How the ideal jamming point illuminates the world of granular media

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The zero temperature properties of frictionless soft spheres near the jamming point have been extensively studied both numerically and theoretically; these studies provide a reliable base for the interpretation of experiments. However, recent work by Ikeda et al. showed that, in a parameter space of the temperature and packing fraction, experiments to date on colloids have been rather far from the theoretical scaling regime. An important question is then whether theoretical results concerning point-J are applicable to any physical/experimental system, including granular media, which we consider here. On the surface, such a-thermal, frictional systems might appear even further from the idealized case of thermal soft spheres. In this work we address this question via experiments on shaken granular materials near jamming. We have systematically investigated such systems over a number of years using hard metallic grains. The important feature of the present work is the use of much softer grains, cut from photoelastic materials, making it possible to determine forces at the grain scale, the details of the contact networks and the motion of individual grains. Using this new type of particle, we first show that the contact network exhibits remarkable dynamics. We find strong heterogeneities, which are maximum at the packing fraction $\phi^*$, distinct from and smaller than the packing fraction $\phi^1$, where the average number of contacts per particle, $z$, starts to increase. In the limit of zero mechanical excitation, these two packing fractions converge at point J. We also determine dynamics on time scales ranging from a small fraction of the shaking cycle to thousands of cycles. We can then map the observed system behavior onto results from simulations of ideal thermal soft spheres. Our results indicate that the ideal jamming point indeed illuminates the world of granular media.

1 Introduction

In a loose sense, jamming describes everyday situations where particles, objects, or people become dense, slow and rigid: one thinks of systems as different as sand piles, foams, or traffic jams as jammed systems. Significant progress was achieved in the field about ten years ago, when frictionless soft spheres at zero temperature were introduced as a minimal and seminal model for jamming. This system has been extensively studied and now provides a paradigm which gives jamming a precise meaning. Specifically, for models where non-zero contact forces are represented by particle overlaps, the jamming transition occurs when a small compression requires overlaps. Hence, jamming involves satisfying geometric constraints, and indeed, an algorithmic description based on this principle has been established. For athermal systems, the jamming transition is intrinsically out-of-equilibrium, which means that the protocol used to evolve a system must be precisely characterized. However, many features of the transition appear to be protocol independent, and for a given protocol applied to an infinite system, the jamming transition is simply controlled by the packing fraction. Jamming occurs at “point J”, and coincides with the onset of isostaticity. This latter property corresponds to a condition that the number of steric and mechanical constraints imposed at the contacts exactly matches the number of degrees of freedom available to the particles.

A number of geometrical and mechanical properties exhibit scaling laws or other singular behavior that involve the distance to point-J. One prominent signature of jamming is the singular behavior of the average number of contacts per particle; above jamming, $z - z_J \propto (\phi - \phi_J)^\alpha$, where $\alpha \approx 0.5$. The value $z_J$ is equal to $2d$, for frictionless particles, and lies between $d + 1$ and $2d$ for frictional particles. Here, $d$ is the space dimension, and $\phi_J$ is the packing fraction at point J. At jamming, the distribution of the gaps between particles displays a delta function at zero and square root decay for increasing gaps, which is at the root of the singular behavior of the average contact number.
The framework that has developed, based on studies of frictionless particles, has provided key insights into the nature of rigidity, and the structure/mechanics of disordered soft matter systems, such as emulsions, foams and grains. Of course, this idealized model misses some of the key features of real systems, such as friction for dry systems, interface effects for multiphase systems, or hydrodynamic interactions for suspensions. In particular, several studies have shown that the jamming scenario for static packing becomes more complex when friction comes into play.

Furthermore, the above discussion applies to static packing. And, many systems of interest are not static: colloidal suspensions undergo thermal agitation, vibrated or flowing granular systems undergo mechanical agitation, and so on. Whether the jamming framework is relevant in the presence of driving or agitation was until recently a matter of debate. Several issues are at play for non-static systems. On the one hand, the singular nature of the jamming point may be blurred. On the other hand, anomalous dynamics may occur because the particle response to agitation may be influenced by the proximity of the singular jamming point (see Fig. 1). A recent numerical study of harmonic spheres, in the presence of agitation was until recently a matter of debate. Several studies have shown that the jamming scenario for static packing becomes more complex when friction comes into play.

The authors demonstrate that: (1) there is no singularity at finite temperatures; (2) there is a critical region in the vicinity of the jamming point, where vibrational dynamics is maximally heterogeneous; (3) in the temperature-packing-fraction parameter space, there is a crossover along lines, originating at point \( J \), between harmonic and non-harmonic regimes. On the basis of the dynamical behavior reported in the literature, Ikeda et al. placed existing colloidal experiments in the temperature-packing-fraction parameter space. Their main conclusion is that these experiments are actually rather far from the critical regime of point \( J \).

In the past ten years, we have systematically investigated systems of horizontally shaken grains in the vicinity of jamming. For systems of rigid brass disks, we observed very large heterogeneities of the dynamics, consisting of very small displacements, on the order of \( 5 \times 10^{-11} \) grain diameters. In this work, we conjectured that these heterogeneities were tied to the dynamics at the contact scale, something that we later confirmed using soft photoelastic disks. For the softer particles, the signature of the dynamical heterogeneities was less sharp. But, the contact network exhibited remarkable dynamics, with strong heterogeneities, that were maximum at the packing fraction \( \phi^* \). This packing fraction was distinct from and smaller than the packing fraction \( \phi^*_J \), for which the average number of contacts per particle started to increase. Furthermore, by varying the vibration frequency, we observed that these two cross-overs merged in the limit of zero mechanical excitation.

The apparent similarities shared by these experimental results and those reported in the numerical study of thermal soft spheres call for further investigation. The goal of the present work is to determine the extent of overlap between model systems of thermal harmonic spheres and systems of shaken granular packings. To address these questions, we present results for the dynamics of photoelastic disks that are novel in the breadth of their time scales, spanning times that are a fraction of a vibration cycle to thousands of cycles. To achieve the shortest times, we use a stroboscopic technique. The longest times studied here are comparable to those in previous studies.

From the present study, we obtain a reasonably complete picture of the dynamics, the forces and the contacts close to jamming in the presence of mechanical agitation. Salient conclusions from this work include: (i) a reconciliation of hard and soft grain experiments; (ii) placement of granular experiments in a “temperature”-packing fraction phase diagram. From (ii), we establish the relevance of the jamming framework for describing driven granular systems. We conclude that our granular experiments do probe the same critical regime as those described in ref. 1. Conversely, this validates the use of soft sphere models to describe such experimental systems close to jamming.

The paper is organized as follows. In Section 2, we describe the experimental set up in detail, emphasizing the two modes of data acquisition, a fast one and a slower stroboscopic one. This dual approach allows us to explore the dynamics over six orders of magnitude in the timescales. Section 3 demonstrates that the force network is essentially isotropic and Section 4 characterizes the dynamics of the contact network. This section summarizes the results already reported in ref. 27 and supplements these results with the dynamical properties of the contacts at short timescales. Section 5 presents results for mean square particle displacements, and explicitly details the data processing required to obtain a meaningful computation of these displacements. Section 6 analyzes the dynamical heterogeneities of the displacement field, relates them to those of the contact dynamics, and shows that they are embedded in the structural properties of the contact network. Finally, Section 7 summarizes our observations, and relates them to the previous study of brass disk experiments performed in the same set-up. These results allow us to address the correspondence...
between thermal soft-sphere models and experiments on vibrated grains, in terms of dynamical behavior in the vicinity of point $J$.

## 2 Setup and protocol

We first review details of the experimental set-up, which was modified from earlier apparatus in order to allow for the use of photo-elastic grains and the detection of contacts. We also review different acquisition techniques, emphasizing in particular the use of fast imaging in order to characterize the dynamics within one vibration cycle. By contrast, previous studies involved the acquisition of one image per vibration cycle.

### 2.1 Setup

The experimental setup is sketched in Fig. 2. A bi-disperse mixture of ~8000 4 mm and 5 mm photoelastic disks (PSM-4) (a) lies on a glass plate, and is confined laterally in a cell (b), whose area can be tuned with a piston (c). The piston is attached to a force sensor (d) and a micro-metric stepper motor (e). The packing fraction, $\phi$, can be fine-tuned from 0.795 to 0.83, with a resolution of $\Delta \phi = 5 \times 10^{-6}$. Below the glass sheet, an LED backlight device, covered with a polarizing sheet, provides an intense, large, thin and uniform source of circularly polarized light. The glass sheet and the light are embedded in a frame (f), whose area can be tuned laterally in a cell (b).

