Vibration Transmission Reduction Through Multi-Element Multi-Path Structural Design in Thin Beams and Cylindrical Shells

by

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Dr. Wilkins Aquino

Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mechanical Engineering and Materials Science in the Graduate School of Duke University 2015
ABSTRACT

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Abstract

Unwarranted vibrations create adverse effects that can compromise structural integrity, precision and stability. This thesis explores an attenuation technique that rethinks the design of simple lightweight flexible structures, providing an alternative to the current methods of active controllers and heavy damping. Through multi-element/multi-path (MEMP) design, a structure is divided up into several constituent substructures with separate, elastically coupled, wave transmission paths that utilize the inherent dynamics of the system to achieve substantial wide-band reductions in the low frequency range while satisfying constraints on static strength and weight. Attenuation is achieved through several processes acting in concert: different subsystem wave speeds, mixed boundary conditions at end points, interaction through elastic couplings, and stop band behavior. The technique is first introduced into thin beams coupled with discrete axial and torsional springs, resulting in wide-band attenuation and agreement between analytical simulations and experimental studies. MEMP design is then implemented into concentric thin cylindrical shells, providing a more three-dimensional study with axially discrete azimuthally-continuous elastic connectors. By employing a modal decomposition of the governing shell equations, simulations reveal more opportunities for attenuation when subjected to various forcing conditions. Future work examines the effect of MEMP shell design on acoustic scattering reduction.
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## List of Abbreviations and Symbols

### Symbols

A bar over a symbol indicates a nondimensional variable (example: $\bar{x}$ and $\bar{y}$). A subscript on an MEMP structural parameter property indicates which substructure it defines (example: $R_1$ and $R_2$)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>Beam transverse deflection</td>
</tr>
<tr>
<td>$x$</td>
<td>Axial coordinate</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Azimuthal coordinate</td>
</tr>
<tr>
<td>$t$</td>
<td>Time variable</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Driven frequency</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus of elasticity</td>
</tr>
<tr>
<td>$I$</td>
<td>Area moment of inertia</td>
</tr>
<tr>
<td>$A$</td>
<td>Cross-sectional area</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$q$</td>
<td>Transverse load</td>
</tr>
<tr>
<td>$V$</td>
<td>Shear force</td>
</tr>
<tr>
<td>$M$</td>
<td>Bending moment</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>Structural natural frequency</td>
</tr>
<tr>
<td>$L$</td>
<td>Length</td>
</tr>
<tr>
<td>$k$</td>
<td>Axial wavenumber</td>
</tr>
<tr>
<td>$F$</td>
<td>Input force</td>
</tr>
<tr>
<td>$s$</td>
<td>Number of discrete couplings</td>
</tr>
</tbody>
</table>
$K$ Structural stiffness

$K_s$ Coupling extensional stiffness

$K_t$ Coupling torsional stiffness

$\alpha$ Material damping

$\alpha_v$ Coupling dashpot damping

$\beta$ Elasticity ratio

$\gamma$ Density ratio

$R$ Shell radius

$h$ Shell thickness

$h$ Poisson’s ratio

$u$ Longitudinal deflection

$v$ Tangential deflection

$w$ Radial deflection

$p_x, p_\phi, p_r$, Longitudinal, tangential and radial pressure

$Q_x, Q_\phi$ Bending shear stress resultant

$N_x, N_\phi$ Normal stress resultant

$N_{x\phi}, N_{\phi x}$ Shear stress resultant

$M_x, M_\phi, M_{x\phi}, N_{\phi x}$ Moment resultant

$T_x$ Tangential shear force

$S_x$ Radial shear force

$n$ Azimuthal mode number

$\zeta$ Deflection eigenvector with components $U, V, W$

$C$ Free constants

$L$ Flugge shell linear mapping

$L_D$ Donnell shell linear mapping

$L_{D,co}$ Donnell shell linear mapping at axial cutoff
$\omega_{co}^g$  Axial cutoff frequency ($g$ specifies which cutoff)

$P_{mn}$  Pressure polynomial constant $m$ for mode $n$

$\delta$  Dirac delta function

**Abbreviations**

BC  Boundary condition

MEMP  Multi-element multi-path

TF  Transfer function, specifically that of the force transfer
Acknowledgements

The author would like to acknowledge his advisor, Dr. Donald Bliss, for his guidance and mentorship along with his parents, Maribel and Ricardo Raudales, for their continuous support during the past two years.
1 Introduction

1.1 Problem Statement

Simple lightweight and flexible structures such as beams and shells provide support and act as building blocks for larger, more intricate systems. When subject to dynamic loads, the structures transmit vibrations that, due to their inherent elasticity, compromise the integrity, stability and control of not only the excited structure itself, but also of its connecting members. External periodic sources of unwanted noise can vary from mechanical components such as motors, fans and other oscillators to natural occurrences caused by the structure’s environment such as fluid and acoustic loadings. The vibrations, while small in amplitude, can lead to fatigue over time and eventual failure if not properly addressed.

While reducing vibrations is a study in dynamics, the design of the structure has consequential effects on its static behavior. In the fields of aerodynamics and underwater research especially, the mass of the structure is a critical design parameter. An increase in mass can lead to a significant penalty in engineering cost, as in the example of an aircraft which must generate enough lift to overcome the total weight.
of its components. Additionally, the static strength, or stiffness, must be considered in relation to its application. The fluid loading on an aircraft’s wings in conjunction with its elastic properties for instance can cause the wing to deflect and eventually shear, an effect known as divergence.

The problem at hand is thus to modify a simple structure in order to reduce its vibration transmission while adhering to design constraints on mass and static strength. Specifically, the focus is on lightweight, flexible structures undergoing steady state periodic motion from discrete and distributed excitations. The modified structure will be compared to the unmodified reference structure, both equal in mass, in its attenuation along a broad frequency range with wideband results desired. By lowering the transmission through these elemental building blocks, the aim is to provide overall reductions on larger systems.

1.2 Current Methods of Vibration Reduction

Current methods for reducing the vibration transmission through simple structures can be divided into two categories: passive and active methods.

Passive methods are defined as isolation techniques implemented during the design process prior to testing that do not require active tuning. Generally, they are the more commonly used and affordable approach. One such method is to introduce viscoelastic damping, either distributed or discrete at a boundary location, as a form of energy dissipation [Chen and Zhou (2000)]. The damping term can be used to stabilize the structure through impedance matching techniques and lower the response at resonance frequencies. However, sufficient damping for wideband attenuation involves adding bulky material to the structure which can significantly increase its weight and is thus not applicable for lightweight systems.

Another passive design approach is to tune the dynamics of the structure through the inclusion of external masses, springs and resonators. Mechanical attachments
act as vibration absorbers that can be used to target troublesome frequencies and
generate stopband attenuations [Den Hartog (1985)]. An example of this behavior
occurs when a Helmholtz resonator is secured to a pipe, eliminating the transmitted
wave through the pipe at the absorber’s resonant frequency. A structure can also
be isolated through a suspension by elastically separating the excitation from the
receiver, resulting in more wideband transmission loss [Beard et al. (1994)]. This
can be helpful in reducing vibrations on and stabilizing payloads. Similarly, it can
also be shown that with multilayered suspensions of beams, the cutoff frequencies
of the different layers act towards drastically reducing the lower frequency range
vibrations by producing evanescent wavemodes [Thompson (2008)].

Active approaches involve control techniques to actively tune parameters of the
structure. As mentioned, the resonant frequencies of a structure coupled with me-
chanical attachments can be shifted to produce stopbands in correspondence with
the driving force. Active methods allow for the shifting of the stopband in order
to maximize the attenuation given a frequency range. Modern engineering of smart
materials has provided various solutions to tuning elastic properties and thus shifting
the resonance locations, one such example being light activated shape memory poly-
mers [Li et al. (2015)]. While active methods have been shown to produce effective
reductions, they come with several drawbacks. Introducing an electrical component
to a mechanical system increases the potential for error while raising both the en-
ergy and cost required for maintenance. Since a passive technique can be improved
through active control and is simpler in implementation, the desired attenuation
solution will need to passively modify the structure.

1.3 Multi-Element Multi-Path Approach

This thesis will present multi-element multi-path (MEMP) approach as a means of
reducing the vibration transmission in lightweight flexible structures. The method
consists of replacing a single element structure, deemed the reference, with an elastically coupled set of similar substructures. The substructures will not necessarily have the same mechanical properties but for consistency will both be equal in length to the single element reference. To remain lightweight, the summed masses of the coupled MEMP structure will equal the mass of the reference structure.

Similar to the aforementioned passive methods of reduction, the MEMP substructures do not require active control. The techniques are implemented during the design process, however, damping will not be essential to their efficacy. Instead, the MEMP design utilizes physical phenomena not evident in single element structures to create opportunities for attenuation. One such example is in wave path augmentation. For a single element structure excited at one end boundary condition and fixed at the other, there is only one reflected and transmitted path for the propagated waves to follow. In the MEMP structure, the discrete elastic couplings generate multiple wave paths and allow for interference to occur. The mechanical properties of the substructures in the MEMP configuration can be designated such that the waves interact in a destructive manner, canceling out the vibration and attenuating transmission.

