Triple Products of Eisenstein Series

by

Anil Venkatesh

Department of Mathematics
Duke University

Date: _______________________
Approved:

_________________________
Richard Hain, Supervisor

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Jayce Getz

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William Pardon

_________________________
Leslie Saper

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2015
Abstract

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Abstract

In this thesis, we construct a Massey triple product on the Deligne cohomology of the modular curve with coefficients in symmetric powers of the standard representation of $\text{SL}_2(\mathbb{Z})$. This result is obtained by constructing a Massey triple product on the extension group of variations of mixed Hodge structures over the modular curve, which induces the desired construction on Deligne cohomology. The result extends Brown’s construction [1] of the cup product on Deligne cohomology to a higher cohomological product.

Massey triple products on Deligne cohomology have been previously investigated by Deninger [3], who considered Deligne cohomology with trivial real coefficients. By working over the reals, Deninger was able to compute cohomology exclusively with differential forms. In this work, Deligne cohomology is studied over the rationals, which introduces an obstruction to applying Deninger’s results. The obstruction arises from the fact that the integration map from the de Rham complex to the Eilenberg-MacLane complex of $\text{SL}_2(\mathbb{Z})$ is not an algebra homomorphism. We compute the correction terms of the integration map as regularized iterated integrals of Eisenstein series, and show that these integrals arise in the cup product and Massey triple product on Deligne cohomology.
This dissertation is dedicated to my parents, Cecily and Santosh, and to my sisters, Saraswathi and Vidya.
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Finally, I thank my parents, Cecily and Santosh, and my sisters, Saraswathi and Vidya, for their love and support.
In this thesis, we construct Massey triple products of Eisenstein series. The products take place in the Deligne cohomology of the modular curve with coefficients in symmetric powers of the standard representation of $\text{SL}_2(\mathbb{Z})$. The Massey triple product of three elements of degree 1 is a subset of the elements of degree 2. If $G_{2k}$ is the Eisenstein series of weight $2k$, then the Massey triple product $\langle G_{2l}, G_{2m}, G_{2n} \rangle$ is represented by the modular symbol of a cusp form. In particular, $\langle G_4, G_4, G_8 \rangle$ and $\langle G_6, G_6, G_4 \rangle$ are represented by the modular discriminant $\Delta$.

We begin by recalling the definition of the Massey triple product. Let $A$ be a differential graded algebra with differential $d$, and let $H^\bullet(A)$ be the cohomology of $A$. Let $u, v, w \in A$ be closed elements of degree 1, and let $[u], [v], [w]$ denote their cohomology classes. Unlike the cup product it generalizes, the Massey triple product results in a (possibly empty) subset of $H^2(A)$ rather than a unique element. Specifically, $\langle [u], [v], [w] \rangle$ is non-empty if and only if there exist elements $s, t \in A$ such that $ds = uv$ and $dt = vw$. Provided such elements exist, the following definition holds:

$$\langle [u], [v], [w] \rangle = \{ [sw + ut] \mid ds = uv, dt = vw \}.$$
If $X$ is an affine (orbi) curve and $V$ is a polarized variation of Hodge structure over $X$, the Deligne cohomology (properly Deligne-Beilinson cohomology) $H_D^j(X, V)$ is an extension

$$0 \longrightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^{j-1}(X, V)) \longrightarrow H_D^j(X, V) \longrightarrow \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^j(X, V)) \longrightarrow 0 \tag{1.1}$$

of the Hodge classes in $H^j(X, V)$ by $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^{j-1}(X, V))$, the group of 1-extensions in the category MHS of mixed Hodge structures over $\mathbb{Q}$. In particular, we examine the case where $X = \mathcal{M}_{1,1}$, the moduli space of elliptic curves. Let $\mathbb{H}$ be the $\mathbb{Q}$-local system over $X$ whose fiber over the elliptic curve $E$ is $H^1(E; \mathbb{Q})$. This underlies a polarized $\mathbb{Q}$-variation of Hodge structure over $X$. In the case where $V = S^{2n}\mathbb{H}(r)$, the Eichler-Shimura isomorphism and the Manin-Drinfeld theorem imply that

$$H_D^1(X, V) = \begin{cases} \mathbb{Q}G_{2n+2} & r = 2n + 1 \\ 0 & r \neq 2n + 1 \end{cases}$$

where $G_{2n+2}$ is the Eisenstein series of weight $2n + 2$. In order to discuss the construction of $H_D^1(\mathcal{M}_{1,1}, V)$, it is necessary to consider the cohomology of $\text{SL}_2(\mathbb{Z})$. Accordingly, we first fix a presentation of $\text{SL}_2(\mathbb{Z})$ in the following proposition.

**Proposition 1.** Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The group $\text{SL}_2(\mathbb{Z})$ has presentation $\langle S, T \mid S^4 = 1, (ST)^6 = 1 \rangle$.

Let $\overline{X}$ be the compactification of $\mathcal{M}_{1,1}$ to the cusp at infinity, and let $\mathcal{H}$ be the canonical extension of $\mathbb{H} \otimes \mathcal{O}_X$ to $\overline{X}$. Chapter 2 contains a thorough construction of $\mathcal{H}$ and its pullback bundle $\mathcal{H}_D$ to the $q$-disk $D$. This has trivialization $\mathcal{O}_D a \oplus \mathcal{O}_D w$ where $a$ has Hodge weight 0 and $w$ has Hodge weight 1. Put $\mathcal{V}_n = S^{2n}\mathbb{H} \otimes \mathbb{Q}(2n+1)$, and let $\mathcal{V}_n = S^{2n}\mathcal{H} \otimes \mathbb{Q}(2n+1)$. Let $E^* (\overline{X} \log D; \mathcal{V}_n)$ denote the de Rham complex of holomorphic forms on $\overline{X}$ with at worst logarithmic singularities at the cusp and coefficients in the bundle $\mathcal{V}_n$. Let $C^*(\text{SL}_2(\mathbb{Z}), \mathcal{V}_n)$ denote the Eilenberg-MacLane
cochain complex of the group \(SL_2(\mathbb{Z})\), with coefficients in the fiber over \(\mathcal{H}\) of the variation of Hodge structure \(\mathbb{V}_n\).

For each cusp form \(g\) of weight \(2n+2\), let \(\psi_g\) be the element of the degree-1 part of the \(C^\infty\) log complex \(E^1(X, \log D; S^{2n}H)\) given by \([2\pi i]^{2n+1}g(\tau)w^{2n}d\tau\). The Eichler-Shimura isomorphism implies that the element \(\{\psi_g\} : \gamma \mapsto \int_0^\infty \psi_g\) is a cocycle for \(SL_2(\mathbb{Z})\). Evaluating \([\psi_g]\) on \(S \in SL_2(\mathbb{Z})\) corresponds to integrating \(\psi_g\) along the imaginary axis in \(\mathcal{H}\), which is a convergent integral because \(g(iy)\) vanishes like \(e^{-y}\) as \(y \to \infty\). The value \([\psi_g](S)\) is termed the modular symbol of the cusp form \(g\).

If \(g\) is not a cusp form, the analogous notion of modular symbol results in a divergent integral because \(g\) does not vanish at the cusp. Therefore, a method of regularization is required in order to extend the notion of modular symbol to non-cuspidal modular forms. Since every non-cuspidal modular form of weight \(2n+2\) decomposes into a sum of \(G_{2n+2}\) with a cuspidal form, it suffices to study the modular symbols of Eisenstein series. Let \(\psi_{2n+2} = (2\pi i)^{2n+1}G_{2n+2}(\tau)w^{2n}d\tau\), and let \(f_n\) denote the regularization of \([\psi_{2n+2}]\). In Chapter 3 we present two methods of regularization of \([\psi_{2n+2}]\) and prove that the methods coincide.

Elements of \(H^1_D(X, \mathbb{V}_n)\) are represented by cocycles

\[
\begin{bmatrix}
  u \\
  \psi \\
  z
\end{bmatrix}
\]

where

\[
\begin{align*}
  \psi & \in F^0W_{-1}E^1(X, \log D; \mathbb{V}_n) \\
  z & \in C^1(SL_2(\mathbb{Z}), \mathbb{V}_n) \\
  u & \in C^0(SL_2(\mathbb{Z}), \mathbb{V}_n \otimes \mathbb{C})
\end{align*}
\]

and \(\delta u = z - [\psi]\). In Chapters 3 and 4 we show that \(f_n\) has a rational representative \(z_n \in C^1(SL_2(\mathbb{Z}), \mathbb{V}_n)\), which implies that there exists a function \(u_n \in C^0(SL_2(\mathbb{Z}), \mathbb{V}_n)\) such that \(z_n - f_n = \delta u_n\). We also discuss why this condition is not satisfied for any choice of \(\psi\) arising from cusp forms. This allows us to represent all elements of
$H^1_D(X, \mathcal{V}_n)$ by scalar multiples of the triple

$$Z_n := \begin{bmatrix} u_n \\ \psi_{2n+2} \\ z_n \end{bmatrix}.$$  

In Chapter 6, this characterization of Deligne cocycles is used to introduce the isomorphism

$$H^1_D(X, \mathcal{V}) \cong \text{Ext}^1_{\text{MHS}(X)}(\mathbb{Q}, \mathcal{V})$$

where $\text{Ext}^1_{\text{MHS}(X)}$ denotes the extension group in the category of admissible variations of mixed Hodge structure over $X$.

In order to construct Massey triple products on $H^1_D(X, \mathcal{V}_n)$, we must first study $Z_m \cup Z_n$. This cup product is an element of $H^2_D(X, \mathcal{V}_m \otimes \mathcal{V}_n)$, which is characterized by Equation (1.1) in the following way. The fact that $\text{SL}_2(\mathbb{Z})$ is virtually free implies that $H^2(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$ vanishes; consequently, Equation (1.1) implies that

$$H^2_D(X, \mathcal{V}_m \otimes \mathcal{V}_n) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^1(D_X, \mathcal{V}_m \otimes \mathcal{V}_n)).$$

Therefore, the cup product $Z_m \cup Z_n$ is represented by an element of $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^1(D_X, \mathcal{V}_m \otimes \mathcal{V}_n))$. For any $\mathbb{Q}$-mixed Hodge structure $A$, the group of 1-extensions $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, A)$ has the structure of a generalized torus [17]:

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, A) \cong W_0A \otimes \mathbb{C}/(F^0W_0A + W_0A).$$

The following obstruction arises when computing cup products in $H^1_D(X, \mathcal{V}_n)$. Since $\psi_{2n+2}$ is a holomorphic 1-form, the product $\psi_{2n+2} \wedge \psi_{2n+2}$ vanishes. However, the cup product $f_m \cup f_n$ of the corresponding group cocycles is nonzero. The correction term is a group cochain $f_{mn}$ whose value on $\gamma \in \text{SL}_2(\mathbb{Z})$ is the iterated integral $\int_0^{\gamma 0} \psi_{2m+2} \psi_{2n+2}$. That is, $\delta f_{mn} + f_m \cup f_n = 0$. In Chapter 4, we construct a rational cochain $z_{mn} \in C^1(\text{SL}_2(\mathbb{Z}), \mathcal{V}_m \otimes \mathcal{V}_n)$ that has the analogous property: $\delta z_{mn} + z_m \cup z_n = 0$. These quantities arise in the cup product on Deligne cohomology in the following way.
Proposition 2. The cup product $Z_m \cup Z_n$ is represented by $z_{mn} - f_{mn} + (u_m \cup z_n) - (f_m \cup u_n)$ in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^1(X, \mathbb{V}_m \otimes \mathbb{V}_n))$.

Brown computed all twice-iterated integrals of Eisenstein series in [1], which by Proposition 2 effectively computes the cup product on Deligne cohomology. The main result of this thesis is an extension of Proposition 2 to the Massey triple product $\langle Z_l, Z_m, Z_n \rangle$ of three Deligne cocycles. On the level of group cohomology, the Massey triple product $\langle f_l, f_m, f_n \rangle$ is well-defined because $H^2(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) = 0$ and so $f_l \cup f_m = 0$ and $f_m \cup f_n = 0$. However, $\langle f_l, f_m, f_n \rangle$ is nonzero, which creates an analogous obstruction as in the case of the cup product. Let $f_{lmn}$ be the group cochain whose value on $\gamma \in \text{SL}_2(\mathbb{Z})$ is $\int_0^\gamma \psi_{2l+2(2m+2(2n+2)}$. In Chapter 4, we use the coproduct property of iterated integrals to show that $\delta f_{lmn} + \langle f_l, f_m, f_n \rangle = 0$. We also construct a rational cochain $z_{lmn}$ with the analogous property: $\delta z_{lmn} + \langle z_l, z_m, z_n \rangle = 0$. These quantities arise in the Massey triple product on $H^1(X, \mathbb{V})$, along with specific functions $u_{lm}$ and $u_{mn}$ constructed in Chapter 6, which are higher analogues of $u_l, u_m, u_n$.

Theorem 3. Suppose $Z_l \cup Z_m = 0$ and $Z_m \cup Z_n = 0$. The Massey triple product $\langle Z_l, Z_m, Z_n \rangle$ is represented by

$$z_{lmn} - f_{lmn} + (u_{lm} \cup z_n) - (f_l \cup u_{mn}) + (u_l \cup z_{mn}) - (f_{lm} \cup u_n)$$

in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, \mathbb{V}_l \otimes \mathbb{V}_m \otimes \mathbb{V}_n)$. It is defined up to an element of

$$Z_l \cup H_D^1(X, \mathbb{V}_m \otimes \mathbb{V}_n) + H_D^1(X, \mathbb{V}_l \otimes \mathbb{V}_m) \cup Z_n.$$ 

This result is obtained by studying Massey triple products on Yoneda extensions and leveraging the isomorphism in Equation 1.2 to induce a Massey triple product on Deligne cohomology. It extends Brown’s construction of the cup product in [1] to a higher cohomological product. However, the value of the thrice-iterated integral
$f_{lmn}$ is not known at the present time. Brown’s computation of $f_{mn}$ uses the Rankin-Selberg method, but his approach does not obviously extend to the case of the thrice-iterated integral. Determining the value of $f_{lmn}$ would resolve the question of whether the Massey triple product vanishes on $H^1_D(X, \mathcal{V})$, and will be the subject of future work.

Massey triple products on Deligne cohomology have been previously investigated by Deninger [3], who considered Deligne cohomology with trivial real coefficients $\mathbb{R}(n)$. By working over the reals, Deninger was able to compute cohomology exclusively with differential forms. Consequently, the obstruction introduced by the fact that the integration map $E^\bullet(X) \to C^\bullet(X)$ is not a ring homomorphism does not arise. For this reason, Deninger’s work does not directly apply in the proof of Theorem 3.

This document is divided thematically into two parts. The first part, consisting of Chapters 2–4, is devoted to presenting the fundamental objects of study along with various technical results and computations relating to these objects. The second part, consisting of Chapters 5 and 6, draws on the material in the previous chapters in order to define the Deligne cohomology group $H^\bullet_D(X, \mathcal{V})$ with coefficients in a local system $\mathcal{V}$. Finally, the Deligne cohomology group is identified with the Yoneda extension group $\text{Ext}_{\text{MHS}(X)}(\mathbb{Q}, \mathcal{V})$ and this correspondence is employed to construct the cup product and Massey triple product on Deligne cohomology.

Chapter 2 reviews the construction of $\mathcal{M}_{1,1}$ as the orbifold quotient of the complex upper half-plane $\mathcal{H}$ by $\text{SL}_2(\mathbb{Z})$. The de Rham complex $E^\bullet(\mathcal{M}_{1,1})$ is introduced by means of working $\text{SL}_2$-equivariantly over $\mathcal{H}$. The cohomology of $\text{SL}_2(\mathbb{Z})$ is reviewed in the context of the Eichler-Shimura isomorphism, which gives a correspondence between cusp forms and (cuspidal) cocycles of $\text{SL}_2(\mathbb{Z})$. Finally, we discuss local systems over $\mathcal{M}_{1,1}$ and show that non-cuspidal period polynomials arise formally in the monodromy representations of certain local systems over $\mathcal{M}_{1,1}$.
In Chapter 3, we discuss in detail the construction of the Deligne canonical extension and use this construction to define a regularized transport function. This method of regularization provides a way to define Eisenstein period polynomials. In [1], Brown uses a different approach to regularization by passing to the universal cover $\mathcal{H}$ of $\mathcal{M}_{1,1}$ and regularizing Manin’s non-commutative modular symbol [13]. We demonstrate that these methods of regularization are equivalent for computing regularized Eisenstein period polynomials. Finally, the regularized modular symbol of $G_{2n+2}$ is computed explicitly using the regularization techniques discussed in the chapter.

In Chapter 4, the rational Eisenstein cocycle $z_n$ is specified by studying the system of linear equations induced by the coboundary map. The relationship between iterated integrals of Eisenstein series and cohomological products is explained, and rational cochains $z_{mn}$ and $z_{lmn}$ are computed. The main result of Chapter 4 is the determination of $z_{lmn}$ in terms of the Massey triple product $\langle z_l, z_m, z_n \rangle$.

In Chapter 5, the Deligne complex is defined and its cohomology investigated. Yoneda extension groups are introduced with a view toward $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,1})}(\mathbb{Q}, \mathbb{V})$. The product of Yoneda extensions is introduced since the Massey triple product on Deligne cohomology is to be constructed by analogy to products of Yoneda extensions.

In Chapter 6, we discuss the mechanics of extensions of variations of mixed Hodge structure and demonstrate the isomorphism in Equation (1.2). We then use this isomorphism and the Yoneda product construction from Chapter 5 to construct cohomological products for Deligne cohomology, resulting in Theorem 3.
2.1 The Moduli Space of Elliptic Curves

In this section, we discuss the construction of the moduli space of elliptic curves with one marked point, denoted $\mathcal{M}_{1,1}$, following Hain’s notes [8]. This is followed by a discussion of modular forms with a view toward modular symbols and the Eichler-Shimura isomorphism. Finally, we review the construction of modular symbols as sections of local systems over $\mathcal{M}_{1,1}$. As in the previous chapter, $\Gamma$ shall denote the modular group $\text{SL}_2(\mathbb{Z})$ unless otherwise specified.

An elliptic curve is a Riemann surface of genus 1. By [8], for each elliptic curve $E$ there is a lattice $\Lambda_E \subset \mathbb{C}$ such that $E \cong \mathbb{C}/\Lambda_E$. In order to construct the moduli space of elliptic curves, it is convenient to first construct the moduli space of frame lattices in $\mathbb{C}$.

**Definition 4.** A framed lattice $\Lambda \subset \mathbb{C}$ is a lattice in $\mathbb{C}$ together with an ordered basis $\mathbf{a}, \mathbf{b}$ such that $\Im(\mathbf{b}/\mathbf{a}) > 0$.

If $\Lambda$ is a framed lattice with basis $\mathbf{a}, \mathbf{b}$, associate to $\Lambda$ the element $\tau := \mathbf{b}/\mathbf{a}$ of the upper half plane $\mathcal{H}$. Moreover, if two lattices $\Lambda, \Lambda'$ correspond to the same $\tau \in \mathcal{H}$,
then there is a complex number $c$ such that $\Lambda' = c\Lambda$. In this case, the two lattices are said to be homothetic.

**Proposition 5.** The moduli space of framed lattices, up to homothety, is given by $\mathfrak{H}$.

*Proof.* It suffices to show that every element of $\mathfrak{H}$ corresponds to a unique homothety class of framed lattices, and every such class of lattices corresponds to a unique element of $\mathfrak{H}$. If $\tau \in \mathfrak{H}$, then $\tau$ corresponds to the homothety class of framed lattice with basis $1, \tau$. Conversely, if $\Lambda$ is a framed lattice with basis $a, b$, then $\Lambda$ corresponds to $b/a \in \mathfrak{H}$; since this correspondence respects homothety, the proposition holds. $\square$

**Definition 6.** A framed elliptic curve is an elliptic curve $(E, P)$ together with an ordered basis $a, b$ of $H_1(E; \mathbb{Z})$ with intersection number $a \cdot b = 1$.

**Proposition 7.** The space of homothety classes of framed lattices is isomorphic to the space of framed elliptic curves.

*Proof.* Let $\Lambda$ be a lattice with framing $a, b$. It follows that $\Lambda$ corresponds to the elliptic curve $(\mathbb{C}/\Lambda, 0)$. Note that $\Lambda$ is naturally isomorphic to $H_1(\mathbb{C}/\Lambda; \mathbb{Z})$ by associating to each $\lambda \in \Lambda$ the path $[0, 1] \to \mathbb{C}$ mapping $t \mapsto \lambda t$. The framing of $\Lambda$ therefore induces a framing of $H_1(\mathbb{C}/\Lambda)$.