The glass sheet and the light are embedded in a frame (f), which vibrates horizontally with an amplitude $a = 1$ cm and frequencies $f = 6.25$, 7.5 and 10 Hz.

Fig. 2 Sketch of the experimental setup. (a) Photoelastic grains lighted by transmission by a polarized backlight. (b) Confining cell. (c) Wall piston. (d) Force sensor. (e) Micrometric stepper motor. (f) Vibrating frame. (g) Stepper motor ensuring vibration. (h) Notched belt transmitting vibration. (i) Shelf. (j) Wall. (k) Translation stages. (l) Stainless steel bars. (m) Optical table. (n) CCD camera. (o) Analyzers located on a rotating wheel. (p) Shelf isolated from vibrations.

The oscillation of the sheet on which the particles rest is driven by a stepper motor (g), a notched belt (h) and an eccentric revolving shaft, which are attached to a rigid base (i), the stability of which is ensured by 300 kg of lead brick ballast and a rigid bracket to the wall (j). The confining cell is mechanically decoupled from the vibration devices. It is embedded in a larger frame, which in turn is attached to four manually adjustable micrometric translation stages (k). This ensures a precise leveling of the confining cell with respect to the oscillating board. The translation stages are attached to stainless steel bars (l), which are screwed to an optical table (m). Also attached to the optical table is a device to trigger the camera which images the grains. The trigger mechanism consists of a reflection photo-transistor/photo-diode device that provides input to a Schmitt trigger. The photo-diode illuminates the revolving shaft; a small piece of non-reflecting black tape on the shaft causes a low light signal to the electronics when it passes in front of the light source. When the tape is out of the light path, the signal from the detector and following electronics is 5 V; when the tape interrupts the light path, the signal falls to 0 V. We choose a falling output signal to trigger the camera, and the tape is located such that the velocity of the plate is zero when the camera is triggered. We choose the taped-to-untaped length ratio such that the transients of the stepper motor (which rotates the second polarizer in and out of the light path – see below) and the exposure times occur separately. The camera is fixed on a shelf (p), lying on an optical table and isolated through a rubber gasket in order to reduce the transmission of vibrations and minimize blur on the pictures.

### 2.2 Data acquisition

We investigate the dynamics, both at short times – for time differences that are a fraction of a vibration period, and at long times – up to several thousand vibration periods. Altogether, the experiment covers seven decades of time steps and, apart from the force sensor (d), all our data come from image acquisition. We thus need to conduct two separate series of experiments, one with a fast camera, running continuously, and one with a standard CCD camera, triggered by the motion of the oscillating plate. In both cases, we access the position of the grains and the photoelastic pattern inside the grains. This cannot be achieved simultaneously, and we need to adapt the acquisition schema in order to be as close as possible to this ideal situation.

To record the displacements and the force network dynamics at short times, we use the following procedure. First, we use a fast camera (2000 frames per s) with a resolution of $1024 \times 1024$ pixels. We record 1361 frames, spanning up to 6 vibration cycles (for the largest vibration frequency of 10 Hz). We successively acquire two movies, with and without introducing an analyzer in the field of view of the camera. Only a few tens of vibration cycles separate the two acquisitions. Since the dynamics are completely frozen on this time scale (see below), the packing barely moves, and by synchronizing the two movies, we can correctly correlate particles that appear in the polarized and unpolarized images.
The long time dynamics are recorded with a high resolution (2048 × 2048) CCD camera (n) triggered in such a way that the images are taken in phase with the motion of the oscillating base. Analyzers (o) located on a low-inertia rotating wheel are inserted in the field of view of the camera once every two cycles. The rotating wheel has alternating analyzers and open disk-shaped regions so that successive forward 60° rotations of the wheel alternately place an analyzer or an open space in the light path to the camera. The rotations are triggered off the rotating belt, and each rotation is fast enough so that there is sufficient time following a rotation to obtain an image with the camera. Thus, every pair of cycles produces one cross-polarized image, and one non-crossed image. It is then straightforward to match pairs of the photoelastic and non-crossed images of the grains. In order to minimize blur, the pictures are taken at the phase during each cycle for which the shaking velocity vanishes. This has direct consequences on the contact number measurement, as we show below.

There are several additional features that improve the experiment. The stepper motor that switches polarizer positions is attached to the ceiling, to avoid camera shake. And, in order to prevent thermal expansion of the grains due to heating, the LED backlight is also triggered on the vibration and flashes only for 6 ms, corresponding to the exposure time of the camera.

From the direct (i.e. without crossed polarizers) images, we extract grain positions, and diameters (black circles in Fig. 3(b)), on which we perform Delaunay triangulation (red lines in Fig. 3(b)) and Voronoi tessellation (blue lines in Fig. 3(b)). We obtain grain positions with a resolution of 0.005d. Once we have determined the positions of the grains, we then estimate the pressure within each grain by integrating the square gradient of the cross-polarized light intensity over the disc area. We denote this estimate of the pressure in grain i by $G_i$. The resolution of the cross-polarized images is not good enough to carry out a force inverse algorithm for the grain scale contact forces.28 However, we can estimate them as follows. For each interparticle contact, we use the two particle positions and the positions of their two common Voronoi vertices to build the triangles typified by a, b, c and d in Fig. 3(b). We then compute the spatial gradient of the associated cross-polarized image (see Fig. 3(a)), and we sum this signal within each of the triangles (see Fig. 3(c)). This defines $G_a^2$, $G_b^2$, $G_c^2$ and $G_d^2$.29 We then estimate the normal force of each link, $F_N$, by $F_N = (G_a^2 + G_b^2 + G_c^2 + G_d^2)/2$. In the same way, we estimate the tangential force of each link by $F_T$, by $F_T = (G_a^2 - G_b^2 + G_d^2 - G_c^2)/2$.

2.3 Calibration and units
To validate our use of $G^2$ as a measure of force, and then to calibrate the overall photoelastic response, we compute $G^2$ summed over the complete experimental space, and compare this average with the force $F$, measured by the force sensor located at the piston (d), (and normalized by $M_g$, the total weight of the grain assembly) as in Fig. 4(a). We observe a linear relationship between $F$ and $G^2$. We then use this same linear relationship to calibrate the local $G_i^2$.

In the following, all pressures and forces computed using the photoelastic images are expressed in units of $M_g$. Lengths are expressed in units of the small grain diameter and time is expressed in units of the microscopic time determined by the stiffness of two compressed discs: $t_0 = (k/m)^{-1/2}$, where $m$ is the mass of a grain ($\sim 3.75 \times 10^{-5}$ kg) and $k$ is the stiffness of two compressed disks ($\sim 1.5 \times 10^3$ N m$^{-1}$).

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2.4 Protocol: obtaining a granular glass

As noted in the Introduction, the jamming transition is intrinsically a $T = 0$, out-of-equilibrium transition, and therefore depends on the preparation protocol of the packing. This situation may still apply in the presence of thermal or mechanical vibration. An additional source of complexity is that, for the packing fractions of interest, most systems undergo dynamical arrest, and exist in non-equilibrium glassy states. The steep increase of the relaxation times associated with glassy behavior seriously hampers experimental work.\textsuperscript{30-33} samples brought to packing fractions near and above jamming are well into the glass phase, and are difficult to manipulate on reasonable timescales. For athermal granular media, the situation is further complicated by the fact that they need the input of mechanical energy to persist in a non-equilibrium steady-state (NESS). As for thermal systems, the preparation requires extremely slow compaction of the sample in order to avoid aging on the experimental timescales.\textsuperscript{34,35} For that reason, many dynamical granular experiments actually probe the glass transition and not the jamming one.\textsuperscript{36-38}

In order to minimize such effects, we perform an annealed compaction (Fig. 4(b)) during which we increased the packing fraction by a constant amount of $\delta \phi = 3 \times 10^{-4}$, with exponentially increasing time steps. Then, the packing fraction is stepwise decreased, and measurements are performed between the decompression steps (Fig. 4(a) and (b)). Lechenault et al.\textsuperscript{24} checked that the dynamics was reversible and stationary on experimental time scales during these decompression steps. As suggested in Fig. 5, the structure of the packing we obtain following the above protocol is frozen: the superimposition of two Voronoi tessellations, separated by a time lag of 5000 vibration cycles, displays very few rearrangements, even for the lowest packing fraction. Such rearrangements are further quantified by $Q^{nn}(\tau)$, the average fraction of neighbor relationships surviving in a time interval $\tau$. $Q^{nn}$ vs. lag time, $\tau$, remains larger than 95% even for the loosest packing fraction, and barely departs from 1 for the densest ones (Fig. 5(b)). In the language of the glass community, “there is no $\alpha$ relaxation”, meaning that the density profile survives on the experimental time-scales, and the system can correctly be considered as a glass, the structure of which is essentially frozen.

