Dynamic Capital Budgeting, Compensation, and Security Design

by

Shiming Fu

Business Administration
Duke University

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Business Administration in the Graduate School of Duke University 2015
Abstract

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This thesis examines how various agency frictions affect corporate financing, capital budgeting, managerial compensation and investment in dynamic settings. In the internal capital budgeting process, the agency issues considered are (i) the division manager privately observes project arrival and quality, and (ii) he can divert allocated capital. The optimal capital budgeting and compensation policies are jointly designed to mitigate agency costs that are endogenously determined. When the division’s financial slack is low, positive NPV investments are possibly forgone and manager’s pay-performance sensitivity is kept small. When the division’s financial slack is high, projects are funded more efficiently and steeper incentives are provided.

In the process of external financing, the key friction considered is that the agent has persistent private information about firm performance. In the optimal contract, the firm is financed by outside equity and a credit line contingent on compliance with a cash flow covenant. The agent is compensated via a combination of equity and stock options. As the level of persistence increases, the agent holds less equity and more stock options; the investors hold more equity. Investment is possibly efficient in the constrained firm and is varying with cash flow in the unconstrained firm.
I dedicate my dissertation to my parents and my wife.
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A fundamental question in corporate finance is: to what extent does capital get allocated to the right firm and right investment projects? To understand the financing and investment behavior of firms, one must consider both the process by which external capital is made available to firms and the process by which raised capital is allocated to the investment projects within the firm. In the frictionless world of Modigliani and Miller (1958), financial structure is irrelevant. Research on how external capital is raised has focused on various market frictions that make different financial structure more or less attractive. The most pervasive and important frictions arise from information asymmetries and agency problems. Also in the frictionless world, modern finance theory prescribes allocating capital according to the net present value (NPV) rule which provides no role for details of the internal capital allocation process. Research on how capital is allocated within firm focuses on decentralized information and incentive problems.

Recent progress in dynamic contracting has proven to be a fruitful approach to analyze the information and incentive issues in corporate finance, because these issues are inherently dynamic. In practice, investors and managers interact repeatedly
over financing events. Headquarters allocate capital to division managers over a sequence of projects. The dynamic approach also has the advantage of simultaneously considering firm financing, investment, compensation and payout. So it improves our understanding of how agency frictions evolve and how optimal corporate policies should be designed over time. However, there are limits to the current literature. For instance, the literature mainly focuses on moral hazard issues that have exogenous and constant incentives. Moreover, the current literature focuses on the compensation choices over time but ignores information links. In other words, private information is typically assumed as independent over time. The goal of this dissertation is to provide new frameworks that explicitly address these limits, enlarging the scope of applying dynamic agency methods in designing corporate policies.

Chapter 2 studies the joint design of capital budgeting and division manager’s compensation when two common agency issues exist. First, the division manager has private information about investment projects’ arrival time and quality. Second, the division manager can obtain private benefits from deploying capital inefficiently. The optimal contract downward distorts pay-performance sensitivity and capital allocation. Positive NPV projects are possibly forgone. As the firm accumulates more financial slack, fewer projects will be forgone, and the optimal contract provides steeper incentives and allocates a larger amount of capital to selected projects. Surprisingly, all distortions disappear at the payout boundary. The firm can implement the optimal project selection, capital allocation, and compensation through a simple budgeting account mechanism. In this implementation, the firm sets up a budgeting account for the division. The division manager can spend the account balance on either investment or consumption. When no project arrives, the HQ depletes the account balance at a designed rate. When a project arrives, the HQ replenishes the account balance depending on the division manager’s report of project quality.

Chapter 3 studies the joint design of firm financial structure, compensation, in-
vestment when an agent (entrepreneur or CEO) privately observes the firm’s cash flows that are subject to persistent shocks. This means by observing today’s cash flow, the agent also obtains private information about the firm’s future performance. This empirically important information link is typically ignored in the existing literature, because considering persistence has proved challenging in dynamic agency models. Using agent’s continuation values contingent on his report today and tomorrow, we analytically characterize the optimal contract in a setting with persistent private information. We also show that the optimal contract can be implemented by a contingent credit line, stock options, and equity. In contrast to the iid case, we find several distinct features: (i) investment is possibly efficient in the constrained firm, and is varying with cash flow in the unconstrained firm; (ii) the firm possibly experiences a longer time of being financially constrained; (iii) the agent receives payments that are less than what he can divert from cash flow and investors hold a larger equity stake; (iv) payment to the agent is via stock options and equity, the combination of which depends on persistence level; (v) the firm’s credit line limits are contingent on complying with a cash flow covenant and are history dependent.
Dynamic Capital Allocation and Managerial Compensation

2.1 Introduction

A vast majority of corporate investment is financed with internal funds\(^1\). Factors arising from information asymmetries and agency problems are among the most pervasive and important frictions that influence the efficiency of capital allocation within firms (Stein (2003)). In large firms, investment is usually delegated to division managers (DM) who have private information about projects\(^2\). Headquarters (HQ) or CEOs potentially face adverse selection problem in allocating capital. In principle, HQs could gather information but it is very costly as firms become large or complex\(^3\). Moreover, in large firms, it is very costly to monitor how DMs deploy capital.

---

\(^1\) Internal funds account for 70%-110\% of total investment for U.S. nonfinancial corporations between 1994-2008 Brealey et al. (2011).

\(^2\) Colom and Delmastro (2004) shows that capital spending decisions are mostly delegated to divisional managers in the surveyed 438 Italian metalworking firms, especially when the task is urgent. Graham et al. (2014) surveys more than 1000 CEOs and finds that CEO delegates investment decisions the most among important corporate policies.

\(^3\) For example, in the model of Aghion and Tirole (1997), more decision-making authorities are delegated down the corporate ladder when the principal is overloaded (such as when he manages a large firm).
DMs could deploy capital inefficiently and gain private benefit. This leads to moral hazard problem in investment. To ensure its long-term health, firms must provide proper incentives to encourage truthful information about investment projects and deter inefficient utilization of resources.

Another important aspect of capital budgeting and associated investment process is their dynamic nature. While in practice HQ and DM always interact repeatedly over a sequence of investment opportunities, the existing literature is mainly focusing on static environments. For instance, Harris and Raviv (1996) study the capital allocation with manager having ‘empire building’ preference; Bernardo et al. (2004) analyzes compensation and capital budgeting mechanisms when managers have private information regarding projects’ qualities. In contrast, this paper explores the optimal capital allocation and compensation in a dynamic setting. The dynamic model highlights that agency costs are endogenous and that the past performance (or financial slackness) of a division plays an important role in the optimal mechanism. In particular, the paper analyzes how do (i) project choice (ii) capital allocation rules (iii) performance-based payments vary with investment quality and the division’s past performance.

To study these questions, this paper focuses on the simplest environment with one risk neutral principal (HQ) and one risk neutral agent (DM). The firm has unlimited access to capital. Its investment opportunities arise stochastically over time. When the firm has an investment project, it can potentially invest in a value-enhancing technology that boosts project return. However, the HQ does not have any information regarding whether the firm has any investment project or not. Moreover, project quality is also unknown to the HQ. The firm has to rely on the DM’s information to discover investment projects and their quality. Once a project is reported, the HQ then allocates capital according to the report. But the deployment of capital is under the DM’s control. This means the DM can divert the allocated capital for
personal consumption.

With asymmetric information and moral hazard, the capital budgeting process faces the trade-off between raising investment efficiency and reducing compensation (information rent). Since project quality is private information, the DM can always misreport project quality downward and divert part of the allocated capital. Hence, inducing higher investment in any project requires increasing compensation to DM who operates all higher quality projects. Compared with the frictionless benchmark, the optimal contract provides flatter incentives and induces lower investment level. This intuition mimics the classic agency problem in Laffont and Tirole (1986)\(^4\).

When the budgeting process repeats, the agency conflicts will be endogenous and will vary over time. The intuition is as follows. In the dynamic setting, the firm has flexibility to pay the DM over time. The contract compensates the DM of information rents by promised future payments until his continuation utility is sufficiently high. Continuation utility represents the value reserved in the firm that funds the future payments to the DM. Continuation utility also summarizes the division’s past performance. Though the DM has to be compensated information rents at the time when investment takes place, the HQ can form expectations of future information rents and extract it from the DM’s continuation utility at any time. In this sense, compensation over time relaxes the constraint on investment and incentive provision imposed by information rents.

However, adjusting compensation over time increases the risk of liquidating the division. The DM is “punished” during the no investment period by lowering his continuation utility. And since the DM can only be “punished” to the extent that limited liability binds, the agency conflicts will not disappear. Importantly, limited liability implies that the level of DM’s continuation utility determines the severity of

\(^4\) The agent in Laffont and Tirole (1986) exerts unobservable but costly effort. The incentive to shirk in their paper is analogous to the incentive to divert capital in this paper.
the agency issues, and hence the optimal policy. After periods of good performance, the DM’s continuation utility will be at a high level and the division is far away from liquidation. It is optimal to design contracts with high-powered incentive since the liquidation risk is low and steeper contract induces more efficient investment. The extreme case is that at the payout boundary investment distortions disappear for all project types. On the contrary, when the DM’s continuation utility is low, the division is close to termination. It is optimal to design contract with low-powered incentive and therefore investment is severely distorted. The extreme case is that low quality projects will be completely excluded from capital allocation under provided technology assumptions.

The key feature of the model is that the agency cost in investment varies over the dimension of project types and the dimension of continuation utility. So the optimal capital allocation and compensation policy also vary over these two dimensions. The paper shows that capital allocation and pay-performance sensitivity exhibit monotone properties over these two dimensions under the provided technology and distribution assumptions. The paper has the following empirical implications for capital budgeting and incentive provisions.

1. The DM is provided steeper incentive and allocated larger amount of capital by reporting higher quality project.

2. The DM is provided steeper incentives and allocated larger amount of capital when the division has more financial slack.

3. With poor performance in the past and low financial slack, the division may forgo low quality projects even though investment in such projects have positive NPV.

4. When the DM gets cash compensation, capital allocation varies little across
reported project qualities.

The continuation utility that shapes the optimal contract in the model can be measured by either the financial slack or the past performance of a division. Using 4080 DM pay contracts from ExecuComp, Alok and Gopalan (2013) finds that the DM payment is less sensitive to the performance of his division during periods of industry distress. This empirical finding is consistent with the implication of this model.

The paper proceeds as follows. Section 2 review the related literature. Section 3 sets up the model. Section 4 summarizes the optimal contract in the frictionless environment. Section 5 provides the presentation of continuation utility and necessary conditions for incentive compatibility. The first part of Section 6 derives the optimal contract heuristically under the necessary conditions for IC. The second part verifies the proposed optimal contract maximizes the HQ’s expected payoff and the IC is satisfied. Section 7 characterizes the dynamic properties of the DM’s continuation utility and optimal policies. Section 8 provides an implementation of the contract using budgeting account mechanism. Section 9 concludes the paper.

2.2 Related Literature

The static trade-off between private information and moral hazard is studied in Laffont and Tirole (1986). They examine how to provide incentives to regulate a monopoly contractor with unobserved cost efficiency and unobserved effort to lower cost efficiency. They show that inducing higher effort level to reduce production cost (resolving moral hazard issue) will make it more costly to induce information revelation (exacerbating adverse selection issue). The effort in Laffont and Tirole (1986) is analogous to the investment in this model. Adopting a dynamic framework, this paper shows that this basic trade-off is endogenously determined and dynamically
evolves, since the severity of the agency conflict is affected by the limited liability constraint.

This paper is related to the continuous-time dynamic contracting literature. While the previous studies mostly focus on moral hazard issues, this paper broadens the framework by considering both hidden action and hidden information. The key friction, moral hazard, in the existing literature is exogenous and constant. For instance, the agent’s stealing ability in DeMarzo and Sannikov (2006b), or the private benefit of shirking in Biais et al. (2010) is an exogenous parameter. And these models largely consider contracts implementing the first best action. On the contrary, the agency issue in this model is endogenously determined by the level of the agent’s continuation utility or the division’s financial slack. Therefore, the optimal incentive provision and investment are time varying, and investment is distorted below the first best level. The hidden information adds another dimension of dynamics over project qualities and generates interesting interaction with the dynamics of investment over time. Using the martingale representation approach over the time dimension and the mechanism design approach (as in Myerson (1981)) over the quality dimension, we provide a rich setting with continuous types and continuous time to explore some interesting dynamic implications.

This paper is also related to the literature on dynamic mechanism design. studies a dynamic costly-state-verification model with an application of capital budgeting process. The DM in his model has private investment information and empire building preference. The HQ can verify the DM’s reports by incurring a fixed cost. The high quality projects are monitored and financed by the HQ. And the low quality projects are financed from the division’s budgeting account without being monitored. The main difference is that the DM in this model can also divert capital. Instead of

5 Exceptions include Zhu (2012) where the optimal contract can implement shirking either as a reward or punishment mechanism, and the moral hazard models with risk averse agent, for example, Sannikov (2008) or Gryglewicz and Hartman-Glaser (2013).
monitoring, the HQ will design a compensation scheme to resolve the agency issues. This paper can also characterize how the capital allocation policy and the slope of compensation vary dynamically, which are not in 

Eso and Szentes (2013) analyzes a discrete time dynamic auction where the agent’s (or the buyer) types are private information and are correlated over time. They show that any implementable allocation provides the seller the same expected revenue as in the case where the seller can observe the agent’s orthogonalized types beyond the first period. So from the revenue’s perspective, only the initial hidden information and its persistence matters, any subsequent orthogonal information is not associated with information rent. Garrett and Pavan (2012) analyze a dynamic contracting model where a firm’s cash flow is determined by its manager’s hidden type (productivity), hidden effort, and an i.i.d noise. The manager’s productivities are correlated over time. In their setting, the dynamics of the optimal policies are entirely driven by the persistence of the manager’s initial private information which is characterized by an impulse response function of information.

Though the investment information in this paper is not persistent, the model clearly shows that all the orthogonalized future information needs to be compensated with information rents. This key difference comes from the limited liability assumption. In the mechanism of Eso and Szentes (2013), the buyer needs to pay a large amount in the first period that is equal to all future expected payments in the case where all future orthogonalized information can be observed. The implicit assumption is that the buyer has deep pockets to make that payment upfront. Garrett and Pavan (2012) uses a linear reward scheme to implement the optimal contract. In their mechanism, the conditional information rents in all future periods are subtracted from the current period compensation. If the manager is not very patient or his outside option is not large enough, he will possibly receive negative compensation. This paper makes an explicit and natural assumption about the DM’s limited
liability. That is the HQ can only compensate the DM with positive cash payments at any time. With limited liability, the DM’s continuation utility or the division’s financial slack will play a key role.

2.3 Model

The model studies a large firm that consists of a HQ and a DM. The firms’ investment decisions are delegated to the DM. Time is continuous and the horizon is infinite. Both the HQ and the DM are risk neutral. The HQ discounts the future cash flows at rate $r$. The DM is more impatient and he discounts future consumptions at rate $\gamma > r$. This assumption rules out the possibility that the HQ delays payment to the DM forever.

2.3.1 Investment Opportunity

A distinct feature in this model is that the division’s investment opportunities are sparse, arriving stochastically over time. In particular, the projects arrive according to a Poisson process $\{N_t : t \geq 0\}$ with intensity $\lambda$. In other words, $N_t$ is the total number of projects arrived before time $t$. Another feature is that projects are heterogeneous in quality which is also stochastic. We use the random variable $J_n$ ($n \in \mathbb{N}$) to characterize the quality of the $n$th project. The qualities $\{J_n, n \in \mathbb{N}\}$ are i.i.d and uniformly distributed over the interval $\Theta = [\theta, \bar{\theta}]$ with $\theta \geq 0$. Moreover, project arrival is independent of project quality. In short, the division’s investment opportunities is summarized by the Compound Poisson process $X_t = \sum_{n=1}^{N_t} J_n$. The evolution of investment projects is easily described by the pair $S_t = (dN_t, dX_t)$: if no project arrives at time $t$, then $dN_t = dX_t = 0$; and if a project with quality $\theta$

---

6 In discrete time models, for example Clementi and Hopenhayn (2006), firms are assumed to have investment opportunity every period.

7 In the Appendix, we consider a more general distribution $F(\theta)$ with $F'(\theta) = f(\theta) > 0$ and provide sufficient conditions for all the analysis to hold.
arrives at time \( t \), then \( dN_t = 1, dX_t = \theta \).

When a project of quality \( \theta \) arrives at time \( t \), the firm will have the opportunity to make investment \( dK_t \geq 0 \). This investment will increase the project return through a value enhancing technology \( R : \mathbb{R}_+ \to \mathbb{R}_+ \). The total return obtained from the project is \( \theta + R(dK_t) \). Capital \( dK_t \) depreciates completely at time \( t \). After time \( t \), the firm has to wait until next project arrives to make another investment. In other words, the firm will not have any opportunity to invest and gets zero return during the no project period. In short, the investment return \( dY_t \) is determined by project arrival, project quality, and investment level in the relation of

\[
dY_t = (dX_t + R(dK_t))dN_t
\]  

(2.1)

The value enhancing technology \( R \) exhibits decreasing return to scale, that is, \( R' > 0, R'' < 0 \). Moreover, to guarantee an interior optimal investment in the frictionless case, we also assume \( R(0) = 0, \lim_{k \downarrow 0} R'(k) = \infty \), and \( \lim_{k \uparrow \infty} R'(k) = 0 \).

2.3.2 Information Frictions and the Mechanism Design

The agency issues arise due to two reasons. First, the DM has private information about the arrival time of projects and their qualities. The DM’s private information about project arrival is crucial for the subsequent investment and compensation decisions in the sense that if the DM does not reveal any investment opportunity to the HQ then the only viable option is to move forward and wait for future projects. In other words, the DM can choose whether the firm can obtain these investment opportunities. Second, the HQ can not enforce the DM to deploy the allocated capital. Once the capital is allocated to the DM, he can either invest or divert the allocated capital for personal consumption. These agency issues are modeled in a principal-agent contracting environment.

By the revelation principle, it suffices to restrict attention to the truth-telling
direct mechanisms. In other words, it is sufficient to consider only the mechanisms in which the DM reports whether there is a project and the project quality if there is one. And the optimal mechanism is designed to induce truthful report.

Denote the probability space as $(\Omega, \mathcal{F}, P)$, and the filtration generated by $\{S_t\}_{t \geq 0}$ as $\{\mathcal{F}_t\}_{t \geq 0}$. The reporting strategy $\hat{S} = \{(d\hat{N}_t, d\hat{X}_t) \in [0,1] \times \Theta\}_{t \geq 0}$ is a $\mathcal{F}$—adapted stochastic process. The direct mechanism $\Gamma$ is described by a triple $\{K, I, \tau\}$ such that $\{K_t\}_{t \geq 0}$ and $\{I_t\}_{t \geq 0}$ are $\hat{S}$ measurable stochastic process, and $\tau$ is a $\hat{S}$ measurable stopping time. Because the DM is subject to limited liability, feasible $I_t$ has to be a nondecreasing process. Moreover, because only positive investment is feasible, the process $K_t$ has to be nondecreasing as well.

The timing of how the sequence of events unfold is illustrated in figure 1. At the beginning of each interval $[t, t + dt]$, the division manager obtains the investment information $S_t$. This means that the DM knows whether a project arrives or not and also the quality of the project when it arrives. Then the DM reports $\hat{S}_t$ to the HQ. Given the report, the mechanism prescribes a capital allocation $dK_t$ to the division. Since the HQ cannot enforce any investment level, the DM can potentially choose a different level $d\hat{K}_t \in [0, dK_t]$. We assume that the DM has no savings $^8$. So the DM does not have any resource to invest a higher amount than $dK_t$. Then

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$^8$ An alternative assumption is that the DM’s saving rate is lower than the HQ’s saving rate. Then it is never optimal for the DM to save. Instead of letting the DM save, the HQ can always save at a higher rate and issue more dividends to the DM.
the return of the investment realizes according to (2.1). Finally, given the report history \( \{ \hat{S}_s : s \leq t \} \) and the return history \( \{ dY_s : s \leq t \} \), the mechanism prescribes an immediate compensation \( dI_t \geq 0 \) to the division manager, and decides whether to terminate the contract.

The outside option of DM is 0 when the contract is terminated. Given a mechanism \( \Gamma \), a reporting strategy \( \hat{S} \), and an investment process \( \hat{K} \), the DM’s expected payoff at time 0 is:

\[
W_0 = E_{\hat{S}} \left[ \int_0^\tau e^{-rt}(dI_t + dK_t - d\hat{K}_t) \right]
\]

Once the division of the firm is terminated, the HQ receives the liquidation value \( L \). The liquidation value and DM’s outside option satisfies \( rL < \lambda E(\theta + R(k^*) - k^*) \), where \( k^* \) is the first best investment level. This assumption means liquidation of the firm is inefficient. The HQ’s expected payoff at time 0 is:

\[
P_0 = E_{\hat{X}} \left[ \int_0^\tau e^{-rt}(dY_t - dI_t - dK_t) + e^{-r\tau}L \right]
\]

### 2.4 Frictionless Contract

Before investigating the optimal policies in the frictional environment, let us consider the first best case with no information friction. That is the HQ knows the arrival of projects and their qualities. Also, the HQ is able to implement any investment level. The frictionless contract has a simple form and is described in the following result.

**Proposition 2.4.1.** In the frictionless contract \( \{ K, I, \tau \} \) that delivers \( W_0 \geq 0 \) to the DM:

(a) investment is constant: \( k_t = k^* \) when there is a project, where \( R'(k^*) = 1 \).

(b) all compensation is paid out at time zero: \( dI_0 = W_0, \ dI_t = 0 \) for \( t > 0 \).
(c) the division of the firm will run without liquidation: $\tau = \infty$

2.5 Continuation Utility and Incentive Compatibility

The challenge in the dynamic setting is the complexity of the contract space. The contract can depend on the entire path of the reports and outputs. As noted in the literature of dynamic contracting, the DM’s continuation utility is a sufficient statistic. It can help analyze the agency issues in a tractable way. In this section, we first use the martingale approach to characterize the continuation utility. Then we characterize the incentive constraints in a set of conditions that typically arise in mechanism design problems.

The information and technology structures imply DM’s incentive has the following features. First, the DM tends to misreport project quality downward. Project quality and capital are substitutes in the firm’s investment technology. To generate the same return, the DM has to invest less capital as project quality increases. And because the HQ observes only the return but not the project type or investment level, the DM can divert part of allocated capital by misreporting a lower quality. Second, the DM can only misreport the project type to the extend that the observed return matches expected return implied by the report and the amount of allocated capital. Otherwise, the HQ can infer that the DM lied. Third, private information about project arrival time implies that the DM always obtain positive information rent by reporting any project. If the contract pays the DM a negative compensation of reporting a project, he can simply forgo the investment opportunity.

When no project arrives at time $t$, there is no return for any level of investment. Reporting a project arrives will be detected by the HQ. It is incentive compatible for the DM to report no project and hence gets no capital allocation. When a project of quality $\theta$ arrives at time $t$, the DM can either report no project or a different quality $\hat{\theta}$. In the former case, the project will be forgone. In the latter case, the DM has to
invest \(dK_t(\hat{\theta}; \theta)\) that satisfies \(\theta + R[dK_t(\hat{\theta}; \theta)] = \hat{\theta} + R(dK_t(\hat{\theta}))\). Otherwise, the HQ will detect the deviation. So the investment level in deviation is:

\[
dK_t(\hat{\theta}; \theta) = R^{-1}[\hat{\theta} + R(dK_t(\hat{\theta})) - \theta] \tag{2.2}
\]

The DM does not have saving and has to choose investment \(dK_t(\hat{\theta}; \theta)\) smaller than the allocated capital \(dK_t(\hat{\theta})\), which implies it only feasible to misreport downward, i.e. \(\hat{\theta} \leq \theta\). Moreover, since investment \(dK_t(\hat{\theta}; \theta)\) is positive, the misreported quality also has to satisfy \(\hat{\theta} \geq \theta - R(dK_t(\hat{\theta}))\). So the set of feasible misreports is \(\Gamma(\theta, dK_t) =: \{\hat{\theta} \in [\underline{\theta}, \bar{\theta}] : \theta - R(dK_t(\hat{\theta})) < \hat{\theta} < \theta\}\).

Given a report history \(\{\hat{S}_s, 0 \leq s \leq t\}\), if the DM reports truthfully after time \(t\) then he must also invest the full amount of the allocated capital. The DM’s continuation utility is the expected total discounted future compensation from the contract:

\[
W_t(\hat{S}) = E_t \left( \int_t^\tau e^{-\gamma(\tau-t)}dI_s \right) \tag{2.3}
\]

In the dynamic setting, the HQ has two ways of providing incentives: paying the DM cash payments \(dI_t\), or promising him future payments summarized by \(dW_t\). The following result provides a convenient representation of the promised future payments \(dW_t\).

**Lemma 2.5.1.** At any time \(0 \leq t \leq \tau\), there exists a \(\mathcal{F}_t\)-adapted sensitivity \(\beta_t\) of the continuation utility such that:

\[
dW_t = \gamma W_t - dt - dI_t + \int_\theta^\hat{\theta} \beta_t(\theta)[N(d(t, \theta)) - \frac{\lambda}{\Delta}d\theta dt] \tag{2.4}
\]

where \(W_{t-} = \lim_{s \uparrow t} W_s\), \(\Delta = \bar{\theta} - \underline{\theta}\).

The term \(N(d(t, \theta))\) indicates whether a project of quality \(\theta\) arrives at time \(t\), therefore reflecting the performance of the division. The DM’s compensation varies
with this performance measure by the magnitude of $\beta_t(\theta)$, which is interpreted as
the pay-performance sensitivity. The important feature is that the pay-performance
sensitivity is not constant but contingent on project type and the performance history
of the division. The intuition is that the DM’s incentive to misreport project quality
changes with the investment policy. In designing the optimal contract, the HQ faces
the following trade off: if the HQ allocates capital to DM reporting a lower type
more efficiently, then it has to pay larger information rent to DM reporting a higher
type.

Lemma 2.5.2. In any incentive compatible contract, the pay for performance sen-
sitivity $\beta_t$ and capital allocation policy $K_t$ satisfy:

(a) $\beta_t(\theta)$ is strictly increasing at $\theta$, if $dK_t(\theta) > 0$.

(b) $\beta'_t(\theta) \geq 1/R'(dK_t(\theta))$, if $dK_t(\theta) > 0$ and $\beta_t(\theta)$ is differentiable.

(c) $\beta_t(\theta) \geq 0$.

Because the DM can always misreport downward, a higher type must obtain a
larger compensation. Moreover, by misreporting a marginally lower type, the DM
can divert the amount $1/R'(dK_t(\theta))$. So raising the investment in type $\theta$ project will
make DM observing a marginally higher type project more willing to misreport as
type $\theta$. Any incentive compatible contract must compensate the DM at least this
amount. In the dynamic setting, this marginal gain is compensated by a higher slope
of the pay-performance sensitivity. The lower bound of $\beta_t$ is implied by the DM’s
private information regarding project arriving time, because the DM can always forgo
the project and get zero compensation.
2.6 Optimal Contract

In this section, we derive firm policy that maximizes the HQ’s value when the DM has private information about investment opportunities. The HQ’s value $P(W_t)$ is a function of the value promised to the DM, $W_t$. We use the dynamic programming approach to determine the most profitable way to deliver this promised value. Note that the marginal cost of compensating the DM can never exceed the marginal cost of immediate cash payment, since the HQ can always provide the DM with a lump-sum cash. So the value function must satisfy $P_p W_t q ¥ 1$ at any $W_t$. The cash payment boundary $\bar{W}$ is the smallest value such that $P(\bar{W}) = -1$. In deriving the optimal contract, we assume that $P(.)$ is concave and strictly concave when $W_t < \bar{W}$. We’ll show it is the case in the verification section.

2.6.1 Optimal Payment

The twin assumptions that, (i) DM is risk neutral (ii) terminating the division is inefficient, jointly determine that cash payments are postponed until the continuation utility reaches threshold $\bar{W}$. Before that, the DM is compensated purely through promised future values.

Lemma 2.6.1. When $W_t < \bar{W}$, no payment is issued to the DM, i.e. $dI_t = 0$. When $W_t \geq \bar{W}$, payment $dI_t = W_t - \bar{W}$ is immediately issued to the DM, and $P(W_t) = P(\bar{W}) - (W_t - \bar{W})$.

Different from the Brownian models, e.g. DeMarzo and Sannikov (2006b), the payment issuance in this model is determined by the jumps in DM’s continuation utility. Lemma 2.5.1 has shown that the DM’s continuation utility immediately jumps to a new level when a project arrives. According to Lemma 2.6.1, cash payment will be issued to the DM if this new level of continuation utility achieves the threshold $\bar{W}$.  

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2.6.2 A Heuristic Derivation

When the continuation utility \( W_t \) hits the DM’s outside option 0, the division will be terminated. The HQ receives the liquidation value \( P(0) = L \). In the interior region \( W_t \in (0, \hat{W}) \), the HQ holds all the investment returns. The HQ’s flow payoff consists of two parts: the net expected return of investment and the expected change in value function induced by the variation in \( W_t \). The net expected return of investment at time \( t \) is:

\[
E_t[(dY_t - dK_t)dN_t] = \frac{\lambda dt}{\Delta} \int_0^\theta \left[ \theta + R(dK_t(\theta)) - dK_t(\theta) \right] d\theta
\]

The HQ’s expected change in contract value is given by:

\[
E_t[dP(W_{t-})] = \left[ \gamma W_{t-} dt - \frac{\lambda dt}{\Delta} \int_0^\theta \beta_t(\theta) d\theta \right] P'(W_{t-})
\]

\[
+ \frac{\lambda dt}{\Delta} \int_0^\theta [P(W_{t-} + \beta_t(\theta)) - P(W_{t-})] d\theta
\]

Because at the optimum the HQ should earn an instantaneous total return of \( rP(W_{t-}) \), the Hamilton-Jacobi-Bellman (HJB) equation has the form \((W_{t-} \in [0, \hat{W}])\):

\[
rP(W_{t-}) = \max_{dK_t \geq 0, \beta_t \geq 0} \frac{\lambda}{\Delta} \int_0^\theta \left[ \theta + R(dK_t(\theta)) - dK_t(\theta) \right] d\theta \quad \text{(HJB)}
\]

\[
+ \left[ \gamma W_{t-} - \frac{\lambda}{\Delta} \int_0^\theta \beta_t(\theta) d\theta \right] P'(W_{t-})
\]

\[
+ \frac{\lambda}{\Delta} \int_0^\theta [P(W_{t-} + \beta_t(\theta)) - P(W_{t-})] d\theta
\]

s.t. \( \theta = \arg \max \left[ \beta_t(\hat{\theta}) + dK_t(\hat{\theta}) - dK_t(\hat{\theta} ; \theta) \right] \quad \text{(IC)}
\]

\[
P(0) = L, P'(\hat{W}) = -1 \quad \text{(BC)}
\]

The constraints (IC) and \( \beta_t \geq 0 \) guarantee that the DM truthfully reveals his private information about investment opportunities. The condition (BC) pins down
the value function and the payout boundary $\bar{W}$. To solve (HJB), we use the first order approach that is typically applied in mechanism design problems (e.g. Myerson (1981)). In particular, we consider a relaxed problem by replacing (IC) with a necessary condition. We then verify the policy derived from the relaxed problem satisfies (IC).

The decision of project selection determines the feasible misreports $\Gamma(\theta, dK_t)$. To replace the (IC) constraint, we have to know which projects will be forgone. When the spread of project qualities ($\Delta$) is sufficiently small, the optimal contract will allocate capital to all projects, i.e. $dK_t(\theta) > 0$ for all $\theta$. This is because the information asymmetry is not severe. In the case of $\Delta \approx 0$, the HQ almost knows the true project quality even without DM’s report. So the firm pays little information rent to the DM. The infinite marginal return of value enhancing investment at $dK_t = 0$ (the Inada condition) implies that selecting all projects is optimal when $\Delta$ is sufficiently small. We will start the analysis with this case. Then we move on to show that as the spread of qualities increases, projects with bottom qualities are possibly forgone.

2.6.3 No Project Exclusion

When no project is excluded, Lemma 2.5.2 provides a necessary condition implied by (IC), that is, $\beta_t(\theta) \geq 1/R'[dK_t(\theta)]$. Using this condition to replace (IC), we can find a necessary condition that characterizes the optimal capital allocation and pay-performance sensitivity. To guarantee that HJB has a unique solution, we assume that $R''R' \leq 3[R'']^2$ holds for the rest of the analysis. In the case of $R(k) = ak^\alpha$, this technical assumption implies $\alpha \in (0, \frac{1}{2}]$.

**Lemma 2.6.2.** In the no payment region ($W_{t-} \in (0, \bar{W})$), if no project is excluded from capital allocation, then the optimal policy satisfies:

$$R'(dK_t(\theta)) - 1 = \frac{R''(dK_t(\theta))}{[R'(dK_t(\theta))]^2} \int_{\theta}^{\bar{\theta}} [P'(W_{t-} + \beta_t(u)) - P'(W_{t-})] \, du$$

(2.5)
Moreover, capital allocation \(dK_t(\theta)\) increases in project quality \(\theta\).

Lemma 2.6.2 characterizes the distortion in investment due to agency cost. The left-hand side of (2.5) is the net return of marginally raising investment in type \(\theta\) project. The strict concavity of the value function \(P\) and the technology \(R\) together implies that the right-hand side of (2.5) is positive and represents the agency cost. It is easy to see that investment in any project is lower than the first best level \(k^*\). Investment distortion in any project arises from the information rents paid to the DM reporting projects of higher qualities. As project quality decreases, more information rent is paid to the DM, and therefore investment distortion is more severe.

An important feature is that the agency cost in capital budgeting is endogenous. The information rents are costly to the firm because they induce jumps in DM’s continuation utility when projects arrive, and downward drift when no project arrives. These variations in continuation utility exacerbate liquidation of the division. In particular, marginal information rent in project \(\theta\) leads to the cost \(P'(W_{t-} + \beta_t(\theta)) - P'(W_{t-})\). So the curvature of the value function and the level of the continuation utility together determine the magnitude of the agency cost. In this respect, this model is very different from static problems such as Laffont and Tirole (1986), and Myerson (1981). In those problems, the agency costs are always exogenously related to the inverse hazard ratio.

2.6.4 Policy with Project Selection

The optimal contract distorts investment downward to economize on information rents. As the spread of project quality increases, investments are severely distorted. When the agency cost is sufficiently high, the extreme policy is to forgo positive NPV investment opportunities. If type \(\theta\) project gets no capital allocation, then DM will not be able to misreport project quality as \(\theta\) when observing higher types. So excluding projects from investment can relax agency issues. We now examine the
general case when project exclusion possibly occurs.

**Proposition 2.6.3.** In the no payout region \(W_{t-} \in (0, W)\), there exists threshold project quality \(\theta^*_t = \theta^*(W_{t-}) \leq \bar{\theta}\) such that

(a) Projects with reported quality above \(\theta^*_t\) receive positive capital allocation and have positive pay-performance sensitivity: \(dK_t(\theta) > 0\), \(\beta_t(\theta) > 0\), and \(\beta'_t(\theta) = 1/R'(dK_t(\theta))\) a.e., for any \(\theta \in (\theta^*_t, \bar{\theta}]\).

(b) Projects with reported quality below \(\theta^*_t\) receive no capital allocation and has zero pay-performance sensitivity: \(dK_t(\theta) = \beta_t(\theta) = 0\), for any \(\theta \leq \theta^*_t\), if \(\theta^*_t > \bar{\theta}\).

Proposition 2.6.3 shows the optimal exclusion exhibits threshold properties. If exclusion ever occurs, projects with low quality, i.e. types lower than the threshold \(\theta^*_t\), will not be allocated capital. Recall Lemma 2.6.2 shows that the capital allocation increases with project type when there is no exclusion. The lowest quality project gets the smallest amount of capital. If we keep the state variable \(W_{t-}\) constant and increase the spread of project quality \(\Delta\), the bottom type will reach zero allocation before other types.

The threshold exclusion policy also implies zero pay-performance sensitivity for types lower than the threshold \(\theta^*_t\). If the DM gets no capital allocation by reporting project type \(\theta\), then it means no type below \(\theta\) will be allocated capital. Hence it is not feasible for the DM with project \(\theta\) to lie as a lower type. It is optimal to compensate zero information rent to the DM, since positive pay-performance sensitivity only adds variation to the DM’s continuation utility and lowers the HQ’s expected value.

The threshold \(\theta^*_t\) is an endogenous object that is moving with the state \(W_{t-}\). This is exactly because the agency cost itself is endogenous. When the endogenous agency cost is high, then only marginal investment in higher quality project can
balance this cost. So the threshold $\theta^*_t$ takes a higher value. Similarly, when the
endogenous agency cost is low, the threshold $\theta^*_t$ takes a lower value.

Whether to exclude any project with quality $\theta$ depends on: (i) the marginal
return of investment at 0 which is $R'(0)$; (ii) the agency cost induced by the marginal
investment. According to (IC), the marginal investment in project $\theta$ will raise the
pay-performance sensitivity by $R''(0)/[R'(0)]^2$, which means (IC) for types above $\theta$
are tighter. The tighter (IC) will induce agency cost to increase by $\frac{\mu_t(\theta)R'(0)}{[R'(0)]^2}$, where
$\mu_t(\theta)$ is the multiplier of (IC). So the optimal exclusion is determined by these two
driving forces.

Lemma 2.6.4. In the optimal contract:

(a) If the value enhancing technology satisfies $\lim_{k \to 0} \frac{R'(k)}{R'(0)^{\frac{3}{2}}} = 0$, then the HQ will
allocate capital to all projects at any state $W_{t-} > 0$, i.e. $\theta^*_t = 0$;

(b) If the value enhancing technology satisfies $\lim_{k \to 0} \frac{R'(k)}{R'(0)^{\frac{3}{2}}} < 0$, then depending
on state $W_{t-}$, the optimal contract possibly excludes low quality projects, i.e.
$\theta^*_t > 0$.

Lemma 2.6.4 illustrates that if the marginal return dominates the agency cost
induced by the marginal investment as capital allocation converges to zero, then
exclusion will not happen at any state. However, if the agency cost induced by the
marginal investment dominates the marginal return as capital allocation converges
to zero, then exclusion of investment possibly occurs, depending on the state $W_{t-}$.

Proposition 2.6.5. The optimal capital allocation policy satisfies

(a) No investment distortion for highest quality project if the division is not liqui-
dated:

$$R'(dK_t(\bar{\theta})) = 1, \quad \text{when } W_{t-} > 0$$
(b) Underinvest in all other projects before payout:

\[ R'(dK_t(\theta)) > 1, \ \forall \ \theta^*_t \leq \theta < \bar{\theta}, \quad \text{when } 0 < W_{t-} < \bar{W} \]

Mover, capital allocation is increasing in project quality.

(c) At the payout boundary, no project is excluded, and investment distortion disappears for all projects:

\[ \theta^*_t = \bar{\theta}; \ R'(dK_t(\theta)) = 1, \ \forall \ \theta, \quad \text{when } W_{t-} \geq W \]

The no investment distortion at highest project quality coincides with the classical result of Mussa and Rosen (1978). Surprisingly, there is also no investment distortion for all types at the highest continuation utility level (\(W\)). In other words, the capital allocation reaches the frictionless benchmark at either the highest project quality or the highest continuation utility. Otherwise, the investment is always distorted downward and varies with both project quality and continuation utility. Theses results arise because the agency cost in this model is endogenous.

Figure 2.2 plots the policy \(\beta_t, K_t\) in a numerical example. In this example, \(\bar{W} = 1.574\), and \(k^* = 0.5\). The right panel shows at the payout boundary \(\bar{W}\) capital allocation to all projects resume the first best level \(k^*\). Given any level of continuation
utility, capital allocation is increasing in project qualities. As continuation utility drops, capital allocation distorts downward. And lower quality projects are distorted more severely. The left panel shows the pay-performance sensitivity is increasing over project types and over continuation utility.

The level of continuation utility indicates the tightness of the limited liability constraint. When the continuation utility is close to the liquidation boundary, the limited liability constraint is very tight. As the continuation utility goes up, the limited liability constraint relaxes and the liquidation concern decreases. At the payout boundary $\bar{W}$, the liquidation concern disappears. Moreover, the liquidation concern determines the trade-off between investment efficiency and information rents. To induce larger investment, the contract has to design larger pay-performance sensitivities, lowering the drift of DM’s continuation utility. When the liquidation concern is sever, it is optimal to lower pay-performance sensitivities which severely distorts capital allocation. When the liquidation concern is low, it is not very costly to design large pay-performance sensitivities. So the optimal contract induces more efficient investments.

2.6.5 Sufficiency

This section verifies that the optimal policy derived from the relaxed problem actually satisfies (IC) and therefore is the optimal policy of the (HJB).

**Lemma 2.6.6.** When $W_{t-} \in (0, \bar{W})$, the policy $K_t^*(\theta)$, $\beta_t^*(\theta)$ satisfies

(i) condition (2.5) and $\beta_t^*(\theta) = 1/R'(dK_t^*(\theta))$, when $\theta \geq \theta_t^*$;

(ii) $\beta_t^*(\theta) = 0$, when $\theta \leq \theta_t^*$;

(iii) $dK_t^*(\theta) = 0$, when $\theta \leq \theta_t^*$ and $\theta_t^* > \bar{\theta}$

is the optimal policy of the (HJB).
In the no exclusion region ($\theta > \theta_t^*$), the monotonicity of capital allocation implies that the envelop condition $\beta_t^*(\theta) = 1/R'(dK_t^*(\theta))$ is sufficient for incentive compatibility. In the exclusion region ($\theta \leq \theta_t^*$), because the DM cannot misreport as lower types, the optimal contract only needs to induce truth telling regarding project arrival time. So zero pay-performance sensitivity is sufficient.

