L2 Index Theory and D-particle Binding

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

(Mathematics)

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Abstract

In this work, we apply L^2-index theory to compute the index of a non-Fredholm elliptic operator. The operator arises in Type I' string theory, and the index is found to be non-zero, thus implying existence of bound states.

Contents

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Chapter 1

Introduction

1.1 Elliptic Differential Operators and Parametrices

The main goal of this work is the computation of the (L^2) index of a first order elliptic differential operator defined over an unbounded domain. Some of the very nice properties of elliptic operators on vector bundles over compact manifolds are lost when the compactness is dropped. For example, existence of Fredholm extensions is no longer guaranteed and the Atiyah-Singer index theorem no longer holds. We will see how to address these issues in the following section. For the moment, we describe some properties that are independent of the compactness criterion.

Theorem 1.1.1. *Existence of Parametrix. If A is an elliptic pseudodifferential operator of order m in $\Omega \subset \mathbb{R}^n$, then there exists a pseudodifferential operator B of order $-m$ such that:*

$$B \circ A = I - R_1,\ A \circ B = I - R_2 \tag{1.1.1}$$

where R_j is order $-\infty$ for $j = 1, 2$. B is unique up to addition of operators with smooth kernels.

Proof. Proof is by construction and is a direct result of the composition formula for pseudodifferential operators and the definition of ellipticity. See [1] □

The existence of such a parametrix directly implies the following:

Corollary 1.1.2. *Elliptic Regularity Let A be an elliptic pseudodifferential operator of order m in $\Omega \subset \mathbb{R}^n$. If $u \in H_s(\Omega)$ for some s satisfies:*

$$Au = f \tag{1.1.2}$$

for $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Proof. See [1] □

1.2 Heat Kernel Index Computation

1.2.1 Compact Case

Let M be a compact manifold, S a smooth vector bundle over M and let $\Gamma(S)$ denote smooth sections of S. For M compact, an elliptic operator $Q : \Gamma(S) \to \Gamma(S)$ of order m, extends to a Fredholm operator $Q : L^2_s(S) \to L^2_{s-m}(S)$ for any s, where $L^2_s(S)$ denotes the Sobolev space of sections with L^2 derivatives to order s. The index is defined as:

$$\mathrm{Ind}(Q) \stackrel{\mathrm{def}}{=} \dim(\ker(Q)) - \dim(\mathrm{coker}(Q)) \tag{1.2.1}$$

This quantity is independent of the choice of extension. Since $\mathrm{coker}(Q) \approx \ker(Q^*)$, we may express the index as:

$$\mathrm{Ind}(Q) = \dim(\ker(Q)) - \dim(\ker(Q^*)) \tag{1.2.2}$$

It is standard to use the small t heat trace to compute the index, because there exist asymptotic expansions of the heat trace for small t. That is, we would like to compute $\mathrm{Ind}\,(Q)$ using:

$$\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{t}Q^*Q}\right) - \mathrm{Tr}\left(\mathrm{e}^{-\mathrm{t}QQ^*}\right) = \sum_{i=0}^{\infty} e^{-t\lambda_i} - \sum_{i=0}^{\infty} e^{-t\mu_i} \tag{1.2.3}$$

Where λ_i and μ_i are the eigenvalues of Q^*Q and QQ^* respectively. Since for $\lambda_i > 0$, $Q/\sqrt{\lambda_i}$ maps the λ_i eigenspace of QQ^* isometrically to the λ_i eigenspace of Q^*Q (and vice-versa) there is a one-one correspondence between the non-zero eigenvalues of these operators. Thus:

$$\sum_{i=0}^{\infty} e^{-t\lambda_i} - \sum_{i=0}^{\infty} e^{-t\mu_i} = \sum_{i=0}^{\dim(\ker(Q))} 1 - \sum_{i=0}^{\dim(\ker(Q^*))} 1 \tag{1.2.4}$$

$$= \dim\left(\ker\left(Q\right)\right) - \dim\left(\ker\left(Q^*\right)\right) \tag{1.2.5}$$

$$= \mathrm{Ind}\,(Q) \tag{1.2.6}$$

1.2.2 Non-compact Case

When the base manifold is not compact, the operators e^{-tQQ^*} and e^{-tQ^*Q} need not be trace class. There is much work in the literature addressing cases where the manifold is not compact, but the operator Q is still Fredholm (cf. [2], [3], [4], [5]). In these cases, one may begin with the above formula involving the heat kernel trace. Note that since there is a gap between zero and the essential spectrum, the above argument applies formally. Summarizing the argument in [4], first define:

$$b\,(t,x) \stackrel{\mathrm{def}}{=} e^{-tQ^*Q}\,(x,x) - e^{-tQQ^*}\,(x,x) \tag{1.2.7}$$

Let Π_+, Π_- denote L^2-orthogonal projection onto $\ker(Q) \cap L^2(S)$, $\ker(Q^*) \cap L^2(S)$ and p_+, p_- their associated kernel functions.

Claim:

$$\lim_{t\to\infty} \int_K b(t,x)\,\mathrm{d}x \to \int_K (p_+(x,x) - p_-(x,x))\,\mathrm{d}x \tag{1.2.8}$$

for any compact set K.

Proof. Note that:

$$\frac{1}{2} d/dt \|e^{-tQ^2}(x,y)\|^2 = -\|Q_x e^{-tQ^2}(x,y)\|^2 \tag{1.2.9}$$

Taking $t \to \infty$, we see that $\lim_{t\to\infty} \|Q_x e^{-tQ^2}(x,y)\|^2 = 0$, so that $\lim_{t\to\infty} e^{-tQ^2}(x,y) \in \ker(Q_x)$ and we have the analogous statement in y. But for finite t, we have that $e^{-tQ^2}(x,y)$ acts as the identity on $\ker(Q)$, so that $\lim_{t\to\infty} e^{-tQ^2}(x,y)$ must be the projection onto $\ker(Q)$. □

Defining the L^2-Index of Q to be:

$$\dim\left(\ker(Q) \cap L^2(S)\right) - \dim\left(\ker(Q^*) \cap L^2(S)\right), \tag{1.2.10}$$

it then follows:

$$L^2\text{-Index}(Q) = \lim_{\alpha\to\infty} \lim_{t\to\infty} \int_{B_\alpha} b(t,x)\,\mathrm{d}x, \tag{1.2.11}$$

for $\{B_\alpha\}$ a compact exhaustion of M. For computational purposes, it is useful to transform the above limit $t \to \infty$ to a limit taking $t \to 0$. This is done as follows:

$$\lim_{t\to\infty} \int_{B_\alpha} b(t,x)\,\mathrm{d}x = \lim_{t_0\to 0} \left(\int_{B_\alpha} b(t_0,x)\,\mathrm{d}x + \int_{t_0}^{\infty} \frac{d}{dt} \int_{B_\alpha} b(t,x)\,\mathrm{d}x\mathrm{d}t \right) \tag{1.2.12}$$

When Q is a Dirac operator, the quantity $\frac{d}{dt}b(t,x)$ turns out to be the total divergence of a vector field V_Q, whose integral may therefore be transformed via the divergence theorem to a boundary integral. In the following section, we recall a few properties of spinor bundles and Clifford algebras that will be useful in the next chapter.

1.2.3 Brief Review of Spinors and Clifford Algebras

We review some relevant definitions and properties necessary to support the computations which follow. These results and more may be found in [6].

Definition 1.2.1. Let V be a finite dimensional vector space (over $\mathbb{R}$, for our purposes) with a quadratic form $q : V \to \mathbb{R}$. Let $\mathscr{T}(V)$ denote the tensor algebra of V and let $\mathscr{I}_q$ denote the ideal generated by elements of the form $v \otimes v + q(V)\,1$. Then we define $\mathscr{C}\ell(V,q) = \mathscr{T}(V)/\mathscr{I}_q$.

As always, the polarization of $q(v)$ gives a bilinear form $q(v,w)$. If we denote multiplication by an element of $\mathscr{C}\ell(V,q)$ by $C(\cdot)$, we have:

$$C(v)\,C(w) + C(w)\,C(v) = -2q(v,w) \tag{1.2.13}$$

for all $v, w \in V$. Note that if $V = \mathbb{R}^n$, and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$, the Clifford algebra $\mathscr{C}\ell(\mathbb{R}^n)$ is spanned by $\{e_1, ..., e_n, e_1e_2, e_1e_3, ..., e_1 \cdots e_n\}$, i.e. all (ordered) Clifford products of the basis $\{e_1, \ldots, e_n\}$. A general element of the algebra is given by:

$$\psi = \sum_{|I| \leq n} a_I e_{i_1} \cdots e_{i_{|I|}} \tag{1.2.14}$$

where I is a multi-index. We define Clifford degree of an element to be the highest $|I|$ for which there is a non-zero a_I. The group Spin_n may be realized as the subgroup of the multiplicative group of units of $\mathscr{C}\ell\left(\mathbb{R}^n\right)$ consisting of elements of the form:

$$\varphi = v_{i_1} \cdots v_{i_r} \tag{1.2.15}$$

where r is even and $v \in \mathbb{R}^n$ with $|v| = 1$. Representations of $\mathscr{C}\ell\left(\mathbb{R}^n\right)$ restrict to representations of Spin_n. We will be dealing with representations of Spin_8. In this dimension, the Clifford multiplication by the volume form ω squares to 1. Here, an irreducible representation

$$\rho : \mathscr{C}\ell\left(\mathbb{R}^8\right) \to \mathrm{Hom}_{\mathbb{R}}\left(W, W\right) \tag{1.2.16}$$

restricts to a reducible representation:

$$\rho_8 : \mathrm{Spin}_8 \to \mathrm{GL}\left(W\right) \tag{1.2.17}$$

such that $\rho_8 = \rho_8^+ \oplus \rho_8^-$ and

$$\rho_8^+ : \mathrm{Spin}_8 \to \mathrm{GL}\left(W^+\right) \tag{1.2.18}$$

$$\rho_8^- : \mathrm{Spin}_8 \to \mathrm{GL}\left(W^-\right) \tag{1.2.19}$$

where W^+ and W^- are the ± 1 eigenspaces of the operator defined by Clifford multiplication by ω. We will define another spinor space, generated by the vector space:

$$\left(W^+ \bigoplus W^+ \bigoplus W^-\right) \tag{1.2.20}$$

The operators we will be concerned with are Dirac operators on a spinor bundle whose fiber is as above. One very useful feature of trace computations involving Clifford bundles is the following (see for example: [4])

Lemma 1.2.2. *Let V be an even dimensional vector space and $\{e_1, ..., e_n\}$ an orthonormal basis of V. Then:*

$$\mathrm{tr} C(e_{i_1}) \ldots C(e_{i_r}) = 0 \tag{1.2.21}$$

for any non-empty $\{i_1, ..., i_r\}$ with all i_j distinct.

Next we describe the physical model which gives rise to the operator we are about to analyze.

1.3 Physical Model

1.3.1 D0 Branes

The following is an extremely curt introduction to the physical model upon which our mathematical contruction is based. For a more thorough review of D-branes and their dynamics see [7]. For more on $D0$-brane bound states, see [8], and for a more complete description of the construction of Type I' strings, see [9],[10].

D-branes are defined as boundary conditions on the ends of open strings, but they can be seen as dynamic objects themselves. At low energy, D0 brane dynamics are governed by the dimensional reduction of 10D $\mathscr{N} = 1$ supersymmetric Yang-Mills to $0+1$ dimensional gauge theory (i.e. supersymmetric quantum mechanics). After dimensional reduction, a system of n D0-branes has hamiltonian:

$$\mathscr{H} = \frac{1}{2n}\mathrm{Tr}\left(p^i p^i\right) - \frac{1}{4n}\sum_{i,j}\mathrm{Tr}\left(\left[x^i, x^j\right]^2\right) - \frac{1}{2n}\mathrm{Tr}\left(\psi\gamma^i\left[x^i, \psi\right]\right) \tag{1.3.1}$$

The ψ's transform in the $SO(9)$ Clifford representation given by the γ's. After factoring out the center of mass action, the gauge group is SU(n); thus, the x's live in the adjoint representation of SU(n). This theory has $\mathscr{N} = 16$ supercharges:

$$Q_\alpha = \frac{i}{2}\gamma^i_{\alpha\beta}\mathrm{Tr}\left(\psi_\beta p^i\right) - \frac{i}{8}\mathrm{Tr}\left(\left[\gamma^i,\gamma^j\right]\psi\left[x^i,x^j\right]\right)_\alpha \tag{1.3.2}$$

which obey the supersymmetry algebra:

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}\left(\mathscr{H} - \mathscr{G}\right) \tag{1.3.3}$$

$$\tag{1.3.4}$$

where $\mathscr{G}$ is the gauge action:

$$\mathscr{G} = i\left[x^i, p^i\right] - \frac{1}{2}\left[\psi_\alpha, \psi_\alpha\right] \tag{1.3.5}$$

In the case $n = 2$, we have the gauge group SU(2). Stern and Sethi [11] have proved the existence of a unique ground state in this model. The present work is concerned with the addition of an $O8$ orientifold plane and further addition of D8-branes as occurs in Type I' string theory.

1.3.2 D0 Branes and O8 orientifolds

Type I' string theory is obtained as a quotient of either IA or IIA:

$$I' = \frac{IIA}{\{1, \Omega R\}}$$

with $\{1, \Omega R\} \approx \mathbb{Z}_2$ and Ω is world sheet parity, R is reflection in the x^9 coordinate. The effect of the quotient operation is to add two 8-dimensional orientifold planes (at positions corresponding to the fixed points of the reflection on the circle), each accompanied by 8 D8 branes and their mirror images. Taking one of the orientifold planes to infinity along with its D8 branes leaves one orientifold and 16 D8 branes to consider. The description of D0 brane dynamics in Type IIA involves bosonic coordinates $x^i \quad i = 1, ..., 9$ living in the adjoint of SU(2), and fermionic coordinates $\Psi_\alpha \;\; \alpha = 1, ..., 16$. The orientifold quotient breaks the gauge symmetry to SO(2), cuts the supersymmetry in half (from $\mathscr{N} = 16$ to $\mathscr{N} = 8$), and causes a projection onto the symmetric and antisymmetric coordinates as follows:

$$\begin{pmatrix} x_1^i & x_2^i \\ x_3^i & -x_1^i \end{pmatrix} \longrightarrow \begin{pmatrix} x_1^i & x_2^i \\ x_2^i & -x_1^i \end{pmatrix} \qquad \text{for } i = 1, ..., 8$$

$$\begin{pmatrix} x_1^9 & x_2^9 \\ x_3^9 & -x_1^9 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & x_3^9 \\ -x_3^9 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \psi_{1\alpha} & \psi_{2\alpha} \\ \psi_{3\alpha} & -\psi_{1\alpha} \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_{1\alpha} & \psi_{2\alpha} \\ \psi_{2\alpha} & -\psi_{1\alpha} \end{pmatrix} \qquad \text{for } \alpha = 1, ..., 8$$

$$\begin{pmatrix} \psi_{1\alpha} & \psi_{2\alpha} \\ \psi_{3\alpha} & -\psi_{1\alpha} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \psi_{3\alpha} \\ -\psi_{3\alpha} & 0 \end{pmatrix} \qquad \text{for } \alpha = 9, ..., 16$$

The supercharges are:

$$Q_\alpha = i\gamma^i_{\alpha\beta}\psi_{N\beta}\frac{\partial}{\partial x^i_N} + i\gamma^{ik}_{\alpha\beta}\psi_{M\beta}\left[x^i, x^k\right]_M \tag{1.3.6}$$

where α now ranges from $9, ..., 16$, and the x's and ψ's obey the above restrictions. The gauge constraint in this model differs from the $D0 - D0$ case not only in the breaking of the symmetry to SO(2), but it also includes a term representing the fermionic strings between the $D0$ branes and the $D8$ branes at the origin.

$$\mathscr{G} = x^i_1\frac{\partial}{\partial x^i_2} - x^i_2\frac{\partial}{\partial x^i_1} + \psi_{1\alpha}\psi_{2\alpha} + \chi^i_1\chi^i_2 \tag{1.3.7}$$

Now, the Q_α are independent of the χ's, but when we perform computations using Q^2 it will be convienient to use the gauge action to replace some of the terms. This is where the χ's will become relevant.

Chapter 2

L^2 Index

2.1 Defining the L^2 Index

Given an involution which anticommutes with Q, we may decompose Q into components $Q_+ : \Gamma(S^+) \to \Gamma(S^-)$ and $Q_- : \Gamma(S^-) \to \Gamma(S^+)$, where S^+ and S^- are the ± 1 eigenspaces of our involution. For Q self-adjoint, we may then write:

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix}.$$

We would like to calculate the index of Q_+, which in the case of a compact manifold is given by:

$$\text{Ind}\,(Q_+) = \dim\left(\ker\left(Q_+\right)\right) - \dim\left(\ker\left(Q_-\right)\right).$$

In the case at hand, however, our manifold is Euclidean space and the essential spectrum of Q^2 contains zero. This means the coker(Q) is not closed, is infinite dimensional and is not equal to ker(Q^*). Thus, the usual (Fredholm) index is not defined. We will instead compute the L^2 index:

Definition 2.1.1. The L^2 index of an elliptic operator $Q_+ : \Gamma(S^+) \to \Gamma(S^-)$ is defined by:

$$L^2 - \text{Index}(Q_+) = \dim(\ker(Q_+)) \cap L^2 - \dim(\ker(Q_-)) \cap L^2.$$

2.2 The Green's Operator

Proposition 2.2.1. *Given a Green's operator G for Q^2, ${}_{L^2-}$Index(Q_+) is given by:*

$$L^2 - \mathit{Index}(Q_+) = tr\left(\tau + \frac{1}{2}[Q, \tau QG]\right),$$

where τ is an involution which anticommutes with Q.

Proof of Proposition. Let G_+ denote the restriction of G to the $+1$ eigenspace of τ, and let $P_\pm$ denote orthogonal projection onto the kernel of $Q_\pm$. Note that

$$G_+ = \frac{1}{2}G(I + \tau).$$

Since $Q^2G = I - P$, we have:

$$QQG_+ = (I - P)\frac{1}{2}(I + \tau) = \frac{1}{2}(I + \tau) - P_+.$$

Similarly:

$$QG_+Q = \frac{1}{2}(I - \tau) - P_-.$$

Thus,

$$\begin{aligned}
L^2 - \text{Index}\,(Q_+) &= \text{tr}P_+ - \text{tr}P_- \\
&= \text{tr}\left(\frac{1}{2}(I+\tau) - QQG_+\right) - \text{tr}\left(\frac{1}{2}(I-\tau) - QG_+Q\right) \\
&= \text{tr}\left((\tau) - [Q, QG_+]\right) \\
&= \text{tr}\left(\tau + \frac{1}{2}[Q, \tau QG]\right). \qquad (2.2.1)
\end{aligned}$$

□

For the computations to follow, we'll need to establish the existence of G as a singular integral operator. Before we begin, let's fix some notation. Here we choose the convention $\psi^2 = 1 = \chi^2 = 1$. This is different from the physics convention which has $\psi^2 = \frac{1}{2}$. Our choice of normalization also leads to a factor of 1 in front of the laplacian in Q^2, as opposed the the factor of 1/2 found in the physics literature.

Recall that our model arises from dimensional reduction of a model describing the dynamics of two $D0$ branes in type I' string theory. This may be realized as an orientifold quotient of the type IIA theory. In type IIA, the reduced model takes the form of supersymmetric matrix quantum mechanics, with the coordinates taking values in the adjoint representation of SU (2). One effect of the (8-dimensional) orientifold plane introduced transverse to the ninth spacial coordinate is a projection onto the symmetric and antisymmetric parts of the coordinate matrices as follows:

$$\begin{pmatrix} x_1^i & x_2^i \\ x_3^i & -x_1^i \end{pmatrix} \to \begin{pmatrix} x_1^i & x_2^i \\ x_2^i & -x_1^i \end{pmatrix} \quad \text{for } i = 1, ..., 8$$

$$\begin{pmatrix} x_1^9 & x_2^9 \\ x_3^9 & -x_1^9 \end{pmatrix} \to \begin{pmatrix} 0 & x_3^9 \\ -x_3^9 & 0 \end{pmatrix}.$$

Thus, we may think of our coordinates as living in $\mathbb{R}^{17}$. The corresponding fermions are:

$$\psi_{1s}, \psi_{2s} \quad \text{for } i = 1, ...8 \quad \text{and } \psi_{3s} \quad \text{for } i = 9, ..., 16.$$

In addition, there are fermions χ_1^i, χ_2^i corresponding to strings stretching between each D0 brane and the i^{th} D8-brane. Our operator Q then acts on sections of a flat spinor bundle over $\mathbb{R}^{17}$ (See [8] and [9]) We denote this bundle S, and its smooth sections by $\Gamma(S)$. S is a trivial bundle whose fibers are the tensor product of the two factors. The first is a Clifford module for the Clifford algebra associated to the vector space spanned by the $\psi_{1s}, \psi_{2s}, s = 1, ..., 8$ and $\psi_{3s}, s = 9, ..., 16$. The module is a pinor representation. The second factor is the spinor representation of the Clifford algebra associated to the vector space spanned by the $\chi' s$. The dimension of this fiber is therefore $2^{12} \times 2^M$, where M is the number of $D8$-branes. For consistency in the physical model, M is taken to be 8, so the dimesion of the fiber of S is 2^{20}. In these coordinates, Q^2 has the form:

$$Q^2 = \Delta + \frac{1}{2} \sum \left| \left[x_A^i, x_B^j \right] \right|^2 + i\gamma_{\alpha\beta}^i x_1^i \psi_{2\alpha} \psi_{3\beta} +$$
$$i\gamma_{\alpha\beta}^i x_2^i \psi_{3\alpha} \psi_{1\beta} + ix_3^9 \left(-\psi_{1s} \psi_{2s} + \chi_1^i \chi_2^i \right), \tag{2.2.2}$$

where the above restrictions apply to the indices i, j. The χ terms have entered Q^2 via the gauge constraint, which is now:

$$\mathscr{G} = x_1^i \frac{\partial}{\partial x_2^i} - x_2^i \frac{\partial}{\partial x_1^i} + \frac{1}{2} \psi_{1\alpha} \psi_{2\alpha} + \frac{1}{2} \chi_1^i \chi_2^i, \tag{2.2.3}$$

under the chosen normalizations for ψ, χ. Note that we will sometimes write:

$$Q^2 = \Delta + V(x) + H_F, \tag{2.2.4}$$

where

$$V(x) = \frac{1}{2} \sum \left| \left[x^i, x^j \right] \right|^2. \tag{2.2.5}$$

Recall that there are 8 operators Q_α, $9 \le \alpha \le 16$ and:

$$Q_\alpha = i\gamma^i_{\alpha\beta}\psi_{A\beta}\frac{\partial}{\partial x^i_A} + \gamma^{ik}_{\alpha\beta}\psi_{B\beta}\left[x^i, x^k\right]_B. \tag{2.2.6}$$

For each α, α', Q^2_α differs from $Q^2_{\alpha'}$ by a gauge transformation. Thus, our computations are independent of α. Fix α_0 for $9 \le \alpha_0 \le 16$. Define $Q = Q_{\alpha_0}$ We are now prepared to state the main theorem of this thesis:

Theorem 2.2.2. *Main Theorem.*

Let $Q : \Gamma(S) \to \Gamma(S)$ be the elliptic operator defined above, with gauge constraint:

$$\mathscr{G} = x^i_1 \frac{\partial}{\partial x^i_2} - x^i_2 \frac{\partial}{\partial x^i_1} + \frac{1}{2}\psi_{1\alpha}\psi_{2\alpha} + \frac{1}{2}\chi^i_1\chi^i_2 \tag{2.2.7}$$

where the index i runs from $1, ..., M$ for M an even integer ≥ 8. Let τ be the involution on S given by $\tau = \prod_{s=1}^{8} \psi_{1s}\psi_{2s}\psi_{3s} \prod_{i=1}^{M} \chi^i_1\chi^i_2$ and decompose:

$$Q = \begin{pmatrix} 0 & Q_+ \\ Q_- & 0 \end{pmatrix}, \tag{2.2.8}$$

so that Q_+ takes the $+1$ eigenspace of τ to the -1 eigenspace and similarly for Q_-. Then:

$$L^2 - \mathit{Index}(Q_+) = 2^{M/2-4}. \tag{2.2.9}$$

Analysis of this operator is made delicate because there are regions where the commutator above vanishes, even as $|x| = r$ gets very large. Let us expand V, to see this more clearly:

$$V(x) = \frac{1}{2} \sum \left| \left[x^i, x^j \right] \right|^2 \tag{2.2.10}$$

$$= \sum_{i=1}^{7} \sum_{j>i} \left(x_1^i x_2^j - x_2^i x_1^j \right)^2 + \sum_{i=1}^{8} \left[\left(x_3^9 x_1^i \right)^2 + \left(x_3^9 x_2^i \right)^2 \right]. \tag{2.2.11}$$

V is zero when either of the following conditions is satisfied:

- $x_1^i x_2^j - x_2^i x_1^j = 0$ for all i, j and $x_3^9 = 0$ or
- $x_1^i = x_2^j = 0$ for all i, j, and x_3^9 is arbitrary.

Calculation of the desired index has 3 main parts

- Proof of the existence of a Green's operator
- Construction of a parametrix that approximates the Green's operator
- Proof that the parametrix approximation computes the index (i.e. our error terms are inconsequential)

For the first item above, we will need the following normed spaces:

Definition 2.2.3. Define $L_Q^2(S)$ to be the completion of $C_0^\infty(S)$ in the norm:

$$\|u\|_{L_Q^2(S)} = \left\| \frac{u}{\sqrt{r^2+1}} \right\|_{L^2(S)}^2,$$

and similarly:

Definition 2.2.4. Define $H_Q^1(S)$ to be the completion of $C_0^\infty(S)$ in the norm:

$$\|u\|_{H_Q^1(S)} = \|Qu\|_{L^2(S)}^2 + \left\| \frac{u}{\sqrt{r^2+1}} \right\|_{L^2(S)}^2 .$$

In this section we prove the existence of an (unbounded) operator $G : L_Q^2(S) \to H^2(S)$ such that:

$$Q^2 G = I - \Pi,$$

where Π is orthogonal projection in the L^2 norm onto ker$Q$. Define the bilinear form $D(u,v)$ on $H_Q^1(S) \times H_Q^1(S)$ so that:

$$D(u,v) = (Qu, Qv)_{L^2(S)} .$$

Theorem 2.2.5. *For any $f \in \left(L_Q^2(S) \cap (\mathrm{ker}Q)^\perp \right)$ with $\sqrt{r^2+1} f \in L^2$, there exists a unique $u \in H_Q^1(S) \cap (\mathrm{ker}Q)^\perp$ such that*

$$Q^2 u = f. \tag{2.2.12}$$

The Lax-Milgram theorem states that if D is a bounded, coercive bilinear functional on a hilbert space $\mathscr{H}$, then for every bounded linear functional f on $\mathscr{H}$, there exists $x_f \in \mathscr{H}$ such that:

$$f(x) = D(x, x_f) . \tag{2.2.13}$$

Therefore, if D is both bounded and coercive in $H_Q^1(S) \cap (\mathrm{ker}Q)^\perp$, then weak solutions to (2.2.12) exist. Boundedness of D is clear from the definition of the H_Q^1 norm, so it suffices to prove the coercive estimate:

Proposition 2.2.6. *For $u \in H_Q^1(S) \cap (\mathrm{ker}Q)^\perp$, there exists $c > 0$ such that*

$$\|Qu\|_{L^2(S)}^2 \geq \left\| \frac{cu}{r} \right\|_{L^2(S)}^2 . \tag{2.2.14}$$

Elliptic regularity then guarantees that our weak solutions (2.2.12) are smooth, for smooth f. In particular, for any fixed y, we have:

$$Q^2 g_y(x) = \delta(x - y) - \Pi(x, y), \tag{2.2.15}$$

where P denotes projection onto $\ker(Q)$ has a solution which is smooth for $x \neq y$. Define:

$$Gf(x) = \int g_y(x) f(y) \, \mathrm{d}y. \tag{2.2.16}$$

This gives our Green's operator as the integral operator represented by the kernel $g_y(x) = g(x, y)$. The proof of the estimate (2.2.14) has two main parts. We will show:

1. For any section $u \in C_0^\infty(\mathbb{R}^{17} \backslash B_R(0), S)$, R large, the estimate (2.2.14) holds.

2. Using Rellich's theorem, we can extend the estimate across the ball by restricting to the subspace $H_Q^1(S) \cap (\ker(Q))^\perp$.