Finally, note that despite the fact that we use the same protocol for each experiment, the initial conditions are different for each run. Also, the system size is finite, and therefore, the jamming transition of each packing fluctuates from one realization to another, a fact that must be kept in mind when comparing different experimental runs.

3 Pressure and contact forces

In the ideal case of soft spheres at zero temperature, the pressure inside a packing exhibits the same basic features as the energy: below jamming, pressure is strictly zero, and above jamming it grows with the packing fraction, according to the interaction force between particles. It is thus of interest to provide insight into the jamming transition in a system with dynamics to examine the dependence of the pressure on the packing fraction.

Fig. 6 displays the pressure measured at the wall as a function of the packing fraction. $P_{\text{TOT}}$ (respectively $P_{\text{STAT}}$) is the pressure measured when the vibration is applied (vibration on) or not applied (vibration off). $P_{\text{STAT}}$ corresponds to the static pressure of the packing, whereas $P_{\text{DYN}} = P_{\text{TOT}} - P_{\text{STAT}}$ is the dynamic part of the pressure that comes from the vibration. The data show a smooth crossover from a constant, but nonzero pressure, to a pressure that increases with the packing fraction. For packing fractions on the high side of the crossover, $P_{\text{TOT}} = P_{\text{STAT}}$ and the pressure, which is mostly static, follows what is expected from the zero temperature prediction. That is, it increases with the packing fraction, according to the particle stiffness. On the low packing fraction side, there is an inherently kinetic character to the pressure, induced by the vibration. As a first estimate, the crossover corresponding to jamming can be identified with the packing fraction where the static pressure...
becomes larger than the kinetic pressure. However, the static part of the pressure is not strictly zero below the cross-over, which we attribute to the mobilization of friction at the contacts, when the vibrations are stopped. We return to this issue and the possible roles of friction in the Discussion section.

The kinetic part of the pressure, which is observed on the loose side of the jamming crossover in these experiments, differs from what occurs in a thermal liquid, where the pressure has a collisional origin. Here, the forcing is periodic, a priori strongly anisotropic, and involves friction with the base. The particles are accelerated along the vibration axis, then compressed along one wall, before being accelerated back in the reverse direction. A clearer idea of the resulting dynamics comes from decomposing the instantaneous system-average contact forces for a few vibration cycles into the vibration and transverse directions: 

\[ F_x = \sqrt{\langle \vec{f}_{ij} \cdot \vec{e}_x \rangle^2} \] 

and 

\[ F_y = \sqrt{\langle \vec{f}_{ij} \cdot \vec{e}_y \rangle^2} \] 

where \( \langle \cdot \rangle \) is the average over space, and where \( \vec{e}_x \) and \( \vec{e}_y \) are unit vectors along the vibration and the transverse directions.

Fig. 7(a) shows that for low packing fractions, there are strong oscillations of the contact forces at the vibration frequency, corresponding to compression and relaxation of the grains at the side walls. These oscillations are in phase, within the temporal resolution of the acquisition; the transfer of momentum, from the vibration direction to the transverse direction, is instantaneous, on the time scales considered here.

For packing fractions above the kinetic-to-static crossover, the oscillations are much pronounced, and the periodicity not so clearly defined. In that regime, the global motion of the grains with respect to the oscillating plate is reduced. Finally, we time-average the photoelastic signal, and plot these data as functions of the packing fraction in Fig. 7(b). We find that the same trends occur as those for the pressure measured at the wall, albeit with larger fluctuations, since the temporal sampling is much smaller. Note that the average force in the direction of vibration is only slightly larger than the average force in the transverse direction, indicating the formation of a rather isotropic force network. We confirm this by a direct inspection of \( G_\alpha \), the pressure field inside each grain. We interpolate this quantity onto a Cartesian grid in order to compute the spatial auto-correlation function, as shown in Fig. 8. Both visual inspection and the auto-correlation function confirm a good level of isotropy of the pressure distribution in the packing.

We conclude this section by considering the probability distribution of the normal and tangential interparticle forces, \( F_N \) and \( F_T \). These are defined by 

\[ \vec{f}_{ij} = F_N \vec{e}_N + F_T \vec{e}_T \] 

where \( \vec{e}_N \) and \( \vec{e}_T \) are the normal (respectively tangential) vectors between the two grains \( i \) and \( j \). To be precise, we consider the distribution of

\[ F_N = \langle F_N(i,j) \rangle/G(i,j) \] 

and

\[ F_T = \langle F_T(i,j) \rangle/G(i,j) \] 

where \( G(i,j) \) is the instantaneous average over the particles and \( G(i,j) \) is the average over time and space. This normalization captures the shape and width of the distributions, irrespective of the temporal variability of the packings and has the advantage of avoiding spuriously large tails in the distributions. Since the normalization is the same for \( F_N \) and \( F_T \), the ratio of \( F_T/F_N \) is close to 0.2, on average, and always smaller than the estimate of 0.7 for the static friction coefficient between the PSM-4 disks. We note an accumulation of contacts close to the threshold value \( \mu_s \), especially at low forces, where a gap in the distribution carefully separates the majority of contacts with \( F_T/F_N = 0.2 \) from a secondary peak of contacts with \( F_T/F_N < \mu_s = 0.7 \). These so-called “critical contacts” are on the verge of slipping. Whether these slipping events are trivial fluctuations, or contain some interesting correlations in the vicinity of the jamming crossover, was the central issue discussed recently by the present authors.

4 Dynamics of the contact network

In order to measure \( z(t) \), the number of contacts of particle \( i \) at time \( t \), one must identify the potential contacts of a particle with its neighbors by thresholding the normal force, \( F_N \), and the

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**Fig. 7** Short time photoelastic response. (a) Average interparticle force decomposed into the vibration \( F_x \) and transverse \( F_y \) directions vs. time \( t \) for packing fractions \( \phi = 0.8079 \) (blue), 0.8123 (green) and 0.8196 (red). (b) Time averaged quantities: \( F_x \) (blue), \( F_y \) (red) and \( F = (F_x + F_y)/\sqrt{2} \) (green) vs. packing fraction \( \phi \). The vibration frequency is \( f = 10 \) Hz.

**Fig. 8** Isotropy of the force network. (a) Interpolated instantaneous force field \( G_\alpha \)'s on a cartesian grid and (b) its associated 2d-autocorrelation. The packing fraction is \( \phi = 0.82 \). The vibration frequency is \( f = 10 \) Hz.
inter-particle distance, s. We have shown\(^\text{27}\) that the overall behavior of \(z(t)\), for both static and dynamic states, remains unchanged when varying the thresholds within a reasonable range. Here, we fix the threshold, and focus on the dynamics of the contact network at both short and long times.

### 4.1 Statics and short time dynamics

Fig. 10(a) shows the average number of contacts \(z\) vs. the packing fraction, computed over stroboscopically acquired images for a vibration frequency of 10 Hz. These data exhibit a clear cusp at a packing fraction \(\phi^\dagger\). For \(\phi > \phi^\dagger\), \(z\) increases with the packing fraction in a way that is similar to what is reported for zero-temperature soft spheres. By contrast, for \(\phi < \phi^\dagger\), \(z\) is non-zero, constant, and lacks a discontinuity at \(\phi^\dagger\), unlike the zero temperature behavior of soft spheres.