$\square$

It is clear that any two framings $a, b$ and $a', b'$ of a given lattice $\Lambda$ are associated by an element of $\text{GL}_2(\mathbb{Z})$. In fact, the linear transformation taking $(a \ b)$ to $(a' \ b')$ has determinant 1 by the condition that $a \cdot b = 1 = a' \cdot b'$. Therefore, the space of framings of any fixed lattice is parametrized by $\text{SL}_2(\mathbb{Z})$. Specify the $\text{SL}_2$-action as follows.

**Definition 8.** If $(\Lambda; a, b)$ is a framed lattice and $\gamma \in \text{SL}_2(\mathbb{Z})$ has matrix representation $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, then $\gamma$ acts on framings by $\gamma(a) := (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) (\begin{smallmatrix} a \\ b \end{smallmatrix}) = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) (\begin{smallmatrix} a \\ b + da \end{smallmatrix})$. 

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Proposition 9. The moduli space of lattices in \( \mathbb{C} \), up to homothety, is given by the quotient of \( \mathfrak{H} \) by the left-action of \( \text{SL}_2(\mathbb{Z}) \) by fractional linear transformation.

Proof. Let \((\Lambda; a, b)\) be a framed lattice corresponding to \( \tau \in \mathfrak{H} \). That is, suppose \( b/a = \tau \). If \( \gamma \in \text{SL}_2(\mathbb{Z}) \) has matrix representation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then \( \gamma \cdot (\Lambda; a, b) = (\Lambda; cb + da, ab + ba) \). Dividing by \( a \), observe that \( \gamma \cdot (\Lambda; a, b) = (\Lambda; c\tau + d, a\tau + b) \) and therefore corresponds to \( \frac{a\tau + b}{c\tau + d} \in \mathfrak{H} \). 

Consequently, the moduli space of lattices modulo homothety is given by \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H} \). The following definition follows from the observation that the set of lattices modulo homothety is isomorphic to the set of elliptic curves with one marked point.

Definition 10. The moduli space of elliptic curves with one marked point is denoted \( \mathcal{M}_{1,1} \), and is given as a set by \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H} \).

Proposition 11 ([18]). The subset of \( \mathfrak{H} \) given by

\[
D := \{ \tau \in \mathfrak{H} : |\tau| \geq 1 \text{ and } -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2} \}
\]

is a fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathfrak{H} \).

Proof. See [18].

Let \( \mathfrak{H}^* \) denote the extended half-plane \( \mathfrak{H} \cup \{\infty\} \cup \mathbb{Q} \). The topology of \( \mathfrak{H}^* \) is specified as follows. For each \( x \in \mathbb{Q} \), the open neighborhoods of \( x \) are the half-disks in \( \mathfrak{H} \) centered at \( x \); the open neighborhoods of \( \infty \) are given by the one-parameter family \( \{ \tau \in \mathfrak{H} : \Im(\tau) > \epsilon \} \) as \( \epsilon \) varies in \( \mathbb{R}_{>0} \). The set of points in \( \mathfrak{H}^* - \mathfrak{H} \) are referred to as the cusps of \( \mathfrak{H}^* \). Note that the set of cusps is stable under the action of \( \text{SL}_2(\mathbb{Z}) \).

In fact, \( \text{SL}_2(\mathbb{Z}) \) acts transitively on the cusps of \( \mathfrak{H}^* \). The compactification \( \overline{\mathcal{M}}_{1,1} \) of \( \mathcal{M}_{1,1} \) is given set-wise by \( \mathcal{M}_{1,1} \cup \{\infty\} = \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}^* \). As an orbifold, \( \overline{\mathcal{M}}_{1,1} \) is obtained
by gluing $\mathcal{M}_{1,1}$ and $C_2 \setminus \mathbb{D}$ together along $C_2 \setminus \mathbb{D}^*$, where $C_2$ denotes the cyclic group of order 2. See [8] for details.

**Fact 12.** The group $\text{SL}_2(\mathbb{Z})$ has presentation $\langle S, T \mid S^4 = 1, (ST)^6 = 1 \rangle$. By convention, put $S := (0 1^{-1} 1)$, $T := (0 1 1)$, and $U := ST = (0 1^{-1})$.

The $\text{SL}_2$ action on $\mathfrak{H}$ is not fixed-point free; for example, the element $S$ fixes $i \in \mathfrak{H}$ and the element $U$ fixes $\rho := e^{2\pi i/3}$. Consequently, the quotient $\text{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ is not smooth at those points. Since de Rham cohomology is central to our investigations, this property makes $\text{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ an unsuitable choice for the moduli space of elliptic curves. In order to resolve the problems arising from the lack of a free group action, it is possible to simply work $\text{SL}_2$-equivariantly over $\mathfrak{H}$, the universal cover of $\mathcal{M}_{1,1}$. This approach is equivalent to studying the orbifold quotient $\text{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$. However, the fact that $\mathfrak{H}$ is simply connected is often convenient in the subsequent investigations, and so we generally prefer to adopt the former perspective. Consequently, the notation $E^*(\mathcal{M}_{1,1}, V)$ shall be understood as $E^*(\mathfrak{H}, V)^\Gamma$, the de Rham complex of $\Gamma$-invariant differential forms on $\mathfrak{H}$ with coefficients in $V$. Likewise, $H^*(\mathcal{M}_{1,1}; V)$ shall denote the cohomology of $E^*(\mathfrak{H}, V)^\Gamma$.

### 2.2 Modular forms and modular symbols

This section contains a brief overview of the space of modular forms and their connection to the cohomology of $\text{SL}_2(\mathbb{Z})$ via the Eichler-Shimura isomorphism.

**Definition 13.** A meromorphic function $g : \mathfrak{H} \to \mathbb{C}$ is termed weakly modular of weight $2n$ if for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, $g$ satisfies functional equation

$$g \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{2n} g(\tau).$$

Putting $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, Equation (2.1) implies that $g(\tau + 1) = g(\tau)$. Consequently, $g$
is periodic with period 1 and therefore has Fourier expansion

$$g(\tau) = \sum_{n=-\infty}^{\infty} a_g(n)e^{2\pi in\tau}.$$ 

Put $q := e^{2\pi i \tau}$. Since $g$ is weakly modular, it follows that $g$ is meromorphic in a neighborhood of $q = 0$. If $g$ extends to a meromorphic function at $q = 0$, it follows that $g$ has a Laurent series expansion in $q$ and we say that $g$ is a modular function of weight $2n$.

**Definition 14 ([18]).** A modular form of weight $2n$ is a modular function of weight $2n$ that is holomorphic on $\mathcal{H}$ and at $q = 0$. If such a function is zero at infinity, it is called a cusp form.

Let $M_{2n}$ denote the set of modular forms of weight $2n$ and let $S_{2n}$ denote the set of cusp forms of weight $2n$. Since the condition of being a modular form is preserved under addition and scalar multiplication, conclude that $M_{2n}$ is a $\mathbb{C}$-vector space. Furthermore, the product of two modular functions of weights $2m$ and $2n$ is a modular function of weight $2m + 2n$, so the space of modular forms has the structure of a graded algebra $M := \bigoplus_{n \geq 0} M_{2n}$.

**Proposition 15.** There are no non-zero modular forms of odd weight.

**Proof.** Let $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and note that $\gamma \tau = \tau$ for all $\tau \in \mathcal{H}$. For any modular form $g$ of weight $n$, Equation (2.1) implies $g(\gamma \tau) = (-1)^n g(\tau)$. If $n$ is odd, this condition is only satisfied if $g$ is identically zero. \(\square\)

**Proposition 16.** If $g$ is a modular form of weight 0, then $g$ is constant.

**Proof.** Equation (2.1) implies that $g(\gamma \tau) = g(\tau)$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$. Therefore, $g$ descends to a holomorphic function on $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H} = \mathcal{M}_{1,1}$. Furthermore, since $g$ extends holomorphically to $\mathcal{H} \cup \{\infty\}$, it follows that $g$ descends to a holomorphic function on $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H} = \mathcal{M}_{1,1}$. Therefore, $g$ is constant. \(\square\)
function on $\mathcal{M}_{1,1} = \mathcal{M}_{1,1} \cup \{\infty\}$. By construction, $\mathcal{M}_{1,1}$ is compact, so the maximum modulus principle implies that $g$ is constant. This requires $\mathcal{M}_{1,1}$ to have a Riemann surface structure, which is obtained below in Lemma 23.

**Corollary 17.** Let $g_1, g_2$ be modular forms of weight $2n$. If $g_1$ and $g_2$ vanish to the same order at every point in $\mathcal{H} \cup \{\infty\}$, then there exists $c \in \mathbb{C}$ such that $cg_1 = g_2$.

**Proof.** By hypothesis, the function $g_1/g_2$ is a modular form of weight 0. By Proposition 16, it is therefore constant.

In the notation of the previous section, let $\rho$ denote $e^{2\pi i/3}$. The dimension of $M_{2n}$ is controlled by the orders of vanishing of modular forms in the fundamental domain of $\mathcal{M}_{1,1}$, as is made precise by the following proposition.

**Proposition 18 ([18]).** Let $v_p(g)$ denote the order of vanishing of $g$ at $p \in \mathcal{H} \cup \{\infty\}$. For any modular form $g$ of weight $2n$, the following condition holds.

$$v_{\infty}(g) + \frac{1}{2}v_i(g) + \frac{1}{3}v_{\rho}(g) + \sum_{p \in \mathcal{M}_{1,1}} \sum_{1 \neq 1 \neq \rho} v_p(g) = \frac{n}{12}$$

**Proof.** See [18]. The proof proceeds by integrating $dg/g$ about the boundary of the fundamental domain of $\mathcal{M}_{1,1}$. Fractional contributions from the points $i, \rho$ are a consequence of the fact that the $\text{SL}_2$ action on $\mathcal{H}$ is not free; otherwise stated, the contour of integration only includes a one-half (respectively one-third) revolution about $i$ (respectively $\rho$).

**Corollary 19.** The dimension of $M_{2n}$ is given by

$$\dim M_{2n} = \begin{cases} \left\lceil \frac{n}{6} \right\rceil & \text{if } n \equiv 1 \pmod{6}, n \geq 0 \\ \left\lceil \frac{n}{6} \right\rceil + 1 & \text{if } n \not\equiv 1 \pmod{6}, n \geq 0 \end{cases}$$

**Proof.** See [18]. Proposition 18 implies for example that every modular form of weight 4 vanishes on $\rho$ and nowhere else in $\mathcal{M}_{1,1}$. In particular, this observation and
Proposition 16 imply \( \dim M_4 = 1 \). Similar methods establish the corollary for all weights.

Proposition 18 implies that weight 2 supports no non-zero modular forms, while weights 4 and 6 each support a single modular form up to scalar multiple. The following definition provides an explicit construction of these.

**Definition 20.** Adopt the following normalization for the Eisenstein series of weight \( 2n \geq 4 \):

\[
G_{2n}(\tau) = \frac{1}{2} \frac{(2n - 1)!}{(2\pi i)^{2n}} \sum_{\lambda \in \Lambda_{\tau} \setminus \{0\}} \frac{1}{\lambda^{2n}} = -\frac{B_{2n}}{4n} + \sum_{k \geq 1} \sigma_{2n-1}(k)q^k
\]

where \( \Lambda_{\tau} \) denotes the lattice \( \mathbb{Z}\tau \oplus \mathbb{Z} \), \( B_{2n} \) is the Bernoulli number of weight \( 2n \), and \( \sigma_n(k) = \sum_{d|k} d^n \).

**Corollary 21.** The space of modular forms \( M \) is generated as a \( \mathbb{C} \)-algebra by the Eisenstein series \( G_4 \) and \( G_6 \).

**Proof.** By counting dimensions given by Corollary 19.

Taking \( E_{2n} := -\frac{4n}{B_{2n}} G_{2n} \) such that \( E_{2n} \) has constant term 1, note that \( E_4^3 - E_6^2 \) is a cusp form of weight 12. In fact, Proposition 18 also implies that the lowest weight supporting a non-zero cusp form is weight 12, and that this weight supports exactly one cusp form up to scalar multiple. The cusp form of weight 12 whose \( q \)-expansion has linear coefficient 1 is \( \Delta(\tau) = \pi^{12}((\frac{1}{3} E_4)^3 - 27(\frac{8}{27} E_6)^2) \). It has \( q \)-expansion \( q - 24q^2 + 252q^3 - 1472q^4 + \cdots \). The coefficients of the \( q \)-expansion of \( \Delta \) are denoted by the Ramanujan tau function \( \tau(k) \).

**Definition 22.** Let \( j = \frac{1728.69^3 (2\pi i)^3 G_4}{\Delta} \) be a modular function of weight zero. This function is known as Klein’s \( j \)-invariant and is the unique modular function with a simple pole at \( \{ \infty \} \) and the values \( j(e^{2\pi i/3}) = 0 \) and \( j(i) = 1728 \).
Lemma 23. The $j$-invariant induces a Riemann surface structure on $\mathcal{M}_{1,1}$.

Proof. See [18].

2.2.1 Cusp forms as cocycles

Let $g$ be a cusp form of weight $2n + 2$. The differential form $\omega_g := g(\tau)(X - \tau Y)^{2n}d\tau$ is holomorphic on $\mathfrak{H}$ and takes values in polynomials over $\mathbb{C}$ in $X$ and $Y$ with homogeneous degree $2n$. With the notation of $S_{2n}^+(X, Y)$ for the $\mathbb{C}$-vector space of polynomials in $X$ and $Y$ with homogeneous degree $2n$, we can write $\omega_g \in \Omega^1(\mathfrak{H}) \otimes S_{2n}^+(X, Y)$. For notational purposes, put $v_\tau := (-Y X) (\tau)$ so that $\omega_g = g(\tau)v_\tau^{2n}d\tau$.

As explained in Section 2.1, the objective is to study $\mathcal{M}_{1,1}$ as an orbifold by working $\Gamma$-equivariantly over $\mathfrak{H}$. The differential form $\omega_g$ on $\mathfrak{H}$ can be used to study $\mathcal{M}_{1,1}$ because it is invariant under the action of $\Gamma$, as is demonstrated in [6] in the following way. In addition to the $\Gamma$-action $\gamma^\ast \omega_g = g(\gamma \tau)v_{\tau^2}(\gamma \tau)$, there is another $\Gamma$-action that is obtained by acting on the coefficients of $\omega_\tau$. Let $\gamma : (-Y X) \mapsto (-Y X) \gamma = (-aY + cX - bY + dX)$, and define $\gamma_\ast \omega_g = g(\tau)(\gamma X - \tau \gamma Y)^{2n-2}d\tau$.

Proposition 24 (Hain). The differential form $\omega_g$ is $\Gamma$-invariant in the sense that $\gamma^\ast \omega_g = \gamma_\ast \omega_g$.

Proof. First consider the $\Gamma$-actions on $v_\tau$. Acting on $\tau$, we have $\gamma^\ast v_\tau = (-Y X) (\gamma \tau) = (c\tau + d)(-Y X) \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)(-Y X) \gamma (\tau)$. Acting on $X$ and $Y$, we have $\gamma_\ast v_\tau = (-Y X) \gamma (\tau)$ Therefore, $\gamma^\ast v_\tau = (c\tau + d)^{-1}\gamma_\ast v_\tau$. Furthermore,

\[ d(\gamma \tau) = \frac{a(c\tau + d) - c(a\tau + b)}{(c\tau + d)^2}d\tau \]
\[ = (c\tau + d)^{-2}(ad - bc)d\tau \]
\[ = (c\tau + d)^{-2}d\tau \]

Consequently, $\gamma^\ast \omega_g = (c\tau + d)^{-2n-2}g(\gamma \tau)\gamma_\ast v_\tau^{2n}d\tau = \gamma_\ast \omega_g$ by the modularity of $g$. □
The Eichler-Shimura isomorphism asserts that cusp forms, in particular the associated differential forms $\omega_g$, can be used to study the homological structure of $SL_2(\mathbb{Z})$. In order to make this statement precise, the cohomology of $SL_2(\mathbb{Z})$ is defined in [2] as follows. Let $G$ be a group, $V$ be a $G$-module, and $C^k(G, V)$ denote the $G$-module of homomorphisms $f : G^{k+1} \to V$. Let $C^\bullet(G, V)$ denote the cochain complex

$$0 \longrightarrow C^0(G, V) \overset{\delta}{\longrightarrow} C^1(G, V) \overset{\delta}{\longrightarrow} C^2(G, V) \longrightarrow \cdots$$

together with a coboundary map $\delta : C^\bullet(G, V) \to C^{\bullet+1}(G, V)$ defined in the following way. Let $g_k \in G^{k+1}$ denote the $(k + 1)$-tuple $(g_0, g_1, \ldots, g_k)$. Geometrically, $g_k$ represents the $k$-simplex with vertices $g_0, g_1, \ldots, g_k$. The boundary $\partial g_k$ is given in the usual way by the alternating sum

$$\partial g_k = \sum_{j=0}^{k} (-1)^j (g_0, g_1, \ldots, \hat{g}_j, \ldots, g_k)$$

where $\hat{g}_j$ denotes an omitted term. The coboundary map $\delta$ is defined accordingly so that $\delta f(g_{k+1}) = f(\partial g_{k+1})$.

**Definition 25.** The cohomology of $G$ with coefficients in $V$ is defined as the cohomology of $C^\bullet(G, V)$.

Consider the $\Gamma$-cochain $f_g \in C^1(\Gamma, S_{2n}^2(X, Y))$ induced by integrating $\omega_g$ between two cusps in $\mathfrak{H}^*$:

$$f_g := (\gamma_0, \gamma_1) \mapsto \int_{\gamma_00}^{\gamma_10} \omega_g.$$  

Since $f_g$ is a $\Gamma$-module homomorphism, $f_g(\gamma_0, \gamma_1) = \gamma_0 f_g(1, \gamma_0^{-1} \gamma_1)$. By abuse of notation, let $f_g(\gamma) := f_g(1, \gamma)$. This notational convention is known as group cohomology with homogeneous coordinates.

**Proposition 26.** For each cusp form $g$ of weight $2n+2$, the 1-cochain $f_g$ is a cocycle for $\Gamma$.  

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Proof. First examine the coboundary map $\delta : C^1(G, V) \to C^2(G, V)$ for any given group $G$ and $G$-module $V$. For any $g_2 = (g_0, g_1, g_2) \in \Gamma^3$, the boundary of $g_2$ is given by $\partial g_2 = (g_1, g_2) - (g_0, g_2) + (g_0, g_1)$. Accordingly, for any 1-cochain $f \in C^1(G, V)$, $\delta f(g_2) = f(g_1, g_2) - f(g_0, g_2) + f(g_0, g_1)$. In homogeneous coordinates, $\delta f(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$. Therefore, the 1-cocycle condition in homogeneous coordinates is $f(g_1 g_2) = f(g_1) + g_1 f(g_2)$. In the case of $f \in C^1(\Gamma, S^n_\mathbb{C}(X,Y)$, we have

$$f(g \gamma_2) = \int_0^{\gamma_1 \gamma_2 - 0} \omega_g = \int_0^{\gamma_1 - 0} \omega_g + \int_{\gamma_1}^{\gamma_1 \gamma_2 - 0} \omega_g = f(g) + \gamma_1 f(g).$$

Note that $f_g(1) = 0$ since this corresponds to integrating over $\{0\}$, a set of measure zero. In discussing the cohomology of $\text{SL}_2(\mathbb{Z})$, it is important to note that for coefficients in any subring $R$ of $\mathbb{C}$ that contains $2^{-1}$ and $3^{-1}$, the cohomology group $H^j(\text{SL}_2(\mathbb{Z}) ; R)$ vanishes for all $j \geq 2$. In particular, we will often draw on the fact that $H^2(\text{SL}_2(\mathbb{Z}), S^n_\mathbb{Q}\mathbb{H})$ vanishes. This fact is confirmed as follows.

Definition 27. For each $N \in \mathbb{Z}_{\geq 1}$, let $r_N : \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ be the reduction map defined coefficient-wise. The kernel of $r_N$ is denoted by $\Gamma[N]$ and is referred to as the congruence subgroup of level $N$.