Proof of 1. It will be necessary to patch together estimates of a lower bound for Q in 3 different regions of $\Omega = \mathbb{R}^{17} \backslash B_R(0)$. We will obtain one estimate away from flat points (exactly what is meant by 'away' will be clarified shortly). In addition we will need estimates near each of the two types of flat points discussed above. 'Near' a flat direction is defined as:

$$V(x) \leq r^{2+\epsilon}, \tag{2.2.17}$$

where $\epsilon > 0$, for now, and an upper bound will be determined momentarily. Let

$$N_F(\epsilon) = \left\{ x \in \Omega : V(x) < r^{2+\epsilon} \right\}. \tag{2.2.18}$$

Define smooth cutoff functions ρ_1, ρ_2 so that

$$\rho_1(x) = \begin{cases} 1 & \text{for } x \in N_F(\epsilon/2) \\ 0 & \text{for } x \in \Omega \backslash N_F(\epsilon) \end{cases} \tag{2.2.19}$$

$$\rho_2(x) = 1 - \rho_1 \tag{2.2.20}$$

and such that $|d\rho_1| \leq C_\rho$ Let $u \in C_0^\infty(\Omega, S)$. Write:

$$u = \rho_1 u + \rho_2 u. \tag{2.2.21}$$

We need to show:

$$\|Qu\|^2 \geq \left\| \frac{cu}{r} \right\|^2 \tag{2.2.22}$$

for some $c > 0$. We have:

$$\begin{aligned} \|Qu\|^2 &= \|Q(\rho_1 u + \rho_2 u)\|^2 \\ &= \|Q(\rho_1 u)\|^2 + \|Q(\rho_2 u)\|^2 + 2(Q(\rho_1 u), Q(\rho_2 u)). \end{aligned} \tag{2.2.23}$$

We will obtain the desired estimate from the first two terms. The last term, the cross term, needs to be controlled. Before proceeding with the estimates, we will simplify this term a bit. As in the introduction we denote Clifford multiplication by forms as $C(\cdot)$:

$$\begin{aligned} (Q(\rho_1 u), Q(\rho_2 u)) &= (C(d\rho_1)u + \rho_1 Qu, C(C(d\rho_2))u + \rho_2 Qu) \\ &= (C(d\rho_1)u, C(d\rho_2)u) + (C(d\rho_1)u, \rho_2 Qu) \\ &+ (\rho_1 Qu, C(d\rho_2)u) + (\rho_1 Qu, \rho_2 Qu) \qquad (2.2.24) \\ &\geq (C(d\rho_1)u, C(d\rho_2)u) + (C(d\rho_1)u, \rho_2 Qu) \\ &+ (\rho_1 Qu, C(d\rho_2)u) \qquad (2.2.25) \end{aligned}$$

$$|(C(d\rho_1)u, C(d\rho_2)u) + (C(d\rho_1)u, \rho_2 Qu) + (\rho_1 Qu, C(d\rho_2)u)|$$

$$\leq |(C(d\rho_1)u, C(d\rho_2)u)| + |(C(d\rho_1)u, \rho_2 Qu)|$$

$$+ |(\rho_1 Qu, C(d\rho_2)u)| \tag{2.2.26}$$

$$= \||d\rho_1||u|\|^2 + |((\rho_1 - \rho_2)Qu, C(d\rho_1)u)| \tag{2.2.27}$$

$$\leq \||d\rho_1||u|\|^2 + \frac{\varepsilon}{2}\|Qu\|^2 + \frac{1}{2\varepsilon}\||d\rho_1||u|\|^2 \tag{2.2.28}$$

$$\leq \frac{2\varepsilon+1}{2\varepsilon}\||d\rho_1||u|\|^2 + \frac{\varepsilon}{2}\|Qu\|^2. \tag{2.2.29}$$

Substituting back into (2.2.23):

$$\|Qu\|^2 \geq \|Q(\rho_1 u)\|^2 + \|Q(\rho_2 u)\|^2 - 2\left(\frac{2\varepsilon+1}{2\varepsilon}\||d\rho_1||u|\|^2 + \frac{\varepsilon}{2}\|Qu\|^2\right), \tag{2.2.30}$$

so that:

$$(1+\varepsilon)\|Qu\|^2 \geq \|Q(\rho_1 u)\|^2 + \|Q(\rho_2 u)\|^2 - \frac{2\varepsilon+1}{2\varepsilon}\||d\rho_1||u|\|^2. \tag{2.2.31}$$

Ultimately, we'll need to show that the last term on the right may be absorbed into the other two terms. First, we obtain estimates for the first two terms.

The estimate is trivial for $\|Q(\rho_2 u)\|^2$. We have:

$$\|Q(\rho_2 u)\|^2 = \left(Q^2(\rho_2 u), \rho_2 u\right) \tag{2.2.32}$$

$$= ((\Delta + V(x) + H_F)\rho_2 u, \rho_2 u). \tag{2.2.33}$$

Recall that H_F is linear in x_A^i, so that

$$((\Delta + V(x) + H_F)\rho_2 u, \rho_2 u) \geq c\left(r^{2+\epsilon}\rho_2 u, \rho_2 u\right) \tag{2.2.34}$$

$$= c\left\|r^{1+\frac{\epsilon}{2}}\rho_2 u\right\|^2. \tag{2.2.35}$$

Before we begin analyzing $\|Q(\rho_1 u)\|^2$, a few comments regarding the different types of flat points are in order. Note that at least one of $x_3^9, x_{1,2}^i$ must be of order

r. In order to be near a flat point, if any of the $x^i_{1,2}$ are of order r, then x^9_3 must be of order $< r^{\epsilon/2}$. We have to split $\rho_1 u$ to cover the two different regions using cutoff functions. However, the region 'in between' the types of flat points:

$$r^{\frac{\epsilon}{2}} < \left|x^9_3\right| < r^{1-\epsilon} \tag{2.2.36}$$

is not part of the flat region, say for $\epsilon < 1/2$. That is, ρ_1 has two disjoint components, one with $x^9_3 \approx r$, the other with $(|x_1|^2 + |x_2|^2)^{1/2} \approx r$. Since the support is disjoint, we may write

$$\rho_1 = \tilde{\rho}_{12} + \tilde{\rho}_9. \tag{2.2.37}$$

We can therefore treat the two situations separately, without additional cutoffs.

First we work in the case $|x^9_3| < r^{\frac{\epsilon}{2}}$. To simplify the computations in this region, we'll apply an SO(3) × SO(9) coordinate action as follows:

Define the map:

$$\begin{aligned} m &: \mathrm{SO}(3) \times \mathbb{R}^9 \times \mathrm{SO}(9) \to \mathbb{R}^{17} \\ &(k, \Lambda, q) \mapsto k\Lambda q, \end{aligned} \tag{2.2.38}$$

with $k \in \mathrm{SO}\,(2) \subset \mathrm{SO}(3), q \in \mathrm{SO}\,(8) \subset \mathrm{SO}(9)$ and

$$\Lambda = \begin{pmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y^2_2 & y^3_2 & y^4_2 & y^5_2 & y^6_2 & y^7_2 & y^8_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^9_3 \end{pmatrix}. \tag{2.2.39}$$

This map is not one-to-one, but we may lift any function on the original space to a function which is constant in the fiber. It is possible then to perform all of our computations in the larger space. We choose tangent vectors to $\mathrm{SO}(3)\times\mathbb{R}^9\times\mathrm{SO}(9)$ which descend to a basis of $\mathbb{R}^{17}$:

$$\frac{\partial}{\partial \tilde{r}_{\text{flat}}}, \frac{\partial}{\partial \tilde{y}_2^i}, \frac{\partial}{\partial \tilde{x}_3^9}, \tilde{X}, \tilde{V}_j, i, j = 2, ..., 8, \tag{2.2.40}$$

such that

$$\tilde{X} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2.41}$$

and V_j is the $\mathfrak{so}(9)$ generator with entries $1j = 1, j1 = -1$ and all others vanishing. We then compute the metric g in terms of the basis

$$\left\{\tilde{V}_2, \tilde{V}_3, ..., \tilde{V}_8, \tilde{X}, \frac{\partial}{\partial \tilde{r}_{\text{flat}}}, \frac{\partial}{\partial \tilde{y}_2^2}, ..., \frac{\partial}{\partial \tilde{y}_2^8}, \frac{\partial}{\partial \tilde{x}_3^9}\right\} \tag{2.2.42}$$

of pushed down vector fields. We decompose g in the following manner:

$$g = \tilde{g} \oplus I_9 \tag{2.2.43}$$

where I_9 is the 9×9 identity matrix, corresponding to the vector fields

$$\left\{\frac{\partial}{\partial \tilde{r}_{\text{flat}}}, \frac{\partial}{\partial \tilde{y}_2^2}, ..., \frac{\partial}{\partial \tilde{y}_2^8}, \frac{\partial}{\partial \tilde{x}_3^9}\right\} \tag{2.2.44}$$

which are chosen to be orthonormal. We obtain:

$$\begin{aligned}
\tilde{g}_{11} &= \mu^2 + |y_2|^2 \\
\tilde{g}_{1i} &= \tilde{g}_{i1} \\
&= 2\mu y_2^i \text{ for } i = 2, ..., 8 \\
\tilde{g}_{ij} &= \tilde{g}_{ji} \\
&= y_2^i y_2^j \text{ for } i = 2, ..., 8\ j > i \\
\tilde{g}_{ii} &= \mu^2 + \left(y_2^i\right)^2 \text{ for } i = 2, ..., 8
\end{aligned}$$

and

$$\sqrt{|\det g|} = \sqrt{|\det \tilde{g}|}$$
$$= r_{\text{flat}}^6 \left(\mu^2 - |y_2|^2\right).$$

In this frame, we obtain the following expression for Q^2:

$$\begin{aligned}
Q^2 = &\left(-\frac{\partial^2}{\partial \left(x_3^9\right)^2} + \Delta_{y_2}\right) - \frac{\partial^2}{\partial \mu^2} + \frac{2y_2^i}{\mu^2 - |y_2|^2}\frac{\partial}{\partial y_2^i} - \frac{8\mu^2 - 6\,|y_2|^2}{\mu\left(\mu^2 - |y_2|^2\right)}\frac{\partial}{\partial \mu} \\
&- \frac{\mu^2 + |y_2|^2}{\left(\mu^2 - |y_2|^2\right)^2}X^2 - \frac{\mu^4 + |y_2|^2\left(|y_2|^2 - \left(y_2^j\right)^2\right) + \mu^2\left(-2\,|y_2|^2 + 3\left(y_2^j\right)^2\right)}{\mu^2\left(\mu^2 - |y_2|^2\right)^2}V_j^2 \\
&+ \frac{2\mu y_2^j}{\left(\mu^2 - |y_2|^2\right)^2}XV_j + \frac{y_2^i y_2^j\left(3\mu^2 - |y_2|^2\right)}{\mu^2\left(\mu^2 - |y_2|^2\right)^2}V_iV_j + \mu^2\left(\left(x_3^9\right)^2 + |y_2|^2\right) \\
&+ \left(x_3^9\right)^2|y_2|^2 + ix_3^9\left(\psi_{1s}\psi_{2s} + \chi_1^i\chi_2^i\right) + iq_{1j}\mu\gamma_{st}^j\psi_{2s}\psi_{3t} + iq_{ij}\gamma_{st}^j\psi_{1s}\psi_{3t}y_2^i,
\end{aligned}$$

where we have set $k = I$, since k is an element of the gauge group and the space of sections we are interested in must be gauge-invariant.

The gauge constraint now looks like (1.3.5):

$$\mathscr{G} = X - \frac{1}{2}\psi_{1s}\psi_{2s} - \chi_1^i\chi_2^i. \tag{2.2.45}$$

In this formulation, the flat directions of the potential may be parametrized locally by the radial coordinate μ and 7 angular coordinates. We label terms in Q^2 as follows:

$$\begin{aligned}
H_m &= \left(-\frac{\partial^2}{\partial \left(x_3^9\right)^2} + \Delta_{y_2}\right) + \mu^2\left(\left(x_3^9\right)^2 + |y_2|^2\right) \\
Y_r &= iq_{1j}\gamma_{st}^j\psi_{2s}\psi_{3t} \qquad (2.2.46) \\
H_F'\left(y_2^j\right) &= iq_{ij}\gamma_{st}^j\psi_{1s}\psi_{3t}y_2^i
\end{aligned}$$

$$
\begin{aligned}
Q^2 = H_m + \mu Y_r - \frac{\partial^2}{\partial \mu^2} + \frac{2y_2^i}{r^2 - |y_2|^2}\frac{\partial}{\partial y_2^i} - \frac{8\mu^2 - 6|y_2|^2}{\mu\left(\mu^2 - |y_2|^2\right)}\frac{\partial}{\partial \mu} \\
- \frac{\mu^2 + |y_2|^2}{\left(\mu^2 - |y_2|^2\right)^2}X^2 - \frac{\mu^4 + |y_2|^2\left(|y_2|^2 - \left(y_2^j\right)^2\right) + \mu^2\left(-2|y_2|^2 + 3\left(y_2^j\right)^2\right)}{\mu^2\left(\mu^2 - |y_2|^2\right)^2}V_j^2 \\
+ \frac{2\mu y_2^j}{\left(\mu^2 - |y_2|^2\right)^2}XV_j + \frac{y_2^i y_2^j\left(3\mu^2 - |y_2|^2\right)}{\mu^2\left(\mu^2 - |y_2|^2\right)^2}V_iV_j + \left(x_3^9\right)^2|y_2|^2 \\
+ ix_3^9\left(\psi_{1s}\psi_{2s} + \frac{1}{2}\chi_1^i\chi_2^i\right) + H_F'.
\end{aligned}
\tag{2.2.47}
$$

We note that in this formulation we have a coordinate singularity at $\mu^2 = |y_2|^2$. This will not cause difficulty, however since this form of Q^2 is useful only for $|y_2|^2 < r^\epsilon$ and $\mu \approx r$. Now we decompose $\rho_1 u$ as a sum:

$$
\rho_1 u = \Pi_0\rho_1 u + (1 - \Pi_0)\rho_1 u \tag{2.2.48}
$$

$$
= \phi_0 + \phi_1 \tag{2.2.49}
$$

where

$$
\phi_0 = f_0(\mu, q)\mu^2 e^{-\mu\left(|y_2|^2 + \left(x_3^9\right)^2\right)/2} \tag{2.2.50}
$$

f_0 is a section of the subbundle defined by $Y_\mu s(q) = -8s(q)$, and Π_0 is projection onto the zero eigenspace of $H_m + \mu Y_\mu$. Note that the lowest mode of ϕ_1 has eigenvalue $10\mu \pm 8\mu \geq 2\mu$ under $H_m + \mu Y_\mu$. To obtain our desired bound on $\|Q(\rho_1 u)\|^2$, we'll need to analyze:

$$
\|Q(\rho_1 u)\|^2 = \|Q\phi_0\|^2 + \|Q\phi_1\|^2 + 2(Q\phi_0, Q\phi_1). \tag{2.2.51}
$$

The first term in (2.2.51) will be computed in section (2.7). It will be shown that

$$\|Q\phi_0\|^2 = (Q\phi_0, Q\phi_0) \tag{2.2.52}$$

$$= \left(Q^2\phi_0, \phi_0\right) \tag{2.2.53}$$

$$\geq \left\| \frac{(3-\epsilon)\,\phi_0}{\mu} \right\|^2 . \tag{2.2.54}$$

Turning to the next term in (2.2.51):

$$\begin{aligned}
\|Q\phi_1\|^2 &= (Q\phi_1, Q\phi_1) \\
&= \left(Q^2\phi_1, \phi_1\right) \\
&= ((H_m + \mu Y_\mu)\,\phi_1, \phi_1) - \left(\frac{\partial^2}{\partial\mu^2}\phi_1, \phi_1\right) + \left(\frac{2}{\mu^2} y_2^i \frac{\partial}{\partial y_2^i}\phi_1, \phi_1\right) \\
&- \left(\frac{8}{\mu}\frac{\partial}{\partial\mu}\phi_1, \phi_1\right) - \left(\frac{1}{\mu^2}\left(X^2 + \sum V_j^2\right)\phi_1, \phi_1\right) \\
&+ \left(\frac{2y_2^j}{\mu^3} X V_j \phi_1, \phi_1\right) + \left(\frac{3y_2^i y_2^j}{\mu^4} V_i V_j \phi_1, \phi_1\right) \\
&+ \left(\left(x_3^9\right)^2 |y_2|^2 \phi_1, \phi_1\right) + \left(i x_3^9 \left(\psi_{1s}\psi_{2s} + \chi_1^i \chi_2^i\right)\phi_1, \phi_1\right) + \left(H_F'\left(y_2^i\right)\phi_1, \phi_1\right).
\end{aligned}$$

We first note that the oscillator part of Q^2 yields a large, positive contribution:

$$\begin{aligned}
\int ((H_m + \mu Y_\mu)\,\phi_1, \phi_1) &\geq c_1 \int \mu\,(\phi_1, \phi_1) \\
&= c_1 \left\|\mu^{1/2}\phi_1\right\|^2 .
\end{aligned} \tag{2.2.55}$$

Because of this large contribution, we need only show that the rest of the terms are easily absorbed by (2.2.55). Next we analyze the action of the y_2 derivative terms:

$$\left(\frac{2}{\mu^2}y_2^i\frac{\partial}{\partial y_2^i}\phi_1,\phi_1\right) = \left(\frac{1}{\mu^2}y_2^i\frac{\partial}{\partial y_2^i}\phi_1,\phi_1\right) + \left(\phi_1,\left(-\frac{7}{\mu^2}-\frac{1}{\mu^2}y_2^i\frac{\partial}{\partial y_2^i}\right)\phi_1\right)$$

$$= -\left(\frac{7}{\mu^2}\phi_1,\phi_1\right)$$

$$= -7\left\|\frac{\phi_1}{\mu}\right\|^2.$$

The next derivative terms, $\left(-\frac{1}{\mu^2}X^2\phi_1,\phi_1\right)$ and $\left(-\frac{1}{\mu^2}V_j^2\phi_1,\phi_1\right)$ are positive. XV_j, V_iV_j enter with coefficients lower than $1/\mu^2$ and so are absorbed into the larger positive terms. The potential term is clearly nonnegative. The last remaining terms in Q^2 are:

$$i\gamma^i_{\alpha\beta}y_2^i\psi_{1\alpha}\psi_{3\beta} + ix_3^9\left(-\psi_{1s}\psi_{2s}+\chi_1^i\chi_2^i\right).$$

Recall that ϕ_1 is supported in a 'small' neighborhood of a flat point where $\left(x_3^9\right)^2 + |y_2|^2 < r^\epsilon$ For $\epsilon < 1$, these terms may be absorbed into (2.2.55). We have now shown that the second estimate holds:

$$\|Q\phi_1\|^2 \geq c_1\left\|\mu^{1/2}\phi_1\right\|^2.$$

The last estimate we need is the cross term, $(Q\phi_0, Q\phi_1) = (Q^2\phi_0,\phi_1)$. To control this term, we 'borrow' from the previous two terms as follows:

$$\left(Q^2\phi_0,\phi_0\right) = (1-\varepsilon)\left(Q^2\phi_0,\phi_0\right) + \varepsilon\left(Q^2\phi_0,\phi_0\right) \tag{2.2.56}$$

$$\geq (1-\varepsilon)\left\|\frac{3\phi_0}{\mu}\right\|^2 + \varepsilon\left(Q^2\phi_0,\phi_0\right) \tag{2.2.57}$$

and similarly for ϕ_1. First we control the second order derivative term in μ. Define:

$$h = \mu^2 e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \tag{2.2.58}$$

so that

$$\phi_0 = hf_0. \tag{2.2.59}$$

We compute:

$$\left(\frac{\partial^2}{\partial\mu^2}\left(hf_0\right), \phi_1\right) = \left(\left(\frac{\partial^2}{\partial\mu^2}\left(h\right)f_0, \phi_1\right) + 2\frac{\partial h}{\partial\mu}\frac{\partial f_0}{\partial\mu} + h\frac{\partial^2}{\partial\mu^2}f_0, \phi_1\right) \tag{2.2.60}$$

$$= \left(\left(\frac{\partial^2}{\partial\mu^2}\left(h\right)\right)f_0, \phi_1\right) + 2\left(\frac{\partial h}{\partial\mu}\frac{\partial f_0}{\partial\mu}, \phi_1\right) \tag{2.2.61}$$

$$= \left(\left(\frac{\partial^2}{\partial\mu^2}\left(h\right)\right)f_0, \phi_1\right) + 2\left(\frac{\partial}{\partial\mu}\left(\frac{\partial h}{\partial\mu}f_0\right) - 2\frac{\partial^2}{\partial\mu^2}hf_0, \phi_1\right). \tag{2.2.62}$$

Now:

$$\frac{\partial h}{\partial\mu} = \left(\frac{2}{\mu} - \frac{1}{2}\left(\left(x_3^9\right)^2 + \left|y_2\right|^2\right)\right)h \text{ and} \tag{2.2.63}$$

$$\frac{\partial^2}{\partial\mu^2}h = \left(\left(\frac{2}{\mu} - \frac{1}{2}\left(\left(x_3^9\right)^2 + \left|y_2\right|^2\right)^2 - \frac{2}{\mu^2}\right)\right)h, \tag{2.2.64}$$

so that:

$$\left(\left(\frac{\partial^2}{\partial\mu^2}(h)\right)f_0,\phi_1\right)+2\left(\frac{\partial}{\partial\mu}\left(\frac{\partial h}{\partial\mu}f_0\right)-\left(\frac{\partial^2}{\partial\mu^2}h\right)f_0,\phi_1\right)$$

$$=-\left(\left(\frac{\partial^2}{\partial\mu^2}(h)\right)f_0,\phi_1\right)+2\left(\frac{\partial}{\partial\mu}\left(\frac{\partial h}{\partial\mu}f_0\right),\phi_1\right)$$

$$=\left(\left(\frac{2}{\mu}-\frac{1}{2}\left(\left(x_3^9\right)^2+|y_2|^2\right)\right)^2hf_0,\phi_1\right)$$

$$+2\left(\frac{\partial}{\partial\mu}\left(\frac{2}{\mu}-\frac{1}{2}\left(\left(x_3^9\right)^2+|y_2|^2\right)hf_0\right),\phi_1\right)$$

$$=\left(\left(\frac{2}{\mu}-\frac{1}{2}\left(\left(x_3^9\right)^2+|y_2|^2\right)\right)^2hf_0,\phi_1\right)$$

$$+2\left(\left(\frac{2}{\mu}-\frac{1}{2}\left(\left(x_3^9\right)^2+|y_2|^2\right)\right)\frac{\partial}{\partial\mu}(hf_0),\phi_1\right)$$

$$+2\left(\left(\frac{-2}{\mu^2}\right)hf_0,\phi_1\right) \tag{2.2.65}$$

$$=\left(\left(-\frac{2}{\mu}\left(\left(x_3^9\right)^2+|y_2|^2\right)+\frac{1}{4}\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\right)hf_0,\phi_1\right)$$

$$+2\left(\left(\frac{2}{\mu}-\frac{1}{2}\left(\left(x_3^9\right)^2+|y_2|^2\right)\right)\frac{\partial}{\partial\mu}(hf_0),\phi_1\right)$$

$$=-\left(\frac{2}{\mu}\left(\left(x_3^9\right)^2+|y_2|^2\right)\phi_0,\phi_1\right)+\frac{1}{4}\left(\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\phi_0,\phi_1\right)$$

$$+4\left(\frac{1}{\mu}\frac{\partial\phi_0}{\partial\mu},\phi_1\right)-\left(\left(\left(x_3^9\right)^2+|y_2|^2\right)\frac{\partial\phi_0}{\partial\mu},\phi_1\right). \tag{2.2.66}$$

Each of these terms may be absorbed into the positive terms in $\|Q\phi_0\|^2$ and

$\|Q\phi_1\|^2$ as follows:

$$\left|\left(\frac{2}{\mu}\left(\left(x_3^9\right)^2+|y_2|^2\right)\phi_0,\phi_1\right)\right| = \left|\left(\frac{2}{\mu}\left(\left(x_3^9\right)^2+|y_2|^2\right)\phi_0,\phi_1\right)\right| \tag{2.2.67}$$

$$\leq \left|\left(\frac{2}{\mu^{1+\epsilon/4}}\phi_0,\mu^{5\epsilon/4}\phi_1\right)\right| \tag{2.2.68}$$

$$\leq \left\|\frac{2}{\mu^{1+\epsilon/4}}\phi_0\right\|\left\|\mu^{5\epsilon/4}\phi_1\right\| \tag{2.2.69}$$

$$\leq \frac{1}{2}\left\|\frac{2}{\mu^{1+\epsilon/4}}\phi_0\right\|^2+\frac{1}{2}\left\|\mu^{5\epsilon/4}\phi_1\right\|^2. \tag{2.2.70}$$

The first term may be absorbed into $\left\|\frac{\phi_0}{\mu}\right\|^2$. The latter term above may be absorbed into $\left\|\mu^{1/2}\phi_1\right\|^2$ for $\frac{5}{4}\epsilon < 1/2$. The size of the next term (dropping the irrelevant constant) is:

$$\left|\left(\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\phi_0,\phi_1\right)\right| = \left|\left(\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\phi_0,\phi_1\right)\right| \tag{2.2.71}$$

$$\leq \frac{\varepsilon}{2}\left\|\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\phi_0\right\|^2+\frac{1}{2\varepsilon}\|\phi_1\|^2 \tag{2.2.72}$$

$$\leq \frac{c\varepsilon}{2}\left\|\frac{\phi_0}{\mu^2}\right\|^2+\frac{1}{2\varepsilon}\|\phi_1\|^2. \tag{2.2.73}$$

Next we have:

$$\left|\left(\frac{1}{\mu}\frac{\partial\phi_0}{\partial\mu},\phi_1\right)\right| \leq \frac{\varepsilon}{2}\left\|\frac{1}{\mu}\frac{\partial\phi_0}{\partial\mu}\right\|^2+\frac{1}{2\varepsilon}\|\phi_1\|^2. \tag{2.2.74}$$

The first term may be absorbed into the $\left\|\frac{\partial\phi_0}{\partial\mu}\right\|^2$ term in $\|Q\phi_0\|^2$ and the second once again may be absorbed into $\left\|\mu^{1/2}\phi_1\right\|^2$.