2.6.6 Verification

In this section we verify that the heuristic characterization does correspond to the optimal contract. Following the standard argument in optimal control, we show that HQ’s value from any incentive compatible mechanism that delivers DM continuation utility $W_0$ is at most $P(W_0)$, which is HQ’s expected payoff from the conjectured mechanism. We also show that the value function is concave as assumed in the heuristic derivation.

**Proposition 2.6.7.** The contract that maximizes the HQ’s expected profit and delivers value $W_0 \in [0, \bar{W}]$ to the DM has the following form:

1. $k_t^*(\theta), \beta_t^*(\theta)$ satisfies all the conditions in Lemma 2.6.6.

2. $W_t$ evolves according to:

$$dW_t = \gamma W_t dt - dI_t + \int_\theta^{\hat{\theta}} \beta_t^*(\theta)[N(d(t, \theta)) - \frac{\lambda}{\Delta} d\theta] dt$$

3. When $W_{t-} \in [0, \bar{W})$, $dI_t = 0$; When $W_{t-} = \bar{W}$, payments $dI_t$ cause $W_t$ to reflect at $\bar{W}$.

4. Liquidation of the division occurs when $W_t$ reaches 0.

5. The HQ’s expected payoff $P(W_{t-})$: matches the objective of the (HJB) evaluated at $k_t^*(\theta), \beta_t^*(\theta)$ on interval $[0, \bar{W}]$; satisfies $P'(W_{t-}) = -1$ when $W_{t-} \geq \bar{W}$, and $P(0) = L.$
6. The value function $P(W)$ is globally concave and strictly concave when $W < \bar{W}$.

2.7 Dynamics of the Optimal Contract

The investment and compensation policy have different properties at different levels of the state, i.e. continuation utility. The investments are distorted downward in general, but the distortion disappears when the continuation utility reaches its upper bound. This clearly shows that the severity of the agency issues are endogenous and therefore the optimal contract should be designed to vary over the past performance of the DM. In this section, we will characterize the dynamics of the model. The optimal policy will be shown to exhibit monotone properties. These properties sharpen the intuition that as continuation utility goes up, the agency cost falls. The DM will be allocated more capital for investment and the DM will also be compensated more.

2.7.1 Evolution of Continuation Utility

Because of the agency issues, the compensation (information rent) to DM has to be positive if the firm decides to invest in the value-enhancing technology. When
project above the exclusion threshold arrives, the DM’s continuation utility will jumps up immediately to reflect this positive compensation. However, the expected amount of these compensations will be taken out from the DM’s continuation utility at other time, as long as the continuation utility is still above 0, DM’s outside option. This mechanism poses a downward force to the continuation utility when no project arrives or project below the exclusion threshold arrives. In this sense the contract “punishes” the DM at time of no capital allocation. This mechanism is feasible without violating limited liability simply because the DM is “punished” by continuation utility not cash. The following result formally shows these intuitions in a large part of the domain.

We use $W^*$ to denote the continuation utility level that maximizes the HQ’s expected payoff, i.e. $P'(W^*) = 0$.

**Proposition 2.7.1.** *In the region $W_{t-} \geq W^*$, the DM’s continuation utility jumps up when projects above the exclusion threshold arrives and it drifts downward during other time.*

From the simulated path of continuation utility in figure 2.4, we can see the patten
in Proposition 2.7.1 holds in more general region of the domain. Cash compensation will be issued when the continuation utility jumps beyond the payout boundary. Cash payment amounts are reflected by the bars on the horizontal axis in figure 2.4. And the cash payments will drive the continuation level immediately back to the payout boundary.

2.7.2 Policy Dynamics

We first examine the investment dynamics around the payout boundary. In particular, capital allocation exhibits monotone property when continuation utility is high in the sense that after investing in one project the DM will be issued cash payment.

Formally, we define a lower bound of continuation utility for each project type as

\[
\hat{W}_p^\theta = \inf_t \{ W_t \leq \hat{W} : W_t + \beta_t(\theta) > \hat{W} \}. 
\]

This definition means if the continuation utility is \(\hat{W}(\theta)\) or higher right before the DM operates a type \(\theta\) project, then his continuation utility level will reach the payout boundary after investing in project \(\theta\). Intuitively, we can interpret these lower bounds as the “one-step” away levels from cash payment for operating each project. Because the compensation in terms of continuation utility is increasing in project type, these one step-away levels are actually decreasing in types. This means the DM gets cash payments earlier if he operates a higher quality project. Moreover, the DM never gets cash payment by operating the lowest quality project.

**Lemma 2.7.2.** The one-step away level of continuation utility:

(a) is decreasing in project type;
(b) is equal to the pay-out boundary for the lowest quality project: \(\hat{W}(\theta) = \hat{W}\).

On one hand, the continuation utility, the compensation \(\beta_t(\theta)\), and the curvature of the value function \(P\) all affect the agency cost and hence optimal capital allocation. On the other hand, the (IC) imposes constraint on compensation by the
level of capital allocation. So in general it is hard to characterize the policy dynamics as DM’s continuation utility varies. We process the characterization by two steps. First, we consider the case when continuation utility is above the one-step away level. In this case, the agency cost is not affected by the jump size in continuation utility. Second, we restrict attention to the Cobb-Douglas form of value-enhancing technology, but consider general levels of continuation utility.

**Proposition 2.7.3.** Capital allocation is increasing in continuation utility when it is above the one-step away level: \(dK_t(\theta)\) is increasing in \(W_{t-}\) when \(W_{t-} \geq \hat{W}(\theta)\).

To better characterize the dynamic properties, we restrict attention to the case where the value enhancing technology is Cobb-Douglas. That is \(R(k) = ak^\alpha\), where \(a > 0\) and \(0 < \alpha < 1\). In Section 2.6, we assume the technology satisfies the condition \(R\theta R' \leq 3(R\theta)^2\). This assumption is equivalent to assume \(0 < \alpha \leq \frac{1}{2}\) in the Cobb-Douglas form. Moreover, by Lemma 2.6.4, we know the exclusion of projects possibly occurs when \(\lim_{k \to 0} \frac{R'(k)}{R(k)} < 0\). And this condition is equivalent to \(\alpha \geq \frac{1}{2}\). We will focus on this more general case where exclusions possibly occur. So in the following results, we restrict attention to the value-enhancing technology \(R(k) = ak^{\frac{1}{2}}\). Using this technology form, we can simplify the HJB and the optimal conditions as a system of differential equations.

**Lemma 2.7.4.** If the value-enhancing technology has the form \(R(k) = ak^{\frac{1}{2}}\), then the optimal policy \(K_t, \beta_t, \theta_t^*\) satisfies the following differential equation system:

\[
a[dK_t(\theta)]^2 = \frac{1}{2}a^2 - \int_0^\theta [P'(W_{t-}) - P'(W_{t-} + \beta_t(u))]du \quad (\theta \geq \theta_t^*)
\]  

(2.6)

\[
rP(W_{t-}) - \gamma W_{t-}P'(W_{t-}) = \frac{\lambda}{\Delta} \int_0^\theta [\theta + a[dK_t(\theta)]^2 - 2dK_t(\theta)]d\theta + \lambda dK_t(\theta_t^*)
\]  

(2.7)

\[
\beta_t'(\theta) = \frac{2}{\alpha} [dK_t(\theta)]^\frac{1}{2} \quad (\theta \geq \theta_t^*)
\]  

(2.8)
Figure 2.5: Policy Dynamics over Continuation Utility

with boundary conditions:

1. \( P(0) = L \) and \( P'(\bar{W}) = -1 \);

2. \( \beta_t(\bar{\theta}) = 0 \) and \( \beta_t'(\bar{\theta}) = 1 \), if \( \theta^*_t = \bar{\theta} \);

3. \( \beta_t(\theta^*_t) = 0 \), \( \beta_t'(\theta^*_t) = 0 \), and \( \beta_t'(\bar{\theta}) = 1 \), if \( \theta^*_t > \bar{\theta} \).

**Proposition 2.7.5.** There exists a continuation utility level \( W_{t-} \in (W^1, \bar{W}) \): the capital allocation \( dK_t(\theta) \), and pay-performance sensitivity \( \beta_t(\theta) \), are increasing in \( W_{t-} \) for all project types; the cutoff project quality for exclusion \( \theta^*_t \) is decreasing in \( W_{t-} \). Moreover, the cutoff value \( W^1 \) satisfies \( P'(W^1) > \frac{a^2}{4\Delta} - 1 \).

**Corollary 2.7.6.** The cutoff value \( W^1 \): i) decreases in the efficiency of investment technology \( a \) increases; ii) increases in the spread of project qualities \( \Delta \).

In figure 2.5, we fix project quality at different levels and plot how the policy varies with continuation utility. In the numerical example, the optimal capital allocation and pay-performance sensitivity exhibit monotone property over most of the domain \((0, \bar{W})\).
2.8 Implementation

In this section, we show that the optimal contract can be implemented by a capital budgeting mechanism. In the mechanism, the firm sets a budgeting account from which the DM can withdraw funds to invest or consume. The HQ can observe the account balance and control the evolution of the balance. According to the DM’s report of project information, the HQ either depletes or replenishes the budgeting account. The budgeting account implementation includes the following elements:

- The firm sets a capital budgeting account with the initial balance of $M_0$.

- The DM can withdraw funds from the budgeting account at any time to either invest in projects or consume.

- The HQ constantly depletes the budgeting account at rate $g(M_t) = \frac{\lambda}{\bar{M}_t} \int_{\theta}^{\theta'} \beta_t^*(\theta) - \gamma$.

- When a project of quality $\theta$ is reported by the DM, the HQ will replenish the account by amount $\beta_t^*(\theta) + dK_t^*(\theta)$.

- When the account balance reaches $\bar{M}$, the DM will receive cash payment for the amount of balance higher than $\bar{M}$.

- Liquidate the division when the balance reaches 0.

According to the this mechanism, the budgeting account balance will evolve as $dM_t = -g(M_t)M_t dt$ when no project arrives, and jumps by the amount $\beta_t^*(\theta)$ when a project of quality $\theta$ arrives. The HQ can adjust the initial balance and cash payout boundary appropriately to induce truth-telling of project information and implement the optimal contract.


**Proposition 2.8.1.** In the capital budgeting mechanism with initial account balance $M_0 = W_0$ and payout boundary $\bar{M} = \bar{W}$, it is incentive compatible for the DM to report truthfully of project information, and to refrain from stealing funds from the budgeting account. Moreover, this mechanism implements the optimal contract.

Under the designed mechanism, the DM has no incentive to divert investment funds because he can always withdraw the same amount from the budgeting account. How can we ensure that the DM will not withdraw funds before the account balance reaches the payout boundary? The DM can withdraw all the funds in the budgeting account and then liquidate the division. From this strategy the DM gets the total gain $M_t$. However, the payoff in deviation is equal to $M_t$, the payoff that the DM obtains from waiting until the account balance reaches $\bar{W}$ to receive cash payment. This is because by design the evolution of the budgeting account balance mimics that of the continuation utility in the optimal contract. Moreover, the funds replenished to the budgeting account when projects arrive guarantee that the DM’s expected compensation increases by $\beta_t^*(\theta)$. Because this increase in compensation satisfies (IC), the DM will always report project information truthfully.

2.9 Conclusion

By constructing a dynamic principal-agent model with continuous time and continuous project qualities, this paper shows how capital budgeting and managerial compensation contracts can be jointly designed to mitigate two important agency issues in conglomerates: asymmetric investment information and capital diversion. The trade off between extracting truthful information and inducing efficient investment is shown to be time varying and endogenously determined by the DM’s continuation utility or the financial slack of the division. When the division has high financial slack, the model predicts that the optimal contract will have high pay-performance.
sensitivity and allocate more capital to the DM. The steep contract makes it less costly to extract DM’s private information about investment opportunities. It is optimal to induce more efficient investment level. The investments in all type of projects are shown to reach the first best level when the division’s financial slack is high enough. When the division has low financial slack, the model predicts that the optimal contract will have low pay performance sensitivity and allocate less capital to the DM. Extracting information will be costly. The problem of underinvestment will be severe in this case. The paper provides both analytical and numerical analysis and illustrates the intuition behind these predictions.
3

Dynamic Security Design with Persistence

3.1 Introduction

There is considerable evidence that funding for firms, especially young firms, is far from efficient, and that firms must grow over time into their optimal size. In particular, financing constraints greatly affect firm size, growth, and a young firm’s prospects.¹ Starting from Jensen and Meckling (1976), a large literature has focused on the conflicts of interests between investors and agents as a key friction that constrains firm financing and investment. Agency problems arise because agents have more information about their firm behavior and their own actions than the outside investors. Financial contracting therefore characterizes the securities and firm policies designed to mitigate agency conflicts.

An influential recent literature analyzes agency problems in dynamic contexts. For instance, Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007b) characterize the optimal long-term contract and firm dynamics when agency frictions

¹ Gertler and Gilchrist (1994) find that manufacturing significantly declines in small firms when monetary policy tightens. Beck et al. (2005) documents that financial constraints create obstacle to the growth of small firms, and that small firms benefit most from financial development.
are involved. These models have advantage in analyzing how firm investment, payout, capital structure and other policies evolve, and how firms grow over time. However, they typically assume by convenience that agents’ private information about firm behavior is iid over time, because considering persistence has proved challenging in this class of models. In practice, firms’ profitability or other economic behavior, which is agents’ private information in these models, exhibits high autocorrelation. For instance, Gomes (2001) calibrated that autocorrelation of firm productivity shocks is 0.62 at the annual frequency. Many other researches show even higher numbers. So having private information about firm behavior today will definitely help agents to better predict firms’ future prospects. Moreover, if we want to take the dynamic agency models seriously or quantify these models, then adopting the persistent assumption becomes crucial.

In this paper, we incorporate persistent private information in a dynamic agency model of firm financing. The firm consists of a risk-neutral agent and risk-neutral investors. The agent has the expertise to operate the firm but does not have funds. The investors provide funds to launch the firm and finance firm’s risky investment in each period. The agency problem is that the cash flows from investment projects are privately observed by the agent. So the agent has incentive to divert firm cash for his own consumption. The key element is that firm cash flows are subject to positively correlated shocks which follow a two-state Markov process.

We develop a method to solve the optimal contract in this environment. When the cash flow shocks are iid, it is well known that promising the agent continuation values contingent on his report today is sufficient for truth telling. However, this

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3 Nikolov and Schmid (2012) estimate a q-theoretic dynamic agency model with iid cash flow shocks. They find that agency conflict between investors and agent needs to be substantial in order to rationalize observed firm financing and investment policies.
approach is no longer sufficient in the persistent environment. This is because agent’s report today also contains private belief about the likelihood of future shocks. To induce truth telling, the contract has to take into account the agent’s private benefit from i) diverting today’s cash flow and ii) distorting investors’ belief about future shocks. We show a sufficient method is to promise the agent different contingent continuation values from tomorrow onward according to his report today. In other words, the continuation values will be contingent on both today and tomorrow’s report. By characterizing and implementing the optimal contract, we find that the persistent environment implies several distinct features regarding firm investment, capital structure, compensation, and growth dynamics.

In the iid case, investment is only determined by the credit availability of the firm. When the firm is young and constrained, its investment is very sensitive to cash flow shock. But investment becomes constant (at the efficient level) when the firm is unconstrained. Such dynamics cannot be reconciled with the evidence shown by Kaplan and Zingales (1997): less financially constrained firms exhibit greater investment to cash flow sensitivity. In our model, besides the financial slack of the firm, investment is also determined by investors’ belief about the likelihood of good shock in the future, which in turn depends on the current cash flow shock. So investment is varying (at efficient levels) with firm cash flow even when it is unconstrained.

A key feature of the implementation in our model is to finance the firm by credit line contingent on maintaining cash flow covenant. On the contrary, the iid models are always implemented by an uncontingent credit line with fixed limit. In our model, because the agent has private belief about future cash flow shocks, the variations in continuation values (or the unused credit lines in the implementation) has to be no less than the current cash flow plus a dynamic information rent. Due to this new element of dynamic information rent (which disappears in the iid case), credit limit
has to be adjusted according to cash flow shocks in order to implement our optimal contract.

Credit line is an important way of firm financing and liquidity management. According to Demiroglu and James (2011), drawdowns of credit line account for 75% of bank lending to firms and 63% of corporate debt. In our model, when the firm violates cash flow covenant at any time, its current credit limit will immediately drop, and its future contingent credit limits will also be reduced. These results very well rationalize how the credit line is provided in practice. Empirically, cash flow based covenants are typically written into the credit line contract. Also most credit lines have material adverse change (MAC) clauses which permit lenders to withhold funds if a borrower’s credit quality deteriorates significantly. According to Sufi (2009), a covenant violation is associated with a 15% to 25% drop in the availability of total line of credit.

The long-run incentive provision and implied compensation scheme in our model are also in stark contrast to the iid models. In dynamic contexts, the variation of future investments and the associated information rents provide incentives for the agent to report cash flow truthfully today. In the iid case, investment and associated information rent become constant when the firm is unconstrained. The only way to provide incentive in the long-run is to issue the agent the same amount of cash payment as what he can divert. So agent’s payment is exogenously determined and linear in firm performance. In the persistent case, since it is efficient to reduce

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4 According to Demiroglu and James (2011), coverage and debt-to-cash flow covenants are the two most common financial covenants. Coverage covenants require that a borrower’s coverage ratio (typically the ratio of EBITDA to fixed charges or interest expenses) remain above a minimum and debt-to-cash flow covenants restrict borrowing if the ratio of debt-to-cash flows exceeds a preset maximum.

5 Both DeMarzo and Samnikov (2006a) and DeMarzo and Fishman (2007b) assume that agent can divert $\lambda \leq 1$ fraction of firm cash flow and hence is compensated this amount by cash when the firm pays off its debt. Clementi and Hopenhayn (2006) and our model correspond to the case of $\lambda = 1$, and can also be extended to the case of $\lambda < 1$. 

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investment level after a bad shock, investors have an added threat to punish the agent if he were to divert cash flow. Hence the agent only receives an endogenous payment less than what he can divert.

The agent’s payment in our model is shown to be convex in firm performance or stock price. In particular, the agent gets larger fraction of firm payouts as firm performance improves. If the persistence level is high, the agent only gets positive payment when good performances are observed in a row or when the highest stock price is achieved. Otherwise, investors obtain all the firm payouts. We show that this convex payment scheme can be implemented by a combination of stock options and equity, the composition of which depends on the persistence level. And as persistence level increases, stock option payoff accounts for a larger portion of the agent’s payment. On the contrary, because compensation is always linear in firm performance in the iid models, agent only holds equity\(^6\) and options never play a role.

Empirically, stock options are a popular way of compensating executives and employees. 71% of the 250 largest U.S. companies in the Standard & Poor’s 500 Index use stock options as incentive grant.\(^7\) According to Larcker (2008), payment from stock options accounts for 27% (the largest component) of CEO compensation in the top 4000 U.S. companies. Bergman and Jenter (2007) also shows that stock options plan is the most common method for employee compensation below the executive rank.

Regarding the relation between firm size and age in the growth dynamics, our model better matches the empirical finding of Hurst and Pugsley (2011): most small

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\(^6\) For instance, the agent in Clementi and Hopenhayn (2006) is the firm’s residual claimant and holds all the equity. Although we also assume the agent can divert all the cash flow, he is not the residual claimant and investors in our model also hold an equity stake.

firms stay in small size for long time and therefore are old firms. Because of persistence, the firm in our model is more likely to receive many bad shocks in a row and becomes more financially constrained. Moreover, investments after bad shocks will be kept low, slowing down firm growth and extending the stage of the firm being constrained. Therefore, the firm in our model possibly experiences longer time of being constrained than in the iid case, and stays in small or inefficient size for longer time.

In our model the constrained firm can temporarily invest at an efficient (first best) level before it becomes unconstrained; while in the iid case, investment is always distorted downward before the firm is unconstrained. This happens when the persistence level is high and a bad shock previously hit the firm. If that is the case, by misreporting the agent can divert little cash today because today’s investment level is low after a bad shock. Moreover, the agent will be punished hard by low level of future investments after reporting a bad shock today. So the agent is possibly better off by reporting truthfully. Or in other words, the incentive compatibility constraint is possibly slack. If this situation happens, the investment will be efficient even before the firm becomes unconstrained.

Consistent with the implications from the iid models, our model also predicts that payouts is delayed until the firm is unconstrained and investments are positively correlated over time. These predictions are robust for all persistence levels. Delaying payments is optimal since it reserves more liquidity in the firm to improve future investment efficiency. And since in the dynamic context firm policies always exhibit history dependence, investments will be positively correlated regardless of the persistence level. But of course, high persistence implies high autocorrelation of investment.

We review the relevant literature in Section 2. Section 3 introduces the model. Section 4 discusses the sequential contracts and our approach of recursively formu-
lating the problem. Section 5 describes the optimal contract for the young firm as it pertains to compensation, investment, and the evolution of equity, while Section 7 looks at the optimal contract for a mature firm. Section 6 shows that every firm will eventually mature (we ignore the possibility that the firm will be liquidated), while Section 8 discusses empirical implications and stylized facts. Section 9 discusses the initialization of the contract, the nature of path dependence, and the possibility of liquidating the firm. Section 10 concludes, and all proofs are in the appendices.

3.2 Related Literature

Our work builds on a literature that studies the financing of firms under asymmetric information, typically assuming that the agent can divert cash flows without the principal’s knowledge.\(^8\) An early and seminal paper in this literature is Bolton and Scharfstein (1990), who study a two-period model, where the threat of early termination provides incentives in the first period. Fully dynamic versions of CFD models are Clementi and Hopenhayn (2006), Biais et al. (2007), and DeMarzo and Fishman (2007a), where the latter two emphasize the implementation of the optimal contract via standard securities. All these papers, regardless of time horizon, consider iid shocks to the output process. We consider the same discrete time economic environment as Clementi and Hopenhayn (2006), except that we allow for persistence in the shocks to the output process.

Infinite horizon (iid) screening models were first studied by Thomas and Worrall (1990), who introduce recursive methods to such problems, and show that by using the utility promised to the agent as a state variable, the optimal contract can be reduced to a Markov decision process for the principal. Although the literature on firm financing has focused on the iid case, there is nonetheless a literature on dynamic screening with Markovian types. The recursive approach is emphasized by

\(^8\) Such models are therefore referred to as cash flow diversion (CFD) models.
Fernandes and Phelan (2000), who note that promised utility alone is inadequate in the Markovian case. To recursively formulate the problem, they use two *ex ante* promised utilities, one from truth-telling and the other from lying. Although we also use a vector of promised utilities, they are *interim*, contingent on the production shock in the period. Our state variables are easier to specify the domain of the dynamic programming, allowing for analytical characterization.

Doepke and Townsend (2006) extends the environment to incorporate both hidden states and hidden actions. They focus on how to reduce constraints by imposing off-path utility bounds and how to numerically solve their model. Instead, we focus on analytical characterization and implementation of the optimal contract. Tchistyi (2013) studies an environment similar as ours except that his model has finite periods and no investment. Although Tchistyi (2013) uses only one ex ante promised utility, a time varying functional has to be defined that transforms the agent’s on-path utility to the utility from lying. Kapička (2013) uses a first-order approach to study an environment with continuum of states that are persistent. If the information rent is monotone, Kapička (2013) shows that the state variables can be reduced to two numbers: continuation utility and marginal continuation utility. However, the validity of the first-order approach is hard to verify.

Battaglini (2005) considers the problem of a principal sells some quantity of a good to a consumer, whose valuation for the good follows a two-point Markov process. Pavan et al. (2014) and Eső and Szentes (2013) study mechanism design in dynamic quasilinear environment where the agent’s persistent private information are described by ”impulse response functions”. The key difference from our model is that our agent has no cash and is subject to limited liability, while the agent in these models has deep pocket, and only has a participation constraint that must be respected at each point in time. With no limited liability constraint, these models have the same feature: the principal’s expected payoff from implementing an alloca-
tion is the same as if he could observe the agent’s orthogonalized private information after the initial period. It is precisely the inability of the principal to extract future information rents that makes our model economically interesting.

While not directly related to the principal-agent literature, Halac and Yared (2014) consider the problem of a government that has time-inconsistent preferences. The government privately observes shocks that follow a two-state Markov process. Using the same techniques as in this paper, Bloedel and Krishna (2014) study the question of immiseration in a problem of risk-sharing where the agent’s taste shock follows a Markov process. Independently, Guo and Hörner (2014) use the same techniques to study mechanism design without monetary transfers.

3.3 Model

A principal with deep pockets has access to an investment opportunity. In order to avail herself of this opportunity, she needs the managerial skills of an agent. The agent has no funds to operate the project and is therefore dependent on the principal’s funds for operational costs. Time is discrete, the horizon is infinite, both the principal and agent are risk neutral, and both discount the future at the common rate $\delta \in (0, 1)$.

The project, which we shall also refer to as the firm, requires an investment $k_t \geq 0$ in every period. Capital depreciates completely, and so cannot be carried over to subsequent periods. The return on capital is random, and is either $R(k)$ or 0, where $R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing, strictly concave, and continuously differentiable function with $R(0) = 0$, $\lim_{k \downarrow 0} R'(k) = \infty$, and $\lim_{k \uparrow \infty} R'(k) = 0$. We shall say a return of $R(k)$ occurs if the random production shock is $\text{good}$, while a return of 0 occurs the random production shock is $\text{bad}$. The probability of having a good shock today is $p_s$ where $s \in \{b, g\} =: S$ was the shock in the last period. The production
shocks follow a Markov process with transition probabilities

\[
\begin{pmatrix}
  b & g \\
  b(1 - p_b & p_b) \\
  g(1 - p_g & p_g)
\end{pmatrix}
\]

where the probability of a good shock in the next period given a bad shock today is \( p_b \), and the probability of a good shock in the next period given a good shock today is \( p_g \). We shall assume that \( \Delta := p_g - p_b \geq 0 \), i.e., the Markov process is persistent. The case where \( \Delta = 0 \) corresponds to the iid case. We also assume that \( p_b, p_g \in (0, 1) \), which ensures that the Markov process has a unique ergodic measure and has neither absorbing nor transient sets.

The agency problem arises because (i) the principal cannot observe the output while the agent can, and (ii) the agent is cash constrained. The agent’s lack of funds implies that he needs, as mentioned above, working capital from the principal. We assume, therefore, that the principal cannot extract payments in excess of revenues from the agent, so the agent is protected by limited liability. These are the twin frictions of the model. If the agent were not cash constrained, the principal could simply sell him the firm. If the output were observable, the principal would pay the agent ‘minimum wage’ of 0, i.e., offer him just enough to stay with the firm, while she retains all the revenue. Thus (and this is true for any degree of persistence), it is the combination of limited liability constraints and privately observed cash flow that gives rise to a non-trivial contracting problem.

The cumulative information available to the principal at time \( t \) consists of the investments the principal has made and the amount of cash that the agent has transferred back to her in all prior periods. A contract (formally defined below) conditions investment and cash transfers (conditional on positive output) in any period on all previous cash transfers by the agent, and all previous investments by
the principal. We assume throughout that the agent cannot save cash made available
to him in any period. In other words, all savings are done on behalf of the agent by
the principal as part of the contract.

The timing runs as follows: At the beginning of time, at $t = 0$, the principal offers
the agent in infinite horizon contract that he may accept or reject. If he rejects the
offer, the principal and agent go their separate ways, and their interaction ends. If
the contract is accepted, it is executed. The agent can leave at any time to an outside
option worth 0 without further penalty. The principal fully commits to the contract.

As mentioned before, in terms of the evolution of the state, the only significant
difference between our model and that studied in Clementi and Hopenhayn (2006) is
that we allow for persistence in the production shocks, while they restrict attention
to the case where production shocks are iid. There is one other, minor, difference.
Clementi and Hopenhayn (2006) allow for the project to be scrapped at any time for
a value of $S$, divided between the principal and the agent according to some formula
that is history dependent and optimally chosen. For simplicity, we set the scrap
value to zero. Our principal results go through in the case of a positive scrap value,
albeit with some straightforward modifications. In particular, the properties of the
mature firm are independent of the existence or level of a scrap value.

3.4 Contracts

A contract conditions investments and cash transfers on the history of all previous
cash transfers and investments. By the Revelation Principle, we may equivalently
think of the agent as reporting the current production as being *good* or *bad* (which
corresponds to positive and zero output respectively), so that a sequence of reports
now constitutes a history. The set of states is $\{b, g\} =: S$, so that a private history is a
sequence of output states $s^t := (s_1, \ldots, s_t) \in S_t$ that is only observed by the agent. A
reporting strategy for the agent is a function $\tilde{s}_t : S_{t-1} \times S \to S$, ie, $\tilde{s}_t(h^{t-1}, s) \in \{b, g\}$,
where \( h^{t-1} := (\tilde{s}_1, \ldots, \tilde{s}_{t-1}) \) is a public history of reports by the agent. Such a history represents public information available at the beginning of period \( t \). Let \( H_{t-1} \) denote the collection of all such period-\( t \) public histories. We are now in a position to describe contracts.

### 3.4.1 Sequential Contracts

A sequential contract is a collection of functions \( k_t : h^{t-1} \rightarrow \mathbb{R}_+ \) and \( m_t : h^{t-1} \times \{b, g\} \rightarrow \mathbb{R} \) for \( t = 1, 2, \ldots \). Here, \( k_t(h^{t-1}) \) specifies the investment in period \( t \) conditional on the public history at the end of period \( t-1 \), while \( m_t(h^{t-1}, \tilde{s}_t) \) specifies the transfer of cash from the agent to the principal in period \( t \) conditional on the (reported) output state \( \tilde{s}_t \) in period \( t \) and the public history of reports \( h^{t-1} \). The net cash flow for the agent at time \( t \) is \( R(k_t(h^{t-1})) - m_t(h^{t-1}, \tilde{s}_t) \). As noted before, the agent cannot save any of this cash. A sequential contract is feasible if for all \( t, m_t(h^{t-1}, g) \leq R(k_t(h^{t-1})) \) and \( m_t(h^{t-1}, b) \leq 0 \). In other words, a sequential contract is feasible if it respects the agent’s limited liability constraints.

Given a contract \((k_t, m_t)\), the expected utility (in terms of expected discounted cash flows) for the agent from a reporting strategy \((\tilde{s}_t)\) is \( V_a((s_t), (\tilde{s}_t), h^0) \), where \((s_t)\) is a sequence of output states observed only by the agent. The contract \((k_t, m_t)\) is incentive compatible if truth-telling is an optimal reporting strategy, in other words, if \( V_a((s_t), (s_t), h^0) \geq V_a((s_t), (\tilde{s}_t), h^0) \) for all alternative reporting strategies \((\tilde{s}_t)\).

Thus, the goal of the principal is to maximize her utility — consisting of expected discounted cash flows — by choosing a sequential contract subject to the contract being feasible and incentive compatible. Unfortunately, working in the space of all sequential contracts is difficult, to say the least. We now show that the principal’s problem has a recursive formulation that can be fruitfully employed.
Before we describe the recursive formulation of the principal’s problem, it is useful to reconsider the recursive formulation in the special case of iid states. Consider a history $h^{t-1}$ before the beginning of period $t$ and a report $\tilde{s}$ in period $t$. Let $w_{\tilde{s}}$ be the agent’s lifetime continuation utility upon a report of $\tilde{s} \in S$, so that the agent enters the next period expecting a lifetime utility of $w_{\tilde{s}}$. Notice that because states are iid, the agent’s preferences over continuation problems are common knowledge — and in particular, are independent of the true state in period $t$ — which implies that by choosing $w_{\tilde{s}}$ suitably, the principal can incentivize the agent to report truthfully.

Consider now the equivalence class of all histories $h^\tau$ such that prior to (observing and) reporting the state in period $\tau + 1$, the agent’s *ex ante* expected utility is $v$. Since the agent’s expected utility beginning with these histories is constant, any optimal contract will also deliver the principal the same expected utility conditional on these histories. Therefore, we may restrict attention to contracts that are constant on any such equivalence class. But this implies that we can let promised utility (prior to entering a period) be a state variable in a recursive problem, and then use standard dynamic programming techniques to derive the optimal contract. Put differently, because states are iid, continuation promised utility is sufficient as an instrument to screen the agent.

However, such an approach is inadequate in the Markovian case precisely because the agent has private information about his preferences over future streams of cash. (Recall that today’s state dictates the probability distribution over tomorrow’s states, and today’s state is only known to the agent.) If the agent has lied in period $t - 1$,

---

9 Because this approach is now well understood, our description will be informal.

10 In a two-period model with iid states, Bolton and Scharfstein (1990) use the threat of termination as a proxy for continuation utility to provide incentives for truthful revelation in the first period. Needless to say, if the contract is not terminated in the first period, there cannot be truthful revelation in the second period.
and if the principal has promised utility $v$ (in the form of future cash flows), the agent will assess a different expected utility from the stream of future cash flows than $v$, and more importantly, the principal does not have enough instruments to successfully screen the agent according to his information.

To screen the agent, the principal needs more instruments. Consider some history $h_{t-1}$, and suppose the agent reports state $\bar{s} \in S$ in period $t$. Rather than give the agent some expected continuation utility, the principal provides a pair of *interim* or *contingent* utilities beginning in the next period, that are conditional on the state in the next period. Such a vector of utilities is $w_{\bar{s}} := (w_{\bar{s}b}, w_{\bar{s}g})$. If the true state in period $t$ is $s$, the agent’s expected utility from such a pair of contingent utilities (obtained by reporting $\bar{s}$) is $(1-p_s)w_{s\bar{s}b} + p_sw_{s\bar{s}g}$. Thus, in spite of preferences over continuation problems not being common knowledge, by using contingent continuation utilities appropriately, the principal can provide the agent with the right incentives so as to induce truth-telling.

Thus, after a history $h_t$, the agent enters period $t + 1$ being promised a pair $v = (v_b, v_g)$ of contingent utilities. Let us consider the equivalence class of all histories such that after any history in this class, the vector $(v_b, v_g)$ of continuation expected utilities are identical. On this equivalence class, the principal’s expected utility must again be constant, and so we may take the vector $(v_b, v_g)$ to be our state variable, along with the previous period’s report. Notice that even if the agent has lied in the last period, we are now able to write down incentive constraints in a meaningful way.

There is one significant difference here from the iid case that needs comment. Suppose the agent enters the period with promised contingent utilities $v = (v_b, v_g)$. If states were iid, his expected utility from this pair is independent of his reports in the past, and in particular, does not depend on whether he lied in the last period. However, in the Markovian case, his expected utility from this pair depends on his belief about the probabilities of the good and bad state today, which in turn depends
on yesterday’s state, which he may not have reported truthfully. But, and this is

crucial, even if the agent lied yesterday, contingent on today’s shock being s, his
lifetime interim utility is still \(v_s\). This is because past information is now rendered
payoff irrelevant (which follows directly from the assumption that states follow a
Markov process). Thus, our formulation ensures that the agent cannot benefit from
double deviations.

It goes without saying that the equivalence of the proposed recursive formulation,
with what we refer to as *contingent* promised utilities, and the sequential contract
needs proof. However, the proof is very similar to the proof offered in Theorem 2.1
of Fernandes and Phelan (2000) and so is omitted.\(^{11}\)

Given a pair of contingent utilities \(v = (v_b, v_g) \in \mathbb{R}^2\) with the last period being
in state \(s\) (at least as far as the principal believes), the principal chooses a capital
advancement policy \(k(v, s) \in \mathbb{R}\), transfers \(m(v, s, \tilde{s}) \in \mathbb{R}\), and continuation contingent
utilities \(w_b = (w_{bb}, w_{bg}) \in \mathbb{R}^2\) and \(w_g = (w_{gb}, w_{gg}) \in \mathbb{R}^2\) subject to the following
promise keeping constraints:

\[
\begin{align*}
  v_b & = -m_b + \delta[(1 - p_b)w_{bb} + p_b w_{bg}] \\
  v_g & = R(k) - m_g + \delta[(1 - p_g)w_{gb} + p_g w_{gg}]
\end{align*}
\]

(PK\(_b\))

(PK\(_g\))

Clearly, the only incentive constraint that need be considered is when the agent
incorrectly reports the state as being bad rather than good, which is written as

\[
R(k) - m_g + \delta[(1 - p_g)w_{gb} + p_g w_{gg}] \geq R(k) - m_b + \delta[(1 - p_g)w_{bb} + p_g w_{bg}] 
\]

(IC)

\(^{11}\) Fernandes and Phelan (2000) have a slightly different formulation, where the state variables are
promised utility and a *threat-point* utility, where the latter evaluates the agent’s expected utility
from cash streams if he has lied in the last period. Notice that both the promised and threat-
point utilities are ex ante utilities, while our contingent utilities are interim in nature. Apart from
this difference, the two approaches are essentially identical. Nevertheless, we shall see below that
contingent utilities are somewhat easier to interpret, and so render themselves more suitable for
the application considered in this paper; see also footnote ??.
The limited liability constraints are
\[ m_g \leq R(k) \quad \text{and} \quad m_b \leq 0 \] (LL)

Throughout we impose the feasibility constraint that \( k \geq 0 \) without comment. Using the promise keeping constraints \((PK_b)\) and \((PK_g)\), the incentive constraint \((IC)\) can be written somewhat more simply as
\[ v_g - v_b \geq R(k) + \delta \Delta (w_{bg} - w_{bb}) \] (IC*)

On the right hand side of the constraint \((IC^*)\), \( R(k) \) is the static information rent while \( \Delta (w_{bg} - w_{bb}) \) is the dynamic information rent which is 0 in the iid case, i.e., if \( \Delta = 0 \). Thus, the constraint \((IC^*)\) crystallizes the effect of Markovian states. If production shocks are iid, \( \Delta = 0 \) and \((IC^*)\) reduces to \( v_g - v_b \geq R(k) \). As we shall see below, we must necessarily have \( w_{bg} \geq w_{bb} \), which implies \( \Delta (w_{bg} - w_{bb}) \geq 0 \), so that with persistence, the incentive constraint is tighter.

It is easy to see that given the promise keeping constraints \((PK_b)\) and \((PK_g)\), the constraints \((IC)\) and \((IC^*)\) are equivalent. In what follows we shall work with both constraints, while being explicit about which version of the incentive constraint is under consideration. Having described the state variables and constraints for our recursive formulation, we now describe more carefully the domain for the principal’s problem.

3.4.3 A Recursive Domain

Given that cash flows for the agent are always non-negative, it is clear that any vector of contingent utilities that can be realized must also be non-negative. But our other constraints impose even more restrictions on the feasible \((v_b, v_g)\). Formally, we say that the tuple \((k, m_i, w_i)_{i=b,g}\) implements \((v_b, v_g)\) if \((k, m_i, w_i)\) satisfies the incentive compatibility, promise keeping, and limited liability constraints.\(^\text{12}\)

\(^{12}\) Strictly speaking, our notion of implementability should also include a transversality condition to ensure that promised utilities are actually delivered. For instance, to show formally that contingent
As noted above, because cash flows are non-negative, the only feasible choices of $w_i$ must lie in $\mathbb{R}^2_+$. However, even with the restriction that $w_i \in \mathbb{R}^2_+$, not every $v \in \mathbb{R}^2_+$ is implementable. To see this, suppose $v = (v_b, 0)$, where $v_b > 0$. Then, (PK$_g$) requires that

$$0 = R(k) - m_g + \delta(1 - p_g)w_{gb} + p_gw_{gg}$$

By (LL), we know that $R(k) - m_g \geq 0$, and by assumption, $w_g \in \mathbb{R}^2_+$, which implies $(1 - p_g)w_{gb} + p_gw_{gg} \geq 0$. Therefore, it must be that $R(k) = m_g$, and $w_g = (0, 0)$. Now notice that by (IC), we obtain

$$0 \geq R(k) - m_b + \delta(1 - p_g)w_{bb} + p_gw_{bg}$$

As noted above, $w_b \in \mathbb{R}^2_+$, and $R(k) \geq 0$. By (LL), we also have $m_b \leq 0$, which implies $0 \geq R(k) - m_b + \delta[(1 - p_g)w_{bb} + p_gw_{bg}] \geq 0$, ie, $R(k) = m_b = k = 0$ and $w_b = (0, 0)$. Therefore, by (PK$_b$), we must have $v_b = -m_b + \delta[(1 - p_b)w_{bb} + p_bw_{bg}] = 0$. But this contradicts our assumption that $v_b > 0$. Thus, $(v_b, 0)$ with $v_b > 0$ is not implementable, or equivalently, is infeasible.

To serve as the domain for a recursive problem, the set of feasible utilities that can be implemented must have the property that the contingent continuation utilities must also lie in this feasible set. In other words, what is required is a set $V \subset \mathbb{R}^2_+$ such that for any $v \in V$, there exists a collection $(k, m_i, w_i)$ that implements $v$ and has $w_i \in V$ for $i = b, g$. Such a set $V$ always exists — take, for instance, $V = \{0\}$. However, there exists a much larger (indeed, a largest), non-trivial set, as described next, that will serve as the domain for our recursive formulation of the principal’s problem.$^{13}$

Utilities can never be negative because of limited liability constraints, one must use a transversality argument. However, because all the contractual variables considered in this paper will actually lie in a compact set, we eschew references to transversality conditions.