Since $\text{Ext}_R^1(F, M)$ vanishes when $F$ is free and $H_1(X)$ is a free abelian group, the universal coefficient theorem implies $H^2(X; M) \cong \text{Hom}(H_2(X), M)$. Moreover, since $X$ is given by a wedge of circles, $H_2(X)$ vanishes and consequently, $H^2(X; M)$ vanishes as well.

**Proposition 29.** If $R$ is any subring of $\mathbb{C}$ that inverts 2 and 3, then $H^2(\text{SL}_2(\mathbb{Z}); M) = 0$ for any $R$-module $M$.

**Proof.** Let $X := K(\Gamma[3], 1)$ and $Y := K(\Gamma, 1)$. Let $p : X \to Y$ be a covering map of index 3, and let $t_j : H_j(Y) \to H_j(X)$ be the transfer map taking $\sum a_k \sigma_k$ to $\sum a_k \sum g \sigma_k$, where $\sum g \sigma_k$ is the sum over all orbits of $\sigma_k$. The composition $p \circ t_j$ corresponds to multiplication by 3; hence, this map is invertible when the coefficient ring contains $3^{-1}$. If $p \circ t_j$ is invertible, it follows that $t_j$ must be injective. In that case, $H_2(X) = 0$ implies that $H_2(Y) = 0$ as well. The universal coefficient theorem states that

$$
0 \longrightarrow \text{Ext}_R^1(H_{n-1}(Y), M) \longrightarrow H^n(Y; M) \longrightarrow \text{Hom}(H_n(Y), M) \longrightarrow 0
$$

which in the case of $n = 2$ implies $\text{Ext}_R^1(H_1(Y), M) \cong H^2(Y; M)$. However, $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, so if $2^{-1}, 3^{-1} \in R$ conclude that $\text{Ext}_R^1(H_1(Y), M) = 0$, and consequently also that $H^2(Y; M) = 0$. 

**2.2.2 Period Polynomials**

As discussed in Section 2.1, $\text{SL}_2(\mathbb{Z})$ has presentation $\langle S, U \mid S^4 = 1, U^6 = 1 \rangle$. Note that since $S^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathfrak{H}$, some authors choose to work over $\text{PSL}_2(\mathbb{Z}) \cong \text{SL}_2(\mathbb{Z})/(S^2 \sim 1)$.

**Proposition 30.** For every $\gamma \in \Gamma$ and every cusp form $g$, the value $f_g(\gamma)$ is determined by the values of $f_g$ on $S$ and $T$. 

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Proof. Let \( \gamma \) be an arbitrary element of \( \Gamma \) and let \( f_g \) be a 1-cocycle for \( \Gamma \) corresponding to some cusp form \( g \). For some nonnegative integer \( r \), there exist integers \( j_1, k_1, j_2, k_2, \ldots, j_r, k_r \) such that \( \gamma = S^{j_1}T^{k_1}S^{j_2}T^{k_2} \cdots S^{j_r}T^{j_r} \). Denote by \( \text{len}(\gamma) \) the smallest nonnegative integer \( r \) for which \( \gamma \) can be expressed in this way. Since \( f_g \) vanishes on \( 1 \in \text{SL}_2(\mathbb{Z}) \), the proposition is trivially satisfied in length 0. Suppose by induction that the proposition holds for all \( \gamma \) with \( \text{len}(\gamma) < R \) and let \( \gamma' \in \text{SL}_2(\mathbb{Z}) \) be an arbitrary element of length \( R \). There exists \( \gamma \) with \( \text{len}(\gamma) = R - 1 \) and integers \( m, n \) such that \( \gamma' = S^j T^k \gamma \). By the cocycle condition, \( f_g(\gamma') = f_g(S^j T^k) + S^j T^k f_g(\gamma) \). Since \( \text{len}(S^m T^n) = 1 < R \), it follows that \( f_g(\gamma') \) can be expressed in terms of the value of \( f_g \) on elements of length less than \( R \).

The value of \( f_g(T) \) is given by the integral \( \int_0^1 \omega_g \) whose path of integration is the semi-circular geodesic in \( \mathfrak{H} \). Descending to the \( q \)-disk, this integral is equivalently expressed as \( \frac{1}{2\pi i} \int_\alpha g(q)(X - \frac{\log q}{2\pi i} Y)^{2n} dq \) where \( \alpha \) is a positively oriented simple closed curve about \( 0 \in \mathbb{D} \). By Cauchy’s integral formula, this integral is given by the residue of \( \omega_g \) at \( q = 0 \), which is zero since \( \omega_g \) is holomorphic on the \( q \)-disk. Consequently, the fact that \( g \) is a cusp form extends Proposition 30 to a stronger result.

**Proposition 31.** For every cusp form \( g \), the cocycle \( f_g \) is determined by its value on \( S \).

**Proof.** Since \( f_g(U) = f_g(S) + S f_g(T) \) and \( f_g(T) = 0 \), the proposition follows. \( \square \)

Let \( f_g(S) \) be denoted by \( r_g(X,Y) \). Since \( S \cdot 0 = i \infty \), this polynomial is defined by an integral along the imaginary axis: \( \int_0^{i \infty} g(\tau) v_\tau^{2n} d\tau \).

**Definition 32.** The **period polynomial** of a cusp form \( g \) of weight \( 2n + 2 \) is the element of \( S^{2n}_\mathbb{C}(X,Y) \) denoted by \( r_g(X,Y) \).

As explained in [12], the period polynomial \( r_g(X,Y) \) is subject to a system of linear equations that are induced by the cocycle condition on \( f_g \). Since \( \delta f_g = 0 \), it
follows that \( f_g(S) + Sf_g(S) = 0 \). Similarly, \( f_g(U) + Uf_g(U) + U^2f_g(U) = 0 \). These identities can also be seen by considering the corresponding domains of integration. Finally, Proposition 31 gives that \( f_g(U) = f_g(S) \), so the result of these identities is a linear system for \( f_g(S) = r_g(X,Y) \).

\[
(I + S)r_g(X,Y) = 0 \\
(I + U + U^2)r_g(X,Y) = 0
\] (2.2)

The period polynomial construction can be viewed as a map \( S_{2n+2} \rightarrow S_{2n}^2(X,Y) \), where \( S_{2n} \) denotes the space of cusp forms of weight \( 2n \). The image of this map is restricted by the relations in Equation (2.2) to the space \( W \) defined below.

**Definition 33.** Let \( W \subset S_{2n}^2(X,Y) \) be defined by \( W := \ker(1 + S) \cap \ker(1 + U + U^2) \). Let \( W^+ \) denote the subspace of \( W \) whose elements have even degree, and let \( W^- \) denote the subspace of \( W \) whose elements have odd degree.

**Theorem 34** (Eichler-Shimura). The map \( r^- : S_{2n+2} \rightarrow W^- (\mathbb{C}) \) is an isomorphism. The map \( r^+ : S_{2n+2} \rightarrow W^+ (\mathbb{C}) \) is an isomorphism onto \( W^+_0 \subset W^+ \) is a subspace of codimension 1, defined over \( \mathbb{Q} \), and not containing the element \( X^{2n} - Y^{2n} \).

**Example 35** ([11]). For certain real constants \( c^+, c^- \), the period polynomial of \( \Delta(\tau) \) is given by \( c^- r^-_\Delta(X,Y) + ic^+ r^+\Delta(X,Y) \) where

\[
\begin{align*}
 r^+_\Delta(X,Y) &= \frac{36}{691} (X^{10} - Y^{10}) + X^2 Y^2 (X^2 - Y^2)^3 \\
 r^-_\Delta(X,Y) &= 4X^9 Y - 25X^7 Y^3 + 42X^5 Y^5 - 25X^3 Y^7 + 4XY^9
\end{align*}
\]

**Observation 36.** For every cusp form \( g \) of weight \( 2n + 2 \), the even-degree terms of \( r_g(X,Y) \) are purely imaginary and the odd-degree terms of \( r_g(X,Y) \) are purely real.

**Proof.** The sum of even-degree terms \( r^+_g(X,Y) \) is obtained by

\[
\sum_{k=0}^{n} \binom{2n}{2k} X^{2n-2k} Y^{2k} \int_{0}^{\infty} g(\tau)(-\tau)^{2k} d\tau.
\]
Each term in this sum contains an even power of \( \tau \). Since \( \tau \) runs over purely imaginary values in the given domain of integration, this corresponds to a sum of even powers of \( i \), while \( d\tau \) contributes one copy of \( i \). As a result, \( r_g^+(X, Y) \) is a sum of purely imaginary terms. The proof that the odd-degree terms of \( r_g(X, Y) \) are purely real proceeds analogously.

The period polynomial \( r_\Delta(X, Y) \) from the corresponding linear system by way of illustrating the general case. The actions of \( S \) and \( U \) on \( S^{10}(X, Y) \) have matrix representations

\[
S_* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
U_* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -9 & 45 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 8 & -36 & 120 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -7 & 28 & -84 & 210 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -36 & 210 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 210 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -28 & -36 & 10 \\
0 & -1 & 2 & -3 & 4 & -5 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
\end{pmatrix}
\]

Solving the linear system in Equation (2.2) with these matrix representations gives the result in Example 35 modulo the term \( \frac{36}{691}(X^{10} - Y^{10}) \). This term is not detected by Equation (2.2) as a consequence of Theorem 34. In order to compute
this remaining term, Kohnen and Zagier use the additional structure of the Hecke action on $S_{12}$ [11].

Chapter 1 introduces motivations for extending the concept of period polynomial to non-cuspidal modular forms. The following sections review the topological framework in which non-cuspidal period polynomials arise through parallel transport on vector bundles over $\mathcal{M}_{1,1}$.

2.3 Local system interpretation

In this section, we review the topological framework presented in [6] from which non-cuspidal period polynomials naturally arise. Let $X = \mathcal{M}_{1,1}$ be the moduli space of elliptic curves, and let $\overline{X}$ be its completion, so $\overline{X} - \{x\} = X$. Since $X$ is not a smooth curve, we will generally work $\text{SL}_2$-equivariantly over its universal cover $\tilde{\mathcal{H}}$.

For each $\tau \in \tilde{\mathcal{H}}$, let $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ be a lattice in $\mathbb{C}$, and let $E_\tau$ denote the elliptic curve $\mathbb{C}/\Lambda_\tau$. Let $H_\tau = H^1(E_\tau; \mathbb{Z})$ and let $\mathbb{H}_{\tilde{\mathcal{H}}}$ be the local system over $\tilde{\mathcal{H}}$ with fiber $H_\tau$ over each $\tau \in \tilde{\mathcal{H}}$. Since $\Lambda_\tau$ is canonically isomorphic to $H_1(E_\tau; \mathbb{Z})$, there is a well-defined basis of $H_1(E_\tau; \mathbb{Z})$ corresponding to $1, \tau \in \Lambda_\tau$; denote this basis by $a(\tau), b(\tau)$.

Denote the dual basis of $H^1(E_\tau)$ by $\tilde{a}, \tilde{b}$. By Poincaré duality, identify $\tilde{b}$ with $a$ and $\tilde{a}$ with $-b$.

Since $\tilde{\mathcal{H}}$ is simply connected, $\mathbb{H}_{\tilde{\mathcal{H}}}$ is isomorphic to $H_0 \times \tilde{\mathcal{H}}$ for some fixed fiber $H_0$. The group $\text{SL}_2(\mathbb{Z})$ acts on $H_0 \times \tilde{\mathcal{H}}$ in the following way. Suppose $\gamma \in \text{SL}_2(\mathbb{Z})$ with matrix representation $(a \ b; c \ d)$. The group action is given by $\gamma : (z, \tau) \mapsto \left(\frac{1}{c\tau + d}z, \gamma \tau\right)$.

This can be seen by the observation that $\gamma$ takes $(\bar{1})$ to $(c\tau + d)^{-1}\gamma(\bar{1})$; accordingly, the $\gamma$ action on $z \in H_0$ takes $z$ to $(c\tau + d)^{-1}z$. The local system $\mathbb{H}$ over $X$ is given by taking the quotient of $\mathbb{H}_{\tilde{\mathcal{H}}}$ over $\tilde{\mathcal{H}}$ by $\text{SL}_2(\mathbb{Z})$.

We will use the Deligne canonical extension in order to extend from $X$ to $\overline{X}$. To accomplish this, it is useful to descend to a local system over $\mathbb{D}^*$ under the projection $\tau \mapsto e^{2\pi i \tau}$, and extend over $0 \in \mathbb{D}$. However, the choice of frame $a, -b$ is not well...
suited for this approach because $-b$ does not descend to a well defined section of the corresponding local system over $D^\ast$. While $a$ is invariant under the map $\tau \mapsto \tau + 1$, this map results in $-b \mapsto -b - a$.

In order to construct a frame for $H_{D^\ast}$, define $\omega_\tau$ as the section of $H_{D^\ast}$ whose image on each $H_\tau$ is the unique holomorphic differential taking value 1 on $a$, and consequently taking value $\tau$ on $b$. Therefore, $\omega_\tau = \dot{a} + \tau \dot{b} = \tau a - b$ under Poincaré duality. Let $\omega$ denote the section $\tau \mapsto \omega_\tau$.

**Proposition 37.** The section $\omega$ descends to a well defined section of $H_{D^\ast}$.

**Proof.** Since $\omega_{\tau + 1} = (\tau + 1) \cdot a(\tau + 1) - b(\tau + 1) = \tau a + a - b - a = \tau a - b$, conclude that $\omega$ descends to a well defined section of $H_{D^\ast}$. \qed

Finally, define the section $w := (2\pi i)\omega$ for normalization purposes. Since the intersection number $\langle a, w \rangle = 2\pi i\tau \langle a, a \rangle + 2\pi i\langle a, b \rangle = 2\pi i$ is nonzero, $a, w$ is a frame for $H_{D^\ast}$ over $D^\ast$. Since $\Gamma$ acts on the left of $H^1(E_\tau)$, it acts on the right of frames. This can be seen by the following calculation. The change of basis $\gamma : \begin{pmatrix} \tau \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a \tau + b \\ c \tau + d \end{pmatrix}$ corresponds to the change of frame $(a \quad -b) \mapsto (d a + c b \quad -(ba + ab)) = (a \quad -b) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore,

$$(a(\gamma \tau) \quad -b(\gamma \tau)) = (a(\tau) \quad -b(\tau)) \gamma^{-1},$$

or equivalently, $\gamma : (a(\tau) \quad -b(\tau)) \mapsto (a(\gamma \tau) \quad -b(\gamma \tau)) \gamma$.

Since $\omega = \tau a - b$, $\omega(\gamma \tau)$ is given by $\gamma \tau a(\gamma \tau) - b(\gamma \tau) = (a(\gamma \tau) \quad -b(\gamma \tau)) \begin{pmatrix} \gamma \tau \\ 1 \end{pmatrix}$.  


Using the previous result, this expression simplifies as

\[
\begin{pmatrix}
(a(\gamma \tau) & -b(\gamma \tau))
\end{pmatrix}
\begin{pmatrix}
\gamma \\
1
\end{pmatrix}
= \begin{pmatrix}
(a(\tau) & -b(\tau))
\end{pmatrix}
\begin{pmatrix}
\gamma^{-1} \\
1
\end{pmatrix}
= \frac{1}{c\tau + d}
\begin{pmatrix}
(a(\tau) & -b(\tau))
\end{pmatrix}
\begin{pmatrix}
\tau \\
1
\end{pmatrix}.
\]

Therefore, \( \Gamma \) acts on \( \omega \) by \( \gamma : \omega(\tau) \mapsto (c\tau + d)\omega(\gamma \tau) \). The same result holds for the action of \( \Gamma \) on \( w \). In order to compute the \( \Gamma \) action on the frame \( a, w \) of \( \mathbb{H}_D \), first note that \( b = \tau a - w/(2\pi i) \). Therefore,

\[
a(\gamma \tau) = da(\tau) + cb(\tau)
= da(\tau) + c\tau a(\tau) - \frac{c}{2\pi i}w(\tau)
= (c\tau + d)a(\tau) - \frac{c}{2\pi i}w(\tau)
\]

Therefore,

\[
\begin{pmatrix}
(a(\gamma \tau) & w(\gamma \tau))
\end{pmatrix}
= \begin{pmatrix}
a(\tau) & w(\tau)
\end{pmatrix}
\begin{pmatrix}
(c\tau + d) & 0 \\
-c & (c\tau + d)^{-1}
\end{pmatrix}.
\]

Equivalently,

\[
\gamma : \begin{pmatrix}
a(\tau) & w(\tau)
\end{pmatrix} \mapsto \begin{pmatrix}
a(\gamma \tau) & w(\gamma \tau)
\end{pmatrix}
\begin{pmatrix}
(c\tau + d)^{-1} & 0 \\
-c & (c\tau + d)
\end{pmatrix}.
\]

The factor of automorphy associated to the trivialization \( \mathcal{H}_D \cong (Ca \oplus Cw) \times \mathcal{S} \) is the matrix \( \begin{pmatrix}
(c\tau + d)^{-1} & 0 \\
-c & (c\tau + d)
\end{pmatrix} \), which shall be denoted by \( M_\gamma \). Since \( a \) and \( w \) are invariant under \( \tau \mapsto \tau + 1 \), the vector bundle \( \mathcal{H}_D \) descends to a vector bundle \( \mathcal{H}_{D^*} \cong (Ca \oplus Cw) \times \mathbb{D}^* \) over \( \mathbb{D}^* \). Let \( \mathcal{H}_D := \overline{\mathcal{H}_{D^*}} \) be the extension of \( \mathcal{H}_{D^*} \) to the fiber over \( 0 \in \mathbb{D} \). In Chapter 3 we will discuss the technical aspects of this extension.

Denote the connection on \( \mathcal{H}_D \) by \( \nabla_0 \). Since \( w = 2\pi i(\tau a - b) \), it follows that

\[
\nabla_0 (a \ w) = (a \ w)
\begin{pmatrix}
0 & 2\pi i \\
0 & 0
\end{pmatrix}
d\tau.
\]

On \( \mathcal{H}_D \), this connection is \( \nabla_0 = d + a\frac{\partial}{\partial q} \).
2.3.1 Period Polynomials from Extensions of Local Systems

In this section, period polynomials are constructed as solutions to parallel transport on an extension of the local systems discussed in the previous section. This construction is shown to be equivalent to the classical formulation of period polynomials as integrals of polynomial-valued differential forms on $\mathfrak{H}$. However, the period polynomials treated in this section are associated not to cusp forms, but to Eisenstein series. Since Eisenstein series do not vanish at the cusps of $\mathfrak{H}$, their Eichler-Shimura integrals are divergent. Chapter 3 introduces methods of regularization that will be used to resolve this obstacle.

Let $\mathcal{V} = S^{2n}\mathbb{H}_D$ be the local system over $\mathbb{D}$ given by taking symmetric powers of the local system $\mathbb{H}_D$ from Section 2.3. Let $\mathcal{V}$ be the corresponding vector bundle $S^{2n}\mathbb{H}_D$ with flat connection $\nabla_0 = d + a \frac{dq}{q}$. Let $e_0 = -a \frac{dq}{q}$, and let $\mathcal{E}$ be the extension of local systems given by the exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{E} \rightarrow \mathbb{Z} \rightarrow 0$$

The corresponding vector bundle is $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{V}$. Let $\psi_{2n+2} = (2\pi i)^{2n}G_{2n+2}(q)w^{2n}dq/q$. Let the connection $\nabla$ on $\mathcal{E}$ be given by $\nabla = \nabla_0 - \psi_{2n+2}$. Finally, introduce the notation $\Omega = e_0 \frac{dq}{q} + \psi$ so that $\nabla = d - \Omega$.

**Proposition 38.** The connection $\nabla = d - \Omega$ is well-defined and flat.

*Proof.* We first establish that $\psi_{2n+2}$ descends to an element of $H^0(\Omega^1_X/(\log D) \otimes \mathcal{V})$. This holds by the observation that Proposition 24 applies equally in the case of non-cuspidal modular forms. It remains to be shown that $\nabla$ is flat. Since $\nabla^2 = d\Omega + \Omega \wedge \Omega$, the condition $\nabla^2 = 0$ is satisfied because $\Omega$ is closed and the wedge of two holomorphic 1-forms is zero. 

