Revealing Asset Quality: Liquidity Signaling and Optimal Stress Tests

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Business Administration in the Graduate School of Duke University

2015
Abstract

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In my first chapter, I present a model in which sellers can signal the quality of an asset both by retaining a fraction of the asset and by choosing the liquidity of the market in which they search for buyers. Although these signals may seem interchangeable, I present two settings which show they are not. In the first setting, sellers have private information regarding only asset quality, and I show that liquidity dominates retention as a signal in equilibrium. In the second setting, both asset quality and seller impatience are privately known, and I show that both retention and liquidity operate simultaneously to fully separate the two dimensions of private information. Contrary to received theory, the fully separating equilibrium of the second setting may contain regions where market liquidity is increasing in asset quality. Finally, I show that if sellers design an asset-backed security before receiving private information regarding its quality, then the optimality of standard debt is robust to the paper’s various settings.

In my second chapter, I explore the question of how informative bank stress tests should be. I use Bayesian persuasion to formalize stress tests and show that regulators can reduce the likelihood of a bank run by performing tests which are only partially informative. Optimal stress tests give just enough failing grades to keep passing grades credible enough to avoid runs. The worse the state of the banking system, the more stringent stress tests must be to prevent runs. I find that optimal stress tests, by reducing the probability of runs, reduce the optimal level of banks’
capital cushions. I also examine the impact of anticipated stress tests on banks’ ex ante incentive to invest in risky versus safe assets.
I dedicate this dissertation to my dear friend, Kaila Marianne Brown.
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1

Search, Liquidity, and Retention:
Signaling Multidimensional Private Information

1.1 Introduction

Models of asymmetric information in finance have identified two main signals of asset quality: retention and illiquidity. Leland and Pyle (1977) and DeMarzo and Duffie (1999) have shown that sellers with private information regarding their asset quality can signal that quality by retaining a fraction of the security. For example, an entrepreneur may retain a high equity stake in his firm in order to signal confidence in his venture to outside investors. In a more recent literature, Guerrieri, Shimer, and Wright (2010) and Chang (2012) present competitive search models in which the liquidity induced by an asset’s price can also sort asset qualities in equilibrium. In their model, holders of high quality assets are willing to set a high price that induces a lower probability of sale, because failing to sell a good asset is not as costly as failing to sell a bad asset. The intuition for both sorting channels is similar: retaining part of an asset and selling an asset in a lower liquidity market are costly to all sellers, but less costly for those with high quality assets because they enjoy a higher payoff.
from their unsold holdings.

At first blush, retention and illiquidity seem to be interchangeable signals of asset quality; a key point of this paper, however, is that they are not. I illustrate not only how liquidity may in fact dominate retention as a signal of asset quality, but also how liquidity and retention may operate simultaneously to separate multiple dimensions of private information.

The paper first modifies the signaling model of DeMarzo and Duffie (1999), so that sellers attempt to sell their securities on a competitive search market. Rather than selecting the fraction of the asset to trade, sellers are forced by assumption to offer their entire asset, and instead price the asset so as to induce a certain probability of sale (liquidity). In the equilibrium of the basic model, sellers signal higher asset quality by selling at a lower liquidity price. I show that as buyer search costs converge to zero, this equilibrium converges to an equilibrium equivalent to that of DeMarzo and Duffie (1999). In particular, the equilibrium liquidity of an asset in my modified model converges to the equilibrium fraction sold of the same asset in DeMarzo and Duffie’s model. This equivalence result comports with our intuition that illiquidity is analagous to retention; in either case, sellers expect some probability of owning at least part of the asset by the end of the trading game.

My next result, however, illustrates a major difference between the two signals. I extend the model to allow sellers to choose not only the liquidity of the selling market, but also the fraction of the asset sold; that is, both liquidity and retention are available signals for the seller. Given the analogy between illiquidity and retention, we might suppose that sellers would be indifferent between the two signals. However, my second result shows that in the Pareto optimal separating equilibrium, retention shuts down: all sellers retain none of the asset, distinguishing themselves solely by the liquidity of their chosen market. The intuition is that selling a lower quantity—i.e., retaining more of the asset—is costly to both buyers and sellers, whereas lower
liquidity markets are costly to sellers but beneficial to buyers, permitting buyers to pay a higher price.

The third result of the paper shows how both liquidity and retention may operate simultaneously to signal the seller type. I consider a model in which not only the asset quality but also the seller’s impatience are private information. It is natural to consider these two attributes together: it seems reasonable that sellers vary in their willingness to sell, and when buyers observe a low asset price, they may struggle to discern if the asset is low quality or if the seller simply has an especially urgent need for cash. Ordinarily, fully separating equilibria over two dimensions are intractable, but I show how both liquidity and retention work together to fully separate both dimensions of the seller’s private information: not only asset quality, but also seller impatience. Including seller impatience in the private information may reverse the usual direction of a key comparative static. In most liquidity signaling games, high quality asset holders sell with lower liquidity, because they are more willing to risk failing to sell than are low-quality asset holders. However, I show that if not only asset quality but also seller impatience are private information, and if both dimensions may be signalled using liquidity and retention, then the Pareto optimal fully separating equilibrium may contain regions where higher quality assets sell with higher liquidity. Intuitively, higher quality assets may be more liquid because buyers find them more attractive, but high quality sellers can offer a fraction low enough to offset the high liquidity and deter low quality types from mimicking. In this way, separation is still maintained, because although liquidity may be increasing in asset quality, the expected fraction sold (the product of liquidity and fraction offered) is always decreasing in asset quality.

The structure of these models, building as they do on the retention model of DeMarzo and Duffie (1999), lend themselves well to analysis of security design. In particular, if a seller designs a security backed by his underlying asset before ac-
quiring private information, we might ask whether the surprising features of the equilibria described above result in a nonstandard security design. However, I show that standard debt is optimal in all of the settings set forth in this paper, regardless of the presence or absence of search frictions, liquidity signaling, and privately known seller patience.

**Relation to Literature.** This section gives a broad overview of the relation between my paper and the preceding finance literature on signaling, but the reader should note that in my conclusion, I include a figure which depicts these connections visually and uses the model’s notation to show my contribution more precisely.

This paper unites and extends two strands of finance theory which approach the problem of asymmetric information and sorting in different ways. On the one hand, the retention signal is emphasized by Leland and Pyle (1977) and DeMarzo and Duffie (1999), who show that sellers with higher quality assets retain a greater fraction of the security. A more recent paper by Hartman-Glaser (2012) explores the interaction of retention with reputation in a dynamic setting.

On the other hand, the liquidity channel is illustrated by Guerrieri, Shimer, and Wright (2010), who combine competitive search with adverse selection and find an equilibrium in which liquidity is decreasing in asset quality. Chang (2012) applies the work of Guerrieri, Shimer, and Wright (2010) to a continuum of types, allowing her to characterize the equilibrium with a differential equation, and utilizes a mechanism design approach to solve for the equilibrium, a strategy my paper draws on. The model in Section 3.2 of my paper incorporates the possibility of both of these signals, but shows how retention shuts down, leaving liquidity to function as the only signal. This section also draws on the solution strategy of Viswanathan (1987), who studies the firm’s choice of signalling instrument in a dynamic model where managers have private information about the firm’s future prospects.

Chang (2012) and Guerrieri and Shimer (2012) both explore multidimensional
private information, but unlike my paper, neither allow for multiple signals, and hence focus on pooling equilibria. Section 6 of this paper, however, not only considers multidimensional private information, but also incorporates the second signal of retaining a portion of the asset as in DeMarzo and Duffie (1999). This allows me to obtain a fully separating equilibrium over the two-dimensional seller type space. Also, in contrast to all of these papers, my paper is the first to apply the adverse selection based competitive search theory to security design.

The setting in my model is static, so I formalize liquidity as the probability of selling an asset within a single period, but several papers present dynamic settings in which speed of sale can sort seller types; in this way, speed of sale is the dynamic version of liquidity. Chang (2012) and Guerrieri and Shimer (2012) apply the competitive search framework in a dynamic setting, finding that higher quality assets sell with lower liquidity (slower speed). Fuchs and Skrypacz (2013) examine a dynamic model in which sellers can signal high quality by waiting longer to sell. Similarly, Daly and Green (2012) and Kremer and Skrzypacz (2007) present dynamic settings in which high quality sellers wait longer to sell so that positive news is revealed. Varas (2014) presents a dynamic model in which sellers can signal by both waiting to sell and retaining fractional holdings in the asset. Of all these dynamic papers, only Chang (2012) considers private information about seller patience, and only Varas (2014) considers the signal retention. My setting is unique for considering both asset quality and seller patience as private information along with both retention and liquidity as signals, showing how the two signals can fully separate the two dimensions of private information.

The paper’s main technical contribution is to find an equilibrium in which types with two dimensions of private information fully separate using two signals. Engers (1987) establishes the existence of such equilibria, but does not provide a characterization. Rochet and Choné (1999) consider multidimensional screening in which
the dimension of signals and of types are equal, but they focus on profit maximization rather than full separation. As mentioned above, Chang (2012) and Guerrieri and Shimer (2014) consider multidimensional private information, but both papers consider only the signal liquidity and hence focus on partial pooling equilibria. Edmans and Mann (2015) study a model of asset sales in which private information consists of (binary) quality and (continuous) synergy, but they find equilibria which fully separate at most only the quality dimension. He (2009) extends Leland and Pyle’s (1977) signaling model to multiple dimensions and solves a fully separating equilibrium, but the symmetry which allows him to simplify and solve the resulting partial differential equations is not present in my setting. My contribution is to find a multidimensional fully separating equilibrium in a new setting, one in which the dimensions of private information may be transformed to consist of (1) buyer value and (2) seller value. Because agents on one side of the market care directly about only one dimension of private information, the seller’s incentive compatibility constraint collapses to a single dimension, allowing me to characterize the equilibrium with an ordinary differential equation.

Because the empirical literature regarding asymmetric information is still emerging, there are not many stylized facts to which the theory of this paper and others may be directly mapped. Downing, Jaffee, and Wallace (2009) show that mortgage originators are more likely to sell ex-post worse mortgages; Begley and Purnanandam (2013) show that RMBS deals with a higher level of equity tranche have significantly lower foreclosure rates, controlling for observable risk factors. These findings are consistent with the prediction of many theory papers, including this one, that sellers retain more of high quality assets than low quality assets. Other papers illustrate more broadly the significance of asymmetric information in financial transactions. For example, Kelly and Ljungqvist (2012) show that information asymmetry (brokerage closure) causes stock prices and uninformed (retail) investor demand to fall.
Piskorski, Seru, and Witkin (2013) show that lenders charge a higher interest rate on misrepresented RMBS relative to otherwise similar loans, but not enough to reflect default risk. The hope is that this paper, along with other related theory, will help to guide future empirical exploration of this topic.

Section 2 describes the setting of the economy, and Section 3 explores both the one- and two-signal models where only asset quality is privately known, and also considers security design. Section 4 augments the private information to both asset quality and seller impatience, solves the fully separating equilibrium, compares it to an equilibrium with partial pooling, and discusses the implications of higher dimensions for security design. Section 5 revisits connections to the literature, and Section 6 concludes.

1.2 Environment

The environment consists of a unit mass of risk-neutral sellers and an endogenous mass of risk-neutral buyers. The model operates for one period, which begins at date 0 and ends at date 1. Each seller owns an asset that generates future cash flows \( f \in [\underline{f}, \bar{f}] \subset \mathbb{R}_{++} \), and the measure of sellers holding assets of quality less than or equal to \( f \) is quantified by distribution function \( G : \mathbb{R} \rightarrow [0, 1] \). The quality \( f \) of each asset is privately known by its owner. The seller discounts future cash flows at rate \( \delta \in (0, 1) \) which is fixed across sellers and publicly known, so the seller’s value of retaining the assets is therefore \( \delta f \). This discounting gives the seller incentive to raise cash by selling some portion of the asset.

All buyers discount at rate normalized to 1, and the wedge between buyers’ and sellers’ discount rates allows gains from trade. Buyers must pay cost \( k \) in order to search for a seller. After paying \( k \), the buyer advertises his desire to buy an asset at price \( p \), which defines a market. Sellers select the price (market) \( p \) from the list \( P \) of posted prices at which to search for a buyer.
1.2.1 Search Market

The buyers and sellers trade in a continuum of competitive search markets, indexed by prices $p$, where each market is uniquely distinguished by the price $p$ at which trade occurs. I let $\theta \equiv \frac{b}{s}$ be the ratio of buyers to sellers in a given market $p$. When sellers select a price $p$ at which to sell, they take the ratio $\theta(p)$ as given, so that a single (infinitesimal) seller cannot alter the ratio $\theta$ in that market. The number of buyers in each market will be pinned down by a zero-profit condition so that buyers are indifferent between markets.

When a seller trades in market $p$, he encounters search frictions which impede his ability to sell, so that even if a market contains equal numbers of buyers and sellers, not every agent will be able to transact. I parameterize this friction with a matching function $m(\theta)$, which maps the market tightness $\theta$ to the seller’s probability of sale $m$. If $n(\theta)$ is the probability that a given buyer will be able to buy in market $\theta$, then pairwise matching requires $b \cdot n(\theta) = s \cdot m(\theta)$, so $n(\theta) = m(\theta)/\theta$. I assume that $m(\theta)$ (Figure 1.2.1) is strictly increasing, strictly concave, and $0 \leq m(\theta) \leq \min[\theta, 1]$, which guarantees that both $m$ and $n$ are in $[0, 1]$. I also assume that $m(0) = 0$, that $m(\theta) \to 1$ as $\theta \to \infty$, and that $m(\theta)/\theta \to 1$ as $\theta \to 0$.

1.3 1-D Private Information: Asset Quality

1.3.1 One Signal: Liquidity

Both buyers and sellers have rational expectations about the equilibrium market tightness $\theta(p)$ associated with each price $p$; both agents take the function $\theta(p)$ as given when choosing a price to maximize their own payoffs, but the function will be derived in equilibrium. Buyers also form expectations $\mu(f|p)$ about which types of sellers will search for an asset of price $p$. So given a market tightness function $\theta(p)$ and beliefs $\mu(p)$, the buyer’s problem is
The fraction $m(\theta(p))$ is the probability that the buyer searching for an asset of price $p$ succeeds in purchasing an asset. If the buyer succeeds in purchasing the asset, he must pay price $p$ and expects a payoff of $E[\tilde{f}|p] = \int \tilde{f} d\mu(\tilde{f}|p)$, the buyer’s expected quality of an asset purchased at price $p$. Finally, search costs $k$ must be paid in order for the buyer to search, regardless of whether he succeeds at matching with the seller.

The buyers move first, selecting prices to post given market tightness $\theta(p)$ and beliefs $\mu(f|p)$ about seller types. I let $P$ denote the set of all posted prices $p$. Each seller surveys the array $P$ of prices $p$ posted by the buyers and selects a $p \in P$ to
maximize his own payoff \( m(\theta(p))p + (1 - m(\theta(p)))\delta f \). Collecting terms in the seller’s payoff function which depend on his choice variable \( p \), I have the seller’s problem:

\[
\max_{p \in P} m(\theta(p))[p - \delta f].
\]

In this formulation, the seller selects \( p \in P \) to maximize his net profit. Because I have assumed that sellers pay no search costs, his profit only deviates from zero if he succeeds at selling his asset, in which case he receives price \( p \) but must give up the value \( \delta f \) of retaining his asset.

I assume throughout the paper that regardless of seller type, ex-post gains from trade \( (1 - \delta)f \) exceed the search costs \( k \).

**Assumption 1.**

1. If \( \delta \) is fixed for all sellers, then \( \forall f \in [\hat{f}, \bar{f}], (1 - \delta)f > k \).

2. If \( \delta \) varies across sellers, then \( \forall (f, \delta) \in [\hat{f}, \bar{f}] \times [\hat{\delta}, \bar{\delta}], (1 - \delta)f > k \).

This simply guarantees that for any seller, there exist terms of trade such that trade with the seller is more efficient than refraining from trade. If buyers know the types of all sellers, as is the case in any fully separating equilibrium, then for any seller type, there always exists some market tightness \( \theta \) and price \( p \) that will induce a buyer to trade with that seller. Without this assumption, there may exist some sellers which cannot attract any buyers, regardless of the terms of trade.

**Complete Information**

As a benchmark, consider the case in which buyers know the seller’s types. Under complete information, the buyer’s profit is as expressed in (1.1), except that the conditional expected value \( E_P[\hat{f}|p] \) in the buyer’s zero-profit condition may be replaced directly by the asset’s fundamental value \( f \), yielding the seller’s complete information problem:
\[
\max_{p \in P} \quad m(\theta(p))[p - \delta f] \\
\text{s.t.} \\
\frac{m(\theta(p))}{\theta(p)}[f - p] - k = 0
\]

Substitute \( p = f - k\theta(p)/m(\theta(p)) \) in the objective function to obtain:

\[
\max_{p > 0} \quad m(\theta(p))[f - k \frac{\theta(p)}{m(\theta(p))} - \delta f]
\]

or

\[
\max_{\theta > 0} \quad m(\theta)(1 - \delta)f - k\theta \tag{1.2}
\]

This representation of the seller’s objective function illustrates the tradeoff that sellers face. Choosing a price which raises the ratio of buyers to sellers has the benefit of increasing the probability of sale \( m(\theta) \), which is more valuable the larger the ex-post gains from trade \((1 - \delta)f\). This leads to relatively more buyers paying search costs \( k \), but relatively fewer sellers to compensate them for these costs so that buyers break even in expectation; i.e., each seller must bear a greater portion \( k\theta \) of the aggregate search costs in their particular market. The seller’s first-order condition, together with the buyer’s free entry condition, leads directly to the complete information equilibrium \((\theta_{CI}^*(f), p_{CI}^*(f))\) for asset quality \( f \):

\[
m'(\theta_{CI}^*(f)) = \frac{k}{(1 - \delta)f} \\
p_{CI}^*(f) = f - k \frac{\theta_{CI}^*(f)}{m(\theta_{CI}^*(f))}
\]

The complete information equilibrium \( \theta_{CI}^* \) is illustrated in Figure 1.3.1. The equilibrium liquidity is increasing in asset quality \( f \), because as shown in Equation (1.2), as \( f \) rises, so does the potential gain from trade \((1 - \delta)f\), which yields a higher
marginal benefit of liquidity. Crucially, I have assumed that the buyer’s search cost $k$ is constant regardless of market tightness $\theta$ or asset quality $f$, so that the seller’s marginal cost of liquidity is also constant.

**Asymmetric Information**

Now assume that the buyer doesn’t know the seller’s type, but can only form some expectation of it, conditional on the market $p$ in which the seller trades. Here I invoke an equilibrium concept developed in Guerrieri, Shimer, and Wright (2010), who are the first to develop a general framework for analyzing competitive search with adverse selection. The scope of their paper is one dimensional private information and one dimensional signaling, which is the setting of this section, and they show that there always exists a separating equilibrium. Define the fully separating equilibrium as follows:

**Definition 1.** A fully separating equilibrium for the liquidation game is

1. a set $P$ of prices $p \in \mathbb{R}_+$,
2. a market tightness function $\theta(p) : \mathbb{R}_+ \to \mathbb{R}_+$, and
3. buyer beliefs $\mu(f|p) : [\underline{f}, \bar{f}] \times \mathbb{R}_+ \to [0, 1]$

such that

1. Full separation: For all $p \in P$, there exists a unique $f \in [\underline{f}, \bar{f}]$ such that $\mu(f|p) > 0$.
2. Seller’s problem: For all $p \in P$ and all $f \in [\underline{f}, \bar{f}]$, $\mu(f|p) > 0$ implies

$$p \in \arg\max_{p' \in P} m(\theta(p'))(p' - \delta f).$$

3. Buyer’s problem:
(a) Free entry: For all $p \in P$ and all $f \in [\underline{f}, \bar{f}]$,

$$\frac{m(\theta(p))}{\theta(p)} (E[\bar{f}|p] - p) - k = 0$$

(b) No profitable deviation: There does not exist a $p \notin P$ such that

$$\frac{m(\theta(p))}{\theta(p)} (E[\bar{f}|p] - p) - k > 0.$$

I refine the equilibria using the off-equilibrium belief structure of Chang (2012) and Guerrieri et al. (2010), which resembles the refined Walrasian general equilibrium approach in Gale (1992). Let the equilibrium profit of type $f$ seller be given by $\Pi(f)$. When a buyer contemplates posting a price $p$ off the equilibrium path, i.e. $p \notin P$, he forms beliefs about the types that he will attract at that price. Then for any $p \notin P$ define

$$\theta(p, f) = \inf\{\theta > 0 : m(\theta)(p - \delta f) \geq \Pi(f)\}$$

$$\theta(p) = \inf_{f \in [\underline{f}, \bar{f}]} \theta(p, f) \quad T(p) = \arg \inf_{f \in [\underline{f}, \bar{f}]} \theta(p, f).$$

I then place the following restriction on the off-equilibrium beliefs $\mu(f|p)$:

For any price $p \notin P$ and type $f$, $\mu(f|p) = 0$ if $f \notin T(p)$.

I apply the mechanism design solution strategy of Chang (2012) in order to characterize the equilibrium. The next proposition characterizes the fully separating equilibrium, which is a differential equation similar to that of Chang (2012).

**Proposition 1.** If private information consists only of the single dimension asset quality $f \in [\underline{f}, \bar{f}]$, and sellers signal only with liquidity $\theta$, then the separating equilibrium $(p^*(f)), \Theta^*(p), \mu(f|p))$ is characterized as follows:
\( (i) \) \( p^*(f) = f - k \frac{\theta^*(f)}{m(\theta^*(f))} \)

\( (ii) \) \( \theta^*(p) = \Theta^*(p^{*-1}(p)) \)

\( (iii) \) \( \mu(f|p) = \mathbb{I}(p^*(f) = p) \)

where \( \Theta^*(f) \) solves

\[
\left( m'(\Theta)(1 - \delta)f - k \right) \frac{d\Theta}{df} = -m(\Theta), \tag{1.3}
\]

with \( \Theta^*(f) = \Theta_{CI}(f) \) and \( \Theta^*(f) < 0 \).

The equilibrium \( \Theta^* \) identified in Proposition 1 is illustrated in Figure 1.3.1, where it is compared to the full information equilibrium \( \Theta_{CI}^* \). We can see that the equilibria are strictly monotone, but in opposite directions. The intuition is that asymmetric information incentivizes sellers with higher quality assets to distinguish themselves from those with lower quality. Sellers with higher quality securities can afford to trade in markets of lower liquidity, and they choose to do so in order to prevent lower quality sellers from mimicking them. The lowest quality seller, however, has no one beneath him to mimic him, so downward distortion is not necessary and he trades in the same market \( \theta \) as with full information.

The asymmetric information price, on the other hand, is higher for each security \( f \) than under complete information. Note that the buyer’s free entry condition implies that each price is composed of the value of the security \( f \) minus a discount \( k\theta/m(\theta) \), which represents search costs divided by the buyer’s probability of a successful purchase. Figure 1.3.1 shows that asymmetric information implies a lower liquidity \( \theta \) per security \( f \), lowering the discount \( k\theta/m(\theta) \). The intuition is that asymmetric information requires higher quality sellers to trade in lower liquidity markets, so that buyers have a higher probability of purchase for a given \( f \), and therefore require a lower discount.
Figure 1.2: Complete information market tightness $\Theta_{CI}^*$ and asymmetric information equilibrium market tightness $\Theta^*$.

The equilibrium of Proposition 1 is similar to the equilibrium of DeMarzo and Duffie (1999), in that liquidity is decreasing in security quality $f$, but in my paper there is a price discount due to search costs. The following theorem shows just how similar these two equilibria are: as search costs $k$ in my model converge to zero, the equilibrium converges to an equilibrium equivalent to that in DeMarzo and Duffie (1999).

**Theorem 1.** As search costs $k \to 0$, 

(i) $p^*(f) \to f$
\[(ii) \ m(\Theta^*(f)) \rightarrow \left(\frac{f}{q}\right)^{\frac{1}{1+\delta}}\]

\[(iii) \ P^*(m(\theta)) \rightarrow \frac{f}{m(\theta)^{1-\delta}},\]

which is equivalent to the separating equilibrium in DeMarzo and Duffie (1999) if

\(m(\theta)\) is replaced by fraction sold \(q\).

The first item of the theorem is intuitive; as search costs go to zero, buyers require no discount in order to compensate them for searching, so prices converge to the inherent value of the asset. The second item illustrates the equivalence of liquidity (probability of sale) in my model to retention (fraction sold) in the model of DeMarzo and Duffie. A seller with any particular asset quality \(f\) will sell the asset, in the limit, with the same probability \(m(\Theta^*(f))\) as his fraction sold \(Q^*(f)\) in the model of DeMarzo and Duffie. The third item follows from the first two, and is included simply for completeness in order to show that the relation between sale probability \(m(\theta)\) and price \(P^*(m(\theta))\) is the same as DeMarzo and Duffie’s formula relating price to fraction sold \(q\).

The theorem is reassuring because it corresponds with our intuition that signaling with liquidity (probability of sale) is analogous to signaling with retention (fraction unsold); in both cases, sellers have some probability of being stuck with at least part of the asset at the end of the period. This prospect is less onerous for sellers with good assets, so these sellers take on either a lower probability of sale or a lower fraction sold in order to signal that quality to buyers. The next section, however, illustrates how these two seemingly identical signals differ.

1.3.2 Two Signals: Liquidity and Retention

Complete Information

Now suppose that the seller chooses not only the price \(p\) (and therefore \(\theta\)) on which to sell, but also the fraction \(q\) of the security he wishes to sell. So sellers now potentially
have two channels available with which to signal their asset quality. First consider the complete information problem:

\[
\max_{\theta > 0, q \in [0,1]} m(\theta(p, q))q (p - \delta f)
\]

s.t.

\[
\frac{m(\theta(p, q))}{\theta(p, q)} q (f - p) - k = 0
\]

Rearrange the free-entry condition to isolate \(p\) and plug into the seller’s objective function to obtain:

\[
\max_{\theta > 0, q \in [0,1]} m(\theta)q(1 - \delta)f - k\theta,
\]

which is clearly solved by \(q_{C1}(f) = 1\) and \(\Theta_{C1}(f)\) solves \(m'(\theta)(1 - \delta)f = k\). So the complete information allocation is unchanged when sellers can choose both the price \(p\) and the fraction \(q\).