$^{13}$ Proposition 3.4.1 is an analogue of Lemma 2.2 in Fernandes and Phelan (2000). In the terminology of Abreu et al. (1990), $V$ is self-generating. Indeed, the proof of Proposition 3.5.1 consists of showing that $V$ is the (largest) fixed point of an appropriate mapping.
Proposition 3.4.1. There exists a largest set $V \subseteq \mathbb{R}_+^2$ such that every $v \in V$ is implemented by some $(k, m_i, w_i)$ with $w_i \in V$ for $i = b, g$. In particular, $V := \{(v_b, v_g) \in \mathbb{R}_+^2 : v_g \geq v_b\}$ (see figure 3.1).

In what follows, for each $(v, s) \in V \times S$, let

$$\Gamma(v, s) := \{(k, m_i, w_i) : (k, m_i, w_i) \text{ implements } v \text{ and } w_i \in V\}$$

that is, $\Gamma(v, s)$ denotes the set of feasible contractual variables $(k, m_i, w_i)$ that satisfy (PK$_b$), (PK$_g$), (IC), and (LL) and have $w_i \in V$. Because $\Gamma(v, s)$ is independent of $s$, we shall, when there is no cause for confusion, denote this set by $\Gamma(v)$.

3.4.4 Optimal Contracts

An optimal contract is a solution to the principal’s problem. However, instead of solving the principal’s maximization problem, we note that transfers are linear, and so don’t affect social surplus. Therefore, we shall consider the recursive problem of maximizing firm surplus (which in this setting is precisely the social surplus), and use the agent’s contingent utilities (and the previous period’s state) as state variables for our dynamic program. Towards this end, we let $Q(v, s)$ denote the surplus of the
firm when the previous period’s shock was \( s \), and when the agent enters the period with contingent utility \( v = (v_b, v_g) \). Because \( Q(v, s) \) is the firm surplus and \( v \) is the agent’s contingent utility, the principal’s expected utility at the beginning of each period is \( Q(v, s) - (1 - p_s)v_b - p_s v_g \).

In what follows, we shall denote \( \partial Q/\partial v_b \) by \( Q_b \) and \( \partial Q/\partial v_g \) by \( Q_g \). An optimal contract is a solution to the firm’s recursive maximization problem. We shall denote an optimal contract by the optimal policy \( (k, m_i, w_i) \), \( (i = b, g) \), of the firm’s problem. Our first result establishes the existence of the firm’s value function, as well as some of its properties. It also shows that an optimal contract exists by virtue of being the policy function for the firm’s problem.

**Theorem 3.4.2.** The firm’s discounted surplus under an optimal contract \( (k, m_i, w_i) \) is given by a unique, concave, and continuously differentiable function \( Q : V \times S \to \mathbb{R} \) that satisfies

\[
Q(v, s) = \max_{(k, m_i, w_i)} \left[ -k + p_s(R(k) + \delta Q(w_g, g)) + (1 - p_s)\delta Q(w_b, b) \right] \tag{VF} \]

subject to \( (k, m_i, w_i) \in \Gamma(v, s) \). The contract \( (k, m_i, w_i) \) is continuous in \( (v, s) \). Moreover, the following are true:

(a) \( Q(0, s) = 0, Q_g((v, v), s) = \infty \) and \( Q_b((0, v), s) = \infty \) for all \( v \geq 0 \).

(b) There exists \( M > 0 \) such that \( 0 \leq Q(v, s) \leq M \) for all \( (v, s) \).

(c) \( Q_g(v, s) \geq 0 \) for all \( (v, s) \in V \times S \), though \( Q_b(v, s) \) is sometimes negative.

(d) \( Q(v, g) \geq Q(v, b) \) for all \( v \in V \).

(e) \( Q(w_g(v, s), g) \geq Q(w_b(v, s), b) \).

(f) \( Q(\cdot, s) \) is supermodular in \( v \) for all \( s \in S \).
The existence, uniqueness, concavity and differentiability properties of the surplus function $Q$ are standard, as is the continuity of the policy function. By never investing and immediately paying the agent his expected promised utilities, which effectively shuts down the firm, the principal can always reduce the value of the firm to 0, which is then a lower bound on the value of the firm. If the principal could operate the firm herself or equivalently, if she could observe cash flows, she would invest the efficient amount $\bar{k}_s$ (which solves $p_uR'(\bar{k}_s) = 1$) in each period, and retain all cash flows. The resulting expected (discounted) cash flows clearly provide an upper bound to the value of the firm.

It is easy to see that $\Gamma(0, s) = \{(k = 0, m_i = 0, w_i = 0) : i = b, g\}$. In other words, with promised contingent utilities of $(0, 0)$, the principal can neither invest in the current period nor promise utilities in the future. Therefore, $Q(0, s) = 0$.

The values of the partial derivatives deserve comment, because they arise fundamentally from the structure of the incentive constraints, and because we are working with contingent promised utilities as opposed to (ex ante) promised utilities which arise in the iid case. Part (a) of Theorem 3.4.2 says that $Q_g(0, s) = \infty$. To understand this, notice that if we consider contingent promises of $(0, \varepsilon)$, the principal can ensure production of $R(k) = \varepsilon$ by promising a move to the state $(\varepsilon/(\delta p_g)(1, 1), g)$ in the event of a success, and $((0, 0), b)$ in the event of a failure. Intuitively, this allows the principal to increase production by a small amount in every period with positive probability. The proof shows that the marginal value of this increase to the firm is actually infinite, and relies crucially on the assumption that $R'(0) = \infty$.

Similarly, consider the claim in part (a) that $Q_b((0, v), s) = \infty$. To understand this, notice that in the event of zero output, $(PK_b)$ requires that $w_b = 0$, so the state in the following period is $(0, b)$. But the argument above now shows that because $Q_g(0, g) = \infty$, we must also have $Q_b((0, v), s) = \infty$. The other partial derivatives are established in a similar fashion.
The main observations with regards to contingent (promised) utilities are the following: First, part (c) says that increasing $v_g$ is always beneficial to firm value, while an increase in the contingent utility $v_b$ does not always increase firm value. If $v_b$ is very low relative to $v_g$, there is little effect on $(IC^*)$, but we can raise $w_b$ as dictated by $(PK_b)$. If $v_b$ is sufficiently high, then the primary impact of raising $v_b$ is on tightening $(IC^*)$. Thus, increasing $v_b$ may be beneficial, but it can also reduce the value of the firm because it constrains feasible levels of output. Second, as part (e) notes, success in the present period increases firm value in the next period. Third, in part (f) which says that for each $s, Q(v, s)$ is supermodular in $v$, ie, $v_b$ and $v_g$ are complementary instruments for the firm. For a fixed $v_g$, increasing $v_b$ reduces the downside risk to the firm, because the smaller $v_b$ is, the lower the size of the firm in the next period. On the other hand, increasing $v_b$ tightens the incentive constraint $(IC^*)$. Intuitively, this second effect is not less pronounced when $v_g$ is higher, so $Q(v, s)$ is supermodular in $v$.

As noted above, an optimal contract is a solution to the firm’s recursive maximization problem $(VF)$. Theorem 3.4.2 says that an optimal contract $(k, m_i, w_i)$ exists and that it is continuous in $(v, s)$, but says little more. In the next section, we shall study in greater detail the structure of the optimal contract.

Before analyzing the structure of the contract, it is worthwhile to consider the set of contingent utilities that ensure perpetual efficient investment and production. Intuitively, there exist threshold levels of contingent utility $\bar{v}_s$ so that once the agent’s continuation utility reaches these threshold levels, private information no longer matters.

The following proposition describes sets of efficient contingent utility: the firm becomes financially unconstrained and reaches first best surplus when contingent utility locates in these sets. Before doing so, let us consider the first best (efficient) firm surplus. This is precisely the case where there are no agency problems and
the principal operates the firm. Then, the efficient surplus level in state \( s \) is \( \bar{Q}(s) \), \( s = b, g \), where

\[
\bar{Q}(b) = -\bar{k}_b + p_b \left[ R(\bar{k}_b) + \delta \bar{Q}(g) \right] + (1 - p_b) \delta \bar{Q}(b) \tag{3.1}
\]

\[
\bar{Q}(g) = -\bar{k}_g + p_g \left[ R(\bar{k}_g) + \delta \bar{Q}(g) \right] + (1 - p_g) \delta \bar{Q}(b) \tag{3.2}
\]

The efficient investment level \( \bar{k}_s \) solves \( p_s R'(\bar{k}_s) = 1 \). These two equations allow us to explicitly calculate \( \bar{Q}(b) \) and \( \bar{Q}(g) \). What is relevant for us is that \( \bar{Q}(s) \) represents an upper bound for the value of the firm in state \( s \), and that it entails perpetual efficient investment and production.

**Proposition 3.4.3.** For each \( s = b, g \), there exist closed sets \( E_s \subseteq V \), called the unconstrained sets or efficient sets, that satisfy:

1. there exists \( \bar{v}_s \in E_s \) such that \( v \in E_s \) implies \( v \geq \bar{v}_s \).
2. for each \( v \in E_s \), \( Q_b(v, s) = Q_g(v, s) = 0 \).
3. for each \( v \in E_s \), \( Q(v, s) = \bar{Q}(s) \), and \( k(v, s) = \bar{k}_s \).
4. for any \( v \in V \setminus E_s \), \( Q(v, s) < \bar{Q}(s) \).
5. for each \( v \in E_s \), \( v_g - v_b \geq R(\bar{k}_s) + \delta \Delta \max \left[ \frac{\delta v_{bg} - v_b}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right] \).

Intuitively, \( \bar{v}_s \) represents the lowest levels of contingent utility that the agent must have in order to obtain *perpetual* efficient investment in state \( s \). In other words, \( \bar{v}_s \) is the smallest level of contingent utility needed so that financing constraints no longer bind. Indeed, if \( v < \bar{v}_s \), then \( v \notin E_s \). That output must be perpetually efficient follows from the fact that \( Q|_{E_s} = \bar{Q}(s) \); if investment were ever to be inefficient at some date and after some history, then the present discounted value of the firm would be strictly less than the value of the efficient firm, namely \( \bar{Q}(s) \).
Lemma 3.7.1 in Section 3.7 below explicitly describes $\bar{v}_s$, which in turn gives us the unconstrained sets $E_s$ explicitly. For now, what is important is the observation that we get efficient investment and firm surplus is thereby maximized if (i) levels of contingent utility in state $s$ are sufficiently high, reflected in the requirement that $v \geq \bar{v}_s$, and (ii) $v_g$ is sufficiently greater than $v_b$. It is clear that (i) is a necessary property, because if contingent utility is too low, then by (IC*), production cannot be efficient. Requirement (ii) is peculiar to our formulation in terms of contingent utilities. This says that the difference $v_g - v_b$ must also be sufficiently great, because this relaxes the incentive constraint (IC*), thereby permitting efficient investment.

Having described threshold levels of contingent utility, we now turn our attention to the properties of the optimal contract in the early stages of the contract, when the firm is young, i.e., prior to reaching the threshold levels of contingent utility.

3.5 Optimal Contract — The Young Firm

An important consequence of the characterization in Theorem 3.4.2 is that the firm’s value function $Q(v, s)$ is the value for a concave programming problem (after an ap-
propriate change of variables). Thus, an optimal contract is a solution to the relevant first order conditions (which are necessary and sufficient). In what follows, \(\eta_b(v, s)\) and \(\eta_b(v, s)\) are the Lagrange multipliers for the promise keeping constraints (PK\(_g\)) and (PK\(_b\)), \(\lambda(v, s)\) is the Lagrange multiplier for the incentive compatibility constraint (IC), and \(\mu_b(v, s)\) and \(\mu_g(v, s)\) are the multipliers for the liquidity constraints (LL) when the current period’s state is reported to be \(b\) or \(g\) respectively. This leads us to the first order conditions

\[
R'(k) = 1/[p_s - \eta_g(v, s) + \mu_g(v, s)] \quad \text{(FOCk)}
\]

\[
(1 - p_s)Q_b(w_b, b) = \eta_b(v, s)(1 - p_b) + \lambda(v, s)(1 - p_g) \quad \text{(FOCw}_{gb})
\]

\[
(1 - p_s)Q_g(w_g, b) = \eta_b(v, s)p_b + \lambda(v, s)p_g \quad \text{(FOCw}_{bg})
\]

\[
p_sQ_b(w_g, g) = \eta_g(v, s)(1 - p_g) - \lambda(v, s)(1 - p_g) \quad \text{(FOCw}_{gb})
\]

\[
p_sQ_g(w_g, g) = \eta_g(v, s)p_g - \lambda(v, s)p_g \quad \text{(FOCw}_{gg})
\]

By an adaptation of Lemma B.1.6 in the appendix, the first order condition for optimal investment of capital can be rewritten as

\[
R'(k) = 1/[p_s - \lambda(v, s)] \quad \text{(FOCk)}
\]

Thus, the agency problem which arises due to private information is the financing constraint, and the intensity of the financing constraint is measured by \(\lambda(v, s)\). In addition, we also have the following envelope conditions

\[
Q_b(v, s) = \eta_b(v, s) \quad \text{(Env}_b)
\]

\[
Q_g(v, s) = \eta_g(v, s) \quad \text{(Env}_g)
\]

The optimal contract determines repayment, capital advancement, as well as the evolution of contingent utility. We shall consider these in turn in the early stages of the contract, when the firm is young.
3.5.1 Optimal Transfer

The following proposition gives us some insight into the nature of the contract when contingent utilities are below the threshold levels. Before stating the proposition, let us define the set

\[ A_{1,s} := \{(v, s) \in V \times S : v_b < \bar{v}_{sb}, \delta[p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] \leq v_g < \bar{v}_{sg}\} \]

**Proposition 3.5.1.** For any optimal contract \((k, m, w)\), suppose \(v < \bar{v}_s\). Then, (i) \(m_b(v, s) = 0\), and (ii) \((v, s) \notin A_{1,s}\) implies \(m_g(v, s) = R(k(v, s))\). Moreover, \(v \in A_{1,s}\) if, and only if, \(w_g(v, s) \in E_g\). Finally, there exist maximal rent contracts such that the agent’s contingent utilities are never greater than \(\bar{v}_s\) following the shock \(s\), and for \(v \in A_{1,s}\), \(v_g > \delta[p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}]\) implies \(R(k(v, s)) > m_g\).

Proposition 3.5.1 says that we may regard \(A_{1,s}\) as a one-step away set in the sense that it contains all contingent utility levels such that a good shock in the present period will send the agent’s equity to the efficient set \(E_g\), and in a maximal rent contract, to the threshold contingent utility level \(\bar{v}_g\). It also says that if the agent’s contingent utility levels are sufficiently low (and in particular, are outside the one-step away set \(A_{1,s}\)), then incentives are provided exclusively through adjustments to contingent utility. On the other hand, if \(m_g(v, s) > 0\), then it must be that \(w_g(v, s) = \bar{v}_g\).

Put differently, Proposition 3.5.1 says that all rents are back loaded — the principal initially keeps all the revenue from production, and only eventually does the agent get a share of the proceeds. The proposition also shows that there exists a useful class of contracts, namely, the maximal rent contracts. These contracts have the feature that they involve the earliest possible payment to the agent, which results in contingent utility levels never rising above \(\bar{v}_s\) in state \(s\).

The most important feature of Proposition 3.5.1 is that the back loading of rents
holds regardless of the degree of persistence. In particular, it also holds in the iid case; see, for instance, Proposition 3 of Clementi and Hopenhayn (2006).

This suggests that back loading of rents is a property that does not depend on the persistence (or lack thereof) of the process generating the shocks. Indeed, there is a more fundamental property at play here. The principal is a monopolist and incentivizes the agent by investing inefficient amounts of capital and making promises of equity. By withholding cash payments and instead by adjusting contingent utility levels, the principal ensures that whatever utility accrues to the agent (in the form of contingent utility) stays within the relationship, and is therefore available for the principal to use in the future (to draw down or raise). If however, the principal makes a cash payment to the agent, then that cash is lost forever because the agent does not save. In other words, when the agent’s contingent utility is low (which results in inefficient production), it is cheaper for the principal to adjust contingent utility rather than make a cash payment. We note that this is no longer true once contingent utility reaches a threshold level.\footnote{Back loading of rents also crucially depends on the twin assumptions that the principal and agent are risk neutral and that they share a common discount factor.}

The optimal contract has another feature that should be remarked on. Because all incentives are provided via dilutions or concentrations of the agent’s contingent utility, the optimal contract also features \textit{debt forgiveness} as well as \textit{debt rollover}.

### 3.5.2 Evolution of Contingent Utility

If $v \notin A_{1,s}$, i.e., if $v$ is not in the one-step away set and if $v < v_s$, Proposition 3.5.1 says that the agent does not enjoy any instantaneous rents. But the promise keeping constraints $(PK_b)$ and $(PK_g)$ imply that $v_b = \delta [(1-p_b)w_{b_b} + p_b w_{b_g}]$ and $v_g = \delta [(1-p_g)w_{g_b} + p_g w_{g_g}]$. Notice that $w_g$ does not enter any of the other constraints, so in determining the optimal contract, we can first compute the optimal $w_g$ by
solving
\[
\Psi(v_g) := \max \delta Q(w_g, g)
\]
\[
s.t. \quad w_g \in V, \quad v_g = \delta [(1 - p_g) w_{y_g} + p_g w_{g_g}]
\]

This allows us to put some structure on the evolution of contingent utility.

**Proposition 3.5.2.** The optimal contingent utilities satisfy:

1. \( w_{y_b} < v_g / \delta < w_{y_g} \).

2. \( w_{b_b} < v_b / \delta < w_{b_g} \).

3. \( w_g \) is increasing in \( v \) for each \( s \in S \).

The first two parts of the proposition follow immediately from Proposition 3.5.1 and the promise keeping constraints. The most important observation here is that \( w_g \) is independent of \( v_b \) and \( s \). This follows immediately from the fact that the constraint set \( \Gamma(v, s) \) is independent of \( s \), which implies \( \Psi(v_g) \) is independent of \( v_b \) and \( s \). The supermodularity of \( Q(w_g, g) \) in \( w_g \) means that increasing \( v_g \) (pointwise) increases \( w_g \).

### 3.5.3 Optimal Financing

Even in the absence of private information, optimal capital advancement is stochastic. Indeed, in that case, \( \bar{k}_s \) satisfies, \( p_s R'(\bar{k}_s) = 1 \), and because \( s \) follows a Markov process, \( \bar{k}_s \) follows the same Markov process. The presence of private information, indeed, private information that follows a Markov process, means that capital advancement may well be inefficient. This is intuitive because the size of investment determines the rents for the agent, and if contingent utility is low, then investment (and hence rents) cannot be high.

We shall say that capital advancement is perpetually efficient, if, starting at some date, capital advancement is efficient at each subsequent date and in each subsequent state \( s \in S \).
Proposition 3.5.3. Let \((v^0, s)\) be such that the optimal financing is inefficient, i.e., \(k(v^0, s) < \bar{k}_s\). Also, let \(w_i(v^0, s)\) be the optimal levels of continuation contingent utility. For a given \(p_b\), there exists a critical level \(\varphi(p_b)\) such that:

1. If \(\Delta < \varphi(p_b)\) and \(w_i(v^0, s) \notin E_i\), then \(k(w_i(v^0, s), i) < \bar{k}_i\) for \(i = b, g\).

2. If \(\Delta \geq \varphi(p_b)\) and \(w_g(v^0, s) \notin E_g\), then \(k(w_g(v^0, s), g) < \bar{k}_g\).

3. There exists a neighborhood of \(v^b\) such that if \(\Delta \geq \varphi(p_b)\) and \(w_b(v^0, s) \notin E_b\), then \(k(w_b(v^0, s), b) < \bar{k}_b\) unless \(v\) is in said neighborhood of \(v^b\), in which case \(k(w_b(v^0, s), b) = \bar{k}_b\).

In the iid case, if capital advancement is efficient at any date, then it is optimal from every date thereafter. This need not hold in the Markovian case. To see this, consider the contingent utility level \((0, v_g)\), where \(v_g > \bar{v}^* g_g > R(\bar{k}_s)\). Promise keeping \(\text{PK}_b\) implies \(w_b = 0\). Then, by \((\text{IC}^*)\), we see that \(v_g > R(\bar{k}_s)\), so that the incentive constraint is slack. It follows immediately from \((\text{FOC}k)\) that \(k((0, v_g), s) = \bar{k}_s\). However, if the state in the current period is bad, then \(w_b = 0\) implies that capital advancement is then 0 in perpetuity. By letting \(v_b \approx 0\), we see that a similar argument holds, although capital advancement following a bad state will be small and positive, and not zero.

Of course, it is reasonable to ask if contingent utility levels of the form \((0, v_g)\) considered above can ever arise along the optimal contract’s path. We show in the appendix that it cannot. The other property that is peculiar to the Markovian setting is that we can compare capital advancement levels as a function of the last period’s state, or more precisely, the principal’s belief about the last period’s state.

Lemma 3.5.4. For any \(v < \bar{v}_b\), \(k(v, g) \geq k(v, b)\).

Thus, the lemma says that conditional on having contingent utility level \(v < \bar{v}_b\), capital advancement is higher if the last period’s state was \(g\) instead of \(b\). Intuitively,
this is because if the last period’s state was $g$, then the probability of high output is greater in the current period because of our assumption that $p_g \geq p_b$. (This is the only effect because $\Gamma(v, s)$ is independent of $s$.)

To recapitulate, the qualitative properties of the early stages of the optimal contract are exactly the same as those of the contract in the iid case. This is because, the short run properties arise exclusively from the back loading of rents, and as noted above, back loading stems two factors: (i) the agent is risk neutral, and (ii) by back loading and paying the agent in equity, the principal can provide stronger incentives because she can always adjust the equity position by dilution or concentration. Thus, increased equity ameliorates the liquidity constraints, an observation that is independent of the degree of persistence.

Of course, with persistence, one would suspect, and our numerical calculation show this to be true for some parameter values, that increased persistence leads to increased volatility in investment. This is natural because with a success in the current period, for instance, the probability of success in the next period cannot go down, but can be strictly higher. Indeed, with high persistence, because the states are highly correlated, investment is also highly correlated across periods. While these observations are intuitive, these properties are nevertheless difficult to establish analytically in our discrete-time framework.

In the next section, we describe in detail the long-run properties of the optimal contract.

3.6 Long-run Properties — Maturing of the Firm

Recall again the Principal’s (recursive) problem given by the functional equation

$$Q(v, s) = \max_{(k, m_i, w_i)} \left[ -k + p_s (R(k) + \delta Q(w_g, g)) + (1 - p_s) \delta Q(w_b, b) \right] \quad (VF)$$
where \((k, m_i, w_i) \in \Gamma(v, s)\). In the iid model of Clementi and Hopenhayn (2006), where ex ante promised utility is the state variable, the long-run properties of the contract are uncovered using the observation that the derivative of the value function is a martingale. This observation was first made (also in an iid setting) by Thomas and Worrall (1990).

As we are working with interim, contingent utilities, the relevant martingale is a little more subtle, though still intuitive. First, a definition. For a fixed \(s \in S\), the directional derivative of \(Q\) at \(v\) in the direction \((1, 1)\) is \(D_{(1,1)}Q(v, s) := \lim_{h \to 0} [Q(v + (h, h), s) - Q(v, s)]/h\). Theorem 3.4.2 ensures that \(Q\) is differentiable everywhere, which in turn implies that \(D_{(1,1)}Q(v, s) = \langle DQ, (1, 1) \rangle = Q_b(v, s) + Q_g(v, s)\). We can now state the main result of this section.

**Theorem 3.6.1.** An optimal contract induces a process \(D_{(1,1)}Q = Q_b + Q_g\) that is a non-negative martingale. The martingale \(D_{(1,1)}Q\) converges to 0 in finite time almost surely. Thus, the sets \(E_s\) represent threshold levels of equity in the sense that once in these sets, equity levels never leave these sets. In a maximal rent contract, contingent utility converges to \(\bar{v}_g\), and then cycles on the set \(\{\bar{v}_b, \bar{v}_g\}\), with transitions according to the Markov process on \(S\).

Theorem 3.6.1 states that from any initial level of contingent utility \(v^0 \in V\) in state \(s\), an optimal contract converges to contingent utility levels \(\bar{v}_g\) in finite time almost surely. This last part echoes the iid case, in that the only way to reach the efficient sets is by experiencing one final good production shock. In other words, if the agent has not yet achieved the threshold levels of contingent utility, then a bad shock will never place him in the efficient sets \(E_s\).

Another implication of Theorem 3.6.1 and part (c) of Proposition 3.4.3 is that once the optimal contract promises contingent utilities in the sets \(E_s\), the agent never leaves these sets. It is this property that justifies the nomenclature 'efficient sets'. Of
course, since we are restricting attention to maximal rent contracts (see Proposition 3.5.1), any transition to the threshold sets means that contingent utility transitions initially to $\bar{v}_g$, and then cycles on the set $\{\bar{v}_g, \bar{v}_b\}$ according to the Markov process on states $S$. This last part of Theorem 3.6.1 is a major difference between the iid case and the Markovian case with persistence. In the former, $\bar{v}_b = \bar{v}_g$, and so the cycling is trivial. However, in the Markovian case, the existence of non-trivial dynamics even after reaching the efficient sets leads to interesting conclusions and testable implications, which we discuss in Section ?? below.

To see why the process $D_{(1,1)} Q$ is a non-negative martingale, recall first from Theorem 3.4.2 that even though $Q_g \geq 0$, for some $(v, s)$, we have $Q_b(v, s) < 0$. Nonetheless, Theorem 3.6.1 says that $Q_b + Q_g \geq 0$ for all $(v, s)$. This is because if we start the optimal contract at a point $(v^0, s)$ where $Q_b(v^0, s) > 0$, then, along the path induced by an optimal contract, we will always have $Q_b(\cdot, s) \geq 0$. This implies $Q_b + Q_g$ is always strictly positive — recall part (c) of Theorem 3.4.2, which tells us that $Q_g \geq 0$ everywhere — until it takes the value 0, and this occurs if, and only if, $Q_b = Q_g = 0$, which happens precisely on the sets $E_s$.

The martingale property of $D_{(1,1)} Q(v, s)$ is easy to see. From the envelope conditions (see Section 3.5), we see that $D_{(1,1)} Q(v, s) = \eta_b(v, s) + \eta_g(v, s)$. From the first order conditions, we obtain $(1 - p_s) D_{(1,1)} Q(w, b) + p_s D_{(1,1)} Q(w, g) = \eta_b(v, s) + \eta_g(v, s)$, ie, $D_{(1,1)} Q(v, s)$ is a martingale.

To understand why $D_{(1,1)} Q = Q_b + Q_g$ must be a martingale, let us reason as Thomas and Worrall (1990) do. Consider an increase in $v$ by $(\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. One way to accomplish this is by increasing $w_i$ by $(\varepsilon/\delta, \varepsilon/\delta)$ for each $i = b, g$. It is easy to see that such a change to continuation contingent utilities is incentive compatible and also satisfies promise keeping, and the resulting change in the value of the firm from this increase in contingent utilities is precisely $(1 - p_s) D_{(1,1)} Q(w, b) + p_s D_{(1,1)} Q(w, g)$. An envelope argument shows that this change in $w_i$ is locally
optimal, and so the change in firm value is $D_{(1,1)} Q(v, s) = Q_b(v, s) + Q_g(v, s)$, ie, $D_{(1,1)} Q$ is a martingale. Doob’s Martingale Convergence Theorem ensures that $D_{(1,1)} Q$ converges to a non-negative and integrable random variable. The proof of Theorem 3.6.1 shows that along almost every path, $D_{(1,1)} Q$ converges to 0.

While such a conclusion is also drawn in the iid case, we hasten to point out an important difference. In the iid case, when using ex ante promised utility as a state variable, it is the derivative of the resulting value function that is a martingale. If the value function is strictly concave, there is a one-to-one relationship between the derivative and ex ante promised utility. (Even if strict concavity doesn’t hold, there is nevertheless a tight relationship.) In the case of persistence, with interim equities as state variables, knowing $D_{(1,1)} Q(v, s) = c$ for some $c > 0$ does not pin down $v$. Instead, it only gives us a set of points (typically a curve in $V$) where the directional derivative is $c$. This makes the convergence argument, which essentially requires us to show that $D_{(1,1)} Q(v, s)$ cannot converge to a strictly positive value, rather more subtle.

Theorem 3.6.1 also states that convergence occurs in finite time almost surely. This last part is established as follows: Recall the set $A_{1,s}$, which is the one-step away set. If $(v, s) \in A_{1,s}$, then a good outcome in the present period will place contingent utility at $\bar{v}_g$ in the following period. The proof shows that the set $A_{1,g}$ have a non-empty interior, and because almost every path converges to $\bar{v}_g$, it cannot spend infinite amounts of time in an open subset of $A_{1,s}$, and hence in a neighbourhood of $\bar{v}_g$. Thus, all convergence occurs in finite time almost surely.

3.7 Optimal Contract — The Mature Firm

A mature firm is one where the equity holders get dividend payments. The structure of the mature firm is stark in the iid case. The agent is the residual claimant of the firm, investment is constant over time, and the ratio of debt-to-equity is constant over
time. These implications are seldom seen in practice. As we shall see below, when output displays persistence, none of these conclusions hold. Moreover, the empirical implications of our model with persistence are in consonance with the data.

Our results will depend on the degree of persistence $\Delta$. To capture the impact of persistence on the optimal contract for a mature firm, let us define the following sets:

$$B_+ := \{(p_b, p_g) : R(\bar{k}_b) < R(\bar{k}_g)\delta p_g/(1 + \delta p_g)\}$$

$$B_- := \{(p_b, p_g) : R(\bar{k}_b) = R(\bar{k}_g)\delta p_g/(1 + \delta p_g)\}$$

$$B_- := \{(p_b, p_g) : R(\bar{k}_b) > R(\bar{k}_g)\delta p_g/(1 + \delta p_g)\}$$

(3.3)

To understand the sets, let us fix $p_g$ and suppose $p_b$ is sufficiently small. Notice that $R(\bar{k}_g)\delta p_g/(1 + \delta p_g)$ is independent of $p_b$, and that $R(\bar{k}_b)$ increases with $p_b$ because $\bar{k}_b$ does. For sufficiently small $p_b$, we have $R(\bar{k}_b) < R(\bar{k}_g)\delta p_g/(1 + \delta p_g)$. The set $B_+$ therefore delineates all probabilities $(p_b, p_g)$ such that $p_g - p_b = \Delta$ is sufficiently large. Similarly, $B_-$ denotes the set of all probabilities $(p_b, p_g)$ such that $p_g - p_b$ is not too large. In particular, if $p_b = p_g$ and we are in the iid case, then $(p_b, p_g = p_b) \in B_-.$

Even for a mature firm, the promise keeping conditions (PK$_b$) and (PK$_g$) must hold. In the appendix, we show that for $s = b$, $(p_b, p_g) \in B_+$ if, and only if, (IC) holds with equality and (LL) holds with inequality, while $(p_b, p_g) \in B_-$ if, and only if, (IC) holds with inequality and (LL) holds with equality. Clearly, the set $B_-$ is the boundary between $B_-$ and $B_+$, ie, is the intersection of their closures. We also show in the appendix that in case of failure, there are no transfers to or from the agent, and if the previous period had a success, then conditional on a good shock in the current period, the agent keeps some of the output and the incentive constraint (IC) holds as an equality. Given the binding constraints at various values of $(p_b, p_g)$, we are now in a position to explicitly describe the threshold levels of contingent utilities $v_s.$
Lemma 3.7.1. The threshold levels of contingent utility are as follows:

1. Suppose \((p_b, p_g) \in B_+\). Then,

\[
\bar{v}_b = (\bar{v}_{bb}, \bar{v}_{bg}) = \left( \frac{\delta p_b \bar{v}_{bg}}{1 - \delta (1 - p_b)}, \frac{\delta p_g R(\bar{k}_g)}{1 + \delta p_g \left[ 1 - \delta p_g - \frac{\delta^2 p_b (1 - p_b)}{1 - \delta (1 - p_b)} \right]} \right)
\]

\[
\bar{v}_g = (\bar{v}_{gb}, \bar{v}_{gg}) = \left( \frac{\delta p_b \bar{v}_{bg}}{1 - \delta (1 - p_b)}, \bar{v}_{bg} + \frac{R(\bar{k}_g)}{1 + \delta p_g} \right)
\]

2. Suppose \((p_b, p_g) \in B_-\). Then,

\[
\bar{v}_b = (\bar{v}_{bb}, \bar{v}_{bg}) = \left( \frac{\delta p_b R(\bar{k}_b)}{(1 - \delta)(1 - \delta \Delta)}, \frac{1 - \delta (1 - p_b)}{(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_b) \right)
\]

\[
\bar{v}_g = (\bar{v}_{gb}, \bar{v}_{gg}) = \left( \frac{\delta p_b R(\bar{k}_b)}{(1 - \delta)(1 - \delta \Delta)}, R(\bar{k}_g) + \frac{\delta (p_g - \delta \Delta)}{(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_b) \right)
\]

The maximal rent contract for a mature firm is described next.

Theorem 3.7.2. Suppose the firm is mature and consider a maximal rent contract, so that contingent utility levels are always \(\bar{v}_s\) if the previous period’s shock was \(s \in \{b, g\}\). The contract \((k, m, w)\) takes the form:

\[
k(\bar{v}_b, b) = \bar{k}_b, \quad k(\bar{v}_g, g) = \bar{k}_g
\]

\[
w_g(\bar{v}_s, s) = \bar{v}_g, \quad w_b(\bar{v}_s, s) = \bar{v}_b
\]

\[
m_g(\bar{v}_b, b) = R(\bar{k}_b), \quad m_g(\bar{v}_g, g) = \frac{\delta p_g R(\bar{k}_g)}{1 + \delta p_g} \quad \text{if } (p_b, p_g) \in B_+
\]

\[
m_g(\bar{v}_b, b) = m_g(\bar{v}_g, g) = \delta p_g \left[ R(\bar{k}_g) - R(\bar{k}_b) \right] \quad \text{if } (p_b, p_g) \in B_-
\]

The most striking feature of the optimal contract is that even if the firm is mature, as long as \(\Delta > 0\), the agent makes positive payments to the principal, thereby
highlighting the sensitivity of the iid model. Theorem 3.7.2 says in particular that if the previous period had a good shock, then in the current period, conditional on another good shock, the agent keeps a positive fraction, but not all, of the output, and transfers some of the output to the principal. In other words, with positive persistence, the principal always gets some part of the output.

Indeed, if $\Delta$ is sufficiently large, then $(p_b, p_g) \in B_+$, and in this case, contingent on the previous period’s shock being bad, the agent gives all of the output to the principal, while retaining only a positive (strictly less than one) fraction of the output only if the previous period’s shock was good. This can be interpreted as a high-water mark contract whereby the agent is paid only if he reaches the previous best level of performance. To see the intuition behind this result, fix $p_g \in (0, 1)$, and suppose $p_b \approx 0$, which implies $\bar{k}_b \approx 0$. Then, if the previous period’s shock was bad, (IC*) does not hold with equality because $\bar{v}_{bg} - \bar{v}_{bg}$ is bounded away from zero, while $R(\bar{k}_b) \approx 0$.

The fact that investment after a bad shock is extremely low gives the principal an added threat: After a good shock, investment will be high, so the agent’s rent should be high. But if the agent lies, his utility in the next period will be very low because investment after a reported bad shock is low, which in turn reduces the agent’s rent in the subsequent period. Thus, the principal can carve out a payment even after a good shock by (rationally) threatening to reduce investment, and thereby rent, if a bad shock is ever reported.

This intuition is for the case where $\Delta$ is sufficiently large relative to $p_b$. But the same tradeoffs are also present for lower levels of $\Delta$. Indeed, as long as $\Delta > 0$, such a tradeoff is always present, which is why as long as there is the slightest bit of persistence, the principal always gets some payment.

It is clear that the discussion above relies on the presence of persistence. If, however, shocks are iid over time (so $p_b = p_g$), we get the following contingent
utilities:
\[ \bar{v}^{iid} = \left( \frac{\delta p}{1 - \delta} R(\bar{k}), \frac{1 - \delta + \delta p}{1 - \delta} R(\bar{k}) \right) \]

which are constant over time and independent of the previous period. Notice that the \textit{ex ante} expected utility from the contingent utility \( \bar{v}^{iid} \) is precisely \( v^{iid} = (1 - p) \left( \frac{\delta p}{1 - \delta} R(\bar{k}) \right) + p \left( \frac{1 - \delta + \delta p}{1 - \delta} R(\bar{k}) \right) = p R(\bar{k}) / (1 - \delta) \), just as in Clementi and Hopenhayn (2006). These generate transfers \( \bar{m}_g^{iid} = \bar{m}_b^{iid} = 0 \), as can easily be seen from the last displayed line in Theorem 3.7.2. In other words, with iid shocks, the agent is the residual claimant.

3.8 Implementation

In this section, we introduce the financial instruments that can be used to implement the optimal contract. Recall that the key features of implementing the iid model are (for example DeMarzo and Sannikov (2006a) and DeMarzo and Fishman (2007b)): (i) the agent holds a fixed fraction of the firm’s equity that is equal to the amount he can divert from cash flow; (ii) the evolution of the agent’s continuation value is induced by the available credit that the firm can draw down on a credit line with constant limit. On the contrary, the distinct features in implementing our model are: (i) the agent holds both an equity stake and stock options; (ii) the firm has a contingent credit line with credit limits contingent on covenant violation. We begin by introducing the elements used in this implementation.

\textit{Equity}: equity holders receive dividend payments made by the firm. Dividends are paid by the available cash or credit and at the discretion of the agent.

\textit{Stock options}: the agent has the option of buying the firm’s stock at the specified strike price and then selling it back to the firm at the market value. The market value of the firm’s stock is defined as the present value of expected cash flows.

\textit{Contingent credit line}: revolving credits provided to the firm with credit limit
contingent maintaining cash flow covenant. Balance on the credit line account is charged a interest rate \( r = 1/\delta - 1 \).

*Compensating balance:* cash deposit required to issue the firm its contingent credit line. The firm earns interest on this balance at interest rate \( r \).

We now illustrate the mechanism and security designs that implement the optimal contract.

### 3.8.1 Equity and Stock Option

Recall that in the constrained stage, the agent does not get paid. In the unconstrained stage, the payment to the agent accounts for a larger fraction of firm cash flow after good-good shocks than after bad-good shocks. This means the payment to the agent is convex in firm performance. And this convex compensation can be implemented by granting the agent a combination of equity share and stock options.

In this part, we assume that the agent pays off the credit line balance first then issue payout. We will show this is the case later. So we only need to specify the stock and strike prices when the firm is unconstrained. Depending on the current and last periods’ shocks, we define the stock prices in the unconstrained stage as \( S_{ib} = S_b \) and \( S_{ig} \). They are jointly determined by:

\[
S_{bg} = R(\bar{k}_b) + \delta[p_g S_{gg} + (1 - p_g) S_b] \tag{3.4}
\]

\[
S_{gg} = R(\bar{k}_g) + \delta[p_g S_{gg} + (1 - p_g) S_b] \tag{3.5}
\]

\[
S_b = \delta[p_b S_{bg} + (1 - p_b) S_b] \tag{3.6}
\]

These stock prices can be explicitly expressed in the following result.

**Lemma 3.8.1.** *When the firm is unconstrained, its equity values, or present values*
expected cash flow, are:

\[
S_{bg} = \frac{R(\hat{k}_b) + p_g \delta [R(\hat{k}_g) - R(\hat{k}_b)]}{1 - p_g \delta - \frac{(1-p_g)p_b \delta^2}{1-\delta(1-p_b)}} \quad (EV_{bg})
\]

\[
S_{gg} = S_{bg} + R(\hat{k}_g) - R(\hat{k}_b) \quad (EV_{gg})
\]

\[
S_b = \frac{p_b \delta}{(1-\delta) + \delta p_b} S_{bg} \quad (EV_b)
\]

The payout in the equity and stock options implementation works as the following. When the agent decides to exercise stock options at strike price \(K\), the firm issues one additional share of stock to the agent and buys it back at current stock price. So the agent gets cash payment of \(S - K\) from exercising the option. The rest of firm cash flow will be issued as dividend to both the agent and investors according to their equity shares. In the following result, we appropriately design the strike price and the fraction of outstanding shares issued to the agent so that payment structure in the implementation is equivalent to that in the optimal contract.

**Proposition 3.8.2.** Payout in the optimal contract can be implemented by: i) issuing equity share of \(\lambda = \frac{R(\hat{k}_b) - \bar{m}_{bg}}{R(\hat{k}_b)}\) to the agent; ii) granting the agent each period stock options that has strike price \(K = S_{bg} + \bar{m}_{gg} - \bar{m}_{bg}\) and expires at the end of the period.

Figure 3.4 shows how the stock price and option payoff change on a simulated path when the firm is unconstrained. If the strike price is set as in Proposition 3.8.2 then the option is in the money only after good-good shock.

Figure 3.5 shows the option and equity payments to the agent when the firm is unconstrained and receives a good shock today. The x-axis measures the level of persistence. We increase the persistence level by raising \(p_g\) and keeping \(p_b\) constant. The left panel shows that at all persistence levels the option is in the money only

15 In the left panel, \(p_b = 0.2, p_g = 0.48\). In the right panel, \(p_b = 0.2, p_g = 0.8\).

16 In both panels, \(p_b\) is fixed at 0.2 and \(p_g\) increases from 0.2 to 0.9.
after good-good shock. The right panel shows that the agent’s equity payment is not contingent on last period’s shock and decreases with persistence level. In the high persistence case, the agent receives no equity payment.

3.8.2 Contingent Credit Line

Another key element in our implementation is the contingent credit line. It works in the following way. Investors provide the firm a credit line associated with cash flow covenant. Investors can adjust the limit of the credit line according to whether the covenant is violated or not. In particular, the covenant requires the firm to have strictly positive cash flow. Or in other words the covenant will be violated when a bad shock occurs. In that case, the bank adjust the credit limit to be \( \bar{v}_b \), which is a constant. On the other hand, if the covenant is not violated or a good shock occurs, then investors provide credit limit \( C^L \), which varies with firm performance history.