A more precise picture of the mechanisms at play behind the shape of the \(z(\phi)\) dependence can be obtained by examining the dynamics of the pressure during a vibration cycle. Fig. 10(b) displays the instantaneous contact number, 

\[
\bar{z}(t) = \frac{1}{N} \sum_{i=1}^{N} z_i(t),
\]

acquired with the fast camera for three different packing fractions. Here, \(N\) is the number of particles. For lower \(\phi\), strong oscillations at the vibration frequency are clearly visible, while they are reduced and less well defined at larger packing fractions. The similarity with the force signals reported in Fig. 7(a) is striking, and is indicative of the fact that the number of contacts is temporarily larger when the grains are compressed against the wall. As a result, the average number of contacts computed from the stroboscopic data depends on the precise phase at which the acquisition is performed, a dependence that is most significant when \(\phi\) is low. This behavior is illustrated in Fig. 10(c), where time averages of \(z\), acquired at different phases, are plotted vs. \(\phi\). In green is the number of contacts, averaged over times which are in phase with the lowest acceleration. In addition, we restrict the average to grains which are away from the walls, which also means that \(z\) is smaller than that for near-wall grains. This situation is indicated by the vertical green dotted lines in Fig. 10(b). At the crossover, the cusp has been replaced by a barely discernible inflection point. In red is the time-averaged contact number, at a phase for which the grains are compressed against one of the walls, making the contact number maximal. This situation is indicated by the vertical red dotted lines in Fig. 10(b). In this case, one recovers the cusp observed in Fig. 10(a), for which the stroboscopic acquisition was in phase with the maximal acceleration and the minimal velocity of the plate. As for the pressure, the signature of the jamming transition is replaced by a crossover, the precise location and shape of which depend on the details of the measurement. Strictly speaking there is no singularity in the

**Fig. 9** Distribution of forces. Distribution of normal (a) and tangential (b) forces. The vibration frequency is \(f \approx 10\) Hz and the packing fractions are the same as in Fig. 5. Joined distribution of tangential \(F_T\) vs. normal \(F_N\) forces for (c) \(\phi = 0.8178\), and (d) \(\phi = 0.8125\); with the same vibration frequency.

**Fig. 10** Static and short time dynamics of the contact network. (a) Average contact number obtained from the stroboscopic data, vs. packing fraction \(\phi\). The dashed line indicates \(\phi^\dagger = 0.8151\). (b) Instantaneous average contact number \(z\) vs. time \(t\), at \(\phi = 0.8079\) (blue), 0.8123 (green) and 0.8196 (red). The green dashed lines indicate the times, \(t^\dagger\), where the contact number is minimal, namely when the grains experience the smallest acceleration. The red dotted lines indicate time frames \(t^\dagger\), where, by contrast, the grains are compressed against a wall. (b) Different temporal averages of the contact number as a function of the packing fraction: in blue is the contact number averaged over all acquisition frames; in green, respectively in red, is the contact number averaged over the time frames \(t^\dagger\), respectively \(t^\dagger\). The vibration frequency is \(f = 10\) Hz.
The results of this section, which concern the long time dynamics of the contact network and the nontrivial correlations that it contains, have been reported previously.

To characterize the dynamics of the contact network, we introduce the contact overlap function, which evaluates how much the contacts have fluctuated between time $t$ and $t + \tau$:

$$Q(t, \tau) = \frac{1}{N} \sum_{i=1}^{N} Q_i(t, \tau),$$  \hspace{1cm} (1)

where

$$Q_i(t, \tau) = \begin{cases} 
1, & \text{if} |z_i(t+\tau) - z_i(t)| \leq 1 \\
0, & \text{if} |z_i(t+\tau) - z_i(t)| > 1
\end{cases}$$  \hspace{1cm} (2)

We have considered other choices of overlap functions and found that the results do not depend substantively on the particular choice. The results below are typical. Fig. 11(a) displays the temporal average of $Q^z(t, \tau)$ for a vibration frequency of 10 Hz and the same set of packing fraction $\phi \in [0.80 - 0.82]$ as in the previous sections. At rather large packing fractions, $Q^z(\tau)$ is essentially constant, with a plateau that is weakly dependent on $\phi$. Hence, there is no relaxation of the contact network on long time scales. The relaxation, which occurs at short times and is responsible for the plateau value, cannot be observed in the present stroboscopic data. However, it is apparently related to the motion of ‘rattlers’, i.e., particles having less than two contacts. This conclusion is suggested by the very strong correlation between the fraction of non-rattling particles and the value of the plateau at short times (see Fig. 11(b)). At lower packing fractions, long time decorrelation occurs. We define the relaxation time of the contact network, $\tau^z$, by $Q^z(\tau^z) = 0.9$. Note that the choice 0.9 in this definition is rather large compared to the commonly used value of 0.5. However, it is the smallest choice which allows a measure of $\tau^z$ for a broad range of $\phi$. For instance, relaxation times measured in the more standard way would be orders of magnitude larger. As shown in Fig. 11(c), left axis, $\tau^z$ increases sharply with the packing fraction, and possibly diverges at the packing fraction $\phi^* = 0.8151$, where the average contact number starts to increase with $\phi$.

Interestingly, the dynamics of the contact network below $\phi^*$ exhibits strong fluctuations and dynamical heterogeneities, albeit of a different kind from those reported in the literature for super-cooled liquids close to their glass transition (see ref. 18). Here, the heterogeneities are relative to the degrees of freedom describing the contacts, not the positions of the particles. To quantify such heterogeneities, we compute the dynamical susceptibility, which estimates the range of spatial correlations in the dynamics of the contact network:

$$\chi^z(\tau) = N \frac{\langle \text{Var}(Q^z(t, \tau)) \rangle}{\langle Q^z(t, \tau) \rangle^2},$$  \hspace{1cm} (3)

Here, $\text{Var}(\cdot)$ denotes the variances sampled over time and $\langle \cdot \rangle_i$ denotes the average over the grains. $\chi^z(\tau)$ has a maximum for $\tau = \tau^z$ (not shown here, see ref. 27), and we have studied how the maximum $\chi^z$ of $\chi^z(\tau)$ depends on both the packing fraction and the vibration frequency. To do so, it was necessary to prepare different packings, and run independent experiments at three different vibration frequencies: $f = 6.25, 7.5$ and 10 Hz. As emphasized in Section 2, the precise values of the jamming transition, and certainly those of the crossovers reported here, depend on the specific packing. Following the methodology of Section 4.1 for each frequency, we identified the structural crossover $\phi^*$. From $\phi^*$, we define the reduced packing fraction $\epsilon = (\phi - \phi^*)/\phi^*$ in order to compare the different experimental runs. Note that for frequencies smaller than $f_0 = 4.17$ Hz, the grains do not slip on the driving plate, and the mechanical excitation is effectively null. Accordingly, we introduce $\gamma = (f - f_0)/f_0$ to quantify the level of mechanical excitation, where $\gamma = 0.5, 0.8$, and 1.4 are the corresponding values for the three driving frequencies.

The results are summarized in Fig. 12 and more details are given by Coulais et al.
the reduced packing fraction, and has a maximum value at a 
negative reduced packing fraction $\epsilon^*$. This indicates a dynam- 
al crossover corresponding to a maximally collective relaxation of the contact network at $\phi$ lower than the structural 
crossover. When $\gamma$ is decreased, (i) $\epsilon^*$ vanishes, i.e., the location of the dynamical crossover moves towards $\phi^*$, and (ii) the magnitude of the maximum $\chi_4^{\gamma^*}$ increases strongly as $1/\gamma$. Hence, we can reasonably conjecture that in the limit of no mechanical excitation, the structural and dynamical crossovers merge, while the length scale associated with the dynamical crossover diverges. This strongly suggests that the experiments lie in the vicinity of a critical point, which in the present case ought to be the jamming transition in the absence of dynamics. In fact, a similar phenomenon occurs for equilibrium systems close to a thermodynamic critical point: thermodynamic susceptibilities exhibit finite maxima along different paths in the parameter space and diverge when these paths intersect at the critical point.\textsuperscript{41,42}

The above results clearly indicate that the mechanical agitation in our experiments blurs the singular nature of the jamming transition. This is similar to the effect of thermal agitation on the jamming transition of soft spheres as reported by Ikeda et al.\textsuperscript{1} One of the remarkable results of that work is a convincing demonstration that all the physics of the soft sphere systems close to jamming can be captured in the time dependence of the particle mean square displacement. Ikeda et al. used this measure as an effective thermometer to place existing colloidal experiments in the temperature-packing fraction parameter space. This raises the question of whether it is possible to extend such an approach to the case of vibrated granular media, and if so, where would the present experiments sit in the equivalent parameter space?

5 Displacement fields

Answering these questions requires determining the particle mean square displacements in the largest possible range of timescales. While this is a straightforward but CPU-costly task in numerical simulations, it requires rather intricate data analysis in the present experiment for several reasons. First, the short time and long dynamics are acquired independently and in different ways. While the long time dynamics is acquired in phase with the vibration, the short time dynamics are acquired within a vibration cycle. The long time acquisition naturally filters the “trivial” motion of the plate, but the short time does not, and we must remove this effect. Second, on long time scales, low-amplitude convection, described below, occurs. Although the resulting flow is never large, it mostly consists of a non-monotonic solid body rotation; we must remove it before computing the mean square displacement.