Asymmetric Information

Now consider the case of asymmetric information. I define a signaling equilibrium in the case of two signals.

**Definition 2.** A fully separating equilibrium for the two-signal game with one-dimensional private information is

1. a set \(M \subset \mathbb{R}_+ \times [0, 1]\) of price-quantity pairs \((p, q)\)

2. a market tightness function \(\theta(p, q) : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}_+\), and

3. buyer beliefs \(\mu(f|p, q) : \left[f, \bar{f}\right] \times \mathbb{R}_+ \times [0, 1] \rightarrow [0, 1]\)

such that

1. Full separation: For all \((p, q) \in M\), there exists a unique \(f \in \left[f, \bar{f}\right]\) such that \(\mu(f|p, q) > 0\).
2. Seller’s problem: For all \((p, q) \in M\) and all \(f \in [\underline{f}, \bar{f}]\), \(\mu(f|p, q) > 0\) implies

\[(p, q) \in \arg \max_{(p', q') \in M} m(\theta(p', q'))q'(p' - \delta f).\]

3. Buyer’s problem:

(a) Free entry: For all \((p, q) \in M\) and all \(f \in [\underline{f}, \bar{f}]\),

\[\frac{m(\theta(p, q))}{\theta(p, q)} q(E[\bar{f}|p, q] - p) - k = 0\]

(b) No profitable deviation: There does not exist a \((p, q) \notin M\) such that

\[\frac{m(\theta(p, q))}{\theta(p, q)} q(E[\bar{f}|p, q] - p) - k > 0.\]

Off equilibrium beliefs are restricted in a manner directly analogous to the one-signal case.

Here I use an approach from Viswanathan (1987) to find the Pareto optimal equilibrium. Viswanathan (1987) considers a setting in which managers have private information about the firm’s future prospects, and can signal the prospects using multiple instruments. Viswanathan’s solution strategy uses a mechanism design to find the set of equilibria, and optimal control to find the Pareto optimal signaling strategy.

The following proposition shows that sellers use only liquidity \(\theta\) to signal asset quality.

**Theorem 2.** If only asset quality \(f\) is private information, the two-signal separating equilibrium \((p^*(f), q^*(f), \theta^*(p, q), \mu^*(f|p, q))\) is given by:

(i) \(p^*(f) = f - \frac{k \theta^*(f)}{m(\theta^*(f))}\)
(ii) \( q^*(f) = 1 \ \forall f \in [\underline{f}, \bar{f}] \)

(iii) \( \theta^*(p, q) = \Theta^*(p^{s-1}(p)) \)

(iv) \( \mu(f|p, q) = \mathbb{I}(\langle p^*(f), q^*(f) \rangle = (p, q)) \)

where \( \Theta^*(f) \) solves

\[
\left( m'(\Theta)(1 - \delta)f - k \right) \frac{d\Theta}{df} = -m(\Theta), \quad \Theta^*(f) = \Theta_{CI}(\underline{f}),
\]

and off-equilibrium \( \theta(p, q) \) and \( \mu(f|p, q) \) are clarified in the proof (Appendix).

The theorem says that the seller will attempt to sell the entire asset regardless of his private information, and will distinguish himself instead by the liquidity of the market he sells in. That is, the retention signal \( q \) shuts down, and higher types signal by selling in a market of lower liquidity. Note that the differential equation characterizing the equilibrium liquidity for each type is the same as it was in the one-signal case, when sellers did not have the retention signal available to them. So, the availability of the additional signal of retention does nothing to change the equilibrium payoff or allocation of the sellers.

This result is consistent with the simpler asset market example in Guerrieri, Shimer, and Wright (2010). In their model, only two seller types exist, and the terms of the contract are two-dimensional: a transfer (price), and conditional on a match, an ex-post trading probability (analogous to my fraction sold \( q \)). They also find that in equilibrium, the ex-post trading probability is 1 for both types, and separation occurs via the matching probability instead; so Proposition 2 confirms their result in a setting with a continuum of types. The continuum setting allows me to use optimal control to solve the equilibrium, which yields a clear intuitive cost-benefit representation (discussed below) of the advantage of signaling with liquidity \( \theta \) rather than fraction sold \( q \). This proposition also serves as a useful contrast to
the next section’s setting with two-dimensional private information, in which almost
every type distorts fraction sold $q$ below 1 in order to fully separate both asset quality $f$ and seller patience $\delta$.

To understand the intuition behind Proposition 2, recall that the seller’s and buyer’s profits, respectively, are given by the following expressions

\[
\text{Seller: } m(\theta)q(p - \delta f) \quad \text{Buyer: } n(\theta)q(f - p) - k = 0
\]

In order to signal higher asset quality $f$, the seller may choose a lower market tightness $\theta$ (and therefore lower sale probability $m(\theta)$) or a lower fraction $q$; both of these signals are not only costly, but are equally costly from the seller’s perspective. Note, however, that lowering $\theta$ and $q$ have inverse effects on the buyer: because buying probability $n(\theta)$ is decreasing in market tightness $\theta$, choosing a market with lower liquidity $\theta$ benefits buyers by resulting in a higher matching probability, whereas selling a lower quantity $q$ is costly to both buyers and sellers. It makes sense that the Pareto optimum would involve raising $q$ to its upper limit so as to benefit both parties, because the signaling can be accomplished solely by distorting market tightness downward; although costly to sellers, lowering $\theta$ is beneficial to buyers, enabling them to pay a higher price.

This heuristic argument can be made more precise if the selection of the Pareto optimal separating equilibrium is framed as an optimal control problem. The proof applies the optimal control setup in a rigorous fashion, but here I broadly illustrate the setup in order to extract some helpful intuition.

Imagine a social planner who receives reports of each seller’s asset quality and offers them a price $p$, a market tightness $\theta$, and a fraction sold $q$. That is, the planner designs a direct revelation mechanism $(p, \theta, q) : [f, \bar{f}] \to \mathbb{R}^2_{++} \times [0, 1]$. The planner must select the mechanism which satisfies both buyer free entry and (by the revelation principle) seller incentive compatibility, and also offers the highest
payoff possible to each seller. Given some mechanism $i$, I can eliminate price $p$ by substituting the buyer’s free entry condition into the seller’s profit function, and then write the type $f$ seller’s profit for reporting $\hat{f}$ as

$$\Pi^i(\hat{f}|f) = m(\theta_i(\hat{f}))q_i(\hat{f})(\hat{f} - \delta f) - k\theta_i(\hat{f}).$$

Local incentive compatibility requires that

$$0 = \Pi^i(f|f) = (m'(\theta_i(f))q_i(f)(1 - \delta)f - k)\theta'_i(f) + m(\theta_i(f))(1 - \delta)f q'_i(f) + m(\theta_i(f))q_i(f). \tag{1.4}$$

In equilibrium, sellers truthfully report their type, so denote the seller’s equilibrium profit by $\bar{\Pi}(f) \equiv \Pi(f|f)$. The envelope condition allows rewriting the local incentive compatibility constraint as

$$\bar{\Pi}'_i(f) = -\delta m(\theta_i(f))q_i(f). \tag{1.5}$$

Now suppose that a seller is trying to decide between two incentive compatible mechanisms $i$ and $j$, which agree up to type $f$ and diverge thereafter. Because the mechanisms are equal at type $f$, the envelope condition (1.5) implies that

$$\bar{\Pi}'_i(f) = -\delta m(\theta_i(f))q_i(f) = \delta m(\theta_j(f))q_j(f) = \bar{\Pi}'_j(f),$$

so the slope of the profit function under both mechanisms is equal at the divergent point $f$. However, the curvature of the profit function may not be. To see this, observe that

$$\bar{\Pi}''_i(f) = -\delta m'(\theta_i(f))q_i(f)\theta'_i(f) - \delta m(\theta_i(f))q'_i(f), \tag{1.6}$$

and although $(\theta_i(f), q_i(f)) = (\theta_j(f), q_j(f))$, there is no guarantee that $(\theta'_i(f), q'_i(f)) = (\theta'_j(f), q'_j(f))$. Therefore, the curvature of the profit function under one mechanism may be unequal to the curvature under another. It is clear that the mechanism which offers the highest curvature must dominate the other mechanism.
The upshot of this reasoning is that the social planner’s problem can be framed as an optimal control problem in which the planner chooses \( \theta'(f) \) and \( q'(f) \) in such a way as to both satisfy incentive compatibility (1.4) and to maximize the curvature of the profit function (1.6).

\[
\max_{-\theta',-q'} \delta m'(\theta)q(-\theta') + \delta m(\theta)(-q') \tag{1.7}
\]

s.t.

\[
m(\theta)q = \left( m'(\theta)q(1 - \delta)f - \delta' \right) (-\theta') + m(\theta)(1 - \delta)f (-q') \tag{1.8}
\]

In this optimal control problem, the states are \((\theta, q)\) and the controls are \((-\theta', -q')\). I have written the controls with a minus sign in order to capture the fact that higher types distinguish themselves by posting a lower value of \( \theta \) or \( q \) than that of lower types, so to signal is to sell with lower liquidity \( \theta \) (i.e., \( -\theta' \) is high) and/or lower quantity \( q \) (i.e., \( -q' \) is high).

The control problem (1.7) and (1.8) makes clearer why the signal \( q \) gets no use in the Pareto optimal equilibrium. The objective function (1.7) weights each signal according to the marginal benefit of using that signal, whereas the IC constraint (1.8) displays the marginal cost of using each signal.

Because both the objective and the IC constraint are linear, the solution is bang-bang, and the signal with the highest benefit-cost ratio does all of the signaling, and the other signal gets no use. It is clear from the coefficients in (1.7) and (1.8) that signalling with liquidity \( \theta \) has the highest benefit-cost ratio because of the search costs \( k \). Were it not for the presence of search costs \( k, \theta \) and \( q \) would have identical benefit-cost ratios and so would be equally preferable from the planner’s perspective.

To signal higher quality, the seller can trade in a market of lower liquidity \( \theta \), or he can simply sell a smaller fraction \( q \) of his security. Both are costly signals because they lower the expected fraction of the security sold. However, trading in a market
of lower liquidity $\theta$ has the benefit of requiring lower search costs. So if a seller trades in a lower market $\theta$, the total search costs per seller are lower. This makes signaling via market liquidity $\theta$ less costly than signaling via fraction sold $q$, so the seller always prefers to signal by trading in a less liquid market $\theta$.

1.3.3 Security Design

Now suppose that before sellers privately learn the quality of their asset, they can design a security backed by that asset. Anticipating the arrival of private information and the equilibrium of the trading game discussed above, the seller designs the security that will result in the highest expected payoff.

The seller’s asset generates future cash flows denoted by $X$, a non-negative bounded random variable. As in the previous section, the seller discounts future cash flows by a factor $\delta \in (0, 1)$, which is fixed at a common value for all sellers and is public knowledge. The buyer’s discount factor is normalized to 1, and the wedge between discount factors generates gains from trade. In order to raise cash, the seller creates an asset-backed security to sell. The payoff of the security $F = \phi(X)$ is contingent on the asset’s cash flows $X$, so $F$ is a real-valued random variable measurable with respect to $X$. Security holders claims are secured solely by the assets, so $0 \leq F \leq X$.

After the design of the security, but before the sale, the seller receives private information relevant to the payoff of the security. Denote the information by random variable $Z \in \mathbb{R}$, so that the issuer’s conditional valuation of the security is $E(F|Z)$. For each security design $F$, the issuer assumes some liquidity schedule $\theta_F(p) : \mathbb{R}_+ \to \mathbb{R}_+$; if the seller posts price $p$, then $\theta_F(p)$ is the market liquidity of the security $F$. Given a security $F$, I can write the seller’s objective as a function of price $p$, and therefore market tightness $\theta_F(p)$ as follows:

$$
\delta E(X - F|Z) + \delta(1 - m(\theta_F(p)))E(F|Z) + m(\theta_F(p))p
$$
\[
\delta E(X|Z) + m(\theta_F(p))[p - \delta E(F|Z)].
\]

After the seller not only designs the security, but also receives private information \( z \), then the seller’s relevant private information is simply the particular outcome \( f = E[F(X)|z] \) of \( E(F|Z) \), and the seller’s liquidation problem is

\[
\Pi_F(f) = \max_{p>0} m(\theta_F(p))[p - \delta f]. \quad (1.9)
\]

The only difference between the seller’s problem in this section and the previous section is that the equilibrium liquidation schedule \( \theta_F(p) \) and therefore profit function \( \Pi_F(f) \) depend on the structure of the security \( F \). Before receiving private information \( Z \), the seller anticipates this dependency, and designs the security \( F \) in order to induce the most favorable profit function \( \Pi(f) \) to maximize his expected profit. Letting \( V(F) = E[\Pi_F(E(F|Z))] \) denote the sellers expected profit contingent on security \( F \), the security design problem is

\[
\sup_F V(F).
\]

I can summarize the timing of the game as follows:

1. The seller designs a security \( F \).
2. The seller receives private information \( Z \), which determines a particular outcome \( f \) of \( E(F|Z) \).
3. Buyers post a set of prices \( P \subset \mathbb{R}_+ \).
4. Each seller selects a sale price \( p \in P \) with an associated liquidity \( \theta_F(p) \), in order to maximize his expected profit.
5. The following period, cash flows are realized and remaining consumption takes place.
Before solving for the optimal $F$, I first define the following restriction on the conditional distribution of $X$ given $Z$:

**Definition 3.** An outcome $z$ of $Z$ is a uniform worst case if, for any other outcome $z$ and any interval $I \subseteq \mathbb{R}_+$ of outcomes of $X$,

1. if $\mu(X \in I|z) > 0$, then $\mu(X \in I|z) > 0$;

2. the conditional of $\mu(\cdot|z)$ given $X \in I$ has first-order stochastic dominance over the conditional of $\mu(\cdot|z)$ given $X \in I$.

Note that the existence of a uniform worst case is weaker than the monotone likelihood ratio property. I am now ready to solve for the optimal security $F$.

**Proposition 2.** If there is a uniform worst case, then among increasing monotone securities, a standard debt contract $F(X) = \min(X, d)$ is an optimal security.

The intuition here is similar to the intuition in DeMarzo and Duffie (1999). At one extreme, riskless debt is the least information sensitive security because it eliminates the problem of asymmetric information, but it also requires sellers to hold a large portion of their future cash flows which they would prefer to sell immediately. At the other extreme, pure equity is the most information sensitive security but allows sellers to receive capital now for all of their future cash flows. Standard debt with a risky portion falls between these two extremes, trading off the lemons cost due to asymmetric information with the cost of holding unsecuritized cash flows.

1.4 2-D Private Information: Quality and Impatience

Suppose that sellers are heterogenous not only in asset quality $f \in [\underline{f}, \bar{f}]$, but also in patience, parameterized by discount factor $\delta \in [\underline{\delta}, \bar{\delta}]$, and that both dimensions are privately known to the seller. This seems reasonable, as different sellers could have
different willingness to sell, and buyers may not know if asset quality or impatience
are affecting the terms of sale. While two dimensional private information is con-
sidered by both Chang (2012) and Guerrieri and Shimer (2013), this section is more
similar to that in Guerrieri and Shimer.

There are several differences between the setup of this section and the setup of
Guerrieri and Shimer (2013), but here I highlight the most salient ones. The must
fundamental difference is that in their model, the sellers must sell the entire asset
or none at all, whereas in my model sellers can signal by selling only a fraction of
the asset. This additional signal allows the sellers to separate both asset quality
and patience, obviating the assumptions made by Guerrieri and Shimer about the
distribution of seller types. A second major difference is the form of the matching
function, which they assume is piecewise linear: \( m(\theta) = \min[\theta, 1] \). My matching
function, on the other hand, is a more general strictly increasing, strictly concave
function bounded between zero and one, and the specific example I explore includes
their matching function as a limiting case. The strictly increasing form of my match-
ing function, though less convenient to work with, turns out to be essential for full
separation. This general form for \( m(\theta) \) also allows me to find conditions under which
liquidity \( \theta \) may be increasing in asset quality. Third, Guerrieri and Shimer let the
agents endogenously decide whether to buy, sell, do both, or neither, whereas in my
model agents are assigned exogenously to be either buyers or sellers.

1.4.1 Complete Information

Denote the seller’s privately known type by \( s = (f, \delta) \in [\bar{f}, f] \times [\bar{\delta}, \delta] \equiv S \subset \mathbb{R}^2_+ \).
Under complete information,

\[
\max_{\theta > 0, \theta \in [0, 1]} m(\theta(p, q))q(p - \delta f)
\]
s.t.

$$\frac{m(\theta(p,q))}{\theta(p,q)} q(f-p) - k = 0$$

Rearrange the free-entry condition to isolate $p$ and plug into the seller’s objective function to obtain:

$$\max_{\theta > 0, q \in [0,1]} m(\theta)q(1 - \delta) f - k\theta,$$

which is clearly solved by $q_{CI}(f) = 1$ and $\Theta_{CI}(f)$ solves $m'(\theta)(1 - \delta) f = k$, as above. Note here that the difference from above is that the discount factor is not fixed across sellers, but ranges from $\tilde{\delta}$ to $\delta$.

1.4.2 Asymmetric Information

Now consider the signaling equilibrium for the case in which the seller’s discount factor $\delta$ and asset quality $f$ are private information.

**Definition 4.** A fully separating equilibrium for the two-signal game with two-dimensional private information is a set $M$ of price-quantity pairs $(p,q)$, a market tightness function $\theta(p,q) : M \rightarrow \mathbb{R}_+$, and buyer beliefs $\mu(f, \delta|p,q) : S \times M \rightarrow [0,1]$ such that

1. **Seller’s problem:** For all $(p,q) \in M$ and all $(f,\delta) \in S$, $\mu(f, \delta|p,q) > 0$ implies

   $$(p,q) \in \arg\max_{(p',q') \in M} m(\theta(p',q')) q'(p' - \delta f).$$

2. **Buyer’s problem:**

   (a) **Free entry:** For all $(p,q) \in M$ and all $(f,\delta) \in S$,

   $$\frac{m(\theta(p,q))}{\theta(p,q)} q(E[f|p,q] - p) - k = 0$$

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(b) No profitable deviation: There does not exist a \((p,q) \notin M\) such that
\[
\frac{m(\theta(p,q))}{\theta(p,q)}q(E[\bar{f}|p,q] - p) - k > 0.
\]
Again, I restrict off-equilibrium beliefs in a manner directly analogous to the one- and two-signal cases in which only asset quality is privately known.

Ordinarily, finding the separating equilibrium of a two-dimensional type space is difficult, if not impossible. In this case, however, the solution is greatly simplified if I reparameterize the type space. Instead of letting the private information be characterized by asset quality \(f = E[F(X)|Z]\) and discount factor \(\delta\), I define a new space \(\tilde{S}\) in which the first dimension is unchanged, but the second dimension is the seller’s discounted asset valuation \(v \equiv \delta f\), so \(\tilde{S} \equiv \{(f,\delta f) : (f,\delta) \in S\}\). Since there is a one-to-one mapping between \(S\) and \(\tilde{S}\), this transformation is without loss of generality.

The solution strategy, as in the case of one-dimensional private information, is to find the Pareto optimal incentive compatible mechanism that satisfies the buyer’s free entry condition and the seller’s individual rationality constraint. I distinguish the mechanism with domain \(\tilde{S}\) from the mechanism with domain \(S\) by placing a tilde \(\tilde{\ }\) over the mechanism with domain \(\tilde{S}\). Let an external market maker design an incentive-compatible mechanism \((\tilde{P}(s),\tilde{\theta}(s),\tilde{q}(s))\), \(s = (f,v) \in \tilde{S}\) that satisfies the conditions of the equilibrium. Note that free-entry implies \(\tilde{P}(\tilde{s}) = \tilde{f} - k\tilde{\theta}(\tilde{s})/m(\tilde{\theta}(\tilde{s}))\tilde{q}(\tilde{s})\). I obtain a convenient expression for the seller’s profit if I eliminate \(\tilde{P}\) by plugging it into the seller’s objective function:
\[
\Pi(\tilde{s}|s) \equiv m(\tilde{\theta}(\tilde{s}))\tilde{q}(\tilde{s})(\tilde{f} - v) - k\tilde{\theta}(\tilde{s})
\]
I seek the Pareto optimal mechanism \((\tilde{\theta}(s),\tilde{q}(s))\) that satisfies:
1. Global incentive compatibility

\[ \Pi(s|s) \geq \Pi(\hat{s}|s) \quad \forall (\hat{s}, s) \in \tilde{S}^2 \quad \text{(GIC)} \]

2. Individual rationality:

\[ \Pi(s|s) \geq 0 \quad \text{(IR)} \]

**Lemma 1.** A mechanism \( (\tilde{\theta}, \tilde{q}) \) satisfies global incentive compatibility (GIC) if and only if

1. For any fixed \( v \), the equilibrium payoff of seller \((f, v)\) is constant in \( f \), and is therefore fully determined by \( v \). For convenience, write the equilibrium payoff to seller \((f, v)\) who tells the truth as \( \Pi((f, v)|(f, v)) = \Pi(v) \).

2. The envelope condition: the equilibrium payoff \( \Pi(v) \) of seller \((f, v)\) is differentiable almost everywhere, and where \( \Pi'(v) \) exists, it is equal to the seller’s expected fraction sold

\[ -\Pi'(v) = m(\tilde{\theta}(f, v))\tilde{q}(f, v) \quad \forall f \in \tilde{S}(v), \quad \text{(ENV)} \]

where \( \tilde{S}(v) = \{ f : (f, v) \in \tilde{S} \} \).

3. Monotonicity: Given \( \hat{s} = (\hat{f}, \hat{v}) \) and \( s = (f, v) \), if \( \hat{v} \succ v \), then \( m(\tilde{\theta}(\hat{s}))\tilde{q}(\hat{s}) \leq m(\tilde{\theta}(s))\tilde{q}(s) \), regardless of \( \hat{f} \) and \( f \).

Part (1) of the lemma states that sellers of common private valuation \( v \) must receive the same profit; so if two sellers have assets of different quality \((f \neq f')\), but value their assets the same \((\delta f = \delta' f')\), they must receive the same profit. If not, then the lower profit seller would always pretend to be the higher profit seller, and there would be no way to punish him for doing so, because the private value he places on his asset is the same as the value which the other seller places on his asset.
Part (2) states for almost every private value $v$, sellers of common private valuation must have the same expected fraction sold. Below, I show that in the Pareto optimal fully separating equilibrium, this must hold for every private value $v$, not just almost every $v$.

**Proposition 3.** Let $f(v) \equiv \inf\{f : (f, v) \in \tilde{S}\}$. The fully separating equilibrium under two dimensional private information and two-dimensional signaling takes the following form:

$$p^*(f, v) = \frac{\Pi(v)}{-\Pi'(v)} + v, \quad q^*(f, v) = \frac{-\Pi'(v)}{m(\theta(f, v))},$$

$$M^* = \{ (p, q) \in \mathbb{R}_+ \times [0, 1] : \exists (f, v) \in \tilde{S} \text{ s.t. } (p^*(f, v), q^*(f, v)) = (p, q) \},$$

$$\mu(f, v|p, q) = \mathbb{I}\{ (p^*(f, v), q^*(f, v)) = (p, q) \}, \quad \text{and} \quad \theta^*(p, q) = \tilde{\theta}((p^*, q^*)^{-1}(p, q)),$$

where $\theta(f, v)$ and $\Pi(v)$ are characterized by

$$\tilde{\theta}(f, v) = -\frac{1}{k}[\Pi(v) + \Pi'(v)(f - v)]$$

$$\Pi'(v) = -m\left(-\frac{1}{k}[\Pi(v) + \Pi'(v)(f(v) - v)]\right), \quad \Pi(v) = \Pi_{Cf}(f(v), v). \quad (1.10)$$

I address off-equilibrium $\theta^*(p, q)$ and $\mu(f, v|p, q)$ in the proof (Appendix).

This structure of seller profit $\Pi$, liquidity $\theta$, and fraction sold $q$ in this equilibrium are illustrated in Figure 1.4.2. The figure shows that isoquants of seller profit $\Pi$ correspond to sets of sellers with common private valuation $v$; i.e., sellers of common private valuation $v$ make the same profit $\Pi(v)$, as asserted by Lemma 1. Price $p(f, v)$ is not pictured in the figure because its isoquants are so similar to those of profit.
Figure 1.3: The Pareto optimal separating equilibrium with two dimensional private information and two signals. Contour plot arrows indicate the direction of increasing contour lines. In the case plotted, there exists a region in which $\theta$ is increasing in asset quality $f$, contrary to received theory.
Π; the proposition shows that the price $p^*$ is independent of asset quality $f$, so its isoquants, as with profit Π, are sets of sellers with common private valuation $v$. This follows by recalling from Lemma 1 that profit $\Pi(f, v) = m(\theta(f, v))q(f, v)(p(f, v) - v)$ is independent of $f$, and so is expected fraction sold $m(\theta)q$. In addition, $p^*(v)$ is invertible, so buyers can infer the private value $v$ of any seller trading at price $p$, regardless of the seller’s fraction sold $q$. In other words, price $p$ reveals seller value $v$, and contingent on price $p$, quantity $q$ reveals asset quality $f$. The structure of this equilibrium will be further explored by a series of corollaries, starting with an examination of how liquidity and retention vary among sellers with common private valuation.