The last piece coming from second order derivatives in μ is:

$$\left|\left(\left(\left(x_3^9\right)^2+|y_2|^2\right)\frac{\partial\phi_0}{\partial\mu},\phi_1\right)\right| \leq \left|\left(\mu^{-\epsilon/4}\frac{\partial\phi_0}{\partial\mu},\mu^{5/4\epsilon}\phi_1\right)\right| \tag{2.2.75}$$

$$\leq \frac{1}{2}\left\|\mu^{-\epsilon/4}\frac{\partial\phi_0}{\partial\mu}\right\|^2+\frac{1}{2}\left\|\mu^{\epsilon}\phi_1\right\|^2. \tag{2.2.76}$$

All other second order terms all vanish, since:

$$\left(X^2\phi_0,\phi_1\right)=-\left(X\phi_0,X\phi_1\right).$$

$X\phi_1$ is orthogonal to ϕ_1 and ϕ_1 lies in the orthogonal complement of the subspace generated by ϕ_0. The remaining second order terms are completely analogous. The first derivative term in μ yields:

$$\begin{aligned}\left|\left(\frac{8}{\mu}\frac{\partial}{\partial\mu}\phi_0,\phi_1\right)\right| &= \left|\left(\frac{1}{\mu}\left(\frac{16}{\mu}-4|y_2|^2+\left(x_3^9\right)^2\right)\mu^2 e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}f_0(\mu,q),\phi_1\right)\right| \\ &= \left|\left(\frac{4}{\mu}\left(|y_2|^2+\left(x_3^9\right)^2\right)\mu^2 e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}f_0(\mu,q),\phi_1\right)\right| \\ &\leq \left\|\frac{f_0(\mu,q)}{\mu^2}\right\|\|\phi_1\| \\ &\leq \left\|\frac{\phi_0}{\mu^2}\right\|\|\phi_1\|.\end{aligned}$$

We now compute the effect of the y derivatives:

$$\begin{aligned}\left|\left(\frac{2}{\mu^2}y_2^i\frac{\partial}{\partial y_2^i}\phi_0,\phi_1\right)\right| &= \left|\left(\frac{2}{\mu^2}|y_2|^2\mu^2 e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}f_0(\mu,q),\phi_1\right)\right| \\ &\leq \left\|\frac{2}{\mu^2}|y_2|^2\mu^2 e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}f_0(\mu,q)\right\|\|\phi_1\| \\ &\leq \left\|\frac{f_0(\mu,q)}{\mu^4}\right\|\|\phi_1\| \\ &= \left\|\frac{\phi_0}{\mu^4}\right\|\|\phi_1\|.\end{aligned}$$

The last two terms are easily absorbed into $\left\|\frac{\phi_0}{\mu}\right\|^2$ and $\left\|\mu^{1/2}\phi_1\right\|^2$. Next we see:

$$\begin{aligned}
&\left|\left(\left(i\gamma^i_{\alpha\beta}y^i_2\psi_{1\alpha}\psi_{3\beta} + ix^9_3\left(-\psi_{1s}\psi_{2s} + \chi^i_1\chi^i_2\right)\right)\phi_0, \phi_1\right)\right| \\
&\leq \frac{1}{2}\left\|\left(i\gamma^i_{\alpha\beta}y^i_2\psi_{1\alpha}\psi_{3\beta} + ix^9_3\left(-\psi_{1s}\psi_{2s} + \chi^i_1\chi^i_2\right)\right)\phi_0\right\|^2 + \frac{1}{2}\left\|\phi_1\right\|^2 && (2.2.77)\\
&\leq \frac{c}{2}\left\|\left(\left(x^9_3\right)^2 + |y_2|^2\right)\phi_0\right\|^2 + \frac{1}{2}\left\|\phi_1\right\|^2 && (2.2.78)\\
&\leq \frac{\tilde{c}}{2}\left\|\frac{\phi_0}{\mu^2}\right\|^2 + \frac{1}{2}\left\|\phi_1\right\|^2. && (2.2.79)
\end{aligned}$$

The only remaining term in Q^2 left to analyze is the $\left(x^9_3\right)^2|y_2|^2$, but that term is handled analogously to the above, so we do not repeat the computations. We now wish to return to (2.2.31). The term we need to be able to control is:

$$\frac{2\varepsilon+1}{2\varepsilon}\left\||d\rho_1|\,|u|\right\|^2. \qquad (2.2.80)$$

Let C_ρ be such that:

$$|d\rho_1| \leq C_\rho. \qquad (2.2.81)$$

We estimate:

$$
\begin{aligned}
\|d\rho_1 u\|^2 &\leq C_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left|\phi_0+\phi_1\right|^2 dwd\mu dq \\
&= \tilde{C}_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left[\left|\phi_0\right|^2+\left|\phi_1\right|^2\right] dwd\mu dq \\
&\leq \tilde{C}_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left[\left|f_0\left(\mu,q\right)\right|^2 \mu^4 e^{-\mu\left(|w|^2\right)}+\left|\phi_1\right|^2\right] dwd\mu dq \\
&\leq \tilde{C}_\rho \int\limits_{|w|>\mu^{\epsilon/2}} \mu^{11} \left|f_0\left(\mu,q\right)\right|^2 e^{-\mu|w|^2} dwd\mu dq \\
&+ \tilde{C}_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left|\phi_1\right|^2 dwd\mu dq \\
&\leq \tilde{C}_\rho \int\limits_{|w|>\mu^{\frac{1+\epsilon}{2}}} \mu^7 \left|f_0\left(\mu,q\right)\right|^2 e^{-|w|^2} dwd\mu dq \\
&+ \tilde{C}_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left|\phi_1\right|^2 dwd\mu dq \\
&\leq \tilde{C}_\rho \int \left|f_0\left(\mu,q\right)\right|^2 \mu^7 \mu^{3(1+\epsilon)} e^{-\mu^{1+\epsilon}} d\mu dq \\
&+ \tilde{C}_\rho \int\limits_{\mu^{\epsilon/2}<|w|<\mu^\epsilon} \mu^7 \left|\phi_1\right|^2 dwd\mu dq.
\end{aligned}
\tag{2.2.82}
$$

The ϕ_1 term may obviously be absorbed into $\left\|\mu^{1/2}\phi_1\right\|^2$. All that remains to be shown is that:

$$
c_0 \left\|\frac{\phi_0}{\mu}\right\|^2 \geq \frac{2\varepsilon+1}{2\varepsilon} \tilde{C}_\rho \int \left|f_0\left(\mu,q\right)\right|^2 \mu^7 e^{-\mu^{1+\epsilon}} d\mu dq.
$$

$$
\begin{aligned}
c_0 \left\| \frac{\phi_0}{\mu} \right\|^2 &= c_0 \int |f_0(\mu, q)|^2 e^{-\mu w^2} \mu^4 \mu^7 dw d\mu dq \\
&= \tilde{c}_0 \int \mu^7 |f_0(\mu, q)|^2 d\mu dq \\
&\geq \frac{2\varepsilon + 1}{2\varepsilon} \tilde{C}_\rho \int |f_0(\mu, q)|^2 \mu^7 \mu^{3(1+\epsilon)} e^{-\mu^{1+\epsilon}} d\mu dq \text{ for } \mu^{3(1+\epsilon)} >> e^{-\mu^{1+\epsilon}}.
\end{aligned}
$$

We note that c_0 can be taken arbitrarily small by restricting the support of ϕ_0 to the complement of a sufficiently large ball. So far, we have established the estimate for the first type of flat point, and shown that the error due to the cutoff may be absorbed. We now treat the second type of flat point. Once again, we use a harmonic oscillator decomposition. Let

$$
H_{x_3^9} = \Delta_{x_{1,2}} + \left(x_3^9\right)^2 \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2 \right) + i x_3^9 \left(-\psi_{1s}\psi_{2s} + \chi_1^i \chi_2^i \right) \tag{2.2.83}
$$

where $\Delta_{x_{1,2}}$ denotes the laplacian in the x_1^i, x_2^i directions.

$$
Q^2 = H_{x_3^9} - \frac{\partial^2}{\partial x_3^{9\,2}} + i\gamma^i_{\alpha\beta} x_1^i \psi_{2\alpha}\psi_{3\beta} + i\gamma^i_{\alpha\beta} x_2^i \psi_{3\alpha}\psi_{1\beta} + \frac{1}{2} \sum \left(x_1^i x_2^j - x_2^i x_1^j \right)^2 \tag{2.2.84}
$$

Note that the $x_1^i \frac{\partial}{\partial x_2^i} - x_2^i \frac{\partial}{\partial x_1^i}$ vanishes on the oscillator ground state. Thus, if gauge invariance is to be preserved, the action of $\psi_1\psi_2 + \chi_1^i\chi_2^i$ on the ground state must vanish as well. Then:

$$
H_{x_3^9} = \Delta_{x_{1,2}} + \left(x_3^9\right)^2 \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2 \right) - 2 i x_3^9 \left(\psi_{1s}\psi_{2s} \right). \tag{2.2.85}
$$

This oscillator has a zero eigenstate. We decompose a section u supported near this type of flat point:

$$
u = \phi_0 + \phi_1 \tag{2.2.86}
$$

where $\phi_0 = \frac{\left(x_3^9\right)^4}{\pi^4} f\left(x_3^9\right) e^{-\left|x_3^9\right| \frac{1}{2} \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right)}$. We will have 3 terms once again. The cross term $(Q^2\phi_0, \phi_1)$ and $(Q^2\phi_1, \phi_1)$ are handled as in the case of the previous oscillator decomposition, and those computations will not be repeated here. Now we compute:

$$\left(Q^2\phi_0, \phi_0\right) = \left(\left(-\frac{\partial^2}{\partial {x_3^9}^2} + i\gamma^i_{\alpha\beta} x_1^i \psi_{2\alpha}\psi_{3\beta} + i\gamma^i_{\alpha\beta} x_2^i \psi_{3\alpha}\psi_{1\beta} + \frac{1}{2}\sum \left(x_1^i x_2^j - x_2^i x_1^j\right)\right)^2 \phi_0, \phi_0\right) \tag{2.2.87}$$

$$= \left(\left(-\frac{\partial^2}{\partial {x_3^9}^2} + \frac{1}{2}\sum \left(x_1^i x_2^j - x_2^i x_1^j\right)\right)^2 \phi_0, \phi_0\right). \tag{2.2.88}$$

To shorten the notation a bit, define:

$$\xi^2 = \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right) \tag{2.2.89}$$

The second order derivative in x_3^9 gives:

$$\left(-\frac{\partial^2}{\partial {x_3^9}^2}\phi_0, \phi_0\right) \geq \left(\frac{\phi_0}{4\left(x_3^9\right)^2}, \phi_0\right). \tag{2.2.90}$$

Thus, we have $\|Q\phi_0\|^2 \geq \left\|\frac{\phi_0}{2r}\right\|^2$. $\square$

We now wish to extend the coercive estimate (2.2.14) to sections (in our weighted space) supported on all of $\mathbb{R}^{17}$. Suppose the estimate (2.2.14) does not hold for $u \in H_Q^1(S) \cap (\ker Q)^\perp$. That is, there exists no constant $c > 0$ such that

$$\|Qu\|^2 \geq c\left\|\frac{u}{r}\right\|^2 \tag{2.2.91}$$

for every $u \in H_Q^1(S) \cap (\ker Q)^\perp$. Recall the definition:

$$D(u, v) = (Qu, Qv)_{L^2(S)}$$

The assertion that no such c exists implies the existence of an infinite orthonormal sequence $\{u_j/r\} \subset (\ker Q)^{\perp} \cap H^1_Q(S)$ such that

$$\frac{D(u_j, u_j)}{\|\frac{u_j}{r}\|} \to 0 \Rightarrow D(u_j, u_j) \to 0.$$

We will argue that no such sequence $\{u_j/r\}$ can exist.

Proposition 2.2.7. *Let $\{u_j\}$ be a sequence in $(\ker Q)^{\perp} \cap H^1_Q(S)$ such that $\{u_j/r\}$ is L^2 orthonormal, and $\|Qu_j\|^2 \to 0$. Then $\{u_j\}$ has a subsequence which converges to 0 on compacta.*

Proof. Let $\{u_j\}$ be a sequence in $(\ker Q)^{\perp} \cap H^1_Q(S)$ such that $\{u_j/r\}$ is L^2 orthonormal. Define a smooth cutoff function ρ so that ρ is identically one on $B_R(0)$ (some $R > 0$), and vanishing on $B_{2R}(0)$ Then $\{\rho u_j\}$ is a bounded sequence in L^2. For each j, Gårding's inequality gives:

$$\|\rho u_j\|_{H^1} \leq K\left(\|Q(\rho u_j)\|_{L^2} + \|\rho u_j\|_{L^2}\right). \tag{2.2.92}$$

Since $\{Q(\rho u_j)\}$ is bounded, this means that $\{\rho u_j\}$ is bounded in H^1, so by the Rellich theorem, $\{\rho u_j\}$ has a subsequence (which we rename ρu_j) that converges to some $u_R \in L^2(S)$. Now, let $\{U_k\}$ be a compact exhaustion of $\mathbb{R}^{17}$. For each k we can find a convergent subsequence $\{\rho_k u_j\}$ that converges on U_k. From this sequence, we can find a subsequence which converges on U_{k+1}. Thus, taking a diagonal subsequence, obtain a sequence convergent on all the U_k. We rename this subsequence u_j. Since $\|Qu_j\|^2 \to 0$ as $j \to \infty$, for any $\varphi \in C^{\infty}_0(S)$, $(Qu_j, \varphi) \to 0$.

Thus:

$$\lim_{j\to\infty}(Qu_j,\varphi) = \lim_{j\to\infty}(u_j,Q\varphi) \tag{2.2.93}$$

$$= \lim_{j\to\infty}(u_j,Q\varphi) \tag{2.2.94}$$

$$= \lim_{j\to\infty}(u,Q\varphi) \tag{2.2.95}$$

$$= \lim_{j\to\infty}(Qu,\varphi) \tag{2.2.96}$$

$$= 0. \tag{2.2.97}$$

So $u \in \ker(Q)$ weakly. Elliptic regularity implies u is smooth, so $u \in \ker(Q)$. Now, since $\{u_j\} \subset (\ker(Q))^{\perp}$, we have:

$$(u_j,u) = 0 \tag{2.2.98}$$

$$= \lim_{j\to\infty}(u_j,u) \tag{2.2.99}$$

$$= (u,u) \tag{2.2.100}$$

$$= 0. \tag{2.2.101}$$

□

Proposition 2.2.8. $\ker Q \cap H_Q^1(S)$ *is finite dimensional.*

Proof. Suppose there exists an infinite orthonormal sequence $u_j \subset \ker Q \cap H_Q^1(S)$. Following exactly the same procedure as in the proof of the preceding proposition, taking a compact exhaustion and constructing a diagonal subsequence, we can construct a subsequence (renamed u_j) which converges on compacta to some $u \in \ker Q \cap H_Q^1(S)$. From this subsequence we can construct a new sequence with terms $u_j - u_{j+1}$ which we renormalize so that terms have L^2 norm 1. This new sequence converges (in the L^2 sense) to zero on compacta. Again, rename this sequence u_j. Let φ be a smooth cutoff function which is identically 1 on the complement of $B_{2R}(0)$, and 0 on $B_R(0)$. Then our estimate on $(B_R(0))^c$ implies:

$$\begin{aligned}\lim_{j\to\infty} D(\varphi u_j, \varphi u_j) &\geq c \lim_{j\to\infty} \left\| \frac{\varphi u_j}{r} \right\|_{L^2(S)}^2 \\ &= c \lim_{j\to\infty} \left\| \frac{u_j}{r} + (\varphi - 1)\frac{u_j}{r} \right\|_{L^2(S)}^2 \\ &\geq c.\end{aligned}$$

Since $u_j \to 0$ on compacta. But

$$\begin{aligned}\lim_{j\to\infty} D(\varphi u_j, \varphi u_j) &= \|Q(\varphi u_j)\|_{L^2(S)}^2 \\ &= \lim_{j\to\infty} \|\varphi Q u_j + [Q, \varphi] u_j\|_{L^2(S)}^2 \\ &= 0.\end{aligned}$$

□

Proposition 2.2.9. *Zero is not in the essential spectrum of* $Q_{|H_Q^1(S)}$.

Proof. Let $\{u_j\}$ be an L_Q^2-orthonormal sequence of sections in $(\ker Q)^\perp \cap H_Q^1(S)$. Suppose $\|Qu_j\|_{L^2} \to 0$. By proposition 2.2.7, we can find a subsequence (rename u_j) which converges to zero on compacta. Let φ be a smooth cutoff function which is identically 1 on the complement of $B_{2R}(0)$, and 0 on $B_R(0)$. Then our estimate on $(B_R(0))^c$ implies:

$$\begin{aligned}\lim_{j\to\infty} D(\varphi u_j, \varphi u_j) &\geq c \lim_{j\to\infty} \left\| \frac{\varphi u_j}{r} \right\|_{L^2(S)}^2 \\ &= c \lim_{j\to\infty} \left\| \frac{u_j}{r} + (\varphi - 1)\frac{u_j}{r} \right\|_{L^2(S)}^2 \\ &\geq c.\end{aligned}$$

Since $u_j \to 0$ on compacta. But

$$\lim_{j\to\infty} D(\varphi u_j, \varphi u_j) = \|Q(\varphi u_j)\|^2_{L^2(S)}$$

$$= \lim_{j\to\infty} \|\varphi Q u_j + [Q,\varphi] u_j\|^2_{L^2(S)}$$

$$= 0.$$

□

We have just shown that there exists no infinite orthonormal sequence $\{u_j\} \subset (\ker Q)^{\perp} \cap H^1_Q(S)$ such that

$$\frac{D(u_j, u_j)}{\left\|\frac{u_j}{r}\right\|} \to 0.$$

Thus, we have established our coercive estimate for all $u \in (\ker Q)^{\perp} \cap H^1_Q(S)$.

2.3 Decay Estimates

Let $L^2_r(S) = \{u \in L^2(S) : (r+1)u \in L^2(S)\}$. The results of the last section along with the Lax-Milgram theorem establish the existence of an operator $G : L^2_r(S) \to H^1_Q(S)$ such that

$$Q^2 G = I - \Pi$$

Where Π is L^2 orthogonal projection onto the kernel of Q. Although it is difficult to construct G explicitly, we may construct an approximation W to G with which an index is easily computable. We then compute:

$$\operatorname{tr}\left(\tau + \frac{1}{2}[Q, \tau Q W]\right). \tag{2.3.1}$$

It will then be necessary to show that our approximation computes the desired index. That is, we will need to show:

$$\operatorname{tr}\left(\tau + \frac{1}{2}\left[Q, \tau Q W\right]\right) = \operatorname{tr}\left(\tau + \frac{1}{2}\left[Q, \tau Q G\right]\right) \tag{2.3.2}$$

$$= L^2 - \operatorname{Index}\left(Q_+\right). \tag{2.3.3}$$

Remark 2.3.1. It is enough to show $\operatorname{tr}[Q, \tau Q(G - W)] = 0$.

Lemma 2.3.2. *Let χ_R be the characteristic function for B_R = ball of radius R. Then*

$$tr\chi_R\left[Q, \tau Q(G-W)\right] = \frac{1}{2}\int_{|x|=R} trie_n \tau Q\left(G-W\right)(x,x)\,dx.$$

We give a sketch of the proof. For full formal treatment, see [4] pp. 241-246.

Sketch of Proof.

$$\begin{aligned}
\operatorname{tr}\chi_R\left[Q, \tau Q(G-W)\right] &= \int_{|x|<R} tr\chi_R\left[Q, \tau Q(G-W)\right](x,x) \\
&= \int_{|x|<R} \frac{\partial}{\partial x^i} \operatorname{tr} i e_i \frac{1}{2} \tau Q\left(G-W\right)(x,x)\,dx \\
&= \frac{1}{2}\int_{|x|=R} \operatorname{tr} i e_n \tau Q\left(G-W\right)(x,x)\,dx \qquad (2.3.4)
\end{aligned}$$

□

Corollary 2.3.3. *If $Q(G-W)(x,x)$ is decaying more rapidly than $1/|x|^{17}$, then $tr\chi_R\left[Q, \tau Q(G-W)\right] \to 0$ as $R \to \infty$.*

Let

$$F(x,y) = (G-W)(x,y). \tag{2.3.5}$$

Then we define the kernel $e(x,y)$ via:

$$Q^2F(x,y) = e(x,y). \tag{2.3.6}$$

Using the methods of [12], we can relate our main estimate (2.2.14) and decay of the kernel e to decay of F.

Theorem 2.3.4. *Let S be a smooth, finite rank spinor bundle over $\mathbb{R}^n$, $n > 2$ and let $Q : L^2(S) \to L^2(S)$ be a Dirac operator on S. Define Ω to be the complement in $\mathbb{R}^n$ of the ball about the origin of radius $R > 0$. Suppose there exists $c \in \mathbb{R}^+$ so $\|Qf\|^2 \geq \left\|c\frac{f}{r}\right\|^2$ for all f with $\operatorname{supp}(f) \subset \Omega$. Then for $F \in H^1_Q(S)$ such that $Q^2F = e$ with $\frac{1}{\epsilon}\left(\|r^{c+1}\ln(r)\rho e\| + \left\|\frac{r^{(c-1)}\rho}{\ln(r)^{\epsilon+1/2}}|F|\right\|^{-1} C_{\rho,\rho'}\right) \in L^2(S)$:*

$$\frac{r^{c-1}}{(ln(r))^{\epsilon}}F \in L^2(S) \quad and \quad \frac{r^c}{(ln(r))^{\epsilon}}QF \in L^2(S) \quad \forall \epsilon > 0.$$

Proof. First note that since $F \in H^1_Q(S)$, we may write F as a limit of a sequence of smooth, compactly supported sections, F_n which converge to F in the H^1_Q norm. Define

$$r_m = \begin{cases} r & \text{for } r \leq m \\ m & \text{for } r > m \end{cases}.$$

For ρ a smooth cutoff function such that for some fixed R:

$$\rho(r) = \begin{cases} 0 & r < R \\ 1 & r > 2R \end{cases}. \tag{2.3.7}$$

Let

$$u_m = \rho\frac{r_m^c}{(\ln(r_m))^{\epsilon}}. \tag{2.3.8}$$

For fixed m, n, we have:

$$\begin{aligned}
\left(Q^2F, u_m^2F_n\right) &= \left(QF, Q\left(u_m^2F_n\right)\right)\\
&= \left(QF, C\left(d\left(u_m^2\right)\right)F_n + u_m^2QF_n\right)\\
&= \left(QF, 2u_mC\left(du_m\right)F_n + u_m^2QF_n\right)\\
&= \left(QF, u_mC\left(du_m\right)F_n + u_mC\left(du_m\right)F_n + u_m^2QF_n\right)\\
&= \left(u_mQF, C\left(du_m\right)F_n + C\left(du_m\right)F_n + u_mQF_n\right)\\
&= \left(Q\left(u_mF\right) - C\left(du_m\right)F, Q\left(u_mF_n\right) + C\left(du_m\right)F_n\right)
\end{aligned}$$

where $C\left(du_m\right)$ denotes Clifford multiplication by du_m. Taking $n \to \infty$, we have:

$$\left(Q^2F, u_m^2F\right) = \||Q\left(u_mF\right)|\|^2 - \|\,|du_m|\,|F|\,\|^2. \tag{2.3.9}$$

Applying the assumed estimate yields:

$$\left\|\frac{cu_mF}{r}\right\|^2 \leq \||C\left(du_m\right)F|\|^2 + \left(u_m^2e, F\right). \tag{2.3.10}$$

Thus:

$$\left\|\frac{c\rho r_m^{c-1}|F|}{\left(\ln\left(r_m\right)\right)^{\epsilon}}\right\|^2 \leq \left\|\frac{\left(r_m^{c-1}\left(\rho\left(c - \frac{\epsilon}{\ln\left(r_m\right)}\right) + r\,|d\rho|\right)\right)|F|}{\left(\ln\left(r_m\right)\right)^{\epsilon}}\right\|^2 + \left(u_m^2e, F\right) \tag{2.3.11}$$

$$\Rightarrow \int_{\mathbb{R}^n} \frac{c^2\rho^2r_m^{2(c-1)}|F|^2}{\left(\ln\left(r_m\right)\right)^{2\epsilon}} \leq \int_{\mathbb{R}^n} \frac{\left(r_m^{2(c-1)}\left(\rho\left(c - \frac{\epsilon}{\ln\left(r_m\right)}\right)\right)\right)^2|F|^2}{\left(\ln\left(r_m\right)\right)^{2\epsilon}} + \left(u_m^2e, F\right) + C_{\rho,\rho'}, \tag{2.3.12}$$

where the last term on the right comes from collecting all terms with a $|d\rho|$ factor and integrating over the compact set $(B_{2R}(0))^c \cap B_R(0)$.

$$\Rightarrow \int_{\mathbb{R}^n} \frac{\rho^2 r_m^{2(c-1)}}{\ln(r_m)^{2\epsilon+1}} (2c) |F|^2 \le \int_{\mathbb{R}^n} \frac{\rho^2 \epsilon}{\ln(r_m)^{2\epsilon+2}} |F|^2 + \frac{1}{\epsilon} \left(\left(u_m^2 e, F \right) + C_{\rho,\rho'} \right) \tag{2.3.13}$$

Since the integral on the right has an extra $\ln(r)$ term in the denominator, it may be absorbed into the left hand side. Passing to the limit, we have:

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{r_m^{2(c-1)} \rho^2}{\ln(r_m)^{2\epsilon+1}} (2c) |F|^2 \le \lim_{m\to\infty} \frac{1}{\epsilon} \left(\left(u_m^2 e, F \right) + C_{\rho,\rho'} \right)$$

$\Rightarrow$

$$\int_{\mathbb{R}^n} \frac{r^{2(c-1)} \rho^2}{\ln(r)^{2\epsilon+1}} (2c) |F|^2 \le \frac{1}{\epsilon} \left(\left(\frac{r^{2c} \rho^2}{(\ln(r))^{2\epsilon}} e, F \right) + C_{\rho,\rho'} \right) \tag{2.3.14}$$

$$\le \frac{1}{\epsilon} \left(\left(r^{c+1} \rho e, \frac{r^{c-1} \rho}{(\ln(r))^{2\epsilon}} F \right) + C_{\rho,\rho'} \right) \tag{2.3.15}$$

$$\le \frac{1}{\epsilon} \left(\left\| r^{c+1} \frac{\rho e}{(\ln(r))^{\epsilon-1/2}} \right\| \left\| \frac{r^{c-1} \rho}{(\ln(r))^{\epsilon+1/2}} F \right\| + C_{\rho,\rho'} \right) \tag{2.3.16}$$

$\Rightarrow$

$$(2c) \left\| \frac{r^{(c-1)} \rho}{\ln(r)^{\epsilon+1/2}} |F| \right\| \le \frac{1}{\epsilon} \left(\left\| r^{c+1} \frac{\rho e}{(\ln(r))^{\epsilon-1/2}} \right\| + \left\| \frac{r^{(c-1)} \rho}{(\ln(r))^{\epsilon+1/2}} |F| \right\|^{-1} C_{\rho,\rho'} \right) \tag{2.3.17}$$

By hypothesis, the right hand side is in $L^2(S)$, so the left hand side is bounded for all $\epsilon > 0$. $\square$

Note that if we exclude the flat regions and consider only sections supported where

$$\sum_{i,j} \left| \left[x^i, x^j \right] \right|^2 > r^{2+\epsilon} \tag{2.3.18}$$

we have an even stronger estimate. Denote:

$$\Omega_{\text{non-flat}} = \mathbb{R}^{17} \backslash B_R(0) \cap \left\{ x : \sum_{i,j} \left| \left[x^i, x^j \right] \right|^2 > r^{2+\epsilon} \right\} \tag{2.3.19}$$

Recall that H_F is linear in the coordinates x^i. Therefore, for $f \in C_0^\infty(S, \Omega_{\text{non-flat}})$:

$$\|Qf\|^2 = \left(\left(\Delta + \sum_{i,j} \left| \left[x^i, x^j \right] \right|^2 + H_F \right) f, f \right) \tag{2.3.20}$$

$$\geq \|\nabla f\|^2 + c \left\| r^{1+\frac{\epsilon}{2}} f \right\|^2 \tag{2.3.21}$$

$$\geq \tilde{c} \|f\|^2 \tag{2.3.22}$$

for some $c \in \mathbb{R}^+$. Let $F \in H^1 \left(S, \left\{ x : \sum_{i,j} \left| [x^i, x^j] \right|^2 > r^{2+\epsilon} \right\} \right)$ satisfy $Q^2 F = e$.