![Figure 1.1: Reflection/transmission wave path generation in single element and MEMP structures](image)

The effects of elastically coupled substructures are explored for various boundary and property configurations to maximize the transmission reduction. The number
of substructures will be limited to two in this discussion which gives four total end boundaries to work with as opposed to the two end boundaries in single element structures.

1.4 Beams and Shells as Simple Structures

The MEMP design process is introduced into high aspect ratio, thin beams subject to discrete loadings. Beams serve as a building block for larger systems and their simplicity allows for basic establishment of the reductions possible in MEMP structures. The beam excitations are in the transverse motion, generating flexural waves that are dispersed through the substructures and modeled with basic Euler-Bernoulli theory. A sort of lumped element algorithm will be written in MATLAB to solve for the highly coupled motion and transfer function of the substructures, yielding quick and easy to access results. To account for the discrete couplings, each substructure is divided into shorter substructures with the couplings serving as discontinuities between elements. Theoretical results are compared to MEMP two beam experimental results attained through a shaker apparatus.

The design will then transition onto thin cylindrical shells, a more three-dimensional structure with highly coupled modes of vibration. The curved surface of the shell leads to interesting results in the dispersion of waves including cutoff behavior with single element structures. A similar algorithm developed for the beams is employed for the MEMP shells, modeled with Flügge cylindrical shell theory.

As simple structures, beams and shells are ubiquitous in the various fields of engineering. Cantilever beams are used everywhere from buildings to microelectromechanical systems. In fluid applications, cylindrical shells are chosen for the bodies of submersibles and transport systems (piping and pressure vessels) while in civil structures they often act as supports in trusses and frameworks. Their lightweight yet robust geometry is a valuable asset to shells and will be illuminated in the MEMP
discussion. The purpose of this investigation is to explore the MEMP design on very basic structures with the hope that reductions will translate when applied to larger systems. It serves as an introduction, with opportunities in parameter optimizations and other studies left for future work.
2.1 Single Thin Beam

The mathematical modeling of the single thin beam is introduced as a reference for the MEMP beam design.

2.1.1 Static Beam Governing Equation

For the mathematical modeling of the beam dynamics, the Euler-Bernoulli theory is chosen for its simplicity and conventional usage. This linear theory of analyzing small deflections in a beam due to transverse loadings neglects shear deformations and assumes that plane cross-sections of the beam remain plane and perpendicular to the neutral axis during bending [Graff (1975)]. These assumptions are appropriate at low frequencies for thin beams. For a beam with constant cross-sectional geometry (area $A$, moment of inertia $I$ and elastic modulus $E$), the stress-deflection relations that define the loading, shear force, bending moment and deflection angle are as
follows:

\[ EI \\frac{\partial^4 y}{\partial x^4} = q(x) \]

\[-EI \frac{\partial^3 y}{\partial x^3} = V(x) \]

\[-EI \frac{\partial^2 y}{\partial x^2} = M(x) \]

\[ \frac{\partial y}{\partial x} = \theta(x) \]  \hspace{1cm} (2.1)

where \( y \) is the vertical deflection of the beam. The beam governing equation has a homogeneous solution, setting the distributed loading to zero \( (q(x) = 0) \), that satisfies the fourth order differential equation \( \frac{\partial^4 y}{\partial x^4} = 0 \). Solutions takes the form of a cubic polynomial:

\[ y = \sum_{n=0}^{3} C_n x^n \]  \hspace{1cm} (2.2)

2.1.2 Dynamic Beam Governing Equation

The dynamic form of the beam equation introduces the inertia:

\[ EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = q(x, t) \]  \hspace{1cm} (2.3)

ignoring the external loading force, the variables and governing equation are nondimensionalized as:

\[ \bar{y} = \frac{y}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = t \omega_n \quad \text{and} \quad \bar{\omega} = \frac{\omega}{\omega_n} \quad \text{where} \quad \omega_n = \sqrt{\frac{EI}{\rho AL^4}} \]

\[ -\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} = -\frac{\partial^2 \bar{y}}{\partial \bar{t}^2} \]

The governing partial differential equation is separable in the time and spatial domain. This thesis will focus on solutions periodic in time due to a harmonic forcing
function, \( e^{i\omega t} \) with frequency \( \omega \). Since the frequency eigenvalue is prescribed, the spatial eigenvalue, or wavenumber, will be determined by examining the dispersion relation.

\[
\bar{k}^4 = \omega^2
\]

\[\implies \bar{k} = \pm \sqrt{\omega}, \pm i\sqrt{\omega}\]

Note the dispersion relation yields both real and imaginary wavenumbers traveling along both directions of the beam. While the real wavenumbers represent the propagating flexural waves (\( e^{i(\bar{\omega} \pm \bar{k}x)} \) functions), the imaginary wavenumbers produce growing and decaying or evanescent waves (\( e^{\pm \bar{k}x} e^{i\omega t} \) functions), which will serve useful in attenuating vibrations under beam configurations mentioned in the latter sections of this thesis. The homogeneous nondimensionalized problem, \( q(x, t) = 0 \), has a solution that can be rewritten in trigonometric functions as:

\[
y = e^{i(\bar{\omega} t)} (C_1 \cos \bar{k}x + C_2 \sin \bar{k}x + C_3 \cosh \bar{k}x + C_4 \sinh \bar{k}x) \quad \text{where} \quad \bar{k} = \sqrt{\omega}
\]

\[ (2.4) \]

2.1.3 Beam Boundary Conditions

Since the highest order derivative in the beam governing equation is four, there will need to be four boundary conditions prescribed in order to ensure uniqueness. Inhomogeneities will generally be applied in the boundary conditions on one end of the beam while the homogeneous set of boundary conditions on the other end will measure the force transfer through the structure. The table below illustrates the combination of Dirichlet and Neumann boundary conditions used in this thesis. Note that in this thesis, all motions both forcing and displacements will be assumed harmonic in time.
Table 2.1: Boundary conditions for beam

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Condition</th>
</tr>
</thead>
</table>
| Fixed              | \[ y|_{x=0} = 0 \]
                     | \[ \frac{\partial y}{\partial x}|_{x=0} = 0 \] |
| Free               | \[ EI \frac{\partial^2 y}{\partial x^2}|_{x=0} = 0 \]
                     | \[ EI \frac{\partial^3 y}{\partial x^3}|_{x=0} = 0 \] |
| Pinned             | \[ y|_{x=0} = 0 \]
                     | \[ EI \frac{\partial^2 y}{\partial x^2}|_{x=0} = 0 \] |
| Forced             | \[ EI \frac{\partial^2 y}{\partial x^2}|_{x=0} = 0 \]
                     | \[ EI \frac{\partial^3 y}{\partial x^3}|_{x=0} = F \] |

2.1.4 Static Results

![Cantilever single beam boundary conditions](image)

Figure 2.1: Cantilever single beam boundary conditions

The cantilever conditions, which will be referenced throughout this thesis, are defined such that one end of the beam is fixed at both the angle and deflection while the opposite end is forced with a shear loading.

\[
y|_{x=0} = 0, \quad \frac{\partial y}{\partial x}|_{x=0} = 0, \quad EI \frac{\partial^2 y}{\partial x^2}|_{x=L} = 0, \quad EI \frac{\partial y^3}{\partial x^3}|_{x=L} = F
\]

or

\[
y|_{x=0} = 0, \quad \frac{\partial y}{\partial x}|_{x=0} = 0, \quad \frac{\partial^2 y}{\partial x^2}|_{x=1} = 0, \quad \frac{\partial y^3}{\partial x^3}|_{x=1} = F
\]

where \[ F = \frac{FL^2}{EI} \]
This common set of boundary conditions is used to determine the overall static stiffness of a member, resulting in a deflection of:

\[ \bar{y} = \frac{\pi^2 (\pi - 3)}{6} \quad \text{with} \quad \bar{y}_{\text{max}} = -\frac{F}{3} \]

Therefore, the static stiffness of a beam can be defined as:

\[ K_{\text{beam}} = \frac{F}{\bar{y}_{\text{max}}} = \frac{3EI}{L^3} \]  \hspace{1cm} (2.5)

and can be used as a metric to determine the strength of a beam structure.