**Corollary 1.** For fixed seller value $v$, liquidity $\theta$ is strictly increasing in asset value $f$ and quantity sold $q$ is strictly decreasing in asset quality $f$.

$$\frac{\partial \hat{\theta}(f, v)}{\partial f} > 0, \quad \frac{\partial \hat{q}(f, v)}{\partial f} < 0.$$  

As explained above, sellers of common private valuation all trade at the same price $p$ and the same expected fraction sold $m(\theta)q$. Also recall that the buyer’s zero profit condition is $n(\theta)q(f - p) = k$, where $n(\cdot)$ is strictly decreasing in liquidity $\theta$. So if, among sellers of common value $v$, sellers with higher quality assets sold a greater fraction $q$, then buyers would find markets with high $q$ more attractive, so market tightness $\theta$ would be driven up until buyers receive zero profit. But this cannot be an equilibrium, because $m(\theta)q$ must be constant across sellers of common $v$. Therefore, it must be that sellers sell a lower fraction $q$ of higher quality assets $f$, with market tightness on the other hand rising with asset quality $f$ to keep expected fraction sold $m(\theta)q$ constant.
Corollary 2. Among sellers of common value \( v \), the seller with the lowest quality asset \( f(v) \) sells the entire asset: \( \bar{q}(f(v), v) = 1 \).

The graph of \( f(v) \) is the left boundary and the upper boundary of \( S \), and Figure 1.4.2 shows that sellers on this boundary sell the entire asset \( (\bar{q}(f(v), v) = 1) \). As in the case where only asset quality \( f \) is private information, because selling a greater quantity \( q \) is beneficial to both buyers and sellers, the Pareto optimal equilibrium will naturally involve the greatest amount \( q \) sold possible. Corollary 1 implies that among sellers of common private value \( v \), the sellers with the lowest quality \( f \) sell the greatest fraction \( q \), so it makes sense that these types will sell everything in the Pareto optimal separating equilibrium.

Corollary 3. Any type \((f, \delta)\) on the left boundary of \( S \) receives the complete information allocation:

1. \( q(f, \delta) = 1 \)

2. \( \theta(f, \delta) \) solves \( n'(\theta(f, \delta))(1 - \delta)f = k \).

The intuition here is similar to the case of one-dimensional private information: the types \((f, \delta)\) on the left boundary hold the worst assets, so no sellers desire to mimic them and it is not necessary to distort their allocation away from the first best in order to prevent mimicry. It is also not necessary to prevent these types from mimicking each other, who vary only in patience \( \delta \), because the buyer is indifferent to seller patience, caring only about asset quality \( f \), so pretending to be more or less patient than the truth confers no advantage.

Note that the features of the Pareto optimal fully separating equilibrium that have been discussed so far are already incompatible with the piecewise linear matching function of Guerrier and Shimer (2012): \( \hat{m}(\theta) = \min[\theta, 1] \). With such a matching
function, the complete information allocation is \( \tilde{\theta}_{CI}(f, v) = \tilde{q}_{CI}(f, v) = 1 \), for all \((f, v) \in \tilde{S}\). So if the left border \((\tilde{f}, v)\) receives the complete information allocation, then \( \tilde{\theta}(\tilde{f}, v) = \tilde{q}(\tilde{f}, v) = 1 \), so \( \tilde{m}(\tilde{\theta}(\tilde{f}, v))\tilde{q}(\tilde{f}, v) = 1 \). However, Lemma 1 indicated that \( m(\tilde{\theta})\tilde{q} \) must be constant for sellers of private valuation \( v \) and Corollary 1 showed that \( \tilde{q}(f, v) \) must be strictly decreasing in asset quality \( f \). If \( \hat{m} = \min[\theta, 1] \), this is impossible, because sale probability \( \hat{m} = 1 \) is at its maximum value, and cannot increase in order to offset a decreasing quantity \( q \).

Now consider how the liquidity \( \theta \) and fraction sold \( q \) vary with asset quality \( f \) and seller patience \( \delta \) in the original untransformed space \( S \). The behavior is liquidity \( \theta \) and fraction sold \( q \) depend on which region of \( S \) the seller occupies, so I first divide \( S \) into partitions. Denote the set of seller valuations \( v \) by \( V = [v, \bar{v}] \). Note that \( v = \delta f \) and \( \bar{v} = \delta \bar{f} \). Partition \( V \) at \( \bar{v} = \delta f \) into a lower region \( V = [v, \bar{v}] \) and a higher region \( \bar{V} = [\bar{v}, \hat{v}] \). The lower region \( V \) is the set of types \( v \) that have the same minimum asset quality \( f(v) = f \). In the higher region \( \bar{V} \), however, higher types \( v \) have a higher minimum asset quality \( f(v) = v/\delta \). Also define a corresponding partition for the original type space \( S \). That is, let \( S \equiv \{(f, \delta) \in S : \delta f \in V\} \) and \( \bar{S} \equiv \{(f, \delta) \in S : \delta f \in \bar{V}\} \).

**Corollary 4.** Suppose \((f, \delta) \in \bar{S} \), so that \( \delta f \in [\delta f, \delta \bar{f}] \). Then

\[
\frac{\partial \theta(f, \delta)}{\partial f} < 0 \quad \frac{\partial \theta(f, \delta)}{\partial \delta} < 0,
\]

so liquidity \( \theta \) is decreasing in both asset quality \( f \) and seller patience \( \delta \).

Figure 1.4.2 shows that in \( \bar{S} \), the upper-right region of \( S \), liquidity falls as either asset quality \( f \) or seller patience \( \delta \) increase.

**Corollary 5.** Suppose \((f, \delta) \in S \), so that \( \delta f \in [\delta f, \delta \bar{f}] \). Then

\[
I conjecture more strongly that if \( m(\theta) = \min[\theta, 1] \), then there does not exist a fully separating equilibrium. This will be further explained in the next draft.
(i) fraction sold $q(f, \delta)$ is decreasing in asset quality $f$ and seller patience $\delta$

$$\frac{\partial q(f, \delta)}{\partial f} < 0, \quad \frac{\partial q(f, \delta)}{\partial \delta} < 0,$$

(ii) liquidity $\theta(f, \delta)$ is decreasing in seller patience $\delta$

$$\frac{\partial \theta(f, \delta)}{\partial \delta} < 0.$$

The figure illustrates that in $S$, the lower left region of $S$, fraction sold $q$ is indeed decreasing in asset quality $f$ and seller patience $\delta$, and liquidity $\theta$ is decreasing in seller patience $\delta$. The corollary does not address how liquidity changes with asset quality $f$, because in $S$, liquidity $\theta$’s response to increased quality $f$ depends on the form of the matching function $m(\cdot)$ and will be discussed below.

I next present a key theorem of the paper, which demonstrates that in this environment, it is possible for liquidity $\theta$ to be increasing in asset quality $f$. Standard assumptions on the matching function $m$ are not sufficient to identify the sign of the partial $\partial \theta/\partial f$ in the lower region $S$, so I assume that $m(\theta) = (1 + \theta^{-r})^{-1/r}$ with $r > 0$. This function satisfies the assumptions on $m$ from Section 2, and also contains a parameter $r$ which controls the efficiency of the matching process. As $r$ goes to zero, both $m(\theta)$ and $m(\theta)/\theta$ converge pointwise to zero, meaning that both sellers’ and buyers’ probability of trade goes to zero, so the matching is perfectly inefficient. On the other hand, as $r$ goes to infinity, $m(\theta)$ converges uniformly to $\min[\theta, 1]$ and $m(\theta)/\theta$ converges uniformly to $\min[1, \theta^{-1}]$; this means that the matching is perfectly efficient, because in a given market, whichever side (buyers or sellers) has fewer agents will trade with probability 1.

**Theorem 3.** Suppose $m(\theta) = (1 + \theta^{-r})^{-1/r}$. 

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(i) For high enough matching efficiency $r$, there exists a positive measure subset of $\mathcal{S}$ for which liquidity $\theta(f, \delta)$ is strictly increasing in asset quality $f$.

(ii) For low enough matching efficiency $r$, liquidity $\theta(f, \delta)$ is strictly decreasing in asset quality $f$ for all $(f, \delta) \in \mathcal{S}$.

Figure 1.4.2 depicts case (i) of the theorem. In the figure, beginning at nearly any point on the left border, liquidity $\theta$ increases with asset quality $f$ until just before crossing into the region $\bar{\mathcal{S}}$. In order to understand the intuition of this theorem, it is helpful to decompose the marginal liquidity of asset quality. Recall that $\theta(f, \delta) = \tilde{\theta}(f, \delta f)$, so

$$\frac{\partial \theta(f, \delta)}{\partial f} = \frac{\partial \tilde{\theta}(f, v)}{\partial f} + \delta \frac{\partial \tilde{\theta}(f, v)}{\partial v}$$ (1.11)

That is, holding seller patience $\delta$ fixed, increasing asset quality affects liquidity $\tilde{\theta}$ through two channels: the value $f$ to the buyer increases, and so does the value $v = \delta f$ to the seller. Also note that analyzing liquidity $\tilde{\theta}(f, v)$ of seller $(f, v)$ is equivalent to analyzing the search costs $k\tilde{\theta}(f, v)$ reimbursed by seller $(f, v)$, and recall that the profit of seller $(f, v)$ may be expressed as

$$\Pi(v) = m(\tilde{\theta}(f, v))\tilde{q}(f, v)(f - v) - k\tilde{\theta}(f, v),$$ (1.12)

$$= m(\tilde{\theta}(f, v))(f - v) - k\tilde{\theta}(f, v),$$ (1.13)

The intuition now may be summarized as follows. An increase in asset quality $f$ impacts liquidity $\theta$ (equivalently search costs $k\theta$) via two channels: buyer value $f$ rises, and seller value $v$ rises. For fixed $v$, the marginal increase of expected gains from trade from raising buyer value $f$ is simply expected fraction sold $m(\tilde{\theta}(f, v))$, so the marginal increase in search costs $k\partial \tilde{\theta}/\partial f$ must equal $m(\tilde{\theta}(f, v))$ in order to keep profits equal across sellers of common $v$. On the other hand, for fixed buyer value $f$, the marginal drop in expected gains from trade from raising seller value $v$
results from the drop in expected fraction sold $\partial m(\tilde{\theta}(f, v))/\partial v$ required to keep low $v$'s from mimicking higher $v$'s, and the drop in search costs $k\tilde{\theta}_v$ exactly offsets this. For high matching efficiency $r$, on the left boundary sellers optimally choose a market tightness $\tilde{\theta}(f, v)$ close to 1, making market tightness $\tilde{\theta}(f, v)$ not very sensitive to seller valuation $v$. Therefore, $k\partial \tilde{\theta}(f, v)/\partial f = m(\tilde{\theta}(f, v))$ is close to 1, and $k\partial \tilde{\theta}(f, v)/\partial v$ is close to zero, so the first channel dominates the second; that is, the rise in search costs (liquidity) from raising buyer valuation $f$ is greater than the drop in search costs (liquidity) from raising seller valuation $v$, so near the left boundary of $S$, liquidity is increasing in asset quality $f$.

1.4.3 Comparison with Partial Pooling

As discussed above, Guerrieri and Shimer (2013) consider two dimensional private information in a similar framework as this model. One crucial difference is that they do not include the retention signal $q$ in their framework, and therefore are unable to fully separate both asset quality and seller impatience. This leads to a partial pooling equilibrium in which sellers are distinguishable only up to their private valuation $\delta f$.

This section compares the fully separating equilibrium of my model to a partial pooling equilibrium analagous to their model. My setting is not identical to the setup in Guerrieri and Shimer (2013), but the environments are sufficiently similar to offer instructive comparisons. I first solve for the Pareto optimal partial pooling equilibrium, and then show that it Pareto dominates full separation; however, I next show that full separation is sustained by a strictly greater set of off-equilibrium beliefs than partial pooling is sustained by, and therefore full separation is more robust.

**Proposition 4.** Suppose that $E[f|\tilde{\delta}f = v]$ is strictly increasing in $v$. The Pareto optimal partial pooling equilibrium under two dimensional private information and two-dimensional signaling takes the following form:
\[ p^*(f, v) = \frac{\Pi(v)}{-\Pi'(v)} + v, \quad q^*(f, v) = 1, \]

where \( \theta(f, v) \) and \( \Pi(v) \) are characterized by

\[ \tilde{\theta}(f, v) = -\frac{1}{k} \left[ \Pi(v) + \Pi'(v) (E[\tilde{f} | \delta \tilde{f} = v] - v) \right], \]

\[ \Pi'(v) = -m \left( -\frac{1}{k} \left[ \Pi(v) + \Pi'(v) (E[\tilde{f} | \delta \tilde{f} = v] - v) \right] \right), \quad \Pi(v) = \Pi_{CI}(f(v), v), \]

(1.14)

and \( M^*, \mu^*(f, v|p, q), \) and \( \theta^*(p, q) \) immediately follow as in Proposition 3.

I discuss the off equilibrium beliefs in the next section on robustness. The assumption of strictly increasing expected asset quality \( E[\tilde{f} | \delta \tilde{f} = v] \) is required in Guerrieri and Shimer (2013), and guarantees that sellers of lower value \( v \) are tempted to mimic sellers of higher value \( v \).

The key feature of this equilibrium is that every type attempts to sell the entire quantity of the asset, so fraction sold \( q \) is equal to 1 for all types. This leaves price \( p \) as the only distinguishing signal; the proposition shows that price is an invertible function of seller value \( v \), so buyers separate sellers only up to \( v \), as in Guerrieri and Shimer (2013). The next proposition compares this partial pooling equilibrium to the fully separating equilibrium.

**Proposition 5.** Let \( \Sigma \) be the Pareto optimal equilibrium in which all types fully separate. Let \( \Phi \) be the Pareto optimal equilibrium in which partial pooling occurs in the following way: sellers with distinct private values \( \delta f \) separate, and sellers with common private values \( \delta f \) pool.

Then partial pooling \( \Phi \) Pareto dominates full separation \( \Sigma \): \( \forall v \succ v, \Pi_{\Phi}(v) > \Pi_{\Sigma}(v) \).
To understand the intuition of the proposition, note that under partial pooling, no retention occurs. Under full separation, however, retention is necessary in order to distinguish between sellers of common private valuation \( v \). This results in lower surplus and therefore lower equilibrium profit to the sellers. Indeed, under full separation, the only types which sell the entire asset \( (q = 1) \) are the worst assets \( f(v) \) for any private value \( v \), and incentive compatibility requires that sellers of the same value \( v \) but with better assets \( f > f(v) \) can do no better than them. In contrast, pooling sellers of common \( v \) allows all of them to sell everything \( (q = 1) \), thereby improving over full separation.

Although partial pooling Pareto dominates full separation, it is less robust to off-equilibrium beliefs. Recall the restriction on off-equilibrium beliefs developed in Gale (1996), and applied by Guerrieri, Shimer, and Wright (2010) and Chang (2012) as follows. First define \( \theta((p, q), s) \equiv \inf\{\tilde{\theta} > 0 : m(\tilde{\theta})q(p - v) \geq \Pi(v)\} \), which is the lowest acceptable market tightness for type \( s \) in equilibrium. Recall that when buyers post off-equilibrium \( (p, q) \) pairs, they expect the type that will accept the lowest probability of trade. Defining \( T(p, q) = \arg\inf_{s \in S} \theta((p, q), s) \), this leads to the following restriction.

For any pair \( (p, q) \notin M \) and type \( s \), \( \mu(s|p, q) = 0 \) if \( s \notin T(p, q) \). (R)

This restriction is not sufficient to choose a unique equilibrium in this context. In particular, there exist beliefs which satisfy the restriction (R) which support both the partial pooling equilibrium \( \Phi \) and the fully separating equilibrium \( \Sigma \). However, the fully separating equilibrium is more robust in the following sense: any off equilibrium beliefs satisfying the restriction (R) support full separation \( \Sigma \), but there exist beliefs satisfying (R) that break partial pooling \( \Phi \).

To make this precise, let \( \Gamma_0 \) be the set of off-equilibrium beliefs satisfying (R).
Let $\Gamma(\Phi)$ be the set of beliefs in $\Gamma_0$ which support the partial pooling equilibrium $\Phi$, and $\Gamma(\Sigma)$ be the set of beliefs in $\Gamma_0$ which support the fully separating equilibrium $\Sigma$.

**Proposition 6.** Full separation $\Sigma$ is more robust than partial pooling $\Phi$ in the following sense: $\Gamma(\Phi) \subset \Gamma(\Sigma) = \Gamma_0$.

That is, there exist beliefs satisfying (R) which break partial pooling $\Phi$ and support full separation $\Sigma$, but not vice versa. (R) only pins down a unique private value $v$ which buyers can expect to be attracted to a contract $(p, q)$. However, sellers of common private value $v$ differ in asset quality $f$. If, for example, buyers expect the best asset quality $\bar{f}(v)$ among these sellers, then offering to buy a slightly smaller fraction $q < 1$, but receiving a significantly better asset $\bar{f}(v)$ is a profitable deviation; and therefore the pooling equilibrium is broken. This difficulty does not occur under full separation, however, because for a fixed equilibrium price $p$, the lowest equilibrium fraction $q$ already corresponds to the highest asset quality $\bar{f}(v)$ among sellers who post $p$; so buyers stand to gain nothing by offering an even lower fraction $q$, which hurts both buyers and sellers.

**1.4.4 Security Design**

Suppose that after the design of security $F$, sellers receive not only private information $Z$ regarding the distribution of the assets $X$ underlying $F$, but also a random, privately known preference shock $\delta \in [\bar{\delta}, \bar{\delta}]$. Again denoting the conditional expected security value by $f = E[F(X)|z]$, the seller’s private information $s$ now is in two dimensions: $s = (f, \delta) \in [\bar{f}, \bar{f}] \times [\bar{\delta}, \bar{\delta}] \equiv S \subset \mathbb{R}^2_{++}$. Assume that $(1 - \delta)f > k$ for all $(f, \delta) \in S$, so that all types find it worthwhile to engage in costly search.

**Proposition 7.** If there is a uniform worst case, then among increasing monotone securities, a standard debt contract $F(X) = \min(X, d)$ is an optimal security.
Although $\delta$ is now stochastic, debt is still an optimal security. This is because the equilibrium profit function $\Pi$ is decreasing in the seller’s private valuation $\delta f$, and therefore decreasing in asset quality $f$ for fixed $\delta$, but increasing in the lowest asset quality $f$. This translates to the same tradeoff as in the case where seller patience $\delta$ was fixed across sellers and public information. That is, sellers seek to design a security which trades off retention costs (low $f$) with lemons costs ($f$ much higher than $f$). So for any fixed $\delta$, a standard debt contract confers the same advantages as in the case of public $\delta$.

1.5 Literature Revisited

Having explored the major results of this paper, it is worthwhile to reflect on their connection with results in related literature. Figure 1.5 illustrates how my paper not only draws on, but departs from other key papers in the literature. I use the symbols from my model to denote analogous variables in other papers, although those papers may use alternative notation.

First consider similar models in which private information consists of only a single dimension: asset quality. On the retention side, DeMarzo and Duffie (1999) show how retaining a fraction $q$ of the asset may function as a signal of asset quality $f$, in which sellers with higher quality assets retain a greater fraction $q$ of the security. On the liquidity side, Guerrieri, Shimer, and Wright (2010) combine competitive search with adverse selection, finding an equilibrium in which liquidity $\theta$ functions as a sorting mechanism and is decreasing in asset quality $f$. The basic model of Chang (2012) has one-dimensional private information, and applies the work of Guerrieri, Shimer, and Wright (2010) to a continuum of types, allowing her to characterize the equilibrium with a differential equation. She utilizes a mechanism design to solve for the equilibrium, a strategy my paper draws on, and finds that sellers with higher quality assets trade with lower liquidity.
DeMarzo and Duffie (1999)

Type: \(f\)
Signal: \(q\)
Equilibrium: separating
Result: \(q \downarrow \text{ in } f\)

Guerrieri, Shimer, and Wright (2010)

Type: \(f\)
Signal: \(\theta\)
Equilibrium: separating
Result: \(\theta \downarrow \text{ in } f\)

Chang (2012)

Type: \((f,\delta)\)
Signal: \(\theta\)
Equilibrium: partial pooling
Result: \(\theta \downarrow \text{ in } f\)
Note: Uses mechanism design, possible upward distortion of \(\theta\).

Williams (2013), § 4

Type: \(f\)
Signal: \((q,\theta)\)
Equilibrium: separating
Result: \(q=1, \theta \downarrow \text{ in } f\)

Guerrieri and Shimer (2012)

Type: \((f,\delta)\)
Signal: \(\theta\)
Equilibrium: separating up to \(\delta f\).
Result: \(\theta \downarrow \text{ in } \delta f\)
Note: Assumes \(E[f|\delta f=v] \uparrow \text{ in } v\).

Williams (2013) § 6

Type: \((f,\delta)\)
Signal: \((q,\theta)\)
Equilibrium: separating
Result: \(q \downarrow \text{ in } f\), non-mon. in \(\delta\)
\(\theta \downarrow \text{ in } \delta\), non-mon. in \(f\).
Note: No distribution assumptions required.

Figure 1.4: Literature overview. Asset quality is denoted by \(f\), seller impatience is denoted by \(\delta\), regardless of notation used in each paper. Fraction sold is \(q\) and liquidity (buyer-seller ratio) is \(\theta\).
These three papers restrict the seller exogenously to the use of a single signal—retention or liquidity, but not both. Section 4 of my paper, however, builds a model in which both retention and liquidity are available to the seller, but only liquidity is used as a signal in equilibrium. That is, even sellers with high asset quality retain none of the asset, instead distinguishing themselves by setting a price which induces a low liquidity in equilibrium. In this manner, seemingly interchangeable signals are shown to be well-ordered: signaling via liquidity dominates signaling via retention, because low liquidity markets are beneficial to buyers, making them willing to pay a higher price.

Now consider similar models in which private information consists of two-dimensions: asset quality and seller impatience. Chang (2012) builds upon her basic model by considering hidden motives of sale in the form a heterogeneous holding cost, which is analogous to seller impatience in my model. She also finds that liquidity $\theta$ is decreasing in asset quality $f$, but shows that hidden motives may create partial pooling; sellers can only separate up to their private value of holding the asset, but not the asset’s fundamental value. In a setup more similar to the one employed in Section 6 of my paper, Guerrieri and Shimer (2012) consider a setting in which sellers are heterogeneous both in asset quality $f$ and discount factor $\delta$. Like Chang (2012), because they only allow for a single signal of liquidity $\theta$, they focus on pooling equilibria in which sellers separate only up to their private valuation, which in their case is the product of the asset’s fundamental value and the seller’s discount factor. This partial pooling equilibrium requires the assumption that, conditional on the seller’s private valuation $v$, the expected asset quality $f$ is increasing in seller value $v$. In this equilibrium, liquidity is decreasing in the seller’s private value $v$.

Unlike Chang (2012) and Guerrieri and Shimer (2012), Section 6 of this paper not only considers multidimensional private information, but also incorporates the second signal of retaining a portion of the asset as in DeMarzo and Duffie (1999).
This allows sellers to use both retention \( q \) and liquidity \( \theta \) to fully separate their asset quality \( f \) and patience \( \delta \). In contrast to all of the papers discussed in this section, I show how the signals retention \( q \) and liquidity \( \theta \) may exhibit nonmonotonicities. The seller’s expected fraction sold \( m(\theta)q \) must still be decreasing in seller valuation \( \delta f = v \), but this monotonicity condition still allows nonmonotonicities of either signal along certain dimensions of private information. In addition, because the equilibrium is fully separating, I have no need to make assumptions about the distribution of seller types.

1.6 Conclusion

The contribution of this paper, broadly speaking, is to develop a unified framework in which both retention and liquidity can act as sorting mechanisms of seller type. Here, the notion of liquidity is search-theoretic, a natural definition of liquidity which draws on recent advances in the literature. I show that for the case in which only asset quality is private information sellers can signal only with liquidity, the equilibrium subsumes DeMarzo and Duffie (1999) as a special limiting case, and that their retention signal shuts down if liquidity signaling is available along with retention. In both of these cases, standard debt is an optimal security.

I show that if private information is multidimensional, including not only asset quality but also seller impatience, it is possible to find a fully separating equilibrium using both liquidity and retention as simultaneous sorting channels. Unlike previous literature that considers liquidity and asymmetric information, this equilibrium may contain regions in which liquidity is increasing in asset quality, and the optimality of debt is robust to privately known seller patience. Future work may explore dynamic considerations, particularly how the possibility of repeated resale on a market plagued by search frictions may influence the design of the security.
Stress Tests and Bank Portfolio Choice

2.1 Introduction

It is widely presumed that investors are better off with more information than less. Much regulation, such as the recent Dodd-Frank Act, is structured in order to help investors become more informed. In the spirit of maximal information disclosure, many presume that stress tests designed to disclose the solvency of banks should be as informative as possible. For example, a recent article in the Wall Street Journal predicts that stress tests performed by the European Central Bank may resolve investor uncertainty about European banks, but with a caveat: “If, that is, the tests are credible... The last tests under the sole supervision of the European Banking Authority were risible; several banks passed them only to require bailouts later.”

The quote laments the inaccuracy of the stress tests, implying that more informative tests are better for investors.

This paper questions the presumption that more informative stress tests improve investor welfare. I show that partially informative stress tests may in fact be optimal.

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Stress tests only need to be credible enough to prevent runs, so even stress tests with some likelihood of issuing a passing grade to bad banks may be optimal.

I explore the structure of optimal stress tests using a model of bank runs which is most similar to that presented by Allen and Gale (1998). A representative bank issues demand deposit contracts to consumers in exchange for deposits. Banks must choose the promised deposit return as well as the proportion of deposits allocated to safe liquid assets and risky illiquid assets. I assume that at the time the consumers deposit their funds at the bank, they are uncertain about their preferred timing of future consumption and that they are intertemporally risk averse. The representative bank offers a deposit contract which acts as intertemporal insurance by smoothing consumption across future dates. At the second date, depositors learn whether they prefer to consume early or late, and they also receive a public signal about the future return of the risky asset. If this signal indicates a low enough future return, then patient investors prefer to withdraw early and the bank is forced to liquidate all assets, which is costly.