5.1 Short time oscillations

The motion of the center of mass $(X_0(t), Y_0(t) = (\langle X(t) \rangle, \langle Y(t) \rangle))$ provides a good indication of how energy is injected into the system at large scales. Fig. 13(a) shows that the center of mass oscillates periodically, with a period equal to the forcing frequency. The amplitude of the motion is much larger in the direction of vibration, but part of the forcing generates motion in the transverse direction too. As shown in Fig. 13(b), the amplitudes of the oscillations $A_x^\gamma$ and $A_y^\gamma$ depend on the packing fraction; they are constant at low packing fractions, typically when $\phi < \phi^*$, and they decrease for larger packing fractions, suggesting that energy injection is less efficient at large packing fractions.

In order to investigate the way the energy is transferred to smaller scales, we compute the averaged spectral density of the position fluctuations. Specifically, we define $(\tilde{X}_i(t) = X_i(t) - X_b(t))$, $(\tilde{Y}_i(t) = Y_i(t) - Y_b(t))$, corresponding to the grain trajectories in the frame of reference of the oscillating center of mass. We next compute $\tilde{X}_i^2(f) = ESD(\tilde{X}_i(t) - \langle \tilde{X}_i(t) \rangle)$, and similarly $\tilde{Y}_i^2(f)$, where ESD denotes the Fourier energy spectral density (ESD). We then average over all grains to obtain the spectra $\tilde{X}^2 = \langle \tilde{X}_i^2(f) \rangle$ and $\tilde{Y}^2 = \langle \tilde{Y}_i^2(f) \rangle$. Fig. 14(a) shows representative data.

The energy evolves to high frequencies, an effect which unfortunately preserves the signature of the periodic forcing in the form of strong harmonics. This indicates that considering

![Fig. 12 Towards zero vibrations. Maximal dynamic susceptibility of the contacts, $\chi_4^{\gamma^*}$, vs. reduced packing fraction, $\epsilon$ and reduced vibration magnitude, $\gamma$.](image)

![Fig. 13 Motion of the center of mass. (a) Center of mass position, in the vibration direction $X_0$ (top), and in the transverse direction $Y_0$ (bottom), vs. time $t$, at $\phi = 0.8089$ (blue), 0.8161 (green) and 0.8196 (red). (b) Amplitudes $A_x^\gamma$ (+) and $A_y^\gamma$ (○) vs. packing fraction, $\phi$. The vibration frequency $f = 10$ Hz.](image)
the motion of the grains in the frame of the center of mass is not sufficient to completely filter out the periodic motion induced by the moving plate. We thus further filter the grain trajectories by applying a Butterworth notch filter centered on each harmonic (up to the fifth) and a low-pass Butterworth filter with a cut-off frequency of 5 times the vibration frequency, on $\tilde{X}_i(t)$ and $\tilde{Y}_i(t)$. The filtered spectra, $\Delta^2_f = \tilde{X}_i^2 + \tilde{Y}_i^2$ (Fig. 14(b)) confirm that the harmonics have been successfully filtered out. The resulting motion $\Delta^2_f$ at the lowest frequency ($f_0 = 0.3$), corresponding to a timescale of a few vibration cycles, is a good indicator of the typical cage size in which the particle vibrates. It strongly depends on the packing fraction and sharply decreases as the jamming transition is crossed. The absolute magnitude corresponding to a timescale of a few vibration cycles, is a good indicator of the typical cage size in which the particle vibrates.

In the remainder of the paper, we apply this filtering procedure on each grain trajectory for the fast camera data prior to computing any statistical property.

5.2 Long time rotation

We now turn to the stroboscopic trajectories of the grains. Fig. 15(a) displays the displacement of all grains in the region of interest (ROI), integrated over a lag time $\tau = 6 \times 10^5$. The inset provides a zoomed view of the trajectories for a few grains at the edge of the ROI. These data show a clear global rotation, which, curiously and fortunately, is essentially solid body motion, as demonstrated by the linear dependence of the azimuthal displacement $R_i(t, t + \tau)$ on the distance $R_i(t, t + \tau) = \frac{\tilde{R}_i(t + \tau) + \tilde{R}_i(t)}{2}$ to the center of the cell (Fig. 15(b)). It is fairly easy to remove this solid body rotation from the grain displacements $\Delta \tilde{R}_i(t) = \tilde{R}_i(t + \tau) - \tilde{R}_i(t)$. We define:

$$\tilde{\Delta} \tilde{R}_i(t) = \Delta \tilde{R}_i(t) - \Delta^0 \tilde{R}_i(t).$$

Fig. 14 Spectral properties. (a) Average Fourier energy spectral density $\tilde{X}_i^2$ (blue) and $\tilde{Y}_i^2$ (red) of the grain position fluctuations for $\phi = 0.8196$. The dotted-dashed lines indicate the first ten harmonics of the excitation frequency. (b) Average Fourier energy spectrum density $\Delta^2_f = \tilde{X}_i^2 + \tilde{Y}_i^2$ of grain position fluctuations at $\phi = 0.8089$ (blue), 0.8161 (green) and 0.8160 (maroon), after filtering the trajectories as described in the text. Inset: low frequency limit ($f_0 = 0.3$) of the average Fourier energy spectrum density $\Delta^2_f$, vs. reduced packing fraction, $\epsilon = (\phi - \phi_1)/\phi_1$. The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$.

Fig. 15 Solid body rotation. (a) Grain displacements over a lag time $\tau = 6 \times 10^5$ at $\epsilon = -0.0948$ (inset: zoom on a few grains at the edge of the region of interest). (b) Orthoradial displacements vs. distance to center for a lag time $\tau = 6 \times 10^5$ at $\epsilon = -0.0948$, and $\gamma = 1.4$. (c) $\Omega \tau$ vs. time, $t$, for different lag times $\tau = 10^5$ (blue), $\tau = 10^6$ (green) and $\tau = 10^7$ (red) at a reduced packing fraction, $\epsilon = -0.0948$. (c) Rotational drift coefficient $\Omega$ vs. reduced packing fraction, $\epsilon$. The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$.

where

$$\Delta^0 \tilde{R}_i(t) = \left( \begin{array}{cc} 0 & \Omega(t) \tau / 2 \\ -\Omega(t) \tau / 2 & 0 \end{array} \right) \left( \tilde{R}_i(t) + \tilde{R}_i(t + \tau) / 2 \right).$$

is the solid rotation deformation field. The values of the angular velocity, $\Omega(t)$, and the center of rotation $\tilde{R}_i(t)$ are explicitly computed from the displacements $\Delta \tilde{R}_i(t)$, by minimizing

$$\min \left\{ \left| \tilde{R}_i(t + \tau) - \tilde{R}_i(t) - \Delta \tilde{R}_i(t) \right| \right\},$$

with respect to $\Omega(t)$ and $\tilde{R}_i(t)$. We find:

$$\Omega(t) = -\sum_{i=1}^{N} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \tilde{R}_i(t + \tau) - \tilde{R}_i(t) \right) \left( \tilde{R}_i(t + \tau) + \tilde{R}_i(t) \right) / 2$$

and

$$\tilde{R}_i(t) = \Omega(t) \frac{1}{N} \sum_{j=1}^{N} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \tilde{R}_i(t + \tau) - \tilde{R}_i(t) \right).$$

Fig. 15(c) reveals that $\Omega \times \tau(t)$, the angular rotation between times $t$ and $t + \tau$, fluctuates around zero, meaning that the solid body rotation has no preferred direction. As a result, there is no statistically systematic drift in any direction. However, for any finite time interval, $[t, t + \tau]$, there is a finite angular displacement, the magnitude of which is controlled by $|\Omega|$.

As shown in Fig. 15(d), the angular displacement sharply decreases as the packing fraction increases through the jamming crossovers.
5.3 Resulting vibration dynamics

Once the short time “trivial” oscillating motion and the long time convection have been filtered out, we are in a position to characterize the vibration dynamics of the grains in the frozen structure of the packing on time scales ranging from a hundredth of a cycle to several thousand cycles. We compute the following estimator of the mean square displacement:

\[
\text{MSD} = \frac{\pi}{2} \left( \left\langle \left| \Delta r \right|^{-1} \right\rangle \right)^{-2}
\]

where \(\langle \cdots \rangle\) denotes the average over time and particles, and \(\Delta r\) is the particle displacement obtained from the filtering procedures described in the previous section. The choice of this estimator is motivated by the fact that it assigns a lower statistical weight to very large moves, such as those of the rattling particles. The factor \(\frac{\pi}{2}\) ensures quantitative matching with the proper mean square displacement in the case of gaussian statistics. Alternatively, we could remove the rattling particles, but that strategy requires additional filtering and/or thresholding. Fig. 16 displays the mean square displacement over the full timescale interval probed in this experimental study. We again emphasize that the data at short times (shorter than \(10^3\)) were obtained from the high speed recording of the grain motion within vibrating cycles, while those at long times were obtained by stroboscopic acquisition in phase with the oscillating driving plate. Each type of acquisition was performed during independent experimental runs. The color encodes the packing fraction. The overlap of the mean square displacement at intermediate time scales is remarkably good, and indicates that high speed and stroboscopic data are mutually consistent.