### 2.1.5 Material Damping

Prior to analyzing the dynamic results, it is both visually and pragmatically important to include damping into the model. Without appropriate damping, the response of the structure experiences singularities at the resonant frequencies. One simple way to prevent this is to make the modulus of elasticity complex with an imaginary portion scaled by a small perturbation \( \bar{\alpha} \).

\[
E (1 + \bar{\alpha}i) I \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0
\]

\[
EI \frac{\partial^4 y}{\partial x^4} + iEI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0
\]

\[
\implies \frac{\partial^4 \bar{y}}{\partial x^4} + \bar{\alpha}i \frac{\partial^4 \bar{y}}{\partial x^4} + \frac{\partial^2 \bar{y}}{\partial t^2} = 0
\]

\[
\implies k^4 = \frac{\omega^2}{1 + \bar{\alpha}i} \approx \omega^2 (1 - \bar{\alpha}i)
\]

This form of damping scales with the wavenumber \( \propto k^4 \), differing from a viscous damper which conventionally scales linearly with the frequency. The effect for a cantilever single beam (derivation in following section) is shown in Figure 2.2. Resonances at higher frequencies are strongly inhibited by the material damping. For the results presented in this thesis, a material damping of five percent \( (\bar{\alpha} = 0.05) \) is used.
2.1.6 Dynamic End Forcing Results

If the forcing on the cantilever beam is dynamic and periodic, $F e^{i\omega t}$, the constants in Equation 2.4 can be determined through linear algebra and yield the following solutions:

$$
\bar{y} = \frac{FL^3}{2EI\bar{k}^3} \left( \sin \bar{k} + \sinh \bar{k} \right) \left( \cos \bar{k}x - \cosh \bar{k}x \right) - \left( \cos \bar{k} + \cosh \bar{k} \right) \left( \sin \bar{k}x - \sinh \bar{k}x \right) \frac{1}{1 + \cos \bar{k} \cosh \bar{k}}
$$

(2.6)

As the frequency approaches the quasistatic limit of zero, the deflection approaches the previously derived static stiffness result. Resonance behavior occurs when the deflection reaches a singularity:

$$
\cos \sqrt{\bar{\omega}} \cosh \sqrt{\bar{\omega}} = -1
$$

It is now necessary to introduce the form of the transfer function which will be commonly used throughout this thesis. The input response will be the applied forcing amplitude, $F$, while the output response will be the reaction force amplitude $(\frac{EI}{L^3} \frac{\partial^3}{\partial x^3} \mid_{x=0})$ at the end of the beam which is fixed:

$$
TF = 20 \log_{10} \left( \frac{\text{Output Force}}{\text{Input Force}} \right) = 20 \log_{10} \left( \frac{\cos \sqrt{\bar{\omega}} + \cosh \sqrt{\bar{\omega}}}{1 + \cos \sqrt{\bar{\omega}} \cosh \sqrt{\bar{\omega}}} \right)
$$

(2.7)
2.1.7 Single Beam on Elastic Suspension

If a single beam is placed on a continuous elastic suspension, the governing equation becomes:

$$-\frac{\partial^4 y}{\partial x^4} - K_s \frac{\partial^2 y}{\partial t^2} = \frac{K_s L^4}{EI}$$ (2.8)

The added stiffness from the suspension affects flexural wave dispersion in the following manner:

$$\kappa^4 + K_s = \omega^2$$

$$\implies \kappa = \pm (\omega^2 - K_s)^{\frac{1}{4}}, \pm i (\omega^2 - K_s)^{\frac{1}{4}}$$ (2.9)
Note that when the forcing frequency matches the suspension natural frequency, \( \bar{\omega}_c = \sqrt{K_s} \), the wavenumber reaches zero and there are no waves propagated through the beam. For frequencies lower than the suspension natural frequency, the wavenumber becomes complex and leads to propagating waves with spatial decay. This frequency is often referred to as the cutoff frequency, \( \bar{\omega}_c \), and will be useful in attenuating the force response on an elastically coupled multi-beam structure. The transfer function for a cantilever beam on a suspension is plotted below with the cutoff behavior clearly visible. The resonant frequencies are higher with the suspension due to the increased overall stiffness. As the suspension stiffness approaches zero, the configuration matches the isolated single beam. As the suspension reaches rigidity, the evanescent waves dominate the propagated waves and the force response becomes more attenuated.

![Transfer Function for Cantilever Beam on Suspension (K_s=400)](image)

**Figure 2.5**: Transfer function for cantilever single seam on suspension (\( \bar{\omega}_c = 20 \))

### 2.2 MEMP Two Beam Configuration

The discussion will now focus on the MEMP beam configuration. For simplicity a two beam system will be used, however the MEMP design can be generalized to apply
to any number of coupled beams. The two beams will be of the same length but not necessarily of the same material nor have the same cross sectional dimensions. Layered over each other, the beams will be coupled by discrete and evenly spaced extensional springs that will guide the propagated flexural waves between the two beams, creating multiple paths for interference.

\[ \begin{align*}
    -E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} - \rho_1 A_1 \frac{\partial^2 y_1}{\partial t^2} - K_s (y_1 - y_2) \delta(x) &= 0 \\
    -E_2 I_2 \frac{\partial^4 y_2}{\partial x^4} - \rho_2 A_2 \frac{\partial^2 y_2}{\partial t^2} - K_s (y_2 - y_1) \delta(x) &= 0
\end{align*} \]  

(2.10)

The governing equations can be integrated around this point to yield the modified boundary conditions. The sifting property is used to evaluate the delta function.

\textbf{Figure 2.6: Multi-element multi-path beam design}
integral. The process for one of the beams is as follows:

\[
\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \left[ -E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} - \rho_1 A_1 \omega^2 y_1 - K_s (y_1 - y_2) \delta(x) \right] dx = 0
\]

\[
\Rightarrow -E_1 I_1 \frac{\partial^3 y_1}{\partial x^3} |_{0+}^{0-} - K_s (y_1(0) - y_2(0)) = 0
\]

(2.11)

Note that the inertial term disappears after the limit of epsilon approaches zero since the displacement and thus its integral are assumed to be continuous along the entire span of the beam. The above equation appears as a resulting force balance, where the jump in shear force along the beam is matched by the spring coupling force. If the spring is removed \((K_s \to 0)\), then the shear force is continuous as it was in the single beam analysis with end forcing. Integrating and taking the limit around zero of the resulting force balance equation three more times formulates the other three boundary conditions at the spring coupling:

\[
-E_1 I_1 \frac{\partial^2 y_1}{\partial x^2} |_{0+}^{0-} = 0
\]

\[
-E_1 I_1 \frac{\partial y_1}{\partial x} |_{0+}^{0-} = 0
\]

\[
-E_1 I_1 y_1 |_{0+}^{0-} = 0
\]

These three boundary conditions dictate the continuity in the beam deflection at the coupling location. The zeroth, first and second spatial derivatives of the deflection are continuous while the third derivative has a discontinuity due to the spring coupling. The process can be repeated for the second beam and nondimensionalized to yield
the set of coupled partial differential equations:

\[
\begin{align*}
\text{Gov Equations:} & \quad -\frac{\partial^4 \gamma_1}{\partial x^4} - \frac{\partial^2 \gamma_1}{\partial t^2} = 0 \quad \text{and} \quad -\frac{\partial^4 \gamma_2}{\partial x^4} - \beta^2 \gamma_2 \frac{\partial^2 \gamma_2}{\partial t^2} = 0 \\
& \quad -\frac{\partial^3 \gamma_1}{\partial x^3} \big|_{0^-}^0 - K_s (\gamma_1(0) - \gamma_2(0)) = 0 \quad \text{and} \quad \frac{\partial^2 \gamma_1}{\partial x^2} \big|_{0^-}^{0+} = \frac{\partial \gamma_1}{\partial x} \big|_{0^-}^{0+} = \gamma_1 \big|_{0^-}^{0+} = 0 \\
& \quad -\frac{\partial^3 \gamma_2}{\partial x^3} \big|_{0^-}^0 - \beta^2 K_s (\gamma_2(0) - \gamma_1(0)) = 0 \quad \text{and} \quad \frac{\partial^2 \gamma_2}{\partial x^2} \big|_{0^-}^{0+} = \frac{\partial \gamma_2}{\partial x} \big|_{0^-}^{0+} = \gamma_2 \big|_{0^-}^{0+} = 0 \\
K_s = \frac{K_s L^3}{E_1 I_1}, \quad \beta^2 = \frac{E_1 I_1}{E_2 I_2}, \quad \gamma^2 = \frac{\rho_2 A_2}{\rho_1 A_1}, \quad \bar{\omega} = \frac{\omega}{\omega_n}, \quad \text{where} \quad \omega_n = \sqrt{\frac{E_1 I_1}{\rho_1 A_1 L^4}}
\end{align*}
\]

Outside the couplings locations, the governing equation for each beam appears as the homogeneous form derived for the single beam. Each beam can be divided up into multiple elements, each element representing the beam segment between couplings and end boundary conditions. This way, the discontinuity is treated as a sort of boundary condition that separates single beam elements. The model for a single coupling is illustrated below.

\[\text{Figure 2.7: Modeling of two beam coupling with multiple elements}\]

Consequently, increasing the number of couplings will then increase the number of sub-elements in each beam. For two beams coupled with \(s\) number of discrete couplings, there will be \(2(s + 1)\) total sub-elements.
2.2.2 Computational Modeling

As before mentioned, the solution form for each multi-beam element will be similar to that of the single beam in Equation 2.4:

\[ \bar{y}_1 = e^{i(\alpha_1 \pi)} \left( C_1^1 \cos k_1 x + C_1^2 \sin k_1 x + C_1^3 \cosh k_1 x + C_1^4 \sinh k_1 x \right) \]

where \( k_1 = \sqrt{\omega} \)

\[ \bar{y}_2 = e^{i(\alpha_2 \pi)} \left( C_2^1 \cos k_2 x + C_2^2 \sin k_2 x + C_2^3 \cosh k_2 x + C_2^4 \sinh k_2 x \right) \]

where \( k_2 = \sqrt{\beta \gamma \omega} \) \hspace{1cm} (2.12)

Assuming that each beam is composed of a homogeneous material with constant properties, the solution forms apply throughout the beams and only the constants must be solved for in accordance with the end boundary conditions. This approach resembles a sort of finite element method, except the governing equations for each element are those of a beam instead of the general three-dimensional elasticity relations.