Choose a weight function $u = \rho r^\alpha$, with ρ defined as above. We have

$$\tilde{c} \|uF\|^2 \leq \|duF\|^2 + \left(u^2 e, F \right) \tag{2.3.23}$$

$$\leq \left\| \left(d\rho r^\alpha + \alpha \rho r^{\alpha-1} \right) F \right\|^2 + \left(u^2 e, F \right) \tag{2.3.24}$$

$$\leq \left\| \left(d\rho r^\alpha + \alpha \rho r^{\alpha-1} \right) F \right\|^2 + (ue, uF) \tag{2.3.25}$$

$$\leq \left\| \left(d\rho r^\alpha + \alpha \rho r^{\alpha-1} \right) F \right\|^2 + \frac{1}{2} \left(\|ue\|^2 + \|uF\|^2 \right) \tag{2.3.26}$$

$$\leq C_{\rho, d\rho} + \left\| \alpha \rho r^{\alpha-1} F \right\|^2 + \frac{1}{2} \left(\|ue\|^2 + \|uF\|^2 \right) \tag{2.3.27}$$

where $C_{\rho,d\rho}$ is a constant depending on $\rho, d\rho$ and the norm of $F_{|_{(B_R(0))^c \cap B_{2R}(0)}}$.

Rearranging terms:

$$\left(\tilde{c} - \frac{1}{2} \right) \|uF\|^2 - \left\| \alpha \frac{u}{r} F \right\|^2 \leq C_{\rho,d\rho} + \frac{1}{2} \|ue\|^2 . \tag{2.3.28}$$

Note that $\tilde{c}$ is proportional to $R^{1+\epsilon}$, and we may choose R arbitrarily large. Thus, for R large enough

$$\tilde{\tilde{c}} \|\rho r^\alpha F\|^2 \leq C_{\rho,d\rho} + \frac{1}{2} \|\rho r^\alpha e\|^2 \tag{2.3.29}$$

for any finite α. Thus, away from flat points, the L^2 decay of F is at least as good as the decay of e. We have just shown:

Lemma 2.3.5. *Let Q, S be as defined in 2.3.4. Suppose $f \in L^2(S)$ such that $supp(f) \subset \Omega_{non\text{-}flat}$. Then if*

$$Q^2 f = e \tag{2.3.30}$$

with $r^\alpha e \in L^2(S)$, we have $r^\alpha f \in L^2(S)$, for any $\alpha > 0$.

2.4 Averaging Over the Gauge Group

In the computations that follow, it will be necessary to account for the gauge symmetry. We will do this by averaging the trace over the gauge group. Recall the gauge action:

$$\mathscr{G} = x_1^i \frac{\partial}{\partial x_2^i} - x_2^i \frac{\partial}{\partial x_1^i} + \psi_{1\alpha}\psi_{2\alpha} + \chi_1^i \chi_2^i. \tag{2.4.1}$$

To compute the trace of an operator $O : L^2(S) \to L^2(S)$ over gauge invariant states, we average over the group. Let $\mathscr{R}_\theta$ denote the gauge rotation in θ. Then:

$$\mathrm{tr} O = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_M e^{\theta\left(\psi_{1\alpha}\psi_{2\alpha} + \chi_1^i\chi_2^i\right)} O(x, \mathscr{R}_\theta x)\, \mathrm{d}x \mathrm{d}\theta. \tag{2.4.2}$$

At a flat point, the gauge symmetry is reduced from S^1 to $\mathbb{Z}_2$. To see this, note that if the commutator vanishes, we can simultaneously diagonalize the matrices x^i:

$$\begin{pmatrix} x_1^i & x_2^i \\ x_2^i & -x_1^i \end{pmatrix} \mapsto \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix} \tag{2.4.3}$$

This means we can choose a basis where the x_2^i all vanish. Using a gauge action we can put our coordinates in the form:

$$\begin{pmatrix} \lambda_i \\ 0 \end{pmatrix} \tag{2.4.4}$$

A rotation then takes:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \lambda_i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta)\lambda_i \\ \sin(\theta)\lambda_i \end{pmatrix} \tag{2.4.5}$$

$$= \pm \begin{pmatrix} \lambda_i \\ 0 \end{pmatrix} \tag{2.4.6}$$

since to preserve the form of (2.4.4), we need either $\theta = 0$ or $\theta = \pi$; so our gauge action is reduced to $\mathbb{Z}_2$.

2.5 Constructing the Parametrix

We use the procedure outlined in [11]. We construct two separate approximate inverses: one within a ball B_R and the other in the complement, and then patch the two together using cutoff functions. Inside the ball B_R, we use the kernel:

$$\int_0^{\beta_0} d\beta Q e^{-\beta Q^2}(x, y). \tag{2.5.1}$$

β_0 will be given in terms of R, so that $\beta_0 \to 0$ as $R \to \infty$. The above kernel is chosen so that the computation inside B_R will be the standard computation for an operator on a compact set. Outside B_R, we construct W_∞ as an approximation to G. Define:

$$A_m(x, y) = \rho_{m,R}(x) \left(\int_0^{\beta_0} d\beta Q e^{-\beta Q^2}(x, y) \right) \rho_{m,2R}(y)$$

$$+ (1 - \rho_{m,R})(x)\, Q W_\infty(x, y) \left(1 - \rho_{m,\frac{R}{2}}\right)(y), \tag{2.5.2}$$

where $\{\rho_{m,R}\}$ denotes a sequence of cutoff functions converging to the characteristic function on a ball of radius R. The index computation is then given by:

$$\operatorname{tr}\left(\tau + \frac{1}{2}\left[Q, \tau Q W\right]\right) = \lim_{\substack{m\to\infty \\ \beta_0\to 0}} \operatorname{tr}\left(\tau + \frac{1}{2}\left[Q, A_m\right]\right). \tag{2.5.3}$$

Compute:

$$\begin{aligned}
-\left[\tau A_m\left(x,y\right), Q\right] = {} & \left[Q, \rho_{m,R}\right]\tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2} \rho_{m,2R} \\
& + \rho_{m,R}\left[Q, \tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2}\right]\rho_{m,2R} \\
& + \rho_{m,R}\tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2}\left[Q, \rho_{m,2R}\right] \\
& - \left[Q, \rho_{m,R}\right]\tau Q W_\infty \left(1 - \rho_{m,\frac{R}{2}}\right) \\
& + \left(1 - \rho_{m,R}\right)\left[Q, \tau Q W_\infty\right]\left(1 - \rho_{m,\frac{R}{2}}\right) \\
& + \left(1 - \rho_{m,R}\right)\tau Q W_\infty \left[Q, \left(1 - \rho_{m,\frac{R}{2}}\right)\right].
\end{aligned}$$

Define:

$$T_1 = \rho_{m,R}\tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2}\left[Q, \rho_{m,2R}\right] + \left(1 - \rho_{m,R}\right)\tau Q W_\infty \left[Q, \left(1 - \rho_{m,\frac{R}{2}}\right)\right].$$

We choose the right cutoffs to be identically one on the support of the left cutoffs. Thus, T_1 will trace to zero. We now have:

$$-\left[\tau A_m\left(x,y\right),Q\right]=\left[Q,\rho_{m,R}\right]\tau\int\limits_0^{\beta_0}d\beta Qe^{-\beta Q^2}\rho_{m,2R}$$

$$+\rho_{m,R}\left[Q,\tau\int\limits_0^{\beta_0}d\beta Qe^{-\beta Q^2}\right]\rho_{m,2R}$$

$$-\left[Q,\rho_{m,R}\right]\tau QW_\infty\left(1-\rho_{m,\frac{R}{2}}\right)$$

$$+\left(1-\rho_{m,R}\right)\left[Q,\tau QW_\infty\right]\left(1-\rho_{m,\frac{R}{2}}\right)+T_1$$

$$=\left[Q,\rho_{m,R}\right]\tau\int\limits_0^{\beta_0}d\beta Qe^{-\beta Q^2}\rho_{m,2R}$$

$$-2\rho_{m,R}\tau\int\limits_0^{\beta_0}d\beta Q^2e^{-\beta Q^2}\rho_{m,2R}$$

$$-\left[Q,\rho_{m,R}\right]\tau QW_\infty\left(1-\rho_{m,\frac{R}{2}}\right)$$

$$+\left(1-\rho_{m,R}\right)\left[Q,\tau QW_\infty\right]\left(1-\rho_{m,\frac{R}{2}}\right)+T_1$$

$$= [Q, \rho_{m,R}]\, \tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2} \rho_{m,2R}$$

$$- [Q, \rho_{m,R}]\, \tau Q W_\infty \left(1 - \rho_{m,\frac{R}{2}}\right) - 2\rho_{m,R} \tau e^{-\beta_0 Q^2} \rho_{m,2R}$$

$$+ (1 - \rho_{m,R}) (\tau - \tau E) \left(1 - \rho_{m,\frac{R}{2}}\right) + T_1$$

where E is the error term given by:

$$Q^2 W_\infty = I - E. \tag{2.5.4}$$

Define:

$$T_2 = -(1 - \rho_{m,R})\, \tau E. \tag{2.5.5}$$

We will construct W_∞ so that $\lim_{R \to \infty} \mathrm{tr} T_2 \to 0$. So far we have shown:

$$\begin{aligned} -[\tau A_m(x,y), Q] = {} & [Q, \rho_{m,R}]\, \tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2} \rho_{m,2R} \\ & - [Q, \rho_{m,R}]\, \tau Q W_\infty \left(1 - \rho_{m,\frac{R}{2}}\right) - 2\rho_{m,R} \tau e^{-\beta_0 Q^2} \rho_{m,2R} \\ & + (1 - \rho_{m,R}) (\tau) \left(1 - \rho_{m,\frac{R}{2}}\right) + T_1 + T_2. \end{aligned} \tag{2.5.6}$$

Returning to (2.3.1) we are left to compute:

$$\begin{aligned} \mathrm{tr}\left(\tau + \frac{1}{2}[Q, \tau Q W]\right) = {} & -\frac{1}{2} \lim_{\substack{m \to \infty \\ \beta_0 \to 0}} \mathrm{tr} \left([Q, \rho_{m,R}]\, \tau \int_0^{\beta_0} d\beta Q e^{-\beta Q^2} \right) \\ & - \frac{1}{2} \lim_{m \to \infty} \mathrm{tr} \left([Q, \rho_{m,R}]\, \tau Q W_\infty \left(1 - \rho_{m,\frac{R}{2}}\right)\right) \\ & - \lim_{\substack{m \to \infty \\ \beta_0 \to 0}} \mathrm{tr} \left(\rho_{m,R} \tau e^{-\beta_0 Q^2} \rho_{m,2R} \right) \end{aligned} \tag{2.5.7}$$

$$= \lim_{\substack{m\to\infty \\ \beta_0\to 0}} \frac{1}{2}\mathrm{tr}\left([Q,\rho_{m,R}]\,\tau \int\limits_0^{\beta_0} d\beta Q e^{-\beta Q^2} - [Q,\rho_{m,R}]\,\tau Q W_\infty + 2\rho_{m,R}\tau e^{-\beta_0 Q^2} \right)$$

$$= \lim_{\substack{m\to\infty \\ \beta_0\to 0}} \frac{1}{2}\mathrm{tr}\left([Q,\rho_{m,R}]\,\tau \int\limits_0^{\beta_0} d\beta Q e^{-\beta Q^2} - [Q,\rho_{m,R}]\,\tau Q W_\infty \right)$$

$$+ \int_{|x|<R} \mathrm{tr}\tau e^{-\beta_0 Q^2}(x,x) \tag{2.5.8}$$

We call the last term the 'principal contribution', and the remaining boundary terms the 'defect'. The contribution from these terms depends upon the manner in which β_0 is taken to zero, which we will see in a moment. We begin by analyzing the first term in (2.5.8):

$$\lim_{m\to\infty} \mathrm{tr}\left([Q,\rho_{m,R}]\,\tau \int\limits_0^{\beta_0} d\beta Q e^{-\beta Q^2} - [Q,\rho_{m,R}]\,\tau Q W_\infty \right)$$

$$= \int_{|x|=R} \left(\mathrm{tr} i e_n \tau \int\limits_0^{\beta_0} d\beta Q e^{-\beta Q^2} - \mathrm{tr} i e_n \tau Q W_\infty \right) \tag{2.5.9}$$

where e_n denotes Clifford multiplication by the outward unit normal to the boundary of B_R. The term:

$$\int\limits_{|x|=R} \mathrm{tr} i e_n \tau \int\limits_0^{\beta_0} d\beta Q e^{-\beta Q^2} \tag{2.5.10}$$

vanishes when we take $\beta_0 \to 0$ more slowly than R^{-4}.

To see this, write:

$$e^{-\beta Q^2} = \frac{-1}{2\pi i} \int\limits_\gamma e^{-\beta z} \left(Q^2 - z\right)^{-1} \mathrm{d}z \tag{2.5.11}$$

where γ is a path enclosing the spectrum of Q^2. Now let:

$$\sigma_z = 4\pi^2 |\eta|^2 + V - z \tag{2.5.12}$$

Define:

$$L = \Delta + 4\pi i \eta_i \frac{\partial}{\partial x_i} + H_F. \tag{2.5.13}$$

Approximating the inverse of $Q^2 - z$:

$$\left(Q^2 - z\right)^{-1} \approx \int e^{2\pi i \eta \cdot (x-y)} \sigma_z^{-1} \sum_{k=0}^{N} (-1)^k \left(L\sigma_z^{-1}\right) \mathrm{d}\eta \tag{2.5.14}$$

$$\stackrel{\text{def}}{=} W_z^N \tag{2.5.15}$$

$$\int_0^{\beta_0} d\beta Q \int_\gamma \frac{e^{-\beta z}}{2\pi i} \left(Q^2 - z\right)^{-1} \mathrm{d}z = \tag{2.5.16}$$

$$\int_0^{\beta_0} d\beta Q \int_\gamma \frac{e^{-\beta z}}{2\pi i} W_z^N \mathrm{d}z + \int_0^{\beta_0} d\beta Q \int_\gamma \frac{e^{-\beta z}}{2\pi i} \left(\left(Q^2 - z\right)^{-1} - W_z^N\right) \mathrm{d}z. \tag{2.5.17}$$

We will see in a moment that the first term does not contribute to the final trace for any $N \geq 1$. Recall from section 2.4, we must average the above over the gauge group.

$$\int_{|x|=R} dx \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathrm{tr} i e_n \tau e^{\theta\left(\chi_1^i \chi_2^i + \frac{1}{2}\psi_{1s}\psi_{2s}\right)} Q \int_\gamma \frac{-e^{-\beta z}}{2\pi i} W_z^N \left(x, \mathscr{R}_\theta x\right) \mathrm{d}z. \tag{2.5.18}$$

Now, recall that any term with non-zero Clifford degree vanishes, so terms in the above expression can contribute only when there are enough Clifford factors to cancel the factor of τ in the trace. One source of Clifford factors is $e_n Q$:

$$e_n Q = \frac{x_A^i}{|x|} \gamma_{r1}^i \psi_{Ar} \left(\gamma_{1s}^j \psi_{Bs} \frac{\partial}{\partial x_B^j} + \gamma_{1s}^{jk} \left[x^j, x^k\right]_B \psi_{Bs}\right). \tag{2.5.19}$$

Here we have two types of terms, one first order in $\frac{\partial}{\partial x_A^i}$ and the other zero order, each contributing either a $\psi_1\psi_2, \psi_1\psi_3$, or $\psi_2\psi_3$. There are also terms in the above which contribute no Clifford factors (when $r = s$ and $A = B$). Those terms will have a higher order of β than the terms we are about to consider, so we will put them aside for now. We first focus on terms coming from the above with Clifford degree 2. There are two other sources: H_F, and the gauge action $e^{\theta\left(\chi_1^i\chi_2^i+\frac{1}{2}\psi_{1s}\psi_{2s}\right)}$. The gauge action cannot supply all the factors on its own, since it does not contain any factors of ψ_3. Any term which does not have vanishing trace must contain some power p of H_F, where $p \geq 7$. This follows simply from a count of the Clifford factors available in any term, and the number of Clifford factors in τ. There are 8 each of the ψ_1, ψ_2 and ψ_3 and as many χ_1 and χ_2 as there are $D8$-branes in the physical model. Further, if we expand the exponential:

$$e^{\theta\left(\chi_1^i\chi_2^i+\frac{1}{2}\psi_{1s}\psi_{2s}\right)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\theta\left(\chi_1^i\chi_2^i + \frac{1}{2}\psi_{1s}\psi_{2s}\right)\right)^k \tag{2.5.20}$$

the first term that will contribute to the trace will be $\left(\theta\left(\chi_1^i\chi_2^i + \frac{1}{2}\psi_{1s}\psi_{2s}\right)\right)^q$ with $q = 11 + M - p$, where M denotes the number of $D8$-branes (and consequently the number of $\chi_1^i\chi_2^i$ terms). For later convenience, let

$$\Psi_{AB}^{ij} = \gamma_{r1}^i\psi_{Ar}\gamma_{1s}^j\psi_{Bs} \tag{2.5.21}$$

$$\tilde{\Psi}_{AB}^{ijk} = \gamma_{r1}^i\gamma_{1s}^{jk}\psi_{Ar}\psi_{Bs}. \tag{2.5.22}$$

Then:

$$e_n Q = \frac{x_A^i}{|x|}\Psi_{AB}^{ij}\frac{\partial}{\partial x_B^j} + \frac{x_A^i}{|x|}\tilde{\Psi}_{AB}^{ijk}\left[x^j, x^k\right]_B. \tag{2.5.23}$$

The coefficient of H_F^p with the lowest power of σ_z^{-1} is:

$$\int\limits_{|x|=R} \mathrm{tr} i e_n \tau \int\limits_0^{\beta_0} d\beta Q \int\limits_\gamma \frac{e^{-\beta z}}{2\pi i} \int e^{2\pi i \eta\cdot(x-y)} \sigma_z^{-p-1} H_F^p \mathrm{d}\eta \mathrm{d}z. \tag{2.5.24}$$

Computing the contour integral:

$$\int\limits_\gamma \frac{-e^{-\beta z}}{2\pi i} \int e^{2\pi i \eta\cdot(x-y)} \sigma_z^{-p-1} H_F^p \mathrm{d}\eta \mathrm{d}z = \frac{\beta^p}{p!} \int e^{-\beta\left(4\pi^2|\eta|^2+V\right)} e^{2\pi i\eta\cdot(x-y)} H_F^p \mathrm{d}\eta \tag{2.5.25}$$

Note that the higher power of σ_z^{-1}, the higher power of β produced by integrating over the contour. Now the η integral:

$$\int \frac{\beta^p}{p!} e^{-4\pi^2\beta|\eta|^2} e^{2\pi i\eta\cdot(x-y)} e^{-\beta V} H_F^p \mathrm{d}\eta = \frac{\beta^p}{p!\,(4\pi\beta)^{17/2}} e^{-\frac{|x-y|^2}{4\beta}} e^{-\beta V} H_F^p. \tag{2.5.26}$$

Substituting this result into (2.5.24):

$$\int\limits_{|x|=R} \mathrm{tr} i e_n \tau \int\limits_0^{\beta_0} d\beta Q \frac{\beta^p}{p!\,(4\pi\beta)^{17/2}} e^{-\frac{|x-y|^2}{4\beta}} e^{-\beta V} H_F^p \tag{2.5.27}$$

$$= -\int\limits_{|x|=R} \mathrm{tr}\tau\Psi_{AB}^{ij} \int\limits_0^{\beta_0} d\beta \frac{\beta^p}{p!\,(4\pi\beta)^{17/2}} \frac{x_A^i}{|x|} e^{-\frac{|x-y|^2}{4\beta}} e^{-\beta V} \left(\left(\frac{\left(x_B^j - y_B^j\right)}{2\beta} + \beta\frac{\partial V}{\partial x_B^j}\right) H_F^p \right.$$

$$\left. + p\frac{\partial H_F}{\partial x_B^j} H_F^{p-1}\right) + \int\limits_{|x|=R} \mathrm{tr}\tilde{\Psi}_{AB}^{ijk}\tau \int\limits_0^{\beta_0} d\beta \frac{\beta^p}{(4\pi\beta)^{17/2}} \frac{x_A^i}{|x|} \left[x^j, x^k\right]_B e^{-\frac{|x-y|^2}{4\beta}} e^{-\beta V} H_F^p. \tag{2.5.28}$$

Finally, we average over the gauge group:

$$-\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\int_{|x|=R}dx\int_{0}^{\beta_0}\mathrm{d}\beta\mathrm{tr}\tau\left(\Psi_{AB}^{ij}e^{\theta\left(\chi_1^i\chi_2^i+\frac{1}{2}\psi_{1s}\psi_{2s}\right)}\frac{\beta^p}{p!\left(4\pi\beta\right)^{17/2}}\frac{x_A^i}{|x|}e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}}e^{-\beta V}\right.$$

$$\left(\frac{\left(x_B^j-\mathscr{R}_\theta x_B^j\right)}{2\beta}+\beta\frac{\partial V}{\partial x_B^j}\right)H_F^p+p\frac{\partial H_F}{\partial x_B^j}H_F^{p-1}$$

$$\left.+\tilde{\Psi}_{AB}^{ijk}\frac{\beta^p}{p!\left(4\pi\beta\right)^{17/2}}\frac{x_A^i}{|x|}\left[x^j,x^k\right]_B e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}}e^{-\beta V}H_F^p\right) \quad (2.5.29)$$

$$\mathscr{R}_\theta\left(x_1^j,x_2^j\right)=\left(\cos(\theta)x_1^j+\sin(\theta)x_2^j,-\sin(\theta)x_1^j+\cos(\theta)x_2^j\right)$$

We rescale $\theta\rightarrow\beta^{3/4}\theta$, and note that:

$$\mathscr{R}_\beta^{3/4}\theta\left(x_1^j,x_2^j\right)=\left(\cos(\beta^{3/4}\theta)x_1^j+\sin(\beta^{3/4}\theta)x_2^j,-\sin(\beta^{3/4}\theta)x_1^j+\cos(\beta^{3/4}\theta)x_2^j\right). \quad (2.5.30)$$

For small β:

$$\left(\cos(\beta^{3/4}\theta)x_1^j+\sin(\beta^{3/4}\theta)x_2^j,-\sin(\beta^{3/4}\theta)x_1^j+\cos(\beta^{3/4}\theta)x_2^j\right)$$

$$=\left(x_1^j+\beta^{3/4}\theta x_2^j,-\beta^{3/4}\theta x_1^j+x_2^j\right) \quad (2.5.31)$$

and

$$\left|x-\mathscr{R}_\beta^{3/4}\theta x\right|^2\approx\sum_{j=1}^{8}\left(x_1^j-x_1^j+\beta^{3/4}\theta x_2^j\right)^2+\left(x_2^j-\beta^{3/4}\theta x_1^j+x_2^j\right)^2 \quad (2.5.32)$$

$$=\left(\beta\right)^{3/2}\theta^2\sum_{j=1}^{8}\left(x_2^j\right)^2+\left(x_1^j\right)^2 \quad (2.5.33)$$

we may write:

$$e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}}\approx e^{-\theta^2\beta^{1/2}\left(\sum_{i=1}^{8}\left(x_1^i\right)^2+\left(x_2^i\right)^2\right)} \quad (2.5.34)$$

for small β. Thus the integral (2.5.29)

$$\approx -\int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \frac{d\theta}{2\pi} \int_{|x|=R} dx \int_0^{\beta_0} \mathrm{d}\beta \mathrm{tr}\tau e^{\beta^{3/4}\theta\left(\chi_1^i\chi_2^i+\frac{1}{2}\psi_{1s}\psi_{2s}\right)} \frac{\beta^p}{p!\,(4\pi\beta)^{17/2}}$$

$$\times \frac{x_A^i}{|x|} e^{-\theta^2\beta^{1/2}\left(\sum_{i=1}^{8}\left(x_1^i\right)^2+\left(x_2^i\right)^2\right)} e^{-\beta V}$$

$$\times \left(\Psi_{AB}^{ij}\left(\left(\frac{\left(x_B^j-\mathscr{R}_\theta x_B^j\right)}{2\beta}+\beta\frac{\partial V}{\partial x_B^j}\right)H_F^p+p\frac{\partial H_F}{\partial x_B^j}H_F^{p-1}\right)\right.$$

$$\left.+\tilde{\Psi}_{AB}^{ijk}\left[x^j,x^k\right]_B H_F^p\right). \quad (2.5.35)$$

Now taking the first term in the expansion of $e^{\theta\left(\chi_i^1\chi_i^2+\frac{1}{2}\psi_{1s}\psi_{2s}\right)}$ that can contribute to the trace:

$$= -\int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \frac{d\theta}{2\pi} \int_{|x|=R} dx \int_0^{\beta_0} \mathrm{d}\beta \mathrm{tr}\tau \beta^{3q/4}\frac{\theta^q}{q!}\left(\chi_1^i\chi_2^i+\frac{1}{2}\psi_{1s}\psi_{2s}\right)^q \frac{\beta^p}{p!\,(4\pi\beta)^{17/2}}$$

$$\frac{x_A^i}{|x|} e^{-\theta^2\beta^{1/2}\left(\sum_{i=1}^{8}\left(x_1^i\right)^2+\left(x_2^i\right)^2\right)} e^{-\beta V}$$

$$\left(\Psi_{AB}^{ij}\left(\left(\frac{\left(x_B^j-\mathscr{R}_\theta x_B^j\right)}{2\beta}+\beta\frac{\partial V}{\partial x_B^j}\right)H_F^p+p\frac{\partial H_F}{\partial x_B^j}H_F^{p-1}\right)\right.$$

$$\left.+\tilde{\Psi}_{AB}^{ijk}\left[x^j,x^k\right]_B H_F^p\right). \quad (2.5.36)$$

Rescaling the above integral taking $x_A^i \to x_A^i \beta^{-1/4}$:

$$= - \int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \frac{d\theta}{2\pi} \int_{|x|=\beta^{1/4}R} dx \int_0^{\beta_0} \mathrm{d}\beta \mathrm{tr}\tau \beta^{3q/4} \frac{\theta^q}{q!} \left(\chi_1^i \chi_2^i + \frac{1}{2}\psi_{1s}\psi_{2s} \right)^q \frac{\beta^p}{p!\,(4\pi\beta)^{51/4}}$$

$$\frac{x_A^i}{|x|} e^{-\theta^2 \left(\sum_{i=1}^{8} \left(x_1^i\right)^2 + \left(x_2^i\right)^2 \right)} e^{-V} \left(\Psi_{AB}^{ij} \left(\frac{\left(x_B^j - \mathscr{R}_\theta x_B^j\right)}{2\beta^{5/4}} + \beta^{1/4} \frac{\partial V}{\partial x_B^j} \right) \beta^{-p/4} H_F^p \right.$$

$$+ \beta^{-\frac{p-1}{4}} p \frac{\partial H_F}{\partial x_B^j} H_F^{p-1}$$

$$\left. + \tilde{\Psi}_{AB}^{ijk} \left[x^j, x^k \right]_B \beta^{-p/4-1/2} H_F^p \right). \quad (2.5.37)$$

The lowest order of β above is

$$\beta^{3(p+q)/4} \beta^{-14} \quad (2.5.38)$$

Recall that $p + q = 11 + M$, where M is the number of $\chi_1^i \chi_2^i$ terms. We have:

$$\beta^{\frac{3}{4}(11+M)-14}, \quad (2.5.39)$$

and after integration over β, for $M \geq 4$, we have at least $\beta^{1/2}$. Thus, each of the terms in the integral above vanishes in $\lim_{\beta_0 \to 0}$, when the limit is taken such that $\beta_0 \to 0$ more slowly than R^{-4}. All terms not computed explicitly have higher powers of β. Briefly examining the error term in (2.5.17):

$$\int_{|x|=R} \mathrm{tr} e_n \tau \int_0^{\beta_0} d\beta Q \int_\gamma \frac{e^{-\beta z}}{2\pi i} \left(\left(Q^2 - z \right)^{-1} - W_z^N \right) \mathrm{d}z \quad (2.5.40)$$

we compute:

$$(Q^2 - z)^{-1} - W_N = (Q^2 - z)^{-1}(I - (Q^2 - z)W_N) \quad (2.5.41)$$

$$= (Q^2 - z)^{-1} E_N \quad (2.5.42)$$

and

$$Q(Q^2 - z)^{-1} - QW_N = Q(Q^2 - z)^{-1}(I - (Q^2 - z)W_N) \tag{2.5.43}$$

$$= (Q^2 - z)^{-1}QE_N. \tag{2.5.44}$$

If $\beta_0 < 1/R^4$, we have $e^{-\beta z}$ is bounded for $Imz < cR^4$. Then we make take our contour γ so that $|\mathrm{Im}z|^2 + \min(\mathrm{Re}z, 0)^2 \approx R^8$. Then:

$$\int\limits_{|x|=R} \mathrm{tr}ie_n\tau \int\limits_0^{\beta_0} d\beta \int\limits_\gamma \frac{e^{-\beta z}}{2\pi i}(Q^2 - z)^{-1}QE_N\mathrm{d}z \tag{2.5.45}$$

easily vanishes as $\beta_0 \to 0$. Now, returning to (2.5.9), we examine the remaining term. Split the region of integration into two parts: $N_F(R) = \partial B_R \cap$ {neighborhood of flat points}, (neighborhood to be the region $\sum\limits_{i,j} \left|[x^i, x^j]\right|^2 < r^{2+\epsilon}$) and its complement:

$$\int\limits_{|x|=R} \mathrm{tr}ie_n\tau QW_\infty = \int\limits_{N_F(R)} \mathrm{tr}ie_n\tau QW_\infty + \int\limits_{(N_F(R))^c} \mathrm{tr}ie_n\tau QW_\infty. \tag{2.5.46}$$

Next, we will construct W_∞ in two pieces:

$$W_\infty = W_\infty^{\text{flat}} + W_\infty^{\text{non-flat}}. \tag{2.5.47}$$

It will be shown, that for a reasonable choice of $W_\infty^{\text{non-flat}}$, this term does not contribute to the trace. We have then:

$$L^2 - \mathrm{Index}\,(Q_+) = tr\left(\tau + \frac{1}{2}[Q, \tau A]\right)$$

$$= \lim_{\substack{R\to\infty \\ \beta\to 0}} \left[\int_{|x|<R} tr\tau e^{-\beta Q^2} - \frac{1}{2}\int_{N_F(R)} \mathrm{tr}ie_n\tau QW_\infty\right], \tag{2.5.48}$$

so that (2.5.48) has the form:

$$L^2 - \mathrm{Index}\,(Q_+) = I_p + I_\delta \tag{2.5.49}$$

where I_p is the standard index computation for an operator defined over a compact manifold, and I_δ is a defect term whose only contribution is in a neighborhood of infinity near a flat point.