The issuance of contingent credit line requires cash deposit in a bank account. And the firm earns interest from its balance, which is called compensating balance. Since the credit limit offered by investors is varying with firm performance, the deposit requirement is also changing with performance history. Investors will design the contingent credit limit and its compensating balance appropriately so that the available credits to firm exactly match the continuation values in the optimal contract.

In the implementation, the agent can draw down any available credit and then either use them to issue dividend or simply divert them. We will show that it is incentive compatible for the agent to refrain from doing so.

Take any time \( t \). Let \( M \) be the balance of credit account if the covenant is violated at time \( t \). Then given the credit limit is \( \bar{v}_b \) in this case, the available credit will be \( \bar{v}_b - M \). If the cash flow covenant is not violated at time \( t \), then the firm has cash flow \( R(k) \). It will use this cash flow to repay the credit balance. So in this case, the
credit line balance will be $M - R(k)$. And the available credit for the firm will be $C^L - M + R(k)$. We define the agent’s expected values from time $t$ onward as the available credits:

$$v_b = \bar{v}_b - M$$

$$v_g = C^L - M + R(k)$$

Let time $t - 1$ shock be $s$ and time $t$ shock be $i = b, g$. In the optimal contract, the policy of continuation values at state $(v_b, v_g, s)$ are $w_{ib}$, $w_{ig}$, which can be considered as functions of $(C^L, M, s)$. The firm will draw down credit to invest $k_i$ in period $t+1$. And the compensating balance will pay interest $D_i$ in period $t + 1$. If the cash flow covenant is violated (bad shock occurs) at time $t$, then the credit account balance at the end of period $t + 1$ will be

$$M_b = (1 + r)M + k_b - D_b$$

if covenant is violated in period $t + 1$, or $M_b - R(k_b)$ if the covenant is satisfied in period $t + 1$. Similarly, if the covenant is violated at time $t$, then the credit account balance at the end of period $t + 1$ will be

$$M_g = (1 + r)(M - R(k)) + k_g - D_g$$

if covenant is violated in period $t + 1$, or $M_g - R(k_g)$ if covenant is satisfied in period $t + 1$.

**Lemma 3.8.3.** Suppose the agent truthfully reports cash flow and does not pay out until credit balance is paid off. Then continuation values in the optimal contract evolve the same way as the available credits under the following mechanism:

$$D_i = r\bar{v}_b - [p_b R(k_i) - k_i] - x_i(C^L, M, s)$$

$$C^L_i = M_i - R(k_i) + w_{ig}(C^L, M, s)$$

where $x_b = p_b[w_{bg} - w_{bb} - R(k_b)]$, and $x_g = p_g[w_{gg} - w_{gb} - R(k_g)] - \Delta(w_{bg} - w_{bb})$. 

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Remark 3.8.4. Our contingent credit limits will collapse to one value as persistence level converges to zero, i.e., $p_g - p_b$ converges to zero. We can see this from (3.7) and (3.8). The credit limit contingent on satisfying the covenant is $C^L = \bar{v}_b + v_g - v_b - R(k)$. In the iid case, i.e., $p_g = p_b$, since the dynamic information rent disappears, and from the binding incentive compatible condition $v_g - v_b - R(k) = 0$. Hence, $C^L = \bar{v}_b$. Similarly, in the iid case, $x_i$ defined in Lemma 3.8.6 will also be zero. So the payments from compensating balance has the same form as in DeMarzo and Fishman (2007b).

The left panel in Figure 3.6 shows how the credit limit evolves on a simulated path. If the covenant is maintained, the credit limit will be the level specified in the upper scatter plot. If the covenant is violated, the credit limit will immediately drop to the lower line (1.462). Moreover, we can see that after bad shock or covenant violation expected credit limit next period also drops.

Theorem 3.8.5. The optimal contract can be implemented by a combination of equity share, stock options, and contingent credit line. The agent holds the fraction $\lambda$ of the outstanding equity, and one period stock options with stick price $K$. The credit line limit is set as $\bar{v}_b$ if cash flow covenant is violated or $C^L$ if the covenant is maintained. Depending today’s shock $i$, the compensating balance pays interest $D_i$ tomorrow and upper limit is set as $C^L_i$ tomorrow.

It is incentive compatible for the agent to truthfully report cash flow and use it to pay credit balance before initiating payout. Once the credit balance is fully repaid, cash flows are issued as option payoff and dividends.

17 $p_b = .4$ and $p_g = .6$. 

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3.9 Discussion and Extensions

We now consider some of the salient features of the optimal contract as well as some extensions.

3.9.1 Initialisation

The initial vector of contingent utility $\mathbf{v} = (\hat{v}_b, \hat{v}_g)$ is chosen by the principal. Clearly, the principal can choose this initial level so that firm value is maximized, i.e., she can choose $\hat{v}^s \in E_s$. However, this clearly gives too much rent to the agent. Therefore, assuming the principal’s belief about the probability of success in the current period is $p_s$ where $s \in \{b, g\}$, the principal chooses initial equity $\hat{v}^s$ so as to maximize

$$D(p_s \hat{v}^s, s) = Q(\hat{v}^s, s) - (1 - p_s)\hat{v}_b + p_s\hat{v}_g$$

where $D(\hat{v}^s, s)$ is the debt that the firm holds. (As noted above, the value of the firm is simply the sum of the debt and the expected equity for the agent.) The next proposition shows that initial debt is higher if the principal believes there is a greater chance of success.

**Proposition 3.9.1.** Initial firm debt is greater if $s = g$, i.e., $D(\hat{v}^g, g) > D(\hat{v}^b, b)$. In addition, initial investment is always inefficient for all $s \in \{b, g\}$.

The dependence of initial debt values of a firm on the surrounding business climate is a subject that has received much attention because of the volatility in venture capital investments — see ?. Proposition 3.9.1 also says that initial funding of firms results in investments that are necessarily suboptimal, also in line with observed funding behaviour.

3.9.2 Path Dependence

As in the iid case, the optimal contract exhibits strong path dependence in that the evolution of $\mathbf{v}$ depends on the sequence of shocks. However, there is a stronger form
of path dependence in the present setting than in, say, the risk sharing model of Thomas and Worrall (1990) who show that with CARA utility, the optimal contract (in an iid setting) only depends on the fraction of good shocks rather than the specific order. In our setting, the order of shocks is crucial. This follows immediately from Theorem 3.6.1, which says that convergence to maturity occurs in finite time almost surely. Therefore, the agent would rather his good shocks come sooner rather than later.

It follows from Theorem 3.6.1 that it takes a good shock for a young firm to reach maturity, i.e., for equity levels to reach the set $E_g$. This is also true in the iid case, as noted by Clementi and Hopenhayn (2006) and Krishna et al. (2013). However, if $\Delta$ is sufficiently large, and this is unique to the Markovian case, it necessarily takes two consecutive good production shocks for the firm to mature. This is formally stated in the next proposition.

**Proposition 3.9.2.** For fixed $p_g$, if $\Delta$ is sufficiently large, then $A_{1,b} = \emptyset$.

The proposition says that the one-step away set is empty if the last period’s shock was bad. This makes intuitive sense because (i) it takes a good shock for the firm to mature, and (ii) if $\Delta$ is sufficiently large, then after a bad shock, the agent’s contingent utility is sufficiently low that it cannot reach $\bar{v}_g$ in one step after a good shock.

### 3.10 Conclusion

In this paper, we explore the question of how a firm is financed when the firm’s output, modulo investment levels, is persistent over time. In particular, we consider a principal who provides an agent with funds to operate a firm. Even though shocks to revenue are persistent, we formulate the mechanism design as a recursive problem, and show that the firm’s value is a solution to a recursive dynamic program where
the agent’s contingent utility (which is a vector) is a state variable, along with the
beliefs about high revenue in the current period.

We show that in the optimal contract, when the firm is young, it faces financ-
ing constraints. Investment in the firm is suboptimal as long as the financing con-
straints bind. The incentive scheme involves the agent being compensated exclusively
through adjustments to his contingent utility, and all payments are backloaded. There exist threshold levels of contingent utility, ie, minimal levels of equity such
that if the agent reaches these levels of equity, the firm no longer faces financing
constraints, and investment is forever hence optimal.

The long-run dynamics of the firm are captured by the observation that the di-
rectional derivative of the firm’s value function in the direction \((1, 1)\) is a martingale.
This allows us to show that with probability one, the agent’s equity reaches the
threshold levels. In other words, with probability one, the firm matures and is then
forever free of financing constraints, and operates at its efficient level.

In contrast to the iid setting, with persistence, the mature firm’s size varies
over time, the agent is no longer the residual claimant, the agent is compensated
through cash as well as adjustments to his contingent utility, and the firm’s debt-
to-equity ratio vary over time. Moreover, the structure of the optimal contract
depends crucially on the degree of persistence. For instance, the model predicts that
in a mature firm, the agent is compensated via a high-water mark contract only if
persistence is sufficiently high. These predictions are in accord with stylized facts
about mature firms.

There are many interesting extensions to our basic model. A first would be to em-
bed our model in a general equilibrium setting, in order to understand how frictions
in firm financing affect the macro-economy. Another interesting direction would be to
allow for the possibility that capital does not depreciate completely between periods.
This would correspond to the setting where the principal provides the agent with
capital sporadically, but is nevertheless active in monitoring the agent’s day-to-day performance. We leave these extensions to future work.
Figure 3.3: Cash Flows and Payments When Good Shock Occurs

Figure 3.4: Stock Prices and Option Payoffs
Figure 3.5: Option and Equity Payments to Agent\textsuperscript{16}

Figure 3.6: Contingent Credit Line and Balance \textsuperscript{17}
Appendix A

Proofs from Chapter 2

A.1 Proofs from Section 2.4

Proof of Lemma 2.4.1: Since both the project information and the investment amount are observable, there will be no information rent paid to the DM. So it is always optimal to invest the first best level for any project. And \( k^* \) can be implemented by a forcing contract. Given any mechanism, DM’s discounted future compensation will be constant over time and, therefore, the division of the conglomerate will never be liquidated. Finally, no liquidation means delay payment to the DM is not optimal, since the DM is more impatient. So the optimal contract will payout \( W_0 \) immediately and always enforce the first best investment. \( \square \)

A.2 Proofs from Section 2.5

Proof of Lemma 2.5.1: Given the contract induce truth telling after time \( t \), we know \( W_t(\hat{X}) \) is also the DM’s continuation utility if \( \hat{X}_s \), \( 0 \leq s \leq t \), were the true project types and the DM reported truthfully. Moreover, since the innovations \( dX_t \) are independent over time. We only need to consider one shot deviation, i.e. the DM
report truthfully after time $t$. Therefore, without loss of generality, we only consider $\hat{X} = X$. In this case, we can define another process
\[ V_t = \int_0^t e^{-\gamma s} dI_s(X) + e^{-\gamma t} W_t(\hat{X}) \] (A.1)
which is a right continuous martingale (with respect to $\mathcal{F}_t$). To see it, note
\[ V_t = E_t \left[ \int_0^\tau e^{-\gamma s} dI_s(X) + e^{-\gamma \tau} R \right] \]
by the definition of $V_t$ and $W_t$. Hence, we get $E_t(V_{t'}) = E_t\{ E_t[\int_0^\tau e^{-\gamma s} dI_s(X) + e^{-\gamma \tau} R]\} = V_t$ for any $t' > t$.

From (A.1), we get
\[ dV_t = e^{-\gamma t}[dI_t - \gamma W_t dt + dW_t] \] (A.2)
Moreover, by the martingale representation theorem there exists a $\mathcal{F}$–adapted process $\beta_t$ \(^1\) such that
\[ dV_t = \int_0^\theta \beta_t(\theta)[N(d(t, \theta)) - \lambda f(\theta)d\theta dt] \] (A.3)
Combining (A.2) and (A.3), we get:
\[ dW_t = \gamma W_t dt - dI_t + \int_0^\theta \beta_t(\theta)[N(d(t, \theta)) - \lambda f(\theta)d\theta dt] \]

\[ \square \]

Proof of Lemma 2.5.2: (a) If the DM report any feasible $\hat{\theta} \in \Gamma(\theta, dK_t)$, the investment has to satisfy:
\[ dK_t(\hat{\theta}; \theta) = R^{-1}[\hat{\theta} + R(dK_t(\hat{\theta})) - \theta] \] (A.4)
\[^1\text{See Theorem 1.13.2 on page 25–26 of Last and Brant (1995).}\]
By misreporting the DM is rewarded by $\beta_t(\hat{\theta})$ in continuation utility term from Lemma 2.5.1, and diverts capital $dK_t(\hat{\theta}) - dK_t(\bar{\theta}; \theta)$. So the DM’s total gain from misreporting is $\beta_t(\hat{\theta}) + dK_t(\hat{\theta}) - R^{-1}[\hat{\theta} + R(dK_t(\hat{\theta})) - \theta]$. It is easy to see that if the DM reports truthfully, the total compensation will be $\beta_t(\theta)$. Incentive compatibility then implies:

$$\beta_t(\theta) \geq \beta_t(\hat{\theta}) + dK_t(\hat{\theta}) - R^{-1}[\hat{\theta} + R(dK_t(\hat{\theta})) - \theta]$$  \hspace{1cm} (A.5)

Take any $\theta \in [\bar{\theta}, \tilde{\theta}]$ such that $dK_t(\theta) > 0$. There exists sufficiently small $\epsilon > 0$ such that $\theta + \epsilon - R(dK_t(\theta)) \leq \theta$. This means $\theta \in \Gamma(\theta + \epsilon, dK_t)$. From (A.5), we know

$$\beta_t(\theta + \epsilon) - \beta_t(\theta) \geq dK_t(\theta) - R^{-1}[R(dK_t(\theta)) - \epsilon] > 0$$  \hspace{1cm} (A.6)

which means $\beta_t(\theta)$ is strictly increasing at $\theta$.

(b) When $\beta'_t(\theta)$ exists, (B.37) further implies

$$\beta'_t(\theta) = \lim_{\epsilon \downarrow 0} \frac{\beta_t(\theta + \epsilon) - \beta_t(\theta)}{\epsilon}$$

$$\geq \lim_{\epsilon \downarrow 0} \frac{dK_t(\theta) - R^{-1}[R(dK_t(\theta)) - \epsilon]}{\epsilon} = \frac{1}{R'(dK_t(\theta))}$$

(c) The DM always gets zero total compensation by reporting no project arrives. If $\beta_t(\theta) < 0$ for any $\theta \in [\bar{\theta}, \tilde{\theta}]$, then the DM will report no project when actually $\theta$ project arrives. Therefore, any incentive compatible contract needs to have $\beta_t(\theta) \geq 0$ to induce the truth telling regarding project arrival.

\[\square\]

A.3 Proof from Section 2.6

\textit{Proof of Lemma 2.6.1:} The concavity of the value function and the definition of $\bar{W}$ implies: $P'(W_t) > -1$ when $W_t < \bar{W}$ and $P'(W_t) = -1$ when $W_t \geq \bar{W}$. In the
region $W_t < \bar{W}$, the marginal cost of compensating the DM through continuation utility is lower than the immediate payment. So $dI_t = 0$. In the region $W_t \geq \bar{W}$, the marginal costs of compensating the DM are the same through continuation utility or immediate payment. Since the DM is more impatient, an immediate payment $dI_t = W_t - \bar{W}$ will be mad. The immediate payment will cause $W_t$ to reflect at $\bar{W}$. So $P(W_t) = P(\bar{W}) - (W_t - \bar{W})$.  

Proof of Lemma 2.6.2: Lemma 2.5.2 shows $\beta_t(\theta)$ is strictly increasing over $[\tilde{\theta}, \bar{\theta}]$. So $\beta_t(\theta)$ is differentiable a.e. Hence, Lemma 2.5.2 future implies $\beta_t'(\theta) \geq 1/R[dK_t(\theta)]$ a.e. Let us consider the relaxed problem

$$
r P(W_t) = \max_{dK_t \geq 0, \beta_t \geq 0} \frac{\lambda}{\Delta} \int_{\theta}^{\bar{\theta}} \left[ \theta + R(dK_t(\theta)) - dK_t(\theta) \right] d\theta \quad (H)
$$

$$
+ \left[ \gamma W_t - \frac{\lambda}{\Delta} \int_{\theta}^{\bar{\theta}} \beta_t(\theta) d\theta \right] P'(W_t)
$$

$$
+ \frac{\lambda}{\Delta} \int_{\theta}^{\bar{\theta}} \left[ P(W_t + \beta_t(\theta)) - P(W_t) \right] d\theta
$$

s.t. $\beta_t'(\theta) \geq 1/R[dK_t(\theta)]$

$$
P(0) = L, P'(\bar{W}) = -1
$$

The derivative of the objective in (H) with respect to $\beta_t(\theta)$ is:

$$
\lambda/\Delta [P'(W_t + \beta_t(\theta)) - P'(W_t)] \leq 0 \quad (A.7)
$$

Since $P$ is strictly concave when $W_t < \bar{W}$, (A.7) holds as strict inequality when $\beta_t(\theta) > 0$. This means the optimal contract will set $\beta_t(\theta)$ as small as possible without violating the constraints. The optimal policy must satisfy $\beta_t(\tilde{\theta}) = 0$ and
\[\beta_t'(\theta) = 1/R'[dK_t(\theta)],\] and therefore solves

\[
\max_{dK_t, \beta_t} \frac{\lambda}{\Delta} \int_0^\theta [\theta + R(dK_t(\theta)) - dK_t(\theta)] d\theta
\]

(H1)

\[
+ \frac{\lambda}{\Delta} \int_0^\theta [P(W_{t-} + \beta_t(\theta)) - P(W_{t-}) - \beta_t(\theta) P'(W_{t-})] d\theta
\]

\[
s.t. \quad \beta_t(\theta) = 1/R'(dK_t(\theta)), \quad \beta_t(\bar{\theta}) = 0
\]

Problem (H1) can be viewed as an optimal control problem with \(dK_t(\theta)\) as the control and \(\beta_t(\theta)\) as the state. The initial state of this optimal control problem \(\beta_t(\theta)\) is given as zero. The terminal state \(\beta(\bar{\theta})\) is free. We will apply the Maximum Principle in the optimal control theory and define the Hamiltonian at each \(W_{t-}\) as:

\[
H_t(\theta, dK_t, \beta_t, \mu_t) = \frac{\lambda}{\Delta} [\theta + R(dK_t(\theta)) - dK_t(\theta)]
\]

\[
+ \frac{\lambda}{\Delta} [P(W_{t-} + \beta_t(\theta)) - P(W_{t-}) - \beta_t(\theta) P'(W_{t-})] + \frac{\mu_t(\theta)}{R'[dK_t(\theta)]}
\]

(A.8)

Define \(g_t(x; \theta) = \frac{\lambda}{\Delta} [R(x) - x] + \frac{\mu_t(\theta)}{R'(x)}\). By the Maximum Principle, the necessary conditions for Problem (H1) are:

\[
\mu_t'(\theta) = -\frac{\partial H_t}{\partial \beta_t} = \frac{\lambda}{\Delta} [P'(W_{t-}) - P'(W_{t-} + \beta_t(\theta))]
\]

(A.9)

\[dK_t(\theta) = \arg \max_{x \geq 0} \{g_t(x; \theta)\}\]

(A.10)

Moreover, \(\mu_t(\bar{\theta}) = 0\) by the fact \(\beta_t(\theta)\) is free. Then from (A.9),

\[
\mu_t(\theta) = \frac{\lambda}{\Delta} \int_0^\theta [P'(W_{t-} + \beta_t(u)) - P'(W_{t-})] du
\]

(A.11)

By (A.10), \(dK_t\) must satisfy

\[
\frac{\lambda}{\Delta} [R'(dK_t(\theta)) - 1] - \frac{\mu_t(\theta) R'(dK_t(\theta))}{[R'(dK_t(\theta))]^2} = 0
\]

(A.12)
Combine (A.11) and (A.12) and rearrange to arrive at (2.5). Moreover, \( g_t(x; \theta) \) satisfies the single crossing property because \( \frac{\partial^2 g_t}{\partial x \partial \theta} = -\frac{\mu'_t(\theta) R'(x)}{[R(x)]^2} > 0 \). The inequality is from \( \mu'_t(\theta) > 0 \) which is implied by (A.9) and the strict concavity of \( P \). Therefore, (A.10) implies the optimal capital allocation \( dK_t(\theta) \) is increasing in \( \theta \).

\[ \text{Lemma A.3.1.} \quad \text{Take types } \theta_1 < \theta_2 < \theta \text{ such that } \theta_1 \in \Gamma(\theta, dK_t) \text{ and } \theta_1 + R(dK_t(\theta_1)) < \theta_2 + R(dK_t(\theta_2)). \text{ Suppose the true project quality at time } t \text{ is } \theta. \text{ Then the DM gets higher compensation by misreporting it as } \theta_2 \text{ than by misreporting it as } \theta_1. \]

\[ \text{Proof.} \text{ By construction, we get the relation} \]
\[
\theta_2 < \theta < \theta_1 + R(dK_t(\theta_1)) < \theta_2 + R(dK_t(\theta_2)) \tag{A.13}
\]

The second inequality is because \( \theta_1 \in \Gamma(\theta, dK_t) \). It is easy to see from (A.13) that \( R(dK_t(\theta_2)) \) is the maximum of \( R(dK_t(\theta_1)) + \theta_2 - \theta, R(dK_t(\theta_1)) + \theta_1 - \theta_2 \). And since \( R^{-1} \) is strictly convex, we know
\[
\frac{1}{2} R^{-1}[R(dK_t(\theta_2))] + \frac{1}{2} R^{-1}[R(dK_t(\theta_1)) + \theta_1 - \theta] > \frac{1}{2} R^{-1}[R(dK_t(\theta_2)) + \theta_2 - \theta] + \frac{1}{2} R^{-1}[R(dK_t(\theta_1)) + \theta_1 - \theta_2] \tag{A.14}
\]

By the definition in (2.2), we can rewrite (A.14) as
\[
dK_t(\theta_2) + dK_t(\theta_1; \theta) > dK_t(\theta_2; \theta) + dK_t(\theta_1; \theta_2) \tag{A.15}
\]
Moreover, by (IC) when the true type is \( \theta_2 \) and (A.15), we get
\[
\beta_t(\theta_2) + dK_t(\theta_2; \theta) - dK_t(\theta_2; \theta) \geq \beta_t(\theta_1) + dK_t(\theta_1; \theta) - dK_t(\theta_1; \theta_2) + dK_t(\theta_2; \theta) - dK_t(\theta_2; \theta) > \beta_t(\theta_1) + dK_t(\theta_1; \theta) - dK_t(\theta_1; \theta) \tag{A.16}
\]

The first inequality is from (IC) at \( \theta_2 \), and the second is from (A.15). Therefore, when the true project quality is \( \theta \), the DM gets higher compensation by misreporting it as \( \theta_2 \) than by misreporting it as \( \theta_1 \). \qed
Lemma A.3.2. If the capital allocation policy satisfies $dK_t(\theta) > 0$ at some $\theta < \bar{\theta}$, then for any arbitrarily small $\varepsilon > 0$ there exists $\hat{\theta} \in (\theta, \theta + \varepsilon)$ such that $dK_t(\hat{\theta}) > 0$.

Proof. Suppose the optimal capital allocation satisfies $dK_t(\theta_2) > 0$ and $dK_t(\theta) = 0$ over $[\theta_2, \theta_2 + \varepsilon]$ for any sufficiently small $\varepsilon$. Take any $\hat{\theta} \in [\theta_2, \theta_2 + \varepsilon]$ such that $R(dK_t(\theta_2)) - (\hat{\theta} - \theta_2) > 0$. Let $\theta_1$ be the optimal lie for type $\hat{\theta}$. The optimal lie exists because the feasible lies $\Gamma(\hat{\theta}, dK_t) \subset [\theta, \theta_2]$. By construction, we know $\theta_1 \leq \theta_2$, and $\hat{\theta} \leq \theta_1 + R(dK_t(\theta_1))$. Suppose $\hat{\theta} = \theta_1 + R(dK_t(\theta_1))$. Then we know $\theta_1 + R(dK_t(\theta_1)) = \hat{\theta} < \theta_2 + R(dK_t(\theta_2))$ and $\theta_1 < \theta_2$. And from Lemma A.3.1, it is strictly better to lie as $\theta_2$ than $\theta_1$, a contradiction. Hence, we must have $\hat{\theta} < \theta_1 + R(dK_t(\theta_1))$.

Raising $dK_t(\hat{\theta})$ to any positive level potentially makes (IC) tighter for types higher than $\hat{\theta}$. However, we will show there exist $k > 0$ such that raising $dK_t(\hat{\theta})$ to $k$ will not change (IC) for any type above $\hat{\theta}$. Now let us consider changing $dK_t(\hat{\theta})$ from 0 to $k > 0$ such that and $R(k) < R(dK_t(\theta_1)) + \hat{\theta} - \theta$. Note that only types $\theta$ that satisfies $\hat{\theta} \leq \theta \leq R(k) + \hat{\theta}$ can feasibly lie as $\hat{\theta}$. We’ll show the incentive compatibility of any such $\tilde{\theta}$ won’t be affected.

By construction, we know $R(dK_t(\theta_1)) + \theta_1 - \hat{\theta} > \max\{R(k), R(dK_t(\theta_1)) + \theta_1 - \hat{\theta}\}$. And since $R^{-1}$ is strictly convex, we know

$$\frac{1}{2}R^{-1}[R(dK_t(\theta_1)) + \theta_1 - \hat{\theta}] + \frac{1}{2}R^{-1}[R(dK_t(\hat{\theta})) + \hat{\theta} - \theta]$$

$$> \frac{1}{2}R^{-1}[R(k)] + \frac{1}{2}R^{-1}[R(dK_t(\theta_1)) + \theta_1 - \hat{\theta}]$$

(A.17)

By the definition in (2.2), we know (A.17) implies:

$$dK_t(\theta_1; \hat{\theta}) + dK_t(\hat{\theta}; \tilde{\theta}) > k + dK_t(\theta_1; \hat{\theta})$$

(A.18)
Moreover, by lying as $\hat{\theta}$, type $\tilde{\theta}$ gets compensation

$$
\beta_t(\tilde{\theta}) + k - dK_t(\hat{\theta}; \tilde{\theta}) \\
= \beta_t(\theta_1) + dK_t(\theta_1) - dK_t(\theta_1; \hat{\theta}) + k - dK_t(\hat{\theta}; \tilde{\theta}) \\
< \beta_t(\theta_1) + dK_t(\theta_1) - dK_t(\theta_1; \hat{\theta})
$$

(A.19)

The equality is by the assumption $\theta_1$ is the best lie for type $\hat{\theta}$. The inequality is by (A.18). By construction, we know $\tilde{\theta} \in R(\theta_1)$. This means it is feasible for type $\tilde{\theta}$ to lie as $\theta_1$. So from (IC), we know

$$
\beta_t(\tilde{\theta}) \geq \beta_t(\theta_1) + dK_t(\theta_1) - dK_t(\theta_1; \hat{\theta})
$$

(A.20)

Combining (A.19) and (A.20) we get:

$$
\beta_t(\tilde{\theta}) \geq \beta_t(\theta_1) + k - dK_t(\hat{\theta}; \tilde{\theta})
$$

(A.21)

The new constraint on $\beta_t(\tilde{\theta})$ imposed by changing $dK_t(\theta)$ from 0 to $k$ is exactly (A.21). But it has be implied by other IC constraints that $\beta_t(\tilde{\theta})$ has to satisfy when $dK_t(\hat{\theta}) = 0$. Hence, raising $dK_t(\hat{\theta})$ from 0 to $k$ will not change $\beta_t(\tilde{\theta})$. In other words, raising $dK_t(\hat{\theta})$ to some sufficiently small but positive level won’t change (IC) but will increase the objective of (HJB). This is a contradiction with $dK_t$ being optimal.

**Proof of Proposition 2.6.3:** (a) Let $\hat{K}_t$, $\hat{\beta}_t$ be the optimal policy. Define the lowest type project with positive capital allocation as $\theta_t^* = \inf\{\theta \leq \bar{\theta} : d\hat{K}_t(\theta) > 0\}$. If $d\hat{K}_t(\theta) = 0$ for all $\theta < \bar{\theta}$, then we simply get $\theta_t^* = \bar{\theta}$. Now let us consider the case $\theta_t^* < \bar{\theta}$. There must exists an interval $(\theta_t^*, \theta_1)$ over which $d\hat{K}_t(\theta) > 0$ a.e. Otherwise, we will get a contradiction with Lemma A.3.2. Then by Lemma 2.5.2, $\hat{\beta}_t(\theta) \geq 1/R(d\hat{K}_t(\theta))$ a.e. over $(\theta_t^*, \theta_1)$. Suppose $d\hat{K}_t(\theta) = 0$ over $(\theta_1, \theta_2)$. And since $\hat{\beta}_t(\theta)$ should be as low as possible if (IC) is satisfied, we know
\[ \hat{\beta}_t(\theta) = 1/R'(d\hat{K}_t(\theta)) \] over \((\theta^*_t, \theta_1)\). Then the optimal policy \(\hat{K}_t, \hat{\beta}_t\) must solve:

\[
\begin{aligned}
\max_{dK_t(\theta), \beta_t(\theta)} & \quad \frac{\lambda}{\Delta} \int_{\theta^*_t}^{\theta_1} \left[ \theta + R(dK_t(\theta)) - dK_t(\theta) \right] d\theta \\
\quad & + \frac{\lambda}{\Delta} \int_{\theta^*_t}^{\theta_1} \left[ P(W_{t-} + \beta_t(\theta)) - P(W_{t-}) - \beta_t(\theta) P'(W_{t-}) \right] d\theta \\
\text{s.t.} & \quad \beta_t'(\theta) = 1/R'(dK_t(\theta)), \quad \beta_t(\theta^*_t) = \hat{\beta}_t(\theta^*_t)
\end{aligned}
\]  

(H2)

This is because the objective of (H2) is part of the objective of (HJB). Note that Problem (H2) has the same structure as Problem (H1). If we consider (H2) and (H1) as optimal control problems, then they differ only by initial states, and start and end points. The same procedure in Lemma 2.6.2 shows that \(d\hat{K}_t(\theta)\) increases over \((\theta^*_t, \theta_1)\). Hence, we know \(d\hat{K}_t(\theta_1) > 0\) and \(d\hat{K}_t(\theta) = 0\) over \((\theta_1, \theta_2)\). This is a contradiction with Lemma A.3.2. So we must have \(d\hat{K}_t(\theta) > 0\) a.e. over \([\theta^*_t, \bar{\theta}]\). The same argument above shows that \(d\hat{K}_t(\theta)\) increases over \([\theta^*_t, \bar{\theta}]\). Hence, \(d\hat{K}_t(\theta) > 0, \hat{\beta}_t(\theta) > 0, \) and \(\hat{\beta}'_t(\theta) = 1/R'(dK_t(\theta))\) a.e. over \([\theta^*_t, \bar{\theta}]\).

(b) Consider the case \(\theta^*_t > \bar{\theta}\). By the definition of \(\theta^*_t\), we know \(d\hat{K}_t(\theta) = 0\) for \(\theta < \theta^*_t\). Since projects below \(\theta^*_t\) do not get capital allocation, any type \(\hat{\theta}\) below \(\leq \theta^*_t\) cannot feasibly lie to be a lower type. Therefore, the optimal contract will set \(\hat{\beta}(\theta) = 0\).

Proof of Lemma 2.6.4: (a) If \(\lim_{x \to 0} \frac{R'(x)}{[R'(x)]^3} = 0\), then from Lemma 2.6.2, we get

\[
\lim_{x \to 0} \frac{\hat{q}_t(x; \theta)}{\hat{x}} = \lim_{x \to 0} \frac{\lambda R'(x)}{\Delta} \left[ 1 - \frac{1}{R'(x)} - \frac{\Delta \mu_t(\theta) R''(x)}{\lambda [R'(x)]^3} \right] = \infty
\]  

(A.22)

for any \(\theta \in [\bar{\theta}, \bar{\theta}]\), since \(\mu_t(\theta)\) is finite. So by (A.10), we always have \(dK_t(\theta) > 0\).
(b) By the definition of $g_t$,
\[
\frac{\partial^2 g_t(x; \theta)}{\partial x^2} = \lambda R''(x)/\Delta
\]
\[
- \frac{\mu_t(\theta)}{[R'(x)]^4} \{ R''(x) [R'(x)]^2 - 2 R'(x) [R''(x)]^2 \} < 0
\]
The inequality is implied by the assumption that $R'' < 0$, $R''' R' < 2(R'')^2$, and $\mu_t(\theta) < 0$. So $g_t(x; \theta)$ is strictly concave. Moreover, if $\lim_{x \to 0} \frac{R'(x)}{[R'(x)]^3} < 0$, then we possibly have:
\[
\lim_{x \to 0} \left[ 1 - \frac{1}{R'(x)} - \frac{\Delta \mu_t(\theta) R''(x)}{\lambda [R'(x)]^3} \right] < 0
\]
when $|\mu_t(\theta)|$ is large. By (A.22) we know $\lim_{x \to 0} \frac{\partial g_t(x; \theta)}{\partial x} < 0$. Then the necessary condition (A.10) implies $dK_t(\theta) = 0$, i.e. project $\theta$ is excluded. From Proposition (2.6.3), we know exclusion possibly occurs at the bottom types.

Proof of Proposition 2.6.5: (a) From equation (2.5), it is easy to get $R'(dK_t(\bar{\theta})) = 0$.

(b) Take any $\theta^* \leq \theta < \bar{\theta}$ and $0 < W_{t-} < \bar{W}$. Since $P(W_{t-})$ and $R$ are strictly concave, and $\beta_t(\bar{\theta}) > 0$ for all $\bar{\theta} > \theta$, we know the right-hand side of (2.5) is positive. This implies $R'(dK_t(\theta)) < 1$.

(c) Take any $\theta$ and $W_{t-} \geq \bar{W}$. Since $P'(W_{t-}) = P'(W_{t-} + \beta_t(\bar{\theta})) = -1$ for any $\bar{\theta} \geq \theta$, we know the right-hand side of (2.5) is zero. This implies $R'(dK_t(\theta)) = 1$.

Since $\theta$ is arbitrary and $dK_t(\theta) > 0$, we must have $\theta^*_t = \theta$.

Proof of Lemma 2.6.6: First, we need to show the necessary condition (2.5) is also sufficient for (H1). Define
\[
H_{t0}^0(\theta, \beta_t, \mu_t) = \max_{dK_t \geq 0} H_t(\theta, \beta_t, dK_t, \mu_t)
\]
Because $P(W)$ is concave, it is easy to see from the definition (A.8) that $H_t$ is concave in $\beta_t$. Also, we have shown in the proof of Lemma 2.6.4 that $g_t(x; \theta)$ is concave in $x$. So $H_t$ is concave in $dK_t$. Hence, $H_t^0(\theta, \beta_t, \mu_t)$ is a concave function of $\beta_t$ for any given $\mu_t$. By Arrow Theorem (see P.222 of Kamien and Schwartz (1991)), $dK_t(\theta)$ and $\beta_t(\theta)$ will maximize (H1).

Second, we need to show the proposed $\beta_t$, $dK_t$ satisfy all the constraints in (HJB). It is obvious that $\beta_t(\theta) \geq 0$ for all $\theta$. We now show (IC) is satisfied.

Take any $\theta, \hat{\theta}$ such that $\theta^*_t < \hat{\theta} < \theta$. Because $dK_t(\theta)$ and $R$ are increasing, we know $\hat{\theta} + R(dK_t(\hat{\theta})) \leq R(dK_t(x)) + x$ for any $x \in [\hat{\theta}, \theta]$. Let $C(x) = R^{-1}(x)$. Since $C$ is convex, we know $C' [\hat{\theta} + R(dK_t(\hat{\theta})) - x] \leq C' [R(dK_t(x))]$. Let us define

$$b_t(x) = \int_{\hat{\theta}}^x 1/R'(dK_t(u))du - dK_t(\hat{\theta}) + dK_t(\hat{\theta}; x)$$

It is easy to see that $b_t(\hat{\theta}) = 0$ and $b'_t(x) = C'[R(dK_t(x))] - C'[\hat{\theta} + R(dK_t(\hat{\theta})) - x] \geq 0$. Hence, $b(\theta) \geq 0$. And by definition we know $b_t(\theta) = \beta_t(\theta) - \beta_t(\hat{\theta}) - dK_t(\hat{\theta}) + dK_t(\hat{\theta}; x) \geq 0$. This means the DM will not misreport the project to be $\hat{\theta}$ when its true quality is $\theta$.

Take any $\theta \leq \theta^*_t$. Because $dK_t(\hat{\theta}) = 0$ for any $\hat{\theta} < \theta$, type $\theta$ can not feasibly lie as $\hat{\theta}$. So $\beta_t(\theta) = 0$ satisfies (IC).

Proof of Proposition 2.6.7: Following the standard argument in optimal control theory, we show that HQ’s value from any incentive compatible mechanism that delivers DM continuation utility $W_0$ is at most $P(W_0)$, which is HQ’s expected payoff from the conjectured mechanism. Let us define

$$G_t = \int_0^t e^{-rt}[(dX_s + R(dK_s) - dK_s)dN_s - dI_s] + e^{-rt}P(W_{t-})$$

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In any incentive compatible contract, $W_t$ evolves according to (2.4). By Ito’s Lemma,

$$e^{rt}E_t(dG_t) = \frac{\lambda}{\Delta} \int_0^\theta (\theta + R(dK_t(\theta)) - dK_t(\theta)) d\theta dt$$

$$+ \left[ \gamma W_t - \frac{\lambda}{\Delta} \int_0^\theta \beta_t(\theta) d\theta \right] P'(W_t) dt$$

$$+ \frac{\lambda}{\Delta} \int_0^\theta \left[ P(W_t + \beta_t(\theta)) - P(W_t) \right] d\theta dt - rP(W_t) dt$$

$$- (1 + P'(W_t)) dI_t$$

Since $P(W_t)$ is constructed from the optimal policies $dK^*_t(\theta), \beta^*_t(\theta)$, the first three lines must be less than or equal to zero. Moreover, $P'(W_t) \geq -1$ implies the last line is also less than or equal to zero. So, $G_t$ is a supermartingale. It is a martingale if and only if $dK_t(\theta) = dK_t^*(\theta), \beta_t(\theta) = \beta_t^*(\theta)$, and $dI_t \geq 0$ only when $W_t \geq \bar{W}$.

Now we evaluate the time zero expected payoff of the HQ from an arbitrary incentive compatible mechanism:

$$E \left[ \int_0^\tau e^{-rs}((dX_s + R(dK_s) - dK_s) dN_s - dI_s) + e^{-rt}L \right]$$

$$= E \left[ G_{t\wedge \tau} + 1_{t\leq \tau} \left( \int_t^\tau e^{-rs}((dX_s + R(dK_s) - dK_s) dN_s - dI_s) + e^{-rt}L - e^{-rt}P(W_t) \right) \right]$$

$$= E(G_{t\wedge \tau}) - e^{-rt}E[1_{t\leq \tau}P(W_t)]$$

$$+ e^{-rt}E \left[ 1_{t\leq \tau} \left( E_t \left( \int_t^\tau e^{-r(s-t)}((dX_s + R(dK_s) - dK_s) dN_s - dI_s) + e^{-r(\tau-t)}L \right) \right) \right]$$

$$\leq G_0 + e^{-rt}E \left[ 1_{t\leq \tau} \left( \frac{\lambda[E(\theta) + R(k^*) - k^*]}{r} - W_t - P(W_t) \right) \right]$$

$$\leq P(W_0) + e^{-rt}E \left[ 1_{t\leq \tau} \left( \frac{\lambda[E(\theta) + R(k^*) - k^*]}{r} - L \right) \right]$$
The first inequality is from: (1) $G_t$ is a supermartingale; (2) HQ’s expected payoff from time $t$ on is smaller than the first best surplus minus promised utility. The second inequality is from the fact that total surplus at any time is at least as large as liquidation value $L$. Therefore, as $t \to \infty$, the second piece in the last line converges to zero. And by definition, $G_0 = P(W_0)$. So we get

$$E \left[ \int_0^\tau e^{-rs}((dX_s + R(dK_s) - dK_s) dN_s - dI_s) + e^{-rs}L \right] \leq P(W_0)$$

Therefore, HQ’s expected payoff under any incentive compatible mechanism is at most the expected payoff obtained from the mechanism in the proposition.

We now show that the value function $P$ must be concave. Let us define

$$S(W_t) = \frac{\lambda}{\Delta} \int_\theta^\theta [P(W + \beta_t(\theta)) - P(W) - \beta_t(\theta)P'(W)] d\theta$$

and consider any incentive compatible policy $K_t, \beta_t$. From (HJB), we know

$$\frac{\lambda}{\Delta} \int_\theta^\theta [\theta + R(dK_t(\theta)) - dK_t(\theta)] d\theta + S(W_t) \leq rP(W) - \gamma WP'(W)$$

$$\leq rP(W) + \gamma W$$

$$\leq \frac{\lambda}{\Delta} \int_\theta^\theta [\theta + R(dK_t(\theta)) - dK_t(\theta)] d\theta$$

The first inequality is from $P'(W) \geq -1$. The second inequality is because the sum of HQ’s flow profit and DM’s flow utility should be less than the expected cash flow. Therefore, we must have $S(W) \leq 0$ for any $W \in [0, W]$. By the continuity of $P'(W)$, we can rewrite $S(W)$ as

$$S(W) = \frac{\lambda}{\Delta} \int_\theta^\theta [P'([\xi(\theta, W))] - P'(W)] \beta^*(\theta, W) d\theta$$

(A.23)

where $\xi(\theta, W) \in [W, W + \beta^*(\theta, W)]$
To show the concavity of $P(W)$, first note that if there is $\hat{W} \in [0, \bar{W}]$ such that $P''(\hat{W}) = 0$, then we must have $P''(W) = 0$ for any $W \geq \hat{W}$. This is because, by (A.23), the above solution satisfies $S(\hat{W}) = 0$, which is the maximum value of $S(\hat{W})$.

