. Mater. Syst. Struct.* **16** 799–807
- [18] Glynne-Jones P, Tudor M J, Beeby S P and White N M 2004 An electromagnetic, vibration-powered generator for intelligent sensor systems *Sensors Actuators A* **110** 344–9
- [19] Roundy S, Wright P K and Rabaey J 2003 A study of low level vibrations as a power source for wireless sensor nodes *Comput. Commun.* **26** 1131–44
- [20] Stephen N G 2006 On energy harvesting from ambient vibration *J. Sound Vib.* **293** 409–25
- [21] Kulkarni S, Koukharenko E, Torah R and Tudor J 2008 Design, fabrication and test of integrated macro-scale vibration-based electromagnetic generator *Sensors Actuators A* **145–146** 336–42
- [22] Saha C R, O'Donnell T, Wang N and McCloskey P 2008 Electromagnetic generator for harvesting energy from human motion *Sensors Actuators A* **147** 248–53
- [23] Williams C B and Yates R B 1996 Analysis of a micro-electric generator for microsystems *Sensors Actuators A* **52** 8–11
- [24] Amirtharajah R and Chandrakasan A P 1998 Self-powered signal processing using vibration-based power generation *IEEE J. Solid-State Circuits* **33** 687–95
- [25] Bedekar V, Oliver J and Priya S 2009 Pen harvester for powering a pulse rate sensor *J. Phys. D: Appl. Phys.* **42** 105105
- [26] Graves K E, Toncich D and Iovenitti P G 2000 Theoretical comparison of motional and transformer EMF device damping efficiency *J. Sound Vib.* **233** 441–53
- [27] Sari I, Balkan T and Kulah H 2008 An electromagnetic micro power generator for wideband environmental vibrations *Sensors Actuators A* **145–146** 405–3
- [28] El-hami M, Glynne-Jones P, White N M, Hill M, Beeby S P, James E, Brown A N and Ross J N 2001 Design and fabrication of a new vibration-based electromechanical power generator *Sensors Actuators A* **92** 335–42
- [29] von Büren T and Tröster G 2007 Design and optimization of a linear vibration-driven electromagnetic micro-power generator *Sensors Actuators A* **135** 765–75
- [30] Yang B and Lee C 2009 Electromagnetic energy harvesting from vibrations of multiple frequencies *J. Micromech. Microeng.* **19** 035001
- [31] Mann B P and Sims N D 2009 Energy harvesting from the nonlinear oscillations of magnetic levitation *J. Sound Vib.* **319** 515–30
- [32] Shahrz S M 2008 Increasing the efficiency of energy scavengers by magnets *J. Comput. Nonlinear Dyn.* **3** 041001
- [33] Daqaq M F, Stabler C, Qaroush Y and Seuaciuc-Osório T 2009 Investigation of power harvesting via parametric excitations *J. Intell. Mater. Syst. Struct.* **20** 545–57
- [34] Erturk A, Hoffmann J and Inman D J 2009 A piezomagnetoelastic structure for broadband vibration energy harvesting *Appl. Phys. Lett.* **94** 254102
- [35] Triplett A and Quinn D D 2009 The effect of non-linear piezoelectric coupling on vibration-based energy harvesting *J. Intell. Mater. Syst. Struct.* **20** 1959–67
- [36] Stanton S C, McGehee C C and Mann B P 2009 Reversible hysteresis for broadband magnetopiezoelectric energy harvesting *Appl. Phys. Lett.* **95** 174103
- [37] Barton D A W, Burrow S G and Clare L R 2010 Energy harvesting from vibrations with a nonlinear oscillator *J. Vib. Acoust.* **132** 021009
- [38] Mann B P and Owens B A 2010 Investigations of a nonlinear energy harvester with a bistable potential well *J. Sound Vib.* **329** 1215–26
- [39] Beeby S P and O'Donnell T 2009 Electromagnetic energy harvesting in ed S Priya and D J Inman *Energy Harvesting Technologies* 1st edn (New York: Springer) p 129
- [40] Woodson H H and Melcher J R 1968 *Electromechanical Dynamics* 1st edn (New York: Wiley)
- [41] Sodano H A, Inman D J and Belvin W K 2006 Development of a new passive-active magnetic damper for vibration suppression *J. Vib. Acoust.* **128** 318–27