The desired computational model must be generalized to account for an arbitrary number of evenly spaced springs with interchangeable end boundary conditions. Taking advantage of the system’s linearity, the unknown deflection constants of each beam element for an MEMP structure with \( s \) number of couplings can be solved for by inverting a matrix which contains the boundary equations and inhomogeneous
forcing terms.

**B** : Boundary Condition/Coupling $8(s + 1) \times 8(s + 1)$ Matrix

**C** : Unknown Beam Constants $8(s + 1) \times 1$ Matrix where $C_{(\text{beam,subelement})}^{(\text{coeff})}$

**F** : Inhomogeneous Forcing $8(s + 1) \times 1$ Matrix

$$B \cdot C = F$$

$$\Rightarrow C = \text{inv}(B) \cdot F$$

$$B = \begin{bmatrix}
\begin{bmatrix} \pi = 0 \text{ BCs} \end{bmatrix} & 0 & 0 \\
\vdots & \text{Coupling Equations} & \vdots \\
0 & 0 & \begin{bmatrix} \pi = 1 \text{ BCs} \end{bmatrix}
\end{bmatrix}$$

and

$$C = \begin{bmatrix}
C_1^{11} \\
C_2^{11} \\
C_3^{11} \\
C_4^{11} \\
\vdots \\
C_1^{2(s+1)} \\
C_2^{2(s+1)} \\
C_3^{2(s+1)} \\
C_4^{2(s+1)}
\end{bmatrix}$$

The boundary condition matrix $B$ consists of cosine, sine, cosh and sinh functions evaluated at $\pi = 0$ and $\pi = \frac{1}{1+s}$ along with the frequency, material properties and spring constants. The forcing matrix $F$ consists of mostly zeros except for in the end boundary condition rows in which the inhomogeneities are applied. In order to compare similar MEMP configurations with different number of couplings, the length used in the nondimensionalization will be the total length of each beam, $L$, instead of the individual beam element length, $L_{(1+s)}$. The evenly spaced couplings and homogeneous beam element assumptions allow for the coupling equations matrix to become recursive and easily be generated through a MATLAB algorithm.
Figure 2.8: MEMP Algorithm

Note that the computational time depends primarily on the size of the $B$ matrix, or the number of elastic couplings $s$, which grows nonlinearly with $s$ and must be iterated for each frequency point. Once the deflection coefficients matrix $C$ is generated, the reaction forces, moments, angles of deflection and deflections can then be calculated to analyze the dynamic response. Observing the quasistatic limit $(\omega \to 0)$ allows for a static analysis. A graphic user interface was created in MATLAB to simplify the process and quickly interchange boundary conditions displayed in the screenshot below:
2.2.3 MEMP Boundary Configurations

While the single beam was examined under the cantilever configuration, the MEMP design introduces two new boundary locations which allow for numerous cantilever-like configurations. Three such combinations are displayed in Figure 2.10 below and will be individually analyzed in the following sections.

![Figure 2.9: MEMP beam graphic user interface in MATLAB](image)
2.2.4 Configuration A: Cantilever Double Beam

The most straightforward boundary configuration consists of both beams being fixed on the receiving end and forced in-phase on the opposing end with equivalent loads $F_0$. Here, the output response in the force transfer function is the summation of both reactions on the fixed ends. Since the reaction side (or wall) is assumed to be a single rigid structure, the absolute value is applied after summing both beam
reactions, taking into account the phase difference.

\[
TF = 20 \log_{10} \left( \left| \frac{E_1 I_1 \frac{\partial^3 y_1}{\partial x^3} \bigg|_{x=0} + E_2 I_2 \frac{\partial^3 y_2}{\partial x^3} \bigg|_{x=0}}{2F_0} \right| \right)
\] (2.13)

The discrete elastic couplings act as transmission paths for the propagated waves and allow for interference, a phenomena not evident in the single cantilever beam.

If the two beams are identical \((\beta = \gamma = 1)\) then there is no interference as the deflection difference between the beams at the coupling locations will be zero and thus the springs will not effect the dynamic response. However, if the wave speed on each beam differs, then interference can occur that leads to interesting behavior. The force transfer functions are plotted to observe this effect with the beam densities altered to vary the wave speed.
The differing wave speeds allow for the transfer function to reach a negative value, signifying that there are frequencies in which the net reaction force is less than the net applied force. This effect will be evident throughout the MEMP discussion and in predictions that MEMP structures are usually most effective when the material properties of the beams, or wave speeds, are not identical. Single element cantilever structures will resemble more the blue curve above (identical wave speeds) with resonance peaks and positive transfer functions.

Note the resonance peaks in the identical beams curve ($\gamma = 1$) of Figure 2.11 and compare it to the two adjacent peaks in the other plots ($\gamma = .7, .8$). When one of the beams is made dissimilar, $\gamma \neq 1$, the resonance mode splits into a set of similar modes, leading to constructive and destructive interference. This phenomenon of "peak splitting" is illustrated in Figure 2.12 which plots the beam amplitudes and phase differences at the peak splitting locations for $\gamma = .7$. At the two peaks, $\overline{\omega} = 22$ and $\overline{\omega} = 32$, one of the beams appears to be resonating. In between the peaks, $\overline{\omega} = 27$, the two beams appear almost completely out-of-phase with each other, resulting in destructive interference and thus the attenuation dip.
Figure 2.12: Beam deflection amplitudes and phase differences at peak splitting locations (Configuration B)

Since the exact geometries of the beams are not prescribed, it is difficult to create a definitive reference single beam structure. One method is to set a rigid coupling stiffness with a large number of springs, reaching the continuous spring distribution limit. In a practical sense, this is as if the two beams are "bolted" together. The total mass of the MEMP beams is equivalent to the mass of the reference single beam. The effectiveness is observed by subtracting the reference single beam transfer function from the MEMP beam transfer function.
As expected, altering the spring stiffness on the identical beam case ($\bar{\gamma} = 1$) does not effect the dynamic results, explaining why the single beam case matches the MEMP case. The two configurations with differing wave speeds ($\bar{\gamma} = .7$ and $\bar{\gamma} = .8$) show both positive and negative differences due to the constructive/destructive interference. The attenuation is concentrated in small frequency bands with the bandwidth depending on the nondimensional input parameters. If a structure were forced in a narrow or discrete band, this MEMP configuration could prove useful. However, the goal is to design an MEMP system with improved attenuation in wideband ranges.
2.2.5 Configuration B: Vibration Absorber

The second configuration consists of a cantilever single beam coupled to an unforced beam with both ends free. The unforced beam can be considered a sort of vibration absorber. Similar to the cantilever double beam, differing wave speeds can create interference between the two beams and lead to attenuation. However since the bottom beam is now free on both ends, attenuation is also possible when both beams share identical properties.

\[ TF = 20 \log_{10} \left( \frac{\left| E_1 I_1 \frac{\partial^2 y}{\partial x^2} \right|_{x=0}}{F_0} \right) \]  

(2.14)

The vibration absorber can be examined similarly to the single beam coupled to a wall by a continuous suspension. From Section 2.1.7, it was observed that below a cutoff frequency, the excited vibration would quickly decay along the beam due to the evanescent behavior of the dispersion relation. If the density ratio of the free beam to the forced beam is made very large, then the free beam appears as a wall with zero deflection. Figure 2.14 below plots the transfer function for two beams as the free beam approaches the rigid wall limit, revealing the evanescent cutoff behavior.
Altering the parameters changes the width and location of the attenuation gap. It has been noted that a heavy free beam leads to extensive attenuation below a cutoff frequency, acting as a high pass filter. This approach however is somewhat contradictory to the MEMP problem statement in which mass reduction is critical to structural design.

2.2.6 Configuration C: Mixed Cantilever Double Beam

The double cantilever beam and vibration absorber configurations produce attenuation bands that can be tuned by the beam and spring parameters. These bands

Figure 2.14: Vibration absorber rigidity with cutoff behavior (Configuration B)
appear in narrow frequency ranges, however a wideband solution is still undetermined. Configuration C consists of a top beam forced on one end and free on the other end. The bottom beam is fixed on the same end in which the top beam is free and is free on the end in which the top beam is forced. Since the free boundary conditions are on opposite end of each beam, this configuration will be referred to as the "mixed cantilever double beam".

\[ TF = 20 \log_{10} \left( \left| \frac{E_2 I_2 \frac{\partial^2 y_2}{\partial x^2} |_{x=0}}{F_0} \right| \right) \]  

(2.15)

The main distinction between this and the other previously described configurations is in the wave path of the excitation towards the fixed, or receiving, end. With configurations A and B, removing the spring couplings results in an analogous single beam case. In the mixed cantilever double beam, the excitation path must propagate through the coupling in order to reach the fixture. Thus, if the spring couplings are removed, \( K_s \rightarrow 0 \), there is no force transmitted onto the fixed end. Of course, the couplings cannot reach this spring value and instead will be limited by physical constraints such as static stiffness and beam interference.