Altogether, one observes three regimes: a ballistic regime at short times \(\tau < \tau_{\text{En}}\) a plateau at intermediate times, \(\tau_{\text{En}} < \tau\), and for low enough packing fraction, a crossover towards a diffusive regime at long times. The plateau regime characterizes the vibrational dynamics we are interested in here. The height of the plateau, \(\Delta^2\), measures the square of the average vibration amplitude of the grains within their cage. It decreases from \(10^{-4}\) to \(10^{-5}\) for increasing packing fractions (Fig. 16(b)), which is consistent with the first estimate of the cage size obtained in Section 5.1 from the low frequency limit of the spectral density of the position fluctuations, \(\Delta^2_0\). The onset of the plateau at short times, estimated by \(\tau_{\text{En}} = (\Delta^2/K)^{1/2}\), where \(K = 10^{-8}\) is obtained from the analysis of the ballistic regime, and typically occurs at \(\tau_{\text{En}} \sim 100\). It decreases slightly with increasing packing fraction (Fig. 16(c)-left axis). This is intuitively reasonable, since the larger \(\phi\) is, the sooner the grains feel their neighbors and enter the vibrational regime.

The above vibrational dynamics is very similar to the one reported by Ikeda et al. for thermal harmonic spheres close to jamming.\(^{44,45}\) In this study, a ballistic regime occurred at short times, followed by a plateau regime, the height of which decreased strongly with \(\phi\), when crossing the jamming point. A plateau exit was also reported in ref. 43, which increases with decreasing quench rate used to prepare the packing. The plateau exit was not reported in ref. 1. However, the maximum lag time there was \(10^4\), and the systems were carefully equilibrated. Hence, for a plateau exit to occur, it would probably be at much larger times than those of the simulation.

Before addressing a more quantitative comparison between thermal soft spheres and mechanically excited grains, we characterize the heterogeneity of the dynamics present in our system. These heterogeneities were first reported in an experiment using brass cylinders\(^{44}\) and more recently in simulations of spheres interacting by harmonic forces.\(^{3}\) They are distinct from those occurring in super-cooled liquids on approaching to the glass transition.\(^{46}\) In the present experiments, the structure is frozen, so that the heterogeneities are not related to the relaxation of the structure. The next section will show how they are related to the heterogeneities of the contact dynamics described in Section 4.2.

6 Dynamical heterogeneities

In this section, we investigate the heterogeneities of the particle displacements. To do so, we focus on the long time stroboscopic
data, with the convective motion subtracted. We will show that the heterogeneities take place at very small scales and are temporally correlated with the heterogeneities of the contact dynamics. Finally, a closer look at the organization of the contacts at short times shows that the heterogeneities are rooted in the short time organization of the contact network associated with the vibrational dynamics of the structure.

6.1 Heterogeneous non-affine dynamics

Characterization of dynamical heterogeneities is now a standard tool for studying the dynamical slowing down of super-cooled liquids and/or colloids approaching their glass transition.\(^4\) It is much less frequently used when probing the jamming transition, but in this case, the same procedure applies.\(^3\) In order to characterize the dynamics, and in particular to probe collective effects, we define a dynamical structure factor for the displacements, \(\Delta f_i(t)\) (defined in eqn (4)):

\[
Q(t, \tau, a) = \frac{1}{N} \sum_i Q_i(t, \tau, a),
\]

where

\[
Q_i(t, \tau, a) = \exp \left( - \frac{\| \Delta f_i(t) \|^2}{2a^2} \right).
\]

This dynamical structure factor characterizes the dynamics at a scale, \(a\), and time \(\tau\): \(Q_i(t, \tau, a) = 0\) (1), when particle \(i\) has moved more (less) than \(a\), during \(\tau\). We then define the dynamic susceptibility:

\[
\chi_s^d(a, \tau) = \frac{N}{\left( \frac{1}{N} \sum_i \text{Var}(Q_i(t, \tau, a)) \right)} \text{Var}(Q(t, \tau, a)),
\]

where \(\text{Var}\) denotes the temporal variance. This susceptibility provides an estimate of the average number of particles which move up to the distance \(a\) during a time \(\tau\) in a correlated manner. In general, \(\chi_s^d(a, \tau)\) has an absolute maximum \(\chi_s^d \ast\) for \(a = a^\ast\) and \(\tau = \tau^\ast\) (see for instance ref. 24).

Fig. 17 illustrates how the dynamical heterogeneities depend on \(\phi\). The most important effect is that \(\chi_s^d \ast\) is non-monotonic and exhibits a clear maximum at precisely reduced packing fraction \(\varepsilon^\ast\) (Fig. 17(b)). The magnitude of \(\chi_s^d \ast\) close to \(\varepsilon^\ast\) is \(\approx 100\), roughly a tenth of the total number of particles. (Even closer to \(\varepsilon^\ast\), the data point indicated by \(\square\) is anomalously low compared to the trend given by the other data points. We believe that this is a signature of the lack of statistics necessary to resolve much larger heterogeneities.) The time, \(\tau^\ast\), where this maximum occurs, is not very sharply defined (note the logarithmic scale for \(\tau\), as is seen from the dependence of \(\chi_s^d(a^\ast, \tau)\) on \(\tau\) (Fig. 17(a))). But it clearly increases significantly when the packing fraction increases and certainly is larger than the times for packing fractions larger than \(\phi^\dag\) (Fig. 17(c)). The length scale, \(a^\ast\), over which the particles move while building up these heterogeneities, decreases with the packing fraction and is of the order of \(10^{-2}d\) (Fig. 17(d)). The same observation made for the system of brass disks\(^3\) lead the authors to conclude that the dynamical heterogeneities observed close to jamming have their origin in the dynamics of the contacts. We are now in a position to confirm this intuition.

6.2 Relation to contact dynamics

The fact that the heterogeneities for the dynamics of the contact and in the displacement fields are both maximal at the same value of the reduced packing fraction \(\varepsilon^\ast\) is already a strong indication that they have a common origin. This is further confirmed by the quantitative comparisons of \(\chi_s^d \ast\) with \(\chi_s^z \ast\) and of \(\tau^\ast\) with \(\tau^z\) provided in Fig. 18(a) and (b). \(\chi_s^d \ast\) and \(\tau^\ast\) are respectively proportional to \(\chi_s^z \ast\) and \(\tau^z\), confirming a strong correlation between the two aspects of the dynamics. As in Fig. 17(b), the \(\square\) data point is significantly off the trend given by the other data points because of the lack of statistics at the jamming crossover (see the previous section). Whereas the timescales are essentially identical, the dynamical susceptibility associated with the displacements is 20 times larger than that associated with the contacts. One must remain cautious in the interpretation of such a factor, since the dynamical susceptibilities are only a single indicator of the number of correlated elements, even when they are properly normalized by the intrinsic fluctuations, since the shape of the spatial correlator also is relevant. With that caveat, such a large relative difference in the two susceptibilities suggests that the spatial organization of the dynamics differs in the two cases. This is confirmed in Fig. 18(c) and (d), which show snapshots of respectively \(Q_i(t, \tau^\ast)\) and \(Q_i(t, \tau^z, a^\ast)\), obtained at the same time. Whereas the
6.3 Short time origin of the heterogeneities

Fig. 17(a) indicates that the non-monotonic dependence of $\chi_4^\gamma$ on $\phi$ applies for times of $\sim \tau^4$, and also at the shortest timescales corresponding to the stroboscopically acquired data, i.e. for $\tau = 1$ cycle to $10^4$ microscopic times. The same holds true for the contacts. Fig. 19(a) and (b) respectively display $\chi_4^\gamma(t)$ and $\chi_4^{c\gamma} = \chi_4(t_0)$, together with $\chi_4^{\gamma^*} = \chi_4(\tau^2)$ as functions of the packing fraction; both are non-monotonic, suggesting that the dynamical heterogeneities of the contact dynamics have roots in the structure of the contact network. Still, $\chi_4^{c\gamma}$ is smaller than $\chi_4^{\gamma^*}$, indicating that the heterogeneities, present at short times, build up progressively via a process which remains to be explained.