The transport function on $\mathcal{E}$ is given by a series of iterated integrals of $\Omega$. We first fix notational conventions for iterated integration as follows.
Definition 39. Let $\omega_1, \omega_2, \ldots, \omega_r$ be smooth 1-forms on a manifold $X$. For any given smooth path $\alpha : [0, 1] \to X$, let $f_i(t) := \alpha^* \omega_i(t)$ be the pullback of $\omega_i$ along $\alpha$. The iterated integral $\int_\alpha \omega_1 \omega_2 \cdots \omega_r$ is defined by

$$
\int_\alpha \omega_1 \omega_2 \cdots \omega_r := \int_{0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1} f_1(t_1)f_2(t_2) \cdots f_r(t_r)dt_1dt_2 \cdots dt_r.
$$

The following is a standard fact from the theory of ordinary differential equations.

Proposition 40 ([9]). For any path $\alpha$ in $\mathbb{D}$, transport on $(\mathcal{E}, \nabla)$ along $\alpha$ is given by the linear transformation

$$
T(\alpha) = 1 + \int_\alpha \Omega + \int_\alpha \Omega \Omega + \int_\alpha \Omega \Omega \Omega + \cdots.
$$

Proof. See [9]. The proposition follows from an application of Picard iteration. $\square$

Proposition 41. The iterated integral $\int_{q_0}^q dq_1 \cdots dq_k$ evaluates to $\frac{1}{k!}(\log q/q_0)^k$.

Proof. Under the change of variables $q \mapsto \tau := \log(q)/(2\pi i)$ and corresponding definition of $\tau_0 := \log(q_0)/(2\pi i)$, the integral $\int_{q_0}^q dq_1 \cdots dq_k$ is equivalently expressed by

$$(2\pi i)^k \int_{\tau_0}^{\tau} d\tau \cdots d\tau = \frac{(2\pi i)^k}{k!}(\tau - \tau_0)^k = \frac{1}{k!}(\log q/q_0)^k.$$

$\square$

Since $\Omega$ is nilpotent in this case, the transport function in Proposition 40 is given by a finite sum. In particular, $\int_{q_0}^q dq_1 \cdots dq_k$ is given by

$$
\left( \begin{array}{c}
\begin{bmatrix}
s_0\ell_q \cdots s_0\ell_q \end{bmatrix} & 0 \\
0 & e_0\ell_q \cdots e_0\ell_q
\end{bmatrix} & e_0\ell_q \cdots e_0\ell_q
\end{array} \right).
$$

Therefore, the transport function along a given path $\alpha$ from $q_0$ to $q$ in $\mathbb{D}$ is given by
the finite sum

\[ T(\alpha) = \left( \sum_{k=0}^{2n} \int_{\alpha} \frac{1}{k} e_0 \frac{dq}{q} \cdots e_0 \frac{dq}{q} \psi_{2n+2} + 1 \right) \left( \sum_{k=1}^{2n} \int_{\alpha} \frac{0}{k} e_0 \frac{dq}{q} \cdots e_0 \frac{dq}{q} \right) \]

\[ = \left( \sum_{k=0}^{2n} \int_{\alpha} \frac{1}{k!} (\log q/q_0)^k e_0 \psi_{2n+2} \right) \left( \sum_{k=0}^{2n} \frac{1}{k!} (\log q/q_0)^k e_0^k \right) \]

\[ = \left( (2\pi i)^{2n} \int_{\alpha} G_{2n+2}(q)(w - (\log q/q_0)a)^{2n} \frac{dq}{q} (q/q_0)^{e_k} \right). \]

Note that the integral arising in \( T(\alpha) \) is the period polynomial of \( G_{2n+2} \) in the indeterminates \( w \) and \( a \). Since \( \lim_{q \to 0} G_{2n+2}(q) = -\frac{B_{2n+2}}{4n+4} \) is not equal to 0, this integral diverges when taken to the cusp. In the following chapter, we introduce methods the regularize integrals of this form and resolve the computation of period polynomials for Eisenstein series.
Chapter 2 demonstrates that periods of Eisenstein series naturally arise in the transport functions of certain vector bundles $\mathcal{V}$ over $\overline{\mathcal{M}}_{1,1}$. Two obstacles arise in computing these period polynomials: the transport function must be extended over the cusp $\infty \in \overline{\mathcal{M}}_{1,1}$, and the corresponding integrals must be regularized. The first obstacle is resolved in Section 3.1 by constructing the Deligne canonical extension of $\mathcal{V}$ over $\overline{\mathcal{M}}_{1,1}$. The second obstacle is resolved in Section 3.4 once the mechanism of regularization is established.

### 3.1 Deligne Canonical Extension

Let $\mathcal{V}$ be a vector bundle over the punctured disk $\mathbb{D}^*$. Let $\pi : \mathfrak{H} \to \mathbb{D}^*$ be the universal covering map $\pi : z \mapsto q := \exp(2\pi i z)$. Since $\pi^* \mathcal{V}$ is a flat bundle over a contractible space, all its fibers are canonically isomorphic. Identify them with one fixed fiber $V_0$, so that $\pi^* \mathcal{V}$ is isomorphic to $V_0 \times \mathfrak{H}$ with connection $\nabla = d$.

Let $h_0 \in \text{Aut}(V_0)$ be the monodromy operator of $\pi^* \mathcal{V}$, and let $N_0 = \log(h_0)/(2\pi i) \in \text{End}(V_0)$. Observe that $\mathcal{V} \to \mathbb{D}^*$ is isomorphic to $(\mathfrak{H} \times V_0) \to \mathfrak{H}$ modulo the equivalence relation $(z+1,v) \sim (z,h_0(v))$. For any $v \in V_0$, the section $\psi$
of \( \pi^*\mathcal{V} \) defined by \( \psi : z \mapsto (z, v \exp(-2\pi iz N_0)) \) is referred to as a Deligne section of \( \pi^*\mathcal{V} \).

**Proposition 42.** The section \( \psi \) of \( \pi^*\mathcal{V} \) descends to a section of \( \mathcal{V} \) over \( \mathbb{D}^* \).

**Proof.** It suffices to show that \( \psi(z+1) = \psi(z) \). This holds by the following reasoning.

\[
\psi(z + 1) = (z + 1, v \exp(-2\pi iz N_0 - 2\pi iN_0)) \\
= (z + 1, v \exp(-2\pi iz N_0 - \log(h_0))) \\
= (z + 1, v \exp(-2\pi iz N_0)h_0^{-1}) \\
= (z, v \exp(-2\pi iz N_0)h_0^{-1}h_0) \\
= (z, v \exp(-2\pi iz N_0)) = \psi(z)
\]

\( \Box \)

Choose a framing \( \{\psi_j\}_j \) of \( \pi^*\mathcal{V} \) consisting of Deligne sections. Since each Deligne section \( \psi \) has the property \( d\psi(z) = \psi(z)(-2\pi i)N_0 dz \), the connection on \( \pi^*\mathcal{V} \) with respect to the Deligne frame is \( d - 2\pi iN_0 dz \). This connection descends to \( d - 2\pi iN_0 dq/q \) on \( \mathcal{V} \). This determines an extension \( \mathcal{V} \) of \( \mathcal{V} \) to \( \mathbb{D} \) whose sections are of the form \( \sum_j f_j\psi_j \) where the \( f_j \) are holomorphic on \( \mathbb{D} \). The connection on \( \mathcal{V} \) has a simple pole at \( q = 0 \).

The canonical extension does not depend on the choice of holomorphic coordinate on \( \mathbb{D}^* \). Suppose \( q \) and \( w \) are two choices of holomorphic coordinate on \( \mathbb{D}^* \), satisfying \( q = wh(w) \) for some holomorphic function \( h \) with \( h(0) \neq 0 \). If \( v \) is a flat section, it satisfies \( \frac{dv}{dq} = v(q)N_0 dq/q \). Under the change of variables, this condition states

\[
wh(w)\frac{1}{h(w)+wh'(w)}\frac{dv}{dw} = v(w)N_0.
\]

Therefore, \( \frac{dv}{dw} = \frac{v}{w}N_0 + \frac{h'}{h}vN_0 \). Since \( \frac{h'(q)}{h(q)}N_0 \) is holomorphic at \( q = 0 \), the residue of the connection form is independent of the change of variables. Since the monodromy of \( (\mathcal{V}, \nabla) \) over 0 is determined by \( \text{Res}_0 \nabla \), conclude that the monodromy over 0 is independent of the change of variables as
well. Since two flat vector bundles over $\mathbb{D}$ are isomorphic if they have the same monodromy on every fiber, conclude that the Deligne extension is independent of the choice of holomorphic coordinate on $\mathbb{D}^*$.

### 3.1.1 Theory of regular singular points

The theory of regular singular points treated in [19] provides a tool for regularizing the transport function of $\overline{V}$. A regular singular point is defined in the following way.

**Definition 43** (Wasow). Let $A$ be matrix-valued holomorphic function on $\mathbb{D}$. The differential equation $qY' = A(q)Y$ is said to have a regular singular point at $q = 0$.

Wasow shows that subject to certain conditions on $A(0)$, the solutions to differential equations with regular singular points can be characterized explicitly. We use a special case of his result in this context. Note that for a square matrix $N$, the notation $q^N$ denotes $\exp((\log q)N)$ in the usual sense of matrix exponentiation.

**Proposition 44** (Wasow). Let $A$ be a matrix-valued holomorphic function on $\mathbb{D}$ with $N := A(0)$ nilpotent. There exists a matrix-valued holomorphic function $P(q)$ with $P(0) = \text{id}$ such that the differential equation $qY' = A(q)Y$ has a fundamental matrix solution of the form $P(q)q^N$.

The function $P(q)$ can be determined termwise from its power series $P(q) = \sum_{r \geq 0} P_r q^r$. Suppose $V(q) = P(q)q^N$ is a fundamental matrix solution for of $qY' = A(q)Y$. Since $\frac{d}{dq}q^N = \frac{N}{q}q^N$, writing $P$ and $A$ in terms of power series results in the following deduction.

$$q \frac{d}{dq}(P(q)q^N) = A(q)P(q)q^N$$

$$q \left( \sum_{r \geq 0} rP_rq^{r-1} + \left( \sum_{r \geq 0} P_rq^{r-1} \right) N \right) q^N = \left( \sum_{r \geq 0} \sum_{s=0}^{r} A_s P_{r-s} q^r \right) q^N$$

$$\left( \sum_{r \geq 0} (rP_r + P_rN)q^r \right) q^N = \left( \sum_{r \geq 0} \sum_{s=0}^{r} A_s P_{r-s} q^r \right) q^N$$
Collecting powers of \( q \), conclude that the power series coefficients of \( P(q) \) are subject to the constraint \( P_r(rI + N) = \sum_{s=0}^{r} A_s P_{r-s} \) for \( r \in \mathbb{Z}_{\geq 0} \) and initial condition \( P_0 = I \). Therefore, we deduce a recurrence relation

\[
P_r(rI + N) = \sum_{s=0}^{r} A_s P_{r-s}
\]

\[
P_r(rI + N) = N P_r + \sum_{s=1}^{r} A_s P_{r-s}
\]

\[
(rI - N) P_r + P_r N = \sum_{s=1}^{r} A_s P_{r-s}
\]

for the power series coefficients of \( P(q) \). This recurrence relation uniquely determines the \( P_r \).

**Example 45** (Wasow). Calculate explicitly, up to the term \( P_2 q^2 \) of the series \( P(q) = \sum_{r \geq 0} P_r q^r \), the solution to the differential equation

\[
q Y'' = \begin{pmatrix} 0 & 1 \\ -q^2 & 0 \end{pmatrix} Y.
\]

**Solution.** Since \( A(q) = \begin{pmatrix} 0 & 1 \\ -q^2 & 0 \end{pmatrix} \) has nilpotent \( N := A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), Proposition 44 applies and the differential equation has a fundamental matrix solution \( P(q)q^N \) for some holomorphic \( P \) with \( P(0) = \text{id} \). Let \( P(q) = \sum_{r \geq 0} P_r q^r \), and note that

\[
q^N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \log q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \log q \\ 0 & 1 \end{pmatrix}.
\]

Putting \( A(q) = \sum_{r \geq 0} A_r q^r \), note that \( A(q) = A_0 + A_2 q^2 \). Consequently, the recurrence relation for \( P_r \) is simply

\[
((2r)I - N) P_{2r} + P_{2r} N = A_2 P_{2r-2}, \quad P_0 = \text{id}
\]

and \( P_{2r-1} = 0 \) for all \( r \geq 1 \). In particular, \( (2I - N) P_2 + P_2 N = A_2 \), from which we
deduce
\[
\begin{pmatrix}
2 & -1 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix}
+ \begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
-1 & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}2p_{11} - p_{21} & 2p_{12} - p_{22} + p_{11} \\
2p_{21} & 2p_{22}
\end{pmatrix}
= \begin{pmatrix}0 & 0 \\
-1 & 0
\end{pmatrix}
\]

It follows that \( P_2 = \begin{pmatrix}-\frac{1}{4} & -\frac{1}{8} \\
-\frac{1}{2} & 0
\end{pmatrix} \), so the solution to the given differential equation is
\[
P(q)q^N = (P_0 + P_2q^2 + \cdots) \begin{pmatrix}1 & \log q \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}1 & \log q \\
0 & 1
\end{pmatrix} + P_2 \begin{pmatrix}1 & \log q \\
0 & 1
\end{pmatrix} q^2 + O(q^4)
= \begin{pmatrix}1 & \log q \\
0 & 1
\end{pmatrix} - \frac{1}{8} \begin{pmatrix}2 & 2\log q + 1 \\
4 & 4\log q
\end{pmatrix} q^2 + O(q^4)
\]

\[\square\]

3.1.2 Regularized transport

The next objective is to define a regularized transport function to the fiber over \( 0 \in \mathbb{D} \).
Let \( v(q) \) be a flat section. Choose a Deligne section \( \phi(q) \) that satisfies \( \phi(1) = v(1) \).
By definition, the regularized value of \( v(q) \) at \( q = 0 \) is \( \lim_{q \to 0} \phi(q) \). In order use this construction in practice, we turn to the theory of regular singular points with the following corollary.

**Corollary 46** (Wasow). Let \( \mathcal{V} \) be a vector bundle over \( \mathbb{D} \) with flat connection \( \nabla = d - A(q) dq / q \), where \( A(q) \) acts on the right of fibers. If \( A(q) \) is holomorphic at \( q = 0 \) and \( N := A(0) \) is nilpotent, then there exists a holomorphic function \( P(q) \) with \( P(0) = \text{id} \) such that every flat section of \( \mathcal{V} \) has the form \( v(q) = vq^N P(q) \).

**Proof.** Every flat section \( v \) of \( \mathcal{V} \) satisfies the differential equation \( \nabla v(q) = 0 \), so \( qv' = v(q) A(q) dq \). Taking transposes, Proposition 44 gives a fundamental matrix so-
lution $tP(q)q^N$. In keeping with the right-action of $A(q)$, this solution is equivalently expressed as $q^NP(q)$.

**Proposition 47.** For any invertible matrix $P$ and nilpotent matrix $N$ of the same dimensions, $\exp(P^{-1}NP) = P^{-1}\exp(N)P$.

**Proof.** Since $N$ is nilpotent, $\exp(N) = 1 + N + \frac{1}{2!}N^2 + \cdots$ represents a finite sum. Since $(P^{-1}NP)^k = P^{-1}N^kP$, it follows that $\exp(P^{-1}NP) = P^{-1}\exp(N)P$. 

Let $h_0 = \exp(2\pi i N)$ be the monodromy operator over $q = 0$, so $N = (\log h_0)/(2\pi i)$. The monodromy operator $h_q$ on the fiber over $q$ is computed as follows.

\[
v(q) = vq^NP(q) = v\exp((\log q)N)P(q) \rightarrow v\exp((\log q)N + 2\pi i N)P(q) = vq^Nh_0P(q) = (vq^N P(q))^1h_0P(q) = v(q)P(q)^{-1}h_0P(q)
\]

Therefore, $h_q = P(q)^{-1}h_0P(q)$. Put $N_q := (\log h_q)/(2\pi i)$. Since $h_q = P(q)^{-1}h_0P(q)$, it follows that $P(q)^{-1}\exp(2\pi i N)P(q) = \exp(2\pi i P(q)^{-1}NP(q))$. Taking logs, conclude that $N_q = P(q)^{-1}NP(q)$. Note that with $v(q) = vq^NP(q)$, we can use Proposition 47 to write $v(q) = vP(q)\exp((\log q)(P(q)^{-1}NP(q))) = vP(q)q^{N_q}$. This format is more conducive to the following investigation.

**Definition 48.** Let $v(q)$ be a flat section of a vector bundle $V$ over $\mathbb{D}$ whose connection $\nabla$ has a simple pole at $q = 0$. Choose a Deligne section $\phi(q)$ with the property that $\phi(1) = v(1)$. The regularized value of $v(q)$ at $q = 0$ is defined to be $\lim_{q \to 0} \phi(q)$.

Since $v(q)$ is a flat section, sections of the form $\phi(q) = v(q)q^{-N_q} = vP(q)$ are a Deligne frame. Furthermore, $\phi(1) = v(1)\exp(-(\log 1)N_1) = v(1)$. The regularized
value of \(v(q)\) at \(q = 0\) is computed in [10] as
\[
\lim_{q \to 0} \phi(q) = \lim_{q \to 0} v(q)q^{-N_q} = \lim_{q \to 0} vP(q) = v.
\]

This definition of regularization depends on the choice of holomorphic coordinate in the following sense.

**Proposition 49.** The regularized transport function depends only to first order on the choice of holomorphic coordinate.

**Proof.** Let \(q = wh(w)\) where \(h\) is holomorphic and \(h(0) = \lambda \neq 0\). Then \(\frac{\partial}{\partial w} = \lambda \frac{\partial}{\partial q}\).

The corresponding \(w\)-Deligne section is
\[
\phi(w) = v(q(w))w^{-N_w} = vP(q)q(w)^{N_q}w^{-N_w}
\]
\[
= vP(q) \exp((\log w)N_q + (\log h(w))N_q) \exp(- (\log w)N_w)
\]
\[
= vP(q) \exp((\log w)(N_q - N_w))h(w)^{N_q}.
\]

Since \(\lim_{w \to 0} q = 0\), it follows that \(\lim_{w \to 0} N_q = \lim_{w \to 0} N_w = N\). Therefore, \(\lim_{w \to 0} \phi(w) = vP(0)h(0)^N = v\lambda^N\). By comparison, \(\lim_{q \to 0} v(q)q^{N_q} = v\). \(\square\)

For the purpose of regularizing an arbitrary transport integral, this process is simplified by the following observation. By [19] the transport of a vector \(v(1) \in T^1X\) from 1 to \(q_0\) is given by evaluating the flat section \(v(q) = vq^NP(q)\) at \(q = q_0\). Since \(N\) is nilpotent, \(q^N\) is a polynomial in \(\log q\). The regularization of the flat section \(v(q)\) is therefore given by setting \(\log q\) equal to 0 in the expression for \(v(q)\), and then putting \(q = 0\), as this gives the correct result \(vP(0) = v\).

**Corollary 50.** Let \(v(q)\) be a flat section of a vector bundle \(V\) over \(\mathbb{D}\) whose connection \(\nabla\) has a simple pole at \(q = 0\). The regularized value of \(v(q)\) at \(q = 0\) is given by formally setting \(\log q\) to 0 in the expression for \(v(q)\), and then setting \(q = 0\).
3.2 Regularization on the universal cover

Brown proposes a different approach to regularization in [1]. For each modular form \( g \) of weight \( 2n+2 \), let \( g(\tau) \) denote the holomorphic differential form \((2\pi i)^{2n+1}g(\tau)(X-\tau Y)^{2n}d\tau \in \Omega^1(\mathfrak{H}) \otimes S^{2n}_C(X,Y)\). Let \( \mathcal{B} \) denote a basis of \( M := \oplus_{k \geq 2} M_{2k} \), the \( \mathbb{C} \)-vector space of modular forms graded by weight. Let \( \Omega(\tau) = \sum_{g \in \mathcal{B}} A_g g(\tau) \) for non-commuting indeterminates \( A_g \), and let define the transport function \( I(\tau_0; \tau_1) := 1 + \int_{\tau_0}^{\tau_1} \Omega + \int_{\tau_0}^{\tau_1} \Omega \Omega + \cdots \), which takes values in \( \mathbb{C}[A_g]_{g \in \mathcal{B}} \). This object is known as Manin’s non-commutative modular symbol [13].