Before examining the configuration C theoretical results, it is helpful to analyze a similar yet much simpler lumped element model consisting of two discrete masses coupled by a spring. The illustration below presents the model in which the input forcing on the first mass \((m_1)\) is transferred onto the second mass \((m_2)\) through the spring coupling.

![Figure 2.15: Two coupled mass system](image-url)
The transfer function for force transmission, derived in Appendix A, is:

\[
TF = \frac{\text{Force on } m_2}{\text{Input Force}} = \frac{m_2 \frac{\partial^2 x_2}{\partial t^2}}{F_0 e^{i\omega t}} = \frac{m}{m + 1 - m \bar{\omega}^2}
\]  

At the quasistatic limit, the response settles to \(TF(\bar{\omega} = 0) = \frac{m}{m + 1}\). After the two mass resonant frequency, \(\bar{\omega} = \sqrt{\frac{1 + m}{m}}\), the transfer function reaches values below \(TF(\bar{\omega} = 0)\) and eventually approaches zero, or complete attenuation as the frequency increases.

Now, plotting the MEMP beam model for configuration C with identical beams and various coupling stiffnesses yields the following results:

**Figure 2.16**: Spring coupling softening for identical beams (Configuration C)

At low frequencies, the beams behave in-phase with each other and the transfer function remains positive. After a structural resonance, dependent on the beam and spring parameters, the peak splitting and interference occurs between the beams, that on average become more attenuated as the frequency increases, similar to the analogous two mass model. In the three cases provided, the transfer function for the configuration with the softest coupling springs (\(K_s = .1\)) reaches the lowest decibel value. Again, this is expected under the mixed cantilever boundary conditions since
removing the coupling or softening the springs to the limit $K_s \to 0$ results in an uncoupling of the beams and thus complete attenuation.

In the above configuration, the beams are identical, yet there are frequency bands with deep attenuation dips. While the structural wave speeds on each element are the same, the boundaries are mixed such that at one end of the MEMP system, one beam is fixed and the other is free. Waves traveling through a medium are reflected at boundary locations and can undergo a phase shift. If the boundary is fixed, or has an impedance of infinity, then the reflected wave experiences a 180 degree phase shift. If the boundary is free, or has an impedance of zero, then there is no phase difference between the incident and reflected wave. This effect, illustrated in Figure 2.17, can lead to interference between reflected/transmitted waves from different boundary configurations.

![Wave behavior at boundary](image)

**Figure 2.17**: Wave behavior at boundary

The reference single beam consists of the same boundary conditions as in the configuration B single beam, both beam ends fixed on one side, forced on the other with rigid couplings. The transfer function for the single beam and MEMP beam are compared.
Figure 2.18: MEMP vs single beam comparison for Configuration C

The comparison reinforces the statement that for the mixed cantilever double beam, the lower coupling stiffness generally produces more attenuation. In the above plots, the two beams in the MEMP configuration were identical. If the beam properties are mismatched and produce differing wavespeeds, more opportunities for peak splitting and interference become present. Configuration C beams with differing stiffnesses ($\beta = .25, 4$) on the lowest stiffness springs curve from Figure 2.16 are plotted below. It is important to note that the correct nondimensional frequency must be used in the comparison to indicate that beam stiffnesses are only increased. Thus for the case in which $\beta = 4$, the frequency plotted is in relation to the second beam, $\omega_2 = (\beta \gamma) \omega$. 
Figure 2.19: Interference for varying beam wavespeeds with low spring stiffness (Configuration C)

As predicted, altering the stiffness ratio of the two beams in the MEMP configuration produces transfer functions with more interference activities (frequency peaks and dips). In the plots, the wavespeeds are varied by increasing the stiffness of one of the beams. The transfer function in which the receiving beam is stiffened (blue curve, $\beta < 1$) outperforms the inverse in which the forcing beam is instead stiffened (green curve, $\beta > 1$), especially at higher frequencies. The hypothesis that increasing the stiffness of the receiving beam over the forcing beam leads to more attenuation will be later evaluated in the empirical results.

Overall, out of the three observed MEMP configurations (A, B, and C) the mixed cantilever double beam setup (case C) reveals the most interesting and effective dynamic behavior. Cases in which the wavespeeds are varied and the coupling stiffness is lowered present robust and wideband dynamic solutions. The implications of converting a single element beam into this MEMP coupled structure will now be discussed.
2.2.7 Mass Conservation and Static Stiffness

In translating between an MEMP and single element system, it is important to consider the overall mass and static strength. One of the benefits of the MEMP design method over other attenuation techniques is that it allows for the conservation of mass in comparison to a reference single beam while reducing the force transfer. Two methods for comparison of the mixed cantilever double beam will be used in this discussion: one in which the two MEMP beams are "bolted" together with rigid couplings and another in which a single beam is forced through cantilever conditions.

As previously mentioned, the attenuation in configuration C depends strongly on the spring coupling strength due to the nature of the wave path between the excitation and fixture. The goal will be to soften the springs as much as physically possible. The constraint on the coupling stiffness will primarily be the static strength of the MEMP beam system. Additionally, depending on the application and magnitude of the forcing excitation it is important to consider the physical interference between the two beams (especially during out-of-phase motion).

The first comparison consists of the two beams equally forced on the same end and fixed on the opposing end with rigid and extensive spring couplings (or bolted), forming what resembles a single beam model in that both beams deflect equally. In this comparison, since the single beam and MEMP model consist of the same beams with different couplings and boundary conditions, the mass is already conserved by definition. The static strength of the MEMP and bolted single beam case can be determined through the dynamic results by examining the quasistatic deflection on the forced end of the beam (see Appendix for definition). The relative static strength is simply then the ratio of deflections. Several stiffness curves are plotted in Figure 2.19 for various coupling stiffness and beam elasticity ratios.
Of course, in this comparison the MEMP case is always going to be less stiff than the reference single beam since the latter is essentially the same beams except with rigid springs. What this reveals is the relationship between the static stiffness and beam parameters, or how the dynamic attenuation affects the overall strength of the configuration.

The other comparison is the cantilever single beam introduced earlier in the chapter. Take for instance the MEMP beam system, forced on beam one, with static stiffness:

$$K_{\text{MEMP}} = \frac{F}{y_{\text{MEMP}}} = \frac{L^3}{E_1 I_1} \frac{F}{y_{\text{MEMP}}} = K_{\text{MEMP}} \frac{y_{\text{MEMP}}}{K_{\text{Beam1}}}$$

The deflection $y_{\text{MEMP}}$ is computationally determined from the MEMP beam algorithm and the input force $F$ is set to unity. The stiffness can be rewritten:

$$K_{\text{MEMP}} = \frac{1}{y_{\text{MEMP}}} = \frac{K_{\text{MEMP}}}{K_{\text{single}}} \left( \frac{K_{\text{single}}}{K_{\text{beam1}}} \right)$$

Finally, matching the stiffness of the MEMP system to the single beam as desired ($\frac{K_{\text{MEMP}}}{K_{\text{single}}} = 1$) regulates what the stiffness of the forced MEMP beam must be in
relation to the single cantilever beam:

\[ \overline{K_{\text{ratio}}} = \frac{K_{\text{beam1}}}{K_{\text{single}}} = \bar{y}_{\text{MEMP}} \]

The metric \( \overline{K_{\text{ratio}}} \) reveals the necessary stiffness of the forced beam in the MEMP mixed cantilever configuration such that the overall MEMP stiffness (given parameters \( \bar{\beta}, K_s \) and \( s \)) matches that of the single beam. Curves for \( s = 10 \) are plotted below.

Figure 2.21: Static strength ratio for \( s = 10 \) using single reference beam (MEMP Configuration C)

Once the static strength of the MEMP and single beam cases are matched, the linear densities are then solved for to match the weight. This process involves substituting dimensional quantities into the nondimensional parameters and determining the correct geometries given material properties so that the constraints (static strength and weight) can be conserved.

\[ \text{2.2.8 Torsional Spring Coupling} \]

The discrete couplings employed in the previous discussions were extensional, proportional to the difference in deflection between beams. Practical springs, however,
are not limited to this form and can store mechanical energy in a variety of different motions. Another common elastic coupling is the torsional spring which produces a moment or torque proportional to the difference in angle of deflection. The application of these couplings can be illustrated with two beams deflected equally at the spring location, unaffected by the extensional spring force. In this scenario, it is possible for the angles of deflection to be out-of-phase with each other and thus influenced by the torsional spring.