The above results suggest that the contact network itself is heterogeneous. Whereas a number of papers discuss the heterogeneities of the force network in terms of the force intensities, we are not aware of a detailed examination of the spatial correlations in the contact network. Fig. 20(a) shows the instantaneous contact network for packing fractions lower than $\phi^\dagger$. The network is highly heterogeneous, with large holes where there are very few contacts. After interpolating the contact number on a grid, we compute the radial dependence of its spatial autocorrelation $G_2^\gamma(r)$. This quantity decays exponentially towards zero (Fig. 20(b)), with a typical decay length $\xi_2^\gamma$ defined as $G_2^\gamma(\xi_2^\gamma) = 0.2$. $\xi_2^\gamma$ is non-monotonic with respect to the packing fraction (Fig. 20(c)), and has a small maximum at $\epsilon^*$; the spatial correlations of the contacts are maximal at $\epsilon^*$.

The static susceptibility

$$\chi_2^\gamma = \frac{N}{\left(\prod_{i=1}^{N} \sum_{z_i} \varphi, z_i \right)}$$

The dynamical heterogeneities of the displacements are organized in well identified large clusters, those of the contacts are more scattered in smaller chain-like clusters. The dynamical correspondence is not simply that the particles moving more than $a^*$ lose or gain contacts. Rather, these results suggest that the loss of contact at one place induces motions on the scale of $a^*$ further away, and in turn, the loss of other contacts.

In Section 4.2, we found that dynamical heterogeneities of the contacts expand when the vibration frequency is reduced towards the zero mechanical excitation limit. One might expect the same to happen for the heterogeneities of the displacements. However, we showed that the system size limits the largest heterogeneities of the displacements. Hence, when we reduce the vibration strength, the displacement heterogeneities saturate, while those for the contacts increase fourfold (Fig. 18(c) and (d)).

**Fig. 18** Dynamical heterogeneities. (a) Maximal dynamical susceptibility of the displacements $\chi_4^\gamma$ vs. maximal dynamical susceptibility of the contacts $\chi_4^{c\gamma}$ and (b) $\gamma^*$ vs. $\tau^4$ in parametric plots, where each point corresponds to a different packing fraction (the same color code as in Fig. 5). The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$. (c and d) Maps of $\chi_4^\gamma(t, \tau)$ (c) and $\chi_4(t, \tau, a_{12})$ (d), for $\epsilon = -0.0015$. (d) Color code spans from yellow ($\chi_4^\gamma(t, \tau) = 0$) to red ($\chi_4^\gamma(t, \tau) = 1$). (d) Color code spans from blue ($\chi_4(t, \tau, a_{12}) = 0$) to red ($\chi_4(t, \tau, a_{12}) = 1$). The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$. (e) The same plot as in (a) but for different values of $\gamma = 0.5$ (blue), 0.8 (green), and 1.4 (red). (f) Peak of the dynamical susceptibility of the displacements $\max(\chi_4^{\gamma^*})$ and peak of dynamical susceptibility of the contacts $\max(\chi_4^{c\gamma})$ vs. $\gamma$.

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**Fig. 19** Dynamical correlations of the contacts at short and long times. (a) Dynamic susceptibility of the contacts $\chi_4^\gamma$ vs. the lag time $\tau$. The same packing fractions as in Fig. 5. (b) Dynamic susceptibility at short times $\chi_4^{c\gamma}$ (x) and maximal dynamical susceptibility $\chi_4^{\gamma^*}$ (■) vs. reduced packing fraction, $\epsilon$. The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$. 
Experimental evidence of such dynamical heterogeneities, disks.

Spatial correlations of the contacts. (a) Instantaneous map of the contact number, for $\epsilon = -0.0091$. The color map varies from white ($z(t) = 0$) to black ($z(t) \approx 6$). Contact links are indicated in red. (b) Spatial correlations of the contacts $G_2^z$ vs. $r$. Inset: $G_2^z$ vs. $r/l_2^z$. The same packing fractions as in Fig. 5. (c) Spatial correlation length of the contacts $z_i^2$ vs. reduced packing fraction $\epsilon$. (d) Contact susceptibility $\chi_2^z$ vs. reduced fraction $\epsilon$. The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$.

where var$z(t)$ is the temporal variance of the instantaneous average number of contacts, $z(t) = \frac{1}{N} \sum_{i=1}^{N} z_i(t)$. $\chi_2^z$ is also a maximum at the packing fraction $\epsilon^*$ (Fig. 20(d)), corresponding to the maximum static correlation.

Altogether, the present results confirm that first, the dynamical heterogeneities observed in the displacement fields are connected to the heterogeneous dynamics of the contacts; second, the heterogeneities are already present in the static properties of the contact network. Such a connection is remarkable, and it would be interesting to see whether something similar exists for thermal soft spheres close to jamming. Also, the mechanism by which the static behavior at short times and the dynamics at longer times are connected remains unclear, and deserves further investigation.

### 7 Discussion

We recall here the motivation which led us to conduct this comprehensive study of vibrated photoelastic disks.

First we sought to confirm our first observations of dynamical heterogeneities in a very dense system of vibrated brass disks. These heterogeneities are unusual in the sense that they concern very small displacements, of the order of $10^{-3}$ grain diameters, and they occur for very large packing fractions, compared to other granular systems. There exists no other experimental evidence of such dynamical heterogeneities, except perhaps in one colloidal experiment, and in other quasi-static experiments by the authors elsewhere. Even so, it remains unclear whether these different experiments probe the same physics. Only recently, similar observations have been reported in numerical simulations of soft spheres, a system that is a priori rather different from vibrated granular media.

Our first set of experiments conducted with soft photoelastic disks confirmed results for a system consisting of brass disks, and led to the observation that similarities with the simulations of thermal soft spheres were stronger than expected. Since the authors of that numerical study argued that existing colloidal experiments are rather far from the critical regime of jamming, either because the packing fractions are too loose, or because the temperature is too high, we chose to decrease the vibration frequency in our system and to explore the vicinity of zero excitation. Indeed, we would like to know whether the behavior of thermal harmonic spheres or dynamical criticality of granular packings can provide mutual information and physical insights.

The Discussion section below is organized as follows. After a brief synthesis of the results, we compare and reconcile the observations obtained for the hard (brass) and the soft (photoelastic disks), before discussing the analogy between the thermal soft spheres and our experimental systems.

#### 7.1 Synthesis

We have conducted systematic experiments of horizontally vibrated grains, decreasing the packing fraction over a very small range of high packing fractions, where the dynamics of both the contacts and of the grain positions are frozen. Despite a strongly anisotropic mechanical forcing at large scales, the system at the grain scale is isotropic; nonlinear mechanisms, together with disorder, redistribute the energy at small scales, causing the system to progressively lose any memory of the forcing anisotropy. This is roughly analogous to the energy cascade in turbulence.

As previously noted, by using fast stroboscopic acquisition, and correcting for short term oscillations and long term convection, we computed the average displacements over more than six temporal decades. For low enough packing fractions, we clearly identified a ballistic regime, followed by a long plateau, eventually followed by a crossover to a very long time diffusive regime. These observations allowed us to measure the size of the cages, $\Delta$, as a function of the packing fraction in several independent ways.

During the time when the grains are trapped in their cages, we observed two distinct crossovers. One is “structural” in the sense that it is revealed by the average number of contacts, which starts increasing sharply at the packing fraction $\phi^*$. The other is “dynamical” in the sense that it is indicated by a maximum of the dynamical heterogeneities of both the contacts and the displacements at a packing fraction $\phi^* \leq \phi$. We have demonstrated that the “dynamical” crossover is tied first to the structure of the contact network, and second to the spatial fluctuations of the contact number. By contrast, the “structural” crossover is given by its average value. Both signatures converge
to a unique packing fraction as the excitation is reduced towards zero. We interpret this packing fraction as the jamming transition for the present experimental system and compression protocol. The critical nature of the transition is suggested by the sharp increase of the dynamical susceptibilities in the zero excitation limit. The two crossovers can be seen as the analogs of the Widom lines reported in the supercritical region of equilibrium phase transitions. The increase of the susceptibilities accounts for the increase of the correlation length, which has been discussed theoretically in several studies. Unfortunately, the scaling of this length scale with the susceptibility is not always simply the dimension of space and, the experimental range of the susceptibilities is too small to extract any significant scaling with the distance to the critical point.