Suppose that $P''(0) \geq 0$. The above observation implies $P''(W) \geq 0$ for any $W \geq 0$. Since the DM is more impatient, it is not optimal to delay payments for ever. So there exists a $\hat{W}$ such that $P'(\hat{W}) = -1$. Then $P'(W) \geq -1$ for any $W \geq 0$ implies the value function is a straight line with $P'(W) = -1$. From the necessary condition (2.5), the optimal effort will always be the first best effort. Also, (HJB) means:

$$rP(W) = -\gamma W + \lambda [E(\theta) + A_{FB} - C(A_{FB})]$$

But this contradicts with $P'(W) = -1$, since $r < \gamma$. Therefore, $P''(0) < 0$. The above argument implies that the value function is concave and strictly so for $W \in [0, \bar{W})$.

\[ \square \]

**Lemma A.3.3.** Let $K_t, \beta_t$ be the optimal policy. The envelope condition of the (HJB) is:

\begin{align}
(r - \gamma) P'(W_t^-) &= \frac{\lambda}{\Delta} \int_{\theta}^{\tilde{\theta}} [P'(W_t^- + \beta_t(\theta)) - P'(W_t^-)] d\theta \\
&\quad + \left[ \gamma W_t^- - \frac{\lambda}{\Delta} \int_{\theta}^{\tilde{\theta}} \beta_t(\theta) d\theta \right] P''(W_t^-) 
\end{align} \tag{A.24}

**Proof.** The objective of the (HJB) is:

\begin{align}
rP(W_t) - \gamma W_t P'(W_t) \\
= \frac{\lambda}{\Delta} \int_{\theta}^{\tilde{\theta}} \{ \theta + R(dK_t(\theta)) - dK_t(\theta) + P(W_t + \beta_t(\theta)) - P(W_t) - \beta_t(\theta) P'(W_t) \} \ d\theta 
\end{align} \tag{A.25}
Differentiate (A.25) with respect to \( W_t \):

\[
(r - \gamma)P'(W) - \gamma WP''(W)
\]

\[
= \frac{\lambda}{\Delta} \int_\theta^{\bar{\theta}} \left\{ \left[ R'(dK) - 1 \right] dK_W + \left[ P'(W + \beta) - P'(W) \right] \beta_W \right\} d\theta + \lambda \int_\theta^{\bar{\theta}} \left\{ P'(W + \beta) - P'(W) - \beta P''(W) \right\} d\theta
\]

(A.26)

where \( dK_W = \frac{\partial dK_t(\theta)}{\partial W}, \) and \( \beta_W = \frac{\partial \beta_t(\theta)}{\partial W}. \) The constraint of (H1) implies

\[
\beta_W = \frac{-R''(dK)}{\left[R'(dK)\right]^2} dK_W
\]

(A.27)

Multiply (A.27) by \( \mu(\theta, W) \) and add it to the right-hand side of (A.26), we get

\[
(r - \gamma)P'(W) - \gamma WP''(W)
\]

\[
= \frac{\lambda}{\Delta} \int_\theta^{\bar{\theta}} \left\{ \left[ P'(W + \beta) - P'(W) \right] \beta_W + P'(W + \beta) - P'(W) - \beta P''(W) \right\} d\theta + \lambda \int_\theta^{\bar{\theta}} \left\{ \left[ \frac{R'(dK)}{\Delta} - \frac{R''(dK)}{\left[R'(dK)\right]^2} \mu \right] dK_W + \mu \beta_W \right\} d\theta
\]

(A.28)

Apply integration by parts to obtain:

\[
\int_\theta^{\bar{\theta}} \mu_\theta(\theta) \beta_W(\theta) d\theta
\]

\[
= \left[ \beta_W(\bar{\theta}) \mu_\theta(\bar{\theta}) - \beta_W(\theta) \mu_\theta(\theta) \right] - \int_\theta^{\bar{\theta}} \mu_\theta(\theta) \beta_W(\theta) d\theta
\]

\[
= - \int_\theta^{\bar{\theta}} \mu_\theta(\theta) \beta_W(\theta) d\theta
\]

(A.29)
Plug (A.29) into (A.28) and rearrange to obtain
\[(r - \gamma)P'(W) - \gamma WP''(W)\]
\[= \lambda \int_\theta^\beta \left[ (P'(W + \beta) - P'(W)) - \mu_\theta \right] \beta_W d\theta + \lambda \int_\theta^\beta \left[ \frac{R'(dK)}{\Delta} - \frac{R''(dK)}{[R'(dK)]^2} \mu \right] dK_W d\theta = 0 \text{ by (A.9)}
+ \frac{\lambda}{\Delta} \int_\theta^\beta \left[ P'(W + \beta) - P'(W) - \beta WP''(W) \right] d\theta = 0 \text{ by (A.12)}

A.4 Proofs from Section 2.7

**Proof of Proposition 2.7.1:** In the region \(P'(W) < 0\), the left-hand side of (A.24) is positive since \(r < \gamma\). The concavity of the value function implies that the first line of (A.24) is smaller or equal to zero. So the second line must be positive, and therefore, \(\gamma W < \lambda \int_\theta^\beta \beta_t f(\theta) d\theta\). From equation (2.4), this simply means the drift of \(dW_t\) is negative.

**Proof of Proposition 2.7.3:** The continuity of \(\beta_t(\theta)\) in \(\theta\) and \(W_t\) implies that \(\hat{W}(\theta)\) is also continuous. Moreover, since \(\beta_t(\theta) = 0\), we must have \(\hat{W}(\theta) = \bar{W}\) from the definition of \(\hat{W}(\theta)\). Pick any \(\theta_2 > \theta_1\). We obtain

\[\hat{W}(\theta_1) + \beta(\theta_2, \hat{W}(\theta_1)) \geq \hat{W}(\theta_1) + \beta(\theta_1, \hat{W}(\theta_1)) \geq \bar{W}\]

The first inequality is from that \(\beta_t(\theta)\) is increasing in \(\theta\). The second inequality is from the definition of \(\hat{W}(\theta_1)\). So we must have \(\hat{W}(\theta_1) \geq \hat{W}(\theta_2)\) from the definition of \(\hat{W}(\theta_2)\).

Take \(\tilde{W} \in (\hat{W}(\theta), \bar{W})\). The continuity and monotonicity of \(\hat{W}(\theta)\) means that there exists \(\theta' \in (\theta, \theta)\) such that \(\hat{W}(\theta') = \tilde{W}\). Hence,

\[\tilde{W} + \beta(\theta', \tilde{W}) \geq \hat{W} + \beta(\theta', \tilde{W}) \geq \bar{W}\]
The first inequality is from that $\beta$ is nondecreasing in $\theta$. The second inequality is from the definition of $\bar{W}(\theta')$. This argument shows that $W + \beta(\theta, W) \geq \bar{W}$ when $W \geq \bar{W}(\theta)$. From the proof in Lemma 2.6.2, we know,

$$g_t(dk; \theta) = \frac{\lambda}{\Delta} \left\{ R(dK) - dK - \frac{(\bar{\theta} - \theta)(1 + P'(W_{t-}))}{R'(dK)} \right\} \tag{A.30}$$

Note that $g_t(dK; \theta)$ satisfies the single crossing property regarding $dK$ and $W_{t-}$, since

$$\frac{\partial^2 g_t}{\partial (dK) \partial W_{t-}} = \frac{(\bar{\theta} - \theta) P''(W_{t-}) R''(dK_t(\theta))}{[R'(dK_t(\theta))]^2} > 0$$

since $R$ and $P$ are both concave. From (A.10), we know $dK(\theta, W)$ is increasing in $W$.

On the optimal path, the Hamiltonian (defined in Lemma 2.6.2) can be characterized as a function of project quality and continuation utility by defining

$$V(\theta, W_t) = H_t(\theta, dK_t(\theta), \beta_t(\theta), \mu_t(\theta)) \tag{A.31}$$

Lemma A.4.1. On the optimal path, the Hamiltonian function satisfies ($\theta > \theta_t^*$):

$$V_t(\theta) = V_t(\theta_t^*) + \theta - \theta_t^* \tag{A.32}$$

Proof. Differentiate (A.31) with respect to $\theta$ to get:

$$\frac{\partial V_t}{\partial \theta} = \frac{\partial H_t}{\partial \theta} + \frac{\partial H_t}{\partial dK_t} \frac{\partial dK_t}{\partial \theta} + \frac{\partial H_t}{\partial \beta_t} \frac{\partial \beta_t}{\partial \theta} + \frac{\partial H_t}{\partial \mu_t} \frac{\partial \mu_t}{\partial \theta} \tag{A.33}$$

where all the partial derivatives are evaluated on the optimal path. By the definition of Hamiltonian (A.8) and the incentive compatibility condition, we know

$$\frac{\partial H_t}{\partial \beta_t} \frac{\partial \beta_t}{\partial \theta} = [P'(W + \beta_t) - P'(W)] \frac{1}{R'(dK_t)} \tag{A.34}$$
Moreover, (A.9) and (A.12) together imply:

\[
\frac{\partial H_t}{\partial \mu_t} \frac{\partial \mu_t}{\partial \theta} = \frac{1}{R'(dK_t)} [P'(W_t) - P'(W_t + \beta_t)]
\] (A.35)

From (A.10), we know \( \frac{\partial H_t}{\partial dK_t} = 0 \). Then plug (A.34) and (A.35) into (A.33) to get:

\[
\frac{\partial V_t}{\partial \theta} = \frac{\partial H_t}{\partial \theta} = 1
\] (A.36)

Finally, integrate (A.36) over \([\theta^*_t, \theta]\) to obtain (A.32).

\[\square\]

Proof of Lemma 2.7.4: Plug the functional form \( R(k) = ak^{1/2} \) into (2.5) and simplify to obtain (2.6). Moreover, using the Cobb-Douglas form we can simplify (A.12) as:

\[
\mu_t(\theta) = \frac{\lambda a}{2\Delta} \left\{ 2[dK_t(\theta)]^{1/2} - a \right\}
\] (A.37)

Plug (A.37) into (A.8), the Hamiltonian on the optimal path can be rewritten as:

\[
V_t(\theta) = \lambda/\Delta [\theta + dK_t(\theta) + P(W_{t-} + \beta_t(\theta)) - P(W_{t-}) - P'(W_{t-})\beta_t(\theta)]
\] (A.38)

Since \( \beta_t(\theta^*_t) = 0 \), we know from (A.38) that

\[
V_t(\theta^*_t) = \lambda/\Delta [\theta^*_t + dK_t(\theta^*_t)]
\] (A.39)

Combining (A.38), (A.39), and (A.32) we obtain:

\[
dK_t(\theta) - P'(W_{t-})\beta_t(\theta) + P(W_{t-} + \beta_t(\theta)) - P(W_{t-}) = dK_t(\theta^*_t)
\] (A.40)

Finally, plug (A.40) into the objective of (HJB) and rearrange to get (2.7).

\[\square\]

Lemma A.4.2. Given \( W_o \in (0, W) \), the optimal policy has the following properties:

(a) If \( P''(W) \) is strictly increasing in at \( W_o \), then \( dK'(\theta^*(W_o), W)_2 \) is locally increasing in \( W \) and \( \theta^*(W) \) is locally decreasing in \( W \) at \( W_o \);

\( ^2 \theta^* \) is fixed at \( \theta^*(W_o) \) when changing continuation utility.
(b) If \(dK(\theta^*(W_o), W)\) is strictly increasing in \(W\) at \(W_o\), then \(P''(W)\) is locally increasing at \(W_o\).

**Proof.** Take some sufficiently small \(\epsilon > 0\) and \(W_o \in (0, \bar{W})\).

(a) For any \(x > 0\), since \(\theta^*(W)\) is continuous, we can define \(\hat{\theta}(x) = \max_{W \in [W_o, W_o + x]} \theta^*(W)\), and \(\tilde{\theta}(x) = \min_{W \in [W_o, W_o + x]} \theta^*(W)\). Pick any \(\eta > 0\) such that \(\eta + \hat{\theta}(\eta) - \tilde{\theta}(\eta) < \varepsilon/2\). Consider any \(\theta_o \in (\hat{\theta}(\eta), \tilde{\theta}(\eta) + \varepsilon/2)\), and \(W \in [W_o, W_o + \eta]\). From Lemma 2.6.6, we know

\[
\beta(\theta_o, W) = \int_{\theta^*(W)}^{\theta_o} \frac{1}{R'(dK(W, W))] du} \\
\leq \theta_o - \theta^*(W) < \hat{\theta}(\eta) + \varepsilon/2 - \tilde{\theta}(\eta)
\]  

(A.41)

The inequality is from the fact that \(1/R'(dK(\theta, W)) \leq 1\). Then (A.41) implies \(W + \beta(\theta_o, W) < W_o + \eta + \hat{\theta}(\eta) - \tilde{\theta}(\eta) + \varepsilon/2 < W_o + \epsilon\). And because \(\theta_o > \hat{\theta}(\eta)\), we know \(\beta(\theta_o, W) > 0\). This means \(W + \beta(\theta_o, W) \in (W_o, W_o + \epsilon)\). Because by assumption \(P''\) is strictly increasing in the neighborhood \([W_o, W_o + \epsilon]\), we know \(P''(W + \beta(\theta_o, W)) > P''(W)\). Note that \(\lambda/\Delta[P''(W + \beta(\theta_o, W)) - P''(W)] > 0\) is the cross partial derivative of the objective of (HJB) with respect to \(\beta(\theta_o, W)\) and \(W\). So the objective of (HJB) satisfies single crossing regarding \(\beta(\theta_o, W)\) and \(W\), when \(W \in [W_o, W_o + \eta]\). And we know \(\beta(\theta_o, W)\) is increasing in \(W\).

Let \(\hat{W}\) be the continuation utility level such that \(\hat{\theta}(\hat{W}) = \theta^*(\hat{W})\). By definition \(\beta(\theta^*(\hat{W}), \hat{W}) = 0\). Suppose that \(\theta^*(\hat{W}) > \theta^*(W_o)\). The above argument implies

\[
\lim_{\theta_o \to \hat{\theta}} \beta(\theta_o, \hat{W}) \geq \lim_{\theta_o \to \hat{\theta}} \beta(\theta_o, W_o)
\]  

(A.42)

By the continuity of policy, (A.42) implies

\[
\beta(\theta^*(\hat{W}), \hat{W}) \geq \beta(\theta^*(\hat{W}), W_o) > \beta(\theta^*(W_o), W_o) = 0
\]
The second inequality is by the assumption that $\theta^*(\hat{W}) > \theta^*(W_o)$. This is a contradiction. Therefore, we must have $\theta^*(\hat{W}) = \theta^*(W_o)$, since by construction $\theta^*(\hat{W}) \geq \theta^*(W_o)$. This means $\theta^*(W)$ is locally decreasing at $W_o$.

Take any $W \in (W_o, W_o + \eta)$. If $dK(\theta^*(W_o), W_o) = 0$, then because the threshold type is locally decreasing, i.e. $\theta^*(W) \preceq \theta^*(W_o)$, we must have $dK(\theta^*(W_o), W) \geq 0$. In the case $dK(\theta^*(W_o), W_o) > 0$, suppose $dK(\theta^*(W_o), W) < dK(\theta^*(W_o), W_o)$. By the continuity of policy, we know there exists $\theta^*(W_o) < \theta < \theta^*(W_o) + \epsilon/2$ such that $dK(\theta, W) < dK(\theta, W_o)$ for $\theta \in (\theta^*(W_o), \theta')$. Then $\beta(\theta', W) < \beta(\theta', W_o)$, which is a contradiction.

\[ (b) \] Suppose that $P''(W)$ is strictly decreasing at $W_o$. Repeating the procedure in (a) shows $\beta(\theta, W)$ is decreasing in $W$ at $W_o$, for $\theta \in (\theta^*(W_o), \theta^*(W_o) + \epsilon/2)$. However, since $dK(\theta^*(W_o), W)$ is strictly increasing in $W$ at $W_o$, there exists some $\theta'' \in (\theta^*(W_o), \theta^*(W_o) + \epsilon/2)$ such that $dK(\theta, W)$ and $\beta(\theta, W)$ are strictly increasing in $W$ at $W_o$, a contradiction.

To ease notation, let us define a function $h(\theta, W)$ as:

$$ h(\theta, W) = P(W + \beta(\theta, W)) - P(W) - P'(W)\beta(\theta, W) \quad (A.43) $$

**Lemma A.4.3.** Suppose $P''(W)$ is increasing over $(W_o, \hat{W})$. If $\beta(\theta, W)$ strictly decreases in $W$ at some $\hat{W} \in (W_o, \hat{W})$, then $h(\theta, W)$ strictly increases in $W$ at $\hat{W}$.

**Proof.** There exists some $\alpha(\theta, \hat{W}) \in [\hat{W}, \hat{W} + \beta(\theta, \hat{W})]$ such that:

$$ P'(\hat{W} + \beta(\theta, \hat{W})) - P'(\hat{W}) - \beta(\theta, \hat{W})P''(\hat{W}) $$

$$ = [P''(\alpha(\theta, \hat{W})) - P''(\hat{W})]\beta(\theta, \hat{W}) \geq 0 \quad (A.44) $$

The inequality is because $P''(W)$ is increasing when $W > \hat{W}$. We know from (A.44) that if $\beta(\theta, W)$ does not change locally at $\hat{W}$, then $h(\theta, W)$ locally increases at
\(\hat{W}\). However, by assumption \(\beta(\theta, W)\) strictly decreases at \(\hat{W}\). And since \(P'(\hat{W} + \beta(\theta, \hat{W})) - P'(\hat{W}) < 0\) by the strict concavity of \(P\), we know the strict increase in \(\beta(\theta, W)\) causes \(h(\theta, W)\) to strictly increase at \(\hat{W}\).

\[\]

**Lemma A.4.4.** Suppose \(P''(W)\) is increasing over \((W_0, \hat{W})\). For any \(\bar{W} \in (W_0, \hat{W})\):

(a) \(\theta^*(W)\) is decreasing at \(\bar{W}\)

(b) there does not exist an interval of types \([\theta^*(\bar{W}), \hat{\theta}]\) such that \(dK(\theta, W)\) decreases in \(W\) at \(\bar{W}\) for all \(\theta\) in this interval and strictly decreases at \(\hat{W}\) for some \(\theta\).

**Proof.** Pick any \(\bar{W}\) in the neighbourhood of \(\hat{W}\) and \(\bar{W} > \hat{W}\).

(a) Suppose there exists an interval of types \([\theta^*(\bar{W}), \hat{\theta}]\) such that \(dK(\theta, \bar{W}) \leq dK(\theta, \hat{W})\) for all \(\theta\) in this interval and \(dK(\theta, \bar{W}) < dK(\theta, \hat{W})\) for some \(\theta\). Since \(dK(\hat{\theta}, \bar{W}) = dK(\hat{\theta}, \hat{W}) = k^\ast\), i.e. no distortion at the top, there must exists some \(\tilde{\theta} \geq \hat{\theta}\) such that \(dK(\tilde{\theta}, \bar{W}) = dK(\tilde{\theta}, \hat{W})\) and \(\beta(\tilde{\theta}, \bar{W}) < \beta(\tilde{\theta}, \hat{W})\). Lemma A.4.3 then implies \(h(\tilde{\theta}, \bar{W}) > h(\tilde{\theta}, \hat{W})\). From (A.40):

\[
dK(\theta^*(\bar{W}), \bar{W}) = dK(\tilde{\theta}, \bar{W}) + h(\tilde{\theta}, \bar{W}) \quad (A.45)
\]

\[
dK(\theta^*(\bar{W}), \bar{W}) = dK(\tilde{\theta}, \hat{W}) + h(\tilde{\theta}, \hat{W}) \quad (A.46)
\]

Hence, from (A.45) and (A.46) we know \(dK(\theta^*(\bar{W}), \bar{W}) < dK(\theta^*(\bar{W}), \hat{W})\).

Suppose \(\theta^*(\bar{W}) > \theta^*(\hat{W})\). Then we know \(dK(\theta^*(\bar{W}), \bar{W}) = 0\). This means \(dK(\theta^*(\bar{W}), \hat{W}) < 0\), a contradiction.

(b) If \(\theta^*(\bar{W}) = \theta^*(\hat{W})\), then the above result means \(dK(\theta^*(\bar{W}), \bar{W}) < dK(\theta^*(\bar{W}), \hat{W})\), a contradiction with our assumption. If \(\theta^*(\bar{W}) < \theta^*(\hat{W})\), then it has to be the case that \(\theta^*(\bar{W}) > \hat{\theta}\). And hence, \(dK(\theta^*(\bar{W}), \hat{W}) = 0\). Moreover, \(dK(\theta^*(\bar{W}), \hat{W}) > dK(\theta^*(\bar{W}), \hat{W}) \geq 0\) by the monotonicity of capital allocation over types. So we also get \(dK(\theta^*(\bar{W}), \hat{W}) < dK(\theta^*(\bar{W}), \hat{W})\), a contradiction.
Lemma A.4.5. Suppose $P''(W)$ is increasing over $(W_o, W)$. Then the optimal policy $dK(\theta, W)$ and $\beta(\theta, W)$ are increasing in $W$ for all $\theta$, and $\theta^*(W)$ is decreasing over $(W_o, W)$.

Proof. Pick $\hat{W} \in (W_o, W)$. Suppose $dK(\hat{\theta}, W)$ is strictly decreasing in $W$ at $\hat{W}$ for some $\hat{\theta}$. By Lemma A.4.4, it must be that $\theta^*(\hat{W}) < \hat{\theta} < \hat{\theta}$. To ease notation, let us define $A(\theta, W) = a[dK(\theta, W)]^{\frac{1}{2}}$. By Lemma A.4.4, there must exist an $\hat{\theta} < \hat{\theta}$ such that $A(\hat{\theta}, W)$ strictly increases in $W$ at $\hat{W}$. Otherwise, there will be an interval of types starting $\theta^*(\hat{W})$ over which $A(\theta, W)$ decreases in $W$ at $\hat{W}$ for all types and strictly decreases in $W$ for some type, a contradiction.

So we have $A_w(\hat{\theta}, \hat{W}) > 0$ and $A_w(\hat{\theta}, \hat{W}) < 0$. This future implies that there is some $\theta_1 < \hat{\theta}$ satisfying $A_w(\theta_1, \hat{W}) = 0$ and $A_w(\theta_1, \hat{W}) < 0$. Moreover, since $A_w(\hat{\theta}, \hat{W}) = 0$, there also exists some $\theta_2 > \hat{\theta}$ satisfying $A_w(\theta_2, \hat{W}) = 0$ and $A_w(\theta_2, \hat{W}) > 0$. Without loss of generality, we consider $(\theta_1, \theta_2)$ to be the interval such that $A_w(\theta, W) < 0$ for all $\theta \in (\theta_1, \theta_2)$. From equation (2.6), we get for any $\theta > \theta^*(\hat{W})$:

$$A_w(\theta, \hat{W}) = P''(\hat{W}) - P''(\hat{W} + \beta(\theta, \hat{W}))(1 + \beta_w(\theta, \hat{W}))$$

(A.47)

Using (A.47) and the fact that $A_w(\theta_1, \hat{W}) < 0 < A_w(\theta_2, \hat{W})$, we get

$$- P''(\hat{W} + \beta(\theta_1, \hat{W}))(1 + \beta_w(\theta_1, \hat{W}))$$

$$< -P''(W + \beta(\theta_2, \hat{W}))(1 + \beta_w(\theta_2, \hat{W}))$$

(A.48)

From (A.47) and the fact $A_{\text{w} \theta}(\theta_2, \hat{W}) > 0$, $P'' \leq 0$, we get $1 + \beta_w(\theta_2, W) > 0$. Moreover, because $A_w(\theta, W) < 0$ for $\theta \in (\theta_1, \theta_2)$ by construction, we know $\beta_w(\theta_1, \hat{W}) > \beta_w(\theta_2, \hat{W})$. Hence,

$$1 + \beta_w(\theta_1, W) > 1 + \beta_w(\theta_2, W) > 0$$

(A.49)

By the assumption $P''(W)$ is increasing when $W > W_o$, we know

$$- P''(\hat{W} + \beta(\theta_1, \hat{W})) \geq -P''(\hat{W} + \beta(\theta_2, \hat{W})) \geq 0$$

(A.50)

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Multiplying (A.49) and (A.50), we obtain

\[ -P' p W \beta p W, W \theta_1, W q p W \beta w(\theta_1, w) \]

\[ \geq -P' p W + \beta(\theta_2, W)(1 + \beta_w(\theta_2, w)) \]  \hspace{1cm} (A.51)

However, (A.48) and (A.51) form a contradiction. Therefore, both \(dK(\theta, W)\) and \(\beta(\theta, W)\) are increasing for all \(\theta\) when \(W > W_o\).

\[ \square \]

Proof of Proposition 2.7.5: (a) From Proposition 2.6.5, \(dK(\theta, \bar{W}) = k^*\) and \(dK(\theta, W) < k^*\) for any \(W < \bar{W}\). So we must have \(dK(\theta, W)\) is strictly increasing in \(W\) over some neighbourhood \((\bar{W}, \bar{W})\). By lemma A.4.2, \(P'(W)\) is increasing over this neighbourhood. Define the lower bound of this interval as: \(W^1 = \inf\{W < \bar{W} : P'(W)\) is increasing over \((\bar{W}, \bar{W})\}\). When \(W \geq W^1\): Lemma A.4.5 then implies policy \(dK(\theta, W), \beta(\theta, W)\) are increasing in \(W\); Lemma A.4.4 implies \(\theta^*(W)\) is decreasing in \(W\).

(b) Now we characterize the property of the threshold level \(W^1\). We first show that \(dK(\theta^*(W^1), W)\) is locally constant at \(W^1\). Take \(\tilde{W} \in (W^1 - \varepsilon, W^1)\) for any sufficiently small \(\varepsilon > 0\). By definition of \(W^1\), we know \(P'(W)\) is strictly decreasing at \(\tilde{W}\). From part (b) of Lemma A.4.2, we know \(dK(\theta^*(\tilde{W}), W)\) is decreasing in \(W\) at \(\tilde{W}\). As \(\varepsilon\) converges to zero, we know \(dK(\theta^*(W^1), W)\) is decreasing in \(W\) at \(W^1\). Since \(dK(\theta^*(W^1), W)\) is also increasing at \(W^1\), it must be constant locally at \(W^1\).

Now we show that \(dK(\theta^*(W^1), W^1) < \frac{a^2}{16}\). Suppose not, then \(\theta^*(W^1) = \underline{\theta}\) and \(dK(\theta, W^1) \geq \frac{a^2}{16}\) for all \(\theta\). Let us define

\[ b(W) = \int_{\theta^*(W)}^{\theta} \left\{ a[dK(\theta, W)]^{\frac{1}{2}} - 2dK(\theta, W) \right\} d\theta \]
Note that \( a(dK)^{\frac{1}{2}} - 2dK \) is decreasing in \( dK \) when \( dK \geq \frac{a^2}{16} \). And because \( dK(\theta, W) \) is increasing in \( W \) at \( W^1 \), \( b(W) \) has to be decreasing at \( W^1 \). Moreover, since \( dK(\theta^*(W^1), W) \) is locally constant at \( W^1 \), we know the right-hand side of (A.52) is decreasing at \( W^1 \). By Lemma A.3.3, we know

\[
(r - \gamma) P'(W) - \gamma W P''(W)
\]

where \( \alpha(\theta, W) \in (W, W + \beta(\theta, W)) \). Since \( P''(W) \) is increasing over \([W^1, W]\), the right-hand side of (A.52) is nonnegative at \( W = W^1 \). This means \( rP(W) - \gamma WP'(W) \), the left-hand side of (2.7), is increasing at \( W^1 \). Hence, both sides of (2.7) have to be constant locally at \( W^1 \), and the right-hand side of (A.52) has to be zero. We also have \( b(w) \) and \( dK(\theta, W) \) are locally constant at \( W^1 \) for all \( \theta \). Since (A.52) equals zeros at \( W^1 \), we know \( P''(W) \) is constant locally at \( W^1 \). Using (2.6), We can rewrite (A.52) as:

\[
(r - \gamma) P'(W) - \gamma WP''(W)
\]

\[
= \frac{\lambda}{\Delta} \left\{ a[dK(\theta^*(W), W)]^{\frac{1}{2}} - \frac{a^2}{2} \right\} - \int_{\theta}^{\tilde{\theta}} \beta(\theta, W)d\theta P''(W) \tag{A.53}
\]

Then we know the right-hand side of (A.53) is locally constant at \( W^1 \). However, the left-hand side of (A.53) is strictly increasing at \( W^1 \), since \( P''(W) \) is negative and locally constant at \( W^1 \). This forms a contradiction. Therefore, \( dK(\theta^*(W^1), W^1) < \frac{a^2}{16} \). From (2.6), this implies

\[
\int_{\theta^*(W^1)}^{\tilde{\theta}} [P'(W^1) - P'(W^1 + \beta(\theta, W^1))]d\theta > \frac{a^2}{4}
\]

Because \( P'(W^1 + \beta(\theta, W^1)) \geq -1 \), we must have \( P'(W^1) > \frac{a^2}{4\Delta} - 1 \).
A.5 Numerical Methods

In this section, we will show the numerical solution of the differential equation system defined by (2.6) to (2.8) and their boundary conditions. We apply the following numerical strategy. Give the value function $P(W)$, (2.6) and (2.8) can be combined to get a second order partial differential equation of $\beta(\theta, W)$:

$$\frac{a^2}{2} \beta_{\theta\theta}(\theta, W) = P'(W) - P'(W + \beta(\theta, W))$$ (A.54)

Given the policy function $\beta(\theta, W)$, (2.7) can be viewed as a first order ordinary differential equation of $P(W)$. The differential equations (2.7) and (A.54) jointly determine $P(W)$ and $\beta(\theta, W)$. Then $dK(\theta, W)$ can be derived from (2.8). The parameter values in the numerical example that we provide here are: $r = .1$, $\gamma = .15$, $\lambda = 2$, $\theta = 0$, $\bar{\theta} = 1$, $R = 0$, $L = 13$.

The value function $P(W)$ is plotted in figure 2.3. It shows that the value function is strictly concave before $\bar{W} = 5.8$. $P'(W)$ declines from a large positive level to $-1$ at $\bar{W}$.

The policy dynamics over DM’s continuation utility are plotted in figure 2.5. Except in a small range around zero, the pay-performance sensitivities and capital allocated to all project types are increasing in DM’s continuation utility. In this example, $W^* = .859$. All the policy functions are increasing in DM’s continuation utility.

---

3 It has two boundary conditions $\beta(\bar{\theta}, W) = 0$ and $\beta_{\theta}(\bar{\theta}, W) = 1$, for all $W$. To solve this second order boundary value problem, we apply the finite difference method illustrated in Chapter 2 of LeVeque (2007).

4 It has a free boundary $\bar{W}$ and two boundary conditions $P(0) = L$ and $P'(\bar{W}) = -1$. To solve the system, we apply the projection method illustrated in Chapter 11 of Judd (1998). In particular, $P(W)$ is approx by a Chebyshev Polynomial of degree 11. So the differential equation system can be transferred to the problem of finding the Chebyshev Polynomial coefficients and $\bar{W}$.

5 In this numerical exercise, we discretize the type interval $[0.01, 0.51]$ into two hundred points. Five types’ dynamics over continuation utility are plotted in figure 2.5.
utility when $W > W^*$. So the numerical result verifies the conclusion of Proposition 2.7.5.

Moreover, the left plot of figure 2.5 shows that the pay-performance sensitivity of the DM operating the lowest quality project is always zero. The right plot shows that the capital allocated to the DM operating the highest quality project always stays at the first best level which is 0.5. The capital allocated to the DM operating projects lower than the highest type are always distorted downward until the DM’s continuation utility reaches $\bar{W}$. At $\bar{W}$ all the capital allocations reach the first best level. The conclusions of Proposition 2.6.5 are verified by these results.

The policy dynamics over project qualities are plotted in figure 2.2. The plots clearly show that both the pay-performance sensitivity and the capital allocation are increasing over project qualities for any fixed continuation utility level. So the conclusions of Lemma 2.6.6 are verified by the numerical results.

Figure A.1 and A.2 show the 3D plot of pay-performance sensitivity and capital allocation over the two dimensions.

Using the numerical solution shown above, we can simulate the evolution of the continuation utility and cash payments from the stochastic differential equation (2.4). Figure 2.4 shows a simulated path. It is easy to see that the simulation has several expected features. The continuation utility drifts downward during the no investment periods and jumps up when a project arrives. This verifies the result of Lemma 2.7.1 and the intuition that the variation in DM’s compensation over time alleviates the agency conflicts. The jump sizes are varying with the level of the continuation utility and the project quality. The bars in the figure represent the amount of cash payments. Cash payments are issued when the continuation utility reaches $\bar{W}$ so that the continuation utility is reflecting at $\bar{W}$. In addition, The downward drift is

\footnote{In this numerical exercise, we choose 200 continuation utility levels between 0 and $\bar{W}$. Five curves are plotted in figure 2.2.}
**Figure A.1:** Pay-performance Sensitivity over Project Qualities and Continuation Utility

**Figure A.2:** Capital Allocation over Project Qualities and Continuation Utility
flatter when the level of the continuation utility is lower. This trend is obvious in the beginning periods when the continuation utility is close to zero. This reflects that the pay-performance sensitivity is designed to be smaller when there is a higher risk of terminating the DM. If the pay-performance sensitivity is constant, then the opposite (i.e. the downward drift is steeper when the continuation utility is low) should be expected.
Appendix B

Proofs from Chapter 3

B.1 Bellman Operator

In this section, we will define the Bellman operator corresponding to maximizing the firm’s surplus and show that the value function satisfies various properties. Let $C(V \times S)$ be the space of continuous functions on the domain $V \times S$ and let $\mathcal{F} := \{ P \in C(V \times S) : 0 \leq P(v,s) \leq \bar{Q}(s) \}$ be endowed with the “sup” metric. Define the Bellman operator $T : \mathcal{F} \rightarrow \mathcal{F}$ as:

$$
(TP)(v,s) = \max_{k,m_i,w_i} \left( -k + p_s[R(k) + \delta P(w_g,g)] + (1-p_s)\delta P(w_b,b) \right) \quad (P1)
$$

s.t. $(k,m_i,w_i) \in \Gamma(v,s)$

It is standard to show that $T : \mathcal{F} \rightarrow \mathcal{F}$ is well defined. In what follows, we shall show that $T$ maps certain subsets of $\mathcal{F}$ to themselves. We begin by showing that $T$ maps concave functions to concave functions.

**Lemma B.1.1.** Let $\mathcal{F}_1 := \{ P \in \mathcal{F} : P(\cdot,s) \text{ is concave for all } s \in S \}$. If $P \in \mathcal{F}_1$, then $TP \in \mathcal{F}_1$.

**Proof.** Let $P \in \mathcal{F}_1$. For any $k \geq 0$, let $R(k) = r$, so that $c(r) = k$, where $c(r) := \ldots$
Because \( R \) is increasing, \( c \) is well defined. The concavity of \( R \) implies that \( c \) is convex. Thus, we can let the choice variables be \((r, m_i, w_i)\), so that the objective becomes

\[
(TP)(v, s) = \max_{r, m_i, w_i} (-c(r) + p_s[r + \delta P(w_g, g)] + (1 - p_s)\delta P(w_b, b))
\]

s.t. \((r, m_i, w_i) \in \Gamma(v, s)\)

It is easy to see that with this transformation, \( \Gamma(v, s) \) is the intersection of finitely many affine sets, and so is convex. Moreover, the objective, \(-c(r) + p_s[r + \delta P(w_g, g)] + (1 - p_s)\delta P(w_b, b)\) is concave in \((r, w_i)\). Standard arguments now imply \( TP(v, s) \in \mathcal{F}_1 \). \( \square \)

Recall that the efficient sets \( E_s \) where the firm is unconstrained are:

\[
E_b := \left\{ v \geq \bar{v}_b : v_g - v_b \geq \frac{R(\bar{k}_b)}{(1 - \delta \Delta)} \right\}
\]

\[
E_g := \left\{ v \geq \bar{v}_g : v_g - v_b \geq R(\bar{k}_g) + \delta \Delta \max \left[ \frac{\delta \bar{v}_{bg} - v_b}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right] \right\}
\]

where \( \bar{v}_b \) and \( \bar{v}_g \) are given in Lemma 3.7.1. In the following proof, we will use the fact that \( \bar{v}_s \in E_s \) and that \( \bar{v}_s \) satisfy:

\[
\bar{v}_{sb} = \delta [p_b \bar{v}_{bg} + (1 - p_b) \bar{v}_{bb}] \quad (B.1)
\]

\[
\bar{v}_{sg} \geq \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] \quad (B.2)
\]

We now show that \( T \) maps the space of functions that achieve \( \bar{Q}(s) \) on the sets \( E_s \) to itself.

**Lemma B.1.2.** Let \( \mathcal{F}_2 := \{ P \in \mathcal{F} : P(v, s) = \bar{Q}(s) \text{ for all } v \in E_s, s \in S \} \). If \( P \in \mathcal{F}_2 \), then \( TP \in \mathcal{F}_2 \).

**Proof.** Suppose \( P \in \mathcal{F} \). We will show that for any \( v \in E_s \), there exists a policy such that \( TP(v, s) = \bar{Q}(s) \).
1. For any \( \mathbf{v} \in E_b \), let us consider the following policy:

\[
\begin{align*}
k &= \bar{k}_b, \ w_g &= \bar{v}_g, \ w_{bg} = \bar{v}_{bg}, \ w_{bb} = \bar{v}_{bb} - \min \{v_g - v_b, \bar{v}_{bg} - \bar{v}_{bb}\} \\
m_g &= R(\bar{k}_b) + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] - v_g, \ m_b &= \delta [p_b w_{bg} + (1 - p_b) w_{bb}] - v_b
\end{align*}
\]

We first show this policy satisfies all the constraints in (P1). Obviously (PK\(_b\)) and (PK\(_g\)) hold by construction. Because \( \mathbf{v} \in E_b \), we have

\[
v_g - v_b \geq R(\bar{k}_b) + \delta \Delta (v_g - v_b) \geq R(\bar{k}_b) + \delta \Delta \min \{v_g - v_b, \bar{v}_g - \bar{v}_b\} = R(\bar{k}_b) + \delta \Delta (w_{bg} - w_{bb})
\]

The first line is from the definition of \( E_b \). So the constructed policy satisfies (IC\(^*\)). By construction, we either have \( w_{bb} = \bar{v}_{bb} \) or \( w_{bb} = v_b - (v_g - \bar{v}_{bg}) \leq v_b \), which implies \( \bar{v}_{bb} \leq w_{bb} \leq v_b \). Then from (B.1) we can obtain:

\[
v_b \geq w_{bb} \geq \delta [p_b \bar{v}_{bg} + (1 - p_b) w_{bb}] = \delta [p_b w_{bg} + (1 - p_b) w_{bb}]
\]

which means the constructed transfer \( m_b \leq 0 \), or (LL) for \( s = b \) is satisfied. Moreover, the constructed transfer \( m_g \) satisfies:

\[
m_g \leq R(\bar{k}_b) + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] - \bar{v}_{bg} \leq R(\bar{k}_b)
\]

The first inequality is from \( \bar{v}_{bg} \leq v_g \), and the second is by (B.2). So (LL) for \( s = g \) is satisfied. Since all constraints of (P1) are satisfied,

\[
TP(\mathbf{v}, b) \geq -\bar{k}_b + p_b [R(\bar{k}_b) + \delta P(\bar{v}_g, g)] + (1 - p_b) \delta P(\mathbf{w}_b, b) \geq \frac{R(\bar{k}_b)}{1 - \delta \Delta}, \text{ because } \mathbf{v}, \bar{v}_b \in E_b.
\]

So we must have \( \mathbf{w}_b \in E_b \). The assumption \( P \in \mathcal{F}_2 \) then implies \( P(\mathbf{w}_b, b) = P(\bar{v}_b, b) \). So the right hand side of (B.3) equals \( \bar{Q}(b) \), which implies \( TP(\mathbf{v}, b) = \bar{Q}(b) \). Therefore \( TP(\mathbf{v}, b) \in \mathcal{F}_2 \).
2. For any \( v \in E_g \), let us consider the following policy:

\[
k = \bar{k}_g, w_g = \bar{v}_g, w_{bg} = \bar{v}_{bg}, w_{bb} = \bar{v}_{bb} - \max \left[ \frac{\delta \bar{v}_{bg} - v_b}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right]
\]

\[
m_g = R(\bar{k}_g) + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] - v_g, m_b = \delta [p_b w_{bg} + (1 - p_b) w_{bb}] - v_b
\]

Obviously (PK\(_b\)), (PK\(_g\)) are satisfied. Because \( v \in E_g \),

\[
v_g - v_b \geq R(\bar{k}_g) + \delta \Delta \max \left[ \frac{\delta \bar{v}_{bg} - v_b}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right] = R(\bar{k}_g) + \delta \Delta (w_{bg} - w_{bb})
\]

which means (IC\({}^*\)) is satisfied. The constructed transfer \( m_g \) satisfies:

\[
m_g \leq R(\bar{k}_g) + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_{gb}] - \bar{v}_{gg} \leq R(\bar{k}_g)
\]

The first inequality is from \( \bar{v}_{gg} \leq v_g \), and the second is by (B.2). So (LL) for \( s = g \) is satisfied. The constructed transfer \( m_b \) satisfies:

\[
m_b = \delta \bar{v}_{bg} - \delta (1 - p_b) \max \left[ \frac{\delta \bar{v}_{bg} - v_b}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right] - v_b \leq 0
\]

So (LL) for \( s = b \) is satisfied. Since all constraints of (P1) are satisfied,

\[
TP(v, g) \geq -\bar{k}_g + p_g [R(\bar{k}_g) + \delta P(\bar{v}_g, g)] + (1 - p_g) \delta P(w_b, b) \quad (B.4)
\]

Moreover, because

\[
w_{bb} \geq \bar{v}_{bb} - \max \left[ \frac{\delta \bar{v}_{bb} - \bar{v}_{bb}}{\delta (1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta \Delta} \right]
\]

\[
\geq \bar{v}_{bb} - (\bar{v}_{bg} - \bar{v}_{bb}) = \bar{v}_{bb}
\]

The first line is from \( \bar{v}_{bb} \leq v_b \). The second line is from \( \frac{\delta \bar{v}_{bb} - \bar{v}_{bb}}{\delta (1 - p_b)} = \bar{v}_{bg} - \bar{v}_{bb} \) (by (B.1)), and \( \frac{R(\bar{k}_b)}{1 - \delta \Delta} \leq \bar{v}_{bg} - \bar{v}_{bb} \). Also by construction, \( w_{bg} - w_{bb} \geq \frac{R(\bar{k}_b)}{1 - \delta \Delta} \). So we know \( w_b \in E_b \). The assumption \( P \in F_2 \) then implies \( P(w_b, b) = P(\bar{v}_b, b) \). So the right hand side of (B.4) equals \( \bar{Q}(g) \). Therefore \( TP(v, g) \in F_2 \).
Lemma B.1.3. Let $\mathcal{F}_3 = \{ P(v, s) \in \mathcal{F} : P \text{ decreases in } v_b \text{ at } ((v, v), s), \forall v > 0 \}$. If $P(v, s) \in \mathcal{F}_3$, then $TP(v, s) \in \mathcal{F}_3$.