One important difference between my model and that of Allen and Gale is that the public signal about the risky asset does not resolve all future uncertainty. The remaining uncertainty creates scope for additional disclosure via a stress test, which I formalize using the Bayesian persuasion framework of Kamenica and Gentzkow (2011). Under this framework, stress tests are conceptualized as a distribution of signals, conditional on the true future return of the risky asset. Both the structure and result of the stress tests are public information, and the test is costless to perform, so neither information asymmetries nor auditing costs exist as frictions in the model.

Despite the frictionless nature of the tests, I still find that partially informative tests are optimal. If the original public signal is low enough to induce a run, regulators may improve consumer welfare by performing a test which reduces the likelihood of runs. It is optimal to design the test so that all good banks and some bad banks pass,
but some bad banks fail. In particular, it is necessary to fail only just enough banks to maintain the credibility of the passing grade, so that a passing grade induces a high enough posterior belief in the risky asset’s return that patient investors do not run. A test of this form improves welfare, because without it, the original public signal was such that a run would have occurred with certainty, so the bank would have been forced to liquidate the risky asset at a discount. Under the optimal test, however, passing grades result in no-run equilibria, so the probability of costly liquidation is reduced.

I also find that the lower the initial public confidence in the risky assets, the more stringent the optimal stress tests must be. In particular, stress tests must give passing grades to bad banks with less and less likelihood. Intuitively, low prior beliefs require very strong signals of quality in order to induce posteriors high enough to prevent runs. The harder a test is to pass, the stronger the signal of quality given by a passing grade.

I examine the impact of anticipated stress tests on the bank’s ex ante portfolio and contract. I find that anticipating optimal stress tests may decrease the bank’s optimal capital cushion and raise the optimal deposit return promised to early withdrawers. Intuitively, optimal stress tests reduce the likelihood of runs. Because the only value of reserving a cushion of safe liquid capital is to lower the probability of runs, optimal stress tests reduce the value of the cushion, so the optimal capital cushion falls. In addition, although raising the promised deposit return to early withdrawers may raise the probability of runs by making early withdrawal more attractive, optimal stress tests reduce that risk by lowering the likelihood of runs, so the optimal deposit return rises.

My model builds on the theory of bank runs, which began most notably with the seminal paper of Diamond and Dybvig (1983), who showed how banks subject to runs can attract deposits by offering intertemporal risk sharing. Allen and Gale
(1998) build on their model by showing how runs may be induced by exogenous signals about fundamentals, and allow banks to decide how much to invest in safe vs risky assets. My model is closely related to the third section of that paper in which the risky asset may be liquidated at a discount, but unlike their paper I assume an exogenous, linear liquidation technology in order to simplify my analysis. The most crucial difference between my paper and theirs is that in my model, uncertainty is not fully resolved until the final date; this creates scope for informative stress tests, which their paper does not consider.

I am aware of only two papers which apply the Bayesian persuasion framework of Kamenica and Gentzkow (2011) to the optimal design of bank stress tests: the first is Gick and Pausch (2012) and the second is Goldstein and Leitner (2013). The most significant feature which distinguishes my paper from these two papers is my examination of the impact of anticipated stress tests on the bank’s asset portfolio, deposit return, and capital cushion. These two papers do not consider these effects of stress tests, and a number of other differences exist. Gick and Pausch assume an exogenous binary form for investor utility, which depends by assumption on the binary quality of the bank and a binary action of the investor. In my model, however, the utility of the investors is determined endogenously from the payoff of the demand deposit contract offered by the bank and the late investor’s decision to withdraw early. They also assume an exogenous form for the regulator’s utility which depends directly on the mass of investors which take a prudent action, whereas my regulator’s objective is simply the endogenous utility of the investors.

The main value of disclosure in Goldstein and Leitner (2013) is that banks can engage in risk-sharing arrangements by pooling bad banks with good banks, in order to protect against a fall in future capital. In my model, bad banks may pool with good banks, but the value of pooling is so that endogenous runs occur less often. Their paper assumes that the bank’s return on cash is a discontinuous function, whereas the
discontinuity in my model is explicitly modeled as the discount suffered from early liquidation during an endogenous run. They examine a case both with and without bank private information, and they find that private information may necessitate more than two disclosure scores; in my model, there is no private information, so two disclosure scores are sufficient.

Section 2 presents the environment and solves for the optimal consumption profile given some particular portfolio and contract terms. Section 3 finds the structure of optimal stress tests and explains how they can improve consumer welfare. Section 4 examines the impact of anticipated stress tests on the bank’s ex ante portfolio decision and contract terms, and Section 5 concludes.

2.2 Environment

The environment consists of two periods, or three dates: 0, 1, and 2. In this environment, there is a consumption good, and two types of assets which transform date 0 consumption to date 2 consumption. The first type of asset is safe and liquid: it gives a certain return of one at date 2 and may be liquidated at date 1 without cost. The second type of asset is risky and illiquid: it gives a risky return $R$ at date 2 and may be liquidated at date 1 only by selling it for a fraction $\theta \in (0, 1)$ of its expected return. I assume the date 2 return on the risky asset takes one of two values: in a good state, the return is $G > 0$, and in a bad state, the return is $B = 0$. I assume that at date 0, the expected return on the risky asset is greater than 1, creating an incentive to invest in the risky asset.

There is a continuum of ex ante identical consumers, with measure normalized to 1. At date 0, consumers have total endowment $E$ of the consumption good, and are uncertain about their future consumption preferences; some will be impatient, preferring to consume at date 1, and the rest will be patient, being indifferent between date 1 and date 2 consumption. At date 0, consumers know only the probabilities
of being patient or impatient, which I assume are equal. At date 1, consumers learn whether they are patient or not, and also receive a public signal $p \sim \text{unif}(0,1)$, which indicates the probability that the risky asset gives a good return $G$ at date 2.

In the spirit of Epstein and Zin (1989), I distinguish between the consumer’s desire to smooth consumption across dates and across states. In particular, I assume that at any particular date, consumers are risk neutral, but they desire to smooth consumption across dates. Let $s_t$ denote the state at date $t$, which may include the public signal $p$, the impatience of each consumer, or the actual realized return $R$ on the risky illiquid asset. Let $c_t(s_t)$ be a consumer’s date $t$ consumption, contingent on the state $s_t$, and let $\beta_t(s_1) \in \{0,1\}$ be the discount factor applied to date 2 consumption by consumer $i$. At date 1, an impatient investor’s utility is given by $u(c_1(s_1))$, and a patient investor’s utility is given by $u(c_1(s_1)) + u(E[c_2(s_2)|s_1])$, where $u(c) = \min[c,k]$. At date 0, before any uncertainty is resolved, a consumer’s utility is given by

$$E[u(c_1(s_1)) + \beta_1(s_1) u(E[c_2(s_2)|s_1])].$$

These preferences are broadly similar to Epstein-Zin preferences, where here the expectation of date 2 consumption is analoguous to a risk-neutral Epstein-Zin certainty equivalent operator, and if the output of this operator is denoted by $z$, then the function $W(c,z) = u(c) + \beta u(z)$ is analoguous to the Epstein-Zin intertemporal aggregator. The concavity of $u$ creates incentives to smooth consumption between dates 1 and 2. So consumers will prefer consumption profiles that offer similar levels of consumption to patient and impatient investors. That is, they desire to insure against intertemporal risk, which creates a role for banks.

Banks may make investments on behalf of consumers. They pool consumers’ assets, and thereby offer insurance to consumers against uncertain liquidity demands.
Free entry forces banks to compete, so they offer deposit contracts which maximize the utility of consumers, and banks fund those contracts by investing deposits into the safe liquid asset and the risky illiquid asset. I assume that the demand deposit contract takes the following form: consumers who withdraw at date 1 receive fixed return $r > 0$ if the bank can afford to pay it; if not, the bank is forced to liquidate all assets and distribute them equally among early withdrawers. Late withdrawers simply receive the remainder of the banks’ assets at date 2, which is zero if the bank is forced to liquidate and distribute all assets at date 1.

I can summarize the timing as follows. At date 0, consumers invest their deposits with banks. Banks invest those deposits in the two assets, allocating $L \geq 0$ to the liquid safe asset and $I \geq 0$ to the illiquid risky asset, and promising early withdrawers a fixed return $r > 0$ if affordable. At date 1, consumers privately learn whether they are patient or impatient and receive a public signal $p \sim \text{unif}[0,1]$ indicating the probability that the risky asset will yield a good return $G$ at date 2. Contingent on this information, patient consumers decide whether to withdraw early or late, and all early withdrawers approach the bank requesting return $r$. If the bank can afford to pay $r$ to all early withdrawers, it does so, and then at date 2 the consumers receive the bank’s remaining assets. If the bank cannot afford to pay $r$ to all early withdrawers, it is forced to liquidate all assets and distribute them across all early withdrawers.

2.2.1 Optimal Consumption Profile

At date 0, the bank receives deposits $E$ from consumers and must decide how to invest those deposits and what consumption profile to offer consumers. Because the bank competes with other banks for the business of consumers, it chooses its portfolio and offered contract to maximize the expected utility $E[u(c_{j1}(p)) + \beta_j u(E[c_{j2}(R, p)|p])]$ of consumers, where $j = 1$ denotes an impatient consumer who prefers to consume
at date 1, and \( j = 2 \) denotes a patient consumer who is indifferent between early and late consumption, so then \( \beta_1 = 0 \) and \( \beta_2 = 1 \).

It is helpful to consider the bank’s problem using backward induction. At date 1, the bank has already selected \( L \) liquid assets and \( I \) illiquid assets, and has promised return \( r \). In addition, consumers privately know their patience level, and all have belief \( p \) that the risky asset will yield a good return \( G \). Given this information, what is the highest aggregate consumption a bank can offer consumers? First, the consumption profile must be feasible, so early withdrawers cannot receive any more consumption than the total value of the early liquidated assets. Denoting the amount of liquidated liquid and illiquid assets by \( l \in \{0,1\} \) and \( i \in \{0,1\} \), respectively, and the total mass of patient consumers who withdraw early (run) by \( \alpha \), I have

\[
c_{11} + \alpha c_{21} \leq lL + i\theta E[R|p]I. \tag{2.1}
\]

Similarly, the late withdrawing investors’ consumption cannot exceed the total remaining assets left over at date 2, which consists of the unliquidated assets plus the unwritten liquidated assets. Taking expectations gives:

\[
(1 - \alpha)E[c_{22}|p] \leq \left((1 - l)L + (1 - i)E[R|p]I\right) + \left(lL + i\theta E[R|p]I - c_{11} - \alpha c_{21}\right)
\]

\[
= L + \left(i\theta + (1 - i)\right)E[R|p]I - c_{11} - \alpha c_{21}. \tag{2.2}
\]

The deposit contract says that early withdrawers can receive no more than the promised fixed return \( r > 0 \), and that if the bank cannot afford to pay this, it is forced to liquidate all of its assets and distribute them equally among early withdrawals. So the deposit contract gives the following two conditions:

\[
\max[c_{11}, c_{21}] \leq r \tag{2.3}
\]

If \( \min[c_{11}, c_{21}] < r \), then \( i = l = 1 \) and

\[
c_{11} + \alpha c_{21} = L + \theta E[R|p]I. \tag{2.4}
\]
Next, note that because investor patience is privately known, all early withdrawers must be treated equally, so

$$\text{if } \alpha > 0, \text{ then } c_{11} = c_{21}. \quad (2.5)$$

Lastly, I consider the patient investor's incentive to run, which occurs only when he expects to do at least as well by running as by waiting.

$$\begin{align*}
\max [u(c_{11}), u(c_{21})] > u(E[c_{22}[p]]) & \implies \alpha = 1 \\
\max [u(c_{11}), u(c_{21})] < u(E[c_{22}[p]]) & \implies \alpha = 0
\end{align*} \quad (2.6)$$

At date 1, then, the bank’s problem is to maximize aggregate utility

$$u(c_{11}(p)) + \left( \alpha(p)u(c_{21}(p)) + (1 - \alpha(p))u(E[c_{22}(R,p)]|p) \right)$$

subject to constraints (2.1) through (2.7). Given $L$, $I$, $r$, and $p$, the optimal consumption profile is the solution to this problem, and is given in the following proposition.

**Proposition 8** (Optimal Consumption Profile). Given $L$, $I$, $r$, and $p$, let $p^*$ be the lowest belief $p$ that does not induce a run:

$$p^* = \inf \{ p \in \mathbb{R} : u(r) \leq u(L - r + E[R|p]I) \}.$$

(i) If $p \geq p^*$, then

- The liquid asset is liquidated, the illiquid asset is not, and all patient investors withdraw late: $l = 1$, $i = 0$, and $\alpha(p) = 0$.
- Impatient investors receive the fixed promised payment, and patient investors receive the remaining assets: $c_{11}(p) = r$ and $c_{22}(R,p) = L - r + RI$.

(ii) If $p < p^*$, then
Figure 2.1: Consumption profiles of early and late withdrawers as a function of public signal $p$, where $p$ indicates the probability of a good return $G$ on the risky illiquid asset.

- Both assets are liquidated and all patient investors withdraw early: $l = i = 1$ and $\alpha(p) = 1$.

- All investors receive an equal portion of the total assets, which portion is strictly less than fixed return $r$: $c_{11}(p) = c_{22}(p) = (L + \theta E[R|p]I)/2 < r$.

I define the parameter $\kappa \equiv L - r$ to be the quantity of the safe liquid asset reserved for the late withdrawers, and I interpret this quantity $\kappa$ as the bank’s capital cushion.

2.3 Stress Tests

Suppose that at date 1, after the public signal $p$ is revealed but before the consumers privately learn their type, a regulator can perform a stress test on the risky asset to reveal additional information about its future return. The testing procedure and results of the test are public information; the bank cannot falsify either. However, the regulator can strategically choose the testing procedure in order to maximize the welfare of the consumers. The result of the stress test is a signal $\sigma \in \{g, b\}$ taking
either a good value $g$ or a bad value $b$. The probability of receiving either signal depends on the underlying true future return $R = \{G, 0\}$ of the risky illiquid asset. Stress tests are formalized as a pair of conditional distributions $\pi(\sigma|R = G)$ and $\pi(\sigma|R = 0)$, which are public information and chosen by the regulator.

By choosing the stress test $\pi(\sigma|R)$, the regulator induces not only a distribution $\mu(\sigma)$ over the signal outcomes $\sigma$, where

$$
\mu(\sigma) = \pi(\sigma|R = G)p + \pi(\sigma|R = 0)(1 - p),
$$

but also posterior beliefs $p_\sigma$ about the probability of a high return $G$, conditional on the revealed signal $\sigma$, where

$$
p_\sigma = \frac{\pi(\sigma|R = G)p}{\mu(\sigma)}.
$$

Choosing the stress test $\pi(\sigma|R)$ is equivalent to choosing distribution $d(p_\sigma) = \mu(\sigma)$ over posterior beliefs $p_\sigma$, provided that the mean posterior belief equals the prior belief $p$:

$$
p = p_g d(p_g) + p_b d(p_b).
$$

Given liquid holdings $L$, illiquid holdings $I$, fixed return $r$ for early withdrawers, let $U(p|L, I, r)$ be the date 1 average consumer utility under the optimal consumption profile $\{c_{11}(p), c_{21}(p), c_{22}(R, p), \alpha(p)\}$:

$$
U(p|L, I, r) = \frac{1}{2} u(c_{11}(p)) + \frac{1}{2} \left( \alpha(p) u(c_{21}(p)) + (1 - \alpha(p)) u(E[c_{22}(R, p)|p]) \right).
$$

The regulator’s disclosure problem is as follows:

$$
V(p|L, I, r) \equiv \max_{p_g, p_b} \quad d(p_b)U(p_b|L, I, r) + d(p_g)U(p_g|L, I, r)
$$

s.t.

$$
p_b, p_g, d(p_b), d(p_g) \in [0, 1]
$$
Proposition 9. Let $p^*(L, I, r)$ be the belief at which the late withdrawer’s expected date 2 consumption equals the kink $k$: $L - r + E[R|p^{**}]I = k$.

1. Suppose the portfolio $(L, I)$ and fixed early return $r$ are such that (a) without a stress test, the public signal $p$ would induce a run: $p < p^*(L, I, r)$; and (b) there exists some feasible belief at which runs are avoided: $p^*(L, I, r) \leq 1$.

Then an optimal stress test $(p_b, p_g)$ must satisfy:

$$p_b = 0, \quad p_g \in [p^*, p^{**}],$$

which implies

$$\pi(g|R = G) = 1, \quad \pi(g|R = 0) = \frac{p(1 - p)}{p_g(1 - p)}.$$
2. Otherwise, an uninformative test is weakly optimal:

\[ p_b = p_g = p, \quad \pi(g|R = G) = \pi(g|R = 0) \]

The proposition explains that in bad times, where \( p < p^* \), the regulator can improve consumer welfare by performing partially informative stress tests. Without the tests, depositors would run with certainty, forcing the bank to liquidate its risky assets at a significant discount. However, with partially informative stress tests, there is a positive probability that the bank will pass the test and patient investors will refrain from running. Stress tests therefore create value by reducing the likelihood of costly liquidation.

Condition (1b) stipulates that in order for stress tests to be valuable at all, there must exist some feasible belief at which investors would avoid runs. The condition is equivalent to \( p^* \leq 1 \), where \( p^*(L, I, r, \kappa) = (u(r) - \kappa)/(GI) \). Intuitively, if the return \( r \) promised to early withdrawers is too high, the capital cushion \( \kappa \) too low, or the risky payoff \( GI \) in a good state too low, then patient investors understand that even in the best case scenario they will do worse by waiting than by withdrawing early, so they run no matter what. As a consequence, costly liquidation is unavoidable, so informative stress tests offer no value.

In explaining optimal stress tests, I focus on the test where \( p_g = p^* \), although the proposition states that a range of posterior beliefs slightly above the run cutoff \( p^* \) could also be optimal. This is simply a consequence of using a piecewise linear utility function, which makes tractable the ex ante portfolio problem explored in the next section. For a strictly concave utility function, \( p_g = p^* \) is uniquely optimal, so I focus on \( p_g = p^* \) because it is optimal in either case.

Now consider the form of the optimal stress test. If conditions (1a) and (1b) hold, then the optimal test passes a good bank with certainty but passes a bad bank
with a lower, but positive, probability. The idea here is that stress tests should be informative, but not too harsh. Because all good banks pass with certainty, depositors know that failing grades can only come from bad banks, so the posterior belief from a failing grade is as low as possible: \( p_b = 0 \). However, because optimal stress tests can produce passing grades for both good and bad banks, a passing grade is less clear, so it’s possible for a passing grade to give depositors less than full confidence: \( p_g < 1 \).

Because runs induce costly liquidation, it is optimal to construct a test which induces runs with the lowest likelihood possible. This is a different objective from designing a test which produces failed grades with the lowest likelihood. Such a test would simply pass every bank, but would be completely uninformative, and if prior beliefs \( p \) were below the run cutoff \( p^* \), then runs would occur with certainty. Therefore, it is necessary to fail banks with some likelihood in order for a passing grade to have any credibility. The optimal test gives a failing grade with just enough likelihood so that passing grades are credible enough to prevent runs.

The following corollary shows how the stringency of the stress tests varies with the public signal \( p \).

**Corollary 6.** Under optimal stress tests, as public prior beliefs \( p \) decrease,

1. \( \pi(g|R = 0) \) decreases, so stress tests become more stringent, and
2. \( d(p_b) \) increases, so failing grades are given with greater frequency.

The corollary highlights a natural intuition, which is that the less confidence consumers have in the economy, the more credible a passing grade must be in order to prevent runs. Credibility is achieved by decreasing the likelihood \( \pi(g|R = 0) \) that bad banks receive a passing grade, as shown in part (1) of the Corollary. Part (2) highlights the fact that the total likelihood of failing grades increases as well, which
results both from the increased stringency of the tests, as well as the lower prior likelihood $p$ that the bank is good.

2.4 Capital Cushion and Portfolio Choice

I now examine how stress tests affect the bank’s optimal portfolio $L$ and $I$ as well as promised early return $r$ and capital cushion $\kappa$. Without loss of generality, I can assume that $r \leq L$, because otherwise the bank would be forced to liquidate the illiquid risky asset in every state, which is clearly not optimal.

2.4.1 No Stress Tests

Suppose the bank does not anticipate stress tests. Then at date 1, the average current and future utility of consumers is captured by the function $U(p|L, I, r, \kappa)$. So by backward induction, the bank chooses its investments, early return, and capital cushion to maximize the expected average date 1 utility $U(p|L, I, r, \kappa)$, where the expectation is taken over realizations of the public signal $p$. I write the bank’s problem with the objective function $U$ expanded to display the run and no-run cases.
as follows:

$$\max_{L,I,r,\kappa} \int_0^{\hat{p}(L,I,r,\kappa)} 2u\left(\frac{L + \theta E[R|p]|I}{2}\right) dp + \int_{\hat{p}(L,I,r,\kappa)}^{1} \left(u(r) + u(\kappa + E[R|p]|I)\right) dp$$

s.t.

$$L + I \leq E$$  \hspace{1cm} (2.8)

$$r + \kappa \leq L$$  \hspace{1cm} (2.9)

$$L \geq 0, \quad I \geq 0, \quad r \geq 0, \quad \kappa \geq 0.$$  \hspace{1cm} (2.10)

where

$$\hat{p} \equiv \max[0, \min[1, p^*]]$$  \hspace{1cm} (2.11)

and

$$p^*(L, I, r, \kappa) = \frac{u(r) - \kappa}{GI}.$$  \hspace{1cm} (2.12)

Constraint (2.8) is simply the bank’s budget constraint: its total investments cannot exceed consumer deposits $E$ because the bank has no other source of funding. Constraint (2.9) highlights the two uses of the safe liquid asset. The first is the fixed return $r$ to early withdrawers if beliefs are optimistic enough to avoid runs. It is optimal to pay early withdrawers with the safe liquid asset when possible because it may be liquidated at no cost. The remaining portion of the safe liquid asset is stored as a capital buffer $\kappa$. Constraint (2.12) defines the minimum belief $p^*$ at which patient depositors will refrain from running, and is the value of $p$ at which the patient investor’s expected consumption from waiting $\kappa + E[R|p]|I$ equals the utility $u(r)$ of withdrawing early. Because the cutoff belief $p^*$ may be greater than 1 or less than 0, the function defined in constraint (2.11) maps the cutoff belief $p^*$ to a
feasible probability value \( \hat{p} \in [0, 1] \) so that the objective function avoids integrating over infeasible values of \( p \).

The utility function \( u(c) = \min[c, k] \) is piecewise linear with a kink at \( k \), so the total deposits \( E \) offered to the bank must fall in a particular range in order for the concavity of \( u \) to be relevant. In particular, I make the following two assumption about the amount of deposits \( E \), relative to the kink \( k \).

**Assumption 2.** \( \frac{E}{2} < k \)

This assumption places an upper bound on the total wealth of the consumers. If it were violated, then the bank could offer consumers the certain utility of \( k \) by simply letting \( L = E, I = 0, \) and \( r = \kappa = E/2 \geq k \). If this were the case, then the deposit contract would have no risk because nothing is invested in the risky asset and because early and late withdrawers receive equal consumption \( E/2 \geq k \), so the maximal utility of \( k \) is guaranteed. Assumption 2 rules out this possibility by capping the total deposits. This creates an incentive to invest in the risky asset in order to benefit from its higher expected return \( E[R] > 1 \).

**Assumption 3.** \( \sqrt{2GE} > k \)

This assumption places a lower bound on the total wealth of consumers. Otherwise, investors only operate on the increasing linear portion of the utility curve, and then are effectively risk neutral, so invest all their assets in the risky asset. Assumption 3 prevents this corner solution by guaranteeing investors have enough wealth for the kink to be relevant, inducing risk aversion.

Although there are feasible portfolios \((L, I)\) and contracts \((r, \kappa)\) which never induce runs, the following proposition asserts that these are never optimal. Denote optimal values with a subscript \( n \) if the bank anticipates no stress tests.

**Proposition 10** (Optimal Runs). *If the bank does not anticipate stress tests, then
the optimal portfolio and contract \((L_n, I_n, r_n, \kappa_n)\) induces runs with some positive probability: \(p^* (L_n, I_n, r_n, \kappa_n) > 0\).

For runs to occur with any positive probability, the promised early return \(r\) must exceed the capital cushion \(\kappa\). It is never optimal for the early return \(r\) to be strictly less than the capital cushion \(\kappa\), because then the late investor is guaranteed to consume more than the early investor, so the contract does not smooth consumption between dates 1 and 2 sufficiently. A better contract would, for a fixed safe liquid investment \(L\), raise \(r\) and lower \(\kappa\) so that early and late investors would consume more similarly.

It is also never optimal for the early return \(r\) to equal the capital cushion \(\kappa\), but for a more subtle reason. For such a contract, there is an intertemporal benefit of raising \(r\) and lowering \(\kappa\) so that \(r > \kappa\), but there is also the cost induced by a run. However, for \(r\) close to \(\kappa\), runs are unlikely, because only very low signals \(p\) can cause expected late consumption \(\kappa + E[R|p]I\) to fall below the early return \(r\). Not only are runs very unlikely for \(r\) close to \(\kappa\), but when they do occur they are not very costly because the risky illiquid asset is not worth very much anyway. Because of this, the benefit from intertemporal smoothing dominates the likelihood and cost of a run, so the bank can improve on \(r = \kappa\) by raising the early return \(r\) while lowering the capital cushion \(\kappa\). As a consequence, some probability of runs occur under the optimal portfolio-contract.

For the remainder of this section, I adopt an additional assumption in order to simplify the analysis.

Assumption 4. \(\frac{GE}{1+G} < k\)

The assumption guarantees that there is no portfolio-contract \((L, I, r, \kappa)\) that satisfies both \(r \geq k\) and \(p^* \leq 1\). This permits me to assume that if there exists
some $p \in [0, 1]$ such that patient depositors refrain from running, then the amount $r$ promised to early investors must be strictly less than the kink $k$.