7.2 Soft vs. hard

In an earlier experiment, using the same apparatus but with hard (brass) disks, the authors reported the first experimental evidence of dynamical heterogeneities involving very small displacements of grains, within a structure that was otherwise almost completely frozen. These dynamical heterogeneities differ from those observed close to the glass transition, and the authors correctly attributed their observation to jamming. However, they could not precisely identify the underlying mechanism responsible for these heterogeneities. The present study has clearly demonstrated that the heterogeneities have their origin in the dynamics of the contact network.

Also, the existence of this maximum in dynamical heterogeneities suggests that the experiment probed both sides of the jamming transition, a puzzling conclusion given the stiffness of the brass disks. The present study with soft disks resolves this apparent contradiction in the following way. There are several signatures of point $f$ at non-zero $\gamma$, and the one associated with the dynamical heterogeneities occurs at a packing fraction, $\phi^*(\gamma)$, that is lower than the one at which the average number of contacts increases, $\phi^*$. In the experiment using brass disks, the authors reported [see Fig. 21(a)] that the maximum of the dynamical heterogeneities occurred for the packing fraction where $P_{\text{DYN}}(\phi)$ and $P_{\text{STAT}}(\phi)$ intersect. This is also the case for the soft disks [see Fig. 21(b)]. The experiment with the brass disks actually probed the dynamical crossover $\phi^*$, both sides of which lie below the structural jamming transition. In the case of brass disks, it is not possible to measure the average number of contacts. However, assuming Hertz' law, the stiffness of two compressed 4 mm height cylinders made of brass (Young modulus, $E = 100$ GPa) is $k_{\text{brass}} \sim 3 \times 10^8$ N m$^{-1}$. By comparison, the stiffness of the force sensor and piston system is $k_{\text{piston}} \sim 6 \times 10^5$ N m$^{-1}$ and the brass grains can be considered as hard. In that case, jamming is the point at which the pressure increases sharply, and the corresponding packing fraction [see Fig. 21(a)], provides a good estimate of the structural crossover $\phi^*$.

We also note that the range of $\phi$ for which the crossovers occur is very different for the systems of soft and hard particles. For the former, the crossovers occur for lower packing fractions and on a broader range than for the latter. This is not so surprising, given that the friction coefficient between the grains and between the grains and the glass plate are different. The soft disks have a larger friction coefficient, so that their jamming transition is expected for lower values of the packing fraction. Frictional particles tend to jam with anisotropic stresses locally, plausibly leading to states, which are different from the isotropic jammed states. The soft disks also have a larger friction coefficient with the glass plate shaking them so that the energy transfer and dissipation are different. It is remarkable that despite these differences, and for different vibration frequencies, the peak of the maximal dynamical susceptibility of the displacements vs. the separation of the dynamical and the structural crossovers (see Fig. 22(a)) is similar for the two types of experiments. This suggests that the hard disks vibrated at a frequency $f = 10$ Hz, i.e. $\gamma = 1.4$ behave like soft disks with a much smaller effective value of $\gamma$; the injection of energy is much less efficient in the case of the hard, less frictional, disks. It also indicates that friction plays a role in

![Fig. 21](image_url)  
**Fig. 21** Hard vs. soft. Piston force (top) and maximal dynamical susceptibility of the displacements (bottom) vs. reduced packing fraction, $\epsilon$, for (a) hard brass disks and (b) soft photoelastic disks. ($\bigcirc$): $P_{\text{TOT}}$, (□): $P_{\text{STAT}}$, (△): $P_{\text{DYN}}$ as in Fig. 6. The vibration frequency $f = 10$ Hz, i.e. $\gamma = 1.4$. Dashed lines indicate $\epsilon^*$ and $\epsilon = 0$.

![Fig. 22](image_url)  
**Fig. 22** Towards zero vibration. (a) Maximum of the dynamical susceptibility of the contact maximum ($\chi^*_c$) for soft grains (×) and hard grains (○), estimated from max ($\chi^*_c$)/20, versus the split $|\epsilon^*|$ between static and dynamic signatures of jamming. (b) MSD plateau vs. density $\epsilon$, for $\gamma = 0.5$ (○), $\gamma = 0.8$ (×), $\gamma = 1.4$ (□) and for hard brass disks at $\gamma = 1.4$ (○).
the absolute value of the packing fraction $\phi_f$, as well as in the efficiency of the mechanical excitation, but not in the physics observed at finite vibration.

We can further confirm that the hard disks behave like the soft disks at a lower level of excitation, by comparing the Debye–Waller factor, i.e. $\Delta^2$ in the plateau regime. For the brass disks, no fast camera acquisitions were conducted, but the displacements over one vibration cycle gives an estimate of the upper bound of the plateau value. Fig. 22(b) displays the Debye–Waller factor for the three experiments using the soft photoelastic disks at three different vibration frequencies, as reported in the present paper, and for the experiment with the hard brass disks. The value of $\Delta^2 \sim 10^{-6}$ is significantly lower for the brass disks, confirming that they are closer to the zero vibration limit.

7.3 A-thermal vs. thermal

We have just seen that the physics of the jamming transition of granular media in the presence of vibration is robust with respect to the specific properties of the grains. However, as stated in the Introduction, although the jamming transition is precisely defined and well characterized for thermal soft spheres, it is not well defined for frictional grains. An important question is then to what extent does it describe the present observations? Specifically, do the concepts and predictions of thermal jamming apply in the granular world?

To answer this question we return to the recent numerical simulations by Ikeda et al. who used the mean square displacement in the plateau regime as a sensitive thermometer close to jamming, in order to compare their observations with experimental colloidal systems. From $\Delta^2(\phi)$ computed for various temperatures, and knowing the range of packing fractions explored by the colloidal experiments, it is straightforward to locate various experiments in the temperature-packing fraction plane. This method does not require any information about the interaction potential, nor knowledge of the kinetic energy. We follow exactly the same procedure to compare our experiments with the simulations. For instance, Ikeda et al. reported that the mean square displacement in the vicinity of jamming decreases from $10^{-3}$ to $10^{-6}$ particle diameters when the temperature is decreased from $T = 10^{-5}$ to $T = 10^{-8}$. However, we can borrow from Ikeda et al. focused on the dynamical behavior of soft spheres, close to jamming, thermalized but approaching the zero temperature. These simulations show the existence of dynamical heterogeneities in the displacements at very small scales. The maximal value of the heterogeneities increases, and the packing fraction at which this peak occurs decreases as the temperature falls towards the $T = 0$ limit. These observations are identical to those reported in the present work. Unlike the simulations, the experiments are not characterized by a well defined and unique value of the jamming packing fraction at zero 'vibration'. That is the jamming packing fraction differs among realizations. Also, the experiments lack a proper definition of effective temperature.

We emphasize the importance of such a conclusion. Shaken granular systems and thermal soft spheres are very different, in large part, due to the effect of dissipation/friction. Shaken granular media are out of equilibrium systems for which detailed balance does not hold. Energy is injected at large scales and dissipated at small scales. In the present case, this ensures the isotropy of the displacements at short times, but it is also responsible for the large convection pattern that we have removed. Furthermore friction not only modifies the counting argument for isostaticity. It also favors locally sheared states and potentially induces history dependence. However our data suggest that there is still a critical point at zero vibration for frictional systems. The position of this point certainly depends on the amount of friction. However we believe that the friction axis becomes “irrelevant” as soon as a small amount of vibrations or thermal agitation is present.

8 Conclusion

In this work, we have demonstrated that, in the presence of agitation, the jamming transition’s singular features are blurred into two crossovers, a structural one, indicated by the increase of the contact number, which is directly inherited from the zero excitation limit case, and a dynamical one, specific to the presence of agitation. The contact network develops heterogeneous dynamics, which in turn induce heterogeneous displacements at very small scale. While the structure of the glass remains essentially frozen, these heterogeneities take place within the vibrational regimes, and are related to structural heterogeneities in the contact network.

These observations match the recent results reported in numerical simulations by Ikeda et al. very well, which leads us...
to conclude that the critical regime of point $J$ is indeed probed by our granular experiments, and that the concepts related to jamming of soft frictionless spheres can provide insight into granular systems. This suggests that similar experiments be conducted in related systems, such as foams or emulsions. The important challenge is to find an appropriate way to “vibrate” them: thermal agitation occurs for particles which are typically smaller than a micron, which makes it a challenge to attain a spatial resolution of the order of a thousandth of the size of a particle if one plans to use thermal agitation as the source of the dynamics. At present, this constraint has prevented colloidal suspensions from being examined, and that the concepts related to jamming of soft colloidal spheres. Finally, one may wonder whether similar conclusions apply in the presence of an additional external driving force, such as weak shear with vibrational excitation, in the spirit of ref. 58.

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