**Proposition 51.** The transport function \( I(\tau_0; \tau_1) \) satisfies the following differential equation:

\[
dI(\tau_0; \tau_1) = I(\tau_0; \tau_1)\Omega(\tau_1) - \Omega(\tau_0)I(\tau_0; \tau_1).
\]

**Proof.** This is a standard fact that follows from Leibniz’s rule. \( \square \)

If \( \tau_0, \tau_1, \tau_2 \in \mathfrak{H} \), the transport function on the composition of paths \( I(\tau_0; \tau_2) \) is given by the product \( I(\tau_0; \tau_1)I(\tau_1; \tau_2) \). This is a consequence of a standard property of iterated integrals, which is reproduced below for clarity.

**Proposition 52.** Let \( \omega_1, \omega_2, \ldots, \omega_r \) be smooth 1-forms on a manifold \( X \). For any given smooth paths \( \alpha, \beta : [0,1] \rightarrow X \), with \( \alpha(1) = \beta(0) \), the following composition property holds:

\[
\int_{\alpha \beta} \omega_1 \omega_2 \cdots \omega_r = \sum_{k=0}^r \int_{\alpha} \omega_1 \cdots \omega_k \int_{\beta} \omega_{k+1} \cdots \omega_r
\]

with the notational convention that the empty integral evaluates to 1. [7]

As seen in the previously in Section 2.3, components of the formal expression for \( I(0; i\infty) \) arise when computing the transport function on a extensions of flat vector bundles such as \( \mathcal{E} := \mathcal{O} \oplus S^{2n}\mathcal{H} \). Brown’s construction of the regularization of \( I(\tau; i\infty) \) resolves the problem of divergence in the following way.
Let \( i_\infty : \mathfrak{H} \to \mathbb{C} \) be the natural inclusion map. View \( i_\infty \) as gluing a copy of \( \mathbb{C} \) to \( \mathfrak{H} \), forming the space \( \mathfrak{H} \cup i_\infty \mathbb{C} \). Note that \( (\mathfrak{H} \cup i_\infty \mathbb{C}, 0) \) is the universal covering space of \( (\mathcal{M}_{1,1} \cup \mathbb{C}^\times, \frac{\partial}{\partial \theta}) \) where \( \Phi : (\mathbb{C}, 0) \to (\mathcal{M}_{1,1}, \infty) \) is an analytic isomorphism with \( d\Phi : \mathbb{C} \to T^\infty \mathcal{M}_{1,1} \) is the identity. By identifying \( \mathfrak{H} \) with its image in \( \mathbb{C} \), we will drop all \( i_\infty \)'s from the notation. If \( g \) is a modular form of weight \( 2n + 2 \), let \( a_g(k) \) be the coefficient of \( q^k \) in the \( q \)-expansion of \( g \), and let \( g^\infty(\tau) = (2\pi i)^{2n+1}a_g(0)(X - \tau Y)^{2n}d\tau \). Similarly, let \( \Omega^\infty \) denote the sum \( \sum_{g \in \mathfrak{B}} A_g g^\infty(\tau) \), and \( I^\infty(\tau_0; \tau_1) := 1 + \int_{\tau_0}^{\tau_1} \Omega^\infty + \int_{\tau_0}^{\tau_1} \Omega^\infty \Omega^\infty + \cdots \). Note that the path composition property for \( I(\tau_0; \tau_1) \) holds for \( I^\infty(\tau_0; \tau_1) \) as well.

**Proposition 53** (Brown). The infinitesimal transport function \( I^\infty(\tau_0; \tau_1) \) satisfies the following differential equation:

\[
dI^\infty(\tau_0; \tau_1) = I(\tau_0; \tau_1)\Omega^\infty(\tau_1) - \Omega^\infty(\tau_0)I(\tau_0; \tau_1).
\]

**Proof.** The proposition holds by the same reasoning in the proof of Proposition 51.

**Definition 54.** The regularized value of \( I(\tau; \infty) \) is defined as

\[
I(\tau; \infty) := \lim_{x \to i\infty} \left( I(\tau; x)I^\infty(x; 0) \right).
\]

**3.2.1 Proof of finiteness of regularization**

For any \( x \in \mathfrak{H} \), define \( RI(\tau; x) := I(\tau; x)I^\infty(x; \tau) \). This notation allows us to decompose the regularized integral \( I(\tau; \infty) \) as the composition \( (\lim_{x \to i\infty} I(\tau; x)I^\infty(x; \tau))I^\infty(\tau; 0) = (\lim_{x \to i\infty} RI(\tau; x))I^\infty(\tau; 0) \).

**Proposition 55** (Brown). \( RI(\tau; x) \) is finite as \( x \to i\infty \).
Proof. Observe that
\[
dRI(\tau; x) = dI(\tau; x)I^\infty(x; \tau) + I(\tau; x)dI^\infty(x; \tau)
\]
\[
= (I(\tau; x)\Omega(x) - \Omega(\tau)I(\tau; x))I^\infty(x; \tau)
+ I(\tau; x)(I^\infty(x; \tau)\Omega^\infty(\tau) - \Omega^\infty(x)I^\infty(x; \tau))
\]
\[
= I(\tau; x)(\Omega(x) - \Omega^\infty(x))I^\infty(x; \tau)
- \Omega(\tau)I(\tau; x)I^\infty(x; \tau) + I(\tau; x)I^\infty(x; \tau)\Omega^\infty(\tau)
\]
In particular, \( \frac{d}{dx}RI(\tau; x) = I(\tau; x)(\Omega(x) - \Omega^\infty(x))I^\infty(x; \tau). \) Since \( I(\tau; x) \) and \( I^\infty(x; \tau) \) are polynomial in \( x \) and \( \Omega(x) - \Omega^\infty(x) \) is of order \( \exp(2\pi i x) \) as \( x \to i\infty \), conclude that \( RI(\tau; x) \) converges as \( x \to i\infty \), for every \( \tau \in \mathfrak{F}. \)

Definition 56. For each \( \tau \in \mathfrak{F} \), let \( RI(\tau) \) denote \( \lim_{x \to i\infty} RI(\tau; x) \). The regularized iterated integral \( I(\tau; \infty) \) is given by the composition \( RI(\tau)I^\infty(\tau; 0) \).

Corollary 57. The restriction of \( I(\tau; \infty) \) to single integrals asserts that for every modular form \( g \), the regularization of \( \int_{\tau}^{x} g \) is \( \int_{\tau}^{i\infty} g^{0} + \int_{\tau}^{0} g^{\infty} \).

Proof. By direct computation,
\[
RI(\tau) = \lim_{x \to i\infty} I(\tau; x)I^\infty(x; \tau)
\]
\[
= \lim_{x \to i\infty} (1 + \int_{\tau}^{x} \Omega + \int_{\tau}^{x} \Omega\Omega + \cdots)(1 + \int_{\tau}^{x} \Omega^\infty + \int_{\tau}^{x} \Omega^\infty\Omega^\infty + \cdots)
\]
\[
= 1 + \int_{\tau}^{i\infty} (\Omega - \Omega^\infty) + h.o.t.
\]
Restricting to terms involving the indeterminate \( A_g \), we have \( I(\tau; \infty) = (1 + \int_{\tau}^{i\infty} g^{0} + \cdots)(1 + \int_{\tau}^{0} g^{\infty} + \cdots). \) Further restriction to single-integral terms gives the desired result. \( \square \)
3.3 Comparison of regularization methods

Adopting the notation from Section 3.2, for each \( n \geq 2 \) let \( g_{2n+2}(\tau) \) denote the differential form \((2\pi i)^{2n+1}G_{2n+2}(\tau)(w - \tau a)^{2n}d\tau\). Under the change of variables \( \tau \mapsto q := \exp(2\pi i \tau) \), we have \( g_{2n+2}(q) = (2\pi i)^{2n+1}G_{2n+2}(q)(w - \log \frac{q}{2\pi i a})^{2n}d\frac{dq}{q} \). Correspondingly, let \( g_{2n+2}^{\infty}(q) = (2\pi i)^{2n}G_{2n+2}(0)(w - \log \frac{q}{2\pi i a})^{2n}d\frac{dq}{q} \) be the polar part of \( g_{2n+2} \), and let \( g_{2n+2}^{0} := g_{2n+2} - g_{2n+2}^{\infty} \). Let \( \alpha \) be a path in \( \mathbb{D} \) from \( q \) to \( x \). From Section 3.1.2, the transport function is

\[
T(q; x) = \left( x, 0, (x/q)^{e_0} \right).
\]

By Section 2.3.1, the value at \( x = 0 \) of the regularized transport function \( T^{reg}(q; x) \) can be computed by setting \( \log x \) to zero in the expression for \( T(q; x) \), and then putting \( x = 0 \). Since \( x^{e_0} = 1 + (\log x)e_0 + \frac{1}{2!}(\log x)^2e_0^2 + \cdots \), the term \( (x/q)^{e_0} \) in \( T(q; x) \) is converted by this process to \( (1/q)^{e_0} = q^{-e_0} \) in \( T^{reg}(q; x) \). The consequence of setting \( \log x \) to zero in the term \( \int_q^x g_{2n+2} \) is revealed by the following observation.

**Observation 58.** The integral \( \int_q^x g_{2n+2}^{\infty} \) is given by

\[
\int_q^x g_{2n+2}^{\infty} = (2\pi i)^{2n+1}G_{2n+2}(0) \sum_{k=0}^{2n} \frac{(-1)^k}{(2\pi i)^k} \binom{2n}{k} \frac{(\log x)^{k+1} - (\log q)^{k+1}}{k+1} a^k w^{2n}.
\]

**Proof.** For all \( k \geq 0 \), the integral \( \int_q^x (\log q)^k \frac{dq}{q} \) is given by \( \frac{(\log x)^{k+1} - (\log q)^{k+1}}{k+1} \). The statement follows from this and the fact that \( g_{2n+2}^{\infty}(q) = (2\pi i)^{2n+1}G_{2n+2}(0)(w - \frac{\log q}{2\pi i a})^{2n}d\frac{dq}{q} \). \( \square \)

Using Observation 58, the \( \log x \) terms of \( \int_q^x g_{2n+2} \) in \( T(q; x) \) can be isolated in the
following way.
\[
\int_q^x g_{2n+2} = \int_q^x g_{2n+2}^0 + \int_q^x g_{2n+2}^\infty
\]
\[
= \int_q^x g_{2n+2}^0 + (2\pi i)^{2n+1} G_{2n+2}(0) \sum_{k=0}^{2n} \frac{(-1)^k}{(2\pi i)^k} \binom{2n}{k} \int_q^x (\log q)^k a^k w^{2n-2} dq \frac{dq}{q}
\]
\[
= \int_q^x g_{2n+2}^0 + (2\pi i)^{2n+1} G_{2n+2}(0) \sum_{k=0}^{2n} \frac{(-1)^k}{(2\pi i)^k} \binom{2n}{k} (\log x)^{k+1} - (\log q)^{k+1} \kappa + 1 a^k w^{2n}
\]

Consequently, the corresponding term of \( T^{\text{reg}}(q; x) \) is given by
\[
\int_q^x g_{2n+2}^0 - G_{2n+2}(0) \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (\log q)^{k+1} - (\log q)^{k+1} \kappa + 1 a^k w^{2n-2}.
\]

**Proposition 59.** The methods of Section 3.1 give the regularized value of \( T(q; x) \) at \( x = 0 \) as
\[
T^{\text{reg}}(q; 0) = \left( \int_q^0 g_{2n+2}^0 + \int_q^\infty g_{2n+2}^\infty q^{-e_0} \right).
\]

**Proof.** The assertion follows from the observation that \( \int_q^\infty g_{2n+2} = (2\pi i)^{2n+1} G_{2n+2}(0) \sum_{k=0}^{2n} \frac{(-1)^k}{(2\pi i)^k} \binom{2n}{k} (\log q)^{k+1} - (\log q)^{k+1} \kappa + 1 a^k w^{2n-2}. \) 

**Corollary 60.** The methods of Section 3.1 coincide with this formulation of \( T^{\text{reg}}(q; 0) \).

**Proof.** The restriction of Brown’s regularization \( I(q; 0) \) to single integral terms is \( \int_q^0 g_{2n+2}^0 + \int_q^\infty g_{2n+2}^\infty \), which coincides with the corresponding term of \( T^{\text{reg}}(q; 0) \).

### 3.4 Computation of Single Eisenstein Integral

This section applies the regularization technique of the previous sections to the prescription of [14] in order to compute the regularized period polynomials of Eisenstein series. For a modular form \( g \) of weight \( 2n \) with Fourier coefficients \( a_g(n) \), let
\[ L(g, s) = \sum_{k \geq 1} a_g(k)k^{-s} \] be the associated \( L \)-function. Let \( \Lambda(f, s) \) be the completed \( L \)-function given by \( \Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s) \). If \( g \) is a cusp form, the numbers \( r_k(g) = \int_0^{i\infty} g(\tau)\tau^k d\tau \) for integers \( k \) ranging from 0 to \( 2n - 2 \) are called the periods of \( g \).

**Proposition 61** (Martin-Royer). The periods \( r_k(g) \) of \( g \) have the property that \( r_k(g) = ik^k \Lambda(g, k + 1) \).

**Proof.** Putting \( q = \exp(2\pi i\tau) \), write \( g(\tau) = \sum_{k \geq 1} a_g(k)q^k \). Under this change of variables, \( r_k(g) \) is given by

\[
 r_k(g) = \frac{1}{(2\pi i)^{k+1}} \int_0^1 \sum_{j \geq 1} a_g(j)q^j (\log q)^k dq \frac{dq}{q} \\
= \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{j \geq 1} a_g(j) j^{k+1} \\
= \frac{(-1)^{k+1} k!}{(2\pi i)^{k+1}} \frac{i^{k+1}}{i^{k+1}} L(g, k + 1) \\
= \frac{i^{k+1} k!}{(2\pi i)^{k+1}} L(g, k + 1)
\]

Since \( \Lambda(g, k + 1) = \frac{\Gamma(k+1)}{(2\pi)^{k+1}} L(g, k + 1) \), this completes the proof.

**Corollary 62.** If \( g \) is a cusp form of weight \( 2n \), the period polynomial \( r_g(X, Y) \) is given by

\[
\sum_{k=0}^{2n-2} (-1)^k \binom{2n-2}{k} i^{k+1} \Lambda(g, k + 1) X^{2n-2-k} Y^k.
\]

**Proof.** This follows from the definition of \( r_g(X, Y) \) as \( \int_0^{i\infty} g(\tau)(X - \tau Y)^{2n-2} d\tau \).}

Using the notation of Section 3.2, \( (2\pi i)^{2n-1} r_g(X, Y) = \int_{i\tau}^{i\infty} g(\tau) + \int_{i\leftarrow}^{i\tau} g(\tau) = (1 - S^*) \int_{i\tau}^{i\infty} g(\tau) \), since \( S \cdot \tau = -1/\tau \). Analogously, denote the (regularized) period polynomial of \( G_{2n} \) by \( r_{G_{2n}}(X, Y) \) such that \( (2\pi i)^{2n-1} r_{G_{2n}}(X, Y) \) is given by the
regularization of \((1 - S^*) \frac{r \Lambda_{1}}{a} q_{2n}(\tau)\). The following identity extends Corollary 62 to the case of \(r_{G_{2n}}(X, Y)\).

**Proposition 63.** For every modular form \(g\) of weight \(2n\), and every \(t_0 > 0\), the completed \(L\)-function of \(g\) is given by

\[
\Lambda(g, s) = \int_{t_0}^{\infty} (g(it) - \hat{g}(0)) t^{s-1} \, dt + \int_{0}^{t_0} \left( g(it) - \frac{\hat{g}(0)}{(it)^{2n}} \right) t^{s-1} \, dt \\
- \hat{g}(0) \left( \frac{t_0^s}{s} + (-1)^n \frac{t_0^{2n-s}}{2n-s} \right).
\]

**Proof.** See [14]. Note that differentiation with respect to \(t_0\) verifies that the identity is independent of the choice of \(t_0\). \(\square\)

**Corollary 64.** Corollary 62 extends to non-cuspidal modular forms. That is, for each \(n \geq 2\), the regularized period polynomial of the Eisenstein series \(G_{2n}\) is given by

\[
r_{G_{2n}}(X, Y) = \sum_{k=0}^{2n-2} (-1)^k \binom{2n-2}{k} i^{k+1} \Lambda(G_{2n}, k+1) X^{2n-2-k} Y^k.
\]

In order to explicitly evaluate \(r_{G_{2n}}(X, Y)\), it is useful to note that the \(L\)-function of \(G_{2n}\) can be expressed as \(L(G_{2n}, s) = \zeta(s) \zeta(s-2n+1)\), as can be seen from the following argument.

**Proposition 65.** The \(L\)-function of \(G_{2n}\) can be expressed as \(L(G_{2n}, s) = \zeta(s) \zeta(s-2n+1)\).

**Proof.** Adopting the normalization of Section 2.3.1, \(L(G_{2n}, s) = \sum_{k=1}^{\infty} \frac{\sigma_{2n-1}(k)}{k^s}\). By definition, \(\zeta(s) \zeta(s-2n+1)\) has series expansion

\[
\sum_{j=1}^{\infty} \frac{1}{j^s} \sum_{k=1}^{\infty} \frac{1}{k^{s-2n+1}} = \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{1}{(k-j)^s j^{s-2n+1}} \\
= \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{j^{2n-1}}{(k-j)(k-j)^s}.
\]

(3.1)
By observation, every term of Equation 3.1 is of the form $\frac{d^{2n-1}}{k}$ for some divisor $d$ of $k$, and each such pair $(d,k)$ arises exactly once in the series expansion. Consequently, $\zeta(s)\zeta(s-2n+1)$ can be rewritten as $\sum_{k=1}^{\infty} \sum_{d|k} \frac{\sigma_{2n-1}(k)}{k^s}$ and the proposition is confirmed. \hfill \Box

**Proposition 66.** For every modular form $g$ of weight $2n$, the following identity holds: $\Lambda(g, s) = i^{2n}\Lambda(g, 2n - s)$. Furthermore, the analytic continuation of $\Lambda(G_{2n}, s)$ at $s = 1$ is given by $-\frac{(2n-2)!}{2i} \frac{\zeta(2n-1)}{(2\pi i)^{2n-1}}$.

**Proof.** The functional equation for $\Lambda(g, s)$ is proven in [14]. Using this fact, note that the analytic continuation of $\Lambda(G_{2n}, s)$ at $s = 1$ is given by

\[
\Lambda(G_{2n}, 1) = i^{2n}\Lambda(G_{2n}, 2n - 1)
\]

\[
= i^{2n} \frac{1}{(2\pi)^{2n-1}} \Gamma(2n - 1) L(G_{2n}, 2n - 1)
\]

\[
= \frac{1}{(2\pi i)^{2n-1}} i^{-1}(2n - 2)! \zeta(2n - 1) \zeta(0)
\]

\[
= -\frac{(2n-2)!}{2i} \frac{\zeta(2n-1)}{(2\pi i)^{2n-1}}.
\]

\hfill \Box

**Corollary 67.** The regularized period polynomial $(2\pi i)^{2n-1} r_{G_{2n}}(X, Y)$ takes the form

\[
(2\pi i)^{2n-1} p_{2n}^0(X, Y) + \frac{(2n-2)!}{2} \zeta(2n - 1) (Y^{2n-2} - X^{2n-2}).
\]

**Proof.** By Corollary 64, the coefficients of $X^{2n-2}$ and $Y^{2n-2}$ in $r_{G_{2n}}(X, Y)$ are $i\Lambda(G_{2n}, 1)$ and $-i\Lambda(G_{2n}, 1)$, respectively, which can be evaluated using Proposition 66. \hfill \Box

The polynomial $p_{2n}^0(X, Y)$ is discussed further in Chapter 4. In particular, it is demonstrated that $p_{2n}^0(X, Y) \in S_{Q}^{2n-2}(X, Y)$. 
4.1 Single Integral Case

In order to characterize non-cuspidal cocycles for $\text{SL}_2(\mathbb{Z})$, it is sufficient to compute each cocycle on $S$ and $T$ only. This follows because the proof of Proposition 30 generalizes without revision from the cuspidal case. However, the fact that non-cuspidal cocycles do not vanish at the cusp implies that the corresponding cocycles for $\text{SL}_2(\mathbb{Z})$ do not vanish on $T$. The value of a non-cuspidal cocycle on $T$ is given by a residue computation, as explained in [1].