**Proposition 11** (Optimal Capital Cushion, No Disclosure). Given endowment $E$ and kink $k$, (i) there exists a good return $G > 0$ satisfying Assumption 3 and liquidation discount $\theta \in [0, 1]$ such it is optimal to hold a positive capital cushion: $\kappa_n > 0$; and (ii) there exists a good return $G > 0$ satisfying Assumption 3 and liquidation discount $\theta \in [0, 1]$ such that it is not optimal to hold a capital cushion: $\kappa_n = 0$.

The proposition states that the optimal presence or absence of a capital buffer $\kappa$ depends on the good return $G$ and the liquidation discount $\theta$. Numerical simulations indicate that a low enough good return $G$ or a low enough liquidation discount $\theta$ imply that it is optimal to hold a positive capital buffer: $\kappa_n > 0$. The intuition is that if the good return $G$ is low, then only very high signals $p$ will assure late investors of consumption exceeding the early return $r$, so costly runs occur with high frequency without reserving some of the safe asset as a capital buffer. Alternatively, if the liquidation discount $\theta$ is low enough, then regardless of the likelihood of a run, they are very costly when they do occur, so lowering their likelihood with a capital cushion $\kappa_n > 0$ is optimal. Alternatively, if the good return $G$ or liquidation discount $\theta$ are high enough, then runs occur with low likelihood or are not very costly, so no capital buffer is needed, and $\kappa_n = 0$.

### 2.4.2 Optimal Stress Tests

If the bank anticipates optimal stress tests, then the date 1 average current and future utility of depositors is given by $V(p|L, I, r, \kappa)$, and the bank maximizes the expected value of $V$ where the expectation is taken over the public signal $p$:

$$\max_{L, I, r, \kappa} \int_0^1 V(p|L, I, r, \kappa) dp$$
Theorem 4 (No Capital Cushion Under Optimal Disclosure). Suppose that the bank anticipates an optimal stress test at date 1. Then it is optimal for the bank to reserve no capital buffer: $\kappa_o = 0$.

Provided Assumption 4 is satisfied, the result of the theorem holds, independent of the good return $G$ or the liquidation cost $\theta$. This is in contrast to the case of no disclosure, in which the optimal capital buffer may be positive given sufficiently low $G$ or $\theta$. Intuitively, optimal stress tests lower the likelihood of a run, so the value of a capital buffer is decreased. In addition, under optimal stress tests, runs occur only when investors are certain the risky illiquid asset is worthless ($p_b = 0$), in which case there is no loss from being forced to liquidate it early. This effectively eliminates the cost of runs, which is the only reason for holding capital buffers in the first place.

Note that although this result technically requires that the risky asset returns nothing in the bad state ($B = 0$), the intuition is similar if the bad state yields a positive return ($B > 0$). In that case, optimal stress tests lower, rather than eliminate, the cost of runs by inducing depositors to run only when they are certain the risky asset will yield the lowest possible return $B > 0$, which implies the liquidation cost $(1 - \theta)BI > 0$ is also the lowest possible value. Therefore, optimal stress tests should lower the optimal size $\kappa_o$ of the capital cushion because the cost of runs have decreased.

Theorem 5. If without stress tests, no capital cushion is optimal ($\kappa_n = 0$), then under optimal stress tests, the return $r_o$ to early withdrawers—and therefore the investment $L_o$ in the safe liquid asset—should rise: $L_o = r_o > L_n = r_n$. 

s.t.

$$L + I \leq E$$ \hspace{1cm} (2.8)

$$r + \kappa \leq L$$ \hspace{1cm} (2.9)

$$L \geq 0, \quad I \geq 0, \quad r \geq 0, \quad \kappa \geq 0.$$ \hspace{1cm} (2.10)
If holding a positive capital cushion is not optimal with or without stress tests, then the liquid asset must be used entirely to fund the return \( r \) to early withdrawers, so \( L_n = r_n \) and \( L_o = r_o \). A low early return \( r \) may allocate too much consumption to the late withdrawers and not enough to the early withdrawers, making the ex ante consumption profile too risky for agents who do not yet know their time preferences. So the value of raising \( r \) (and therefore \( L \)) is that is smooths consumption across dates 1 and 2. The cost of raising \( r \) (and therefore \( L \)) is twofold: (1) depositor beliefs \( p \) must clear a higher bar \( p^* \) to avoid running, so runs are more likely, and (2) less deposits are allocated to the high return illiquid asset. Under optimal stress tests, the cost of runs is eliminated, so the only remaining cost of raising the early return \( r \) is the foregone return \( R \) on the illiquid asset. Therefore, optimal stress tests should result in a higher return \( r_n \) to early withdrawers and therefore a greater investment \( L_n \) in the safe liquid asset.

2.5 Conclusion

The paper presents a model in which stress tests may be optimally designed in order to reduce the probability of a run. Runs are costly because they force banks to sell illiquid assets at a discount, reducing the resources available to withdrawing depositors. It is optimal for stress tests to pass all good banks, but to fail just enough bad banks to maintain the credibility of a passing grade. In particular, passing grades must be credible enough to prevent runs. The less confidence depositors have in the bank, the more stringent stress tests must be to prevent runs. By reducing the probability of runs, optimal stress tests reduce the value of capital cushions, and therefore the optimal level of capital cushions. In addition, the reduced probability of runs permit greater consumption smoothing by allowing a higher return to early withdrawers.
Appendix A

Proofs for Chapter 1

**Proof of Proposition 1.** The solution strategy is to let a market maker design an incentive-compatible mechanism \((P(\hat{f}), \theta(\hat{f}))\) which maximizes the seller’s profit and satisfies the buyer’s free-entry condition. Note that for the buyer’s free-entry condition to be satisfied, we must have:

\[
P(\hat{f}) = \hat{f} - \frac{k\theta(\hat{f})}{m(\theta(\hat{f}))}.
\]

Using the above expression for \(P\), I can write the seller’s payoff for a given mechanism \(\theta\) as

\[
U(\hat{f}|f) = m(\theta(\hat{f})) \left( P(\hat{f}) - \delta f \right)
= m(\theta(\hat{f})) \left( \hat{f} - \delta f \right) - k\theta(\hat{f}),
\]

so that \(U(\hat{f}|f)\) is the payoff of a type \(f\) seller who reports \(\hat{f}\). Define \(\bar{U}(f) \equiv U(f|f)\) to be the profit to a seller who reports the truth, given a particular mechanism \(\theta(\hat{f})\). The following lemma gives necessary and sufficient conditions for global incentive compatibility.
Lemma 2. Global incentive compatibility (GIC), defined as

\[ U(f | f) \geq U(\hat{f} | f) \quad \forall f, \hat{f} \in [\underline{f}, \overline{f}] \]

is equivalent to

(i) Local incentive compatibility (LIC): \( U(f | f) = U_2(f | f) \) or \( U_1(f | f) = 0 \) almost surely, and

(ii) Monotonicity (M): \( U_{21}(\hat{f} | f) \geq 0 \) almost surely.

Proof. Global incentive compatibility implies

\[
\frac{U(f | f) - U(\hat{f} | f)}{f - \hat{f}} \geq \frac{U(\hat{f} | f) - U(\hat{f} | \hat{f})}{f - \hat{f}}
\]

if \( f > \hat{f} \), and the sign reversed if \( f < \hat{f} \). Letting \( f \to \hat{f} \) from above and below immediately gives local incentive compatibility.

Now rewrite (GIC) as follows:

\[
0 \leq U(f | f) - U(\hat{f} | f)
= \left( U(f | f) - U(\hat{f} | f) \right) = \left( U(\hat{f} | f) - U(\hat{f} | \hat{f}) \right)
= \int_{\hat{f}}^{f} U'(s) ds - \int_{\hat{f}}^{f} U_2(\hat{f} | s) ds
= \int_{\hat{f}}^{f} U_2(s | s) ds - \int_{\hat{f}}^{f} U_2(\hat{f} | s) ds \quad \text{(from LIC)}
= \int_{\hat{f}}^{f} \int_{\hat{f}}^{s} U_{21}(t, s) dtds \quad \text{(A.1)}
\]

This immediately gives monotonicity, i.e. \( U_{21}(t, s) \geq 0 \) almost everywhere. If not, then there exists some square \( \overline{[\hat{f}, \hat{f}]^2} \in [\underline{f}, \overline{f}]^2 \) such that \( U_{21}(t, s) < 0 \ \forall (t, s) \in [\hat{f}, \hat{f}]^2 \), violating (A.1). Note that the inequality applies whether \( f \) is greater than or
less than \( \hat{f} \), because the double integral cancels the effect of integrating backwards. So I have shown that (LIC) and (M) are necessary conditions for (GIC).

To show sufficiency, note that (LIC) implies that (GIC) is characterized by (A.1), which is clearly satisfied when (M) holds.

In this setting, where \( U(\hat{f}|f) = m(\theta(\hat{f}))(\hat{f} - \delta f) - k\theta(\hat{f}) \), we have the following characterization for GIC:

1. LIC:
\[
0 = U_1(f|f) = (m'(\theta)(1 - \delta)f - k)\theta'(f) + m(\theta) \quad \text{(A.2)}
\]

2. M:
\[
0 \leq U_{12}(\hat{f}|f) = -\delta m'(\theta)\theta'(\hat{f}), \quad \text{or} \quad \theta'(f) \leq 0 \quad \text{(A.3)}
\]

The function \( \theta(f) \) characterized by (A.2) is sensitive to the initial condition \( \theta(f) \). Note that at \( f = f_{CI} \), the coefficient on \( \theta'(f) \) is zero at \( \theta = \theta_{CI}(f) \). So if \( \theta(f) > \theta_{CI}(f) \), then \( \theta'(f) > 0 \), violating (M). If \( \theta(f) < \theta_{CI}(f) \), then \( \theta'(f) < 0 \), consistent with (M). If \( \theta(f) = \theta_{CI}(f) \), then an increasing or decreasing \( \theta(f) \) is consistent with (A.2).

The above reasoning indicates that \( \theta(f) \) must be less than or equal to \( \theta_{CI}(f) \) in order to satisfy (M). I select the decreasing \( \theta(f) \) with initial condition \( \theta(f) = \theta_{CI}(f) \), as it is the most liquid (highest profit) \( \theta(f) \) which satisfies (LIC) and (M). This completes the proof of part 1 of the proposition, and part 2 follows directly from the buyer’s free-entry condition.

It remains to show that this mechanism is decentralizable; i.e., that buyers cannot profit by offering an off equilibrium \( p \notin P \). First note that \( p^*(f) \) is continuous, with derivative \( p'^*(f) = 1 - k\theta'(f)(m(\theta) - m'(\theta)\theta)/m(\theta)^2 \). Because \( \theta'(f) < 0 \), and the concavity of \( m \) implies \( m(\theta)/\theta > m'(\theta) \), I must have \( p'^*(f) > 0 \), so \( P^* = [p^*(\underline{f}), p^*(\hat{f})] \subseteq \mathbb{R}_+ \). Denote the lower and upper bounds of \( P^* \) by \( \underline{p} \) and \( \bar{p} \), respectively. It suffices to show that \( p < \underline{p} \) and \( p > \bar{p} \) are not profitable deviations for the buyer. Now
suppose that a coalitio of buyers posts \( p > \bar{p} \). Recall that \( \theta(p, f) \equiv \inf \{ \theta \geq 0 : m(\theta)(p - \delta f) \geq \Pi(f) \} \), and that buyers expect types \( T(p) = \arg \inf_f \theta(p, f) \) (if any) and market tightness \( \theta(p) = \inf_f \theta(p, f) \) for posting off-equilibrium \( p \). Then if \( p > \bar{p} \), then \( \theta(p, f) \) satisfies \( m(\theta(p, f))(p - \delta f) = \Pi(f) = m(\theta^*(f))(p^*(f) - \delta f) \). If so, then differentiating both sides with respect to \( f \) and applying the envelope condition shows that \( \theta_2(p, f) \) has the same sign as \( (p^*(f) - \delta f)/(p - \delta f) - 1 \), so if \( p > \bar{p} \), then \( \theta(p, f) \) is minimized by \( \bar{f} \), and \( \theta(p) = \theta(p, \bar{f}) \). Next, note that because \( m(\theta(p))(p - \delta \bar{f}) = \Pi(\bar{f}) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta \bar{f}) \), then \( \theta(p) < \theta^*(\bar{f}) \). Because \( \theta^*(\bar{f}) < \theta_{CI}(\bar{f}) \), and \( \theta(p)(1 - \delta) \bar{f} - k\theta(p) \) is increasing in \( \theta \) for all \( \theta < \theta_{CI}(\bar{f}) \), it must be that \( m(\theta(p))(1 - \delta) \bar{f} - k\theta(p) < m(\theta^*(\bar{f}))(1 - \delta) \bar{f} - k\theta^*(\bar{f}) \). Multiply the zero profit function of a buyer who trades with \( \bar{f} \) in equilibrium by \( \theta^*(\bar{f}) \) to get

\[
0 = m(\theta^*(\bar{f}))(\bar{f} - p^*(\bar{f})) - k\theta^*(\bar{f}) = m(\theta^*(\bar{f}))(\bar{f} - \Pi(f)/m(\theta^*(\bar{f}))) - \delta \bar{f}) - k\theta^*(\bar{f})
\]

\[
= -\Pi(\bar{f}) + m(\theta^*(\bar{f}))(1 - \delta) \bar{f} - k\theta^*(\bar{f}) > -\Pi(\bar{f}) + m(\theta(p))(1 - \delta) \bar{f} - k\theta(p)
\]

\[
=-m(\theta(p))(p - \delta \bar{f}) + m(\theta(p))(1 - \delta) \bar{f} - k\theta(p) = m(\theta(p))(\bar{f} - p) - k\theta(p)
\]

Divide both sides by \( \theta(p) \) to get \( 0 > m(\theta(p))(\bar{f} - p) - k \), so \( p > \bar{p} \) is not a profitable deviation.

On the other hand, if a coalition of buyers post \( p < \bar{p} \), then because \( \theta_2(p, f) \) has the same sign as \( (p^*(f) - \delta f)/(p - \delta f) - 1 \), \( \theta_2(p, f) \) is minimized at \( \bar{f} \), so buyers expect type \( \bar{f} \), and therefore \( \theta(p) = \theta(p, \bar{f}) \). Note that the lowest type \( \bar{f} \) receives the complete information liquidity \( \theta^*(\bar{f}) = \theta_{CI}(\bar{f}) \), which maximizes \( m(\theta)(1 - \delta) \bar{f} - k\theta \). Also note that \( m(\theta(p))(p - \delta \bar{f}) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta \bar{f}) \), so \( \theta(p) > \theta^*(\bar{f}) \). Therefore, \( m(\theta^*(\bar{f}))(1 - \delta) \bar{f} - k\theta^*(\bar{f}) > m(\theta(p))(1 - \delta) \bar{f} - k\theta(p) \), so by the method above for \( p > \bar{p} \), it must be that \( p < \bar{p} \) is not a profitable deviation.

**Proof of Theorem 1.** For ease of notation, I drop the * on all equilibrium functions, use \( \theta(f) \) in place of \( \Theta^*(f) \), and let \( \bar{\theta} = \Theta^*(\bar{f}) \).
**Part (i):** The price function may be written as

\[
P(\theta(f)) = f - \frac{k\theta(f)}{m(\theta(f))},
\]

so I must show that the discount \(k\theta/m(\theta)\) goes to zero as \(k\) goes to zero.

First consider the equilibrium for the lowest type \(f\). Recall that \(\theta\) solves the complete information FOC \(m'(\theta)(1 - \delta)f = k\), so \(\theta \to \infty\) as \(k \to 0\). Using the FOC, I can express the discount for \(f\) as

\[
\frac{k\theta}{m(\theta)} = (1 - \delta)f \frac{\theta m'\theta}{m(\theta)}.
\]

The following lemma guarantees that this discount goes to zero.

**Lemma 3.**

\[
\lim_{\theta \to \infty} \theta m'(\theta) = 0
\]

**Proof.** Observe that due to the concavity of \(m\), for any \(\hat{\theta} > 0\), if \(\theta > \hat{\theta}\), then

\[
m'(\theta) < \frac{m(\theta) - m(\hat{\theta})}{\theta - \hat{\theta}}.
\]

If so, then for any \(\hat{\theta} > 0\),

\[
\lim_{\theta \to \infty} \theta m'(\theta) \leq \lim_{\theta \to \infty} \theta \left[\frac{m(\theta) - m(\hat{\theta})}{\theta - \hat{\theta}}\right] = \lim_{\theta \to \infty} \frac{m(\theta) - m(\hat{\theta})}{1 - \frac{\hat{\theta}}{\theta}} = 1 - m(\hat{\theta})
\]

This holds for any \(\hat{\theta}\), and since \(\sup_{\hat{\theta}} m(\hat{\theta}) = 1\), I must have \(\lim_{\theta \to \infty} \theta m'(\theta) = 0\).

Now recall from monotonicity that \(\theta(f) < \theta\) for any \(f > f\), so as \(k \to 0\),

\[
0 < \frac{k\theta}{m(\theta)} < \frac{k\theta}{m(\theta)} = (1 - \delta)f \frac{\theta m'(\theta)}{m(\theta)} \to 0.
\]
By the squeeze theorem, the discount \( \frac{\theta k}{m(\theta)} \) for any \( f \) goes to zero, and therefore \( P(\theta(f)) \to f \).

**Part (ii):** Let \( \Theta(f, k) \) be the equilibrium \( \theta \) for a given \( f \) and \( k \). Let \( h(\theta, k) \) be the inverse of \( \Theta(f, k) \) so that \( h(\Theta(f, k), k) = f \). If so, then \( \Theta_1(h(\theta, k), k) = \frac{1}{h_1(\theta, k)} \). Then write (1.3) in terms of \( h \):

\[
(m'(\theta)(1 - \delta)h(\theta, k) - k) = -m(\Theta)h_1(\theta, k),
\]

Using integrating factors, I can solve for \( h \) explicitly:

\[
h(\theta, k) = m(\theta)^{-1(1-\delta)} \left( k \int m(\theta)^{-\delta} d\theta + C(k) \right),
\]

where

\[
C(k) = \left. f m(\theta(k))^{(1-\delta)} - k \int m(\theta)^{-\delta} d\theta \right|_{\theta=\theta(k)}
\]

\[
= \left. f m(\theta)^{(1-\delta)} - (1 - \delta) f m'(\theta) \int m(\theta)^{-\delta} d\theta \right|_{\theta=\theta(k)}
\]

Note that because \( \lim_{\theta \to \infty} m(\theta)^{-\delta} = 1 \), it must be that \( \lim_{k \to \infty} \int m(\theta)^{-\delta} d\theta|_{\theta=\theta(k)} = \infty \).

Apply L’Hopital’s Rule first to Lemma 3 and then to the second term of \( C(k) \) to show that the second term converges to zero, which implies \( C(k) \to f \). This implies that as \( k \to 0 \), \( h(\theta, k) \to m(\theta)^{-(1-\delta)} f \equiv h(\theta) \). So then

\[
P(\theta, k) = h(\theta, k) - \frac{k\theta}{m(\theta)} \to h(\theta) = \frac{f}{[m(\theta)]^{1-\delta}}.
\]

**Part (iii):** Note that \( h(\theta) \) is invertible, so let \( \Theta(f) \) be its inverse. Because \( h(\Theta(f, k), k) = f \), it must be that \( \Theta(f, k) \to \Theta(f) \). By the continuity of \( m \),

\[
\lim_{k \to 0} m(\Theta(f, k)) = m(\Theta(f)) = \left( \frac{f}{h(\Theta(f))} \right)^{1-\delta} = \left( \frac{f}{h(\theta)} \right)^{1-\delta}.
\]

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Proof of Theorem 2. The solution strategy again is to design a mechanism $(P(\hat{f}), \theta(\hat{f}), q(\hat{f}))$ which is incentive compatible, satisfies free-entry, and maximizes the seller’s payoff. The buyer’s problem (Definition 2, Part 2) can be rearranged to obtain an expression for $P(\hat{f})$ in terms of $\theta(\hat{f})$ and $q(\hat{f})$:

$$P(\hat{f}) = \hat{f} - \frac{k\theta(\hat{f})}{m(\theta(f))}.$$

I can then plug $P$ into the seller’s objective function (Definition 2, Part 1) to obtain an expression for the type $f$ seller’s profit for reporting $\hat{f}$:

$$U(\hat{f}|f) = m(\theta(\hat{f}))q(\hat{f})(\hat{f} - \delta f) - k\theta(\hat{f}).$$

Applying Lemma 2, which was proved independent of the mechanism structure, I can characterize GIC in this context as

1. LIC:

$$0 = U_1(f|f) = (m'(\theta(f))q(f)(1-\delta)f - k) \theta'(f) + m(\theta(f))(1-\delta)f q'(f) + m(\theta(f))q(f)$$

2. M:

$$0 \leq U_{12}(\hat{f}|f) = -\delta m'(\theta(\hat{f}))q(\hat{f})\theta'(\hat{f}) - \delta m(\theta(\hat{f}))q'(\hat{f}).$$

The problem is now to choose the Pareto optimal GIC equilibrium $(\theta(f), q(f))$. To do this, first observe that if two GIC equilibria $i$ and $j$ agree on $[\tilde{f}, f]$, where $f < \tilde{f}$, but disagree thereafter, then it must be that $\bar{U}_i(f) = \bar{U}_j(f)$ and $\bar{U}_i'(f) = -\delta m(\theta_i(f))q_i(f) = -\delta m(\theta_j(f))q_j(f) = \bar{U}_j'(f)$. Note however, that $\bar{U}_i''(f) = -\delta m'(\theta_i(f))q_i(f)\theta'_i(f) - \delta m(\theta_i(f))q'_i(f)$, and it is not clear that $\theta'_i(f) = \theta'_j(f)$ or that $q'_i(f) = q'_j(f)$, so it may be that $\bar{U}_i''(f) \neq \bar{U}_j''(f)$. Clearly the Pareto optimal equilibrium will have the highest $\bar{U}_i''(f)$ for all $f$, which I can find using optimal control. The problem is to choose
an incentive compatible equilibrium \((\theta, q)\) which maximizes \(\bar{U}(f)\) for all \(f\). The solution given in the proposition has \(\theta'(f) = -\infty\), so I first solve the optimal control problem with the restriction \(\theta' \geq -C\), where \(C\) is a positive constant, and then show that as \(C \to \infty\), the solution converges to that in the proposition. I can now define the optimal control problem:

\[
\begin{align*}
\max_{\theta', q'} & -\delta m'(\theta)q\theta' - \delta m(\theta)q' \\
\text{s.t.} & \quad m'(\theta)q\theta' + m(\theta)q' \geq 0 \\
& \quad (m'(\theta)(1 - \delta)f - k)\theta' + m(\theta)(1 - \delta)fq' + m(\theta)q = 0
\end{align*}
\] (Obj.)

where the states are given by \((\theta, q)\), and the control variables are \((\theta', q')\). It is straightforward to derive the solution for the following cases.

\[
\theta' = \begin{cases} 
\frac{-m(\theta)}{m'(\theta)(1 - \delta)f - k} & \text{if } \theta > 0, q = 1, \text{ and } m'(\theta)(1 - \delta)f - k > m(\theta)/C \\
0 & \text{if } \theta = 0 \text{ or } q = 0 \\
-C & \text{otherwise}
\end{cases}
\]

\[
q' = \begin{cases} 
0 & \text{if } \theta > 0, q = 1, \text{ and } m'(\theta)(1 - \delta)f - k > m(\theta)/C \text{ or } q = 0 \\
\frac{-m(\theta)q + (m'(\theta)q(1 - \delta)f - k)C}{m(\theta)(1 - \delta)f} & \text{if } \theta = 0 \\
-\infty, \infty & \text{otherwise}
\end{cases}
\]

I set the initial condition for \(f\) equal to the complete information case, which maximizes the payoff to the lowest type. So the initial condition is \(q(f) = 1\) and \(\theta(f)\) solves \(m'(\theta)(1 - \delta)f = k\). Denote the solution to the \(q'\) differential equation above as \(\tilde{q}(f, C)\), which can be solved for explicitly as \(\tilde{q}(f, C) \equiv [m(\theta)(f - C)f]^{-1} (M_1 f^{-1/(1-\delta)} - kC)\), where \(M_1\) is a constant chosen so that \(\tilde{q}(f) = 1\). The function \(\tilde{q}(f, C)\) is U-shaped
and has a positive vertical asymptote at \( f = \theta(f)/C \). Denote by \( f^*(C) \) the unique \( f \in (f, \theta(f)/C) \) such that \( \bar{q}(f, C) = 1 \). In otherwords, the function \( \bar{q}(f, C) \) begins at 1, slopes downward and then upward in a U-shape, and then crosses back over \( q = 1 \) at \( f^*(C) \) before asymptoting to \(+\infty\) at \( f = \theta(f)/C \).

Denote the solution to the differential equation for \( \theta' \) above as \( \tilde{\theta}(f, C) \), with initial condition \( \tilde{\theta}(f^*(C), C) = \theta(f) - Cf^*(C) \).

I can now define piecewise the Pareto optimal separating equilibrium, given the restriction that \( \theta' \geq -C \).

\[
q(f, C) = \begin{cases} \bar{q}(f, C) & f \leq f \leq f^*(C) \\ 1 & f^*(C) < f \leq \bar{f} \end{cases}
\]

\[
\theta(f, C) = \begin{cases} \theta(f) - Cf & f \leq f \leq f^*(C) \\ \sigma(f, C) & f^*(C) < f \leq \bar{f} \end{cases}
\]

It remains to be shown that this equilibrium converges to the one identified in the proposition as \( C \to \infty \).