2.6 W_∞ in the Non-flat Region

Suppose $\sum_{i,j} \left|[x^i, x^j]\right|^2 > r^{2+\epsilon}$, for some $\epsilon > 0$. We need to construct a parametrix $W_\infty^{\text{non-flat}}$ such that the error is given as an integral operator with kernel defined by:

$$Q^2 W_\infty^{\text{non-flat}} f(x) = f(x) + E_\infty^{\text{non-flat}} f(x) \tag{2.6.1}$$

$$= f(x) + \int e_\infty^{\text{non-flat}}(x, y)\, f(y)\, \mathrm{d}y \tag{2.6.2}$$

and $e_\infty^{\text{non-flat}}(x, x)$ decays rapidly enough to imply vanishing of 2.3.4. Let

$$\sigma(\eta, x) = 4\pi^2 |\eta|^2 + V(x) \tag{2.6.3}$$

and define:

$$L = \Delta + 4\pi i \eta_i \frac{\partial}{\partial x_i} + H_F. \tag{2.6.4}$$

Then we construct:

$$W_\infty^{\text{non-flat},N} = \int e^{2\pi i \eta\cdot(x-y)} \sigma^{-1} \sum_{k=0}^{N} (-1)^k \left(L\sigma^{-1}\right) \mathrm{d}\eta \tag{2.6.5}$$

The associated error term is:

$$e_\infty^{\text{non-flat},N} = \int e^{2\pi i \eta\cdot(x-y)} (-1)^N \left(L\sigma^{-1}\right)^{N+1} \mathrm{d}\eta. \tag{2.6.6}$$

And now we ask, for

$$V = \sum_{i,j} \left|\left[x^i, x^j\right]\right|^2 \tag{2.6.7}$$

$$> r^{2+\epsilon} \tag{2.6.8}$$

what is the order of ΔV and $|\nabla V|^2$?

$$\sum_{k,C} \left| \frac{\partial V}{\partial x_C^k} \right|^2 = \sum_{k,C} \left| 2 \left[x^i, x^j \right] \frac{\partial \left[x^i, x^j \right]}{\partial x_C^k} \right|^2 \tag{2.6.9}$$

$$\leq cV(x) r^2 \tag{2.6.10}$$

The laplacian of V, on the other hand is proportional to r^2. Now if we consider $(L\sigma^{-1})^{N+1}$, we see that each time σ^{-1} is differentiated, we increase the power of σ^{-1} by 1 and multiply by a derivative of V. This has the effect of increasing the terms' decay in r by $r^{-\epsilon/2}$. Thus, the term with least decay in (2.6.6) is the term which comes from H_F^{N+1}. Estimating the η integral:

$$\int \frac{r^{N+1}}{\left(4\pi |\eta|^2 + V(x)\right)^{N+1}} \mathrm{d}\eta = \int \frac{\rho^{16} r^{N+1}}{\left(4\pi \rho^2 + V(x)\right)^{N+1}} \mathrm{d}\rho \mathrm{d}\omega \tag{2.6.11}$$

$$= C \left(V(x)\right)^{\frac{17}{2} - N - 1} r^{N+1} \tag{2.6.12}$$

$$\leq C r^{17 - 2N - 2 + \tilde{\epsilon}} r^{N+1} \tag{2.6.13}$$

$$= C r^{17 + \tilde{\epsilon} - N - 1}. \tag{2.6.14}$$

Recall from lemma 2.3.5, that away from flat points, the kernel $G - W_\infty^{\text{non-flat}}$ decays (in the L^2 sense) at least as rapidly as the error $e_\infty^{\text{non-flat}}$. If we take N large enough so that $r^\alpha e_\infty^{\text{non-flat}} \in L^2\left(S^{16}(R)\right)$, then $r^\alpha (G - W) \in L^2\left(S^{16}(R)\right)$. If we replace $G - W$ in 2.3.5 with $Q(G - W)$, we see that we have $r^{\alpha+1} Q(G - W) \in L^2\left(S^{16}(R)\right)$. The corresponding L^1 statement is:

$$\|Q(G - W)\|_{L^1(S^{16}(R))} \leq \left\| r^{\frac{17+\epsilon}{2}} Q(G - W) \right\|_{L^2} \left\| r^{-\frac{17+\epsilon}{2}} \right\|_{L^2} \tag{2.6.15}$$

$$\leq C \frac{1}{R^{1+\epsilon}}. \tag{2.6.16}$$

We need the error to have L^2 decay of $1/r^{\frac{16+\epsilon}{2}}$, so we must take $N > 24$.

2.7 W_∞ Near a Flat Point

We now explicitly construct W_∞^{flat}, the approximate Green's function for Q^2 near infinity along a flat direction. Recall that there are two different branches of flat directions, one for which the commutator term $\left|x_1^i x_2^j - x_2^i x_1^j\right|^2$ vanishes and x_3^9 vanishes, and another where all the x_1^i, x_2^i vanish and x_3^9 is arbitrary. We will see later that the latter branch is unimportant to our final computation. We begin with the former branch which requires delicate handling. Recall from (2.2.46) the previous labeling of terms in Q^2:

$$H_m = \left(-\frac{\partial^2}{\partial\left(x_3^9\right)^2} + \Delta_{y_2}\right) + \mu^2\left(\left(x_3^9\right)^2 + |y_2|^2\right)$$

$$Y_\mu = q_{1j}\gamma_{st}^j\psi_{2s}\psi_{3t}$$

$$H_F'(y_2) = q_{ij}\gamma_{st}^j\psi_{1s}\psi_{3t}y_2^i.$$

$$\begin{aligned} Q^2 &= H_m + \mu Y_\mu \\ &-\frac{\partial^2}{\partial\mu^2} + \frac{2}{\mu^2}y_2^i\frac{\partial}{\partial y_2^i} - \frac{8}{\mu}\frac{\partial}{\partial\mu} - \frac{1}{\mu^2}\left(X^2 + \sum V_j^2\right) + \left(x_3^9\right)^2|y_2|^2 \\ &+ ix_3^9\left(\psi_{1s}\psi_{2s} + \chi_1^i\chi_2^i\right) + H_F'(y_2) - 3\frac{y_2^i y_2^j}{\mu^4}V_iV_j \end{aligned} \qquad (2.7.1)$$

As before, we decompose a section supported in the neighborhood of a flat point in terms two subspaces: the zero mode of the harmonic oscillator $H_m + \mu Y_\mu$ and its complement.

$$\varphi = \Pi_0\varphi + \Pi_1\varphi \qquad (2.7.2)$$

$$= \phi_0 + \phi_1 \qquad (2.7.3)$$

where Π_0 denotes projection onto the zero eigenspace of the oscillator and $\Pi_1 = I - \Pi_0$. Once again, ϕ_0 has the form:

$$\phi_0 = f(\mu, q) \frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2 + |y_2|^2\right)/2} \tag{2.7.4}$$

where $f(\mu, q)$ is a section which satisfies:

$$(Y_\mu(q) + 8) f(\mu, q) = 0. \tag{2.7.5}$$

We can then think of $Q^2 = H$ as a matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

so that H_{11} takes the ground state oscillator back to itself, H_{21} takes the ground state to a section in the orthogonal complement and similarly for H_{12} and H_{22}. We first calculate $H_{11} = \Pi_0 H \Pi_0$. The first order derivative terms in μ gives:

$$\begin{aligned}
\Pi_0\left(-\frac{8}{\mu}\frac{\partial \phi_0}{\partial \mu}\right) &= -\frac{8}{\mu}\frac{\partial f}{\partial \mu}\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2 + |y_2|^2\right)/2} \\
&\quad - \Pi_0\left(f\frac{8}{\mu}\frac{\partial}{\partial \mu}\left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2 + |y_2|^2\right)/2}\right)\right) \qquad (2.7.6) \\
&= -\frac{8}{\mu}\frac{\partial f}{\partial \mu}\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2 + |y_2|^2\right)/2} \\
&\quad - \frac{8}{\mu} f \Pi_0\left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2 + |y_2|^2\right)/2}\left(\frac{2}{\mu} - \left(\left(x_3^9\right)^2 + |y_2|^2\right)/2\right)\right) \qquad (2.7.7)
\end{aligned}$$

and now we compute:

$$\Pi_0 \left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \left(\frac{2}{\mu} - \left(\left(x_3^9\right)^2 + |y_2|^2 \right)/2 \right) \right)$$

$$= \int \frac{\mu^4}{\pi^4} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)} \left(\frac{2}{\mu} - \left(\left(x_3^9\right)^2 + |y_2|^2 \right)/2 \right) \mathrm{d}x\mathrm{d}y \tag{2.7.8}$$

$$= \frac{2}{\mu} - \frac{1}{2\mu}\,(8/2) \tag{2.7.9}$$

$$= \frac{2}{\mu} - \frac{2}{\mu} \tag{2.7.10}$$

$$= 0, \tag{2.7.11}$$

so that the contribution from the $\frac{\partial}{\partial\mu}$ term is just:

$$-\frac{8}{\mu}\frac{\partial f}{\partial\mu}\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \tag{2.7.12}$$

Turning to the second order derivative terms in μ:

$$\Pi_0 \left(\frac{\partial^2}{\partial\mu^2}\phi_0 \right) = \Pi_0 \left(-\frac{\partial^2}{\partial\mu^2} f \right) \frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}$$

$$- \Pi_0 f \frac{\partial^2}{\partial\mu^2} \left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \right). \tag{2.7.13}$$

Computing the second term:

$$-\Pi_0 f \frac{\partial^2}{\partial\mu^2} \left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \right)$$

$$= -\Pi_0 f \frac{\partial}{\partial\mu} \left(\frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \left(\frac{2}{\mu} - \left(\left(x_3^9\right)^2 + |y_2|^2 \right)/2 \right) \right) \tag{2.7.14}$$

$$= -\Pi_0 f \frac{\mu^2}{\pi^2} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2} \left(\left(\frac{2}{\mu} - \frac{\left(\left(x_3^9\right)^2 + |y_2|^2 \right)}{2} \right)^2 + \left(-\frac{2}{\mu^2} \right) \right) \tag{2.7.15}$$

and so we need to compute:

$$\int \frac{\mu^4}{\pi^4} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)}$$

$$\left(\left(\frac{2}{\mu^2}-\frac{2}{\mu}\left(\left(x_3^9\right)^2+|y_2|^2\right)\right)+\frac{1}{4}\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\right)\mathrm{d}x\mathrm{d}y \quad (2.7.16)$$

$$= \frac{2}{\mu^2}-\frac{2\cdot 8}{2\mu^2}+\frac{1}{4}\int \frac{\mu^4}{\pi^4} e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)}\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\mathrm{d}x\mathrm{d}y$$

$$= \frac{2}{\mu^2}-\frac{8}{\mu^2}+\int \frac{1}{4\pi^4\mu^2} e^{-\left(\left(x_3^9\right)^2+|y_2|^2\right)}\left(\left(x_3^9\right)^2+|y_2|^2\right)^2\mathrm{d}x\mathrm{d}y \quad (2.7.17)$$

$$= \frac{2}{\mu^2}-\frac{8}{\mu^2}+\frac{5}{\mu^2} \quad (2.7.18)$$

$$= -\frac{1}{\mu^2} \quad (2.7.19)$$

Substituting this result into (2.7.13) yields an overall contribution of:

$$\left(-\frac{\partial^2}{\partial\mu^2}f\right)\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2\right)}+\frac{\phi_0}{\mu^2}. \quad (2.7.20)$$

Now the y derivatives:

$$\Pi_0\frac{2y_2^i}{\mu^2}\frac{\partial\phi_0}{\partial y_2^i}\phi_0 = f\Pi_0\left(\frac{2y_2^i}{\mu^2}\frac{\partial}{\partial y_2^i}\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}\right). \quad (2.7.21)$$

Computing the projection term:

$$\Pi_0\left(\frac{2y_2^i}{\mu^2}\frac{\partial}{\partial y_2^i}\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}\right) = \frac{-2}{\mu^2}\int\frac{\mu^4}{\pi^4}|y_2|^2\,e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)}\mathrm{d}x\mathrm{d}y \quad (2.7.22)$$

$$= \frac{-2}{\mu^2}\int\frac{1}{\pi^4}|y_2|^2\,e^{-\left(\left(x_3^9\right)^2+|y_2|^2\right)}\mathrm{d}x\mathrm{d}y \quad (2.7.23)$$

$$= \frac{-2\cdot 7}{2\mu^2}. \quad (2.7.24)$$

The overall contribution from the y derivative terms is:

$$= \frac{-7}{\mu^2}\phi_0. \tag{2.7.25}$$

Next we consider the angular derivatives V_j. These will act non-trivially on our section $f(\mu, q)$. First let's find a state in the kernel of $Y_\mu(q)+8$. For simplicity, we take $q = I$, so that $Y_\mu = i\gamma^1_{st}\psi_{2s}\psi_{3t}$. Note that the sum is restricted to $s = 1, ..., 8$ and $t = 9, ..., 16$ as a result of the physical model. Now we choose a basis so that γ^1 has the form:

$$\gamma^1 = \begin{pmatrix} 0 & I_8 \\ I_8 & 0 \end{pmatrix}$$

. Then we complexify the ψ's. For $\alpha = 1, ..., 8$, define:

$$b_\alpha = \frac{1}{\sqrt{2}}\left(\psi_{2\alpha} + i\psi_{3\alpha+8}\right) \tag{2.7.26}$$

$$b^\dagger_\alpha = \frac{1}{\sqrt{2}}\left(\psi_{2\alpha} - i\psi_{3\alpha+8}\right). \tag{2.7.27}$$

Then Y_μ has the form:

$$Y_\mu = -\left(b_\alpha b^\dagger_\alpha - b^\dagger_\alpha b_\alpha\right).$$

Choosing $s(q)$ to be a state annihilated by b_α for each $\alpha = 1, ..., 8$, $s(q)$ is in the kernel of $Y_\mu(I) + 8$. Therefore, we compute:

$$P(q)\sum_i V_i^2 P(q)$$

where $P(q)$ is the projection operator:

$$P(q) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z - Y_\mu(q)}dz$$

for Γ a small curve about $z = -8$. Clearly, $P(q)$ projects onto the kernel of $Y_\mu(q) + 8$. By a change of coordinates, it is enough to compute $V_i^2 P(q)$ at $q = I$. Note that $V_i Y_\mu$ takes the -8 eigenspace of Y_μ to the -4 eigenspace, since:

$$\begin{aligned} V_i Y_\mu &= i\gamma^i_{st}\psi_{2s}\psi_{3t} \\ &= i\sum_{s=1}^{8} \pm\psi_{2s}\psi_{3\sigma(s)} \\ &= \sum_{s=1}^{8} \pm\left(b_s^\dagger + b_s\right)\left(b_{\sigma(s)} - b_{\sigma(s)}^\dagger\right) \\ &= \sum_{s=1}^{8} \pm\left(b_s^\dagger b_{\sigma(s)} - b_s^\dagger b_{\sigma(s)}^\dagger + b_s b_{\sigma(s)} - b_s b_{\sigma(s)}^\dagger\right) \\ &= \sum_{s=1}^{8} \pm b_s^\dagger b_{\sigma(s)}^\dagger \quad \text{acting on the ground state,} \end{aligned} \tag{2.7.28}$$

where σ is some permutation of $\{1, ..., 8\}$ with $\sigma(s) \neq s$ for any s. The assertion $\sigma(s) \neq s$ follows from the fact that the gamma matrices $\gamma^1, ..., \gamma^8$ have no diagonal entries. Now consider Y_μ acting on $b_s^\dagger b_{\sigma(s)}^\dagger s(q)$.

$$\begin{aligned} Y_\mu V_i Y_\mu s(q) &= \sum_{\alpha=1}^{8}\sum_{s=1}^{8}\left(b_\alpha b_\alpha^\dagger - b_\alpha^\dagger b_\alpha\right) b_s^\dagger b_{\sigma(s)}^\dagger s(q) \\ &= -8 b_s^\dagger b_{\sigma(s)}^\dagger s(q) - 4\left[\left(b_\alpha b_\alpha^\dagger - b_\alpha^\dagger b_\alpha\right), b_s^\dagger b_{\sigma(s)}^\dagger\right] s(q) \\ &= -8 b_s^\dagger b_{\sigma(s)}^\dagger s(q) + 4 b_s^\dagger b_{\sigma(s)}^\dagger s(q) \\ &= -4 b_s^\dagger b_{\sigma(s)}^\dagger s(q) \end{aligned}$$

Now write:

$$P(q)\sum_i V_i^2 P(q) =$$

$$= P(q)\sum_i V_i^2 \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - Y_\mu(q)}$$

$$= P(q)\sum_i \frac{2}{2\pi i}\int_\Gamma \frac{1}{z - Y_\mu(q)} V_i Y_\mu \frac{1}{z - Y_\mu(q)} V_i Y_\mu \frac{1}{z - Y_\mu(q)} dz$$

$$+ \frac{1}{2\pi i}\int_\Gamma \frac{1}{Y_\mu + 8}\sum_i V_i^2 Y_\mu \frac{1}{Y_\mu + 8} dz. \tag{2.7.29}$$

Note that Y_μ is an eigenfunction of the Laplacian $\sum V_i^2$. The fact that matrix coefficients are eigen under Δ may be derived in general terms as a consequence of the Peter-Weyl theorem. However, in the matter at hand, we can see this a bit more directly:

Let $\{v_j\}_j$ be an orthonormal basis for a representation space of SO(8). A matrix coefficient is a function $T_{ij}(g) = \langle gv_i, v_j\rangle$., for $g \in$ SO(8). In this case, the matrix coefficients are given by the matrix q. Now we compute:

$$VT_{jk} = \left.\frac{d}{dt}\right|_{t=0} \langle g\exp(tV)v_j, v_k\rangle \tag{2.7.30}$$

$$= \langle gVv_j, v_k\rangle \tag{2.7.31}$$

$$= V_{jm}\langle gv_m, v_k\rangle \tag{2.7.32}$$

$$= V_{jm}T_{mk} \tag{2.7.33}$$

So we see that the action of the V_j on q is given simply by matrix multiplication, and

$$V_j^2 q = -q. \tag{2.7.34}$$

Thus, the last term in (2.7.29) is a simple pole of order two and integrates to zero. The remaining term yields:

$$P\left(q\right)\sum_i \frac{2}{2\pi i}\int\limits_{\Gamma} \frac{1}{z-Y_\mu(q)}V_iY_\mu\frac{1}{z-Y_\mu(q)}V_iY_\mu\frac{1}{z-Y_\mu(q)}dz$$

$$=P\left(q\right)\sum_i \left(V_iY_\mu\right)^2\frac{2}{2\pi i}\int\limits_{\Gamma}\frac{1}{(z+8)}\frac{1}{(z+4)}\frac{1}{(z+8)}dz$$

$$=-P\left(q\right)\sum_i\frac{\left(V_iY_\mu\right)^2}{8}$$

$$=\frac{1}{8}P\left(q\right)\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\alpha'\beta'}\psi_{2\alpha'}\psi_{3\beta'}\quad \text{at } q=I$$

$$=\frac{1}{8}P\left(q\right)\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}$$

$$=\frac{1}{8}\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}$$

$$=-\frac{1}{8}\gamma^i_{\alpha\beta}\gamma^i_{\alpha\beta}$$

$$=-\sum_i 1$$

$$=-7$$

and so the contribution from this term is:

$$\frac{-1}{\mu^2}\sum_{j=2}^{8}V_j^2\left(f\left(\mu,q\right)\right)\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}+\frac{7}{\mu^2}\phi_0. \tag{2.7.35}$$

The remaining derivative term is given by X^2. Setting the gauge constraint to zero allows us to replace X^2 with:

$$X^2=\frac{1}{4}\left(\psi_{1s}\psi_{2s}+\chi_1^i\chi_2^i\right)^2.$$

The terms proportional to the identity are:

$$\Pi_0 \frac{1}{\mu^2}\left(\frac{8}{4}+\frac{M}{4}\right)\phi_0 = \frac{2+\frac{M}{4}}{\mu^2}\phi_0 \tag{2.7.36}$$

where M is the number of $\chi_1^i\chi_2^i$ terms. We have cross terms coming from $\left(\chi_1^i\chi_2^i\right)^2$, which look like:

$$F = \frac{1}{4}\sum_{i\neq j}^{M}\left(\chi_1^i\chi_2^i\chi_1^j\chi_2^j\right) \tag{2.7.37}$$

We will hold on to these terms until later in the computation. For now, consider the last term in $\Pi_0 H \Pi_0$:

$$\Pi_0\left(\left(x_3^9\right)^2|y_2|^2\right)\phi_0 = \frac{\phi_0}{2\mu}\frac{7}{2\mu} \tag{2.7.38}$$

$$= \frac{7\phi_0}{4\mu^2} \tag{2.7.39}$$

Thus far, we have computed:

$$\begin{aligned}\Pi_0 H\phi_0 = &-\frac{\partial^\mu}{\partial\mu^\mu}\left(f\left(\mu,q\right)\right)\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}\\ &-\frac{8}{\mu}\frac{\partial f\left(\mu,q\right)}{\partial\mu}\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}\\ &-\frac{1}{\mu^2}V_j^2\left(f\left(\mu,q\right)\right)\frac{\mu^2}{\pi^2}e^{-\mu\left(\left(x_3^9\right)^2+|y_2|^2\right)/2}+\frac{m}{\mu^2}\phi_0\\ &+\text{ lower order terms} + F/\mu^2.\end{aligned} \tag{2.7.40}$$

The μ derivatives together with the 7 angular derivatives V_j^2 are nearly an 8 dimensional laplacian, except that the coefficient of $\frac{1}{\mu}\frac{\partial}{\partial\mu}$ is 8, rather than 7. If we redefine our sections

$$f = \frac{1}{\sqrt{\mu}}\tilde{f} \tag{2.7.41}$$

we can shift one radial derivative as follows:

$$-\left(\frac{\partial^2}{\partial\mu^2}+\frac{8}{\mu}\frac{\partial}{\partial\mu}\right)=-\left(\left(\frac{\partial}{\partial\mu}+\frac{1}{2\mu}\right)^2+\frac{7}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{4\mu^2}\right)$$

$$=-\left(\left(\frac{\partial}{\partial\mu}+\frac{1}{2\mu}\right)^2+\frac{7}{\mu}\left(\frac{\partial}{\partial\mu}+\frac{1}{2\mu}\right)+\frac{1}{4\mu^2}-\frac{7}{2\mu^2}\right)$$

$$=-\frac{1}{\mu^{1/2}}\left(\frac{\partial^2}{\partial\mu^2}+\frac{7}{\mu}\frac{\partial}{\partial\mu}-\frac{13}{4\mu^2}\right)\mu^{1/2}.$$

We can now group together the μ and V_j derivative terms into the laplacian Δ_8 (the laplacian on $\mathbb{R}^8$). Gathering results from (2.7.20),(2.7.25),(2.7.35),(2.7.36), and (2.7.39) we see that the coefficient m in (2.7.40) is:

$$m=1-7+7+2+\frac{M}{4}+\frac{7}{4} \tag{2.7.42}$$

$$=3+\frac{M}{4}+7/4. \tag{2.7.43}$$

We pick up one more $1/\mu^2$ term from shifting the radial derivative:

$$m'=3+\frac{M}{4}+7/4+13/4 \tag{2.7.44}$$

$$=8+\frac{M}{4}. \tag{2.7.45}$$

Now let's look at the terms which raise the eigenvalue of ϕ_0 under $H_m+\mu Y_\mu$:

$$H_{21}=\Pi_1\left(i\gamma^i_{\alpha\beta}y^i_2\psi_{1\alpha}\psi_{3\beta}+ix^9_3\left(-\psi_{1s}\psi_{2s}+\chi^i_1\chi^i_2\right)\right)+\text{lower order terms} \tag{2.7.46}$$

$$=\Pi_1\left(i\gamma^i_{\alpha\beta}y^i_2\psi_{1\alpha}\psi_{3\beta}-ix^9_3\psi_{1s}\psi_{2s}\right)+\Pi_1\left(ix^9_3\chi^i_1\chi^i_2\right)+\text{lower order terms} \tag{2.7.47}$$

where by 'lower order terms' we mean terms which come in with a coefficient in μ of $\leq 1/\mu^2$. There are terms which increase the eigenvalue of ϕ_0 by 4μ coming from the first expression on the right hand side and terms which increase the eigenvalue of ϕ_0 by 2μ coming from the second expression. Let b denote the sum of the first set of terms, and $\tilde{b}$ the latter. We now see that Q^2 has the following form:

$$Q^2\varphi = \begin{pmatrix} \Delta_8 + F/\mu^2 + \frac{8+M/4}{\mu^2} + \frac{c_1}{\mu^3} + \frac{c_2 V_j^2}{\mu^5} & b^t + \tilde{b}^t \\ b + \tilde{b} & H_{22} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{2.7.48}$$

where Δ_8 denotes the 8 dimensional laplacian acting on the flat directions. The following transformation allows us to express Q^2 in upper diagonal form:

$$T = \begin{pmatrix} 1 & 0 \\ \frac{b}{4\mu} + \frac{\tilde{b}}{2\mu} & 1 \end{pmatrix}. \tag{2.7.49}$$