Proof. Suppose $P(v, s) \in \mathcal{F}_3$ and pick any $v > 0$. Let $(k, m_b, m_g, w_b, w_g)$ be the optimal policy at $((v, v), s)$. From (IC*), we must have $k = 0$, $w_{bg} = w_{bb} = v/\delta$. Let $v' = v - \frac{1 - p_b}{1 - p_g} R(\varepsilon)$ for any arbitrary small $\varepsilon > 0$, and $k' = R(\varepsilon)$, $w_{bb}' = \frac{v}{\delta} - \frac{R(\varepsilon)}{\delta(1 - p_g)}$.

Since $(k', m_b, m_g, (w_{bb}', w_{bg}), w_g) \in \Gamma((v', v), s)$, we have

$$TP((v', v), s) - TP((v, v), s) \geq -\varepsilon + p_a R(\varepsilon) + (1 - p_a) \delta \left[ P \left( \left( w - \frac{R(\varepsilon)}{\delta(1 - p_g)}, w \right), b \right) - P((w, w), b) \right] \geq 0$$

The last inequality is implied by the fact that $P \in \mathcal{F}_3$ and $R'(0) = \infty$. Therefore $TP(v, s) \in \mathcal{F}_3$. \hfill $\square$

Lemma B.1.4. Let $\mathcal{F}_4 := \{ P(v, s) \in \mathcal{F} : P(v, g) \geq P(v, b) \}$. If $P(v, s) \in \mathcal{F}_4$, then $TP(v, s) \in \mathcal{F}_4$.

Proof. Suppose $P(v, s) \in \mathcal{F}_4$. Let $(k, m_b, m_g, w_b, w_g)$ be the optimal policy at $(v, b)$.

By (IC), we know

$$v_g \geq R(k) - m_b + \delta[p_g w_{bg} + (1 - p_g) w_{bb}]$$

$$\geq \delta[p_g w_{bg} + (1 - p_g) w_{bb}]$$

The second inequality is from $m_b \leq 0$. Let $m'_g = \delta[p_g w_{bg} + (1 - p_g) w_{bb}] - v_g + R(k)$.

So $m'_g \leq R(k)$. And the policy $(k, m_b, m'_g, w_b, w_g)$ satisfies (??), and (LL) for $s = g$.

Since other constraints do not change, $(k, m_b, m'_g, w_b, w_g) \in \Gamma(v, b)$, which implies

$$TP(v, b) = -k + p_b R(k) + \delta p_b \left[ P(w_g, g) - P(w_b, b) \right] + \delta P(w_b, b)$$

$$\geq -k + p_b R(k) + \delta p_b \left[ P(w_b, g) - P(w_b, b) \right] + \delta P(w_b, b) \quad \text{(B.5)}$$

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From (B.5) we get \( P(w_g, g) - P(w_b, b) \geq P(w_b, g) - P(w_b, b) \geq 0 \). The second inequality is by \( P \in \mathcal{F}_4 \). Moreover, because \((k, m_b, m_g, w_b, w_g) \in \Gamma(v, g)\), we have
\[
TP(v, g) \geq -k + p_g R(k) + \delta p_g [P(w_g, g) - P(w_b, b)] + \delta P(w_b, b)
\]
\[
\geq -k + p_b R(k) + \delta p_b [P(w_g, g) - P(w_b, b)] + \delta P(w_b, b)
\]
\[
= TP(v, b)
\]
The second line is from \( P(w_g, g) - P(w_b, b) \geq 0 \). Therefore \( TP(v, s) \in \mathcal{F}_4 \). \qed

Lemma B.1.5. Let \( \mathcal{F}_5 = \{P(v, s) \in \mathcal{F} : P(v + (\varepsilon, \varepsilon), s) \geq P(v, s), \forall \varepsilon > 0\} \). Then \( TP(v, s) \in \mathcal{F}_5 \). Moreover, \( P(v, s) \in \mathcal{F}_5 \) implies \( m_b(v, s) = 0 \) is optimal in (P1).

Proof. Let \((k, m_b, m_g, w_b, w_g)\) be the optimal policy at state \((v, s)\). Since \((k, m_b - \varepsilon, m_g - \varepsilon, w_b, w_g) \in \Gamma(v + (\varepsilon, \varepsilon), s)\) and \(m_b, m_g\) do not appear in the objective of (P1), we know \( TP(v + (\varepsilon, \varepsilon), s) \geq TP(v, s) \).

Suppose \( m_b < 0 \). Let us consider the policy \((k', m'_b, m'_g, w'_b, w'_g) = (k, 0, m_g, w_b - (m_b, m_b)/\delta, w_g)\). Obviously, \((k', m'_b, m'_g, w'_b, w'_g) \in \Gamma(v, s)\). Since \( P \in \mathcal{F}_6 \), we know \( P(w'_b, b) \geq P(w_b, b) \). Hence the new policy \((k', m'_b, m'_g, w'_b, w'_g)\) at least weakly increases the objective of (P1). So \( m_b = 0 \) is optimal. \qed

Because the optimal contract lies in the interior of the feasible set (in an appropriate sense), the continuous differentiability of \( TP \) and \( P \) follows from standard results as, for instance, in Stokey et al. (1989). To further establish properties of the value function, we will apply the optimality conditions of (P1). In what follows, \( \hat{\eta}_b(v, s), \hat{\eta}_g(v, s) \) are the Lagrange multipliers for (PK\(_b \)), (PK\(_g \)) constraints in (P1), \( \hat{\lambda}(v, s) \) is the Lagrange multiplier for (IC) constraint, and \( \hat{\mu}_b(v, s), \hat{\mu}_b(v, s) \) are the
Lagrange multipliers for (LL) constraints. The first order conditions of (P1) are:

\[ R'(k(v,s)) = 1/[p - \hat{\eta}g(v,s) + \hat{\mu}g(v,s)] \]  
(BFOCk)

\[(1 - p)Pb(w_b(v,s), b) = \hat{\eta}b(v,s)(1 - p) + \hat{\lambda}(v,s)(1 - p) \]  
(BFOCwb)

\[(1 - p)Pg(w_b(v,s), b) = \hat{\eta}b(v,s)p + \hat{\lambda}(v,s)p_g \]  
(BFOCwb)

\[ psPb(w_g(v,s), g) = \hat{\eta}g(v,s)(1 - p_g) - \hat{\lambda}(v,s)(1 - p_g) \]  
(BFOCwg)

\[ pgPb(w_g(v,s), g) = \hat{\eta}g(v,s)p_g - \hat{\lambda}(v,s)p_g \]  
(BFOCwg)

The envelope conditions of (P1) are:

\[(TP)b(v,s) = \hat{\eta}b(v,s) \]  
(BEnvb)

\[(TP)g(v,s) = \hat{\eta}g(v,s) \]  
(BEnvg)

**Lemma B.1.6.** For any \((v, s) \in V \times S\), the Lagrange multipliers in (P1) satisfy:

\[ \hat{\eta}b(v,s) + \hat{\lambda}(v,s) - \hat{\mu}b(v,s) \geq 0, \quad m_b(v,s)[\hat{\eta}b(v,s) + \hat{\lambda}(v,s) - \hat{\mu}b(v,s)] = 0 \]  
(B.6)

\[ \hat{\eta}b(v,s) - \hat{\lambda}(v,s) - \hat{\mu}g(v,s) = 0 \]  
(B.7)

**Proof.** Note that \(m_b, m_g\) only appear in (P1) in a linear way. The term multiplying \(m_b\) in Lagrangian of (P1) is: \(\hat{\eta}b + \hat{\lambda} - \hat{\mu}b\). This term must be nonnegative. Otherwise, at \(s = b\), maximizing Lagrangian means that the optimal transfer is \(m_b = -\infty\), since it is not bounded below. But this simply means the Lagrangian is unbounded above which is a contradiction since \(P\) is bounded. Moreover, if \(\hat{\eta}b + \hat{\lambda} - \hat{\mu}b > 0\) then maximizing Lagrange implies \(m_b = 0\), because \(m_b \leq 0\) by (LL). Hence we always have \(m_b(\hat{\eta}b + \hat{\lambda} - \hat{\mu}b) = 0\).

Now we show (B.7) holds. Since \(m_g\) is also unbounded below, we must have \(\hat{\eta}g - \hat{\lambda} - \hat{\mu}_g \geq 0\). Let \((k, m_b, m_g, w_b, w_g)\) be the optimal policy at any \((v, s)\). For any \(\varepsilon > 0\), consider another policy: \((k', m'_b, m'_g, w'_b, w'_g) = (R^{-1}(R(k) + \varepsilon), m_b, m_g, w_b, w_g + \frac{\varepsilon}{g})\).
By definition,

\[(TP)_g(v, s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [(TP((v_b, v_g + \varepsilon), s) - TP((v_b, v_g), s)]
\]

\[\geq \lim_{\varepsilon \to 0} \varepsilon \bigg\{ p_s R(k') - k' - p_s R(k) + k + p_s \delta \left[ P(w'_g, g) - P(w_g, g) \right] \bigg\}
\]

\[= p_s - \frac{1}{R'(k)} + p_s \left[ P_b(w_g, g) + P_g(w_g, g) \right]
\]

\[= p_s - \frac{1}{R'(k)} + \hat{\eta}_g(v, s) - \hat{\lambda}(v, s) \tag{B.8} \]

The inequality is because \((k', m'_b, m'_g, w'_b, w'_g) \in \Gamma((v_b, v_g + \varepsilon), s)\). The last equality is obtained by adding \((\text{BFOC}w_{gb})\) and \((\text{BFOC}w_{gg})\). Moreover, \((\text{BFOC}k)\), \((\text{BEn}v_g)\), and \((B.8)\) together imply \(\hat{\eta}_g(v, s) - \hat{\lambda}(v, s) - \hat{\mu}_g(v, s) \leq 0\). Hence, (B.7) must hold. \(\square\)

Note that we can simplify \((\text{BFOC}k)\) by using (B.7) as,

\[R'(k(v, s)) = \frac{1}{[p_s - \hat{\lambda}(v, s)]} \tag{BFOCk} \]

Let us say that a function \(f : (0, 1) \to \mathbb{R}\) is \textit{locally increasing} if for any \(x \in (0, 1)\), there exists \(\varepsilon > 0\) such that \(f\) is increasing on \([x, x + \varepsilon]\). (Here, increasing is taken to mean non-decreasing.) The following is a useful lemma that shows that continuous locally increasing functions on an interval are increasing.

**Lemma B.1.7.** Let \(f : (0, 1) \to \mathbb{R}\) be continuous and locally increasing. Then, \(f\) is increasing, ie, for all \(x, y \in (0, 1)\), \(x \leq y\) implies \(f(x) \leq f(y)\).

**Proof.** Let \(x, y \in (0, 1)\) such that \(x < y\). Then, it suffices to show that \(f(x) \leq f(y)\). Because \(f\) is continuous, \(f\) achieves a maximum, \(M\), on \([x, y]\). Moreover, because \(f\) is continuous, the set \(Z := \{z \in [x, y] : f(z) = M\}\) is closed. Let \(z^*\) be the supremum of \(Z\), which is in \(Z\) because \(Z\) is closed. If \(z^* = y\), we are done, because \(f(x) \leq M\). If, however, \(z^* < y\), then by the hypothesis that \(f\) is locally increasing, there exists \(\varepsilon > 0\) (which depends on \(z^*\)) such that \(f\) is increasing on \([z^*, z^* + \varepsilon] \subset [x, y]\). This

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implies \( f(z^* + \varepsilon) = M \), because \( M \) is the maximum value of \( f \) on \([x, y]\), which contradicts the definition of \( z^* \), namely that \( z^* \) is the supremum of the set \( Z \) and that \( z^* < y \). □

**Lemma B.1.8.** Let \( F_6 = \{ P(v, s) \in F_1 \cap F_5 : P_g(v, s) \text{ is increasing in } v_b \} \). If \( P(v, s) \in F_6 \), then \( TP(v, s) \in F_6 \).

**Proof.** Suppose \( P(v, s) \in F_6 \). Then \( m_b(v, s) = 0 \) is optimal by Lemma B.1.5. From (IC*) and (PK_b) at any \((v, s)\), we obtain

\[
\begin{align*}
    w_{bb}(v, s) &\geq p_b R(k(v, s)) + p_g v_b - p_b v_g \\
    w_{bg}(v, s) &\leq \frac{1 - p_b}{\delta \Delta} \left[ v_g - \frac{1 - p_g}{1 - p_b} v_b - R(k(v, s)) \right]
\end{align*}
\] (B.9) (B.10)

If (IC*) holds as equality, then (B.9) and (B.10) both hold as equality. Note that \( w_g(v, s) \) do not vary with \( v_b \), because \( w_{gb}, w_{gg} \) only appear in (PK_g) of (P1) if we consider the incentive compatibility constraint as (IC*). Pick any sufficiently small \( \varepsilon > 0 \) and let \( v' = v + (\varepsilon, 0) \). We will show in two cases that \((TP)_g(v', s) \geq (TP)_g(v, s)\).

(i) (IC*) holds as strict inequality at \((v, s)\).

Complementary slackness implies \( \lambda(v, s) = 0 \). Because \( \varepsilon \) is sufficiently small, continuity of policy imply that \( \lambda(v, s) = \lambda(v', s) = 0 \). From (BFOC\(w_{gg}\)), we must have \( \tilde{\eta}_g(v, s) = \tilde{\eta}_g(v', s) \), since the left hand side of (BFOC\(w_{gg}\)) does not vary with \( v_b \). Then (BEn\(v_g\)) implies \((TP)_g(v', s) = (TP)_g(v, s)\).

(ii) (IC*) holds as equality at \((v, s)\).

We first show \( \hat{\lambda}(v', s) \geq \hat{\lambda}(v, s) \). If \( \hat{\lambda}(v, s) = 0 \), then we simply get \( \hat{\lambda}(v', s) \geq \hat{\lambda}(v, s) \). Suppose that \( \hat{\lambda}(v, s) > 0 \) and \( \hat{\lambda}(v', s) < \hat{\lambda}(v, s) \). By (BFOC\(k\)), we
know \( k(\mathbf{v}', s) > k(\mathbf{v}, s) \). From (B.9) at \((\mathbf{v}, s)\) and \((\mathbf{v}', s)\), we obtain
\[
\begin{align*}
w_{bb}(\mathbf{v}', s) &\geq \frac{p_b R(k(\mathbf{v}', s)) + p_g v'_b - p_b v_g}{\delta \Delta} \\
&> \frac{p_b R(k(\mathbf{v}, s)) + p_g v_b - p_b v_g}{\delta \Delta} = w_{bb}(\mathbf{v}, s)
\end{align*}
\]
The equality is because \((\text{IC}^*)\) holds as equality at \((\mathbf{v}, s)\). From (B.10) at \((\mathbf{v}, s)\) and \((\mathbf{v}', s)\), we obtain
\[
\begin{align*}
w_{bg}(\mathbf{v}', s) &\leq \frac{1 - p_b}{\delta \Delta} \left[ v_g - \frac{1 - p_g}{1 - p_b} v'_b - R(k(\mathbf{v}', s)) \right] \\
&< \frac{1 - p_b}{\delta \Delta} \left[ v_g - \frac{1 - p_g}{1 - p_b} v_b - R(k(\mathbf{v}, s)) \right] = w_{bg}(\mathbf{v}, s)
\end{align*}
\]
The equality is also because \((\text{IC}^*)\) holds as equality at \((\mathbf{v}, s)\). Since \(P_g(\mathbf{v}, s)\) is increasing \(v_b\) and decreasing in \(v_g\) by \(P \in \mathcal{F}_6\), we have \(P_g(w_b(\mathbf{v}', s), b) \geq P_g(w_b(\mathbf{v}, s), b)\). So \((1 - p_a)P_g(w_b(\mathbf{v}', s), b) - p_g \hat{\lambda}(\mathbf{v}', s) > (1 - p_a)P_g(w_b(\mathbf{v}, s), b) - p_g \hat{\lambda}(\mathbf{v}, s)\). And by (BFOC\(w_{bg}\)), we must have \(\hat{\eta}_b(\mathbf{v}', s) > \hat{\eta}_b(\mathbf{v}, s)\), which implies \((TP)_b(\mathbf{v}', s) > (TP)_b(\mathbf{v}, s)\) by (BEnv\(_b\)). However, from Lemma B.1.1, \(TP(\mathbf{v}, s) \in \mathcal{F}_1\) and therefore concave in \(\mathbf{v}\). So \((TP)_b(\mathbf{v}', s) \leq (TP)_b(\mathbf{v}, s)\), a contradiction. Hence \(\hat{\lambda}(\mathbf{v}', s) \geq \hat{\lambda}(\mathbf{v}, s)\).

From (BFOC\(w_{gg}\)), we must have \(\hat{\eta}_g(\mathbf{v}', s) \geq \hat{\eta}_g(\mathbf{v}, s)\), which further implies \((TP)_g(\mathbf{v}', s) \geq (TP)_g(\mathbf{v}, s)\) by (BEnv\(_g\)).

Given the results in (i) and (ii) above, \((TP)_g(\mathbf{v}, s)\) is locally increasing in \(v_b\). By Lemma B.1.7, \((TP)_g(\mathbf{v}, s)\) is increasing in \(v_b\), and therefore \(TP(\mathbf{v}, s) \in \mathcal{F}_6\).

\[\square\]

**Theorem B.1.9.** The unique fixed point of \(T\), which we call \(Q\), lies in \(\bigcap_{i=1}^{6} \mathcal{F}_i\). Therefore, \(Q\) satisfies:

1. \(Q(\mathbf{v}, s)\) is concave in \(\mathbf{v}\);
2. \( Q(v, s) = \bar{Q}(s) \) for any \( v \in E_s \);

3. \( Q(v, s) \) is decreasing in \( v_b \) at \( ((v, v), s) \) for any \( v > 0 \);

4. \( Q(v, g) \geq Q(v, b) \);

5. \( Q(v + (\varepsilon, \varepsilon), s) \geq Q(v, s) \);

6. \( Q_g(v, s) \) is increasing in \( v_b \).

Proof. It is easy to see that \( T \) is monotone (\( P_1 \leq P_2 \) implies \( T P_1 \leq T P_2 \)) and satisfies discounting (\( T(P + a) = T P + \delta a \)), which implies \( T \) is a contraction mapping on \( F \). From Lemma B.1.1 to Lemma B.1.8 we established that if \( P \in \bigcap_{i=1}^{6} F_i \) then \( TP \in \bigcap_{i=1}^{6} F_i \). This implies the unique fixed point of \( T \) also lies in \( \bigcap_{i=1}^{6} F_i \).

B.2 Proofs from Section 3.4

B.2.1 Recursive Domain

We first present the proof of Proposition 3.4.1. It is easy to see that the set of contingent utilities \( v \in \mathbb{R}^2_+ \) that can be implemented by \( (k, m, w_i) \) with \( w_i \in \mathbb{R}^2_+ \) is a closed and convex cone. Therefore, in our search for a suitable domain, it suffices to restrict attention to closed and convex cones.

Let \( \mathcal{K} \) denote the space of closed and convex cones that are subsets of \( \mathbb{R}^2_+ \). We define the operator \( \Phi : \mathcal{K} \to \mathcal{K} \) as follows: for \( C \in \mathcal{K} \), let

\[
\Phi(C) := \{ v \in \mathbb{R}^2_+ : \exists (k, m, w_i) \text{ that implements } v \text{ and has } w_i \in C, i = b, g \}
\]

In other words, \( \Phi(C) \) consists of all implementable contingent utilities \( v \) wherein the continuation contingent utilities \( w_i \) lie in the set \( C \). Clearly, any recursive program must only consider contingent utilities \( v \) that lie in a set \( C \) such that \( C \) is a fixed point of \( \Phi \), so that all present contingent utilities as well as future continuation
contingent utilities lie in the same set. Essentially, Proposition 3.4.1 delineates such a set.

Proof of Proposition 3.4.1. It is easy to see that $\Phi$ is well defined, that is, $\Phi$ maps closed and convex cones to closed and convex cones. Let $\alpha \in [0, 1]$, and define $C_{\alpha} := \{ (v_b, v_g) \in \mathbb{R}^2 : v_g \geq \alpha v_b \}$. Let $v \in \mathbb{R}^2$ be such that $(k, m, w_i)$ implements $v$ with the restriction that $w_i \in C_{\alpha}$. The set of all such $v$ is precisely the set $\Phi(C_{\alpha})$.

By (PK$_b$), we obtain

$$v_b = -m_b + \delta[(1 - p_b)w_{bb} + p_b w_{bg}]$$

$$\geq -(1 - p_b + p_b \alpha)m_b + \delta[(1 - p_b)w_{bb} + p_b \alpha w_{bb}]$$

$$= (1 - p_b + p_b \alpha)(\delta w_{bb} - m_b)$$

where the inequality follows from the assumption that $w_{bg} \geq \alpha w_{bb}$, and (LL) which requires that $m_b \leq 0$. This implies

$$m_b - \delta w_{bb} \geq -v_b / (1 - p_b + p_b \alpha)$$

Notice that (PK$_b$) can be written as $\delta p_b (w_{bg} - w_{bb}) = v_b + (m_b - \delta w_{bb})$, which implies

$$\delta (w_{bg} - w_{bb}) \geq \frac{v_b}{p_b} \left[ 1 - \frac{1 - \alpha}{1 - p_b + p_b \alpha} \right]$$

$$= -v_b \left[ \frac{1 - \alpha}{1 - p_b + p_b \alpha} \right] \quad \text{(B.11)}$$

Plugging this into (IC$^*$), we obtain

$$v_g \geq v_b + R(k) + \delta \Delta (w_{bg} - w_{bb})$$

$$\geq v_b \left[ 1 - \frac{(1 - \alpha) \Delta}{1 - p_b + p_b \alpha} \right]$$

$$= v_b \left[ \frac{1 - p_g + p_g \alpha}{1 - p_b + p_b \alpha} \right]$$

$$=: \alpha' v_b$$

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where the first inequality is merely (IC*) and the second inequality follows from (B.11) and the fact that $R(k) \geq 0$.

Thus, if continuation contingent utilities $w_i$ are constrained to the set $C_\alpha$, then the set of implementable $v$ must lie in the set $C_{\alpha'}$, where $\alpha' = (1-p_g(1-\alpha))/(1-p_b(1-\alpha))$.

We claim that if $\alpha \in [0, 1)$, then $\alpha' > \alpha$. To see this, notice that

$$\frac{1 - p_g(1 - \alpha)}{1 - p_b(1 - \alpha)} > \alpha$$

iff

$$1 - p_g(1 - \alpha) > \alpha - \alpha p_b(1 - \alpha)$$

iff

$$(1 - \alpha)(1 - p_g) > -\alpha p_b(1 - \alpha)$$

iff

$$1 - p_g > -\alpha p_b$$

which always holds because $p_b, p_g \in (0, 1)$ and $\alpha < 1$. Therefore, for any $\alpha \in [0, 1)$, $\Phi(C_\alpha) = C_{\alpha'} \subseteq C_\alpha$. Notice that $\Phi^n(C_0) = \bigcap_{k \leq n} \Phi^k(C_0) = C_{\alpha_n}$, where $\Phi^n(C_0) := \Phi(\Phi^{n-1}(C_0))$, $\alpha_n = \frac{1-p_b(1-\alpha_{n-1})}{1-p_b(1-\alpha_{n-1})}$, and $\alpha_0 = 0$. This means iterating the operator $\Phi$ from $C_0 = \mathbb{R}_+^2$ induces a strictly increasing sequence $\{\alpha_n\}_{n=0}^\infty \in [0, 1)$. It is easy to see that $\lim_{n \to \infty} \alpha_n = 1$, and therefore, $\lim_{n \to \infty} \Phi^n(C_0) = C_1 = V$.

To see that $V := \{(v_b, v_g) \in \mathbb{R}_+^2 : v_g \geq v_b\}$ is a fixed point of $\Phi$, we apply the operator $\Phi$ to $V$. Take any continuation utilities $v \in V$, and consider the policy $(k, m_i, w_i)$ that satisfies $R(k) = m_g = v_g - v_b$, $m_b = 0$, $w_{ig} = w_{ib} = v_i/\delta$. Since $(k, m_i, w_i)$ implements $v$ and $w_i \in V$, we must have $v \in \Phi(V)$, which means $V = \Phi(V)$.

\[\square\]

### B.2.2 Value Function and Efficient Sets

In this part, we show various properties regarding value functions and unconstrained sets.
Proof of Theorem 1. (a) Because the only feasible policy at state \( (0, s) \) is

\[
(k, m_b, m_g, w_b, w_g) = (0, 0, 0, 0, 0)
\]

we must have \( Q(0, s) = 0 \). We first show that \( Q_g(0, s) = \infty \), and \( D_{(1,1)} Q(0, s) = \infty \). Then we use these two facts to show \( Q_b((0, v), s) = \infty \). Note that

\begin{align*}
Q_g(0, s) &= \lim_{\epsilon \to 0} \frac{Q((0, \epsilon), s)}{\epsilon} \\
&\geq \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ -R^{-1}(\epsilon) + p_s \epsilon + \delta p_s Q \left( \frac{0, \epsilon}{\delta p_g}, s \right) \right] \\
&= p_s + \frac{p_s}{p_g} Q_g(0, g) \\
&= \lim_{\epsilon \to 0} \frac{Q((0, \epsilon), s)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{D_{(1,1)} Q((\epsilon, \epsilon), s)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\delta}{\epsilon} \left[ p_s Q \left( \frac{0, \epsilon}{\delta p_g}, s \right) + (1 - p_s) Q \left( \frac{\epsilon}{\delta}, \frac{\epsilon}{\delta}, b \right) \right] \\
&= \frac{p_s}{p_g} Q_g(0, g) + (1 - p_s) D_{(1,1)} Q(0, b) \\
&= \frac{p_s}{p_g} Q_g(0, g) + (1 - p_s) \frac{D_{(1,1)} Q(0, b)}{\epsilon}
\end{align*}

The inequality is because

\[
(k, m_b, m_g, w_b, w_g) = \left( R^{-1}(\epsilon), 0, 0, \left( \frac{\epsilon}{\delta p_g} \right), \left( \frac{\epsilon}{\delta} \right) \right) \in \Gamma((\epsilon, \epsilon), s)
\]

Because \( p_g > 0 \), \( Q_g(0, g) \) has to be \( \infty \) for (B.12) to hold when \( s = g \). Then \( Q_g(0, b) = \infty \) is implied by (B.12) when \( s = b \).

Next,

\[
D_{(1,1)} Q(0, s) = \lim_{\epsilon \to 0} \frac{D_{(1,1)} Q((\epsilon, \epsilon), s)}{\epsilon} \\
&\geq \lim_{\epsilon \to 0} \frac{\delta}{\epsilon} \left[ p_s Q \left( \frac{0, \epsilon}{\delta p_g}, s \right) + (1 - p_s) Q \left( \frac{\epsilon}{\delta}, \frac{\epsilon}{\delta}, b \right) \right] \\
&= \frac{p_s}{p_g} Q_g(0, g) + (1 - p_s) D_{(1,1)} Q(0, b)
\]

The inequality is because

\[
(k, m_b, m_g, w_b, w_g) = \left( 0, 0, 0, \left( \frac{\epsilon}{\delta}, \frac{\epsilon}{\delta} \right), \left( 0, \frac{\epsilon}{\delta p_g} \right) \right) \in \Gamma((\epsilon, \epsilon), b)
\]

Because \( Q_g(0, g) = \infty \), (B.13) at \( s = b \) implies \( D_{(1,1)} Q(0, b) = \infty \). Then (B.13) at \( s = g \) implies \( D_{(1,1)} Q(0, g) = \infty \).
Next, let \((k, 0, m_g, 0, w_g)\) be the optimal policy at \(((0, v), s)\) for any \(v > 0\). The optimal \(m_b = w_{bi} = 0\) is implied by \((PK_b)\) at \(v_b = 0\). Then we have

\[
\left(k, 0, m_g, \left(\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right), w_g\right) \in \Gamma((\varepsilon, v), s)
\]

which implies

\[
Q_b((0, v), s) = \lim_{\varepsilon \to 0} \frac{Q((\varepsilon, v), s) - Q((0, v), s)}{\varepsilon}
\]

\[
\geq \lim_{\varepsilon \to 0} \frac{(1 - p_s)\delta}{\varepsilon} \left[ Q\left(\left(\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right), b\right) - Q(0, b) \right]
\]

\[
= (1 - p_s) D_{(1,1)} Q(0, b) = \infty
\]

Next, let’s define \(Q_b((v, v), s)\) as the left derivative since only \(v_g \geq v_b\) is feasible. Using the same argument as in Lemma B.1.3, for arbitrary small \(\varepsilon > 0\) and \(v' = v - \frac{1 - p_b}{1 - p_g} R(\varepsilon)\), we get

\[
Q((v', v), b) - Q((v, v), b)
\]

\[
\geq -\varepsilon + p_b R(\varepsilon) + (1 - p_b)\delta \left[ Q\left(\left(\frac{w_{bg}, w}, \frac{w_{bg}}{\delta}\right), b\right) - Q\left(\left(w, w\right), b\right) \right] \geq 0 \quad (B.14)
\]

where \(w = \frac{v + m_b}{\delta}\), and \(w_{bg} = w - \frac{R(\varepsilon)}{\delta(1 - p_g)}\). Divide both sides of (B.14) by \(\frac{1 - p_b}{1 - p_g} R(\varepsilon)\), and take the limit as \(\varepsilon\) converges to zero, we obtain:

\[
-Q_b((v, v), b) \geq \frac{p_b(1 - p_g)}{1 - p_b} - Q_b((w, w), b) \quad (B.15)
\]

Because \(Q_b((w, w), s) \leq 0\) by part (c) of Theorem B.1.9, we know \(-Q_b((v, v), b) \geq \frac{p_b(1 - p_g)}{1 - p_b}\). The same argument shows that \(-Q_b((w, w), b) \geq \frac{p_b(1 - p_g)}{1 - p_b}\). So it must be that \(-Q_b((v, v), b) \geq \frac{2p_b(1 - p_g)}{1 - p_b}\). Repeating this procedure, we get the result \(-Q_b((v, v), b) \geq \frac{np_b(1 - p_g)}{1 - p_b}\) for any \(n \in \mathbb{N}\). Hence, we must have \(Q_b((v, v), b) = -\infty\). Now let \((k, m_b, m_g, (w, w), w_g)\) be the optimal policy at
\((v, v), s, v > 0\), where \(w = \frac{v + m_b}{b}\). And let \(w_{b_b} = w - \varepsilon, m_{b_b}' = m_b - (1 - p_b)\delta\varepsilon,\) and \(m_{g'} = m_g - \varepsilon\). Then we have

\[(k, m_{b_b}', m_{g}', (w - \varepsilon, w), w_g) \in \Gamma((v, v + \varepsilon), s)\]

which implies

\[Q_g((v, v), s) \geq \lim_{\varepsilon \to 0} \frac{\delta(1 - p_s)}{\varepsilon} [Q ((w - \varepsilon, w), b) - Q((w, w), b)]\]

\[= -\delta(1 - p_s)Q_b((w, w), b) = \infty\]

(b) Because it is always feasible to advance zero capital at all time, make repayments \(m_g = -v_g, m_b = -v_b\) in the first period and no repayment in all subsequent periods, \(Q(v, s) \geq 0\). Moreover, the surplus \(Q(v, s)\) is uniformly bounded above by the efficient surplus \(\bar{Q}(g)\).

(c) Take any \((v, s) \in V \times S\). Let \((k, m_b, m_g, w_b, w_g)\) be the optimal policy at \((v, s)\), \(v' = v + (0, \varepsilon)\) for any \(\varepsilon > 0\), and \(m_{g'} = m_g - \varepsilon\). Then the policy

\[(k, m_b, m_{g'}, w_b, w_g) \in \Gamma(v', s)\]

because the specified change in states and policy increase both sides of \((PK_g)\) by \(\varepsilon\) and only increase the left hand side of \((IC^*)\). Moreover, because the repayment \(m_g\) does not appear in the objective of \((VF)\), we must have \(Q(v', s) \geq Q(v, s)\), implying \(Q_g(v, s) \geq 0\).

(d) Shown in part (d) of Theorem B.1.9.

(e) Recall that the proof of Lemma B.4 (needs reference) shows that any function in \(F_4\) satisfies this property. Because \(Q \in F_4\), so the result holds.

(f) Shown in part (f) of Theorem B.1.9.
Lemma B.2.1. The efficient supluses of the firm are:

\[
\bar{Q}(b) = \frac{1-p_g \delta}{(1-\delta)(1-\Delta \delta)} [p_b R(\bar{k}_b) - \bar{k}_b] + \frac{p_b \delta}{(1-\delta)(1-\Delta \delta)} [p_g R(\bar{k}_g) - \bar{k}_g]
\]

\[
\bar{Q}(g) = \frac{(1-p_g) \delta}{(1-\delta)(1-\Delta \delta)} [p_b R(\bar{k}_b) - \bar{k}_b] + \frac{1-\delta + p_b \delta}{(1-\delta)(1-\Delta \delta)} [p_g R(\bar{k}_g) - \bar{k}_g]
\]

Proof. We simply solve equations (3.1) and (3.2) that jointly determine \( \bar{Q}(s) \).

To properly define the threshold contingent utilities, we use the following procedure. First, we fix the level of \( v_b \) and find the smallest value of \( v_g \) at which \( Q_g \) becomes zero. This defines a cutoff curve as functions of \( v_b \) along which \( Q_g \) is zero. Second, we find the smallest value of \( v_b \) at which \( Q_b \) becomes zero along the defined cutoff curve.

Lemma B.2.2. For each \( v_b \geq 0 \), there exists \( f_s(v_b) > v_b \) such that \( Q_g(v, s) = 0 \) if \( v_g \geq f_s(v_b) \), and \( Q_g(v, s) > 0 \) if \( v_b \leq v_g < f_s(v_b) \). Moreover, \( f_s(v_b) \) is increasing in \( v_b \).

Proof. First, we show that for any \( v_b \geq 0 \), there exists some \( v_g > v_b \) such that \( Q_g(v, s) = 0 \). Note that part (b) of Theorem B.1.9 shows that there exists some \( \hat{v}_s \) with \( \hat{v}_{sg} > \hat{v}_{sb} \) such that \( Q(\hat{v}_s, s) = \bar{Q}(s) \) in a small neighborhood around \( \hat{v}_s \), implying \( Q_g(\hat{v}_s, s) = 0 \). For any \( v_b \leq \hat{v}_{sb} \), the supermodularity of \( Q \) implies that \( 0 \leq Q_g((v_b, \hat{v}_{sg}), s) \leq Q_g(\hat{v}_s, s) = 0 \). For any \( v_b > \hat{v}_{sb} \), we consider \( v_g = v_b + \hat{v}_{sg} - \hat{v}_{sb} \).

Part (b) of Theorem B.1.9 shows that \( Q(\hat{v}_s, s) = \bar{Q}(s) \) in a small neighborhood around \( \hat{v}_s \), implying \( Q_g(\hat{v}_s, s) = 0 \).

Next, we fix any \( v_b \geq 0 \) and define \( f_s(v_b) := \min\{ v_g : Q_g(v, s) = 0\} \). Because \( Q_g((v_b, v_b), s) = \infty \), we must have \( f_s(v_b) > v_b \). By this definition, \( Q_g(v, s) > 0 \) if \( v_b \leq v_g < f_s(v_b) \). Moreover, the concavity of \( Q \) implies that \( 0 \leq Q_g(v, s) \leq Q_g[(v_b, f_s(v_b)), s] = 0 \) if \( v_g \geq f_s(v_b) \).
Now we show that $f_s(.)$ is increasing. Take any $v_b, v'_b$ such that $0 \leq v_b < v'_b$. Supermodularity of $Q$ implies $0 \leq Q_g[(v_b, f(v'_b)), s] \leq Q_g[(v'_b, f(v'_b)), s] = 0$. So we have $Q_g[(v_b, f_s(v'_b)), s] = 0$, which further implies that $f_s(v_b) \leq f_s(v'_b)$ by the definition of $f_s(.)$.

Lemma B.2.3. For any $v_b \geq 0$, we have $Q_b[(v_b, f_s(v_b)), s] \geq 0$. Moreover, there exists some $\hat{v}_s$ in the interior of $V$ such that $Q_b[(\hat{v}_{sb}, f_s(\hat{v}_{sb})), s] = 0$.

Proof. Note that the definition of $f_s(.)$ means $Q_g[(v_b, f_s(v_b)), s] = 0$. So we have $Q_b[(v_b, f_s(v_b)), s] = Q_b[(v_b, f_s(v_b)), s] + Q_g[(v_b, f_s(v_b)), s] \geq 0$ for any $v_b \geq 0$. The inequality is implied by part (e) of Theorem B.1.9. Moreover, part (b) of Theorem B.1.9 implies that there exists some $\hat{v}_s$ in the interior of $V$ such that $Q_b(\hat{v}_s, s) = Q_g(\hat{v}_s, s) = 0$. By the definition of $f_s(.)$, we know $f_s(\hat{v}_{sb}) \leq \hat{v}_{sg}$. Supermodularity of $Q$ then implies $Q_b[(\hat{v}_{sb}, f_s(\hat{v}_{sb})), s] \leq Q_b(\hat{v}_s, s) = 0$. So it has to be that $Q_b[(\hat{v}_{sb}, f_s(\hat{v}_{sb})), s] = 0$.

We are now ready to define the threshold contingent utilities and the unconstrained sets. Let $\bar{v}_{sb} = \min\{v_b : Q_b((v_b, f_s(v_b)), s) = 0\}$ be the threshold of continuation utility contingent on bad shock. From Theorem 3.4.2, $Q_b((0, f(0)), s) = \infty$. So by definition $\bar{v}_{sb} > 0$. Let $\bar{v}_{sg} = f_s(\bar{v}_{sb})$ be the threshold continuation utility contingent on good shock. The unconstrained sets of contingent utilities are defined as $E_s := \{v \in V : v_b \geq \bar{v}_{sb}, v_g \geq f_s(v_b)\}$.

Proof of Proposition 3.4.3. (a) Take any $v \in E_s$. The definition of $E_s$ simply requires $v_b \geq \bar{v}_{sb}$. Because $f_s(.)$ is increasing, we have $v_g \geq f_s(v_b) \geq f_s(\bar{v}_{sb}) = \bar{v}_{sg}$.