First consider \( q(f, C) \). As explained above, there exists a unique \( f^*(C) \) such that \( f^* \in (f, \theta(f)/C) \) and \( \bar{q}(f^*, C) = 1 \). Because \( \theta(f)/C \to 0 \) as \( C \to \infty \), it must be that \( f^*(C) \to 0 \) as well, implying that \( \forall f \in [f, \bar{f}], q(f, C) \to 1 = Q^*(f) \).

Now consider \( \theta(f, C) \). Because \( f^*(C) \to 0 \) as \( C \to \infty \), the linear portion of \( \theta(f, C) \) occupies a smaller and smaller interval \( [f, f^*(C)] \). However, the slope of that segment also becomes more and more negative, so it is not immediately clear if \( \theta(f, C) \) converges to the convex curve identified in the proposition. I next show that the value of \( \theta(f^*(C), C) \) at the piecewise boundary \( f^*(C) \) converges to \( \theta(f) \), the full information liquidity of the lowest type, which is sufficient to prove convergence to the equilibrium in the proposition.

Now denote \( \hat{\theta} \equiv \theta(f^*(C), C) \) as the value of \( \theta(f, C) \) where \( \bar{q}(f, C) = 1 \) and \( \theta(f, C) \)
transitions from the linear segment to the convex curve defined by the differential equation. For convenience, denote \( \theta \equiv \theta(f) \) to be the initial condition, the full information liquidity to the lowest type. Because \( \hat{\theta} = \theta(f^*(C), C) = \theta - Cf^* \) and \( \tilde{q}(f^*, C) = 1 \), I can write \( \tilde{q}((\theta - \hat{\theta})/C, C) \) to characterize \( \hat{\theta} \). Now substitute \( x \equiv 1/C \) and rearrange to obtain the fixed point problem:

\[
\hat{\theta} = H(\hat{\theta}, x) \equiv \theta - \frac{f}{x}\left[\frac{(m(\theta)x + k)^{1-\delta}}{m(\theta)x + k} - 1\right]
\]

It can easily be shown that \( H_{1,1}(\hat{\theta}, x) < 0 \) for all \( (\hat{\theta}, x) \in \mathbb{R}_+^2 \), that \( H_1(\hat{\theta}, x) = k/(m(\hat{\theta})x + k) < 1 \) for \( x > 0 \), and that \( \lim_{x \to 0} H_1(\hat{\theta}, x) = 1 \). These features of \( H \) imply that for small enough \( x > 0 \), there exists exactly one fixed point \( \hat{\theta} = H(\hat{\theta}, x) \) such that \( \hat{\theta} < \theta \), and that as \( x \to 0 \) it must be that \( \hat{\theta} \to \bar{\theta} \). Therefore, as \( C \to \infty \), I must have that \( \theta(f^*(C), C) = \hat{\theta} \to \bar{\theta} = \theta_{C1}(f) \), and so for all \( f \in [f, \bar{f}] \), \( \theta(f, C) \to \Theta^*(f) \), so the Pareto optimal incentive compatible mechanism has been found. I now check that buyers do not have a profitable off-equilibrium deviation. First observe the equilibrium prices \( p \) occupy the same space \([\underline{p}, \bar{p}]\) as in the one-signal case, but equilibrium \( q \) is equal to 1 everywhere, so \( M^* = [\underline{p}, \bar{p}] \times 1 \). First observe that \( m(\theta(p, q, f))q(p - \delta f) = \Pi(f) \). Following the same reasoning as in the case with one signal, \( \theta_3(p, q, f) \) has the same sign as \( (p^*(f) - \delta f)/(p - \delta f) - 1 \). So if buyers deviate with \( (p, q) \in [\underline{p}, \bar{p}] \times (0, 1) \), then \( \theta(p, q, f) \) is minimized by the unique \( f \in [f, \bar{f}] \) that chooses \( p \) in equilibrium; i.e., \( p^*(f) = p \). Therefore, \( \theta(p, q) = \theta(p, q, p^*-1(p)) \), and I have \( m(\theta(p, q))q(p - \delta f) = \Pi(f) = m(\theta^*(f))(p^*(f) - f) \), where \( f \) is the type which chooses \( p^*(f) = p \) in equilibrium (and therefore \( p - \delta f \) and \( p^*(f) - \delta f \) cancel). This gives \( m(\theta(p, q)) = m(\theta^*(f))/q > m(\theta^*(f)) \), because \( q < 1 \). Therefore, \( \theta(p, q) > \theta^*(f) \), and because \( n(\theta) = m(\theta)/\theta \) is strictly decreasing, I have \( k = n(\theta^*(f))(f - p^*(f)) = n(\theta^*(f))(f - p) > n(\theta(p, q))q(f - p) \), so \( (p, q) \in [\underline{p}, \bar{p}] \times (0, 1) \) is not a profitable deviation. Next, suppose a coalition of buyers post
a price \( p > \bar{p} \) and \( q \in [0, 1] \). First recall that because \( \theta_3(p, q, f) \) has the same sign as \( (p^*(f) - \delta f)/(p - \delta f) - 1 \), for \( p > \bar{p} \equiv p^*(\bar{f}) \), \( \theta(p, q, f) \) must be minimized by \( \bar{f} \). So buyers expect type \( \bar{f} \), and \( \theta(p, q) = \theta(p, q, \bar{f}) \) for \( p > \bar{p} \). Next, observe that \( \theta(p, 1) \) satisfies \( m(\theta(p, 1))(p - \delta \bar{f}) = \Pi(f) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta \bar{f}) \), so then \( \theta(p, 1) < \theta^*(\bar{f}) \).

Because \( \theta^*(\bar{f}) \) is less than the complete information \( \theta \) and \( \theta(p, 1) \) is even lower, I must have \( m(\theta^*(\bar{f}))(\bar{f} - \delta \bar{f}) - k(\theta^*(\bar{f})) > m(\theta(p, 1))(\bar{f} - \delta \bar{f}) - k\theta(p, 1) \). So the zero-profit condition of buyers who trade with \( \bar{f} \) in equilibrium gives

\[
0 = m(\theta^*(\bar{f}))(\bar{f} - p^*(\bar{f})) - k\theta^*(\bar{f}) = m(\theta^*(\bar{f}))(\bar{f} - \Pi(\bar{f}))/m(\theta^*(\bar{f})) - \delta \bar{f})
\]

\[
= -\Pi(\bar{f}) + m(\theta^*(\bar{f}))(\bar{f} - \delta \bar{f}) - k\theta^*(\bar{f}) > -\Pi(\bar{f}) + m(\theta(p, 1))(\bar{f} - \delta \bar{f}) - k\theta(p, 1)
\]

\[
= -m(\theta(p, 1))(p - \delta \bar{f}) + m(\theta(p, 1))(\bar{f} - \delta \bar{f}) - k\theta(p, 1) = m(\theta(p, 1))(\bar{f} - p) - k\theta(p, 1)
\]

Dividing both sides by \( \theta(p, 1) \) gives \( 0 > m(\theta(p, 1))(\bar{f} - p) - k \). Next, note that for \( q \in [0, 1], m(\theta(p, q))q(p - \delta \bar{f}) = m(\theta(p, 1))(p - \delta \bar{f}) \), so \( \theta(p, q) \geq \theta(p, 1) \), and I have \( n(\theta(p, q))q(\bar{f} - p) - k \), so \( (p, q) \in (\bar{p}, \infty) \times [0, 1] \) is not a profitable deviation. Finally, suppose that a coalition posts \( p < \bar{p} \) with \( q \in [0, 1] \). Again, because \( \theta_3(p, q, f) \) has the same sign as \( (p^*(f) - \delta f)/(p - \delta f) - 1 \), if \( (p, q) \) attracts any type, it must be \( \bar{f} \), so \( \theta(p, q) \equiv \theta(p, q, \bar{f}) \). Recall that \( \bar{f} \) receives the complete information liquidity, and because \( m(\theta(p, q))q(p - \delta \bar{f}) = \Pi(f) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta \bar{f}) \) and therefore \( \theta(p, q) > \theta^*(\bar{f}) = \theta_{ci}(\bar{f}) \), it must be that \( m(\theta^*(\bar{f}))(\bar{f} - \delta \bar{f}) - k\theta^*(\bar{f}) > m(\theta(p, q))(\bar{f} - \delta \bar{f}) - k\theta(p, q) \). So applying the same reasoning as for \( p > \bar{p} \), I have \( 0 > m(\theta(p, q))(\bar{f} - p) - k \geq m(\theta(p, q))q(\bar{f} - p - k) \), so \( (p, q) \in [0, \bar{p}] \times [0, 1] \) is not a profitable deviation.

Proof of Proposition 2 First observe that \( \bar{U}_f'(\bar{f}) = -\delta m(\Theta^*(\bar{f}))Q^*(\bar{f}) < 0. \)

Because the profit function is decreasing in \( f \), the proof is identical to the one in DeMarzo and Duffie 1999.

Proof of Lemma 1. First I show that global incentive compatibility implies the
conditions in the lemma. Global incentive compatibility implies that
\[
\Pi(\hat{s}|\hat{s}) \geq \Pi(s|\hat{s})
\]
\[
= \Pi(s|s) + \Pi(s|\hat{s}) - \Pi(s|s)
\]
\[
= \Pi(s|s) - m(\hat{\theta}(s))\bar{q}(s)(\hat{\nu} - \nu)
\]
Switching \(\hat{s}\) and \(s\) and combining inequalities gives
\[
-m(\hat{\theta}(s))\bar{q}(s)(\hat{\nu} - \nu) \leq \Pi(\hat{s}|\hat{s}) - \Pi(s|s) \leq -m(\hat{\theta}(s))\bar{q}(s)(\hat{\nu} - \nu)
\]  \hspace{1cm} (A.4)

Part (i): Clearly, if \(\hat{\nu} = \nu\), then \(\Pi(\hat{s}|\hat{s}) = \Pi(s|s)\), regardless of \(\hat{f}\) and \(f\). Therefore, \(\Pi((f, v)|(f, v))\) is constant in \(f\), and is fully determined by \(v\).

Part (ii): For shorthand, write \(\Pi((f, v)|(f, v)) = \Pi(v)\), and rewrite (A.4) as
\[
-m(\hat{\theta}(s))\bar{q}(s)(\hat{\nu} - \nu) \leq \Pi(\hat{\nu}) - \Pi(\nu) \leq -m(\hat{\theta}(s))\bar{q}(s)(\hat{\nu} - \nu).
\]  \hspace{1cm} (A.5)

The right hand inequality indicates that if \(\hat{\nu} > \nu\), then \(\Pi(\hat{\nu}) \leq \Pi(\nu)\), so \(\Pi\) is decreasing in \(\nu\). Next, for any \(\epsilon > 0\), choose an arbitrary set of disjoint intervals \((a_k, b_k)\) in \(V = [v, \tilde{v}]\) such that \(\sum_{k=1}^{N}(b_k - a_k) < \epsilon\) and set of arbitrary asset qualities \(\{f_k\}\) so that \(f_k \in \tilde{S}(a_k)\). Then
\[
\sum_{k=1}^{N}|\Pi(b_k) - \Pi(a_k)| = -\sum_{k=1}^{N}(\Pi(b_k) - \Pi(a_k))
\]
\[
\leq \sum_{k=1}^{N}m(\tilde{\theta}(f_k, a_k))\bar{q}(f_k, a_k)(b_k - a_k)
\]
\[
\leq \sum_{k=1}^{N}(b_k - a_k) < \epsilon,
\]
so \(\Pi\) is absolutely continuous, and is therefore differentiable almost everywhere. Now if \(\Pi'(v)\) exists, then dividing (A.5) by \((\hat{\nu} - \nu)\) and letting \(\hat{\nu}\) go to \(\nu\) from above and below gives (ENV).

Part (iii): This follows directly from (A.5).
I now show that the conditions in the lemma imply global incentive compatibility. Denote \( m(\hat{\theta}(f, v))\tilde{q}(f, v) \) by \( H(f, v) \). The third condition guarantees that for every \( f \in [\underline{f}, \bar{f}] \), the slice \( H(f, \cdot) \) is measurable in \( \{v : (f, v) \in \tilde{S}\} \), and together with the boundedness of \( H \) means that \( H(f, \cdot) \) is integrable in \( \{v : (f, v) \in \tilde{S}\} \). The second condition indicates that where \( \Pi' \) exists, which is almost everywhere, then for every \( f \in \tilde{S}(v) \), \( \Pi'(v) = H(f, v) \). Now take \((\hat{s}, s) \in \tilde{S}^2\). Then

\[
\Pi(\hat{s}|\hat{s}) = \Pi(\hat{v}) = \Pi(v) - \int_{\hat{v}}^{\tilde{v}} \Pi'(t)dt = \Pi(v) - \int_{\hat{v}}^{\tilde{v}} H(f, t)dt
\]

\[
\geq \Pi(v) - \int_{\hat{v}}^{\tilde{v}} H(f, v)dt = \Pi(v) - m(\hat{\theta}(s))\tilde{q}(s)(\hat{v} - v)
\]

\[
= \Pi(s|s)
\]

whether \( \hat{v} > v \) or \( \hat{v} < v \), so global incentive compatibility holds.

Proof of Proposition 3 The structure of the proof is as follows. I first find the Pareto optimal fully separating mechanism, subject to constraints which are implied by, but not necessarily equivalent to incentive compatibility. This Pareto optimal mechanism turns out satisfy Lemma 1, so it is incentive compatible. Finally, I show that buyers can’t profit from deviating from the equilibrium set of prices \( p \) and quantities \( q \), so the mechanism is decentralizable, and must therefore correspond to the fully separating equilibrium.

First, observe that because \( \Pi(v) \) is convex, the right and left derivatives of \( \Pi \) exist everywhere; I denote them by \( \Pi'_+ \) and \( \Pi'_- \), respectively. Then for any \( v \), the Pareto optimal incentive compatible mechanism is given by the mechanism which maximizes \( \Pi'_+ \) subject to constraints implied by incentive compatibility. I can frame the problem using optimal control as follows:

Given \( v \in V, \Pi(v) > 0, \) and \( \Pi'_-(v) \in \mathbb{R} \), choose \( \Pi'_+(v), \theta(f, v) \) and \( q(f, v) \) to solve
the following program:

$$\max_{\Pi'_+ \in \mathbb{R}, \theta \geq 0, q \in [0,1]} \Pi'_+$$

s.t.

$$-\Pi'_-(v) \geq m(\theta(f, v))q(f, v) \geq -\Pi'_+(v) \geq 0 \quad \forall f \in [\underline{f}(v), \bar{f}(v)] \quad (A.6)$$

$$\Pi(v) = m(\theta(f, v))q(f, v)(f - v) - k\theta(f, v) \quad \forall f \in [\underline{f}(v), \bar{f}(v)] \quad (A.7)$$

$$\Pi(\cdot) \text{ is continuous and convex} \quad (A.8)$$

where (A.6) and (A.8) follow from (A.5). Note that $$\Pi'_+(v)$$ is constant for all $$f \in [\underline{f}(v), \bar{f}(v)]$$, whereas $$\theta$$ and $$q$$ may vary across $$f$$. I first prove two useful lemmas which characterize the constraint set for $$f = \underline{f}(v)$$, and then show that this leads to a unique solution which satisfies the constraints for all $$f \in [\underline{f}(v), \bar{f}(v)]$$.

**Lemma 4.** Given $$\Pi > 0$$, and $$v \in V$$, let $$\psi(\Pi, v)$$ be the set of $$(\theta, q) \in \mathbb{R}_+ \times [0,1]$$ that attain $$\Pi$$ at $$(\underline{f}(v), v)$$:

$$\psi(\Pi, v) \equiv \{ (\theta, q) \in \mathbb{R}_+ \times [0,1] : \Pi = m(\theta)q(\underline{f}(v) - v) - k\theta \}.$$  

Let $$\theta(\Pi, v)$$ be the lowest $$\theta$$ that attains $$\Pi$$ when $$q = 1$$:

$$\theta(\Pi, v) \equiv \inf\{ \theta \geq 0 : \Pi = m(\theta)(\underline{f}(v) - v) - k\theta \}.$$  

Then

(i) $$\psi(\Pi, v)$$ is nonempty iff $$\Pi \leq \Pi_{Cl}(\underline{f}(v), v).$$

(ii) $$m(\theta)q$$ is minimized over $$\psi(\Pi, v)$$ by $$(\theta, q) = (\theta(\Pi, v), 1).$$
Proof. Part (i): The function \( m(\theta)q(f(v) - v) - k\theta \) is continuous, unbounded below, and attains its maximum at \( \Pi_{Cl}(f(v), v) \). Therefore, there exist \((\theta, q) \in \mathbb{R}_+ \times [0, 1]\) which set \( m(\theta)q(f(v) - v) - k\theta \) equal to \( \Pi \) iff \( \Pi \) is in the range of \( m(\theta)q(f(v) - v) - k\theta \), which is \((-\infty, \Pi_{Cl}(f(v), v)]\).

Part (ii): The program is to minimize \( m(\theta)q \) over \( \psi(\Pi, v) \). If I solve for \( q \) in the definition of \( \psi \), I may rewrite the program with \( q \) eliminated as follows:

\[
\min_{\theta \geq 0} \frac{\Pi + k\theta}{f(v) - v}
\]

s.t.

\[
\frac{\Pi + k\theta}{m(\theta)(f(v) - v)} \in [0, 1] \quad (A.9)
\]

The objective function is increasing in \( \theta \), and the left-hand side of \( (A.9) \) goes to \( \infty \) as \( \theta \to 0 \). Therefore, the program is solved when the left-hand side of \( (A.9) \) (i.e., \( q \)) is equal to 1, and \( \theta \) equals \( \bar{\theta}(\Pi, v) \).

\[\square\]

**Lemma 5.** For all \( v \in V \), \(-\Pi'_+(v)\) is bounded below by \( m(\bar{\theta}(\Pi, v)) \).

**Proof.** Suppose there exists a \( v \in V \) such that \(-\Pi'_+(v) < m(\bar{\theta}(\Pi, v)) \). Let \( \epsilon = m(\bar{\theta}(\Pi, v)) - (-\Pi'_+(v)) \). By the continuity of \( \Pi(\cdot, \cdot), \bar{\theta}(\cdot, \cdot), \) and \( m(\cdot) \), there exists a \( \delta > 0 \) such that for all \( \hat{v} \in (v, v + \delta) \), \( |m(\bar{\theta}(\Pi(v), v)) - m(\bar{\theta}(\Pi(\hat{v}), \hat{v}))| < \epsilon \). Also, by the convexity of \( \Pi \), I know that \(-\Pi'_+(v) \geq -\Pi'_-(\hat{v}) \). Combining these two facts with \( (A.6) \) at \( \hat{v} \) yields:

\[
m(\bar{\theta}(\Pi(\hat{v}), \hat{v})) \geq m(\bar{\theta}(\Pi(v), v)) + \epsilon = -\Pi'_+(v) \geq -\Pi'_-(\hat{v}) \geq m(\bar{\theta}(\hat{v}))q(\hat{v})
\]

By Part (ii) of the prior lemma, this is a contradiction.

\[\square\]
The lemma says that for any \( v \in V \), there is no IC mechanism in which the right-derivative of the profit function exceeds \(-m(\theta(\Pi(v), v))\). Furthermore, Part (i) of Lemma 4 implies that there is no IC mechanism in which the initial value \( \Pi(v) \) exceeds the complete information profit \( \Pi_{CI}(f(v, v)) \). Therefore, if there exists a mechanism that satisfies (A.6) - (A.8), with \( \Pi_1(v) = \Pi(\theta(\Pi(v), v)) \) at every \( v \in V \), and initial value \( \Pi(v) = \Pi_{CI}(f(v, v)) \), it must be Pareto optimal. I now construct such a mechanism and show it is unique.

If \( \Pi'_+(v) = -m(\theta(\Pi(v), v)) \) for all \( v \in V \), then by the continuity of \( m(\theta(\Pi(\cdot), \cdot)) \), \( \Pi'_+(v) \) must be continuous in \( V \). This, together with the continuity of \( \Pi \), implies that \( \Pi \) is differentiable everywhere, so for all \( v \in V \), \( \Pi'_+(v) = \Pi'_+(v) \), and (A.6) must hold with equality. Now for all \( f \in [f(v), \hat{f}(v)] \), equations (A.6) and (A.7) uniquely pin down \( \theta(f, v) \) and \( q(f, v) \) as follows:

\[
\theta(f, v) = \frac{1}{k} \left[ -\Pi(v) - \Pi'(v)(f - v) \right], \quad q(f, v) = \frac{-\Pi'(v)}{m(\theta(f, v))},
\]

where \(-\Pi'(v) = m(\theta(\Pi(v), v))\). Recalling the definition of \( \theta(\Pi(v), v) \), write

\[
\theta(\Pi(v), v) = \frac{1}{k} \left[ -\Pi(v) - \Pi(\Pi(v), v)(f(v) - v) \right],
\]

where \( \theta(\Pi(v), v) \) is the lowest fixed point of the right-hand side. Apply \( m(\cdot) \) to both sides and substitute \(-\Pi'(v) = m(\theta(\Pi(v), v))\) to get the expression in the proposition:

\[
-\Pi'(v) = m \left( \frac{1}{k} \left[ -\Pi(v) - \Pi'(v)(f(v) - v) \right] \right) \quad \text{with} \quad \Pi(\cdot) = \Pi_{CI}(f(v), v),
\]

where \( \Pi'(v) \) is the lowest fixed point of the right-hand side of the ODE. The solution \( \Pi(\cdot) \) to this initial value problem is unique, strictly convex, and of course continuous, so the proposed mechanism satisfies (A.8), and by construction it satisfies (A.6) and (A.7). It remains to be shown that \( \theta \) and \( q \) are within their proper bounds.
Substituting $-\Pi'(v) = m(\bar{\theta}(\Pi(v), v))$ in the expression for $\theta(f, v)$ and noting $f \geq \bar{f}(v)$, I have

$$\theta(f, v) \geq \frac{1}{k} \left[ - \Pi(v) - m(\bar{\theta}(\Pi(v), v))(\bar{f}(v) - v) \right] = \theta(\Pi(v), v) > 0,$$

where the last inequality is strict as long as $\Pi > 0$, which is true everywhere in the unique solution to the ODE. This expression implies

$$q(f, v) = \frac{-\Pi'(v)}{m(\bar{\theta}(f, v))} = \frac{m(\bar{\theta}(\Pi(v), v))}{m(\theta(f, v))} \in [0, 1],$$

so both $\theta(f, v)$ and $q(f, v)$ are in their feasible sets, and I have found the solution to the optimal control problem. Note that although the constraints of the optimal control problem were implied by but not necessarily equivalent to incentive compatibility, it can be easily shown that this solution satisfies the assumptions in Lemma 1, so it is the Pareto optimal incentive compatible fully separating mechanism.

The form for $p^*$ and $q^*$ in the Proposition follow directly from the form of the Pareto optimal mechanism characterized above. Because the mechanism is IC, it is clear that the strategies $p^*$ and $q^*$ are optimal to the seller, and by construction the mechanism satisfies the buyer’s zero-profit condition. It remains to check that buyers cannot profit from offering a $(p, q) \notin M^*$.

Observe that because $p^*(f, v)$ is independent of $f$, continuous, and strictly increasing in $v$, the set of prices posted in equilibrium is the closed interval $[\underline{p}, \bar{p}]$, where I let $\underline{p} \equiv p^*(v)$ and $\bar{p} \equiv p^*(\bar{v})$. Next, recall that in the Pareto optimal mechanism, $q(\bar{f}(v), v) = 1$ for all $v \in V$, and observe that $q^*(f, v)$ is continuous and strictly decreasing in $f$. Therefore, for any $p \in [\underline{p}, \bar{p}]$, the set of quantities $q$ posted is the closed interval $[q(p), 1]$, where $q(p) \equiv q^*(\bar{f}(v), v)$ and $p = p^*(v)$. Because $M^*$ takes this form, it suffices to show that $p > \bar{p}$ and $p < \underline{p}$ are not profitable deviations, and that if $p$ is an element of $[\underline{p}, \bar{p}]$, then $q < q(p)$ is not a profitable deviation.
Suppose that a coalition of buyers consider posting a \((p, q)\) pair where \(p \in [\bar{p}, \tilde{p}]\), but \(q < \tilde{q}(p)\). First define \(\theta((p, q), s) \equiv \inf\{\tilde{\theta} > 0 : m(\tilde{\theta})q(p - v) \geq \Pi(v)\}\), which is the lowest acceptable market tightness for type \(s\) in equilibrium. Recall that when buyers post off-equilibrium \((p, q)\) pairs, they expect the type that will accept the lowest probability of trade. Because \(\theta((p, q), s)\) depends strictly on \(v\), buyers’ off-equilibrium beliefs are a distribution over \(T(p, q) = \arg\inf_{s \in S} \theta((p, q), s)\). Noting that \(m(\theta(p, q, v))q(p - v) = \Pi(v)\), differentiating by \(v\) and substituting the expression for \(p^*(v)\) shows that \(\theta(p, q, v)\) is proportional to \((p^*(v) - v)/(p - v) - 1\), so \(\theta(p, q, v)\) is clearly minimized over \(V\) by \(p^{* -1}(p)\), the unique type \(v\) which selects \(p\) in equilibrium. So buyers off-equilibrium beliefs about asset quality \(f\) are a distribution over \(\tilde{S}(v)\).

Also recall that for off-equilibrium \((p, q)\), \(\theta(p, q) \equiv \inf_{s \in S} \theta((p, q, s))\). So if \(v\) is the minimizing value, then \(m(\theta(p, q))q(p - v) = \Pi(v) = m(\theta(f, v))q(f, v)(p^*(v) - v)\), where \(v\) is the type which chooses \(p^*(v) = p\) in equilibrium (and therefore \((p - v)\) and \((p^*(v) - v)\) cancel), and \(f \in \tilde{S}(v)\). This gives \(m(\theta(p, q)) = m(\theta(f, v))q(f, v)/q > m(\theta(f, v))\), because \(q < \tilde{q}(p) \leq q(f, v)\). Therefore, \(\theta(p, q) > \theta(f, v)\), and because \(n(\theta) = m(\theta)/\theta\) is strictly decreasing, for any \(f \in \tilde{S}(v)\), I have \(k = n(\theta(f, v))q(f, v)(f - p) = n(\theta(\tilde{f}(v), v))q(\tilde{f}(v), v)(\tilde{f}(v) - p) > n(\theta(p, q))q(E[\tilde{f}[p, q] - p])\), where I have used the fact that the buyer’s expectation over \(\tilde{S}(v)\) cannot exceed \(\max\{\tilde{S}(v)\} = \tilde{f}(v)\).