For notational brevity, let $f_n$ denote the Eichler cocycle for $\psi_{2n+2} := (2\pi i)^{2n+1} G_{2n+2}(b - \tau a)^{2n} d\tau$. Observe that $f_n(T)$ lies in $S^{2n}(\mathbb{H})_{\pi^{2n+1}\mathbb{Q}}$. By contrast, we have seen in the previous section that under a suitable regularization, $\int_0^{i\infty} G_{2n+2}(\tau) d\tau \in \zeta(2n + 1)\mathbb{Q}$ which is not known to lie in $\pi^{2n+1}\mathbb{Q}$. The following question therefore presents itself: can we construct a rational cocycle representative $z_n$ of $f_n$ that takes values in $S^{2n}(\mathbb{H})_{\pi^{2n+1}\mathbb{Q}}$? If such a $z$ exists, it must have the following familiar properties.

Since $z_n$ is a cocycle, it must satisfy $\delta z(S, S) = 0$. Therefore, we have $z_n(S) + \ldots$
\[ S z_n(S) = 0. \] Likewise, \( \delta z_n(U, U) = 0 \) and \( \delta z_n(U^2, U) = 0. \) Equivalently, \( z_n(U) + U z_n(U) - z_n(U^2) = 0 \) and \( z_n(U^2) + U^2 z_n(U) = 0. \) Combining these conditions, we have \( z_n(U) + U z_n(U) + U^2 z_n(U) = 0. \) In this way, the coboundary map induces the following linear system for \( z_n. \)

\[
(I + S) z_n(S) = 0 \\
(I + U + U^2) z_n(U) = 0
\]

In order to solve for \( z_n(S), \) we use the cocycle property of \( z_n \) to replace the second linear equation by one involving \( z_n(S). \) Since \( z_n(U) = z_n(ST) = z_n(S) + S z_n(T), \) it follows that:

\[
(I + S) z_n(S) = 0 \\
(I + U + U^2) z_n(S) = -(I + U + U^2) S z_n(T).
\]

The linear system determines \( z_n(S) \) up to a cuspidal cocycle. By comparing terms we establish that \( f_n(S) - z_n(S) = \frac{(2n)!}{2} \zeta(2n + 1)(a^{2n} - b^{2n}). \) In particular, putting \( u_n = \frac{(2n)!}{2} \zeta(2n + 1)b^{2n} \in H^0(SL_2(\mathbb{Z}), S^{2n} \mathbb{H}_C), \) we have \( f_n(S) - z_n(S) = \delta u_n. \)

### 4.2 Cup products

The objective of studying cup products and Massey products of Eisenstein series naturally leads us to consider iterated integrals of Eisenstein series. In particular, consider the map from \( SL_2(\mathbb{Z}) \) to \( S^{2m} \mathbb{H} \otimes S^{2n} \mathbb{H} \) given by the iterated integral

\[ \gamma \mapsto f_{mn}(\gamma) := \int_0^{\gamma} \psi_{2m+2} \psi_{2n+2}. \]

Unlike the classical case of a single integral, this map no longer has the cocycle property. Instead, it has a related property that proves useful to the investigation.

**Remark 68.** The 1-cochain \( f_{mn} \) is a primitive for the cup product \( -(f_m \cup f_n). \)
This fact follows from the coproduct property of iterated integrals given in Proposition 52. In particular, \( f_{mn}(\gamma_1 \gamma_2) \) has coproduct decomposition

\[
\psi_{2m+2} \psi_{2n+2} + \psi_{2m+2} \psi_{2n+2} + \psi_{2m+2} \psi_{2n+2}.
\]

That is,

\[
f_{mn}(\gamma_1 \gamma_2) = f_{mn}(\gamma_1) + \gamma_1 f_{mn}(\gamma_2) + f_m(\gamma_1) f_n(\gamma_2).\]

Note that \( f_m(\gamma_1) f_n(\gamma_2) = (f_m \cup f_n)(\gamma_1, \gamma_2) \).

**Remark 69.** The coboundary of \( f_{mn} \) is computed as

\[
\delta f_{mn}(\gamma_1, \gamma_2) = f_{mn}(\gamma_1) - f_{mn}(\gamma_1 \gamma_2) + \gamma_1 f_{mn}(\gamma_2).
\]

Comparing \( \delta f_{mn} \) with the coproduct formula, conclude \( \delta f_{mn}(\gamma_1, \gamma_2) \) equals

\[
-f_m(\gamma_1) f_n(\gamma_2) = -(f_m \cup f_n)(\gamma_1, \gamma_2).
\]

**Proposition 70.** For all \( m, n \in \mathbb{Z}_{\geq 2} \), \( f_{mn} \) is determined by its values on \( S \) and \( T \) and by the values of cocycles \( f_m \) and \( f_n \).

**Proof.** In the terminology of Proposition 30, suppose by induction that the proposition holds on all \( \gamma \in \Gamma \) with \( \text{len}(\gamma) < R \). Let \( \gamma' \in \Gamma \) be an arbitrary element of length \( R \) and let \( j, k \) be integers such that \( \gamma' = S^j T^k \gamma \). The coproduct property of \( f_{mn} \) implies that

\[
f_{mn}(\gamma') = f(S^j T^k) + S^j T^k f(\gamma) + (f_m \cup f_n)((S^j T^k, \gamma)
\]

which expresses \( f_{mn}(\gamma') \) in terms of \( f_{mn}(\gamma) \) and a cup product of Eichler-Shimura cocycles. Since \( (f_m \cup f_n)((S^j T^k), \gamma) = f_m(S^j T^k) \cdot S^j T^k f_n(\gamma) \), it follows that \( f_{mn}(\gamma') \) is expressed in terms of the values of \( f_{mn} \) on elements of length less than \( R \), and the cocycles \( f_m \) and \( f_n \). Since \( f_{mn}(1) = 0 \), the induction proof is complete.

The computation of \( f_{mn}(T) \) is given by analogy to the previous case by a residue computation in [1]. Since \( f_{mn}(T) \) lies in \( S^{2m+2n}(\mathbb{H}_{2m+2n+2} \mathbb{Q}) \), we can once again investigate whether there exists a rational cochain \( z_{mn} \) of \( f_{mn} \) that takes values in \( S^{2m+2n}(\mathbb{H}_{2m+2n+2} \mathbb{Q}) \) for all \( \gamma \in \text{SL}_2(\mathbb{Z}) \). If such a \( z_{mn} \) exists, it is subject to a linear

45
system induced by the coboundary operator, just as in the classical case.

\[(I + S)z_{mn}(S) = -(z_m \cup z_n)(S, S)\]

\[(I + U + U^2)z_{mn}(U) = -(z_m \cup z_n)(U, U) - (z_m \cup z_n)(U^2, U)\]

In order to solve for \(z_{mn}(S)\), we use the coproduct property of \(z_{mn}\) to replace \(z_{mn}(U)\) by an expression in terms of \(z_{mn}(S)\) and \(z_{mn}(T)\).

\[z_{mn}(U) = z_{mn}(ST)\]

\[= z_{mn}(S) + Sz_{mn}(T) + (z_m \cup z_n)(S, T)\]

This substitution results in the following linear system for \(z_{mn}(S)\):

\[(I + S)z_{mn}(S) = -(z_{2m} \cup z_n)(S, S)\]

\[(I + U + U^2)z_{mn}(S) = -(z_m \cup z_n)(U, U) - (z_m \cup z_n)(U^2, U)\]

\[- (I + U + U^2)(Sz_{mn}(T) + (z_m \cup z_n)(S, T))\].

In this way, \(z_{mn}(S)\) is determined up to a cuspidal cocycle.

### 4.3 Massey triple products

In order to extend the results of the previous section to triple-iterated integrals of modular forms, we first introduce the Massey triple product, which extends the notion of cup product. Let \(A^\bullet\) be a differential graded algebra with decomposition \(A^\bullet = \bigoplus_{k \geq 0} A^k\) and differential \(d : A^\bullet \to A^{\bullet+1}\).

**Definition 71.** For any \(u, v, w \in A^1\) with the property that \(0 = du = dv = dw\), the Massey triple product \(\langle u, v, w \rangle\) is defined as \(\{sw + ut \mid ds = uv, dt = vw\} \subseteq A^2\).

The Massey triple product descends to cohomology in the following way. For any choice of \(u, v, w \in A^1\) representing cohomology classes \([u], [v], [w] \in H^1(A^\bullet)\), the Massey triple product \(\langle [u], [v], [w] \rangle\) on cohomology is given by \(\{[sw + ut] \mid ds = uv, dt = vw\}\). Note that \(\langle u, v, w \rangle\) is empty unless \(uv\) and \(vw\) are exact in
$A^2$. In the case that $uv$ and $vw$ are exact, it is possible to choose $s, t \in A^1$ such that $ds = uv$ and $dt = vw$. However, these choices are unique only up to $\ker(d : A^1 \to A^2)$. Consequently, the elements of $\langle [u], [v], [w] \rangle$ comprise the quotient group $H^1(A^*)/( [u]H^1(A^*) + H^1(A^*)[w] )$.

**Corollary 72.** The Massey triple product $\langle [u], [v], [w] \rangle$ determines a unique element if $H^1(A^*) \otimes H^1(A^*) \to H^2(A^*)$ vanishes.

Therefore, the Massey triple product is uniquely determined on $H^1(\Gamma)$ when working over $\mathbb{Q}$. Let

$$f_{2l} := \gamma \mapsto \gamma \int_0^{\gamma - 0} \psi_{2l+2} \psi_{2m+2} \psi_{2n+2},$$

be a 1-cochain on $SL_2(\mathbb{Z})$ with values in $S^{2l} \otimes S^{2m} \otimes S^{2n}$. 

**Remark 73.** The 1-cochain $f_{lmn}$ is a primitive for the Massey triple product $-(f_{lm} \cup f_n) - (f_l \cup f_{mn})$.

As in the case of twice-iterated integrals, this fact follows from the coproduct property of iterated integrals. Specifically, $f_{lmn}(\gamma_1 \gamma_2)$ has coproduct decomposition

$$\int_0^{\gamma_1 - 0} \psi_{2l+2} \psi_{2m+2} \psi_{2n+2} + \int_0^{\gamma_1 - 0} \psi_{2l+2} \psi_{2m+2} \int_{\gamma_1 - 0}^{\gamma_2 - 0} \psi_{2n+2} + \int_0^{\gamma_1 - 0} \psi_{2l+2} \int_{\gamma_1 - 0}^{\gamma_2 - 0} \psi_{2m+2} \psi_{2n+2} + \int_{\gamma_1 - 0}^{\gamma_2 - 0} \psi_{2l+2} \psi_{2m+2} \psi_{2n+2}.$$ That is, $f_{lmn}(\gamma_1 \gamma_2) = f_{lmn}(\gamma_1) + (f_l \cup f_n)(\gamma_1, \gamma_2) + (f_l \cup f_{mn})(\gamma_1, \gamma_2) + \gamma_1 f_{lmn}(\gamma_2)$.

**Remark 74.** The coboundary of $f_{lmn}$ is given by $\delta f_{lmn}(\gamma_1, \gamma_2) = f_{lmn}(\gamma_1) - f_{lmn}(\gamma_1 \gamma_2) + \gamma_1 f_{lmn}(\gamma_2)$.

Comparing $\delta f_{lmn}$ with the coproduct formula, conclude $\delta f_{lmn}(\gamma_1, \gamma_2) = -(f_{lm} \cup f_n)(\gamma_1, \gamma_2) - (f_l \cup f_{mn})(\gamma_1, \gamma_2)$. For notational convenience, put $\langle f_l, f_m, f_n \rangle$ equal to $(f_{lm} \cup f_n) + (f_l \cup f_{mn})$.

**Proposition 75.** For all $l, m, n \in \mathbb{Z}_{\geq 2}$, the cochain $f_{lmn}$ is determined by its values on $S$ and $T$ and by the values of $f_l, f_m, f_n, f_{lm}, f_{mn}$. 47
Proof. This follows from an analogous induction proof to Propositions 30 and 70. Suppose by induction that the proposition holds for all $\gamma \in \Gamma$ of length less than $R$. Let $\gamma' \in \Gamma$ have length $R$, so there exist integers $m, n$ such that $\gamma' = S^jT^k\gamma$. The coproduct property of $f_{2l,2m,2n}$ implies that

$$f_{lmn}(\gamma') = f_{lmn}(S^jT^k) + S^jT^kf_{lmn}(\gamma) + \langle f_l, f_m, f_n \rangle (S^jT^k, \gamma)$$

which expresses $f_{lmn}(\gamma')$ in terms of $f_{lmn}(\gamma)$ and a Massey triple product of Eichler-Shimura cocycles. Since $\langle f_l, f_m, f_n \rangle$ decomposes as $f_l \cup f_{mn} + f_{lm} \cup f_n$, and since $f_{lmn}(1) = 0$, the induction proof is complete. \qed

The computation of $f_{lmn}(T)$ is given by analogy to the previous cases by a residue computation in [1]. Since $f_{lmn}(T)$ lies in $S^{2l+2m+2n}(\mathbb{H}_{\pi^2l+2m+2n+3Q})$, we can once again investigate whether there exists a rational cochain $z_{lmn}$ of $f_{lmn}$ that takes values in $S^{2l+2m+2n}(\mathbb{H}_{\pi^2l+2m+2n+3Q})$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$. If such a $z_{lmn}$ exists, it is subject to a linear system induced by the coboundary operator, just as in the previous cases.

$$(I + S)z_{lmn}(S) = -\langle z_l, z_m, z_n \rangle (S, S)$$

$$(I + U + U^2)z_{lmn}(U) = -\langle z_l, z_m, z_n \rangle (U, U) - \langle z_l, z_m, z_n \rangle (U^2, U)$$

In order to solve for $z_{lmn}(S)$, we use the coproduct property of $z_{lmn}$ to replace $z_{lmn}(U)$ by an expression in terms of $z_{lmn}(S)$ and $z_{lmn}(T)$.

$$z_{lmn} = z_{lmn}(ST)$$

$$= z_{lmn}(S) + Sz_{lmn}(T) + \langle z_l, z_m, z_n \rangle (S, T)$$

This substitution results in the following linear system for $z_{lmn}(S)$:

$$(I + S)z_{lmn}(S) = -\langle z_l, z_m, z_n \rangle (S, S)$$

$$(I + U + U^2)z_{lmn}(S) = -\langle z_l, z_m, z_n \rangle (U, U) - \langle z_l, z_m, z_n \rangle (U^2, U)$$

$$- (I + U + U^2)(Sz_{lmn}(T) + \langle z_l, z_m, z_n \rangle (S, T)).$$

In this way, $z_{lmn}(S)$ is determined up to a cuspidal cocycle.
5

Deligne Cohomology and Yoneda Extensions

5.1 The Deligne Complex

Denote $\mathcal{M}_{1,1}$ by $X$ and $\overline{\mathcal{M}}_{1,1}$ by $\overline{X}$. Let $V$ denote a constant Hodge structure; in practice, we will put $V = \mathbb{Q}(2n + 1)$. Let $E^\bullet(\overline{X} \log D; S^{2n}\mathcal{H}) \otimes V$ denote the de Rham complex of holomorphic $j$-forms on $\overline{X}$ with at worst logarithmic singularities and coefficients in the Hodge bundle $S^{2n}\mathcal{H} \otimes V$. As in previous chapters, $E^\bullet(\overline{X} \log D)$ should be understood as notation for $E^\bullet(\mathfrak{f} \log D)^\Gamma$. Let $C^\bullet(\Gamma, S^{2n}\mathbb{H}) \otimes V$ denote the Eilenberg-MacLane complex of the group $\Gamma$, with coefficients in the variation of Hodge structures $S^{2n}\mathbb{H} \otimes V$. The Deligne complex $A^\bullet$ is given in degree $j$ by triples

$$
\begin{bmatrix}
c \\
\omega \\
z
\end{bmatrix} : \left\{ \begin{array}{l}
\omega \in F^0W_{-j}E^j(\overline{X} \log D; S^{2n}\mathcal{H}) \otimes V \\
z \in C^j(\Gamma, S^{2n}\mathbb{H}) \otimes V \\
c \in C^{j-1}(\Gamma, S^{2n}\mathbb{H}) \otimes V
\end{array} \right\
$$

Note that the integration map associates to $\omega \in F^0W_{-j}E^j(\overline{X} \log D; S^{2n}\mathcal{H}) \otimes V$ the element $[\omega] \in C^j(\Gamma, S^{2n}\mathbb{H}) \otimes V$ defined by $\gamma \mapsto \int_\gamma \omega$ for each $\gamma \in \Gamma$. Under this notation, the Deligne coboundary is defined as follows.
Definition 76. The coboundary map for $A^\bullet$ is given by

$$\delta \begin{bmatrix} c \\ \omega \\ z \end{bmatrix} = \begin{bmatrix} -\delta c + z - [\omega] \\ d\omega \\ \delta z \end{bmatrix}.$$ 

Proposition 77. The map $\delta$ defined in Definition 76 has the requisite property $\delta^0 = 0$.

Proof. By direct computation,

$$\delta^2 \begin{bmatrix} c \\ \omega \\ z \end{bmatrix} = \delta \begin{bmatrix} -\delta c + z - [\omega] \\ d\omega \\ \delta z \end{bmatrix} = \begin{bmatrix} -\delta(-\delta c + z - [\omega]) + \delta z - [d\omega] \\ d^2\omega \\ -\delta z + \delta [\omega] + \delta z - [d\omega] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

By Stokes’ theorem, $[d\omega] = \delta[\omega]$, which confirms that $\delta^2 = 0$. \qed

Let $H^\bullet_D(\mathcal{M}_{1,1}, S^{2n-2}_C(2n-1))$ denote the cohomology of the Deligne complex $A^\bullet$. The Deligne cohomology groups are subject to an exact sequence that relates them to the classical cohomology groups $H^\bullet(X, \mathcal{V})$.

Proposition 78. Let $\mathcal{V} = S^{2n}_C \otimes X$. For each $m \geq 1$, the Deligne cohomology group $H^m_D(X, S^{2n}_C \otimes \mathcal{V})$ is subject to the short exact sequence

$$0 \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^{m-1}(X, \mathcal{V})) \to H^m_D(X, \mathcal{V}) \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^m(X, \mathcal{V})) \to 0.$$ 

Corollary 79. Proposition 78 implies that $H^1_D(X, \mathcal{V}) \cong \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1(X, \mathcal{V}))$ and $H^2_D(X, \mathcal{V}) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^1(X, \mathcal{V}))$.

Proof. For a fixed $n \in \mathbb{Z}_{>0}$, let $c \in C^0(X, S^{2n}_C)$. It follows that $c$ is a cocycle if and only if $c$ is $\Gamma$-invariant. But the $\mathcal{S}$-invariant elements of $S^{2n}_C$ are
given by span\(\{a^j b^{2n-j} - (-1)^j a^{2n-j} b^j, 0 \leq j \leq 2n\}\) and the \(T\)-invariant elements of \(S^{2n} \HH\) are spanned by \(a^{2n}\). Consequently, there are no non-trivial 0-cocycles and so \(H^0(X, V) = 0\). By Proposition 78, conclude that \(H^1_D(X, V) \cong \Hom_{\text{MHS}}(Q, H^1(X, V))\). By Proposition 29, the cohomology of \(\text{SL}_2(\mathbb{Z})\) vanishes in degree greater than 1. Consequently, \(\Hom_{\text{MHS}}(Q, H^2(X, V)) = 0\) and Proposition 78 implies that \(H^2_D(X, V) \cong \Ext^1_{\text{MHS}}(Q, H^1(X, V))\).

For any \(Q\)-mixed Hodge structure \(A\), the group of 1-extensions \(\Ext^1_{\text{MHS}}(Q, A)\) has the structure of a generalized torus [17]: \(\Ext^1_{\text{MHS}}(Q, A) \cong (W_0 A \otimes \mathbb{C})/(F^0 W_0 A + W_0 A).\) Accordingly, the extension group \(\Ext^1_{\text{MHS}}(Q, H^1(X, V))\) is given by a generalized torus \(W_0 H^1(X, V \otimes \mathbb{C})/(F^0 W_0 H^1(X, V) + W_0 H^1(X, V)).\) This fact can be established from first principles by the following observation. Since \(H^2(X, S^{2n} \HH) = 0\), every element of \(H^2_D(X, S^{2n} \HH) \otimes V\) can be represented by \(\begin{bmatrix} c \\ 0 & 0 \end{bmatrix}\) for some \(c \in C^1(X, S^{2n} \HH) \otimes V\) with the property that \(\delta c = 0\). The choice of \(c\) not unique; since \(\delta \begin{bmatrix} 0 \\ \omega & z \end{bmatrix} = \begin{bmatrix} d\omega - [\omega] \delta z \\ \delta z \end{bmatrix}\), it follows that \(H^2_D(X, S^{2n} \HH) \otimes V\) is given by \(H^1(X, S^{2n} \HH) \otimes \mathbb{C}\) modulo the indeterminacy \(F^0 W_0 H^1(X, V) + W_0 H^1(X, V)\). This results in the formula for \(\Ext^1_{\text{MHS}}(Q, H^1(X, S^{2n} \HH)) \otimes V\).