(b) We first show that $Q_b((\bar{v}_{sb}, v_g), s) = 0$ if $v_g \geq \bar{v}_{sg}$. For each $v_b \leq \bar{v}_{sb}$, because $f_s(.)$ is increasing, we know $f_s(v_b) \leq f_s(\bar{v}_{sb}) = \bar{v}_{sg}$. This means $Q_g(v, s) = 0$ when $v_b \leq \bar{v}_{sb}$ and $v_g \geq \bar{v}_{sg}$. Take any $\hat{v}_g > \bar{v}_{sg}$ and any small $\varepsilon > 0$. We have
\[ Q((\bar{v}_{ab}, \hat{v}_g), s) = Q((\bar{v}_{ab}, \bar{v}_{sg}), s) \] and \[ Q((\bar{v}_{ab} - \varepsilon, \hat{v}_g), s) = Q((\bar{v}_{ab} - \varepsilon, \bar{v}_{sg}), s), \]

which implies

\[ Q_b((\bar{v}_{ab}, \hat{v}_g), s) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [Q((\bar{v}_{ab}, \hat{v}_g), s) - Q((\bar{v}_{ab} - \varepsilon, \hat{v}_g), s)] \]

\[ = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [Q((\bar{v}_{ab}, \bar{v}_{sg}), s) - Q((\bar{v}_{ab} - \varepsilon, \bar{v}_{sg}), s)] \]

\[ = Q_b(\bar{v}_s, s) = 0 \]

Now take any \( v \in E_s \). Concavity of \( Q \) implies \( Q_b(v, s) \leq Q_b((\bar{v}_{ab}, v_g), s) = 0 \). The definition of \( f_s(\cdot) \) and \( E_s \) simply implies that \( Q_g(v, s) = 0 \). Moreover, because \( Q_b(v, s) + Q_g(v, s) \geq 0 \) by part (e) of Theorem B.1.9, \( Q_g(v, s) = 0 \) further implies \( Q_b(v, s) \geq 0 \). Therefore, it has to be \( Q_b(v, s) = 0 \).

(c) Take any \( v \in E_s \). Since \( Q \) is concave, \( Q_b(v, s) = Q_g(v, s) = 0 \) implies \( Q(v, s) \) achieves its maximum \( Q(s) \). Suppose \( k(v, s) < \bar{k}_s \). By the definition of \( Q(v, s) \) and \( \bar{Q}(s) \), we have:

\[ Q(v, s) = -k(v, s) + p_s [R(k(v, s)) + \delta Q(w_g(v, s), g)] + (1 - p_s) \delta Q(w_b(v, s), b) \]

\[ < -\bar{k}_s + p_s [R(\bar{k}_s) + \delta \bar{Q}(s)] + (1 - p_s) \delta \bar{Q}(s) = \bar{Q}(s) \]

which is a contradiction. So \( k(v, s) = \bar{k}_s \).

(d) Take any \( v \in V \setminus E_s \). There are two possible cases. First, if \( v_g < f_s(v_b) \), then by the definition of \( f_s \), \( Q_g(v, s) > 0 \). Second, if \( v_b < \bar{v}_{ab} \) and \( v_g \geq f_s(v_b) \), then the supermodularity of \( Q \) implies \( Q_b(v, s) \geq Q_b([v_b, f_s(v_b)], s) > 0 \). The strict inequality is from the definition of \( \bar{v}_{ab} \). So we must have either \( Q_g(v, s) > 0 \) or \( Q_b(v, s) > 0 \), which implies \( Q(v, s) < \bar{Q}(s) \).

(e) First, take any \( v \in E_b \). We show that \( v_g - v_b \geq R(\bar{k}_b) + \delta \Delta \frac{R(\bar{k}_b)}{1 - \delta \Delta} \) is necessary to obtain efficient firm surplus at \((v, b)\). Let \((k, m_i, w_i)\) be the optimal policy at \((v, b)\). We know \( k = \bar{k}_b \) from part (c). The constraint \((IC^*)\) at \((v, b)\)
implies $v_g - v_b \geq R(\bar{k}_b) + \delta \Delta (w_{bg} - w_{bb}) \geq R(\bar{k}_b)$. Moreover, we must also have $w_b \in E_b$. Otherwise, $Q(v, b)$ will be smaller than the first best surplus $\bar{Q}(b)$, a contradiction. The constraint (IC*) at $(w_b, b)$ implies that $w_{bg} - w_{bb} \geq R(\bar{k}_b)$. So we have $v_g - v_b \geq R(\bar{k}_b) + \delta \Delta R(\bar{k}_b)$. Repeating this procedure we obtain $v_g - v_b \geq (1 + \delta \Delta + \delta^2 \Delta^2 + \ldots) R(\bar{k}_b) = R(\bar{k}_b) + \delta \Delta \frac{R(\bar{k}_b)}{1-\delta \Delta}$.

Second, take any $v \in E_g$ and let $(k, m_i, w_i)$ be the optimal policy at $(v, g)$. Similar argument shows that $k = \bar{k}_g$, and $w_b \in E_b$. So (IC*) at $(v, g)$ implies that $v_g - v_b \geq R(\bar{k}_g) + \delta \Delta (w_{bg} - w_{bb}) \geq R(\bar{k}_g) + \delta \Delta \frac{R(\bar{k}_b)}{1-\delta \Delta}$, because $w_{bg} - w_{bb} \geq \frac{R(\bar{k}_b)}{1-\delta \Delta}$ from the first step.

Third, take any $v \in E_s$ for $s = b, g$. Let $(k, m_i, w_i)$ be the optimal policy at $(v, s)$. We show that $w_{bg} - w_{bb} \geq \frac{\delta v_{bg} - v_b}{\delta (1-p_b)}$. Suppose not. Then we can derive

\[ w_{bg} - w_{bb} < \frac{\delta v_{bg} - v_b}{\delta (1-p_b)} = \frac{\delta v_{bg} + m_b - \delta[p_b w_{bg} + (1-p_b) w_{bb}]}{\delta (1-p_b)} \] (B.16)

The equality is from (PK). Rearranging (B.16) we get $w_{bg} < \bar{v}_{bg} + \frac{m_b}{\delta} \leq \bar{v}_{bg}$.

This means $w_b \in V \setminus E_b$ by part (a). Hence, $Q(w, b) < \bar{Q}(b)$, implying $Q(v, b) < \bar{Q}(s)$, a contradiction with $v \in E_s$. Since $k = \bar{k}_s$, (IC*) at $(v, s)$ implies that $v_g - v_b \geq R(\bar{k}_s) + \delta \Delta \frac{R(\bar{k}_b)}{\delta(1-p_b) (1 - \delta \Delta)}$.

Combing the above results, we conclude that it is necessary to satisfy $v_g - v_b \geq R(\bar{k}_s) + \delta \Delta \max \left[ \frac{\delta v_{bg} - v_b}{\delta (1-p_b), \frac{R(\bar{k}_b)}{1-\delta \Delta}} \right]$ for any $v \in E_s$.

\[ \square \]
B.3 Auxiliary Problem

To proceed the proof in Section 5 and beyond, it is convenient to define an auxiliary problem:

$$\Psi(y, s) = \max_{x_g \geq x_b \geq 0} \delta Q(x, s)$$

subject to

$$y \geq \delta(p_s x_g + (1 - p_s)x_b)$$

where $$y \geq 0$$ and $$s = b, g$$. To ease notation, let $$x^*(y, s)$$ be a solution of problem (P3). Also let $$\bar{y}_s = \delta[p_s \bar{v}_{sg} + (1 - p_s)\bar{v}_{sb}]$$.

**Lemma B.3.1.** Function $$\Psi(y, s)$$ defined in (P3) has the following properties:

(a) $$\Psi(y, s)$$ is increasing and concave in $$y$$.

(b) $$\Psi_y(y, s) = D_{(1,1)} Q(x^*(y, s))$$.

(c) $$\Psi(y, s) = \delta \bar{Q}(s)$$ and $$x^*(y, s) \in E_s$$ when $$y \geq \bar{y}_s$$.

(d) $$\Psi(y, s)$$ is strictly increasing in $$y$$, and the constraint of (P3) binds when $$y < \bar{y}_s$$.

**Proof.** (a) Since raising $$y$$ always relaxes the constraint in problem (P3), we have $$\Psi_y(y, s) \geq 0$$. Moreover, because (P3) has concave objective (shown in part (a) of Theorem B.1.9) and convex constraint, $$\Psi(y, s)$$ is concave in $$y$$.

(b) Theorem 3.4.2 implies that $$x^*$$ lie in the interior of $$V$$. Let $$\gamma_s$$ be the Lagrange multiplier of (P3). The first order conditions and the envelope condition for problem (P3) are:

$$Q_b(x^*(y, s), s) = (1 - p_s)\gamma_s, \quad Q_g(x^*(y, s), s) = p_s \gamma_s \quad (B.17)$$

$$\Psi_y(y, s) = \gamma_s \quad (B.18)$$

It is easy to see from (B.17) and (B.18) that $$\Psi_y(y, s) = D_{(1,1)} Q(x^*(y, s), s)$$.

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(c) When \( y \geq \bar{y}_s \), \( x = \bar{v}_s \) is feasible in (P3). And because \( Q(., s) \) reaches its upper bound \( \bar{Q}(s) \) at \( \bar{v}_s \), we must have \( \Psi(y, s) = \delta \bar{Q}(s) \). Proposition 3.4.3 implies that \( x^*(y, s) \in E_s \).

(d) Suppose there exists some \( \bar{y}_s < \bar{y}_s \) such that \( \Psi_g(\bar{y}_s, s) = 0 \). Concavity of \( \Psi \) implies that \( \Psi(\bar{y}_s, s) = \Psi(\bar{y}_s, s) = \delta \bar{Q}(s) \). From Proposition 3.4.3, we know \( x^*(\bar{y}_s, s) \geq \bar{v}_s \). However, this implies \( \bar{y}_s \geq \bar{y}_s \), a contradiction. Therefore, \( \Psi_g(y, s) \) is strictly increasing in \( y \) when \( y < \bar{y}_s \). The envelope condition is: \( \Psi_g(y, s) = \gamma_s \). So when \( y < \bar{y}_s \), we know \( \gamma_s > 0 \) and therefore the constraint of (P3) binds by complementary slackness. \( \square \)

**Lemma B.3.2.** The optimal policy \( w_h(v, s), w_g(v, s) \) satisfies:

(a) \( w_g(v, s) \) is only a function of \( v_g \), and is a solution of (P3) at \((v_g, g)\).

(b) \( w_g(v, s) \in E_g \) when \( v_g \geq \bar{y}_g \).

(c) If \( \lambda(v, s) = 0 \), then \( w_h(v, s) \) is the solution of (P3) at \((v_b, b)\), hence only a function of \( v_b \).

In this section we show that \( Q(v, s) \) is strictly concave on the set \( H := \{(v, s) \in V \times S : Q_b(v, s) > 0, Q_g(v, s) > 0\} \). As we will show later, the reason of restricting attention to the states in set \( H \) is that the optimal contract always stays in \( H \) before the firm becomes unconstrained. Let \( k^t(h^{t-1}; v, s) \) be the optimal investment, and \( w^t_i(h^{t-1}; v, s) \) be the optimal contingent utilities generated from policy functions starting at state \((v, s)\) after history \( h^{t-1} \).

**Lemma B.3.3.** For \((\bar{v}, s), (\hat{v}, s) \in V \times S \) with \( \bar{v} \neq \hat{v} \), and \( \theta \in (0, 1) \), if \( k^t(h^{t-1}; \bar{v}, s) \neq k^t(h^{t-1}; \bar{v}, s) \) for some history \( h^{t-1} \), then \( Q(\theta \hat{v} + (1 - \theta)\bar{v}, s) > \theta Q(\bar{v}, s) + (1 - \theta)Q(\bar{v}, s) \).
Proof. As in Lemma B.1.1, we change the control variable by letting \( r = R(k) \), so that \( C(r) := R^{-1}(r) = k \). Then \( C(.) \) is strictly convex and all the constraints are linear in \( r \). Define the average policies as:

\[
\bar{k}^t(h^{t-1}) = \theta k^t(h^{t-1}; \hat{\lambda}, s) + (1 - \theta) k^t(h^{t-1}; \bar{\lambda}, \bar{s})
\]

\[
\bar{w}_t^t(h^{t-1}) = \theta w_t^t(h^{t-1}; \hat{\lambda}, s) + (1 - \theta) w_t^t(h^{t-1}; \bar{\lambda}, \bar{s})
\]

To ease notation, let \( \hat{r}^t(h^{t-1}) = r^t(h^{t-1}; \hat{\lambda}, s) \), \( \bar{r}^t(h^{t-1}) = r^t(h^{t-1}; \bar{\lambda}, s) \), \( \bar{w}_t^t(h^{t-1}) = w_t^t(h^{t-1}; \bar{\lambda}, \bar{s}) \), and \( \bar{w}_t^t(h^{t-1}) = w_t^t(h^{t-1}; \bar{\lambda}, \bar{s}) \). Now consider iterating \( T \) times of the Bellman operator starting from states \( (\hat{\lambda}, s) \) and \( (\bar{\lambda}, s) \) respectively.

\[
Q(\hat{\lambda}, s) = \sum_{t=0}^{T} \delta^t E_0[-C(\hat{r}^t(h^{t-1})) + p_{s^{t-1}} \hat{r}^t(h^{t-1})]
\]

\[
+ \delta^{T+1} E_0[p_{s^{T}} Q(\bar{w}_T^T(h^{T-1}), g) + (1 - p_{s^{T}}) Q(\bar{w}_b^T(h^{T-1}), b)]
\]

\[
Q(\bar{\lambda}, s) = \sum_{t=0}^{T} \delta^t E_0[-C(\bar{r}^t(h^{t-1})) + p_{s^{t-1}} \bar{r}^t(h^{t-1})]
\]

\[
+ \delta^{T+1} E_0[p_{s^{T}} Q(\hat{w}_T^T(h^{T-1}), g) + (1 - p_{s^{T}}) Q(\hat{w}_b^T(h^{T-1}), b)]
\]

Averaging for large enough \( T \), we obtain

\[
\theta Q(\hat{\lambda}, s) + (1 - \theta) Q(\bar{\lambda}, s)
\]

\[
= \sum_{t=0}^{T} \delta^t E_0[-\theta C(\hat{r}^t(h^{t-1})) - (1 - \theta) C(\bar{r}^t(h^{t-1})) + p_{s^{t-1}} \bar{r}^t(h^{t-1})]
\]

\[
+ \theta \delta^{T+1} E_0[p_{s^{T}} Q(\hat{w}_T^T(h^{T-1}), g) + (1 - p_{s^{T}}) Q(\hat{w}_b^T(h^{T-1}), b)]
\]

\[
+ (1 - \theta) \delta^{T+1} E_0[p_{s^{T}} Q(\bar{w}_T^T(h^{T-1}), g) + (1 - p_{s^{T}}) Q(\bar{w}_b^T(h^{T-1}), b)]
\]

\[
\leq \sum_{t=0}^{T} \delta^t E_0[-C(\hat{r}^t(h^{t-1})) + p_{s^{t-1}} \hat{r}^t(h^{t-1})]
\]

\[
+ \delta^{T+1} E_0[p_{s^{T}} Q(\hat{w}_T^T(h^{T-1}), g) + (1 - p_{s^{T}}) Q(\hat{w}_b^T(h^{T-1}), b)]
\]

\[
\leq Q(\theta \hat{\lambda} + (1 - \theta) \bar{\lambda}, s)
\]
The strict equality follows from the facts that \( \tilde{r}^t(h^{t-1}) \neq \bar{r}^t(h^{t-1}) \) for some history \( h^{t-1} \) and that the residual term is close to zero after a large enough \( T \). The weak equality follows from the fact that the average plan \( \{\bar{r}^t, \bar{w}^t\}_{t=0}^T \) satisfy the constraints of the Bellman equation at every step of the iteration starting from state \( (\theta \bar{v} + (1 - \theta)\bar{v}, s) \).

**Lemma B.3.4.** For any \( (v, s) \in H \), \( Q(v, s) \) is strictly concave in both \( v_b \) and \( v_g \).

**Proof.** Take any \( (\tilde{v}, s), (\hat{v}, s) \in H \) with \( \tilde{v}_g < \hat{v}_g \) and \( \tilde{v}_b = \hat{v}_b \). Then \( Q(\hat{v}, s) > Q(\tilde{v}, s) \), because \( Q_g(., s) > 0 \). This further implies that \( k^t(h^{t-1}; \hat{v}, s) \neq k^t(h^{t-1}; \tilde{v}, s) \) for some history \( h^{t-1} \). Otherwise, the firm surplus will be the same starting at \( (\tilde{v}, s) \) and \( (\hat{v}, s) \). Then Lemma B.3.3 implies \( Q(v, s) \) is strictly concave in \( v_g \) for \( (v, s) \in H \). The same argument shows \( Q(v, s) \) is strictly concave in \( v_b \) for \( (v, s) \in H \).

**Lemma B.3.5.** For any \( (v, s) \in H \) and \( v_g \geq \bar{y}_g \), investment \( k(v, s) \) decreases in \( v_b \), and strictly increases in \( v_g \).

**Proof.** Take any \( (v, s) \in H \) and \( v_g \geq \bar{y}_g \). The left hand side of (FOC\( w_{gg} \)) is zero, because \( w_g(v, s) \in E_g \) by Lemma B.3.2. Moreover, \( \eta_b(v, s) = Q_g(v, s) \) increases in \( v_b \), and strictly decreases in \( v_g \). The former is by the supermodularity of \( Q \) and the latter is by Lemma B.3.4. Then we can see from (FOC\( w_{gg} \)) that \( \lambda(v, s) \) increases in \( v_b \), and strictly decreases in \( v_g \). From (FOC\( k \)), \( k(v, s) \) decreases in \( v_b \), and strictly increases in \( v_g \).

**Proposition B.3.6.** For \( (\tilde{v}, s), (\hat{v}, s) \in H \) with \( \tilde{v} \neq \hat{v} \) and \( \theta \in (0, 1) \), \( Q(\theta \tilde{v} + (1 - \theta)\hat{v}, s) > \theta Q(\tilde{v}, s) + (1 - \theta)Q(\hat{v}, s) \).

**Proof.** Take any \( (\tilde{v}, s), (\hat{v}, s) \in H \) with \( \tilde{v} \neq \hat{v} \). By Lemma B.3.3, it suffices to show that \( k^t(h^{t-1}; \tilde{v}, s) \neq k^t(h^{t-1}; (\tilde{v}, s)) \) for some history \( h^{t-1} \). Suppose not. The firm surplus and investment must be the same after any history starting at the two initial
states \( (\bar{v}, s) \) and \( (\tilde{v}, s) \). This means we must have either \( \tilde{v}_b > \hat{v}_b \) and \( \tilde{v}_g < \hat{v}_g \), or \( \tilde{v}_b < \hat{v}_b \) and \( \tilde{v}_g > \hat{v}_g \). Otherwise, \( Q(\tilde{v}, s) \neq Q(\bar{v}, s) \), because \( Q_b(., s), Q_g(., s) > 0 \) on \( H \). Without loss of generality, we assume \( \hat{v}_b > \tilde{v}_b, \tilde{v}_g < \hat{v}_g \). At history \( h^0 = \{s, g\} \), we know \( Q(\hat{v}, h^0), g) = Q(\tilde{v}, h^0), g) \), which implies \( \Psi(\hat{v}, g) = \Psi(\tilde{v}, g) \) by Lemma B.3.2. If \( \tilde{v}_g < \bar{y}_g \), then Lemma B.3.1 implies that we must have \( \hat{v}_g = \bar{v}_g \), since \( \Psi(y, g) \) is increasing overall and strictly increasing in \( y \in [0, \delta[p_g \bar{v}_g g + (1 - p_g) \bar{v}_g]) \), a contradiction. If \( \bar{v}_g = \bar{y}_g \), then \( k(\bar{v}, s) > k((\hat{v}_b, \bar{v}_g), s) \geq k(\tilde{v}, s) \) by Lemma B.3.5. This forms a contradiction.

\[ Q_p(\hat{v}, s) Q_p(\bar{v}, s), Q_p(\tilde{v}, s) \]

Lemma B.3.7. For any \( y < \bar{y}_s \), \( \Psi(y, s) \) is strictly concave in \( y \).

Proof. Take any \( \hat{y}, \tilde{y} < \bar{y}_s \) and \( \hat{y} \neq \tilde{y} \). From Lemma B.3.1, the constraint of (P3) binds at both \( y = \hat{y} \) and \( y = \tilde{y} \). So \( \hat{y} \neq \tilde{y} \) implies \( x^*(\hat{y}, s) \neq x^*(\tilde{y}, s) \). Moreover, from (B.17) and (B.18), we can see that \( Q_b(x^*(y, s), s), Q_g(x^*(y, s), s) > 0 \) when \( y < \bar{y}_s \), because \( \Psi(y, s) > 0 \). This means \( (x^*(\hat{y}, s), s), (x^*(\tilde{y}, s), s) \in H \). Hence,

\[
\theta \Psi(\hat{y}, s) + (1 - \theta) \Psi(\tilde{y}, s) = \delta[\theta Q(x^*(\hat{y}, s), s) + (1 - \theta) Q(x^*(\tilde{y}, s), s)]
\]

\[
< \delta Q(\theta x^*(\hat{y}, s), s) + (1 - \theta) x^*(\tilde{y}, s), s) \]

\[
\leq \Psi(\theta \hat{y} + (1 - \theta) \tilde{y}, s)
\]

The second line is implied by Proposition B.3.6.

B.4 Properties of the Directional Derivative

In this section, we show the directional derivative of the firm’s value function is a nonnegative martingale and how it evolves in the optimal contract. We also show the directional derivative must split at the good state.

Lemma B.4.1. The process \( D_{(1,1)} Q(v, s) = Q_b(v, s) + Q_g(v, s) \) induced by the optimal contract is a nonnegative martingale.
Proof. Take any \((v, s) \in V \times S\). Adding the first order conditions (FOC\(_{wb}\)) to (FOC\(_{gg}\)), and using envelope conditions (Env\(_b\)) and (Env\(_g\)) to substitute \(\eta_i(v, s)\), we get

\[
(1 - p_s) D_{(1,1)} Q[w_b(v, s), b] + p_s D_{(1,1)} Q[w_g(v, s), g] = D_{(1,1)} Q(v, s) \quad (B.19)
\]

Moreover, by part (e) of Theorem B.1.9, \(D_{(1,1)} Q(v, s) = \lim_{\varepsilon \to 0} [Q(v + (\varepsilon, \varepsilon), s) - Q(v, s)] \geq 0\). So the process \(D_{(1,1)} Q\) is a nonnegative martingale. \(\square\)

Using the martingale relation (B.19) and first order conditions we can characterize the evolution of directional derivative martingale \(D_{(1,1)} Q\) on any optimal path in the following Lemma.

**Lemma B.4.2.** In the optimal contract starting at state \((v, s)\), the martingale \(D_{(1,1)} Q\) evolves according to the following relations, if "good-good", "good-bad", "bad-good", and "bad-bad" shocks occur:

\[
D_{(1,1)} Q(w^g_i, g) = D_{(1,1)} Q(w_g, g) - \frac{1}{p_g} \lambda(w_g, g) \quad (B.20)
\]

\[
D_{(1,1)} Q(w^g_i, b) = D_{(1,1)} Q(w_g, g) + \frac{1}{1 - p_g} \lambda(w_g, g) \quad (B.21)
\]

\[
D_{(1,1)} Q(w^b_i, g) = D_{(1,1)} Q(w_b, b) - \frac{(1 - p_s) \lambda(w_b, b) - \Delta \lambda(v, s)}{p_b(1 - p_s)} \quad (B.22)
\]

\[
D_{(1,1)} Q(w^b_i, b) = D_{(1,1)} Q(w_b, b) + \frac{(1 - p_s) \lambda(w_b, b) - \Delta \lambda(v, s)}{(1 - p_b)(1 - p_s)} \quad (B.23)
\]

where \(w_i = w_i(v, s), i = b, g\), are the states induced by the optimal contract at \((v, s)\);

\(w^g_i = w_i(w_g, g), w^b_i = w_i(w_b, b)\) are the states induced by the optimal contract at \((w_g, g), (w_b, b)\) respectively.

Proof. First, from (Env\(_b\)) and (FOC\(_{gg}\)), we get:

\[
p_s[\eta_g(w_g, g) - \lambda(w_g, g)] = p_s Q_g(w_g, g) - p_s \lambda(w_g, g) \quad (B.24)
\]

\[
= p_g[\eta_g(v, s) - \lambda(v, s)] - p_s \lambda(w_g, g)
\]
Add \((\text{FOC}_{w_g})\) and \((\text{FOC}_{w_g})\) at state \((v, s)\) to get:

\[ p_s \mathcal{D}_{(1,1)} Q(w_g, g) = \eta_g(v, s) - \lambda(v, s) \]

(B.25)

Add \((\text{FOC}_{w_g})\) and \((\text{FOC}_{w_g})\) at state \((w_g, g)\) to get:

\[ p_g \mathcal{D}_{(1,1)} Q(w^g_g, g) = \eta_g(w_g, g) - \lambda(w_g, g) \]

(B.26)

Then combine (B.24), (B.25), (B.26) and rearrange to get (B.20).

Next, from \((\text{En}_{w_b})\) and \((\text{FOC}_{w_b})\), we get

\[ (1 - p_s)[\eta_b(w_b, b) + \lambda(w_b, b)] \]

(B.27)

\[ = (1 - p_s)Q_b(w_b, b) + (1 - p_s)\lambda(w_b, b) \]

\[ = (1 - p_b)[\eta_b(v, s) + \lambda(v, s)] - \Delta\lambda(v, s) + (1 - p_s)\lambda(w_b, b) \]

Add \((\text{FOC}_{w_b})\) and \((\text{FOC}_{w_b})\) at state \((v, s)\) to get:

\[ (1 - p_s) \mathcal{D}_{(1,1)} Q(w_b, b) = \eta_b(v, s) + \lambda(v, s) \]

(B.28)

Add \((\text{FOC}_{w_b})\) and \((\text{FOC}_{w_b})\) at state \((w_b, b)\) to get:

\[ (1 - p_b) \mathcal{D}_{(1,1)} Q(w^b_b, b) = \eta_b(w_b, b) + \lambda(w_b, b) \]

(B.29)

Then combine (B.27), (B.28), (B.29) and rearrange to get (B.23).

Finally, (B.19) at state \((w_g, g)\) implies:

\[ (1 - p_g) \mathcal{D}_{(1,1)} Q(w^g_g, b) + p_g \mathcal{D}_{(1,1)} Q(w^g_g, g) = \mathcal{D}_{(1,1)} Q(w_g, g) \]

(B.30)

(B.19) at state \((w_b, b)\) implies:

\[ (1 - p_b) \mathcal{D}_{(1,1)} Q(w^b_b, b) + p_b \mathcal{D}_{(1,1)} Q(w^b_b, g) = \mathcal{D}_{(1,1)} Q(w_b, b) \]

(B.31)

Then combine (B.30), (B.20) and rearrange to obtain (B.21). Similarly, combine (B.31), (B.23) and rearrange to obtain (B.22).
We show in the following that the directional derivative must split (goes down after a good shock and goes up after a bad shock) if last period had a good shock.

**Lemma B.4.3.** For any \((v, s) \in V \times S\), we must have either \(w_g(v, s) \in H\) or \(w_g(v, s) \in E_g\).

**Proof.** Take any \((v, s) \in V \times S\). From the first order conditions (FOC\(_{w_g}\)) and (FOC\(_{w_{gg}}\)), we know that \(p_gQ_b(w_g(v, s), g) = (1 - p_g)Q_g(w_g(v, s), g)\). So we must either have \(Q_b(w_g(v, s), g) > 0\) and \(Q_g(w_g(v, s), g) > 0\) or have \(Q_b(w_g(v, s), g) = 0\) and \(Q_g(w_g(v, s), g) = 0\). The former case means that \(w_g(v, s) \in H\) by definition and the latter case means that \(w_g(v, s) \in E_g\) by Proposition 3.4.3.

**Lemma B.4.4.** Suppose the optimal contract starts at \((v, s)\) and evolves to the state \((w_g, g)\) satisfying \(D_{(1,1)} Q(w_g, g) > 0\) after a good shock. Then the directional derivative goes down after another good shock and goes up after another bad shock, i.e., \(D_{(1,1)} Q(w_g^g, g) < D_{(1,1)} Q(w_g, g)\) and \(D_{(1,1)} Q(w_g^b, b) > D_{(1,1)} Q(w_g, g)\).

**Proof.** We shall show in two cases that \(D_{(1,1)} Q(w_g^g, g) < D_{(1,1)} Q(w_g, g)\). The conclusion that \(D_{(1,1)} Q(w_g^b, b) > D_{(1,1)} Q(w_g, g)\) then simply follows from the martingale equation (B.19) at the state \((w_g, g)\). First, we consider the case of \(w_{gg} < \bar{y}_g\). From (PK\(_g\)), we know that \(v_g \leq \delta[p_g w_{gg} + (1 - p_g)w_{gb}] \leq \delta w_{gg} < w_{gg}\), implying \(v_g < w_{gg} < \bar{y}_s\). By Lemma B.3.7, we have \(\Psi_y(w_{gg}, g) < \Psi_y(v_g, g)\), because \(\Psi(\cdot, g)\) is strictly concave on \((0, \bar{y}_s)\). By Lemma B.3.1 and Lemma B.3.2, we know that \(D_{(1,1)} Q(w_g, g) = \Psi_y(v_g, g)\) and \(D_{(1,1)} Q(w_g^g, g) = \Psi_y(w_{gg}, g)\). So the conclusion follows. Second, we consider the case of \(w_{gg} \geq \bar{y}_g\). From Lemma B.3.2, we know that \(w_g^g \in E_g\). So the left hand side of (FOC\(_{w_{gg}}\)) at \((w_g, g)\) is zero implying \(\eta_b(w_g, g) = \lambda(w_g, g)\). The condition \(D_{(1,1)} Q(w_g, g) > 0\) implies that \((w_g, g) \in H\) by Lemma B.4.3. Hence, \(\lambda(w_g, g) = Q_g(w_g, g) > 0\). And equation (B.20) simply means \(D_{(1,1)} Q(w_g^g, g) < D_{(1,1)} Q(w_g, g)\). 

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B.5 Proofs from Section 3.5

In this section we show the optimal repayments, one-step set, and various properties regarding investments.

The condition derived in Lemma B.1.6 regarding Lagrange multipliers in problem (P1) also holds for Lagrange multipliers in problem (VF). This is because function \( P \) in (P1) satisfies all the properties of function \( Q(v, s) \) in (VF). In particular, we know

\[
\eta_b(v, s) + \lambda(v, s) - \mu_b(v, s) \geq 0, \quad m_b(v, s)[\eta_b(v, s) + \lambda(v, s) - \mu_b(v, s)] = 0 \quad (B.32)
\]

\[
\eta_g(v, s) - \lambda(v, s) - \mu_g(v, s) = 0
\]

Moreover, we can use (B.33) to rewrite (FOCk) as:

\[
R'(k(v, s)) = 1/[p_s - \lambda(v, s)]
\] (FOCk)

Proof of Proposition 5.1.  
(a) If \( \mu_b(v, s) > 0 \), then complementary slackness implies \( m_b(v, s) = 0 \). If \( \mu_b(v, s) = 0 \), then \( \eta_b(v, s) + \lambda(v, s) - \mu_b(v, s) > 0 \), because \( \eta_b(v, s) > 0 \) when \( v < \bar{v}_s \) and \( \lambda(v, s) \geq 0 \). Then (B.32) implies \( m_b(v, s) = 0 \).

(b) Given \( v < \bar{v}_s, (v, s) \notin A_{1,s} \) implies \( m_g(v, s) = R(k(v, s)) \)

Since \( v < \bar{v}_s, (v, s) \notin A_{1,s} \), we know \( v_g < \delta[p_g\bar{v}_{gg} + (1-p_g)\bar{v}_b] \). By Lemma B.3.2, \( w_g(v, s) \) is the solution of (P3) at \( (v_g, g) \). And by lemma B.3.1, the constraint of (P3) at \( (v_g, g) \) must bind when \( v_g < \delta[p_g\bar{v}_{gg} + (1-p_g)\bar{v}_b] \). This means \( \delta[p_gw_{gg}(v, s) + (1 - p_g)w_{gb}(v, s)] = v_g \). Therefore (PK_g) implies \( m_g(v, s) = R(k(v, s)) \).

(c) Suppose \( (v, s) \in A_{1,s} \). Given the optimal policy \( (k, m_b, m_g, w_b, w_g) \) at \( (v, s) \), let \( w'_g = \bar{v}_g, m'_g = R(k(v, s)) + \delta[p_g\bar{v}_{gg} + (1-p_g)\bar{v}_b] - v_g \). Then \( m'_g \leq R(k) \) because \( v_g \geq \delta[p_g\bar{v}_{gg} + (1 - p_g)\bar{v}_b] \) by the assumption \( (v, s) \in A_{1,s} \). So
\[(k, m_b, m'_b, w_b, w'_b) \in \Gamma(v, s)\]. Moreover, because \(Q(w'_g, g) = \bar{Q}(v, s)\), changing the policy to \((k, m_b, m'_b, w_b, w'_g)\) at least weakly increases the objective of (VF). So \(w(v, g) = \bar{v}_g \in E_g\).

Suppose \(w_g(v, s) \in E_g\). That is the contingent utilities reach the efficient region after a good shock. By Proposition 3.4.3, \(w_{gg}(v, s) \geq \bar{v}_g^g\) and \(w_{gb}(v, s) \geq \bar{v}_b\).

Then by (PK\(_g\)) at \((v, s)\), we know \(v_g = \delta[p_g w_{gg}(v, s) + (1 - p_g)w_{gb}(v, s)] \geq \delta[p_g \bar{v}_g^g + (1 - p_g)\bar{v}_b]\). Therefore, \((v, s) \in A_{1,s}\).

(d) When \(v \in A_{1,s}\) and \(v_g > \delta[p_g \bar{v}_g^g + (1 - p_g)\bar{v}_b]\), part (c) implies \(w_g(v, s) \in E_g\). So in the maximum rent contrat, \(w_g(v, s) = \bar{v}_g\). Then (PK\(_g\)) implies \(R(k(v, s)) - m_g(v, s) = v_g - \delta[p_g \bar{v}_g^g + (1 - p_g)\bar{v}_b] > 0\).

We will show in the following proof that: (a) for any \(v < \bar{v}_b, k(v, g) \geq k(v, b)\); (b) \(k(v, s)\) is decreasing in \(v_b\) for any \((v, s) \in V \times S\); (c) \(k(v, s)\) is increasing in \(v_g\) for any \((v, s) \in V \times S\).

**Proof of Lemma 3.5.4.** 1. Let \(\hat{k}, \hat{w}_i\) and \(k, w_i\) be the optimal policies at states \((v, b)\) and \((v, g)\) respectively. Suppose \(\hat{k} > k\). Since \(\Gamma(v, g) = \Gamma(v, b)\), optimality at \((v, g)\) implies

\[
-k + p_g[R(\hat{k}) + \delta Q(w_g, g)] + (1 - p_g)\delta Q(w_b, b) \\
\geq -\hat{k} + p_g[R(\hat{k}) + \delta Q(\hat{w}_g, g)] + (1 - p_g)\delta Q(\hat{w}_b, b) \quad (B.34)
\]

Since \(w_g = \hat{w}_g\) by Lemma (B.3.2), (B.34) then implies \(Q(w_b, b) > Q(\hat{w}_b, b)\).

Moreover, optimality at \((v, b)\) implies

\[
-k + p_b[R(\hat{k}) + \delta Q(\hat{w}_g, g)] + (1 - p_b)\delta Q(\hat{w}_b, b) \\
\geq -\hat{k} + p_b[R(\hat{k}) + \delta Q(\hat{w}_g, g)] + (1 - p_b)\delta Q(\hat{w}_b, b) \quad (B.35)
\]

Add (B.34), (B.35) and rearrange to get:

\[Q(\hat{w}_b, b) - Q(w_b, b) \geq R(\hat{k}) - R(k) > 0\]

which is a contradiction with \(Q(w_b, b) > Q(\hat{w}_b, b)\).
2. Take any $(v, s), (v', s) \in V \times S$ with $v'_g = v_g, v'_b > v_b$. By Lemma B.3.2 we know $w(v, s) = w(v', s)$. Moreover, $Q_g(v', s) > Q_g(v, s)$ by result (g) of Theorem B.1.9. Then from (FOC$_{w_{gg}}$), we must have $\lambda(v', s) > \lambda(v, s)$. Hence, (FOC$_k$) implies $k(v', s) < k(v, s)$.

3. Take any $(v, s), (v', s) \in V \times S$ with $v'_b = v_b, v'_g > v_g$. Suppose that $k(v, s) > k(v', s)$. From (FOC$_k$), $\lambda(v', s) > \lambda(v, s)$. From (IC*) and (PK$_b$) ($m_b(v, s) = m(v', s) = 0$ are optimal), we obtain

$$w_{bb}(v', s) = \frac{p_b R(k(v', s)) + p_g v_b - p_b v'_g}{\delta \Delta}$$

$$< \frac{p_b R(k(v, s)) + p_g v_b - p_b v_g}{\delta \Delta} \leq w_{bb}(v, s)$$

and

$$w_{bg}(v', s) = \frac{1 - p_b}{\delta \Delta} \left[ v'_g - \frac{1 - p_b}{1 - p_b} v_b - R(k(v', s)) \right]$$

$$\geq \frac{1 - p_b}{\delta \Delta} \left[ v_g - \frac{1 - p_b}{1 - p_b} v_b - R(k(v, s)) \right] \geq w_{bg}(v, s)$$

The equalities are because $\lambda(v', s) > 0$. Then we know $Q_g(w_b(v, s), b) \geq Q_g(w_b(v', s), b)$ because $Q_g$ is increasing in the first coordinate and decreasing in the second coordinate by Theorem B.1.9. Moreover, $\eta_b(v, s) = Q_b(v, s) \leq Q_b(v', s) = \eta_b(v', s)$ by Theorem B.1.9. From (FOC$_{w_{bg}}$), we will have $\lambda(v', s) \leq \lambda(v, s)$, a contradiction. Therefore, $k(v, s) \leq k(v', s)$.

To proceed the proof of Proposition 5.3, we first show that given a certain value of $v_b$, if $v_g$ is sufficiently large, then efficient investment will be achieved. Moreover, for a certain value of $v_b$, there exists a threshold value of $v_g$ such that investment is efficient if $v_g$ is above the threshold and inefficient if below the threshold.

Lemma B.5.1. For any state $(v, s) \in V \times S$ that satisfies $v_g \geq R(\bar{k}_s) + \frac{p_g v_b}{p_b}$, we have $\lambda(v, s) = 0, k(v, s) = \bar{k}_s$.
Proof. By (PK_b), \( \delta[p_b(w_{bg} - w_{bb}) + w_{bb}] \leq v_b + m_b \leq v_b \). So we have \( w_{bg} - w_{bb} \leq \frac{v_b}{\delta p_b} \).

Then the right hand side of (IC*) is smaller than \( R(\tilde{k}_s) + \frac{\Delta v_b}{p_b} \). Since \( v_g \geq R(\tilde{k}_s) + \frac{p_a v_b}{p_b} \), we know \( v_g - v_b \geq R(\tilde{k}_s) + \frac{\Delta v_b}{p_b} \). This means for all feasible policies at \((v, s)\), (IC*) will not bind. Therefore, we must have \( \lambda(v, s) = 0 \), \( k(v, s) = \tilde{k}_s \). □

**Lemma B.5.2.** For any \( v_b \geq 0 \), there exists \( h_s(v_b) > v_b \) such that \( v_g \geq h_s(v_b) \) implies \( \lambda(v, s) = 0 \), and \( v_b \leq v_g < h_s(v_b) \) implies \( \lambda(v, s) > 0 \). Moreover, \( h_s(v_b) \) is increasing and satisfies \( h(0) = R(\tilde{k}_s), h(\bar{v}_{sg}) \leq \bar{v}_{sg} \).

**Proof.** Take any \( v_b > 0 \). If \( v_g \) is sufficiently close to \( v_b \), then investment \( k(v, s) \) will be sufficiently close to zero by (IC*). We know \( \lambda(v, s) > 0 \) by (FOC_k). If \( v_g \) is sufficiently large, by Lemma B.5.1, we know \( \lambda(v, s) = 0 \). Hence, for a certain value of \( v_b \) we can define the smallest value of \( v_g \) such that \( \lambda(v, s) = 0 \) as \( h_s(v_b) = \inf\{v_g \geq v_b : \lambda(v, s) = 0\} \). By Proposition 5.x and (FOC_k) that \( \lambda(v, s) \) is increasing in \( v_b \) and decreasing in \( v_g \). So we have \( \lambda(v, s) = 0 \) if \( v_g \geq h_s(v_b) \).

Then, \( h(0) = R(\tilde{k}_s) \), since the only feasible contingent utility vector \( w_{b} \) is \( 0 \) at \((0, R(\tilde{k}_s))\). And since \( \lambda(\bar{v}_s, s) = 0 \), we know \( h(\bar{v}_b) \leq \bar{v}_{sg} \). Now we show \( h_s(v_b) \) is increasing. Take any \( v_b, v_b' \) with \( v_b' > v_b \). We know \( 0 < \lambda((v_b, h_s(v_b) - \varepsilon), s) \leq \lambda((v_b', h_s(v_b) - \varepsilon), s) \) for any small \( \varepsilon > 0 \). The first inequality is by the definition of \( h_s(v_b) \), and the second is by \( \lambda(v, s) \) is increasing in \( v_b \). By definition, \( 0 = \lambda((v'_b, h_s(v'_b)), s) < \lambda((v'_b, h_s(v_b) - \varepsilon), s) \). So \( h(v'_b) \geq h_s(v_b) - \varepsilon \), which implies \( h(v'_b) \geq h_s(v_b) \). □

**Lemma B.5.3.** Take \((v, s), (\hat{v}, s) \in V \times S \) such that \( v = (v_b, h_s(v_b)), \hat{v} = (v_b, \hat{v}_g) \) with \( \hat{v}_g < h_s(v_b) \). If \( \lambda(w_b(v, s), b) > 0 \), then \( \lambda(w_b(\hat{v}, s), b) > 0 \).