Applying this to Q^2:

$$Q^2 T = \begin{pmatrix} \Delta_8 + F/\mu^2 + \frac{8+M/4}{\mu^2} + \frac{c_1}{\mu^3} + \frac{c_2 V_j^2}{\mu^5} & b^t + \tilde{b}^t \\ b + \tilde{b} & H_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{b}{4\mu} - \frac{\tilde{b}}{2\mu} & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{2.7.50}$$

$$= \begin{pmatrix} \Delta_8 + F/\mu^2 + \frac{8+M/4}{\mu^2} + \frac{c_1}{\mu^3} + \frac{c_2 V_j^2}{\mu^5} & b^t + \tilde{b}^t \\ b & \Delta_8 + H_m + \mu Y_\mu + \text{l.o.t.} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 \\ -\frac{b}{4\mu} - \frac{\tilde{b}}{2\mu} & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{2.7.51}$$

Note that b raises the eigenvalue of ϕ_0 by 4, and $\tilde{b}$ raises the eigenvalue of ϕ_0 by 2. Thus, excluding terms with coefficents in μ of lower order than $1/\mu^2$, we obtain the following upper triangular form:

$$= \begin{pmatrix} \Delta_8 + \frac{F}{\mu^2} + \frac{8+M/4}{\mu^2} - \frac{b^t b}{4\mu} - \frac{\tilde{b}^t \tilde{b}}{2\mu} + \frac{c_1}{\mu^3} + \frac{c_2 V_j^2}{\mu^5} & b^t + \tilde{b}^t \\ \text{l.o.t.} & \Delta_8 + H_m + \mu Y_\mu + \text{l.o.t.} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}. \tag{2.7.52}$$

To compute $-\frac{b^t b}{4\mu} - \frac{\tilde{b}^t \tilde{b}}{2\mu}$ we consider:

$$\left(\gamma^i_{\alpha\beta} y^i_2 \psi_{1\alpha}\psi_{3\beta} - i x^9_3 \psi_{1s}\psi_{2s} + i x^9_3 \chi^i_1 \chi^i_2\right)^2 \tag{2.7.53}$$

and take terms which do not contribute any ψ_2 or ψ_3 factors as such factors will raise the eigenvalue of the ground state and cannot appear in the product $b^t b$. We have 7 coordinates y_2^i, and one x_3^9, and 8 each of ψ_2, ψ_3.

$$\frac{b^t b}{4\mu} + \frac{\tilde{b}^t \tilde{b}}{2\mu} = \frac{8 \cdot 8}{2\mu 4\mu} + \frac{M}{2\mu 2\mu} + \frac{4F}{2\mu 2\mu} \tag{2.7.54}$$

$$= \frac{8 + M/4 + F}{\mu^2}. \tag{2.7.55}$$

The last term contributes $-\left(8 + M/4\right)/\mu^2 - F/\mu^2$, and all of the $\frac{1}{\mu^2}$ terms cancel. Now that we have a simplified expression for Q^2 near a flat point, we move on to calculate W_∞^{flat}. Define:

$$\begin{pmatrix} \tilde{H}_{11} & a \\ \text{l.o.t.} & \tilde{H}_{22} \end{pmatrix} = \begin{pmatrix} \Delta_8 + \frac{c_1}{\mu^3} + \frac{c_2 V_j^2}{\mu^5} & b^t + \tilde{b}^t \\ \text{l.o.t.} & \Delta_8 + H_m + \mu Y_\mu + \text{l.o.t.} \end{pmatrix} \tag{2.7.56}$$

As mentioned at the beginning of this section, W_∞^{flat} will be constructed in two pieces. The first is the inverse of $\tilde{H}_{11}$, called previously W_{11}, which we rename W_1. The second piece is the inverse of $\tilde{H}_{22} = W_{22} = W_2$. I.e. we will construct:

$$W_\infty^{\text{flat}} = \begin{pmatrix} W_1 & -W_1 \left(b^t + \tilde{b}^t\right) W_2 \\ 0 & W_2 \end{pmatrix} \tag{2.7.57}$$

with error defined by:

$$Q^2 \left(W_\infty^{\text{flat}}\right) = \begin{pmatrix} \tilde{H}_{11} & a \\ \text{l.o.t.} & \tilde{H}_{22} \end{pmatrix} \begin{pmatrix} W_1 & -W_1 \left(b^t + \tilde{b}^t\right) W_2 \\ 0 & W_2 \end{pmatrix} \tag{2.7.58}$$

$$= I - \begin{pmatrix} e_1^{\text{flat}} & 0 \\ 0 & e_2^{\text{flat}} \end{pmatrix} \tag{2.7.59}$$

$$= I - e_\infty^{\text{flat}}. \tag{2.7.60}$$

First, we construct W_2. First write:

$$\tilde{H}_{22} = \Delta_8 + H_m + \mu Y_\mu + \varphi \tag{2.7.61}$$

where φ stands for the lower order terms in $\tilde{H}_{22}$. Let $L = H_{22} + 4\pi i \eta_i \frac{\partial}{\partial x^i} + \varphi$ Now define:

$$W^{k,M} = \int \frac{e^{2\pi i \eta(x-y)}}{4\pi^2 |\eta|^2 + k\mu} \sum_{\ell=1}^{M} (-1)^{\ell} \left(L \left(4\pi^2 |\eta|^2 + k\mu\right)^{-1} \right)^{\ell}. \tag{2.7.62}$$

Let Π_k denote projection onto the $k\mu$ eigenspace of $H_m + \mu Y_\mu$ and define

$$W_M = \sum_{k=1}^{\infty} W^{k,M} \Pi_k. \tag{2.7.63}$$

Then we have:

$$\tilde{H}_{22} W_M = \tilde{H}_{22} \sum_{k=1}^{\infty} W^{k,M} \Pi_k \tag{2.7.64}$$

$$= \sum_{k=1}^{\infty} \Pi_k - \sum_{k=1}^{\infty} e_{k,M} \Pi_k. \tag{2.7.65}$$

First we estimate $e_{k,M}$:

$$\tilde{H}_{22} W^{k,M} \Pi_k = \tilde{H}_{22} \int \frac{e^{2\pi i \eta(x-y)}}{4\pi^2 |\eta|^2 + k\mu} \sum_{\ell=1}^{M} (-1)^{\ell} \left(L \left(4\pi^2 |\eta|^2 + k\mu\right)^{-1} \right)^{\ell} \Pi_k \tag{2.7.66}$$

$$= I - \int e^{2\pi i \eta(x-y)} (-1)^{M} \left(L \left(4\pi^2 |\eta|^2 + k\mu\right)^{-1} \right)^{M+1} \Pi_k. \tag{2.7.67}$$

We note that since $4\pi^2 |\eta|^2 + k\mu$ depends only on μ, the only derivatives contributing to the above expression are the derivatives in μ. Let $\tilde{\varphi}$ denote terms in φ that do not contain derivatives. Then:

$$e_{k,M} = \int e^{2\pi i \eta(x-y)} (-1)^{M} \left(\left(\Delta + 4\pi^2 \eta_i \frac{\partial}{\partial x_i} + \varphi \right) \left(4\pi^2 |\eta|^2 + k\mu\right)^{-1} \right)^{M+1} \Pi_k \tag{2.7.68}$$

$$= \int e^{2\pi i\eta(x-y)} (-1)^M$$

$$\times \left(\left(-\frac{\partial^2}{\partial\mu^2} - \frac{7}{\mu}\frac{\partial}{\partial\mu} + 4\pi^2\eta_i\frac{x_i}{\mu}\frac{\partial}{\partial\mu} + \tilde{\varphi}\right)\left(4\pi^2|\eta|^2 + k\mu\right)^{-1}\right)^{M+1} \Pi_k \qquad (2.7.69)$$

Noting that $|\tilde{\varphi}| \leq c\frac{1}{\mu^3}$, the term with least decay in μ is:

$$\int e^{2\pi i\eta(x-y)} (-1)^M \left(\left(\pi^2\eta_i\frac{x_i}{\mu}\frac{\partial}{\partial\mu}\right)\left(4\pi^2|\eta|^2 + k\mu\right)^{-1}\right)^{M+1} \Pi_k. \qquad (2.7.70)$$

Expanding $\left(\left(\pi^2\eta_i\frac{x_i}{\mu}\frac{\partial}{\partial\mu}\right)\left(4\pi^2|\eta|^2 + k\mu\right)^{-1}\right)^{M+1}$ yields the following highest order term in μ:

$$\int e^{2\pi i\eta(x-y)} (-1)^M \left(\left(\pi^2\eta_i\frac{x_i}{\mu}\frac{\partial}{\partial\mu}\right)\left(4\pi^2|\eta|^2 + k\mu\right)^{-1}\right)^{M+1} \Pi_k$$

$$\leq \qquad c\int \frac{|k\eta|^{M+1}}{\left(4\pi^2|\eta|^2 + k\mu\right)^{2(M+1)}}\Pi_k. \qquad (2.7.71)$$

Taking $M > 3$:

$$\int \frac{|k\eta|^{M+1}}{\left(4\pi^2|\eta|^2 + k\mu\right)^{2M+2}}\mathrm{d}\eta \leq \frac{2^{\frac{5-M}{2}}}{(2(M+1))!}k^{\frac{7-M}{2}}\mu^{\frac{5-3M}{2}}\Gamma\left(\frac{3M-5}{2}\right)\Gamma\left(\frac{M+9}{2}\right). \qquad (2.7.72)$$

Choosing $M \geq 8$ gives decay $|e_{k,8}| \leq \frac{c_k}{\mu^9}$. Since $e_{k,8}$ is summable over k, the series $\sum_{k=1}^{\infty} e_{k,M}\Pi_k$ converges and has decay $\leq \frac{c_k}{\mu^9}$.

We now turn to the construction of W_1.

2.7.1 Construction of W_1

We have:

$$\tilde{H}_{11} = \Delta + V(x) \qquad (2.7.73)$$

where

$$\Delta = -g_{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_j} \tag{2.7.74}$$

with x^i coordinates on the 8 dimensional flat directions, and $g_{ij} = \delta_{ij} + \ell_{ij}(x)$, and $|\ell_{ij}| \leq 1/\mu^5$. The potential term is:

$$V(x) = \frac{b}{\mu^3} + \text{ lower order terms }. \tag{2.7.75}$$

We construct the parametrix for W_1 using a Neumann series for $(\Delta + V)^{-1}$. Convergence of the series will require the following:

Proposition 2.7.1. *Let $V = \tilde{V}\tilde{V'}$ satisfy*

$$\tilde{V}(y) \leq c|y|^{-3/2} \text{ and} \tag{2.7.76}$$

$$\tilde{V'}(y) \leq c|y|^{-3/2} \tag{2.7.77}$$

suppose $\tilde{V}, \tilde{V'}$ are supported on $B_R(0)^c$. Consider the integral operator defined on $L_2(\mathbb{R}^n)$ by

$$Kf(x) = \int \frac{\tilde{V}(x)\tilde{V'}(y)f(y)}{|x-y|^{n-2}}dy. \tag{2.7.78}$$

Then K defines a bounded operator from L_2 to L_2, with L^2 bound C/R, for some constant C.

Proof. We note that if

$$\tilde{V}(y) \leq c|y|^{-3/2} \text{ and} \tag{2.7.79}$$

$$\tilde{V'}(y) \leq c|y|^{-3/2} \tag{2.7.80}$$

then

$$\int \frac{\tilde{V}(x)\tilde{V}'(y)}{|x-y|^{n-2}}dy \leq \int \frac{1}{|x|^{3/2}|x-y|^{n-2}|y|^{3/2}}dy \tag{2.7.81}$$

$$= \int \frac{1}{|x|}\frac{1}{|x|^{1/2}|x-y|^{n-2}|y|^{3/2}}dy. \tag{2.7.82}$$

By lemma 2.1 of [13], kernel $\frac{1}{|x|^{1/2}|x-y|^{n-2}|y|^{3/2}}$ defines a bounded operator on L^2, Thus, the result follows. □

Now consider a Neumann series for $(\Delta + \tilde{V}^2)$:

$$(\Delta + \tilde{V}^2)^{-1} = \Delta^{-1}(1 + \tilde{V}^2\Delta^{-1})^{-1} = \Delta^{-1}\sum_{k=0}^{\infty}(-1)^k(\tilde{V}^2\Delta^{-1})^{-k}$$

$$= \Delta^{-1} - \Delta^{-1}\tilde{V}^2\Delta^{-1} + \Delta^{-1}\tilde{V}(\tilde{V}\Delta^{-1}\tilde{V})\tilde{V}\Delta^{-1} - \Delta^{-1}\tilde{V}(\tilde{V}\Delta^{-1}\tilde{V})^2\tilde{V}\Delta^{-1} + \cdots$$

$$= \Delta^{-1} - \Delta^{-1}\tilde{V}\sum_{k=0}^{\infty}(-1)^k(\tilde{V}\Delta^{-1}\tilde{V})^k\tilde{V}\Delta^{-1}.$$

The inner sum converges in L_2. Let $\tilde{V} = \left(\frac{a}{r^3}\right)^{1/2}$ and $\tilde{V}' = V\left(\tilde{V}\right)^{-1}$. Then

$$V = \left(\frac{1}{r^3}\right)^{1/2}\left(r^{3/2}V\right) \tag{2.7.83}$$

$$= \tilde{V}\tilde{V}' \tag{2.7.84}$$

where both $\tilde{V}$ and $\tilde{V}'$ meet the requirements for the above proposition. For a fixed $N > 3$, define

$$W_1 = \Delta^{-1} - \Delta^{-1}\tilde{V}\sum_{k=0}^{N}(-1)^k(\tilde{V}'\Delta^{-1}\tilde{V})^k\tilde{V}'\Delta^{-1}. \tag{2.7.85}$$

Every power of $\tilde{V}'\Delta^{-1}\tilde{V}$ has kernel with singularity dropping by a power of $|x-y|^2$, so the terms are smooth for $k > 3$. The error associated to W_1 is:

$$E_{W_1} = (-1)^N \tilde{V}(\tilde{V}'\Delta^{-1}\tilde{V})^N \tilde{V}'\Delta^{-1} \tag{2.7.86}$$

In the following section, we make some observations about $\operatorname{tr}(QW_1)$, and verify that the above error has the required decay rate.

2.7.2 Integral Estimates

Consider the following:

$$|K(x,y)| := |QW_1(x,y)| \tag{2.7.87}$$

$$\leq |x-y|^{1-n} + \sum_{N=1}^{\infty} \int |x-y_1|^{1-n} \rho(y_1) |y_1|^{-3} |y_2 - y_1|^{2-n} \ldots$$

$$\rho(y_N) |y_N|^{-3} |y_N - y|^{2-n} dy_1 \ldots dy_N \tag{2.7.88}$$

where ρ is a smooth cutoff function that is identically one outside $B_{2R}(0)$ and vanishes on $B_R(0)$. In section 2.9, we will need to evaluate the above expression at $y = x$ and at $y = -x$. Eventually, we will see that the only term which contributes to our index computation is the very first term, evaluated at $y = -x$. This result follows from a combination of integral decay estimates, and algebraic cancellations which occur when tracing over the Clifford algebra. In this section, we will prove decay of the terms with $y = x, N > 3$ and with $y = -x, N > 0$. That is:

Proposition 2.7.2. *1. For $N > 3$,*

$$\lim_{R\to\infty} \int\limits_{|x|=R} |x-y_1|^{1-n} \frac{\rho(y_1)}{|y_1|^3} |y_2 - y_1|^{2-n} \ldots \frac{\rho(y_N)}{|y_N|^3} |y_N - x|^{2-n} dy_1 \ldots dy_N = 0 \tag{2.7.89}$$

and

2. *for* $N > 0$,

$$\lim_{R\to\infty}\int\limits_{|x|=R} |x-y_1|^{1-n}\frac{\rho\left(y_1\right)}{|y_1|^3}|y_2-y_1|^{2-n}\dots\frac{\rho\left(y_N\right)}{|y_N|^3}|y_N+x|^{2-n}dy_1\dots dy_N = 0. \tag{2.7.90}$$

Proof. We will require estimates of integrals of the form:

$$\int |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}|y_i-y_{i+1}|^{m-n}dy_i \tag{2.7.91}$$

where $m \le n, k \le n$ and $k \le p$. We will obtain an estimate for this form of integral and then iterate to find an estimate for an integral of the form (2.7.89):

$$\int |x-y_1|^{1-n}\rho\left(y_1\right)|y_1|^{-3}|y_2-y_1|^{2-n}\dots\rho\left(y_N\right)|y_N|^{-3}\left|y_N-y\right|^{2-n}dy_1\dots dy_N. \tag{2.7.92}$$

Returning to (2.7.91), we split the region of integration:

$$\begin{aligned}
&\int |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}|y_i-y_{i+1}|^{m-n}dy_i\\
&= \int_{|y_{i+1}-y_i|<|y_{i+1}/2|} |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}|y_i-y_{i+1}|^{m-n}dy_i\\
&+ \int_{|y_{i+1}-y_i|>|y_{i+1}/2|} |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}|y_i-y_{i+1}|^{m-n}dy_i \qquad (2.7.93)\\
&\le \frac{2^p}{|y_{i+1}|^p}\int_{|y_{i+1}-y_i|<|y_{i+1}/2|} |y_{i-1}-y_i|^{k-n}\rho(y_i)|y_i-y_{i+1}|^{m-n}dy_i\\
&+ 2^{n-m}|y_{i+1}|^{m-n}\int_{|y_{i+1}-y_i|>|y_{i+1}/2|} |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i \qquad (2.7.94)
\end{aligned}$$

In the first integral, we take $y_i \mapsto y_i + y_{i+1}$, and split the second integral:

$$= \frac{2^p}{|y_{i+1}|^p} \int\limits_{|y_i|<|y_{i+1}/2|} |y_{i-1} - y_{i+1} - y_i|^{k-n} \rho(y_i + y_{i+1}) |y_i|^{m-n} dy_i$$

$$+ 2^{n-m} |y_{i+1}|^{m-n} \int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2| \\ |y_{i-1}-y_i|>|y_{i-1}/2|}} |y_{i-1} - y_i|^{k-n} \frac{\rho(y_i)}{|y_i|^p} dy_i$$

$$+ 2^{n-m} |y_{i+1}|^{m-n} \int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2| \\ |y_{i-1}-y_i|<|y_{i-1}/2|}} |y_{i-1} - y_i|^{k-n} \frac{\rho(y_i)}{|y_i|^p} dy_i \tag{2.7.95}$$

Label the terms $I_1 + I_2 + I_3$. First consider I_1:

$$\frac{2^p}{|y_{i+1}|^p} \int\limits_{|y_i|<|y_{i+1}/2|} |y_{i-1} - y_{i+1} - y_i|^{k-n} \rho(y_i + y_{i+1}) |y_i|^{m-n} dy_i$$

$$= \frac{2^p}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|<\left|\frac{(y_{i+1}-y_{i-1})}{2}\right|}} |y_{i-1} - y_{i+1} - y_i|^{k-n} \rho(y_i + y_{i+1}) |y_i|^{m-n} dy_i$$

$$+ \frac{2^p}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_{i-1} - y_{i+1} - y_i|^{k-n} \rho(y_i + y_{i+1}) |y_i|^{m-n} dy_i \tag{2.7.96}$$

$$\leq \frac{2^{n-m+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})| \\ <\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_{i-1} - y_{i+1} - y_i|^{k-n} \rho(y_i + y_{i+1}) |y_{i-1} - y_{i+1}|^{m-n} dy_i$$

$$+ \frac{2^{n-k+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_{i-1} - y_{i+1}|^{k-n} \rho(y_i + y_{i+1}) |y_i|^{m-n} dy_i \tag{2.7.97}$$

$$\leq \frac{2^{n-m+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i+y_{i-1}-y_{i+1}|<|y_{i+1}/2| \\ |y_i|<\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_i|^{k-n}\rho(y_i+y_{i-1})|y_{i-1}-y_{i+1}|^{m-n}dy_i$$

$$+ \frac{2^{n-k+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_{i-1}-y_{i+1}|^{k-n}\rho(y_i+y_{i+1})|y_i|^{m-n}dy_i \tag{2.7.98}$$

$$\leq \frac{2^{n-m+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i+y_{i-1}-y_{i+1}|<|y_{i+1}/2| \\ |y_i|<\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_i|^{k-n}\rho(y_i+y_{i-1})|y_{i-1}-y_{i+1}|^{m-n}dy_i$$

$$+ \frac{2^{n-k+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right| \\ |y_{i-1}-y_{i+1}|<\left|\frac{y_i}{2}\right|}} |y_{i-1}-y_{i+1}|^{k-n}\rho(y_i+y_{i+1})|y_i|^{m-n}dy_i$$

$$+ \frac{2^{n-k+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right| \\ |y_{i-1}-y_{i+1}|>\left|\frac{y_i}{2}\right|}} |y_{i-1}-y_{i+1}|^{k-n}\rho(y_i+y_{i+1})|y_i|^{m-n}dy_i \tag{2.7.99}$$

$$\leq \frac{2^{n-m+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i+y_{i-1}-y_{i+1}|<|y_{i+1}/2| \\ |y_i|<\left|\frac{(y_{i-1}-y_{i+1})}{2}\right|}} |y_i|^{k-n}\rho(y_i+y_{i-1})|y_{i-1}-y_{i+1}|^{m-n}dy_i$$

$$+ \frac{2^{n-k-m+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right| \\ |y_{i-1}-y_{i+1}|<\left|\frac{y_i}{2}\right|}} |y_{i-1}-y_{i+1}|^{k+m-n}\rho(y_i+y_{i+1})|y_i|^{-n}dy_i$$

$$+ \frac{2^{n-k+p}}{|y_{i+1}|^p} \int\limits_{\substack{|y_i|<|y_{i+1}/2| \\ |y_i-(y_{i-1}-y_{i+1})|>\left|\frac{(y_{i-1}-y_{i+1})}{2}\right| \\ |y_{i-1}-y_{i+1}|>\left|\frac{y_i}{2}\right|}} |y_{i-1}-y_{i+1}|^{k-n}\rho(y_i+y_{i+1})|y_i|^{m-n}dy_i \tag{2.7.100}$$

$$\leq \frac{2^{n-m+p}C_1'}{|y_{i+1}|^p}|y_{i-1}-y_{i+1}|^{k+m-n} \tag{2.7.101}$$

$$+ \frac{2^{n-k-m+p}C_2'}{|y_{i+1}|^p}|y_{i-1}-y_{i+1}|^{k+m-n}$$

$$+ \frac{2^{n-k-m+p}C_3'}{|y_{i+1}|^p}|y_{i-1}-y_{i+1}|^{k+m-n} \tag{2.7.102}$$

$$\leq \frac{2^{n-k-m+p}C_1}{|y_{i+1}|^p}|y_{i-1}-y_{i+1}|^{k+m-n} \tag{2.7.103}$$

Now for I_2:

$$2^{n-m}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|}}|y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i$$

$$=2^{n-m}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|>|y_i/2|}}|y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i$$

$$+2^{n-m}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|<|y_i/2|}}|y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i \tag{2.7.104}$$

$$\leq 2^{2n-m-k}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|>|y_i/2|}}|y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i$$

$$+2^{2n-m-k}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|<|y_i/2|}}|y_{i-1}|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i \tag{2.7.105}$$

$$\leq 2^{2n-m-k}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|>|y_i/2|}}|y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i$$

$$+2^{3n-m-2k}|y_{i+1}|^{m-n}\int\limits_{\substack{|y_{i+1}-y_i-y_{i-1}|>|y_{i+1}/2|\\|y_{i-1}-y_i|>|y_{i-1}/2|\\|y_{i-1}-y_i|<|y_i/2|}}|y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i \tag{2.7.106}$$

$$\leq\left(2^{2n-m-k}+2^{3n-m-2k}\right)c_2|y_{i+1}|^{m-n}R^{k-p} \tag{2.7.107}$$

and finally, I_3:

$$2^{n-m}|y_{i+1}|^{m-n} \int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2| \\ |y_{i-1}-y_i|<|y_{i-1}/2|}} |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}dy_i$$

$$\leq 2^{n-m+p}|y_{i+1}|^{m-n} \int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2| \\ |y_{i-1}-y_i|<|y_{i-1}/2|}} |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_{i-1}|^p}dy_i \qquad (2.7.108)$$

$$\leq 2^{n-m+p}|y_{i+1}|^{m-n} \int\limits_{\substack{|y_{i+1}-y_i|>|y_{i+1}/2| \\ |y_i|<|y_{i-1}/2|}} |y_i|^{k-n}\frac{\rho(y_i+y_{i-1})}{|y_{i-1}|^p}dy_i \qquad (2.7.109)$$

$$\leq 2^{n-m+p-k}c_3|y_{i+1}|^{m-n}|y_{i-1}|^{k-p}\rho(y_{i-1}) \text{ for } p \geq k \qquad (2.7.110)$$

We have:

$$\int |y_{i-1}-y_i|^{k-n}\frac{\rho(y_i)}{|y_i|^p}|y_i-y_{i+1}|^{m-n}dy_i \leq I_1+I_2+I_3$$

$$\leq \frac{2^{n-m+p-k}c_1}{|y_{i+1}|^p}|y_{i-1}-y_{i+1}|^{k+m-n}$$

$$+\left(2^{2n-m-k}+2^{3n-m-2k}\right)c_2|y_{i+1}|^{m-n}R^{k-p}$$

$$+\,2^{n-m-k+p}c_3\left(|y_{i+1}|^{m-n}|y_{i-1}|^{k-p}\right) \qquad (2.7.111)$$

Applying this estimate to the inner integral in (2.7.92):

$$\int |x-y_1|^{1-n}\frac{\rho(y_1)}{|y_1|^3}|y_1-y_2|^{2-n}dy_1$$

$$\leq \frac{2^n c_1}{|y_2|^3}\left(\rho(y_2)|x-y_2|^{3-n}\right)$$

$$+\left(2^{2n-3}+2^{3n-5}\right)c_2|y_2|^{2-n}R^{-2}$$

$$+\,2^n c_3\left(|y_2|^{2-n}|x|^{-2}\right) \qquad (2.7.112)$$

Now, for $N=1$, evaluating at $y_2=-x$ yields a decay rate of $1/|x|^n$. Clearly, decay is increasing with iteration, and there are no singular terms for $y_i=-x$.