In order to clarify the structure of \(H^1_D(X, S^{2n} \HH) \otimes V\), consider more closely the group \(\Hom_{\text{MHS}}(Q, H^1(X, V))\). The Eichler-Shimura isomorphism gives \(H^1(X, V)\) as an extension of mixed Hodge structures

\[
0 \longrightarrow H^1_{\text{cusp}}(X, S^{2n} \HH) \longrightarrow H^1(X, S^{2n} \HH) \longrightarrow Q \longrightarrow 0
\]

where \(H^1_{\text{cusp}}(X, S^{2n} \HH)\) consists of the cocycles \(\omega_g = (2\pi i)^{2n+1} g(\tau) w^{2n} d\tau\) associated to cusp forms \(g\) in the sense of Chapter 2, as well as the anti-holomorphic counterparts \(\omega_g\) of these forms [6]. Since \(w^{2n}\) has Hodge weight \(2n\) and \(d\tau\) contributes weight 1, \(H^1_{\text{cusp}}(X, S^{2n} \HH)\) is a Hodge structure consisting of types \((2n + 1, 0)\) and \((0, 2n + 1)\).
The quotient term is generated by the cocycle associated to $G_{2n+2}$, as computed in Chapters 3 and 4. It is a Hodge structure of type $(2n + 1, 2n + 1)$.

By the Manin-Drinfeld theorem, this extension is split [6]. Consequently, $H^1(X, S^{2n}\mathbb{H}) \cong H^1_{\text{cusp}}(X, S^{2n}\mathbb{H}) \oplus \mathbb{Q}$ is a mixed Hodge structure. If $f : \mathbb{Q}(0) \to V$ is a morphism of mixed Hodge structures, then it follows that $f$ is trivial unless $V$ has Hodge weight 0. Therefore, $\text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1(X, S^{2n}\mathbb{H}(r)))$ is trivial except when $r = 2n + 1$, in which case $\text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1(X, S^{2n}\mathbb{H}(r))) \cong \text{Hom}_{\text{MHS}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$.

**Corollary 80.** If $r = 2n + 1$, then the Deligne cohomology group $H^1_D(X, S^{2n}\mathbb{H}) \otimes \mathbb{Q}(r)$ is generated by the Eisenstein series of weight $2n + 2$. If $r \neq 2n + 1$, then $H^1_D(X, S^{2n}\mathbb{H}) \otimes \mathbb{Q}(r)$ vanishes.

### 5.2 Yoneda Extensions

In order to investigate the cup product and higher order products on Deligne cohomology, we will first express Deligne cohomology in terms of Yoneda extensions of variations of mixed Hodge structures over $\mathcal{M}_{1,1}$. We will then use the Yoneda product to construct homological products for $H^1_D(\mathcal{M}_{1,1}, V)$. This section draws on [15] for foundational properties of Yoneda extensions.

**Definition 81.** Let $M, N$ be objects in an abelian category $\mathcal{C}$. An extension $N$ by $M$ is a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$  

Two extensions of $E, E'$ of $N$ by $M$ are said to be equivalent if there exists an isomorphism $\phi : E \to E'$ such that following diagram commutes:

$$
\begin{array}{ccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
& & \parallel & & \downarrow{\phi} & & \parallel & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0.
\end{array}
$$
The abelian group \( \text{Ext}(M, N) \) is defined as the set of such extensions under this equivalence relation. Define \( \text{Ext}^n(M, N) \) accordingly as the set of long exact sequences

\[
0 \longrightarrow N \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0
\]

under an equivalence relation that is explained below. It is however instructive to first introduce the concept of Yoneda product.

**Definition 82.** Given extensions

\[
0 \longrightarrow N \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0
\]

and

\[
0 \longrightarrow P \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0
\]

lying respectively in \( \text{Ext}^m(M, N) \) and \( \text{Ext}^n(N, P) \), define the Yoneda product \( \text{Ext}^n(N, P) \otimes \text{Ext}^m(M, N) \rightarrow \text{Ext}^{m+n}(M, P) \) by the following composition.

Let \( E \) and \( E' \) be extensions \( 0 \rightarrow N \rightarrow E \rightarrow B \rightarrow 0 \) and \( 0 \rightarrow B' \rightarrow E' \rightarrow M \rightarrow 0 \) together with a morphism \( \beta : B \rightarrow B' \). As illustrated by the following diagram, the Yoneda products \( \beta^* E \circ E' \) and \( \beta_* E' \circ E \) do not coincide in general:
**Definition 83.** Let $E, F \in \text{Ext}^2(M, N)$ be given by the diagrams

$$
\begin{align*}
0 &\longrightarrow N \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0 \\
0 &\longrightarrow N \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.
\end{align*}
$$

$E$ and $F$ are said to be equivalent if they sit in a diagram of the form of Equation 5.1. That is, there must exist objects $B, B'$ and a morphism $\beta : B \to B'$ such that $E_1 = \beta^*F_1$ and $F_0 = \beta_*E_0$.

In general, two extensions in $\text{Ext}^n(M, N)$ are said to be equivalent if there is a chain of 1-extension and 2-extension equivalences connecting them. The identity element of $\text{Ext}^n(M, N)$ is represented by the trivial $n$-extension given by connecting 1-extensions of the form $0 \to B \to B \to 0$. The notion of Yoneda product can be extended by taking tensor products to pairings of extensions that do not naturally admit composition.

**Lemma 84.** Let $A, B, C$ be objects in $\mathcal{C}$. The Yoneda extension group $\text{Ext}^1(A, B)$ is isomorphic to $\text{Ext}^1(A \otimes C, B \otimes C)$.

**Corollary 85.** Given extensions

$$
\begin{align*}
0 &\longrightarrow N' \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow N \longrightarrow 0
\end{align*}
$$
and

\[ 0 \rightarrow M' \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \]

lying respectively in \( \text{Ext}^m(M, M') \) and \( \text{Ext}^n(N, N') \), define the Yoneda product of these extensions by \( \text{Ext}^n(N \otimes M', N' \otimes M') \otimes \text{Ext}^m(M, M') \rightarrow \text{Ext}^{m+n}(M, N \otimes M') \).

**Proposition 86** (Yoneda’s Lemma). Let \( E_1 \in \text{Ext}^1(C, B) \) and \( E_2 \in \text{Ext}^1(A, C) \) be extensions in the category \( C \). The Yoneda product \( E_1 \circ E_2 \) is trivial if and only if there exists an object \( V \) in \( C \) with a filtration \( 0 \subseteq B \subseteq K \subseteq V \) such that \( V/K \cong A \).

**Lemma 87.** Given the filtration \( 0 \subseteq B \subseteq K \subseteq V \) with \( V/K \cong A \), the zero map \( \beta : 0 \rightarrow K/B \) has pullback \( \beta^*K = B \) and pushout \( \beta_*A = V/B \).

**Proof.** By observation of the following maps of extensions:

\[
\begin{array}{ccc}
0 & \rightarrow & B \\
\downarrow & & \downarrow \\
0 & \rightarrow & K \\
\downarrow^\beta & & \downarrow \\
0 & \rightarrow & V/B \\
\downarrow & & \\
0 & \rightarrow & A.
\end{array}
\]

**Lemma 88.** Given a filtration \( 0 \subseteq B \subseteq K \subseteq V \), the extension

\[ 0 \rightarrow B \rightarrow K \rightarrow V/B \rightarrow A \rightarrow 0 \]

is Yoneda trivial.
Proof. By definition, the following commutative diagram is an equivalence of 2-extensions:

\[
\begin{array}{ccccccc}
0 & \rightarrow & B & \rightarrow & \beta^* K & \rightarrow & A & \rightarrow & A & \rightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & & & \rightarrow & & \beta & & \\
& & & & & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & K/B & & & & \\
& & & & & & \rightarrow & & \beta A & & A & \rightarrow & 0.
\end{array}
\]

By Lemma 87, the first extension is Yoneda trivial. \hfill \Box

Proof of Proposition 86. Together with Lemma 88, putting \( V/B := E_2 \) and \( K := E_1 \) proves the proposition. \hfill \Box

5.2.1 Trivializing Yoneda Products

Given 1-extensions \( E_1, E_2 \), Yoneda’s lemma states that the product \( E_1 \circ E_2 \) is trivial if and only if there is an object \( V \) with certain properties. In this section, we show that \( V \) is itself a 1-extension. We then use this construction to define the Massey triple product of Yoneda extensions. For \( j \in \{1, 2, 3\} \), let \( E_j \in \text{Ext}^1(P_j, P_{j-1}) \) given by the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & P_{j-1} & \rightarrow & E_j & \rightarrow & P_j & \rightarrow & 0.
\end{array}
\]

The Yoneda products \( E_1 \circ E_2 \) and \( E_2 \circ E_3 \) are then given by the following diagrams.

\[
\begin{array}{ccccccc}
0 & \rightarrow & P_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & P_2 & \rightarrow & 0 \\
0 & \rightarrow & P_1 & \rightarrow & E_2 & \rightarrow & E_3 & \rightarrow & P_3 & \rightarrow & 0.
\end{array}
\]
Proposition 89. The Yoneda product $E_1 \circ E_2$ is trivialized by an extension $E_{12} \in \text{Ext}^1(P_2, E_1)$. The Yoneda product $E_2 \circ E_3$ is trivialized by and extension $E_{23} \in \text{Ext}^1(E_3, P_1)$.

Proof. By Yoneda’s lemma, $E_1 \circ E_2$ is trivialized by an object $V$ with a filtration $0 \subseteq P_0 \subseteq E_1 \subseteq V$ and the property that $V/E_1 \cong P_2$. Therefore, $V$ is an extension satisfying

$$0 \longrightarrow E_1 \longrightarrow V \longrightarrow P_2 \longrightarrow 0.$$

By Yoneda’s lemma, $E_2 \circ E_3$ is trivialized by an object $V'$ with a filtration $0 \subseteq P_1 \subseteq E_2 \subseteq V'$ and the property that $E_3 \cong V'/P_1$. Therefore, $V'$ is an extension satisfying

$$0 \longrightarrow P_1 \longrightarrow V' \longrightarrow E_3 \longrightarrow 0.$$

\[\square\]

Proposition 90. The Yoneda products $E_{12} \circ E_3$ and $E_1 \circ E_{23}$ are simultaneously trivialized by an extension $E_{123} \in \text{Ext}^1(E_3, E_1)$.

Proof. The Yoneda product $E_{12} \circ E_3$ is given by

$$0 \longrightarrow E_1 \longrightarrow E_{12} \longrightarrow E_3 \longrightarrow P_3 \longrightarrow 0.$$

By Yoneda’s lemma, this product is trivialized by an object $V$ with filtration $0 \subseteq E_1 \subseteq E_{12} \subseteq V$, such that $E_3 \cong V/E_{12}$. Therefore, $E_{123}$ is an extension satisfying

$$0 \longrightarrow E_1 \longrightarrow V \longrightarrow E_3 \longrightarrow 0.$$

The Yoneda product $E_1 \circ E_{23}$ is given by

$$0 \longrightarrow P_0 \longrightarrow E_1 \longrightarrow E_{23} \longrightarrow E_3 \longrightarrow 0.$$

By Yoneda’s lemma, this product is trivialized by an object $V'$ with filtration $0 \subseteq P_0 \subseteq E_1 \subseteq V'$ with the property that $V'/E_1 \cong E_3$. Therefore, $V'$ is an extension...
satisfying
\[ 0 \to E_1 \to V' \to E_3 \to 0. \]

\[ \square \]

**Corollary 91.** The Yoneda products $E_{12} \circ E_3$ and $E_1 \circ E_{23}$ fit into the following commutative diagram.

\[
\begin{array}{cccccc}
0 & \to & P_0 & \to & E_1 & \to & E_{23} & \to & E_3 & \to & 0 \\
& & | & & | & & | & & | & & \\
0 & \to & E_1 & \to & E_{12} & \to & E_3 & \to & P_3 & \to & 0.
\end{array}
\]

**Proposition 92.** Let $E_1 \circ E_2$ and $E_2 \circ E_3$ be trivialized respectively by $E_{12}$ and $E_{23}$. The Massey triple product $\langle E_1, E_2, E_3 \rangle$ is an element of $\text{Ext}^2(P_3, P_0)$ that is trivialized by $E_{123}$. 

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Extensions of Variations of Mixed Hodge Structure

This chapter introduces extensions of variations of mixed Hodge structure over $\mathcal{M}_{1,1}$. The Deligne cohomology group $H^1_b(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}(2n + 1))$ is shown to be isomorphic to a group of such extensions, and that equivalence is employed to construct cohomological products by analogy to Yoneda products of extensions.

6.1 Extensions of Mixed Hodge Structures

Let $A$ be a mixed Hodge structure, and let $Q = Q(0)$ be of type $(0, 0)$. Consider an extension of mixed Hodge structure $E$ given by the exact sequence

$$
0 \rightarrow A \rightarrow E \rightarrow Q \rightarrow 0
$$

This extension implies the existence of an exact sequence

$$
0 \rightarrow A_Q \rightarrow E_Q \rightarrow Q \rightarrow 0
$$

where $A_Q$ and $E_Q$ are the $\mathbb{Q}$-structures of $A$ and $E$, respectively. Since this is an exact sequence of $\mathbb{Q}$-modules, we can choose a splitting $1_B : \mathbb{Q} \rightarrow E_Q$ that is well defined.
up to $A_\mathbb{Q}$. Taking tensor powers by $\mathbb{C}$, we obtain another short exact sequence:

$$0 \longrightarrow A_\mathbb{C} \longrightarrow E_\mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0$$

In this case, choose a splitting $1_{DR} : \mathbb{C} \to A_\mathbb{C}$ that respects the Hodge filtration. As before, $1_{DR}$ is well defined up to $A_\mathbb{C}$. However, since $\mathbb{C} = F^0\mathbb{C}$, we can require $1_{DR} : F^0\mathbb{C} \to F^0E_\mathbb{C}$. Such a splitting is well defined up to $F^0A_\mathbb{C}$. Accordingly, the difference $1_B - 1_{DR}$ between Betti and de Rham lifts is an element of $A_\mathbb{C}/(A_\mathbb{Q} + F^0A_\mathbb{C})$.

In subsequent investigations, it will be of particular interest to consider the case when $A$ has negative Hodge weight, so that $F^0A_\mathbb{C} = 0$ and $1_B - 1_{DR} \in A_\mathbb{C}/A_\mathbb{Q}$.

The choice of lift corresponds to a choice of basis for the extension $E$. The linear transformation that changes from the de Rham basis to the Betti basis has the following matrix representation

$$A^{DR} = 1_{DR} A^{DR}$$

$$f = 1_{DR} - 1_B$$

where $f$ is the period matrix that changes the basis of $A$ from the de Rham basis to the Betti basis.

6.1.1 Example

Consider an extension of Hodge structures $E$ given by the exact sequence

$$0 \to \mathbb{Z}(1) \to E \to \mathbb{Z}(0) \to 0$$

where $\mathbb{Z}(1)$ is the Hodge twist of $\mathbb{Z}$ with weight $-2$. The Betti splitting $1_B : \mathbb{Z}(0) \to E$ is determined up to an element of $\mathbb{Z}(1)$, and the de Rham splitting $1_{DR} : \mathbb{C} \to E$ is uniquely determined since $F^0\mathbb{Z}(1) = 0$. The linear transformation changing from the de Rham to the Betti basis has matrix representation

$$\begin{pmatrix}
1_{DR} & A^{DR} \\
1_B & f \\
0 & P
\end{pmatrix}$$
The extension $E$ is a mixed Hodge structure with weight filtration $\mathbb{Z}(1) = W_{-2} \subseteq W_0 = \mathbb{Z}(1) \oplus \mathbb{Z}(0)$. The Hodge filtration is $E = F^{-1}E \subset F^0E = \mathbb{C}(0)$ and $F^pE = 0$ for all $p > 1$. Since $\mathbb{Z}(1)$ has underlying $\mathbb{Z}$-module structure of $2\pi i \mathbb{Z}$, the period matrix $P$ is given by multiplication by $(2\pi i)^{-1}$.

### 6.2 Extensions of Variations of Mixed Hodge Structure

Let $\overline{X}$ be a curve and $X = \overline{X} - D$ for some divisor $D$. Let $A$ be a variation of mixed Hodge structure over $X$. Consider an extension $V$ of variations of mixed Hodge structure given by the exact sequence

$$0 \longrightarrow A \longrightarrow V \longrightarrow \mathbb{Q} \longrightarrow 0$$

Fix a point $p \in X$. Suppose the fibers of $A$ and $V$ over $p$ are $A_0$ and $V_0$, respectively. The corresponding extension of $\mathbb{Q}$-local systems over $p$ is

$$0 \longrightarrow A_0 \longrightarrow V_0 \longrightarrow \mathbb{Q} \longrightarrow 0$$

Since $A_0$ and $V_0$ are free, there is a splitting $1_B : \mathbb{Q} \to V_0$. This splitting is well defined modulo $A_0$. The monodromy of this extension of $\mathbb{Q}$-local systems has matrix representation $\begin{pmatrix} 1 & 0 \\ z & \rho \end{pmatrix}$ where $z : \pi_1(X, p) \to A_0$ is a group cochain and $\rho$ is the monodromy representation $\rho : \pi_1(X, p) \to \text{Aut} A_0$. In fact, the group law on the monodromy representation implies that $z$ has the cocycle condition. Let $\gamma, \mu \in \pi_1(X, p)$. Since $T(\gamma\mu) = T(\gamma)T(\mu)$, it holds that

$$\begin{pmatrix} 1 & 0 \\ z(\gamma) & \rho(\gamma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z(\mu) & \rho(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z(\gamma) + z(\gamma)z(\mu) & \rho(\gamma)\rho(\mu) \end{pmatrix}.$$
The cocycle $z$ is called the *extension data* of the Betti realization of the extension of local systems.

Taking tensor powers by $\mathcal{O}_X$, we form the exact sequence of flat vector bundles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O} \longrightarrow 0$$

with connections $\nabla_\mathcal{A}$, $\nabla_\mathcal{V}$, and $\nabla_\mathcal{O} = d$. Supposing these connections have regular singular points, we can extend to vector bundles over $\overline{X}$ using the canonical extension technique of Chapter 3. Let us consider the case where $\mathcal{V} = \mathcal{O} \oplus \mathcal{A}$ and $F^0\mathcal{A} = 0$. Under these conditions, there is a unique lift $1_{DR} : \mathcal{O} \rightarrow F^0\mathcal{V}$ since $F^0\mathcal{O} = \mathcal{O}$.

The connection on $\mathcal{V}$ is $\nabla_\mathcal{V} := \nabla_\mathcal{A} + \psi$, where $\psi$ is a global section of the sheaf $\Omega^1_X(\log) \otimes \mathcal{A}$. In particular, Griffiths transversality implies that $\psi \in \Omega^1_X(\log) \otimes F^{-1}\mathcal{A}$.

Let $\Omega$ be the connection form of $\mathcal{V}$. Using Proposition 40, the parallel transport function $T$ on $\mathcal{V}$ is given by the iterated path integral of $\Omega$ as given in Proposition 40. Accordingly, the monodromy representation of the de Rham realization of $\mathcal{V}$ has matrix representation $(f_0)$ for some cocycle $f$ of $\pi_1(\overline{X}, x_0)$.

$\mathcal{V}$ is not guaranteed to be an admissible variation of mixed Hodge structure. Although the fibers of $\mathcal{V}$ have a rational structure, the monodromy operator on each fiber may not be defined over $\mathbb{Q}$. Therefore, the question of whether $\mathcal{V}$ is an admissible variation of MHS reduces to whether the monodromy representation of the de Rham realization is conjugate to the monodromy representation of the Betti realization.