The inclusion of the torsional spring (spring constant $K_t$) in the mathematical model is similar to that of the extensional spring except with the discontinuity in the moment equation.

\[
\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \left[ -E_1 I_1 \frac{\partial^3 y_1}{\partial x^3} - K_t \left( \frac{\partial y_1}{\partial x} - \frac{\partial y_2}{\partial x} \right) \delta(x) \right] dx = 0
\]

\[
\implies -E_1 I_1 \frac{\partial^2 y_1}{\partial x^2} \bigg|_{0^+}^{0^-} - K_t \left( \frac{\partial y_1(0)}{\partial x} - \frac{\partial y_2(0)}{\partial x} \right) = 0 \quad (2.17)
\]

**Figure 2.22: Torsional coupling effect**
The revised coupling boundary conditions are, including the extensional spring:

\[
\begin{align*}
\text{BCs} - \frac{\partial^3 y_1}{\partial x^3}(0^+) - K_s(y_1(0) - y_2(0)) &= 0 \\
- \frac{\partial^3 y_1}{\partial x^3}(0^-) - K_t\left(\frac{\partial y_1(0)}{\partial x} - \frac{\partial y_2(0)}{\partial x}\right) &= 0 \quad \text{and} \quad \frac{\partial y_1}{\partial x}(0^-) = y_1(0^-) = 0 \\
- \frac{\partial^3 y_2}{\partial x^3}(0^-) - \beta^2 K_s(y_2(0) - y_1(0)) &= 0 \\
- \frac{\partial^3 y_2}{\partial x^3}(0^-) - \beta^2 K_t\left(\frac{\partial y_2}{\partial x}(0^-) - \frac{\partial y_1}{\partial x}(0^-)\right) &= 0 \quad \text{and} \quad \frac{\partial y_2}{\partial x}(0^-) = y_2(0^-) = 0 \\
\text{ND} \quad K_t = \frac{K_t L^2}{E_1 I_1}
\end{align*}
\]

The dynamic and static effect of the coupling is examined by revisiting the mixed cantilever double beam configuration and adjusting the torsional spring.

![Dynamic and Static Effect of Torsional Couplings for Mixed Cantilever Beam](image)

**Figure 2.23**: Dynamic and static effect of torsional coupling \((s = 10, \beta = 1.00, \gamma = 1.00, K_s = 0.10)\) (MEMP Configuration C)

The added torsional stiffness helps to lower \(K_{\text{ratio}}\), but has a dramatic negative effect on the dynamic attenuation. Figure 2.23 illustrates the spring activation at out-of-phase MEMP motion, during which destructive interference occurs. Restricting
this behavior thus limits the depth of the attenuation gaps in the transfer function. Alternate torsional spring couplings can be designed such that the static improvement is far more significant [Su (2013)].

2.2.9 Viscous Damping

In Section 2.1.5, damping was introduced into the single beam through the dispersion relation. The imaginary portion of the wavenumber appeared as material damping and used primarily to remove the transfer function singularities at the resonance frequencies. Another form of damping not yet discussed is the viscous damper, commonly referred to as a dashpot, that is proportional to the velocity. Since this relies on differences in velocities (or deflections) and consequently requires two structures, it is inapplicable to the single beam case. The dashpot (damping constant $\alpha_v$) can be placed within the discrete spring couplings of the MEMP beams as follows:

$$-E_1 I_1 \frac{\partial^3 y_1}{\partial x^3} \bigg|^{0+}_{0-} - K_s (y_1 - y_2) - \alpha_v \left( \frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} \right) = 0$$

$$-E_2 I_2 \frac{\partial^3 y_2}{\partial x^3} \bigg|^{0+}_{0-} - K_s (y_2 - y_1) - \alpha_v \left( \frac{\partial y_2}{\partial t} - \frac{\partial y_1}{\partial t} \right) = 0$$

**Coupling BCs**  \[\rightarrow\]  $$- \frac{\partial^3 y_1}{\partial x^3} \bigg|^{0_+}_{0_-} - K_s (+i \alpha_v \omega) (\bar{y}_1 - \bar{y}_2) = 0$$

$$- \frac{\partial^3 y_2}{\partial x^3} \bigg|^{0_+}_{0_-} - \beta^2 K_s (1 + i \alpha_v \omega) (\bar{y}_2 - \bar{y}_1) = 0$$

**ND**  \[\alpha_v = \frac{\alpha_v \omega_n}{K_s}\]

The damping term can be inserted by modifying the spring constant to become a complex number with the imaginary portion dependent on frequency and the damping constant. The effect on the dynamic transfer function is plotted below for various damping constants on the mixed cantilever double beam.
As expected, increasing the damping brings down the level of the resonance peaks that would otherwise ascend to infinity, leading to more attenuation. It is also evident though that the transfer function dips, or destructive interference frequencies, are less efficient with higher damping. If the motion of the coupled beams is limited due to a viscous damper, then the opportunities for interference, both constructive and destructive, are restricted.

Comparison between the viscous and material damping involves discussing the applicability of each. The material damping is more of a mathematical construct that simply disallows the resonances from reaching singularity. Practically speaking, it would be difficult to include such a damping into the beam material itself. On the other hand, the viscous damper is commonly employed by placing viscoelastic material in between the two beams and is another advantage of MEMP design.

2.3 Experimental MEMP Double Beam Results

The analytical results from the mathematical modeling reveal that wideband attenuation is possible in MEMP double beam systems. The beams can be configured with
a variety of different boundary conditions to employ physical phenomena not evident in single beam models. After exploring the three configurations (cantilever double beam, vibration absorber and mixed cantilever double beam) it could be concluded that configuration C, the mixed cantilever double beam, had the most potential for vibration attenuation. In this configuration, the path of the propagated wave from the forced first beam must travel directly through the elastic couplings in order to reach the fixture of the second beam. It is now of great interest to experimentally test this result and compare it to the mathematical models.

2.3.1 Experimental Configuration and Setup

The experiment consisted of sending an excitation onto a MEMP double beam and single beam structure and measuring the force transfer from the excitation onto the fixture. The general schematic is illustrated below:

![Experimental setup diagram](image)

**Figure 2.25: Experimental setup**

A dSPACE DS1104 controller board in junction with the Simulink, a MATLAB programming tool, was used to send and receive the analog signals during the experiment. The shaker, a PASCO sf-9324 Mechanical Vibrator, was able to produce a vertical excitation linear response between .1 and 5000 Hz. The force was measured through Omega force transducers attached in two locations: at the shaker end and on top of the beam on the receiving end. The fixture was a heavy stand composed...
of steel to ensure that the displacement of the receiving end beam was fixed and isolated from the table.

Sending the excitation consisted of generating a white noise signal in Simulink, transmitting the signal from the dSPACE board to an external amplifier and feeding the amplified signal to the shaker. The white noise was generated for one minute to ensure a clean and steady state response. During the excitation, force transducers measured the transfer of force along the beams which was filtered and converted into the frequency domain through a Fourier transform in MATLAB.

The two beams for the MEMP structure were composed of 6063 aluminum alloy and had a rectangular cross sectional area with a large width to height ratio (1 1/2 inch width to 1/8 inch thickness), complying with Euler-Bernoulli beam theory. The length to width ratio was high as well (> 15). There were 54 evenly spaced holes drilled near the edges to secure the springs between the beams. The couplings were lightweight elastic clips which secured tightly into the drilled holes on each beam and could be removed with ease to alter the overall stiffness of the MEMP system. Industrial earplugs were fit in between the couplings to add damping. It is important to note that the couplings behaved in a wide range of motions with different elasticities. Other than the simple extensional stiffness, there was a torsional stiffness that varied depending on the angle in which the clips were placed on the beams.

![Figure 2.26: Discrete elastic couplings](image)

**Figure 2.26:** Discrete elastic couplings
2.3.2 Dimensional Parameters

In order to compare the results to the mathematical model, the dimensional parameters must be obtained. For the beam and spring combination, these parameters are the beam cross sectional dimensions (width and height), the beam length, the beam Young’s modulus, the beam mass density and the spring stiffness. The extensional and torsional stiffness of the clips were evaluated through a tensile testing machine in which the load and deflection (or angle of deflection) was measured under tension/compression. A linear regression was computed on the deflection curves (see Appendix C) and averaged over several clips to obtain the mean stiffness values of $K_s = 4710 \text{ N/m} (\sigma = 213)$ and $K_t = 0.19 \text{ N*m/rad} (\sigma = 0.05)$.

The beam density was measured using a scale and the geometric dimensions. The Young’s modulus was obtained through a tensile testing machine by measure the end deflection on a cantilever point load with a fixed end boundary condition. While the material specifications sheet listed the modulus as $\approx 70 \text{ MPa}$, it was believed that drilled holes and modifications made to the beam significantly altered the its elasticity. Once measured, the mean value was obtained to be $58.9 \text{ MPa}$.