Thus, we have proved the proposition for $y = -x$. Evaluating at $y_2 = x$ results in a singular term, but as we will see in 2.9, this term vanishes due to algebraic considerations. If $N > 1$, we continue, integrating with respect to y_2:

$$\begin{aligned}
&\int \Bigg(\frac{2^n c_1}{|y_2|^3} |x - y_2|^{3-n} \\
+ \quad & \left(2^{2n-3} + 2^{3n-5}\right) c_2 |y_2|^{2-n} R^{-2} + 2^n c_3 \left(|y_2|^{2-n}|x|^{-2}\right)\Bigg) \frac{\rho(y_2)}{|y_2|^3} |y_2 - y_3|^{2-n} dy_2 \\
&\leq \int \Bigg(\frac{2^n c_1}{|y_2|^3} |x - y_2|^{3-n} \\
&+ \left(2^{2n-3} + 2^{3n-5}\right) c_2 |y_2|^{2-n} R^{-2} + 2^n c_3 \left(|y_2|^{2-n}|x|^{-2}\right)\Bigg) \frac{\rho(y_2)}{|y_2|^3} |y_2 - y_3|^{2-n} dy_2
\end{aligned} \tag{2.7.113}$$

The last two terms have the form:

$$\frac{1}{R^2}\int \frac{\rho(y_2)}{|y_2|^{n+1}}|y_2-y_3|^{2-n}\,dy_2 = \frac{1}{R^2}\int\limits_{|y_2-y_3|>|y_3/2|}\frac{\rho(y_2)}{|y_2|^{n+1}}|y_2-y_3|^{2-n}\,dy_2$$

$$+\frac{1}{R^2}\int\limits_{|y_2-y_3|<|y_3/2|}\frac{\rho(y_2)}{|y_2|^{n+1}}|y_2-y_3|^{2-n}\,dy_2 \quad (2.7.114)$$

$$\leq \frac{1}{R^2}\int\limits_{|y_2-y_3|>|y_3/2|}\frac{2^{n-2}\rho(y_2)}{|y_2|^{n+1}}|y_3|^{2-n}\,dy_2$$

$$+\frac{1}{R^2}\int\limits_{|y_2-y_3|<|y_3/2|}\frac{2^{n+1}\rho(y_2)}{|y_3|^{n+1}}|y_2-y_3|^{2-n}\,dy_2$$

$$(2.7.115)$$

$$\leq \frac{1}{R^2}\int\limits_{|y_2-y_3|>|y_3/2|}\frac{2^{n-2}\rho(y_2)}{|y_2|^{n+1}}|y_3|^{2-n}\,dy_2$$

$$+\frac{1}{R^2}\int\limits_{|y_2|<|y_3/2|}\frac{2^{n+1}\rho(y_2+y_3)}{|y_3|^{n+1}}|y_2|^{2-n}\,dy_2 \quad (2.7.116)$$

$$\leq \frac{1}{R^2}\frac{2^{n-2}}{R}|y_3|^{2-n}+\frac{1}{R^2}\frac{2^{n-2}}{|y_3|^{n-1}} \quad (2.7.117)$$

If $N=2$, we can replace $|y_3|$ with $|x|$ and obtain decay on the order of $1/R^{n+1}$. For $N>2$, at the next step we would multiply by $\rho(y_3)/|y_3|^3|y_3-y_4|^{2-n}$ and integrate in y_3. This has the same form as the y_2 integral above, with increased decay. The remaining term in (2.7.113) is:

$$\int\left(\frac{2^n c_1}{|y_2|^3}|x-y_2|^{3-n}\right)\frac{\rho(y_2)}{|y_2|^3}|y_2-y_3|^{2-n}dy_2 \quad (2.7.118)$$

Applying the estimate (2.7.111):

$$\int \left(\frac{2^n c_1}{|y_2|^3} |x - y_2|^{3-n} \right) \frac{\rho(y_2)}{|y_2|^3} |y_2 - y_3|^{2-n} dy_2$$

$$\leq \frac{2^{2n+1} c_1'}{|y_3|^6} |x - y_3|^{5-n}$$

$$+ \left(2^{3n-5} + 2^{4n-8}\right) c_2' |y_3|^{2-n} R^{-3}$$

$$+ 2^{2n+1} c_3' \left(|y_3|^{2-n} |x|^{-3}\right) \tag{2.7.119}$$

Our estimate is only valid upon iteration for terms where $|y_i - x|$ is raised to a negative power. However, when a term reaches the threshold $k - n > 0$, we will have $p > k$, so that these (smooth) terms have decay $< 1/R^n$. Each iteration decreases the order of singularity by 2, thus, after 4 iterations (recall that $n = 8$) we have smooth terms with decay $< 1/R^n$. Integration over the sphere $S^{n-1}(R)$ and taking $R \to \infty$ yields zero as desired. □

Note that the above proposition also implies decay of the kernel associate to the error term (2.7.86) for $N > 3$. We have one more issue to address before the final index computation.

2.7.3 Transverse Branch

In this section we handle the parametrix construction for Q^2 near flat directions such that x_3^9 is arbitrary (but outside $B_R(0)$) and the $x_{1,2}^i$ are small. (Recall that near a flat point means $V(x) < r^{2+\epsilon}$.) As we did for the estimates in this region, we decompose Q^2 as follows. Define:

$$H_{x_3^9} = \Delta_{x_{1,2}} + \left(x_3^9\right)^2 \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right) + ix_3^9 \left(-\psi_{1s}\psi_{2s} + \chi_1^i\chi_2^i\right) \qquad (2.7.120)$$

$$= \Delta_{x_{1,2}} + \left(x_3^9\right)^2 \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right) + ix_3^9 \left(-\psi_{1s}\psi_{2s} + \chi_1^i\chi_2^i\right) \qquad (2.7.121)$$

where $\Delta_{x_{1,2}}$ denotes the laplacian in the x_1^i, x_2^i directions.

$$Q^2 = H_{x_3^9} - \frac{\partial^2}{\partial {x_3^9}^2} + i\gamma^i_{\alpha\beta}x_1^i\psi_{2\alpha}\psi_{3\beta} + i\gamma^i_{\alpha\beta}x_2^i\psi_{3\alpha}\psi_{1\beta} + \left(x_2^i x_2^j - x_1^j x_2^i\right)^2 \qquad (2.7.122)$$

For a section u supported near this type of flat point, define:

$$u = \phi_0 + \phi_1 \qquad (2.7.123)$$

where $\phi_0 = \frac{\left(x_3^9\right)^4}{\pi^4} f\left(x_3^9\right) e^{-\left|x_3^9\right| \frac{1}{2} \sum_{i=1}^{8}\left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right)}$. Let Π_0 denote L^2 orthogonal projection onto the zero eigenspace of $H_{x_3^9}$ and $\Pi_1 = (1 - \Pi_0)$. We write:

$$Q^2 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \qquad (2.7.124)$$

$$= \begin{pmatrix} \Pi_0 Q^2 \Pi_0 & \Pi_0 Q^2 \Pi_1 \\ \Pi_1 Q^2 \Pi_0 & \Pi_1 Q^2 \Pi_1 \end{pmatrix} \qquad (2.7.125)$$

and now we compute $\Pi_0 Q^2 \Pi_0$. To shorten the notation a bit, define:

$$\xi^2 = \sum_{i=1}^{8} \left(\left(x_1^i\right)^2 + \left(x_2^i\right)^2\right) \qquad (2.7.126)$$

The second order derivative in x_3^9 gives:

$$\left(-\frac{\partial^2}{\partial x_3^{9\,2}}\phi_0,\phi_0\right) = \left(\left(-\frac{12}{\left(x_3^9\right)^2}+\frac{4\xi^2}{\left(x_3^9\right)}-\frac{\xi^4}{4}\right)\phi_0,\phi_0\right)$$

$$-\left(\frac{\left(x_3^9\right)^4}{\pi^4}\frac{\partial^2}{\partial\left(x_3^9\right)^2}\left(f\left(x_3^9\right)\right)e^{-\left|x_3^9\right|\xi^2/2},\phi_0\right) \tag{2.7.127}$$

$$= \left(\left(-12+32-18\right)\frac{\phi_0}{\left(x_3^9\right)^2},\phi_0\right)$$

$$-\left(\frac{\left(x_3^9\right)^4}{\pi^4}\frac{\partial^2}{\partial\left(x_3^9\right)^2}\left(f\left(x_3^9\right)\right)e^{-\left|x_3^9\right|\xi^2/2},\phi_0\right) \tag{2.7.128}$$

$$= \left(2\frac{\phi_0}{\left(x_3^9\right)^2},\phi_0\right)$$

$$-\left(\frac{\left(x_3^9\right)^4}{\pi^4}\frac{\partial^2}{\partial\left(x_3^9\right)^2}\left(f\left(x_3^9\right)\right)e^{-\left|x_3^9\right|\xi^2/2},\phi_0\right) \tag{2.7.129}$$

The contribution from the $\frac{1}{2}\sum\left(x_1^i x_2^j - x_2^i x_1^j\right)^2$ term is:

$$\left(\frac{1}{2}\sum\left(x_1^i x_2^j - x_2^i x_1^j\right)^2\phi_0,\phi_0\right) = \left(\frac{7\cdot 8}{4\left(x_3^9\right)^2}\phi_0,\phi_0\right) \tag{2.7.130}$$

$$= \left(\frac{14}{\left(x_3^9\right)^2}\phi_0,\phi_0\right) \tag{2.7.131}$$

So, we may write:

$$T_{11} = -\frac{\partial^2}{\partial\left(x_3^9\right)^2}+\frac{16}{\left(x_3^9\right)^2} \tag{2.7.132}$$

We may express (2.7.125) in a more simplified form by a change of basis

$$\begin{pmatrix} T_{11} & b^t \\ b & T_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{b}{4x_3^9} & 1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_{11} & b^t \\ \text{l.o.t.} & T_{22} \end{pmatrix} \tag{2.7.133}$$

and $\tilde{T}_{11} = T_{11} - bb^t/4x_3^9$. Now consider:

$$\Pi_0 \frac{bb^t}{4x_3^9} = \Pi_0 \frac{1}{4\left(x_3^9\right)} \left(\gamma^i_{\alpha\beta} x_1^i \psi_{2\alpha}\psi_{3\beta} + \gamma^i_{\alpha\beta} x_2^i \psi_{3\alpha}\psi_{1\beta}\right)^2 \tag{2.7.134}$$

$$= \frac{1}{4\left(x_3^9\right)} \left(\frac{16 \cdot 8}{2x_3^9}\right) \tag{2.7.135}$$

$$= \frac{16}{\left(x_3^9\right)^2} \tag{2.7.136}$$

So that

$$\tilde{T}_{11} = T_{11} - bb^t/4x_3^9 \tag{2.7.137}$$

$$= -\frac{\partial^2}{\partial\left(x_3^9\right)^2} + \frac{16}{\left(x_3^9\right)^2} - \frac{16}{\left(x_3^9\right)^2} \tag{2.7.138}$$

$$= -\frac{\partial^2}{\partial\left(x_3^9\right)^2} \tag{2.7.139}$$

Now, T_{22} has the large lower bound $\|T_{22}\phi_1\|^2 \geq c \left\| \left(x_3^9\right)^{1/2} \phi_1 \right\|$, so construction of a parametrix for T_{22} is straightforward. Furthermore, its parametrix cannot contribute to trace terms, as it will have exponential decay. Now, if we construct a parametrix P_1 for T_{11}, it looks a priori as though we could have a non-zero contribution to the trace. We note, however, that this contribution is finite, since P_1 would be at most linear in x_3^9. (This is simply the observation that harmonic functions of $\mathbb{R}$ are of the form $ax_3^9 + b$). Then $QP_1 \approx C\left(dx_3^9\right)a$. This term does not contribute to the final trace, as we will see in section 2.9. Now, we also need to see that $C\left(dx_3^9\right) Q\left(\left(T^{-1}\right)_{11} - P_1\right)$ has trace which vanishes as $R \to \infty$. Now, we have chosen G so that $r^{-1}Gr^{-1}$ is a bounded operator on L^2. The kernel $k\left(t,s\right)$ for $\left(T^{-1}\right)_{11} - P_1$ should have the form $a\left(t+s\right)+b$, where $t^{-1}\left(a\left(t+s\right)+b\right)s^{-1}$ is bounded on $L^2(\mathbb{R})$. Let $\phi(s)$ be a bump function of L^2 norm one supported in a small neighborhood of 0. Consider $\phi_R(s) = \phi(s-R)$ supported near R. Then:

$$\int t^{-1}\left(a\left(t+s\right)+b\right)s^{-1}\phi_R\left(s\right)\mathrm{d}s = \int as^{-1}\phi_R\left(s\right)\mathrm{d}s + \int t^{-1}a\phi_R\left(s\right)\mathrm{d}s + \int t^{-1}bs^{-1}\phi_R\left(s\right)\mathrm{d}s \tag{2.7.140}$$

The second and third terms are bounded, but the first term is unbounded, so we see that $a \to 0$ as $R \to 0$.

2.8 Patching W_∞ and Estimating the Error

Define:

$$\rho_{N_\epsilon}\left(x\right) = \begin{cases} 1 & \text{for } \sum_{i,j}\left|\left[x^i, x^j\right]\right|^2 < r^{2+\frac{\epsilon}{2}} \\ 0 & \text{for } \sum_{i,j}\left|\left[x^i, x^j\right]\right|^2 > r^{2+\epsilon} \end{cases} \tag{2.8.1}$$

and let

$$W_\infty = \rho_{N_\epsilon}\left(x\right)W_\infty^{\text{flat}}\left(x,y\right)\rho_{N_{2\epsilon}}\left(y\right) + \left(1-\rho_{N_\epsilon}\right)\left(x\right)W_\infty^{\text{non-flat}}\left(x,y\right)\left(1-\rho_{N_{\frac{\epsilon}{2}}}\left(y\right)\right) \tag{2.8.2}$$

Note that the left cutoffs are identically 1 on the support of the right cutoffs. This is chosen so that the errors associated to cutoffs trace to zero:

$$Q^2 W_\infty = Q^2 \rho_{N_\epsilon}(x) W_\infty^{\text{flat}}(x,y) \rho_{N_{2\epsilon}}(y) \tag{2.8.3}$$

$$+ Q^2 (1-\rho_{N_\epsilon})(x) W_\infty^{\text{non-flat}}(x,y) \left(1-\rho_{N_{\frac{\epsilon}{2}}}(y)\right) \tag{2.8.4}$$

$$= \rho_{N_\epsilon}(x) Q^2 W_\infty^{\text{flat}}(x,y) \rho_{N_{2\epsilon}}(y) \tag{2.8.5}$$

$$+ (1-\rho_{N_\epsilon})(x) Q^2 W_\infty^{\text{non-flat}}(x,y) \tag{2.8.6}$$

$$+ 2\nabla\rho_{N_\epsilon}(x) \cdot \nabla W_\infty^{\text{flat}}(x,y) \rho_{N_{2\epsilon}}(y) \tag{2.8.7}$$

$$- 2\nabla\rho_{N_\epsilon}(x) \cdot \nabla W_\infty^{\text{non-flat}}(x,y) \left(1-\rho_{N_{\frac{\epsilon}{2}}}(y)\right) \tag{2.8.8}$$

$$+ 2\Delta\rho_{N_\epsilon}(x) W_\infty^{\text{flat}}(x,y) \rho_{N_{2\epsilon}}(y) \tag{2.8.9}$$

$$- \Delta\rho_{N_\epsilon}(x) W_\infty^{\text{non-flat}}(x,y) \left(1-\rho_{N_{\frac{\epsilon}{2}}}(y)\right) \tag{2.8.10}$$

So that:

$$Q^2 W_\infty = \rho_{N_\epsilon}(x) Q^2 W_\infty^{\text{flat}}(x,y) \rho_{N_{2\epsilon}}(y) \tag{2.8.11}$$

$$+ (1-\rho_{N_\epsilon})(x) Q^2 W_\infty^{\text{non-flat}}(x,y) + E_1(x,y) \tag{2.8.12}$$

Putting this together with our parametrix inside B_R, we see that:

$$Q^2 (G-W)(x,y) = e(x,y) + e_{B_R}(x,y) \tag{2.8.13}$$

$$= e_\infty(x,y) + e_{B_R}(x,y) + E_1 \tag{2.8.14}$$

$$= e_\infty^{\text{flat}} + e_\infty^{\text{non-flat}} + e_{B_R}(x,y) + E_1 \tag{2.8.15}$$

where $e_{B_R}(x,y)$ is a compact error term. We have already seen that e_∞^{flat} and $e_\infty^{\text{non-flat}}$ are small. Now the last part of the error that needs to be handled is that due to E_1 in (2.8.15). Note that because of the choice that the right cutoffs are identically one on the support of the left, $E_1(x, y)$ vanishes at $x = y$. When $x \neq y$, E_1 is made up of terms with W_∞^{flat},$W_\infty^{\text{non-flat}}$ and their derivatives, multiplied by derivatives of the cutoffs. $E_1(x, y)$ can have either x or y (but not both), near a flat direction. But the support of $E_1(x, y)$ is located where the gaussian associated to the ground state of the oscillator is small, so these terms have good decay rates.

By theorem 2.3.4, if $r^{c+1}e \in L^2(S)$, then $r^{c-\epsilon}(G - W) \in L^2(S)$. Substituting $Q(G - W)$ for F in theorem 2.3.4 then gives that $r^{c+1-\epsilon}Q(G - W) \in L^2(S)$. In the non-flat regions, we have an even stronger result, since there $\|Qu\|^2 \geq \tilde{c}\left\|r^{1/2}u\right\|$. We have the same estimate in the flat regions when our section is in the orthogonal complement of the oscillator zero eigenspace. Hence, decay of $Q(G - W)$ is sufficient for its trace to vanish everywhere both in the non-flat directions, and the orthogonal complement of the oscillator zero eigenspace. But consider:

$$H_{11} = \Delta_8 + \varphi \tag{2.8.16}$$

where Δ_8 is the 8 dimensional laplacian along the flat directions, and φ consists of terms with coefficient $1/r^3$ or lower. We have the Poincare estimate:

$$\|\nabla u\|^2 \geq \left(\frac{(n-2)}{2}\right)^2 \left\|\frac{u}{\mu}\right\|^2 \tag{2.8.17}$$

This means our $c = 3 - \epsilon$. We cannot obtain the required decay by applying theorem 2.3.4. However, we can obtain an improved constant c' by considering $(G - W_1)(v, w)$'s dependence on both variables. For the remainder of this section, we redefine F to mean:

$$F := \Pi_0 (G - W) \Pi_0 \tag{2.8.18}$$

where Π_0 is L^2 orthogonal projection onto the oscillator zero eigenspace. This introduces cutoff errors, of course, but they are supported away from flat directions. Define:

$$\tilde{Q}^2 = Q_v^2 + Q_w^2 \tag{2.8.19}$$

where v is the variable in the first factor of the product space, and w is the variable in the second factor. Then $\left\|\tilde{Q}F\right\|^2_{L^2(\mathbb{R}^{16})} \geq \|(7-\epsilon)\,F/\nu\|^2$, for $|\nu|^2 = |v|^2 + |w|^2$, $v = \nu\cos(\theta)$, $w = \nu\sin(\theta)$. Setting $F = G - W$, we have

$$\tilde{Q}^2 F = 0 \tag{2.8.20}$$

We define a cutoff function ρ so that ρ vanishes for $|v| < R$ and for $|w| < R$, and is identically one for $|v|, |w| > 2R$. Then $d\rho$ is supported in a region where either $R < |v| < 2R$ or $R < |w| < 2R$. Let $p = \nu^\alpha \rho$, for a fixed α such that $2c \geq \alpha > 0$. Then, estimating p as a sequence of bounded functions and writing F as a limit of C_0^∞ section as in the proof of theorem 2.3.4, we can integrate by parts to obtain:

$$\left\|\frac{c' pF}{\nu}\right\|^2 \leq \|dpF\|^2 \tag{2.8.21}$$

We compute:

$$|dp|^2 = \left(\frac{\alpha p}{\nu} + \nu^\alpha \frac{\partial\rho}{\partial\nu}\right)^2 + \left(\nu^\alpha \frac{\partial\rho}{\partial\theta}\right)^2 \tag{2.8.22}$$

$$= \left(\frac{\alpha p}{\nu}\right)^2 + 2\frac{\alpha p}{\nu}\nu^\alpha \frac{\partial\rho}{\partial\nu} + \left(\nu^\alpha \frac{\partial\rho}{\partial\nu}\right)^2 + \frac{1}{\nu^2}\left(\nu^\alpha \frac{\partial\rho}{\partial\theta}\right)^2 \tag{2.8.23}$$

Considering first the terms where ρ is differentiated, we need to estimate:

$$\int \left(2\frac{\alpha p}{\nu}\nu^\alpha \frac{\partial\rho}{\partial\nu} + \left(\nu^\alpha \frac{\partial\rho}{\partial\nu}\right)^2 + \frac{1}{\nu^2}\left(\nu^\alpha \frac{\partial\rho}{\partial\theta}\right)^2\right) |F|^2 \,\mathrm{d}v\mathrm{d}w$$

$$\leq C_\rho \int\limits_{R<|v|<2R} \nu^\alpha |F|^2 \,\mathrm{d}v\mathrm{d}w + C_\rho \int\limits_{R<|w|<2R} \nu^\alpha |F|^2 \,\mathrm{d}v\mathrm{d}w \tag{2.8.24}$$

The first integral is over a region bounded in v, the second over a region bounded in w. Since $|v|^{\alpha/2} F \in L^2$ and $|w|^{\alpha/2} F \in L^2$, we see that the contribution from ρ derivative terms is bounded. Now, the remaining term in $|dp|^2$ is just:

$$\left(\frac{\alpha p}{\nu}\right)^2 \tag{2.8.25}$$

Now, returning to our original inequality:

$$\left\|\frac{c' pF}{\nu}\right\|^2 \leq \|dpF\|^2 \tag{2.8.26}$$

$$\leq \int \frac{\alpha^2 p^2}{\nu^2} |F|^2 + B \tag{2.8.27}$$

where B is the contribution of the term containing derivatives of ρ, and is bounded so long as $|v|^{\alpha} F \in L^2$ and similarly for w. Thus:

$$\int \left(c'^2 - \alpha^2\right) \frac{p^2 |F|^2}{\nu^2} \leq B \tag{2.8.28}$$

The integral is bounded from below only for $\alpha^2 \leq c'^2$. Thus, it looks as though we have not improved our decay rate. However, since the integral measure in the product space is $\nu^{2n-1}\mathrm{d}\nu$ versus $|v|^{n-1}\,\mathrm{d}v$ in the one variable case, we actually do have a higher decay rate.

Proposition 2.8.1. *Let $\tilde{Q}$ and F be as defined above, with $\tilde{Q}^2 F = 0$ for $|v| > R$ and $|w| > R$ and $\nu = \sqrt{|v|^2 + |w|^2}$. Then:*

$$|F(v, w)| \leq \frac{C}{\nu^{n+1-\delta}} \tag{2.8.29}$$

for some $0 < \delta < 1$.

Proof. Define $\xi = \nu/2$. Fix a point (v_0, w_0), with $\nu_0^2 = |v_0|^2 + |w_0|^2$ and let let $B_\xi(v_0, w_0)$ denote the ball centered at the point (v_0, w_0) with radius ξ, and let $S_\xi(v_0, w_0)$ denote its boundary. Let $\partial_{\mathscr{N}}$ denote the unit normal vector field on $S_\xi(v_0, w_0)$. Choosing coordinates so that $\tilde{\xi} = \sqrt{|\tilde{v} - v_0|^2 + |\tilde{w} - w_0|^2}$ and taking the domain $\Omega_\epsilon = B_\xi(v_0, w_0) \backslash B_\epsilon(v_0, w_0)$, we begin with Green's identity:

$$\int\limits_{\partial\Omega_\epsilon} \frac{1}{\tilde{\xi}^{2n-2}} \partial_{\mathscr{N}} F - \partial_{\mathscr{N}} \left(\frac{1}{\tilde{\xi}^{2n-2}} \right) F \mathrm{d}\sigma$$

$$= \int\limits_{\Omega_\epsilon} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F - F \Delta \frac{1}{\tilde{\xi}^{2n-2}} \mathrm{dvol} \tag{2.8.30}$$

Since $\partial_{\mathscr{N}} \left(\frac{1}{\tilde{\xi}^{2n-2}} \right) = (2-2n)\, \xi^{1-2n}$ on $S_\xi(v_0, w_0)$, we have:

$$0 = \frac{-1}{\xi^{2n-2}} \int\limits_{S_\xi(v_0,w_0)} \partial_{\mathscr{N}} F \mathrm{d}\sigma + \int\limits_{S_\xi(v_0,w_0)} (2-2n)\, \xi^{1-2n} F \mathrm{d}\sigma + \frac{1}{\epsilon^{2n-2}} \int\limits_{S_\epsilon(v_0,w_0)} \partial_{\mathscr{N}} F$$

$$+ \frac{1}{\epsilon^{2n-2}} \int\limits_{S_\epsilon(v_0,w_0)} (2-2n)\, F \mathrm{d}\sigma + \int\limits_{\Omega_\epsilon} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F \mathrm{dvol} \tag{2.8.31}$$

So that:

$$F(v_0, w_0) = \frac{1}{\xi^{n-2}} \int\limits_{S_\xi(v_0,w_0)} \partial_{\mathscr{N}} F \mathrm{d}\sigma + \int\limits_{S_\xi(v_0,w_0)} (2-2n)\, \xi^{1-2n} F \mathrm{d}\sigma$$

$$+ \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2n-2}} \int\limits_{S_\epsilon(v_0,w_0)} \partial_{\mathscr{N}} F + \lim_{\epsilon \to 0} \int\limits_{\Omega_\epsilon} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F \mathrm{dvol} \tag{2.8.32}$$

Now, if F were harmonic, $\partial_{\mathscr{N}} F$ would integrate to zero over the sphere, the last term would vanish and we would obtain the usual mean value theorem for harmonic functions. Instead we have:

$$\lim_{\epsilon\to 0}\left(\frac{1}{\xi^{n-2}}\int\limits_{S_\xi(v_0,w_0)}\partial_{\mathcal{N}}F\mathrm{d}\sigma+\frac{1}{\epsilon^{2n-2}}\int\limits_{S_\epsilon(v_0,w_0)}\partial_{\mathcal{N}}F\mathrm{d}\sigma\right)=\frac{1}{\xi^{2n-2}}\int\limits_{B_\xi(v_0,w_0)}\Delta F\mathrm{d}\sigma \tag{2.8.33}$$

and so

$$F(v_0,w_0)=\frac{1}{\xi^{2n-2}}\int\limits_{B_\xi(v_0,w_0)}\Delta F\mathrm{d}\sigma+\int\limits_{S_\xi(v_0,w_0)}(2-2n)\,\xi^{1-2n}F\mathrm{d}\sigma$$

$$+\int\limits_{B_{\xi(v_0,w_0)}}\frac{1}{\tilde{\xi}^{2n-2}}\Delta F\mathrm{dvol} \tag{2.8.34}$$

Finally, using $\tilde{Q}^2F=0\Rightarrow\Delta F=\varphi F$, (where φ is just the collection of bounded terms in $\tilde{Q}^2$ with coefficients of order $1/\xi^3$ or smaller) we have:

$$|F(v_0,w_0)|\leq\left|\int\limits_{S_\xi(v_0,w_0)}(2-2n)\,\xi^{1-2n}F\mathrm{d}\sigma\right|+\frac{1}{\xi^{2n-2}}\left|\int\limits_{B_\xi(v_0,w_0)}\varphi F\mathrm{dvol}\right|$$

$$+\left|\int\limits_{B_{\xi(v_0,w_0)}}\frac{1}{\tilde{\xi}^{2n-2}}\Delta F\mathrm{dvol}\right| \tag{2.8.35}$$

$$\leq(2n-2)\,\xi^{1-2n}\left(\int\limits_{S_\xi(v_0,w_0)}\left|\tilde{\xi}\right|^{4-\delta}F^2\mathrm{d}\sigma\right)^{1/2}\left(\int\limits_{S_\xi(v_0,w_0)}\frac{1}{\left|\tilde{\xi}\right|^{4-\delta}}\mathrm{d}\sigma\right)^{1/2}$$

$$+\frac{1}{\xi^{2n-2}}\left(\int\limits_{B_\xi(v_0,w_0)}\left|\tilde{\xi}\right|^{10-\delta}|\varphi F|^2\,\mathrm{dvol}\right)^{1/2}\left(\int\limits_{B_\xi(v_0,w_0)}\tilde{\xi}^{-10+\delta}\mathrm{dvol}\right)^{1/2}$$

$$+\left|\int\limits_{B_{\xi(v_0,w_0)}}\frac{1}{\tilde{\xi}^{2n-2}}\Delta F\mathrm{dvol}\right| \tag{2.8.36}$$

$$\leq (2n-2)\,\xi^{1-2n} C_1 \xi^{n-2+\delta} + C_2 \frac{\xi^{n-5}}{\xi^{2n-2}} + \left| \int\limits_{B_{\xi(v_0,w_0)}} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F \mathrm{dvol} \right| \tag{2.8.37}$$

$$\leq (2n-2)\,\xi^{-n-1+\delta} C_1 + C_2 \frac{1}{\xi^{n+3}} + \left| \int\limits_{B_{\xi(v_0,w_0)}} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F \mathrm{dvol} \right| \tag{2.8.38}$$

The first two terms have the desired decay. The last term:

$$\left| \int\limits_{B_{\xi}(v_0,w_0)} \frac{1}{\tilde{\xi}^{2n-2}} \Delta F \mathrm{dvol} \right| = \left| \int\limits_{B_{\xi}(v_0,w_0)} \frac{1}{\tilde{\xi}^{2n-2}} \varphi \left| F \right| \mathrm{dvol} \right| \tag{2.8.39}$$

requires more delicate handling. We will first prove an L^p decay estimate for F:

Claim:

$$\nu^{b-1} |F|^{2p+1} \in L^2 \tag{2.8.40}$$

for any $b/4p < 7$.