So regardless of the buyer’s off-equilibrium belief distribution over asset qualities \(f \in \tilde{S}(v), (p, q) \in [\bar{p}, \tilde{p}] \times [0, q(p)]\) is not a profitable deviation.

Next, suppose that a coalition posts \(p > \tilde{p}\), with any \(q \in [0, 1]\). As explained above, \(\theta_3(p, q, v)\) has the same sign as \((p^*(v) - v)/(p - v) - 1\), so \(\theta(p, q, v)\) must be minimized by \(v = \tilde{v}\). Note that \(\tilde{S}(v) = \{\tilde{f}\}\), the highest possible asset quality in \(\tilde{S}\), not just \(\tilde{S}(v)\), so buyers expect type \((\tilde{f}, \tilde{v})\). Next, observe that \(\theta(p, 1)\) satisfies \(m(\theta(p, 1))(p - \tilde{v}) = \Pi(\tilde{v}) = m(\theta(\tilde{f}, \tilde{v}))(p^*(\tilde{v}) - \tilde{v})\), where I have used \(q(\tilde{f}, \tilde{v}) = 1\); so then, \(\theta(p, 1) < \theta(\tilde{f}, \tilde{v})\). Because \(\theta(\tilde{f}, \tilde{v})\) is less than the complete information \(\theta\), and
θ(p, 1) is even lower, I must have $m(\theta(\tilde{f}, \tilde{v}))(\tilde{f} - \tilde{v}) - k\theta(\tilde{f}, \tilde{v}) > m(\theta(p, 1))(\tilde{f} - \tilde{v}) - k\theta(p, 1)$ so the zero-profit condition of buyers who trade with $(\tilde{f}, \tilde{v})$ in equilibrium gives

$$0 = m(\theta(\tilde{f}, \tilde{v}))(\tilde{f} - p^*(\tilde{v})) - k\theta(\tilde{f}, \tilde{v}) = m(\theta(\tilde{f}, \tilde{v}))(\tilde{f} - \Pi(\tilde{v})/m(\theta(\tilde{f}, \tilde{v})) - \tilde{v}) - k\theta(\tilde{f}, \tilde{v})$$

$$= -\Pi(\tilde{v}) + m(\theta(\tilde{f}, \tilde{v}))(\tilde{f} - \tilde{v}) - k\theta(\tilde{f}, \tilde{v}) > -\Pi(\tilde{v}) + m(\theta(p, 1))(\tilde{f} - \tilde{v}) - k\theta(p, 1)$$

$$= -m(\theta(p, 1))(p - \tilde{v}) + m(\theta(p, 1))(\tilde{f} - \tilde{v}) - k\theta(p, 1) = m(\theta(p, 1))(\tilde{f} - p) - k\theta(p, 1).$$

Dividing both sides by $\theta(p, 1)$ gives $0 > n(\theta(p, 1))(\tilde{f} - p) - k$. Next, note that for $q \in [0, 1]$, $m(\theta(p, q))q(p - \tilde{v}) = m(\theta(p, 1))(p - \tilde{v})$, so $\theta(p, q) \geq \theta(p, 1)$, and I have $0 > n(\theta(p, q))q(\tilde{f} - p) - k$, so $(p, q) \in (\bar{p}, \infty) \times [0, 1]$ is not a profitable deviation.

Suppose that a coalition posts $p < \bar{p}$, with $q \in [0, 1]$. Because $\theta_3(p, q, v)$ has the same sign as $(p^*(v) - v)/(p - v) - 1$, if $(p, q)$ attracts any type, it must be $\bar{v}$ and therefore $(\tilde{f}, \bar{v})$. If so, then $q(p - \bar{v}) > m(\theta(p, q))q(p - \bar{v}) = \Pi(\bar{v}) = m(\theta(\tilde{f}, \tilde{v})) \cdot 1 \cdot (p^*(\bar{v}) - \bar{v})$. Therefore, $q > m(\theta(\tilde{f}, \tilde{v}))(p^*(\bar{v}) - \bar{v})/(p - \bar{v}) > m(\theta(\tilde{f}, \tilde{v}))$. Also require that $p - \bar{v} > m(\theta(\tilde{f}, \tilde{v}))(p^*(\bar{v}) - \bar{v})$. Otherwise, no types are attracted, $\theta(p, q) = \infty$, and the buyer’s profit is $-k$. With these restrictions on $q$ and $p$, recall that $(\tilde{f}, \bar{v})$ receives the complete information allocation, so $m(\theta(\tilde{f}, \bar{v})(\tilde{f} - \bar{v}) - k\theta(\tilde{f}, \bar{v}) \geq m(\theta(p, q))q(\tilde{f} - \bar{v}) - k\theta(p, q)$. So the zero-profit condition of buyers who trade with $(\tilde{f}, \bar{v})$ in equilibrium gives

$$0 = m(\theta(\tilde{f}, \bar{v}))(\tilde{f} - p^*(\bar{v})) - k\theta(\tilde{f}, \bar{v}) = m(\theta(\tilde{f}, \bar{v}))(\tilde{f} - \Pi(\bar{v})/m(\theta(\tilde{f}, \bar{v})) - \bar{v}) - k\theta(\tilde{f}, \bar{v})$$

$$= -\Pi(\bar{v}) + m(\theta(\tilde{f}, \bar{v}))(\tilde{f} - \bar{v}) - k\theta(\tilde{f}, \bar{v}) \geq -\Pi(\bar{v}) + m(\theta(p, q))q(\tilde{f} - \bar{v}) - k\theta(p, q)$$

$$= -m(\theta(p, q))(p - \bar{v}) + m(\theta(p, q))q(\tilde{f} - \bar{v}) - k\theta(p, q) = m(\theta(p, q))q(\tilde{f} - p) - k\theta(p, q).$$

Dividing both sides by $\theta(p, q)$ gives $0 \geq n(\theta(p, q))q(\tilde{f} - p) - k$, so $(p, q) \in [0, \bar{p}] \times [0, 1]$ is not a profitable deviation, and the proof is complete. □

**Proof of Corollary 1** These follow directly from differentiating the expressions for $\tilde{\theta}$ and $\tilde{q}$ in Proposition 3. □
Proof of Corollary 2 This is stated in the form of the equilibrium.

Proof of Corollary 3 Part (i): Follows from Corollary 2. Part (ii): Recall from the proof of Proposition 3 that $-\Pi'(v) = m(\theta(f_v, v))q(f_v, v) = m(\theta(f, v))$, where the second equality follows from Corollary 2. Differentiate both sides with respect to $v$ to obtain

$$-\Pi''(v) = m'(\theta(f_v, v))\theta_v(f_v, v)$$

$$= -m'(\theta(f_v, v))\Pi''(v)(f_v - v)/k,$$  \hfill (A.10)

where the second equality follows from differentiating the expression for $\theta(f_v, v)$ in Proposition 3. The strict convexity of $\Pi$ implies that $\Pi''(v) > 0$ and may be cancelled, yielding $m'(\theta(f_v, v))(f_v - v) = k$, which characterizes the complete information liquidity $\theta(f_v, v)$. \hfill \square

Proof of Corollary 4 Denote the liquidity and fraction sold over domain $\tilde{S}$ by $\tilde{\theta}$ and $\tilde{q}$, and over domain $S$ by $\theta$ and $q$. Then $\theta(f, \delta) = \tilde{\theta}(f, \delta f)$ and $q(f, \delta) = \tilde{q}(f, \delta f)$. This gives

$$\theta_1 = \bar{\theta}_1 + \bar{\theta}_2 \delta$$

$$= -\frac{1}{k}[\Pi'(v) + \delta \Pi''(v)(f_v - v)].$$

Also note that differentiating (1.10) with respect to $v$ gives

$$[m'(\bar{\theta}(v))(f_v - v) - k] \Pi''(v) = -\Pi'(v)m'(\bar{\theta}(v))f'(v),$$  \hfill (A.12)

where I denote $\bar{\theta}(v) \equiv \tilde{\theta}(f_v, v)$ as the liquidity of the lowest quality asset $f$ for a given private valuation $v$, and therefore the argument of $m'$ in (A.12) and of $m$ in (1.10) is $\bar{\theta}(v)$. Use (A.12) to solve for $\Pi''(v)$, note that in $\tilde{S}$, $f_v(v) = v/\delta$, and rearrange the above equation to get

$$\theta_1 = \frac{-\Pi'(v)}{k[m'(\bar{\theta}(v))(1 - \delta)v/\delta - k]}[-(\delta - \delta)m'(\bar{\theta}(v))v/\delta - k] < 0.$$
And finally,

$$\theta_2 = \tilde{\theta}_2 f = \frac{-\Pi''(v)(f - v)f}{k} < 0.$$ 

\[\square\]

**Proof of Corollary 5** First consider \(\tilde{q}_2(f, v)\) in the region \(v \in V\). Differentiate the expression for \(\tilde{q}\) in Proposition 3 with respect to \(v\):

$$\tilde{q}_2(f, v) = \frac{\Pi''(v)}{m(\theta)k} \left( \frac{-\Pi'(v)}{m(\theta)} m'(\tilde{\theta})(f - v) - k \right)$$  \hspace{1cm} \text{(A.13)}

This has the same sign as the expression in parentheses. Note that \(-\Pi'(v) = m(\theta(v))\), and observe that

$$-\Pi'(v)(f - v) - k\tilde{\theta}(f, v) = \Pi(v) = -\Pi'(v)(f - v) - k\tilde{\theta}(v),$$

which implies that \(f = f + (\tilde{\theta}(f, v) - \tilde{\theta}(v))k/m(\theta(v))\), so \(\tilde{q}(f, v)\) has the same sign as

$$\frac{-\Pi'(v)}{m(\theta)} m'(\tilde{\theta})(f - v) - k = \frac{m(\theta(v))}{m(\tilde{\theta})} \left[ \frac{k}{m(\tilde{\theta})} (\tilde{\theta} - \theta) + f - v \right] - k$$

Dividing by \(k\) gives

$$\frac{m(\theta(v))}{m(\tilde{\theta})} m'(\tilde{\theta}) \left[ \frac{\tilde{\theta} - \theta}{m(\tilde{\theta})} + \frac{f - v}{k} \right] - 1 = \frac{m'(\tilde{\theta})}{m(\tilde{\theta})} (\tilde{\theta} - \theta) + \frac{m(\theta)m'(\tilde{\theta})}{m(\tilde{\theta})} - 1$$

$$< \frac{m(\tilde{\theta}) - m(\theta)}{m(\tilde{\theta})} + \frac{m(\theta)m'(\tilde{\theta})}{m(\tilde{\theta})} - 1 = \frac{m(\theta)}{m(\tilde{\theta})} \left[ \frac{m'(\tilde{\theta})}{m(\tilde{\theta})} - 1 \right] < 0,$$

so \(\tilde{q}_2 < 0\) when \(v \in V\). Therefore \(q_2(f, \delta) = \tilde{q}_2(f, \delta f) f < 0\) in \(S\).

Also, observe that

$$\tilde{q}_1(f, v) = -\frac{1}{k} \left( \frac{\Pi'(v)}{m(\theta)} \right)^2 < 0,$$

so \(q_1(f, \delta) = \tilde{q}_1(f, v) + \tilde{q}_2(f, v)\delta < 0\).
Finally, \( \bar{\theta}_2(f, v) = -\Pi''(v)(f - v)/k < 0 \), so \( \bar{\theta}_2(f, \delta) = \bar{\theta}_2(f, v) = 0 \).

**Proof of Theorem 3.** Consider \( \theta_1(f, \delta) \), which is the market tightness partial at a point on the left border of \( S \). Recall that \( k\theta(f, \delta) = -\Pi(\delta f) - \Pi'(\delta f)(f - \delta f) \). Differentiating with respect to \( f \) gives

\[
k\theta_1(f, \delta) = -\Pi'(\delta f) - \Pi''(\delta f)(1 - \delta)\delta f
\]

(A.14)

For a given \( v \), denote the liquidity associated with the lowest type \( f \) as \( \bar{v}(v) = \bar{\theta}(v) \). Then I can compute \( \Pi''(v) \) by noting that \( \Pi'(v) = -m(\theta(v)) \). Recalling that in \( S \), \( m(\bar{\theta}(v))(\bar{f} - v) = k \), I have \( \Pi'(v) = -m'(\bar{\theta}(v))\theta'(v) = -\frac{m'(\bar{\theta}(v))^2}{m''(\bar{\theta}(v))(\bar{f} - v)} \). Returning to (A.14), I have

\[
k\theta_1(f, \delta) = m(\theta(v)) + \delta \frac{m'(\theta(v))^2}{m''(\theta(v))}.
\]

Dividing both sides by \( m(\theta(v)) \) and incorporating the assumed form of \( m(\theta) = (1 + \theta^{-r})^{-1/r} \), I have that \( \theta_1(f, \delta) > 0 \) if and only if \( (1 + r)\theta(v)^{-r} > k \). Use \( m(\bar{\theta}(v))(\bar{f} - v) = k \) to solve for \( \bar{\theta}(v) \), plug in the previous inequality, and rearrange to get that \( \theta_1(f, \delta) > 0 \) if and only if

\[
\left( \frac{f(1 - \delta)}{k} \right)^{-\frac{1}{r+1}} > \frac{\delta}{1 + r} + 1.
\]

(A.15)

Assumption 1 guarantees that \( f(1 - \delta)/k > 1 \), so as \( r \to \infty \), the limit of the left hand side of (A.15) is strictly greater than the limit of the righthand side, and therefore \( \theta_1(f, \delta) > 0 \) for high enough \( r \). \( \Pi''(v) \) is continuous in \( V \), and so (A.14) indicates that \( \theta_1(f, \delta) \) is continuous on \( S \), so there must exist a neighborhood around \( (\bar{f}, \delta) \) for which \( \theta_1(f, \delta) > 0 \).

Now consider (A.15) for the case \( \delta = \bar{\delta} \). As \( r \to 0 \), the left hand side of (A.15) converges to 1, and the right hand side converges to \( \bar{\delta} + 1 > 1 \), so (A.15) is violated for low enough \( r \). Due to the monotonicity of both sides, this implies that (A.15) is
violated for all $\delta \in [\delta, \delta]$. Therefore, for low enough $r$, $\theta_1(f, \delta) < 0$ for all $\delta \in [\delta, \delta]$. Now for any $(f, \delta) \in \mathcal{S}$, let $v = \delta f$ and write

$$k\theta_1(f, \delta) = -\Pi'(v) - \Pi''(v)(1 - v/f)v$$

$$\leq -\Pi'(v) - \Pi''(v)(1 - v/f)v$$

$$= k\theta_1(f, \delta f/f)$$

$$< 0,$$

and the theorem is proved. $\square$

**Proof of Proposition 4** The first part of the proof is nearly identical to the proof of Proposition 3, but with $(f - v)$ in (A.7) replaced with $(E[\tilde{f}|\tilde{f} = v] - v)$. Making this substitution, it is easy to show that the equilibrium identified in Proposition 4 corresponds to the Pareto optimal partial pooling mechanism.

To show that buyers are not motivated to deviate from the equilibrium set $M^*$ of prices $p$ and quantities $q$, first note that this set takes the form $M^* = [p, \bar{p}] \times 1 \subset \mathbb{R}^2_+$. The proof that $(p, q)$ with $p \notin [p, \bar{p}]$ is not a profitable deviation is the same as in the proof of Proposition 3, so now consider a deviation $(p, q)$, with $p \in [p, \bar{p}]$ and $q < 1$. As discussed in the proof of Proposition 3, the buyer’s beliefs must be distributed over the types $(f, v) \in \tilde{S}$ where $v$ is the unique $v$ for which $p^*(v) = p$. The literature do not restrict beliefs beyond that, so suppose that (in accordance with Guerrieri and Shimer (2013)), the buyer’s belief corresponds to the actual distribution of sellers with private value $v$, so his expected asset quality is $E[\tilde{f}|\tilde{f} = v]$. Recall as in the proof of Proposition 3 that $m(\theta(p, q))q(p - v) = m(\theta(v))(p - v)$, so $\theta(p, q) > \theta(v)$. If so, then $k = n(\theta(v))(E[\tilde{f}|\tilde{f} = v] - p) > n(\theta(p, q))q(E[\tilde{f}|\tilde{f} = v] - p)$, so $(p, q)$ is not a profitable deviation, and the Proposition is proved. $\square$

**Proof of Proposition 5** Recall that under full separation $\Sigma$, the profit of any seller who privately values his asset as $v = \delta f$ may be characterized by the following ODE:
\[ \Pi'(v) = -m \left( -\frac{1}{k} \left[ \Pi(v) + \Pi'(v)(\tilde{f}(v) - v) \right] \right), \quad \Pi(v) = \Pi_{CI}(\tilde{f}(v), v), \quad (A.16) \]

where \( \tilde{f}(v) \) is the lowest quality asset among sellers with common private value \( v \), and \( \Pi_{CI}(f, v) \) is the complete information profit of seller \( (f, v) \). The optimal control argument which generates the ODE may be similarly applied to the case of partial pooling \( \Phi \), yielding the following characterization:

\[ \Pi'(v) = -m \left( -\frac{1}{k} \left[ \Pi(v) + \Pi'(v)(E[\tilde{f}\tilde{\delta} = v] - v) \right] \right), \quad \Pi(v) = \Pi_{CI}(\tilde{f}(v), v), \quad (A.17) \]

Clearly, both equilibria have the same initial condition \( \Pi(v) = \Pi_{CI}(\tilde{f}(v), v) \), and the only difference between (A.16) and (A.17) is that \( E[\tilde{f}\tilde{\delta} = v] \) has been substituted for \( \tilde{f}(v) \). As long as some sellers of value \( v \) have assets better than \( \tilde{f}(v) \), it must be that \( E[\tilde{f}\tilde{\delta} = v] > \tilde{f}(v) \).

Now suppose that the two equilibria have the same profit \( \Pi(v) \) for some \( v \), and consider how \( \Pi'(v) \) differs under the two equilibria. Let \( g \) be a placeholder for either expression \( E[\tilde{f}\tilde{\delta} = v] \) or \( \tilde{f}(v) \), and let \( \alpha \) be a placeholder for \( \Pi'(v) \) and write

\[ \alpha = -m \left( -\frac{1}{k} \left[ \Pi(v) + \alpha(g - v) \right] \right). \quad (A.18) \]

Now fix \( v \) and \( \Pi(v) \), and consider how \( \alpha \) changes as \( g \) increases from \( \tilde{f}(v) \) to \( E[\tilde{f}\tilde{\delta} = v] \). Differentiate both sides of (A.18) with respect to \( g \), and solve for \( \alpha'(g) \) to obtain

\[ \alpha'(g) = \frac{m'(\ldots)(-\alpha)}{m'(\ldots)(g - v) - k} > 0, \quad (A.19) \]

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where the inequality is due to the fact that $\alpha = \Pi'(v)$ is negative and the denominator $m'(\ldots)(g - v) - k$ is positive. Therefore, $\Pi'(v)$ is higher under partial pooling where $g = E[\tilde{f}|\tilde{f} \delta = v] > f(v)$ than under full separation where $g = f(v)$, so wherever $\Pi_\Phi(v)$ crosses $\Pi_\Sigma(v)$, it must be that $\Pi'_\Phi(v) > \Pi'_\Sigma(v)$.

Finally, suppose there exists a $v$ at which the profit function $\Pi_\Phi(v)$ under pooling is less than or equal to that under full separation $\Pi_\Sigma(v)$. Then because the two equilibrium profit functions are equal at the initial condition $\Pi(p,v) = \Pi_{CI}(f,v),v$, so wherever $\Pi_\Phi(v)$ crosses $\Pi_\Sigma(v)$, it must be that $\Pi'_\Phi(v) > \Pi'_\Sigma(v)$. Therefore, for all $v > \_\_\$, $\Pi_\Sigma(v) < \Pi_\Phi(v)$, and Part (i) is proved.

**Proof of Proposition 6** The proof of the fully separating equilibrium shows that $\Sigma$ is robust to any belief satisfying (R), so therefore $\Gamma(\Sigma) \equiv \Gamma_0$. By definition, any belief in $\Gamma(\Sigma)$ must satisfy (R), so $\Gamma(\Sigma) \subseteq \Gamma_0$. Combining these two relations, $\Gamma(\Sigma) = \Gamma_0$.

By definition, $\Gamma(\Phi) \subseteq \Gamma_0 = \Gamma(\Sigma)$, so what remains to be shown is that $\Gamma(\Phi)$ is strictly less than $\Gamma(\Sigma)$. To do this, let $\hat{T}(p,q) \equiv \{(f(v),v) : (f,v) \in T(p,q)\}$. I propose that any off-equilibrium beliefs with support restricted to $\hat{T}(p,q)$ support full separation $\Sigma$ but break partial pooling $\Phi$. The proof of Proposition 3 shows that if $p$ is in the set of equilibrium prices, then $T(p,q) = \{(f,v) \in \tilde{S} : p^*(v) = p\}$. This indicates that as defined, $\hat{T}(p,q) \subset T(p,q)$. Therefore, beliefs which are restricted to have support no larger than $\hat{T}(p,q)$ satisfy restriction (R) and are therefore in $\Gamma(\Sigma)$. However, such beliefs are not in $\Gamma(\Phi)$. To see this, note that under partial pooling $\Phi$, (A.17) indicates that all sellers choose $q = 1$, so $q < 1$ is off equilibrium. Suppose that buyers post $(p,q)$, where $p$ is in the equilibrium set of prices, but $q < 1$ and therefore off-equilibrium. Then $m(\theta(p,q))q(p-v) = \Pi(v) = m(\theta(p,1))(p-v)$, where $p^*(v) = p$. If so, then off-equilibrium $\theta(p,q)$ satisfies $m(\theta(p,q)) = m(\theta(p,1))/q$, and is therefore continuous in $q$. So letting $\epsilon = \tilde{f} - E[\tilde{f}|v]$, let $\epsilon' = n(\theta(p,1))\epsilon/(\tilde{f}(v)-p)$. Then by the
continuity of $\theta(p, q)$ in $q$, there exists a $q < 1$ such that $n(\theta(p, q))q > n(\theta(p, 1)) - \epsilon'$. If so, then $n(\theta(p, q))q[\bar{f}(v) - p] > [n(\theta(p, 1)) - \epsilon'][\bar{f}(v) - p] = n(\theta(p, 1))[\epsilon + E[\bar{f}]v - p] - \epsilon'[\bar{f}(v) - p] = n(\theta(p, 1))[E[\bar{f}]v - p] + n(\theta(p, 1))\epsilon - \epsilon'(\bar{f}(v) - p) = k$, so $(p, q)$ is a profitable deviation, and the beliefs do not support $\Phi$ and are therefore not in $\Gamma(\Phi)$.

Proof of Proposition 7 Using a strategy identical to the proof of Proposition 10 in DeMarzo and Duffie (1999), I can show that given any increasing security $G$ with $g = E[G(X)|z]$, then if $F(X) = \min[X, d]$ is a standard debt contract with $f = g$, then for all $z$, $g = E[G(X)|z] \geq E[F(X)|z] = f$. Since $\Pi(\delta(\cdot), \underline{f})$ is decreasing for any $\delta \in [\underline{\delta}, \bar{\delta}]$, I have $\Pi(\delta g, \underline{g}) = \Pi(\delta g, \underline{f}) \leq \Pi(\delta f, \underline{f})$. Because this inequality holds for any $\delta$ and any $z$, take expectations to get $V(F) \geq V(G)$. So standard debt is an optimal security.
Appendix B

Proofs for Chapter 2

Proof of Proposition 8 The piecewise linearity of the utility function, while improving tractability of the date 0 portfolio problem, makes the date 1 consumption problem more complicated than in Allen and Gale (1998). Throughout the proof, I abbreviate \( c_{11} \) by \( c_1 \). I also assume that \( r \leq L \), because otherwise costly liquidation occurs for all \( p \in [0, 1] \), which is clearly not optimal.

First note that for any \( (L, I, r, p), l = 1 \) weakly dominates \( l = 0 \). To see this, observe that for any \( \{c_1, c_{21}, \alpha, c_{22}, i, l = 0\} \) satisfying constraints (2.1) through (2.7), by (2.4) I must have \( c_1 = r \). So then if \( l \) is changed so \( l = 1 \), the only changed condition is (2.1), which is clearly satisfied. Therefore, any consumption allocation attainable by \( l = 0 \) is attainable by \( l = 1 \), so \( l = 1 \) is weakly optimal.

I next prove two lemmas which will be useful in completing the proof of the proposition.

Lemma 6. If \( l = 1 \) and \( i = 0 \), then

1. \( c_1 = r \)

2. \( L + pGI - r \geq u(r) \)
3. $\alpha = 0$

4. $E[c_{22}] = L + pGI - r$.

5. Aggregate utility is $u(r) + u(L + pGI - r)$.

**Proof:** By (2.4), $c_1 = r$ so item 1 holds. If item 2 is strict, then $\alpha > 0$ requires 
(by (2.5)) $c_{21} = r$ and (by (2.7)) $u(E[c_{22}]) \leq u(r)$, which is clearly improved on by 
$\alpha = 0$, $u(E[c_{22}]) = u(L + pGI - r) > u(r)$. If item 2 holds with equality, then by 
(2.2) raising $\alpha$ will weakly decrease $E[c_{22}]$, so $\alpha = 0$ is optimal. This shows that 
item 2 implies items 3 through 5.