2.3.3 Mixed Cantilever Double Beam Results
Prior to testing the MEMP configuration, the reference single beam case needed to be designed and analyzed. The single beam was formed by bolting the two beams together at regular intervals, similar to that of the mathematical model, forming a rigid coupling between the beams. Once the transfer function was generated from the input and output force transducers, the result was filtered and calibrated at the quasistatic limit ($\frac{F_{\text{out}}}{F_{\text{in}}} \to 1$ as $\omega \to 0$).

![Experimental Force Transfer Function for Reference Single Beam Case](image.png)

**Figure 2.27**: Experimental transfer function for reference single beam (two beams bolted together with 14 screws)

The transfer function is examined between 0 and 800 Hz to encapsulate several resonance peaks. Qualitatively, the result appears reasonable in that a single element thin beam should only experience peaks with no interference or dips. After about 500 Hz, the transfer function reaches a negative value indicating that there is less force on the receiving end than on the excited end. This is most likely attributed to damping within the structure, similar to the artificial material damping used in the mathematical modeling.

The MEMP beam was tested under the mixed cantilever double beam configuration for three coupling stiffnesses (18, 28 and 54 springs, aligned on the edges of both
beams) by altering the number of springs, effectively changing the overall suspension stiffness. The beams in this testing are identical to each other in both material and geometry. Additionally, the viscoelastic dampers were placed in between the springs to observe its effect. The results are plotted below along with the reference single beam.
Figure 2.28: Experimental transfer function for MEMP mixed cantilever double beam with 18, 28 and 54 springs (damping and no damping)

From first observing the plots, there is wideband attenuation in all three spring
configurations, beginning after a clearly defined resonance at around 45 Hz. This frequency indicates the point in which both beams are resonating together prior to interference and is roughly at the same frequency for the three cases. The least stiff spring configuration, 18 springs, exhibits the lowest frequency attenuation as the transfer function begins to reach a negative value at 110 Hz. This behavior agrees with the notion from Section 2.2.6 that softening the couplings leads to lower band results in the mixed cantilever double beam configuration.

Deep, narrow attenuation wells appear in each configuration and most significantly in the 54 spring case in the 200-300 Hz range. As mentioned in the MATLAB results, multi layered structures such as this allow for destructive interference when both beams are behaving out-of-phase with each other, providing opportunities for robust attenuation gaps not evident in single element structures. In all three configurations, the transfer function at the interference attenuation gap reaches a remarkable -40 dB (515 Hz for 18 springs, 215 Hz for 28 springs and 270 Hz for 54 springs).

Another defining feature of the MEMP results is the increased number of resonant peaks. Even though both beams are identical, the boundary conditions are mixed leading to a variety of different modal behaviors. One example can be seen by the reference structure resonance at 680 Hz, indicated by the peak in the black curve. The MEMP spring configurations each contain two resonance peaks nearby this single beam resonance, suggesting the occurrence of peak splitting.
A single viscoelastic earplug was placed in between each coupling to act as a dashpot damper with results indicated by the blue curve. The damping is effective at frequencies in which the beam deflections are out-of-phase with each other, explaining why at the first resonance peak, where the beams behave as two lumped masses, the transfer function is unaffected. In the rest of the frequency range, the peaks are brought down and the attenuation dips are brought up, thus leveling out the overall transfer function and resulting in wider but less intensive reductions.

2.3.4 Experimental and Mathematical Modeling Comparison

With the previously listed dimensional parameters, the experimental results can be compared to the mathematical modeling results obtained in MATLAB. The algorithm had to be slightly modified to include the elongated ends of each beam which secured the force transducers. Additionally, while the fixture end of the structure was prevented from displacing/rotating, the forced end, connected to the mechanical shaker, did not entirely match the forced boundary condition from the modeling.
From examining the structure, the bolt did not completely hinder the rotation. To match the experiment to the model, a torsional spring was included between at the excited end in the modeling with a spring constant determined through an ad hoc procedure. This was a reasonable adjustment to make since in real world applications boundary conditions are often not exactly what is modeled in the mathematical construct.

The results for the single beam comparison along with the three spring configurations are plotted below in dimensional frequency (Hz). A material damping of $\bar{\alpha} = 1.5\%$ was included in the simulation:
Figure 2.30: Experimental and theoretical transfer function for MEMP mixed cantilever double beam with 18, 28 and 54 springs

The experimental and simulation reference single beam transfer functions (black
and green curves, respectively) each contain the four resonance peaks within the observed range but vary slightly in frequency location (\(<25\) Hz). It would be ideal for the two plots to overlay each other, however uncertainties in the measurements, thin beam approximations and unaccounted for lumped elements (force transducer weight and stiffness) allow for variation between the two. Discrepancies between levels occur primarily and systematically in the frequency range of 300-500 Hz due to undetermined damping and interference encountered in the experimental setup. Even so, observing the experimental and simulation MEMP transfer function for each spring case reveals compelling similarities in terms of attenuation events. The peak splitting, interference gaps and resonance peaks discussed in the experimental results are present at roughly the same frequencies.

Subtracting the experimental and theoretical MEMP transfer functions from their respective reference single beam cases reveals the MEMP effectiveness in attenuating the vibration transmission. As predicted, the 18 spring configuration, in both the experiment and simulation, experienced the most attenuation averaged from 0-800 Hz due to its low stiffness suspension. Additionally, this configuration showed very promising wideband reductions between 50-700 Hz, reaching positive transfer function. The experimental 28 spring configuration, or medium stiffness, reached the highest level of attenuation at -60 dB.
Figure 2.31: Experimental and Theoretical Transfer Function Difference for MEMP Mixed Cantilever Double Beam with 18, 28 and 54 Springs
While the previous experimental results were performed on identical beams, theoretical analysis suggests that assigning differing beam properties could provide more opportunities for interference and consequently, attenuation. In order to achieve this, a stiffening beam, composed of the same material as the MEMP beams, was bolted onto the MEMP system with two configurations: bolting the stiffener onto the forced beam (configuration F, $\bar{\beta} > 1$) and bolting it onto the receiver beam (configuration R, $\bar{\beta} < 51$). This was modeled by adjusting the moment of inertia and linear density of the stiffened beam. The transfer functions of the experimental and simulation results are plotted along with the previously introduced reference single beam.
There are similarities between the experimental and simulation results in that both contain identical events in comparable frequency locations, however the correlation is not as strong due to the "roughness" of the stiffening beam modeling. Comparing the two plots, configuration R begins to reach higher attenuation than configuration F near the end of the observed frequency range. This agrees with predictions made in the simulations from Figure 2.18 in that stiffening the receiving beam is the more effective solution. The transfer function differences for the two configurations along with the removed stiffener case are plotted as well, clearly show-
ing the effect that altering the beam wavespeeds has on the transfer function. The difference is prominent in the 400-800 Hz range where both stiffener configurations consistently attenuate more vibrations than the identical beam case.

**Figure 2.34**: Experimental Transfer Function Difference for MEMP Mixed Cantilever Double Beam with Added Stiffener (28 Springs)
The theoretical and experimental results from the MEMP thin beam analysis served as a one-dimensional introduction into multi-element multi-path design and illuminated the advantages of coupled elements over single element systems. However, a more three-dimensional and practical structure is desired. The following chapters will thus focus on thin cylindrical shells, a ubiquitous structure found in various fluid applications such as underwater submersibles, aircrafts and flow transports.

![Figure 3.1: Cylindrical shell-like structures (McDonnell- Douglas MD-115 [Gunston and Badrocke (1999)], Nautilus SSN 571 [Polmar and Moore (2004)] and Lockheed Martin F-16)]](image)

The cylindrical shell can be thought of as a flat plate rolled and connected on its ends. The inclusion of curvature is essential to shell theory and will lead to interesting results and deformations not evident in simple flat beams. Similar to the...
prior chapters, discussion will begin with the single shell and progress to elastically coupled shells, delving into the physics behind the highly coupled deformations and their effects on vibration attenuation.

3.1 Single Thin Shell

3.1.1 Stress Resultants

The shell geometry is setup in cylindrical coordinates: the radius $R$, the azimuth $\phi$ and the axial direction $x$ with the origin located in the cross sectional center at an end of the shell.

![Cylindrical shell geometry](Flugge (1960))

The Euler-Bernoulli stress governing equations for the thin beam are rather simple, with beam motion constrained to one dimensional deflection and an intuitive dispersion relation. The cylindrical shell on the other hand will consist of three highly coupled deflection motions: displacement in the longitudinal ($u$), tangential ($v$) and radial ($w$) direction. Prior to stating the stress relations, assumptions must be made regarding the shell deformation, as were with beam theory. The middle surface can be taken as the circular line drawn at the half thickness of the cross sectional area. After deformation, it will be assumed that points normal to this middle surface remain normal and that the stress normal to the surface in the radial direction ($\sigma_z$)
is negligible in comparison to those in the axial and azimuthal directions ($\sigma_x$ and $\sigma_\phi$) [Flugge (1960)]. Deflections are also presumed to be small to maintain linearity.