### 6.2.1 Example

Consider the local system $\mathcal{V}$ from Section 2.3.1. In this case, $\mathcal{V} = S^{2n}\mathbb{H}_D$ is generated by symmetric powers of $\mathbb{H}_D \cong \mathbb{Q}a \oplus \mathbb{Q}w$. Put a Hodge structure on $\mathcal{V}$ by setting the Hodge weights of $a$ and $w$ to 0 and 1, respectively. That is, $\mathbb{H}_D \otimes \mathbb{C} = F^0\mathbb{H}_D \subset F^1\mathbb{H}_D \cong \mathbb{C}w$ and $F^p\mathbb{H}_D = 0$ for all $p > 1$. The corresponding Hodge filtration on $\mathcal{V}$
has $F_p \mathcal{V} = \bigoplus_{k \geq p} \mathbb{C} a^k w^{2n-k}$. This makes $\mathcal{V}$ a variation of Hodge structure of weight $2n$. As in the general case, we wish to specialize to extensions of $\mathbb{Q}$ by $\mathcal{V}$ in which $F^0 \mathcal{V} = 0$. Therefore, take the Hodge twist $\mathcal{V}(2n+1)$ of $\mathcal{V}$, which is a variation of Hodge structure of weight $-2n-2$.

Let $E$ be an extension of mixed Hodge structures given by

\[
0 \longrightarrow \mathcal{V}(2n+1) \longrightarrow E \longrightarrow \mathbb{Q} \longrightarrow 0.
\]

Let $e$ be a basis for $\mathbb{Q}$. Since the exact sequence splits, write $E = \mathcal{V}(2n+1) \oplus \mathbb{Q} e$. Over each point $p$, define a MHS on the fiber of $E$ by giving $e$ type $(0,0)$ and $a^k w^{2n-k}$ type $(-k-1, k-2n-1)$. This specifies the weight filtration as $W_{-2n-2} = \mathcal{V}(2n+1) \otimes \mathbb{Q}$ and $W_0 = E \otimes \mathbb{Q}$. The Hodge filtration is given by $F_p E = \mathbb{C} e \oplus \bigoplus_{k \geq p-2n-1} \mathbb{C} a^k w^{2n-k}$ for $p \geq 0$. Let $\text{SL}_2(\mathbb{Z})$ act on $E$ by acting trivially on $e$ and by the factor of automorphy in Section 2.3 on $a$ and $w$.

**Proposition 93.** The Hodge and weight filtrations on the fibers of $E$ are invariant under the action of the factor of automorphy of Section 2.3.

**Proof.** Since all monomials in $a$ and $w$ lie in the same weight and $\text{SL}_2(\mathbb{Z})$ sends none of these to $e$, the action preserves the weight filtration. In order to prove that the Hodge filtration is also invariant under the group action, first observe that since $\text{SL}_2(\mathbb{Z})$ fixes $e$, it follows that $F^0 E = \mathbb{C} e$ is invariant under the action. Furthermore, for $p \leq -1$ it holds that $F_p E = \mathbb{C} e \oplus \mathbb{C} w^{2n} \oplus \cdots \mathbb{C} a^{p+1} w^{2n-p+1}$. In order to prove that $F_p E$ is invariant under the group action, it suffices to establish that the action does not introduce powers of $a$. Since the factor of automorphy from Section 2.3 takes $(a \ w)$ to $(a \ w) \begin{pmatrix} (c \tau + d)^{-1} & 0 \\ \frac{c}{2 \pi i} & (c \tau + d) \end{pmatrix}$, it follows that the Hodge filtration is also invariant. \qed

Since the mixed Hodge structure on the fibers of $E$ is invariant under the action of the factor of automorphy, it descends to a mixed Hodge structure on the quotient
over $\mathcal{M}_{1,1}$. Specializing the monodromy representation in Section 2.3.1 to the loop in $\mathcal{M}_{1,1}$ corresponding to $\gamma \in \Gamma$, we have

$$T_{dR}(\gamma) = \begin{pmatrix} 1 & 0 \\ f_n(\gamma) & \rho(\gamma) \end{pmatrix}$$

where $f_n$ is the cocycle corresponding to $G_{2n+2}$ derived in Chapters 3 and 4 and $\rho : \Gamma \to \text{End } S^{2n}H$ is given by the usual action of $\text{SL}_2(\mathbb{Z})$ on the fibers of $V$. In order for $E$ to be an admissible variation of MHS over $\mathcal{M}_{1,1}$, we must confirm that $T_{dR}$ is defined over $\mathbb{Q}$. This fact is confirmed below in Section 6.2.2.

### 6.2.2 Relation to Deligne Cohomology

As discussed in the previous section, an extension of variations of mixed Hodge structures $E$ may not be a VMHS, depending on the behavior of its monodromy representation. In particular, even though the individual fibers of $E$ have a mixed Hodge structure, $E$ may fail to be a VMHS if its monodromy representation is not defined over $\mathbb{Q}$ and hence does not preserve the rational structure of the fibers of $E$. This section recalls the 1-extension example from Chapter 2 of $E = \mathbb{Q} \oplus S^{2n}H$ and illustrates the correspondence between this and the Deligne cohomology group $H^1_D(\mathcal{M}_{1,1}, S^{2n}H(2n + 1))$.

Let $V$ denote the vector bundle $E \otimes \mathcal{O}$. By the results of Section 2.3.1, the $V$ has connection form $\Omega := \begin{pmatrix} 0 & 0 \\ \psi_{2n+2} & e_0 \frac{dq}{q} \end{pmatrix}$ and monodromy representation $T(\gamma) = \begin{pmatrix} 1 & 0 \\ f_n(\gamma) \rho(\gamma) \end{pmatrix}$.

The corresponding representation for the Betti realization is $\begin{pmatrix} 1 & 0 \\ z_n(\gamma) \rho(\gamma) \end{pmatrix}$.

**Lemma 94.** Let $u \in S^{2n}H_C$ and $U := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$. For each $\gamma \in \Gamma$, the conjugate of $T(\gamma)$ by $U$ is $\begin{pmatrix} 1 & 0 \\ f_n(\gamma) - (\delta u)(\gamma) \rho(\gamma) \end{pmatrix}$.

**Proof.** View $T(\gamma)$ as the element $(T, \gamma) \in \Gamma \ltimes \text{Aut}(A)$. In this sense, conjugation of
Corollary 95. The Betti and de Rham realizations of the monodromy representation of \( V \) are conjugate if and only if there exists \( u \in C^0(\Gamma, S^{2n}\mathbb{H}_\mathbb{C}) \) such that \( f_n - z_n = \delta u \).

Sections 3.4 and 4.1 together establish that \( f_n(S) - z_n(S) \) is given by the complex coboundary \( \frac{(2n)!\zeta(2n+1)}{2(2n)!}\delta^0 Y \). Since this is precisely the condition for the monodromy representation of \( E \) to be defined over \( \mathbb{Q} \), conclude that the 1-extension \( E \) is always a VMHS. Furthermore, this observation implies the following equivalence.

Proposition 96. For all \( r \in \mathbb{Z} \), the Deligne cohomology group \( H^1_D(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}(r)) \) is isomorphic to \( \text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(\mathbb{Q}, S^{2n}\mathbb{H}(r)) \).

Proof. See [6]. By Definition 76, Deligne cocycles correspond to pairings \((\omega, z)\) of a closed form \( \omega \in F^0\mathcal{W}_-j\mathcal{E}^j(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \otimes V \) and a cocycle \( z \in S^j(\Gamma, S^{2n}\mathbb{H}) \otimes V \) with the property that \([\omega]\) and \( z \) differ by a coboundary. By Corollary 95, the extension of VMHS \( E = \mathbb{Q} \oplus S^{2n}\mathbb{H} \) is admissible if an only if the Betti and de Rham realizations of the monodromy representation are conjugate. As illustrated in Lemma 94, these conditions are equivalent.

6.3 Higher Extension Criteria

In this section, the cup product and higher cohomological products for Deligne cohomology are constructed using Proposition 96 and the technique of Yoneda product in Section 5.2.
6.3.1 Cup Product

\( V_k \) denote the VMHS over \( M_{1,1} \) given by \( S^{2k}\mathbb{H}(2k+1) \). Let \( Z_m \in H_D^1(M_{1,1}, \mathbb{V}_m) \) and \( Z_n \in H_D^1(M_{1,1}, \mathbb{V}_n) \). By Proposition 96, \( Z_m \) and \( Z_n \) correspond to Yoneda extensions \( \mathbb{E}_m \in \text{Ext}^1_{\text{MHS}(M_{1,1})}(\mathbb{Q}, \mathbb{V}_m) \) and \( \mathbb{E}_n \in \text{Ext}^1_{\text{MHS}(M_{1,1})}(\mathbb{Q}, \mathbb{V}_n) \).

**Lemma 97.** The Yoneda product \( \mathbb{E}_m \circ \mathbb{E}_n \) is trivialized by an extension \( \mathbb{E}_{mn} \) given by

\[
0 \to \mathbb{E}_m \otimes \mathbb{V}_n \to \mathbb{E}_{mn} \to \mathbb{Q} \to 0.
\]

**Proof.** Using Lemma 84, identify \( \mathbb{E}_m \) with an element in \( \text{Ext}^1_{\text{MHS}(M_{1,1})}(\mathbb{V}_n, \mathbb{V}_m \otimes \mathbb{V}_n) \). The Yoneda product \( \mathbb{E}_m \circ \mathbb{E}_n \) is given by Section 5.2.1 as the 2-extension

\[
0 \to \mathbb{V}_m \otimes \mathbb{V}_n \to \mathbb{E}_m \otimes \mathbb{V}_n \to \mathbb{E}_n \to \mathbb{Q} \to 0
\]

and the product is trivialized by an extension \( \mathbb{E}_{mn} \in \text{Ext}^1_{\text{MHS}(M_{1,1})}(\mathbb{Q}, \mathbb{E}_m \otimes \mathbb{V}_n) \) by Proposition 89.

**Lemma 98.** For each \( \gamma \in \Gamma \), the monodromy representation \( T(\gamma) \) of \( \mathbb{E}_{mn} \) has matrix representation

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \rho(\gamma) & 0 \\
0 & 0 & \rho(\gamma)
\end{pmatrix}
\]

and the corresponding representation for the Betti realization is

\[
\begin{pmatrix}
1 & 0 & 0 \\
\rho(\gamma) & 0 & 0 \\
\rho(\gamma) & 0 & 0
\end{pmatrix}
\]

**Proof.** In the notation of Section 2.3.1, the extension \( \mathbb{E}_{mn} \) has connection form

\[
\Omega := \begin{pmatrix}
0 & 0 & 0 \\
\psi_{2m+2} & e_0 \frac{dq}{q} & 0 \\
0 & 0 & \psi_{2m+2} \frac{dq}{q}
\end{pmatrix}
\].
Applying Proposition 40, the associated transport function is given by $T(\alpha) = 1 + \int_\gamma \Omega + \int_\gamma \Omega \Omega + \cdots$.

Note that while $E_{mn}$ is not split, it has graded quotients isomorphic to $E_m$ and $E_n$ whose individual monodromy representations of the form $\left( \begin{array}{cc} 1 & 0 \\ f_k(\gamma) & \rho(\gamma) \end{array} \right)$ are non-trivially interleaved in $T(\gamma)$. The Deligne cup product $Z_m \cup Z_n$ is trivial if and only if $E_{mn}$ is an admissible VMHS over $\mathcal{M}_{1,1}$. Consequently, we now compute the criterion under which $E_{mn}$ is admissible. Specifically, this is the criterion under which the monodromy representation of $E_{mn}$ is defined over $\mathbb{Q}$.

**Lemma 99.** Let $u_m \in S^{2m} \mathbb{H}$, $u_n \in S^{2n} \mathbb{H}$, and $u_{mn} \in (S^{2m} \otimes S^{2n}) \mathbb{H}$. Let $U = \left( \begin{array}{ccc} 1 & 0 & 0 \\ u_n & 1 & 0 \\ u_{mn} & u_m & 1 \end{array} \right)$. For each $\gamma \in \Gamma$, the conjugate of $T(\gamma)$ by $U$ is

$$
\left( \begin{array}{ccc}
1 & 0 & 0 \\
\frac{f_n(\gamma) - (\delta u_2)(\gamma)}{f_m(\gamma) + u_m f_n(\gamma)} & \frac{\rho(\gamma)}{\rho(\gamma)} & 0 \\
(\delta u_m)(\gamma) - (\delta u_m)(\gamma) & (\delta u_m)(\gamma) & 0 \\
(\delta u_m)(\gamma) & (\delta u_m)(\gamma) & \rho(\gamma) \end{array} \right).
$$

**Corollary 100.** The Betti and de Rham realizations of the monodromy representation of $\mathcal{V}$ are conjugate if and only if there exist $u_m, u_n, u_{mn} \in C^0(\Gamma, S^{2n} \mathbb{H}_\mathbb{C})$ such that

$$
\delta u_{mn} = -f_{mn} + z_{mn} - (f_m \cup u_n) + (u_m \cup z_n) \quad (6.1)
$$

**Proof.** The result follows from Lemma 99 and the observations that $(\delta u_k)(\gamma) = \gamma u_k - u_k$ and $f_j(\gamma) \gamma u_k = (f_j \cup u_k)(\gamma)$.

**Corollary 101.** Let $Z_k \in H^1_D(\mathcal{M}_{1,1}, \mathcal{V}_k)$ correspond to $G_{2k+2}$. The cup product $Z_m \cup Z_n$ on Deligne cohomology is represented by $z_{mn} - f_{mn} + (u_m \cup z_n) - (f_m \cup u_n)$ in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^1(X, \mathcal{V}_m \otimes \mathcal{V}_n))$.

**Proof.** This follows from Equation (6.1).
6.3.2 Massey Triple Product

As before, let $V_k$ denote the VMHS over $\mathcal{M}_{1,1}$ given by $S^{2k}\mathbb{H}(2k + 1)$. For each $k \in \mathbb{Z}_{\geq 2}$, let $Z_k$ be an element of $H^1_D(\mathcal{M}_{1,1}, V_k)$. By Proposition 86, $Z_k$ corresponds to a Yoneda extension $E_k \in \text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(\mathbb{Q}, V_k)$. Let $E_{lm} \in \text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(\mathbb{Q}, E_l \otimes V_m)$ be a primitive of $E_l \circ E_m$ in the sense of Section 6.3.1, and let $E_{mn}$ be similarly defined.

**Lemma 102.** Suppose the Yoneda products $E_l \circ E_m$ and $E_m \circ E_n$ are trivial extensions trivialized respectively by $E_{lm}$ and $E_{mn}$. The products $E_{lm} \circ E_n$ and $E_l \circ E_{mn}$ are trivialized by an extension $E_{lmn}$ given by

$$
0 \longrightarrow E_l \otimes V_m \otimes V_n \longrightarrow E_{lmn} \longrightarrow E_n \longrightarrow 0.
$$

**Proof.** Using Lemma 84, identify $E_m$ with an element in $\text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(V_n, V_m \otimes V_n)$ and $E_l$ with an element in $\text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(V_m \otimes V_n, V_l \otimes V_m \otimes V_n)$. The Massey triple product $\langle E_l, E_m, E_n \rangle$ is trivialized by an extension $E_{lmn} \in \text{Ext}^1_{\text{MHS}(\mathcal{M}_{1,1})}(E_n, E_l \otimes V_m \otimes V_n)$ by Proposition 92. 

**Definition 103.** The Massey triple product $\langle E_l, E_m, E_n \rangle$ is trivialized by the $\mathbb{Q}$-local system $E_{lmn}$ of Lemma 102.

**Lemma 104.** For each $\gamma \in \Gamma$, the monodromy representation $T(\gamma)$ of $E_{lmn}$ has matrix representation

$$
T(\gamma) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \rho(\gamma) & 0 & 0 \\
0 & 0 & \rho(\gamma) & 0 \\
0 & 0 & 0 & \rho(\gamma)
\end{pmatrix}
$$

and the corresponding representation for the Betti realization is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\rho(\gamma) & 0 & 0 \\
0 & \rho(\gamma) & 0 \\
0 & 0 & \rho(\gamma)
\end{pmatrix}
$$

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Proof. In the notation of Section 2.3.1, the extension $E_{lmn}$ has connection form

$$\Omega := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi_{2n+2} & e_0 \frac{dq}{q} & 0 & 0 \\ 0 & \psi_{2m+2} & e_0 \frac{dq}{q} & 0 \\ 0 & 0 & \psi_{2l+2} & e_0 \frac{dq}{q} \end{pmatrix}. \tag{105}$$

Note that while $E_{lmn}$ is not split, it has graded quotients isomorphic to $E_{lm}$ and $E_{mn}$ whose individual monodromy representations of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ f_k(\gamma) & \rho & 0 \\ f_{jk}(\gamma) & f_j(\gamma) & \rho(\gamma) \end{pmatrix}$$

are non-trivially interleaved in $T(\gamma)$. By Proposition 96, the Massey triple product $\langle Z_l, Z_m, Z_n \rangle$ on Deligne cohomology is trivial if and only if $E_{lmn}$ is an admissible VMHS over $\mathcal{M}_{1,1}$. Consequently, we now compute the criterion under which $E_{lmn}$ is admissible. Specifically, this is the criterion under which the monodromy representation of $E_{lmn}$ is defined over $\mathbb{Q}$.

Lemma 105. Let $u_l \in S^2 \mathbb{H}_\mathbb{C}$, $u_m \in S^{2m} \mathbb{H}_\mathbb{C}$, $u_n \in S^{2n} \mathbb{H}_\mathbb{C}$. Let

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_n & 1 & 0 & 0 \\ u_{mn} & u_m & 1 & 0 \\ u_{lmn} & u_{lm} & u_l & 1 \end{pmatrix}. \tag{106}$$

For each $\gamma \in \Gamma$, the conjugate of $T(\gamma)$ by $U$ has extension data

$$f_{lmn}(\gamma) + u_l f_{mn}(\gamma) - f_{lm}(\gamma) \gamma u_n + u_{lm} f_n(\gamma) - u_l f_m(\gamma) \gamma u_n + f_i(\gamma) \gamma (u_m u_n) - f_i(\gamma) \gamma u_{mn} - (\delta u_{lmn})(\gamma) - (\delta u_l)(\gamma) \gamma (u_m u_n) + (\delta u_l)(\gamma) \gamma u_{mn} + (\delta u_{lm})(\gamma) \gamma u_n.$$
Corollary 106. The Betti and de Rham realizations of the monodromy representation of $\mathcal{V}$ are conjugate if and only if there exist $u_l \in S^2 H_{\mathbb{C}}$, $u_m \in S^2 H_{\mathbb{C}}$, $u_n \in S^2 H_{\mathbb{C}}$, such that

$$\delta u_{lmn} = -f_{lmn} + z_{lmn} - (f_l \cup u_{mn}) + (u_l \cup z_n) - (f_m \cup u_n) + (u_l \cup z_{mn}) \quad (6.2)$$

Proof. The result follows from Lemma 105 and the observations that $(\delta u_k)(\gamma) = \gamma u_k - u_k$ and $f_j(\gamma)\gamma u_k = (f_j \cup u_k)(\gamma)$. □

Corollary 107. Let $Z_k \in H^1_D(M_{1,1}, \mathcal{V}_k)$ correspond to $G_{2k+2}$. If $Z_l \cup Z_m = 0$ and $Z_m \cup Z_n = 0$, the Massey triple product $\langle Z_l, Z_m, Z_n \rangle$ on Deligne cohomology is represented by $z_{lmn} - f_{lmn} + (u_l \cup z_n) - (f_l \cup u_{mn}) + (u_l \cup z_{mn}) - (f_m \cup u_n)$ in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, \mathcal{V}_l \otimes \mathcal{V}_m \otimes \mathcal{V}_n)$. It is defined up to an element of $Z_l \cup H^1_D(X, \mathcal{V}_m \otimes \mathcal{V}_n) + H^1_D(X, \mathcal{V}_l \otimes \mathcal{V}_m) \cup Z_n$.

Proof. This follows from Equation (6.2). □
Bibliography


$^1$ http://www.math.uchicago.edu/~may/misc/torext.pdf

$^2$ http://dukespace.lib/duke/edu/dspace/handle/10161/1281
Biography

Anil Venkatesh was born on September 4, 1987, in Philadelphia, PA. He earned a B.A. in mathematics and a B.S.E. in electrical engineering from the University of Pennsylvania in 2009. He pursued graduate study at Duke University and earned a Ph.D. in mathematics in 2015.

He received a National Science Foundation Graduate Research Fellowship in 2010, and a Dean’s Award for Excellence in Teaching from the Duke University Graduate School in 2014. He begins his new appointment as Assistant Professor of Mathematics at Ferris State University in August 2015.