Now suppose item 2 is violated. Then by (2.2) raising $\alpha$ will strictly decrease 
$E[c_{22}]$, so for all $\alpha \in [0,1]$, $u(E[c_{22}]) \leq u(L + pGI - r) < u(r)$, and so by (2.6), 
$\alpha = 1$. This gives, by (2.5), $c_{21} = r$, and (2.1) implies $2r \leq L$. But this contradicts 
my assumption that item 2 is violated. Therefore, item 2 must hold, and so must 
items 3 through 5.

Define $W \equiv L + \theta pGI$ to be the total wealth if all assets are liquidated.

**Lemma 7.** If $l = 1$ and $i = 1$, then

1. If $W - r \geq u(r)$, then aggregate utility is $u(r) + u(W - r)$, and $\alpha = 0$, $c_1 = r$, 
   and $E[c_{22}] = W - r$ is an optimal allocation.

2. If $W - r < u(r)$, then aggregate utility is $2u(W/2)$, $\alpha = 1$, and $c_1 = c_{21} = W/2 < r$.

**Proof:** First note that if $c_1 < r$, then $\alpha = 1$. To see this, observe that (2.4) and 
(2.2) imply $(1 - \alpha)E[c_{22}] = 0$. If $\alpha < 1$, then by (2.6) I have $u(c_1) < u(E[c_{22}])$ and 
$E[c_{22}] = 0$, which is a contradiction.

(Part 1) Suppose $r \geq k$. Then letting $c_1 = r$, $\alpha = 0$, and $E[c_{22}] = W - c_1 = W - r \geq u(r)$ implies $u(E[c_{22}]) = u(W - r) \geq u(r)$, so this is a feasible allocation.
Also, the aggregate utility under this allocation is \( u(r) + u(W - r) = 2k \), which is the unconstrained maximum utility, so this allocation must be an optimal one.

Suppose \( r < k \). If \( c_1 < r \), then as noted above, \( \alpha = 1 \), and (2.4) implies that \( r > c_1 = c_{21} = W/2 \). This contradicts the assumption of Part 1, so \( c_1 = r \). Given this, if \( W - r > r \), then \( \alpha > 0 \) requires by (2.5) \( c_{21} = r \) and by (2.7) \( u(E[c_{22}]) \leq u(r) \), which is clearly improved on by \( \alpha = 0 \), \( u(E[c_{22}]) = u(W - r) > u(r) \). And if \( W - r = r \), then raising \( \alpha \) will leave \( E[c_{22}] \) unchanged, so \( \alpha = 0 \) is an optimal value.

This gives \( c_1 = r, \alpha = 0 \), and \( E[c_{22}] = W - r \) as an optimal allocation, with aggregate utility of \( u(r) + u(W - r) \).

(\text{Part 2}) If \( c_1 = r \), then by (2.2) raising \( \alpha \) will strictly decrease \( E[c_{22}] \), so for all \( \alpha \in [0, 1] \), \( u(E[c_{22}]) \leq u(W - r) < u(r) \), and so by (2.6), \( \alpha = 1 \). This gives, by (2.5), \( c_{21} = c_1 \) and by (2.1), \( 2c_1 = 2c_{21} \leq W < u(r) + r \leq 2r \), which is a contradiction. So \( c_1 < r, \alpha = 1, c_1 = c_{21} \) and (2.4) gives \( c_1 = W/2 \). \( \square \)

Armed with Lemmas 6 and 2, I am ready to prove Proposition 8. I have already shown above that \( l = 1 \) weakly dominates \( l = 0 \).

\text{Part (i):} If \( i = 0 \), then Lemma 6 indicates that aggregate utility is \( u(r) + u(L - r + rG) \). If \( i = 1 \), then Lemma 2 indicates that if \( W - r \geq u(r) \), then aggregate utility is \( u(r) + u(W - r) \leq u(r) + u(L - r + pG) \). If \( W - r < u(r) \), then \( W < r + u(r) \leq 2r \), and by Lemma 2 aggregate utility is \( 2u(W/2) \leq 2u(r) \leq u(r) + u(L - r + pG) \). So clearly \( i = 0 \) is optimal, and by Lemma 6, I have \( c_1 = r, \alpha = 0, E[c_{22}] = L - r + pG \).

\text{Part (ii):} If \( p < p^* \), then \( L - r + pG < u(r) \), so by Part 2 of Lemma 6 I must have \( i = 1 \) is optimal. Also, this implies \( W - r = L + \theta pG - r < L + pG - r < u(r) \), so by Lemma 2 I have \( \alpha = 1, c_1 = c_{22} = W/2 \), and the proposition is proved. \( \square \)

\textbf{Proof of Proposition 9} \text{Part 1:} Denote the left limit of \( U(p) \) by \( U_-(p) \), and note that the jump discontinuity at \( p^* \) implies \( U_-(p^*) < U(p^*) \). If \( p_g = p^* \) and \( p_b = 0 \), then the constraints of the disclosure problem imply \( d(p_g) = p/p^* \) and \( d(p_b) = 1 - p/p^* \), so the objective function is \( (1 - p/p^*)U(0) + (p/p^*)U(p^*) > (1 - \)
the disclosure objective function is strictly decreasing in $p$ with strict inequality if $p = p^*$ and without loss of generality set $p = p^*$, then the first term is zero, so (B.1) is negative. If $p_g < p^*$, then by the linearity of $U$ over $[0, p^*)$, the objective function takes value equal to $U(p)$, which I have shown is not optimal. So the optimal test must have $p_g \geq p^*$ and $p_b \leq p < p^*$. The constraints imply that $d(p_b) = (p_g - p)/(p_g - p_b)$ and $d(p_g) = (p - p_b)/(p_g - p_b)$. Substituting into the objective function and differentiating with respect to $p_g$ gives

$$FOC(p_g) : \frac{U'(p_g)(p_g - p_b) - (U(p_g) - U(p_b))}{(p_g - p_b)^2},$$

which has the same sign as $U'(p_g) - (U(p_g) - U(p_b))/(p_g - p_b)$, or

$$u'(L - r + p_g GI) GI - \frac{u(r) + u(L - r + p_g GI) - 2u((L + \theta p_b GI)/2)}{(p_g - p_b)}. \quad (B.1)$$

Because $U(\cdot)$ is strictly increasing below $p^*$ and weakly increasing thereafter, and $p_b < p < p^* \leq p_g$, the second term must be positive. If $p_g > p^{**}$, then the first term is zero, and (B.1) is negative. Now suppose $p_g \in [p^*, p^{**}]$. If $r \geq k$, then the first term is zero, so (B.1) is negative. If $r < k$, then (B.1) reduces to $GI - GI(p_g - \theta p_b)/(p_g - p_b) \leq 0$, with strict inequality if $p_b > 0$ and equality otherwise. So for any $p_b < p$ and $p_g \geq p^*$, the disclosure objective function is strictly decreasing in $p_g$ for $p_g > p^{**}$ and strictly decreasing in $p_g$ for $p_g \in [p^*, p^{**}]$ if $p_b$ is nonzero. So for any $p_b < p$, an optimal $p_g$ must satisfy $p_g \in [p^*, p^{**}]$.

Now differentiate the objective with respect to $p_b$ to get

$$FOC(p_b) : \frac{U'(p_b)(p_g - p_b) - (U(p_g) - U(p_b))}{(p_g - p_b)^2},$$

and without loss of generality set $p_g = p^*$, which I have shown is optimal for any $p_b < p$. The $FOC(p_b)$ has the same sign as $U'(p_b) - (U(p^*) - U(p_b))/(p^* - p_b) <$
\[ U'(p_b) - (U_-(p^*) - U(p_b))/(p^* - p_b) = 0, \] where the equality follows from the linearity of \( U \) over \([0, p^*] \). So \( \text{FOC}(p_b) \) is strictly negative, and therefore \( p_b = 0 \) is uniquely optimal.

Using Bayes rule, I can back out the stress test \( \pi(\sigma|R) \). Note that 0 = \( p_b = (1 - \pi(g|G))p/\mu(b) \) implies \( \pi(g|G) = 1 \), and \( p_g = \pi(g|G)p/(\pi(g|G)p + \pi(g|0)(1 - p)) \) implies \( \pi(g|0) = p(1 - p_g)/(p_g(1 - p)) \).

**Part 2:** If \( p^* > 1 \) or if \( p^* \leq 0 \), then \( U(\cdot) \) is weakly concave over \([0, 1] \). So for any \( d(p_b), d(p_g), p_b, \) and \( p_g \) satisfying the constraints, \( d(p_b)U(p_b) + d(p_g)U(p_g) \leq U(d(p_b)p_b + d(p_g)p_g) = U(p) \). Since \( U(p) \) can be achieved by an uninformative signal \( p_b = p_g = p \), it must be weakly optimal.

**Proof of Corollary 6**

**Part 1:** The optimal posterior belief \( p_g \) is independent of \( p \), and \( \pi(g|0) = p(1 - p_g)/(p_g(1 - p)) \), which decreases as \( p \) decreases. **Part 2:** Since \( p_b = 0 \), the constraint \( p = d(p_b)p_b + d(p_g)p_g \) implies \( d(p_g) = p/p_g \). Therefore, \( d(p_b) = 1 - d(p_b) = 1 - p/p_g \), which decreases as \( p \) decreases.

**Proof of Proposition 10**

Because the objective function is weakly increasing in \( I \) and \( \kappa \), without loss of generality, Equations (2.8) and (2.9) bind. So the choice variables can be reduced to \( L \in [0, E] \) and \( \kappa \in [0, L] \). Denote the objective function under no stress tests by \( O_n(L, \kappa) \). The objective function is differentiable only in certain regions, with kinks identified by the dotted lines in Figure B.1. The figure is drawn assuming \( k < E \), but this is not necessary for the proofs. If \( k \geq E \), then regions \( A_1 \) and \( A_7 \) are simply eliminated. Where \( O_n \) is differentiable, I have
The strategy is to show that the optimum must be in region $A_4$, and then to show

\[
\frac{\partial O_n}{\partial L} = \int_0^{\hat{\rho}} u' \left( \frac{L + \theta pGI}{2} \right) (1 - \theta pG) dp + \int_{1}^{1} \left[ u'(L - \kappa) - u'(\kappa + \hat{\rho}GI) \right] dp
\]

\[
- \frac{\partial \hat{\rho}}{\partial L} \left[ u(L - \kappa) + u(\kappa + \hat{\rho}GI) - 2u \left( \frac{L + \theta \hat{\rho}GI}{2} \right) \right] \tag{B.2}
\]

\[
\frac{\partial O_n}{\partial \kappa} = \int_{\hat{\rho}}^{1} \left[ - u'(L - \kappa) + u'(\kappa + pGI) \right] dp
\]

\[
- \frac{\partial \hat{\rho}}{\partial \kappa} \left[ u(L - \kappa) + u(\kappa + \hat{\rho}GI) - 2u \left( \frac{L + \theta \hat{\rho}GI}{2} \right) \right] \tag{B.3}
\]
that it does not lie on the upper boundary \( p^* = 0 \) of \( A_4 \). First consider region \( A_1 \), where \( \kappa > k \). In this region, \( p^* < 0 \), so \( \hat{p} = 0 \), and \( L - \kappa < k \). Therefore, \( \partial \hat{p} / \partial \kappa = 0 \), so \( \partial O_n / \partial \kappa = -1 \), and in region \( A_1 \), the optimum must lie along \( \kappa = k \). In region \( A_3 \), \( \hat{p} > 1 \), \( \kappa < k \), and \( p^* < 0 \), so \( \partial O_n / \partial L = 1 - G/2 < 0 \), which implies the optimum must lie along \( \bar{p} = 1 \). In region \( A_2 \), \( p^* < 0 \), \( \kappa < k \), and \( \hat{p} < 1 \), so \( \partial O_n / \partial \kappa = -1 + \bar{p} < 0 \), which implies the optimum in \( A_2 \) must lie along \( \bar{p} = 1 \) or \( p^* = 0 \). Also, in region \( A_2 \), \( \partial O_n / \partial L = 1 - G\bar{p}^2/2 \). Let \( \kappa(L) \) be the value of \( \kappa \) which solves \( \bar{p} = 1 \) for a given \( L \). Then \( dO_n(L, \kappa(L)) / dL = \partial O_n(L, \kappa(L)) / \partial L + (\partial O_n(L, \kappa(L)) / \partial \kappa) \kappa'(L) = 1 - G\bar{p}^2/2 + (-1 + \bar{p}) \kappa'(L) = 1 - G/2 + 0 < 0 \), so \( O_n \) is decreasing along \( \bar{p} = 1 \) as \( L \) increases, which implies the optimum in \( A_2 \) must lie on \( p^* = 0 \). Because \( A_2 \) includes the optima of \( A_1 \) and \( A_3 \), the optimum of \( (A_1, A_2, A_3) \) together must lie on \( p^* = 0 \), with \( \bar{p} \leq 1 \).

I next establish a lemma which simplifies the computation of (B.2) and (B.3) when \( p^* \geq 0 \).

**Lemma 8.** Suppose \( p^* \geq 0 \). Then the consumption \( c_{11}(p) \) under a run \( (p < p^*) \) does not exceed the kink \( k \).

**Proof.** If \( p < p^* \), then \( c_{11}(p) = (L + \theta \hat{p} GI) / 2 < (L + \theta \hat{p} GI) / 2 \). In regions \( A_6 \) and \( A_7 \), \( p^* > 1 \), so \( \hat{p} = 1 \), and also \( L > k \). Then \( L + \theta \hat{p} GI = L + \theta G (E - L) \). If \( \theta G \leq 1 \), then \( L + \theta G (E - L) \leq E < 2k \). If \( \theta G > 1 \), then \( L + \theta G (E - L) = \theta GE - (\theta G - 1)L < \theta GE - (\theta G - 1)(E - k/G) = \theta k + E - k/G < \theta k + k < 2k \). In regions \( A_4 \) and \( A_5 \), where \( p^* < 1 \) and \( L - \kappa < k \), I have \( L + \theta \hat{p} GI = L + \theta p^* GI < L + p^* GI = L + L - 2\kappa = 2(L - \kappa) < 2k \). So in regions \( A_4 \) through \( A_7 \), \( c_{11}(p) = (L + \theta p GI) / 2 < k \), and the lemma is proved.

Now consider the region \( (A_4, A_5, A_6, A_7) \), where \( p^* \geq 0 \). In regions \( A_6 \) and \( A_7 \), I have \( \hat{p} > 1 \) and \( p^* > 1 \), so \( \hat{p} = 1 \) and then \( \partial \hat{p} / \partial \kappa = 0 \). Therefore, (B.3) is 0, the objective function is constant in \( \kappa \), so an optimal point of \( (A_6, A_7) \) must lie on the
upper boundary $p^* = 1$. In the interior of $A_5$, I have $p^* < 1$ but $\bar{\varphi} = 1$, so (B.3) is

$$\int_{p^*}^1 [-1 + \theta] - (-2/GI) (1 - \theta)(L - 2\kappa) > 0; \text{ therefore } \partial O_n/\partial \kappa > 0,$$

and the optimum must lie on the upper boundaries $\bar{\varphi} = 1$ or $p^* = 0$. Now consider (B.2) in $A_5$, which is

$$\int_0^1 (1 - \theta_p G) dp + \int_{p^*}^1 (1 - pG) dp - (E - 2\kappa)/(GI^2)(1 - \theta)(L - 2\kappa).$$

Along $p^* = 0$, I have $0 = p^* = (L - 2\kappa)/GI$, which implies $L = 2\kappa$, so along $p^* = 0$, (B.2) reduces to $1 - G/2 < 0$ and (B.3) reduces to 0. Let $\kappa(L) = L/2$, so that for any $L$, $\kappa(L)$ solves $p^* = 0$. Then along $p^* = 0$, I have $dO_n(L, \kappa(L))/dL = \partial O_n/\partial L + \partial O_n/\partial \kappa \cdot \kappa'(L) = 1 - G/2 + 0 \cdot 1/2 < 0$, so $O_n$ is decreasing along $p^* = 0$ as $L$ increases, and the optimum of $A_5$ must lie along $\bar{\varphi} = 1$. Because $A_5$ contains an optimal point of $(A_6, A_7)$, I must have that the optimal point of $(A_5, A_6, A_7)$ lies along $\bar{\varphi} = 1$. Because $A_4$ contains the optimal point of $(A_1, A_2, A_3)$ along $p^* = 0$ and also contains the optimal point of $(A_5, A_6, A_7)$ along $\bar{\varphi} = 1$, the global optimum must lie in $A_4$.

In the region $A_4$, where $p^* \in [0, 1]$ and $\bar{\varphi} \leq 1$, (B.2) and (B.3) may be written as

$$\frac{\partial O_n}{\partial L} = 1 - \frac{G}{2} p^2 + \frac{G}{2} (1 - \theta) p^2 \bar{\varphi} - \frac{E - 2\kappa}{GI^2} (1 - \theta)(L - 2\kappa) \quad \text{(B.4)}$$

$$\frac{\partial O_n}{\partial \kappa} = \bar{\varphi} - 1 + \frac{2}{GI} (1 - \theta)(L - 2\kappa). \quad \text{(B.5)}$$

Along $p^* = 0$, $L - 2\kappa = 0$, so (B.4) reduces to $1 - \bar{\varphi} G/2$ and (B.5) reduces to $\bar{\varphi} - 1 < 0$. Therefore, along any point of the interior of $p^* = 0$, lowering $\kappa$ is an improvement. Now check the endpoints of $p^* = 0$ by examining how $O_n$ changes along $p^* = 0$. Letting $\kappa(L) = L/2$ as above so that $\kappa(L)$ solves $p^* = 0$ for any $L$, observe that $dO_n(L, \kappa(L))/dL = \partial O_n/\partial L + \partial O_n/\partial \kappa \cdot \kappa'(L) = 1 - \bar{\varphi} G/2 + (\bar{\varphi} - 1)/2$. At the right-most endpoint of $p^* = 0$ in $A_4$, $\bar{\varphi} = 1$, so $dO_n(L, \kappa(L))/dL = 1 - G/2 < 0$, so moving slightly to the interior of $p^* = 0$ is an improvement. At the left-most endpoint of $p^* = 0$, $L = \kappa = 0$, so $\bar{\varphi} = k/(GE)$, so then Assumption 3 implies $\partial O_n/\partial L = 1 - \bar{\varphi} G/2 = 1 - (k/(GE))^2 G/2 > 1 - (\sqrt{2} G/G)^2 G/2 = 0$. Therefore, raising $L$ is an improvement over $L = \kappa = 0$, so the left-most endpoint of $p^* = 0$
cannot be optimal, and therefore the optimal point of $A_4$ must satisfy $p^* > 0$. The optimal point of $A_4$ is the global optimum, so the Proposition is proved.

\textbf{Proof of Proposition 11} In region $A_4$, the objective function $O_n$ is strictly concave in $\kappa$, which can be seen by nothing that (B.5) is strictly decreasing in $\kappa$. Therefore, the optimum must lie on the set of points where (B.5) is equal to zero, which is a positive-sloped straight line intersecting $(L = (GE - k)/(G + 2(1 - \theta)), \kappa = 0)$, as well as the upper right corner of $A_4$. At upper-right corner of $A_4$, I have $p^* = 0$ (so $L - 2\kappa = 0$) and $\bar{p} = 1$, so (B.4) reduces to $1 - G/2 < 0$. Letting $\kappa(L)$ denote the value $\kappa$ that sets (B.5) equal to zero, I have $dO_n(L, \kappa(L))/dL = \partial O_n/\partial L + \partial O_n/\partial \kappa \cdot \kappa'(L) = \partial O_n/\partial L + 0 \cdot \kappa'(L) = \partial O_n/\partial L$ along $\kappa(L)$. At the upper right corner of $A_4$, $\partial O_n/\partial L = 1 - G/2 < 0$, so the optimum cannot lie on the upper right corner of $A_4$.

Next, I show that (B.4) is strictly decreasing in $L$. Because $\bar{p}$ is increasing in $L$, the sum of the first two terms are decreasing in $L$. The second two terms can be combined as $- (1 - \theta)p^2 G((E - 2\kappa)/(L - 2\kappa) - 1/2)$, and differentiating with respect to $L$ gives $-(E - 2\kappa)^2 p^* / (G^2 (L - 2\kappa)) < 0$, so the sum of the last two terms is also strictly decreasing in $L$. Therefore, $O_n$ is strictly concave in $L$. I have shown above that along $\kappa(L)$, $dO_n(L, \kappa(L))/dL = \partial O_n/\partial L$, so by the $L$-concavity of $O_n$, the optimal $\kappa$ is nonzero if and only if $\partial O_n/\partial L > 0$ at the intersection of the graph of $\kappa(L)$ and $\kappa = 0$, which occurs at the point $(L = (GE - k)/(G + 2(1 - \theta)), \kappa = 0)$. Evaluating (B.4) at this point, and multiplying by $2(G + 2(1 - \theta))^2$ to eliminate fractions indicates that (B.4) has the same sign as

$$
(1 - \theta) E[EG^2 + 4(1 - \theta)(E + k) + kG] + k^2[-(G - 2\theta)^2 - 2G + 7\theta - 3].
$$

(B.6)

If $\theta = 1$, then (B.6) reduces to $-k^2G(G - 2)$, which is negative for $G > 2$, so by the continuity of (B.6) there exists a neighborhood of $(\theta = 1, G = 2)$ which contains a point $\theta < 1, G > 2$ in which (B.6) is negative, so that $\kappa = 0$ is optimal.
On the other hand, if \((\theta, G) = (0, 2)\), then (B.6) reduces to \(12E^2 + k(6E - 11k) > 12E^2 + E/2(6E - 11k) > 12E^2 + E/2(6E - 22E) = 4E^2 > 0\), where the two inequalities follow from Assumptions 2 and 3 respectively. Therefore, there exists a neighborhood of \((\theta, G) = (0, 2)\) that contains a point with \(\theta > 0, G > 2\) so that \(\kappa > 0\) is optimal. 

**Proof of Theorem 4** I first show that the optimum must lie in \(A_4\). Denote the objective function under optimal stress tests by \(O_o(L, \kappa)\). By Proposition 9, an uninformative test is weakly optimal if \(p^* \leq 0\) or if \(p^* > 1\), so \(O_n = O_o\) in all but regions \(A_4\) and \(A_5\), and the optimum must lie in \(A_4\) or \(A_5\). Also note that if \(p \geq p^*\), then \(V(p) = U(p)\), but if \(p < p^*\), then \(V(p) = 2u((L + pGI)/2)\), which is simply \(U(p)\) with \(\theta = 1\). Therefore, the derivative of \(O_o\) in regions \(A_4\) and \(A_5\) may be obtained by simply substituting \(\theta = 1\) into (B.2) and (B.3). For region \(A_5\), then, (B.3) is equal to zero, so an optimal point of \(A_5\) must lie on the upper boundaries \(\bar{p} = 1\) or \(p^* = 0\). Let \(\kappa(L) = L/2\), so that for any \(L, \kappa(L)\) solves \(p^* = 0\). Then along \(p^* = 0\), I have \(dO_n(L, \kappa(L))/dL = \partial O_n/\partial L + \partial O_n/\partial \kappa \cdot \kappa'(L) = 1 - G/2 + 0 \cdot 1/2 < 0\), so \(O_n\) is decreasing along \(p^* = 0\) as \(L\) increases, and the optimum of \(A_5\) must lie along \(\bar{p} = 1\). This boundary is shared by \(A_4\), so the optimum must lie in \(A_4\).

In the region \(A_4\), the \(\kappa\) derivative of \(O_o\) is given by (B.5) with \(\theta = 1\), which reduces to \(\bar{p} - 1\). This value is strictly negative everywhere but the boundary \(\bar{p} = 1\), so the optimal point of \(A_4\) must lie on either the lower boundary \(\kappa = 0\) or the right boundary \(\bar{p} = 1\). However, the \(L\) derivative of \(O_o\) is given by (B.4) with \(\theta = 1\), which reduces to \(1 - p^2G/2\). This value is strictly negative on the boundary \(\bar{p} = 1\), so the optimum cannot lie on the boundary \(\bar{p} = 1\), and must lie on the lower boundary \(\kappa = 0\). 

**Proof of Theorem 5** If \(\kappa_n = 0\), then the optimal \(L_n\) without stress tests lies along the lower boundary where the \(L\)-derivative equals zero. Therefore, rearranging
the last two terms of (B.4), I have that at $L_n$,

$$0 = \frac{\partial O_o}{\partial L} \bigg|_{L=L_n} = 1 - \frac{G}{2} p^2 + \frac{G}{2} (1 - \theta) p^2 - \frac{E - 2\kappa}{GI^2} (1 - \theta) (L - 2\kappa) \quad (B.7)$$

$$= 1 - \frac{G}{2} p^2 - (1 - \theta) p^2 G \left( \frac{E - 2\kappa}{L - 2\kappa} - \frac{1}{2} \right) \quad (B.8)$$

$$< 1 - \frac{G}{2} p^2 = \frac{\partial O_o}{\partial L} \bigg|_{L=L_n}. \quad (B.9)$$

By the $L$-concavity of $O_o$, I must have $L_o > L_n$, and therefore $r_o = r_o + \kappa_o = L_o > L_n = r_n + \kappa_n = r_n$. 

□
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Biography

My full name is Basil Joseph Williams. I was born on July 12, 1983 in Provo, Utah. I earned a B.S. in Mathematics and a B.S. in Statistics from Brigham Young University, both in 2009. I earned a PhD in Business Administration with an emphasis in Finance from the Fuqua School of Business at Duke University in 2015.

My job market paper, which is Chapter 1 of this dissertation, was awarded the 2014 PhD Outstanding Paper Award in Honor of Stuart I. Greenbaum from Olin Business School, Washington University in St. Louis. I also received the 2015 Cubist Systematic Strategies PhD Candidate Award for Outstanding Research. In August of 2015, I will begin working as an Assistant Professor in the Department of Economics in the Faculty of Arts and Science at New York University.