Proof. We have $\tilde{Q}^2 F = 0$. Define a weight function u such that

$$u = \nu^b \left| F \right|^{2p} \tag{2.8.41}$$

Consider the following L^2 inner product:

$$\langle \tilde{Q}^2 F, u^2 F \rangle \tag{2.8.42}$$

We may write ν^b as the limit of a sequence of bounded functions and F as the limit of a sequence in C_0^∞ as in the proof of 2.3.4. Then we may integrate by parts to obtain:

$$\langle \tilde{Q}^2 F, u^2 F \rangle = \|\tilde{Q}(uF)\|^2 - \|u' F\|^2 \geq \|\nabla(uF)\|^2 - \|u' F\|^2 \tag{2.8.43}$$

Thus,

$$0 \geq \|\nu^b|F|^{2p}\nabla F\|^2 + 2\langle \nabla(\nu^b|F|^{2p})F, \nu^b|F|^{2p}\nabla F\rangle \tag{2.8.44}$$

$$\geq \frac{1}{(2p+1)^2}\|\nu^b\nabla|F|^{2p+1}\|^2 + \frac{4p}{(2p+1)^2}\|\nu^b\nabla|F|^{2p+1}\|^2$$

$$+ 2\langle \nabla(\nu^b)|F|^{2p}F, \nu^b|F|^{2p}\nabla F\rangle \tag{2.8.45}$$

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1}) - b\nu^{b-1}|F|^{2p+1}\|^2 + 2\langle \nabla(\nu^b)|F|^{2p}, \nu^b|F|^{2p+1}\nabla|F|\rangle \tag{2.8.46}$$

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - 2\frac{4p+1}{(2p+1)^2}\langle \nu^b\nabla(|F|^{2p+1}), b\nu^{b-1}|F|^{2p+1}\rangle$$

$$- 2b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2 + \frac{4p+1}{(2p+1)^2}\|b\nu^{b-1}|F|^{2p+1}\|^2$$

$$+ 2\langle \nabla(\nu^b)|F|^{2p}, \nu^b|F|^{2p+1}\nabla|F|\rangle \tag{2.8.47}$$

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{4p+1}{(2p+1)^2}\langle \nabla_r|F|^{4p+2}, b\nu^{2b-1}\rangle$$

$$- 2b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2 + \frac{1}{4p+2}\langle 2b\nu^{2b-1}, \nabla_r|F|^{4p+2}\rangle \tag{2.8.48}$$

Once again, we integrate by parts:

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{4p+1}{(2p+1)^2}\langle |F|^{4p+2}, b\left(-\nabla_\nu - \frac{15}{\nu}\right)\nu^{2b-1}\rangle$$

$$- b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2$$

$$+ \frac{1}{4p+2}\langle 2b\left(-\nabla_\nu - \frac{15}{\nu}\right)\nu^{2b-1}, |F|^{4p+2}\rangle \tag{2.8.49}$$

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{2p}{(2p+1)^2}\langle |F|^{4p+2}, b\left(-\nabla_\nu - \frac{15}{\nu}\right)\nu^{2b-1}\rangle$$

$$- b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2 \tag{2.8.50}$$

$$= \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{2p}{(2p+1)^2}\langle|F|^{4p+2}, b\,(-2b-14)\,\nu^{2b-2}\rangle$$

$$- b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2 \tag{2.8.51}$$

and so we have:

$$0 \geq \frac{4p+1}{(2p+1)^2}\|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{2p}{(2p+1)^2}\langle|F|^{4p+2}, b\,(-2b-14)\,\nu^{2b-2}\rangle$$

$$- b^2\frac{4p+1}{(2p+1)^2}\||F|^{2p+1}\nu^{b-1}\|^2$$

$$\Rightarrow$$

$$0 \geq \|\nabla(\nu^b|F|^{2p+1})\|^2 - \frac{2p}{(4p+1)}\langle|F|^{4p+2}, b\,(-2b-14)\,\nu^{2b-2}\rangle \tag{2.8.52}$$

$$- b^2\||F|^{2p+1}\nu^{b-1}\|^2 \tag{2.8.53}$$

$$= \|\nabla(\nu^b|F|^{2p+1})\|^2 + \frac{2pb\,(2b+14)}{(4p+1)}\||F|^{2p+1}\nu^{b-1}\|^2 - b^2\||F|^{2p+1}\nu^{b-1}\|^2 \tag{2.8.54}$$

This implies boundedness of the terms on the right, so long as:

$$\frac{2pb\,(2b+14)}{(4p+1)} - b^2 \geq 0 \tag{2.8.55}$$

We must have:

$$\frac{b}{4p} \leq 7 \tag{2.8.56}$$

□

Returning to the integral (2.8.39):

$$\int\limits_{B_\xi(v_0,w_0)} \frac{1}{\tilde{\xi}^{2n-2}}\,|\varphi F| \leq \int\limits_{B_\xi(v_0,w_0)} \frac{1}{\tilde{\xi}^{2n-2}}\xi^{-3}\,|F| \tag{2.8.57}$$

$$\leq \left\|\frac{1}{\tilde{\xi}^{2n-2}}\right\|_{L^q(B_1(v_0,w_0))} \left\|{\nu_0}^{-3}\,|F|\right\|_{L^s\left(B_\xi(v_0,w_0)\right)} \tag{2.8.58}$$

for $1/s + 1/q = 1$. For $q < (2n)/2(n-1) = 8/7$, $\left\| \frac{1}{\xi^{2n-2}} \right\|_{L^q(B_1(v_0,w_0))} < \xi$, so we choose $s > 8$. For the second factor:

$$\left\| \nu_0{}^{-3} |F| \right\|_{L^s\left(B_\xi(v_0,w_0)\right)} = \nu_0^{-3} \left\| |F| \right\|_{L^s\left(B_\xi(v_0,w_0)\right)} \tag{2.8.59}$$

$$= \nu_0{}^{-3} \left(\left\| |F|^{s/2} \right\|^{2/s}_{L^2\left(B_\xi(v_0,w_0)\right)} \right) \tag{2.8.60}$$

$$= \nu_0{}^{-3-a} \left\| \nu_0^{\frac{sa}{2}} |F|^{s/2} \right\|^{2/s}_{L^2\left(B_\xi(v_0,w_0)\right)} \tag{2.8.61}$$

By the preceding claim, $\left\| \nu_0^{\frac{sa}{2}} |F|^{s/2} \right\|^{2/s}_{L^2\left(B_\xi(v_0,w_0)\right)}$ is bounded for $(sa/2+1)/(4(s/4-1/2)) < 7$, so that:

$$\frac{(sa+2)}{(s-2)} < 14 \tag{2.8.62}$$

$$a < (14(s-2)-2)/s \tag{2.8.63}$$

$$a < 14 - 30/s \tag{2.8.64}$$

Choosing $s > 30$, we may set $a = 12$. Substituting this result into (2.8.38), we have:

$$|F(v_0,w_0)| \leq (14)\,\xi^{-9+\delta} C_1 + C_2 \frac{1}{\xi^{11}} + C_3 \frac{1}{\nu_0{}^{14}} \qquad \Rightarrow$$

$$|F(v_0,w_0)| \leq (14)\,\xi^{-9+\delta} C_1 + C_2 \frac{1}{\xi^{11}} \tag{2.8.65}$$

$$\leq \frac{C}{\nu_0{}^{9-\delta}} \tag{2.8.66}$$

□

This gives us sufficient decay for the trace of $Q(G - W_1)$ to vanish, and so the parametrix W may be used to compute the L^2-index of Q_+.

2.9 Defect Contribution

Let us now return to the trace we must compute:

$$\frac{1}{2}\int\limits_{|x|=R} \mathrm{tr} i e_n \tau Q W_\infty = \frac{1}{2}\int\limits_{|x|=R} \mathrm{tr} i e_n \tau Q \left(W_\infty^{\text{non-flat}} + W_\infty^{\text{flat}}\right) \tag{2.9.1}$$

We must average over the gauge group, as discussed in section 2.4. As noted in that section, at flat points the S^1 action is reduced to the discrete action $\mathbb{Z}_2$. We have two terms:

$$\frac{1}{2}\sum_{\sigma\in\mathbb{Z}_2}\int\limits_{|x|=R} \mathrm{tr} i e_n \tau \sigma^* Q W_\infty (x,y)_{|y=\sigma x} \tag{2.9.2}$$

where σ^* represents the gauge action on spinors. This action is:

$$\exp\left(\theta\left(\frac{1}{2}\psi_{1s}\psi_{2s} + \frac{1}{2}\chi_1^i\chi_2^i\right)\right) \tag{2.9.3}$$

The residual Z_2 action arises from $\theta = 0$ and $\theta = \pi$. Thus, for $\sigma = I$, $\sigma^* = 1$ and for $\sigma = -I$, $\sigma^* = \prod\limits_{s=1}^{8}\psi_{1s}\psi_{2s}\prod\limits_{i=1}^{8}\chi_1^i\chi_2^i$. We obtain:

$$\frac{1}{2}\sum_{\sigma\in\mathbb{Z}_2}\int\limits_{|x|=R} \mathrm{tr}\tau\sigma^* i e_n Q W_\infty (x,y)_{|y=\sigma x}$$

$$= \frac{1}{2}\int\limits_{|x|=R} \mathrm{tr}\tau \prod_{s=1}^{8}\psi_{1s}\prod_{i=1}^{8}\chi_1^i\chi_2^i i e_n Q W_\infty (x,y)_{|y=-x} + \frac{1}{2}\int\limits_{|x|=R} \mathrm{tr} i e_n \tau Q W_\infty (x,y)_{|y=x} \tag{2.9.4}$$

Recall that the trace of any term with non-zero Clifford degree is zero. Away from flat directions, the trace is over the full Clifford algebra, and it is clear from the construction (2.6.5) that terms which are able to contribute enough Clifford factors to saturate the trace will have enormous decay. Near flat points, on the zero eigenspace of the oscillator $H_m + \mu Y_\mu$, we have (for fixed r):

$$i\psi_{2r}\psi_{3r}\phi_0(x) = -\phi_0(x). \qquad (2.9.5)$$

Thus, for $y = x$, we can replace τ with $\prod_{s=1}^{8}\psi_{1s}\prod_{i=1}^{8}\chi_1^i\chi_2^i$. We therefore must have at least the first 7 terms (including the leading term) of $QW_1(x,x)$ vanish in the trace. We have already seen that the trace of these terms vanishes due to proposition 2.7.2.

Now, for $y = -x$, proposition 2.7.2 also gives that the terms for $N > 1$ vanish in the trace. The leading term does contribute in this case. Before proceeding to the computation of this term, we note that some care must be taken with the expression for W_1. When we computed H_{11} in section 2.7, we chose a particular point on the sphere to calculate the contribution from the action of the angular laplacian V_j^2, acting on the spinors. We need to account for this in our trace computation at $y = -x$. First define η_0 so that:

$$i\gamma^1_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\eta_0 = -8\eta_0. \qquad (2.9.6)$$

I.e. η_0 has eigenvalue -8 under $Y_\mu(x_0)$. Now we define:

$$\eta_0(x) = g_x\eta_0 g_x^{-1} \qquad (2.9.7)$$

where $g_x \in \mathrm{Spin}(8)$ and $g_x x_0 g_x^{-1} = x$. Then:

$$Y_\mu(x)\,\eta_0(x) = g_x i x_0^1\gamma^1_{\alpha\beta}g_x^{-1}\psi_{2\alpha}\psi_{3\beta}g_x\eta_0 \qquad (2.9.8)$$

$$= g_x i x_0^1\gamma^1_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\eta_0 \qquad (2.9.9)$$

$$= -8g_x\eta_0 \qquad (2.9.10)$$

$$= -8\eta_0(x). \qquad (2.9.11)$$

Now we need to find g_{-x}. By definition:

$$g_{-x}x_0g_{-x}^{-1} = -x_0 \tag{2.9.12}$$

so that g_{-x} must map to $-1 \in \mathrm{SO}(8)$, under the usual double cover map. We have:

$$Y_\mu(-x_0)\,\eta_0(-x_0) = -Y_\mu(x_0)\,g_{-x}\eta_0. \tag{2.9.13}$$

We just check that $g_{-x} = \prod \psi_{3s}$ gives $\eta_0(-x_0) = g_{-x}\eta_0$ with the correct eigenvalue under $Y_\mu(-x_0)$.

$$Y_\mu(-x_0)\prod_{s=9}^{16}\psi_{3s}\eta_0 = -Y_\mu(x_0)\prod_{s=9}^{16}\psi_{3s}\eta_0 \tag{2.9.14}$$

$$= \prod_{s=9}^{16}\psi_{3s}Y_\mu(x_0)\,\eta_0 \tag{2.9.15}$$

$$= -8\prod_{s=9}^{16}\psi_{3s}\eta_0. \tag{2.9.16}$$

Note that we could also have chosen $g_{-x} = \prod \psi_{2s}$. The redundancy is not a problem, since for any x, $\eta_0(x)$ is eigen under $\psi_{2r}\psi_{3r}$, for any fixed r. Hence, in the trace, $\prod \psi_{2s}$ and $\prod \psi_{3s}$ are interchangeable. Thus, τ may be replaced with

$\prod_{s=1}^{8} \psi_{1s}\psi_{2s} \prod_{i=1}^{8} \chi_1^i \chi_2^i$. Now we have:

$$\frac{1}{2} \int_{|x|=R} \operatorname{tr} \prod_{s=1}^{8} \psi_{1s}\psi_{2s} \prod_{i=1}^{8} \chi_1^i \chi_8^i \prod_{s=1}^{8} \psi_{1s}\psi_{2s} \prod_{i=1}^{8} \chi_1^i \chi_2^i i e_n Q W_\infty (x,y)_{|y=-x}$$

$$+ \frac{1}{2} \int_{|x|=R} \operatorname{tr} \prod_{s=1}^{8} \psi_{1s} \prod_{i=1}^{8} \chi_1^i \chi_2^i i e_n Q W_\infty (x,y)_{|y=x} \tag{2.9.17}$$

$$= \frac{1}{2} \int_{|x|=R} \operatorname{tr} i e_n Q W_\infty (x,y)_{|y=-x}$$

$$+ \frac{1}{2} \int_{|x|=R} \operatorname{tr} \prod_{s=1}^{8} \psi_{1s} \prod_{i=1}^{8} \chi_1^i \chi_2^i i e_n Q W_\infty (x,y)_{|y=x} \tag{2.9.18}$$

$$= \frac{1}{2} \int_{|x|=R} \operatorname{tr} i e_n Q W_\infty (x,y)_{|y=-x} \tag{2.9.19}$$

We also must consider the flat directions along x_3^9. This part of W_∞^{flat} cannot contribute to the trace. Note that for ϕ_0 in the zero eigenspace of the oscillator associated to the flat direction along x_3^9, we have that $\chi_1^i \chi_2^i \phi_0 = -8\phi_0$. So we can replace the χ fermions in τ with a constant. The gauge action is identically zero on ϕ_0, so we could replace the $\psi_1\psi_2$ terms with $\chi_1\chi_2$ terms. But we cannot get rid of the ψ_3 terms, so the trace vanishes.

Now consider:

$$W_\infty^{\text{flat}} = \begin{pmatrix} W_1 & -W_1 \left(b^t + \tilde{b}^t\right) W_2 \\ 0 & W_2 \end{pmatrix}. \tag{2.9.20}$$

Away from $x = y$, W_2 is smooth, and satisfies $\|QW\|^2 > c \left\|r^{1/2} W\right\|^2$ so $QW_2|_{y=-x}$ decays much more rapidly than the required $1/r^9$ and this term traces to zero in the limit $R \to \infty$.

We have left only to compute the term associated to W_1. Recall from (2.7.85):

$$W_1 = \Delta^{-1} - \Delta^{-1}\tilde{V}\sum_{k=0}^{\infty}(-1)^k(\tilde{V'}\Delta^{-1}\tilde{V})^k\tilde{V'}\Delta^{-1} \tag{2.9.21}$$

Let ω_7 denote the volume of S^7. Then Δ^{-1} has the kernel:

$$\frac{1}{6\omega_7 g\left(x-y,x-y\right)^3} \tag{2.9.22}$$

where g^{ij} denotes the inverse of the metric g_{ij} and $g_{ij} = \delta_{ij} + \ell_{ij}\left(x\right)$, and $|\ell_{ij}| \leq 1/\mu^5$. In section 2.7.2 we have shown that the terms in W_1 with $k > 3$ must vanish in the trace. For $k < 3$, the only term with enough Clifford factors to saturate the trace is the very first term. Now, since $QW_1\left(x,y\right)$ must be evaluated at $y=-x$:

$$\int\limits_{|x|=R} \mathrm{tr} i e_n Q W_1\left(x,-x\right)\mathrm{d}x$$

$$= \int\limits_{|x|=R} \mathrm{tr} i e_n Q\left(\frac{1}{6\omega_7\left|x-y\right|^6}\right)\Bigg|_{x=-x} + \int\limits_{|x|=R} \mathrm{d}x \mathrm{tr} i e_n\; Q\left(W_1 - \frac{1}{6\omega_7\left|x-y\right|^6}\right)\Bigg|_{x=-x} \tag{2.9.23}$$

The second term vanishes in the limit $R\to\infty$. Now, the defect term is:

$$\begin{aligned} I_\delta &= \int\limits_{|x|=R} \mathrm{tr} i e_n Q\left(\frac{1}{6\omega_7\left|x-y\right|^6}\right)\Bigg|_{x=-x} \\ &= 2^{3+M/2}\frac{1}{\omega_7}\int\limits_{S^7(r)}\frac{1}{|2x|^7} \\ &= 2^{M/2-4} \end{aligned}$$

and so finally:

$$I_\delta = 2^{M/2-4} \tag{2.9.24}$$

where M is the number of $\chi_1^i\chi_2^i$ terms.

2.10 Calculating the Principal Contribution

Note: in this section, we rename $\beta_0 = \beta$ to simplify notation. The principal contribution is:

$$\lim_{\beta\to 0}\int_{|x|<R} \mathrm{tr}\tau e^{-\beta Q^2} P_{\mathscr{G}} dx, \tag{2.10.1}$$

where $P_{\mathscr{G}}$ denotes projection onto gauge-invariant states. This computation is analogous to the boundary term computed in section 2.5. As before, we write:

$$e^{-\beta Q^2} = \int_\gamma \frac{e^{-\beta z}}{2\pi i}\left(Q^2 - z\right)^{-1} \mathrm{d}z \tag{2.10.2}$$

Now let:

$$\sigma_z = 4\pi^2 \left|\eta\right|^2 + V - z \tag{2.10.3}$$

Define:

$$L = \Delta + 4\pi i \eta_i \frac{\partial}{\partial x_i} + H_F \tag{2.10.4}$$

Approximating the inverse of $Q^2 - z$:

$$\left(Q^2 - z\right)^{-1} \approx \int e^{2\pi i\eta\cdot(x-y)}\sigma_z^{-1}\sum_{k=0}^{N}(-1)^k\left(L\sigma_z^{-1}\right)^k \mathrm{d}\eta \qquad \stackrel{\mathrm{def}}{=} W_z^N \tag{2.10.5}$$

$$\int_{|x|<R} \mathrm{tr}\tau \int_\gamma \frac{e^{-\beta z}}{2\pi i}\left(Q^2 - z\right)^{-1}\mathrm{d}z = \tag{2.10.6}$$

$$\int_{|x|<R} \mathrm{tr}e_n\tau \int_\gamma \frac{e^{-\beta z}}{2\pi i} W_z^N \mathrm{d}z$$

$$+\int_{|x|<R} \mathrm{tr}e_n\tau \int_\gamma \frac{e^{-\beta z}}{2\pi i}\left(\left(Q^2 - z\right)^{-1} - W_z^N\right)\mathrm{d}z \tag{2.10.7}$$

We retain the same definitions of p, q, and M as in section 2.5. We have M representing the number of χ^i terms, p the number of H_F terms and q the number of gauge terms. Also note that the error term associated to this contribution vanishes by exactly the same argument as in the case of the boundary term (2.5.8). So, focusing our attention to the first integral above, we again note that the trace is zero whenever a term does not have Clifford degree 0. Thus, we compute only terms which saturate the trace with Clifford factors. In this case, the only terms which contribute Clifford factors are H_F and the gauge action. Thus, we'll need $p + q = 12 + M$. The term lowest order in β that may contribute to the trace is:

$$\int\limits_{|x|<R} \mathrm{tr}\tau \int\limits_{\gamma} \frac{e^{-\beta z}}{2\pi i} \int e^{2\pi i\eta\cdot(x-y)} \sigma_z^{-p-1} H_F^p P_{\mathscr{G}} \mathrm{d}\eta \mathrm{d}z \tag{2.10.8}$$

Computing the contour integral:

$$\int\limits_{\gamma} \frac{e^{-\beta z}}{2\pi i} \int e^{2\pi i\eta\cdot(x-y)} \sigma_z^{-p-1} H_F^p \mathrm{d}\eta \mathrm{d}z = \beta^p \int e^{-\beta\left(4\pi^2|\eta|^2+V\right)} e^{2\pi i\eta\cdot(x-y)} H_F^p \mathrm{d}\eta \tag{2.10.9}$$

and now the η integral:

$$\int \beta^p e^{-4\pi^2\beta|\eta|^2} e^{2\pi i\eta\cdot(x-y)} e^{-\beta V} H_F^p \mathrm{d}\eta = \frac{\beta^p}{(4\pi\beta)^{17/2}} e^{-\frac{|x-y|^2}{4\beta}} e^{-\beta V} H_F^p \tag{2.10.10}$$

Again, we average over the gauge group:

$$\int\limits_{|x|<R} dx \int\limits_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathrm{tr}\tau e^{\theta\left(\chi_1^i\chi_2^i+\psi_{1s}\psi_{2s}\right)} \frac{\beta^p}{(4\pi\beta)^{17/2}} e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}} e^{-\beta V} \tag{2.10.11}$$

Rescaling $\theta \to \beta^{3/4}\theta$ and replacing

$$e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}} \approx e^{-\theta^2\beta^{1/2}\left(x_1^2+x_2^2\right)} \tag{2.10.12}$$

for small β, we obtain:

$$\int_{|x|<R} dx \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathrm{tr}\tau e^{\theta\left(\chi_1^i\chi_2^i+\psi_{1s}\psi_{2s}\right)} \frac{\beta^p}{(4\pi\beta)^{17/2}} e^{-\frac{|x-\mathscr{R}_\theta x|^2}{4\beta}} e^{-\beta V} H_F^p$$

$$\approx \int_{|x|<R} dx \int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \beta^{3/4}\frac{d\theta}{2\pi} \mathrm{tr}\tau e^{-\theta^2\beta^{1/2}\left(x_1^2+x_2^2\right)} e^{\beta^{3/4}\theta\left(\chi_1^i\chi_2^i+\psi_{1s}\psi_{2s}\right)} \frac{\beta^p}{(4\pi\beta)^{17/2}} e^{-\beta V} H_F^p \tag{2.10.13}$$

Expanding $e^{\theta\left(\chi_i^1\chi_i^2+\psi_{1s}\psi_{2s}\right)}$ and considering the lowest order contributing term:

$$\int_{|x|<R} dx \int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \beta^{3/4}\frac{d\theta}{2\pi} \mathrm{tr}\tau e^{-\theta^2\beta^{1/2}\left(x_1^2+x_2^2\right)} \left(\beta^{3/4}\theta\right)^q$$

$$\times \left(\chi_1^i\chi_2^i+\psi_{1s}\psi_{2s}\right)^q \frac{\beta^p}{(4\pi\beta)^{17/2}} e^{-\beta V} H_F^p \tag{2.10.14}$$

Rescaling $x^i \to x^i\beta^{1/4}$:

$$\int_{|x|<R\beta^{1/4}} dx \int_{-\pi/\beta^{3/4}}^{\pi/\beta^{3/4}} \beta^{12}\frac{d\theta}{2\pi} \mathrm{tr}\tau e^{-\theta^2\left(x_1^2+x_2^2\right)} \left(\beta^{3/4}\theta\right)^q$$

$$\times \left(\chi_1^i\chi_2^i+\psi_{1s}\psi_{2s}\right)^q \frac{\beta^{3p/4}}{(4\pi)^{17/2}} e^{-V} H_F^p \tag{2.10.15}$$

Setting $p+q = 12+M$ and counting $\beta's$:

$$\beta^{\frac{3(p+q)}{4}-12} = \beta^{\frac{3(M+12)}{4}-12} \tag{2.10.16}$$

$$= \beta^{3M/4-3} \tag{2.10.17}$$

From the above, we see that the principal term vanishes when β is taken to zero more slowly than R^{-4}, for $M > 4$. Putting this result together with the defect term calculated in the previous section, we obtain:

$$L^2 - \text{Index}\,(Q_+) = I_p + I_\delta \tag{2.10.18}$$

$$= 0 + 2^{M/2-4} \tag{2.10.19}$$

$$= 2^{M/2-4} \tag{2.10.20}$$

and we have just proved the main theorem, 2.2

Chapter 3

Conclusion

We computed the L^2 index of our operator Q_+. Since the index is non-zero, we conclude that bound states do exist. There are a few other observations that we would like to note.

First, the number of χ^i terms being at least 8 is necessary to prove the vanishing of the boundary term corresponding to the principal contribution. If there are fewer than 8 terms our $\beta \to 0$ limit does not exist. Once we have 8 terms, the principal contribution vanishes and contributions to the index come solely from the flat directions. This is different from the related problem of calculating the L^2 index of the operator associated to $D0$ brane binding in the absence of the orientifold. There, the principal term contributes, but its contribution is non-integral. In that case the non-integral principal term is complemented by a non-integral defect term, which of course gives an integral index.

Another interesting feature of both this model and the related $D0$ case is that when Q^2 is decomposed in terms of oscillators transverse to flat directions, the

$1/r^2$ terms always exactly cancel. In this model, the additional flat branch along the x_3^9 direction also has this property, provided there are exactly 8 χ terms.

We note in conclusion, that the increased symmetry of this problem (as compared to the $D0 - D0$ case) could allow for an explicit description of the kernel of Q.

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Biography

Janice McCarthy was born on May 3, 1966 in New York City. She received a B.S. from SUNY Stony Brook in May, 1995 with majors in Mathematics and Biochemistry. From July 1995, until September 1997, Janice was a graduate student in the Department of Biochemistry at Duke University. While in this program, she was awarded graduate fellowships from the Duke Program in Biophysics and the Program in Mathematics and Molecular Biology. She received a Master's degree in 1997. After a brief hiatus as a system administrator in the Pratt School of Engineering at Duke, she entered the graduate program in Mathematics. During her tenure as a mathematics graduate student she produced (with the help of her husband, Martin Steinmeyer) what will surely turn out to be the greatest accomplishment of her life: 2 beautiful boys. Zachary Benjamin McCarthy, born February 9, 2001, and Julian Marek Steinmeyer, born July 1, 2004.