**Proof.** Note that \( Q(w_b(v, s), b) \geq Q(w_b(\hat{v}, s), b) \), since \( w_b(v, s), w_b(\hat{v}, s) \) are both feasible in problem (P3) at \((v, b)\) and \( w_b(\hat{v}, s) \) is the maximizer (because by construction \( \lambda(v, s) = 0 \)). Suppose \( w_{bg}(\hat{v}, s) - w_{bb}(\hat{v}, s) > w_{bg}(v, s) - w_{bb}(v, s) \). Then
there exists $k > k(\tilde{v}, s)$ such that
\[
\hat{v}_g - v_b \geq R(k(\tilde{v}, s)) + \delta \Delta [w_{bg}(\tilde{v}, s) - w_{bb}(\tilde{v}, s)] \\
= R(k) + \delta \Delta [w_{bg}(v, s) - w_{bb}(v, s)] \tag{B.36}
\]

Let $m_b = \delta[p_b w_{bg}(v, s) + (1 - p_s) w_{bb}(v, s)] - v_b$. Then (B.36) means
\[
(k, m_b, m_g(\tilde{v}, s), w_b(v, s), w_g(\tilde{v}, s)) \in \Gamma(\tilde{v}, s)
\]
so that
\[
-k(\tilde{v}, s) + p_s R(k(\tilde{v}, s)) + (1 - p_s) \delta Q(w_b(\tilde{v}, s), b) \\
\geq -k + p_s R(k) + (1 - p_s) \delta Q(w_b(v, s), b)
\]
which further implies $Q(w_b(v, s), b) < Q(w_b(\tilde{v}, s), b)$, a contradiction. So $w_{bg}(\tilde{v}, s) - w_{bb}(\tilde{v}, s) \leq w_{bg}(v, s) - w_{bb}(v, s)$. Since $m_b(v, s) = m_b(\tilde{v}, s) = 0$ are optimal and from (PK$_b$) at both states $(v, s)$ and $(\tilde{v}, s)$, we know $w_{bg}(\tilde{v}, s) \leq w_{bg}(v, s)$ and $w_{bb}(\tilde{v}, s) \geq w_{bb}(v, s)$. So we have,
\[
w_{bg}(\tilde{v}, s) \leq w_{bg}(v, s) < h_b(w_{bb}(v, s)) \leq h_b(w_{bb}(\tilde{v}, s))
\]
The second inequality is by assumption $\lambda(w_b(v, s), b) > 0$. The last inequality is by the fact that $h_s$ is increasing from Lemma B.5.2. Therefore, $w_{bg}(\tilde{v}, s) < h_b(w_{bb}(\tilde{v}, s))$ means $\lambda(w_b(\tilde{v}, s), b) > 0$.

We will show in the following proof that:

(a) $\lambda(w_g(v, s), g) > 0$ if $w_g(v, s) \notin E_g$;

(b) $\lambda(w_b(v, s), b) > 0$, for any $(v, s) \in V \times S$ with $v_b < \frac{\delta p_b R(k_b)}{(1 - \delta p_b)(1 - \delta)}$;

(c) If $(p_b, p_g) \in \{p : p_b \geq \phi(p_g)\}$, then $\lambda(w_b(v, s), b) > 0$ if $w_b(v, s) \notin E_b$.

(a) means after a good shock investment is inefficient. (b) means if $v_b$ is not sufficiently close to $\bar{v}_b^s$, then investment is inefficient after a bad shock. (c) means if persistence is low, then we always have investment is inefficient after a bad shock.
Proof of Proposition 5.3.  (a) If \( \mathbf{w}_g(v, s) \in A_{1,g} \), then the left hand side of (FOC\( w_{gg} \))

at \( (\mathbf{w}_g(v, s), g) \) is zero, and hence from its right hand side, \( \lambda(\mathbf{w}_g(v, s), g) = \eta_g(\mathbf{w}_g(v, s), g) > 0 \). Consider \( \mathbf{w}_g(v, s) \notin A_{1,g} \). Let \( \mathbf{w}'_g = \mathbf{w}_g(\mathbf{w}_g(v, s), g) \). Lemma B.4.4 shows that \( D_{(1,1)} Q(\mathbf{w}_g(v, s), g) > D_{(1,1)} Q(\mathbf{w}'_g, g) \). And by (B.20), we must have \( \lambda(\mathbf{w}_g(v, s), g) > 0 \).

(b) Note it suffices to show \( \lambda(\mathbf{w}_b(\hat{v}, s), b) > 0 \) if \( \hat{v}_b < \frac{\delta p_b R(\bar{k}_b)}{(1-\delta)(1-\delta\Delta)}, \hat{v}_g = h_s(\hat{v}_b) \). If it is true, then: (1) \( \lambda(\mathbf{w}_b(\hat{v}', s), b) > 0 \) for any \( (\hat{v}', s) \) with \( v'_b = v_b, v'_g < \hat{v}_g \), by Lemma B.5.3; (2) \( \lambda(\mathbf{w}_b(\hat{v}', s), b) = \lambda((\mathbf{w}_b(v, s), b) > 0 \) for any \( (\hat{v}', s) \) with \( v'_b = v_b, v'_g > \hat{v}_g \), because \( \mathbf{w}_b(\hat{v}', s) = \mathbf{w}_b(v, s) \) by Lemma B.3.2.

Suppose \( \lambda(\mathbf{w}_b(\hat{v}, s), b) = 0 \). And by construction \( \lambda(\hat{v}, s) = 0 \). Then (B.23) implies that \( D_{(1,1)} Q(\mathbf{w}_b', b) = D_{(1,1)} Q(\mathbf{w}_b(\hat{v}, s), b) \), where \( \mathbf{w}_b' = \mathbf{w}_b(\mathbf{w}_b(\hat{v}, s), b) \).

By Lemma B.3.1, we know that \( \Psi_1(\hat{v}_b, b) = D_{(1,1)} Q(\mathbf{w}_b(\hat{v}, s), b) \), and \( \Psi_1(w_{gb}(\hat{v}, s), b) = D_{(1,1)} Q(\mathbf{w}_b', b) \). So \( \Psi_1(\hat{v}_b, b) = \Psi_1(w_{gb}(\hat{v}, s), b) \). The strict concavity of \( \Psi_1(Z, b) \) in \( Z \), by Lemma B.3.7, implies \( w_{gb}(\hat{v}, s) = \hat{v}_b \). Then we know \( \mathbf{w}_b' = \mathbf{w}_b(\hat{v}, s) \), since by Lemma B.3.2 \( \mathbf{w}_b' \) is only function of \( w_{gb}(\hat{v}, s) \), and \( \mathbf{w}_b(\hat{v}, s) \) is only a function of \( \hat{v}_b \). From (PK\( b \)) and (IC\( * \)) at state \( (\mathbf{w}_b(\hat{v}, s), b) \):

\[
\begin{align*}
\mathbf{w}_{gb}(\hat{v}, s) & \geq \delta[p_b w_{gb}(\hat{v}, s) + (1 - p_b) w_{bb}(\hat{v}, s)] & (B.37) \\
\mathbf{w}_{gb}(\hat{v}, s) - w_{bb}(\hat{v}, s) & \geq R(\bar{k}_b) + \delta\Delta[w_{gb}(\hat{v}, s) - w_{bb}(\hat{v}, s)] & (B.38)
\end{align*}
\]

(B.37) and (B.38) together imply \( w_{gb}(\hat{v}, s) \geq \frac{\delta p_b R(\bar{k}_b)}{(1-\delta)(1-\delta\Delta)} \), and \( w_{gb}(\hat{v}, s) \geq \frac{1-\delta(1-p_b)}{(1-\delta)(1-\delta\Delta)} R(\bar{k}_b) \). Then from (PK\( b \)) at \( (\hat{v}, s) \), we know

\[
\hat{v}_b \geq \delta[p_b w_{gb}(\hat{v}, s) + (1 - p_b) w_{bb}(\hat{v}, s)] \geq \frac{\delta p_b R(\bar{k}_b)}{(1-\delta)(1-\delta\Delta)}
\]

This is a contradiction with the assumption that \( \hat{v}_b < \frac{\delta p_b R(\bar{k}_b)}{(1-\delta)(1-\delta\Delta)} \).
(c) If \((p_b, p_g) \in \{p : p_b \geq \phi(p_g)\}\), then \(\bar{v}_b = \frac{\delta p_b R(k_b)}{(1-\theta)(1-\theta)}\). And \(w_b(v, s) \not\in E_b\) means \(v_b < \bar{v}_b\). So from part (b), \(\lambda(w_b(v, s), b) > 0\).

\[\text{B.6 Proofs from Section 3.6}\]

Recall from section \(\ref{section:E}\) of the Appendix that the set \(H\) is defined as \(H = \{(v, s) \in V \times S : Q_b(v, s) > 0, Q_g(v, s) > 0\}\). Define the process \(\{(v^{(t)}, s_{t-1})\}_{t=0}^{n}\) to be the states induced by the optimal contract starting at some \((v^{(0)}, s_0) \in V \times S\). To establish the convergence results in Theorem 3.6.1, we first show some useful properties regarding \(H\). Denote the closure of \(H\) as \(\text{cl}(H)\).

**Lemma B.6.1.** The set \(H\) has the following properties:

1. \((v, s) \in H\) implies \(v_b < \bar{v}_{sb}\) and \(v_g < \bar{v}_{sg}\).

2. For any \((v, s) \in \text{cl}(H)\), \(D_{(1,1)} Q(v, s) = 0\) implies \(v = \bar{v}_s\).

**Proof.**

1. Take any \((v, s) \in H\). By the definition of \(H\), \(Q_b(v, s) > 0\) and \(Q_g(v, s) > 0\). Suppose \(v_b \geq \bar{v}_{sb}\). Then from part (e) of Theorem B.1.9, we know \(Q_b(v, s) \leq 0\), a contradiction. Suppose \(v_g \geq \bar{v}_{sg}\) and \(v_b < \bar{v}_{sb}\). From part (b) of Proposition 3.4.3, we know \(Q_g((\bar{v}_{sb}, v_g), s) = 0\) because \((\bar{v}_{sb}, v_g) \in E_s\). Supermodularity of \(Q\) then implies \(0 \leq Q_g(v, s) \leq Q_g((\bar{v}_{sb}, v_g), s) = 0\), a contradiction. So we must have \(v_b < \bar{v}_{sb}\) and \(v_g < \bar{v}_{sg}\).

2. Let \(A := \{(v, s) \in V \times S : v \leq \bar{v}_s\}\). Part (a) implies \(H \subseteq A\). Since \(A\) is a closed set, we know \(\text{cl}(H) \subseteq A\). Take any \((\bar{v}, s) \in \text{cl}(H)\) such that \(D_{(1,1)} Q(\bar{v}, s) = 0\).

The assumption \((\bar{v}, s) \in \text{cl}(H)\) implies \(\bar{v} \leq \bar{v}_s\) and \(Q_b(\bar{v}, s) \geq 0, Q_g(\bar{v}, s) \geq 0\). Then \(D_{(1,1)} Q(\bar{v}, s) = 0\) implies \(Q_b(\bar{v}, s) = 0, Q_g(\bar{v}, s) = 0\). From Proposition 3.4.3 we know \(\bar{v} \in E_s\), and hence, \(\bar{v} \geq \bar{v}_s\). So we must have \(\bar{v} = \bar{v}_s\). \(\square\)

In the high peresistence case the optimal contract may need enough good shocks (at least two) to reach efficient sets. So to establish the result that efficient sets are
achieved in finite time, we need to consider sequences with two good shocks in a row infinitely often.

**Lemma B.6.2.** The sets \( \{ s_t = \alpha \text{ i.o.} \} \), where \( \alpha \in S \), have full measure. Similarly, the sets \( \{ (s_{t-1}, s_t) = (\alpha, \beta) \text{ i.o.} \} \), where \( \alpha, \beta \in S \), have full measure.

Intuitively, the lemma says that bad or good shocks occur infinitely often with probability one. Moreover, consecutive ‘bad-bad’, ‘bad-good’, ‘good-bad’, and ‘good-good’ shocks also occur infinitely often with probability one.

**Proof of Theorem 3.6.1.** (a) We show that \( D_{(1,1)} Q(\bar{v}(t), s_{t-1}) \) converges to 0 almost surely.

By Lemma B.4.1, the process \( D_{(1,1)} Q(\bar{v}(t), s_{t-1}) \) is a nonnegative martingale. So Doob’s Martingale Convergence Theorem ensures that \( D_{(1,1)} Q(\bar{v}(t), s_{t-1}) \) converges almost surely to a non-negative and integrable random variable. Consider a path with the property that \( \lim_{t \to \infty} D_{(1,1)} Q(\bar{v}(t), s_{t-1}) = a > 0 \) and good shock occurs infinitely many times. There exists a large \( T \) such that \( D_{(1,1)} Q(\bar{v}(t), s_{t-1}) > 0 \) for \( t \geq T \).

Now let us consider the subsequence that has only good shocks. Note that when \( t \geq T \), this subsequence must stay in the set \( H \). This is because by part (c) of Lemma B.6.1, after good shocks if directional derivative is strictly positive then the state induced by the optimal contract has to be in \( H \). Moreover, by part (a) of Lemma B.6.1 we know this subsequence with only good shocks is bounded and must have a converging subsequence \( \{(v_{\tau_i}, g)\}_{i=0}^{\infty} \) with limit \( (\bar{v}, g) \).

By construction we have \( s_{\tau_i-1} = g \). And it is either the case that \( s_{\tau_i} = b \) infinitely often (i.e. bad shock occurs infinitely often after the chosen good
shocks), or the case that $s_{\tau_t} = g$ infinitely often. Suppose $s_{\tau_t} = b$ for infinitely many $t$. And let $\{(v^{(\tau'_t)}, g)\}_{t=0}^{\infty}$ be a subsequence of $\{(v^{(\tau_t)}, g)\}_{t=0}^{\infty}$ that has $s_{\tau'_t} = g$ and $s_{\tau'_t} = b$. By construction, $v^{(\tau'_t+1)} = w_b(v^{(\tau'_t)}, g)$. And hence, $\lim_{t \to \infty} v^{(\tau'_t+1)} = w_b(\hat{v}, g)$ because $w_b$ is continuous. And by the continuity of $D_{(1,1)} Q$, we get $\lim_{t \to \infty} D_{(1,1)} Q(v^{(\tau'_t)}, g) = D_{(1,1)} Q(\hat{v}, g) = a$, and $\lim_{t \to \infty} D_{(1,1)} Q(v^{(\tau'_t+1)}, b) = D_{(1,1)} Q(w_b(\hat{v}, g), b) = a$. However, from Lemma B.4.4 the directional derivative martingale must strictly increase after a bad shock if $D_{(1,1)} Q(\hat{v}, g) > 0$, which means $D_{(1,1)} Q(\hat{v}, g) < D_{(1,1)} Q(w_b(\hat{v}, g), b)$. This forms a contradiction.

In the case that $s_{\tau_t} = g$ for infinitely many $t$, we can use the same argument to show that $D_{(1,1)} Q(\hat{v}, g) = D_{(1,1)} Q(w_g(\hat{v}, g), g) = a$. It also contradicts Lemma B.4.4 because $D_{(1,1)} Q(\hat{v}, g) > D_{(1,1)} Q(w_g(\hat{v}, g), g)$ if $D_{(1,1)} Q(\hat{v}, g) > 0$. Therefore, we must have $\lim_{t \to \infty} D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0$. By Lemma B.6.2, paths with only finitely many good shocks have measure zero, so it must be that $\lim_{t \to \infty} D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0$ almost surely.

(b) We show that once contingent utilities enter $E_s$, they will never leave these sets.

Take any $(v, s) \in E_s$. Suppose that $w_g(v, s) \notin E_g$. By part (c) of Proposition 3.4.3, $Q(w_g(v, s), g) < Q(g)$. This implies $Q(v, s) < Q(s)$, by the objective function defining $Q(v, s)$. And by part (a) of Proposition 3.4.3, we must have $(v, s) \notin E_s$, which is a contradiction. Therefore, $w_g(v, s) \in E_g$. The same argument shows we must also have $w_b(v, s) \in E_b$.

(c) We show that $D_{(1,1)} Q(v^{(t)}, s_{t-1})$ converges to 0 in finite time almost surely.

Consider a path with the property that $D_{(1,1)} Q(v^{(t)}, s_{t-1}) > 0$ for any finite $t$ and good-good shock occurs infinitely many times. Take a subsequence
\{ (v^{(\gamma_t)}, s_{\gamma_t-1}) \}_{t=0}^{\infty} \) that has only good-good shocks, i.e. \( s^{\gamma_{t-1}} = s^{\gamma_t} = g \). Note that the sequence \( \{ (v^{(\gamma_t)}, s_{\gamma_t-1}) \}_{t=0}^{\infty} \) is in set \( H \). This is because by part (c) of Lemma B.6.1, after good shocks if directional derivative is strictly positive then the state induced by the optimal contract has to be in \( H \).

Since \((\bar{v}_s, s)\) are the only points in the closure of \( H \) that have zero directional derivative by Lemma B.6.1, \( \lim_{t \to \infty} D_{(1,1)} Q(v^{(\gamma_t)}, g) = 0 \) implies \( \lim_{t \to \infty} v^{(\gamma_t)} = \bar{v}_g \). This means there exists a large \( T \) and sufficiently small \( \varepsilon > 0 \) such that \( v_{gg}^{(\gamma_t)} < \varepsilon \) for \( t \geq T \). So \( (v^{(\gamma_T)}, s_{\gamma_T-1}) \in A_1 \). And since \( s_T = g \) by construction, we must have \( v^{(\gamma_{T+1})} \in E_g \) and \( D_{(1,1)} Q(v^{(\gamma_{T+1})}, g) = 0 \), a contradiction.

Since the path with only finitely many good-good shocks has measure zero by Lemma B.6.2, the argument above shows \( D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0 \) for some finite \( t \) almost surely. From part (b), we know if \( D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0 \), then \( D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0 \) for any \( t \geq T \). Therefore, \( D_{(1,1)} Q(v^{(t)}, s_{t-1}) \) converges to 0 in finite time almost surely.

(d) We show in a maximum rent contract that contingent utilities reach \( \bar{v}_g \) in finite time almost surely and cycle between \( \bar{v}_b, \bar{v}_g \).

Consider a path in the maximum rent contract along which \( D_{(1,1)} Q(v^{(t)}, s_{t-1}) \) converges to zero in finite time. This means on the path that we consider, there exists \( T = \min \{ t : D_{(1,1)} Q(v^{(t)}, s_{t-1}) = 0 \} \). From part (b) of Proposition 3.4.3, we know that \( v^{(T)} \in E_s \) where \( s = s_{T-1} \). And since the contingent utilities reach efficient sets only after good shocks, we must have \( s_{T-1} = g \) and \( v^{(T)} = \bar{v}_g \) in the maximum contract. Moreover, part (c) shows that the path that we are considering has measure one. Therefore, in the maximum rent contract, the contingent utilities must reach \( \bar{v}_g \) in finite time almost surely.

Since part (b) shows that \((v, s)\) never leave the efficient sets once reach there, we know \( v^{(t)} = \bar{v}_g \) if \( s_{t-1} = g \) and \( v^{(t)} = \bar{v}_b \) if \( s_{t-1} = b \) for \( t \geq T \).
B.7 Proofs from Section 3.7

In this section, we show the level of threshold contingent utilities and repayments of the mature firm.

For a given \( p_g \), we can define a cutoff level \( \phi(p_g) \) that satisfies: \( \phi(p_g)R'(k^\phi) = 1 \) and \( R(k^\phi) = \frac{\delta p_g R(k_g)}{1+\delta p_g} \). With this cutoff, we can partition the parameter space as: \( B_+ = \{p: \phi(p_g) < p_b < p_g\} \), \( B_0 = \{p : p_b = \phi(p_g)\} \), and \( B_- = \{p : p_b < \phi(p_g)\} \). The parameter space with \( p_b \geq \phi(p_g) \) characterizes the case of low persistence. And the parameter space with \( p_b \leq \phi(p_g) \) characterizes the case of high persistence. The first result shows that the high and low persistence cases are separated by an increasing boundary \( \phi(p_g) \).

**Lemma B.7.1.** (a) \( 0 < \phi(p_g) < p_g \), and \( \phi(p_g) \) is increasing in \( p_g \);

(b) \( p \in B_- \) if, and only if \( \delta p_g[R(\bar{k}) - R(\bar{k}_b)] < R(\bar{k}_b) \);

(c) \( p \in B_0 \) if, and only if \( \delta p_g[R(\bar{k}) - R(\bar{k}_b)] = R(\bar{k}_b) \);

(d) \( p \in B_+ \) if, and only if \( \delta p_g[R(\bar{k}) - R(\bar{k}_b)] > R(\bar{k}_b) \).

**Proof.** (a) Because \( \frac{\delta p_g}{1+\delta p_g} < 1 \), by the definition of \( \phi(.) \), \( R(k^\phi) < R(\bar{k}) \), and hence \( k^\phi < \bar{k} \). Concavity of \( R \) then implies \( R'(k^\phi) > R'(\bar{k}) \). Moreover, by the definition of \( \phi(.) \), we have \( \phi(p_g)R'(k^\phi) = p_gR'(\bar{k}) = 1 \). So \( R'(k^\phi) > R'(\bar{k}) \) implies \( \phi(p_g) < p_g \).

\[
R(k^\phi) = \frac{\delta p_g R(\bar{k}_g)}{1+\delta p_g}
\]

is increasing \( p_g \). So \( k^\phi \) is increasing in \( p_g \). Hence, \( \phi(p_g) = 1/R'(k^\phi) \) is increasing in \( p_g \).

(b) By the definition of \( \phi(.) \), we have \( \phi(p_g)R'(k^\phi) = p_bR'(\bar{k}_b) = 1 \). Then \( p \in B_- \) implies \( R'(k^\phi) > R'(\bar{k}_b) \). Concavity of \( R \) implies \( k^\phi < \bar{k} \). Moreover, by the definition of \( \phi(.) \), we have \( \frac{\delta p_g R(\bar{k}_g)}{1+\delta p_g} = R(k^\phi) < R(\bar{k}_b) \). Rearrange to obtain
that \( \delta p_g[R(\bar{k}_g) - R(\bar{k}_b)] < R(\bar{k}_b) \). If we know \( \delta p_g[R(\bar{k}_g) - R(\bar{k}_b)] < R(\bar{k}_b) \), then
\[
R(\bar{k}_b) > \frac{\delta p_g R(\bar{k}_g)}{1 + \delta p_g} = R(k^\phi) \text{ which implies } \bar{k}_b > k^\phi. \]
By concavity of \( R \), we know
\[
R'(k^\phi) > R'(\bar{k}_b). \]
So by definition of \( \phi(.) \), we have \( \phi(p_g) < p_b \), meaning \( p \in B_\sim \).

(c) Similar argument as in (b) shows the result.

(d) Similar argument as in (b) shows the result. \( \square \)

**Lemma B.7.2.** At \((\bar{v}_s, s)\), either (IC*) or (LL) for \( s = g \) or both must hold as equality.

**Proof.** Suppose not. Then (IC) and (LL) for \( g \) both hold as inequality at \((\bar{v}_s, s)\).
Since \( Q_b(\bar{v}, s) = Q_g(\bar{v}, s) = 0 \), we know \((\bar{v}^s, s) \in \text{cl}(H)^1 \). By continuity of policies, there exists \((\bar{v}, s) \in H \) such that (IC) and (LL) for \( g \) both hold as inequality. Complementary slackness then implies \( \mu_g(\bar{v}, s) = 0 \) and \( \lambda(\bar{v}, s) = 0 \). And from (B.33), we know \( \eta_g(\bar{v}, s) = Q_g(\bar{v}, s) = 0 \), contradicted with \((\bar{v}, s) \in H \). \( \square \)

**Lemma B.7.3.** (a) \( \bar{m}_b^s = 0 \); (b) \( \bar{v}_g^b = \bar{v}_b^b, \bar{v}_g^b < \bar{v}_g^g \).

**Proof.** (a) From Proposition (3.5.1), \( m_b(v, s) = 0 \). So by continuity \( \bar{m}_b^s = m_b(\bar{v}_s, s) = 0 \).

(b) The right hand side of (PK\(_b\)) at \((\bar{v}^s, s)\) is not contingent on \( s \) because \( \bar{m}_b^s = 0 \).

So we must have \( \bar{v}_b^b = \bar{v}_b^g \). Then we can rewrite (IC) at \((\bar{v}_s, s)\) as
\[
\bar{m}_g^s \leq \delta p_g(\bar{v}_g^g - \bar{v}_b^g). \quad (\text{B.39})
\]

Moreover, by (PK\(_g\)) for \((\bar{v}_g^g, g)\) and \((\bar{v}_g^g, b)\):
\[
\bar{v}_g^g - \bar{v}_b^b = R(\bar{k}_g) - R(\bar{k}_b) - (\bar{m}_g^g - \bar{m}_g^b) \quad (\text{B.40})
\]

\(^1\) Recall set \( H \) is defined in Section 4 of the Appendix as \( \{(v, s) \in V \times S : Q_b(v, s) > 0, Q_g(v, s) > 0\} \).
Suppose \( \tilde{\nu}_g^a \leq \tilde{\nu}_g^b \). Then (B.39) implies \( \tilde{m}_g^a, \tilde{m}_g^b \leq 0 \). Also, (B.40) implies \( \tilde{m}_g^a - \tilde{m}_g^b \geq R(\tilde{k}_g) - R(\tilde{k}_b) > 0 \). So we must have \( \tilde{m}_g^b < \tilde{m}_g^a \leq \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \leq 0 \). This means \( \tilde{m}_g^b < R(\tilde{k}_b) \) and \( \tilde{m}_g^a > \delta p_g (\tilde{v}_g^a - \tilde{v}_b^b) \), which contradicts with Lemma B.7.2.\(\square\)

**Lemma B.7.4.** In the maximum rent contract, the debt repayment satisfies:

(a) At state \((\tilde{v}_g, g)\): \( \tilde{m}_g^a = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) < R(\tilde{k}_g) \);

(b) At state \((\tilde{v}_b, b)\): if \( p \in B_- \), then \( \tilde{m}_g^b = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \);

(c) At state \((\tilde{v}_b, b)\): if \( p \in B_+ \), then \( \tilde{m}_g^b = R(\tilde{k}_b) \);

(d) At state \((\tilde{v}_b, b)\): if \( p \in B_- \), then \( \tilde{m}_g^b = R(\tilde{k}_b) = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \).

**Proof.** (a) We will show that \( \tilde{m}_g^a < R(\tilde{k}_g) \). Suppose \( \tilde{m}_g^a = R(\tilde{k}_g) \). From (B.40), \( \tilde{m}_g^b - R(\tilde{k}_b) = \tilde{v}_g^a - \tilde{v}_g^b > 0 \). This means \( \tilde{m}_g^b > R(\tilde{k}_b) \), which violates (LL) for \( b \), a contradiction. Then \( \tilde{m}_g^a < R(\tilde{k}_g) \) implies (IC) must hold as equality by Lemma B.7.1. Therefore, from (B.39), we know \( \tilde{m}_g^a = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \).

(b) We will show that when \( p \in B_- \), we have \( \tilde{m}_g^b < R(\tilde{k}_b) \). Suppose not. \( \tilde{m}_g^b = R(\tilde{k}_b) \). By (B.40), we know \( \tilde{v}_g^a - \tilde{v}_g^b = R(\tilde{k}_g) - \tilde{m}_g^a = R(\tilde{k}_g) - \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \). The last equality is from part (a) that \( \tilde{m}_g^a = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \). So \( \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) = \frac{\delta p_g R(\tilde{k}_b)}{1 + \delta p_g} < R(\tilde{k}_b) \). The last inequality is from Lemma B.7.1 when \( p \in B_- \). So \( \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) < R(\tilde{k}_b) = \tilde{m}_g^b \). However, comparing with (B.39), we know (IC) at \((\tilde{v}_b, b)\) is violated. Therefore, we must have \( \tilde{m}_g^b < R(\tilde{k}_b) \). By Lemma B.7.2, (IC) must hold as equality. So \( \tilde{m}_g^b = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \).

(c) We will show that when \( p \in B_+ \), we have \( \tilde{m}_g^b = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \). Suppose not. \( \tilde{m}_g^b = \delta p_g (\tilde{v}_{gg} - \tilde{v}_b) \). Then we know from part (a) that \( \tilde{m}_g^b = \tilde{m}_g^a \). And from (B.40), \( \tilde{v}_g^a - \tilde{v}_g^b = R(\tilde{k}_g) - R(\tilde{k}_b) \). So \( \tilde{m}_g^b = \delta p_g [R(\tilde{k}_g) - R(\tilde{k}_b)] > R(\tilde{k}_b) \). The
last inequality is from Lemma B.7.1 when \( p \in B_+ \). However, this means (LL) for \( g \) at \((\bar{v}_b, b)\) is violated. Therefore, we must have \( \bar{m}_b^g < \delta p_g (\bar{v}_{gg} - \bar{v}_{bg}) \), which means the (IC) holds as inequality. So (LL) for \( g \) must hold as equality by Lemma B.7.2, which means \( \bar{m}_b^g = R(\bar{k}_b) \).

(d) When \( p \in B_- \), the same procedure as in part (b) and (c) shows that \( \bar{m}_b^g = R(\bar{k}_b) \) and \( \bar{m}_g^b = \delta p_g (\bar{v}_{gg} - \bar{v}_{bg}) \) imply one another. Therefore, by Lemma B.7.2, they must both hold. \( \Box \)

**Proof of Theorem 3.** When the firm is mature, the contingent utility levels in a maximum rent contract are \( w_g (\bar{v}_s, s) = \bar{v}_g \) if a good shock occurs, and \( w_b (\bar{v}_s, s) = \bar{v}_b \) if a bad shock occurs. From (PK\(_b\)) and (PK\(_g\)) at \((\bar{v}_b, b)\) and \((\bar{v}_g, g)\) respectively, we have

\[
\begin{align*}
\bar{v}_b &= \delta [p_g \bar{v}_{bg} + (1 - p_g) \bar{v}_b] \quad \text{(B.41)} \\
\bar{v}_{bg} &= R(\bar{k}_b) - \bar{m}_b^g + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_b] \quad \text{(B.42)} \\
\bar{v}_{gg} &= R(\bar{k}_g) - \bar{m}_g^b + \delta [p_g \bar{v}_{gg} + (1 - p_g) \bar{v}_b] \quad \text{(B.43)}
\end{align*}
\]

where \( \bar{v}_{gb} = \bar{v}_{bb} = \bar{v}_b \), because \( \bar{m}_s^b = 0 \).

(a) When \( p_b \geq \phi(p_g) \), or equivalently \( p \in B_- \) or \( B_+ \), Lemma B.7.4 implies:

\[
\bar{m}_b^g = \bar{m}_g^b = \delta p_g (\bar{v}_{gg} - \bar{v}_{bg}) \quad \text{(B.44)}
\]

Combining (B.41) to (B.43), and (B.44), we obtain the solution:

\[
\begin{align*}
\bar{v}_b &= \frac{\delta p_b R(\bar{k}_b)}{(1 - \delta)(1 - \delta \Delta)} \\
\bar{v}_{bg} &= \frac{1 - \delta(1 - p_b)}{(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_b), \quad \bar{v}_{gg} = R(\bar{k}_g) + \frac{\delta(p_g - \delta \Delta)}{(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_b)
\end{align*}
\]

(b) When \( p_b \leq \phi(p_g) \), or equivalently \( p \in B_- \) or \( B_+ \), Lemma B.7.4 implies:

\[
\bar{m}_b^g = R(\bar{k}_b), \quad \bar{m}_g^b = \delta p_g (\bar{v}_{gg} - \bar{v}_{bg}) \quad \text{(B.45)}
\]
Comibine (B.41) to (B.43), and (B.45), we obtain the solution:

\[
\bar{v}_b = \frac{\delta^2 p_g p_g R(\bar{k}_g)}{(1 + \delta p_g)(1 - \delta)(1 - \delta \Delta)}
\]

\[
\bar{v}_{bg} = \frac{\delta p_g [1 - \delta (1 - p_b)]}{(1 + \delta p_g)(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_g)
\]

\[
\bar{v}_{gg} = R(\bar{k}_g) + \frac{\delta^2 p_g (p_g - \delta \Delta)}{(1 + \delta p_g)(1 - \delta)(1 - \delta \Delta)} R(\bar{k}_g)
\]

\[\square\]

B.8 Proofs from Section 3.8

Proof of Lemma 3.8.1: We simply solve \(S_{bg}, S_{gg}, \) and \(S_b\) from (3.4) to (3.6) and obtain \((EV_{bg})\) to \((EV_b)\).

Proof of Proposition 3.8.2. Note that \(K \geq S_{bg} > S_b\). The first inequality is because \(\bar{m}_{gg} \geq \bar{m}_{bg}\). And the second inequality is from \((EV_b)\). So the stock option is out of the money when either bad shock occurs today or bad-good shock occurs. And from \((EV_{gg})\), we get

\[
S_{gg} - K = [R(\bar{k}_g) - \bar{m}_{gg}] - [R(\bar{k}_b) - \bar{m}_{bg}] > 0 \tag{B.46}
\]

because \(R(\bar{k}_g) \geq \bar{m}_{gg}\) and \(R(\bar{k}_b) \geq \bar{m}_{bg}\) by the limited liability constraints. So the stock option is in the money when good-good shock occurs.

Now let us consider the total payment to the agent. When a bad shock occurs today, the firm has zero cash flow and the stock option is out of money. So the agent gets zero payment. When a good shock occurs today, depending on yesterday’s shock \(i\), the agent’s total payment is

\[
\lambda [R(\bar{k}_i) - \max(S_{ig} - K, 0)] + \max(S_{ig} - K, 0)
\]

\[\lambda R(\bar{k}_i) + (1 - \lambda) \max(S_{ig} - K, 0) \tag{B.47}
\]
We show in the following cases that (B.47) is equal to the agent’s total payment in the optimal contract: $R(\tilde{k}_i) - \bar{m}_{ig}$.

If $i = b$, then since the option is out of money, (B.47) reduces to $\lambda R(\tilde{k}_b)$ which equals $R(\tilde{k}_b) - \bar{m}_{bg}$. If $i = g$, then by (B.46), we can rewrite (B.47) as

$$\lambda R(\tilde{k}_g) + (1 - \lambda)[R(\tilde{k}_g) - \bar{m}_{gg} - (R(\tilde{k}_b) - \bar{m}_{bg})]$$

$$= R(\tilde{k}_g) + (1 - \lambda)[\bar{m}_{bg} - \bar{m}_{gg} - R(\tilde{k}_b)]$$

(B.48)

In the case of low persistence, we have $\bar{m}_{bg} = \bar{m}_{gg}$. So (B.48) reduces to $R(\tilde{k}_g) - (1 - \lambda) R(\tilde{k}_b) = R(\tilde{k}_g) - \bar{m}_{gg}$. In the case of high persistence, we have $R(\tilde{k}_b) = \bar{m}_{bg}$. So $\lambda = 0$ and (B.48) reduces to $R(\tilde{k}_g) - \bar{m}_{gg}$. Therefore, in all cases, the agent’s total payment is equivalent to that in the optimal contract.

Proof of Lemma 3.8.3. Suppose the time $t$ continuation values $v_b, v_g$ are in (3.7) and (3.8). If bad shock occurs at time $t$, then from (3.9) and (3.11) we know, the credit balance at time $t + 1$ will be $M_b = \bar{v}_b - w_{bb}$, or $M_b - R(k_b)$, if bad or good shock occurs. So the available credit at time $t + 1$ is $w_{bb}$, or $w_{bg}$. The latter is obtained by (3.12).

If good shock occurs at time $t$, then from (3.10) and (3.11), we know the credit balance at time $t + 1$ will be $M_g = \bar{v}_g - w_{gb}$, or $M_g - R(k_g)$ if bad or good shock occurs. So the available credit at time $t + 1$ is $w_{gb}$, or $w_{gg}$. Then the induction argument, the continuation values equal to the available credits in each period after $t + 1$.

Proof of Theorem 3.8.5. We only need to show that the agent has no incentive to draw down available credit to payout or misreport cash flow. First, at any time $t$, the agent will not draw down available credit to either pay out or consume. He at most get $C^L - M + R(k)$ or $\bar{v}_b - M$ depending on which shock occurs. But these contingent values are equal to the payoffs that the agent can obtain from waiting until the credit balance is fully repaid before issuing payout: $v_b$, or $v_g$.

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Second, the agent has no incentive to misreport cash flow as well. By misreporting at time $t$, the agent can divert $R(k)$. Since the account balance and credit limit will evolve according to (3.9), and (3.12), the available credit in period $t + 1$ will be $(w_{bb}, w_{bg})$. So the expected continuation value of misreporting is $R(k) + \delta[p_g w_{bg} + (1 - p_g) w_{bb}]$. And by the incentive compatible condition in the optimal contract, the agent must get no less by truthfully reporting cash flow.

B.9 Proofs from Section 3.9

In this section, we show the initial debt value is higher at the good state.

At the beginning of the contracting relation, the principal chooses $v \in V$ to maximize the expected debt value: $Q(v, s) - p_s v_g - (1 - p_s) v_b$. Let $\hat{v}^s$ be the optimal contingent utilities at initial stage, when the state is $s$. Let $D(\hat{v}^s, s) = Q(\hat{v}^s, s) - p_s \hat{v}_g^s - (1 - p_s) \hat{v}_b^s$.

Proof of Proposition 9.1. Let $(k, m_b, m_g, w_b, w_g)$ be the optimal policy at state $(\hat{v}^b, b)$. The optimal debt value at the bad state is:

$$D(\hat{v}^b, b) = Q(\hat{v}^b, b) - p_b \hat{v}_g^b - (1 - p_b) \hat{v}_b^b$$

$$= -k + p_b R(k) + \delta p_b [Q(w_g, g) - Q(w_b, b)]$$

$$+ \delta Q(w_b, b) - p_b (\hat{v}_g^b - \hat{v}_b^b) - \hat{v}_b^b$$

$$= -k + p_b R(k) + \delta p_b [Q(w_g, g) - Q(w_b, b)] - p_b (\hat{v}_g^b - \hat{v}_b^b)$$

$$+ \delta [Q(w_b, b) - p_b w_{bg} - (1 - p_b) w_{bb}] + m_b$$

$$\leq -k + p_b R(k) + \delta p_b [Q(w_g, g) - Q(w_b, b)] - p_b (\hat{v}_g^b - \hat{v}_b^b)$$

$$+ \delta [Q(w_b, b) - p_b w_{bg} - (1 - p_b) w_{bb}]$$

which implies

$$p_b \{ R(k) + \delta [Q(w_g, g) - Q(w_b, b)] - (\hat{v}_g^b - \hat{v}_b^b) \}$$

$$\geq D(\hat{v}^b, b) - \delta [Q(w_b, b) - p_b w_{bg} - (1 - p_b) w_{bb}] + k$$  \quad (B.49)
If $Q(w_b, b) - p_bw_{bg} - (1 - p_b)w_{bb} \leq 0$, then the right hand side of (B.49) is positive. If $Q(w_b, b) - p_bw_{bg} - (1 - p_b)w_{bb} > 0$, then because $\delta < 1$ and because $\hat{v}^b$ maximizes debt value when $s = b$, we still have the right hand side of (B.49) is positive. Hence,

$$R(k) + \delta[Q(w_g, g) - Q(w_b, b)] - (\hat{v}_g^b - \hat{v}_b^b) > 0 \quad (B.50)$$

Then we know

$$D(\hat{v}^g, g) - D(\hat{v}^b, b)$$

$$\geq Q(\hat{v}^b, g) - p_g\hat{v}_g^b - (1 - p_g)\hat{v}_b^b - [Q(\hat{v}^b, b) - p_b\hat{v}_b^b - (1 - p_b)\hat{v}_b^b]$$

$$= Q(\hat{v}^b, g) - Q(\hat{v}^b, b) - \Delta(\hat{v}_g^b - \hat{v}_b^b)$$

$$\geq \Delta\{R(k) + \delta[Q(w_g, g) - Q(w_b, b)] - (\hat{v}_g^b - \hat{v}_b^b)\} > 0$$

The first inequality is from the fact that $(k, m_b, m_g, w_b, w_g) \in \Gamma(\hat{v}^b, g)$, and the second inequality is from (B.50).

To see that investment is inefficient, notice that because $\hat{v}^s$ maximizes debt value, we have

$$Q_b(\hat{v}^s, s) = 1 - p_s, \quad Q_g(\hat{v}^s, s) = p_s \quad (B.51)$$

By (B.51), $1 = D_{(1,1)} Q(\hat{v}^s, s) = \Psi(\delta(p_s\hat{v}_g^s + (1 - p_s)\hat{v}_b^s), s)$. Suppose investment is efficient at the initial stage which implies $\lambda(\hat{v}^g, g) = 0$ by (FOC$k$). By adding up (FOC$w_{gb}$) and (FOC$w_{gg}$), we get $p_gD_{(1,1)} Q(w_g, g) = Q_g(\hat{v}^g, g) - \lambda(\hat{v}^g, g) = Q_g(\hat{v}^g, g) = p_g$, and hence $1 = D_{(1,1)} Q(w_g, g) = \Psi(\hat{v}_g^g, g)$. By the strict concavity of $\Psi$, we know $\hat{v}_g^g = \delta(p_g\hat{v}_g^g + (1 - p_g)\hat{v}_b^g) \geq \delta\hat{v}_g^g > \hat{v}_g^g$, a contradiction. \hfill \Box
Bibliography


Biography

Shiming Fu was born in Shijiazhuang, China on June 15, 1982. He graduated from Nankai University with a Bachelor of Economics in 2004. He also earned a master of economics from Arizona State University in 2009. He will graduate with a Doctor of Philosophy in Business Administration (Finance) from Duke University and join Simon School of Business at University of Rochester as an Assistant Professor of Finance in July 2015.