**Figure 3.3**: Stress resultants and moments on cylindrical shell [Flugge (1960)]

The governing equations for the shell are derived in terms of the "stress resultant", a force per unit distance derived from the stress integration along a direction of the shell. These stress resultants along with their moments are balanced in the shell governing equations by the inertial, adhering to Newton’s Second Law. The normal stress resultants ($N_x$ and $N_\phi$) and shear stress resultants ($N_{x\phi}$ and $N_{\phi x}$) appear along with the bending shear resultants normal to the middle surface ($Q_x$ and $Q_\phi$).

**longitudinal balance**
\[
R \frac{\partial N_x}{\partial x} + R \frac{\partial N_{\phi x}}{\partial \phi} + R p_x = \rho Rh \frac{\partial^2 u}{\partial t^2} \tag{3.1}
\]

**tangential balance**
\[
\frac{\partial N_\phi}{\partial \phi} + R \frac{\partial N_{x\phi}}{\partial x} - Q_\phi + R p_\phi = \rho Rh \frac{\partial^2 v}{\partial t^2} \tag{3.2}
\]

**radial balance**
\[
- \frac{\partial Q_\phi}{\partial \phi} - R \frac{\partial Q_x}{\partial x} - N_\phi + R p_r = \rho Rh \frac{\partial^2 w}{\partial t^2} \tag{3.3}
\]

Two additional equations, derived from moment balance equilibrium, relate the bending terms to their respective bending moments:

**longitudinal bending**
\[
\frac{\partial M_\phi}{\partial \phi} + R \frac{\partial M_{x\phi}}{\partial x} - R Q_\phi = 0 \tag{3.4}
\]

**tangential bending**
\[
R \frac{\partial M_x}{\partial x} + \frac{\partial M_{\phi x}}{\partial \phi} - R Q_x = 0 \tag{3.5}
\]
Note that if the curvature is removed, the angular derivatives are set to zero and the governing equations become:

\[
\frac{\partial N_x}{\partial x} + p_x = \rho h \frac{\partial^2 u}{\partial t^2}
\]

\[
-\frac{\partial Q_x}{\partial x} + p_r = \rho h \frac{\partial^2 w}{\partial t^2}
\]

\[
\frac{\partial M_x}{\partial x} = Q_x
\]

Without curvature, the structure becomes familiar. These equations are the three-dimensional equivalence of the governing equations describing longitudinal and transverse motion on a beam with bending. The importance of curving the surface of the structure is essential to shell theory and will be emphasized when examining its dynamic characteristics.

Utilizing strain relations and integrating across the thickness of the shell, the stress resultants can be written as approximate functions of the deformations derived
from curved bars and flat plate theories [Flugge (1960)].

\[
N_\phi = \frac{Eh}{(1-\nu^2)} \left( \frac{1}{R} \frac{\partial v}{\partial \phi} + \frac{w}{R} + \nu \frac{\partial u}{\partial x} + \frac{h^2}{12R^2} \left( \frac{1}{R} \frac{\partial^2 w}{\partial \phi^2} + \frac{w}{R} \right) \right) 
\]  
(3.6)

\[
N_\nu = \frac{Eh}{1-\nu^2} \left( \frac{\partial u}{\partial x} + \nu \frac{w}{R} + \frac{\partial v}{\partial \phi} - \frac{h^2}{12R^2} \left( \frac{\partial^2 w}{\partial x^2} \right) \right) 
\]  
(3.7)

\[
N_{\phi x} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial v}{\partial \phi} + \frac{1}{R} \frac{\partial u}{\partial \phi} + \frac{h^2}{12R^2} \left( \frac{1}{R} \frac{\partial u}{\partial \phi} + \frac{\partial^2 w}{\partial \phi \partial x} \right) \right) 
\]  
(3.8)

\[
N_{x\phi} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial v}{\partial x} + \frac{1}{R} \frac{\partial u}{\partial x} + \frac{h^2}{12R^2} \left( \frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial \phi \partial x} \right) \right) 
\]  
(3.9)

\[
M_\phi = \frac{Eh^3}{12(1-\nu^2)} \left( \frac{1}{R^2} \frac{\partial^2 w}{\partial \phi^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) 
\]  
(3.10)

\[
M_\nu = \frac{Eh^3}{12(1-\nu^2)} \left( \frac{\nu}{R^2} \frac{\partial^2 w}{\partial \phi \partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R} \frac{\partial u}{\partial x} - \frac{1}{R^2} \frac{\partial v}{\partial \phi} \right) 
\]  
(3.11)

\[
M_{\phi x} = \frac{Eh^3}{12(1+\nu)} \left( \frac{1}{R} \frac{\partial^2 w}{\partial \phi \partial x} + \frac{1}{2R^2} \frac{\partial u}{\partial \phi} - \frac{1}{2R} \frac{\partial v}{\partial x} \right) 
\]  
(3.12)

\[
M_{x\phi} = \frac{Eh^3}{12(1+\nu)} \left( \frac{1}{R} \frac{\partial^2 w}{\partial \phi \partial x} - \frac{1}{R} \frac{\partial v}{\partial x} \right) 
\]  
(3.13)

### 3.1.2 Governing Equations and General Solution

Substituting in the stress-deformation relations into the three shell governing equations yields the coupled linear partial differential equations that define the shell deformation.
Through separation of variables, each deflection \(u, v\), and \(w\) can be separated in the two independent directional domains \(\phi\) and \(x\) and in the time domain \(t\).

The eigenvector \((\zeta)\) with its components \((U, V, W)\) determines the relative coupling motions between the deflections. As with the beam, solutions will be assumed steady state and harmonic in time, \(e^{i\omega t}\), with the transient solution already decayed. The separated directional solutions will be evaluated through the method of undetermined coefficients, forming the propagating solutions \(X = e^{-ikx}\) and \(\Phi = e^{-im\phi}\) along with the respective dispersive wavenumbers \(k\) and \(n\). In the thin beam analysis, the vibration was forced at a given frequency in the transverse beam motion.
For the shell analysis, the frequency along with the azimuthal wavenumber, $n$, will be specified in the forcing term. The qualification for $n$ is that it must be a whole number to preserve angular symmetry along the azimuth ($\Phi(0) = \Phi(2\pi)$). The full solution will appear as:

$$u = \sum_{n=-\infty}^{\infty} C_n U_n e^{i(\omega t - k_n x - n \phi)}$$  \hspace{1cm} (3.20)

$$v = \sum_{n=-\infty}^{\infty} C_n V_n e^{i(\omega t - k_n x - n \phi)}$$  \hspace{1cm} (3.21)

$$w = \sum_{n=-\infty}^{\infty} C_n W_n e^{i(\omega t - k_n x - n \phi)}$$  \hspace{1cm} (3.22)

The azimuthal distribution of the displacements and forcing terms along the shell cross section are illustrated below:

![Shell Azimuthal Distribution](image)

**Figure 3.4**: Modes of shell vibration

For $n = 0$, the cross section deforms uniformly in the radial and azimuthal di-
rections, signifying axisymmetric motion. For \( n = 1 \), the shell behaves as a dipole similar to the transverse deflections of a beam. As will be further discussed, the shell motions can generally be grouped into the axisymmetric, beam-like and higher order modes \((n > 1)\).

The shell governing equations are nondimensionalized:

Gov Eqns

\[
\bar{R}^2 \frac{\partial^2 \bar{u}}{\partial x^2} + \nu \frac{\bar{R}}{\partial x} \frac{\partial \bar{w}}{\partial x} + \frac{1 - \nu}{2} \frac{\partial^2 \bar{u}}{\partial \phi^2} + \frac{\bar{R} (1 + \nu)}{2} \frac{\partial^2 \bar{v}}{\partial x \partial \phi} + \ldots
\]

\[
\frac{\bar{h}}{12} \left( \frac{1 - \nu}{2} \frac{\partial^2 \bar{w}}{\partial \phi^2} - \frac{\bar{R} (1 - \nu)}{2} \frac{\partial^3 \bar{w}}{\partial x^3} + \frac{\bar{R} (1 - \nu)}{2} \frac{\partial^3 \bar{w}}{\partial x^2 \partial \phi} \right) + \frac{\bar{p}_x (1 - \nu^2)}{\bar{h}^2} = \frac{\partial^2 \bar{v}}{\partial t^2} \tag{3.23}
\]

\[
\frac{\partial^2 \bar{v}}{\partial \phi^2} + \frac{\partial \bar{w}}{\partial \phi} + \frac{\bar{R}^2 (1 - \nu)}{2} \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\bar{R} (1 + \nu)}{2} \frac{\partial^2 \bar{w}}{\partial x \partial \phi} + \ldots
\]

\[
\frac{\bar{h}}{12} \left( \frac{3 \bar{R}^2 (1 - \nu)}{2} \frac{\partial^2 \bar{v}}{\partial x^2} - \frac{\bar{R}^2 (3 - \nu)}{2} \frac{\partial^3 \bar{w}}{\partial x^2 \partial \phi} \right) + \frac{\bar{p}_\phi (1 - \nu^2)}{\bar{h}^2} = \frac{\partial^2 \bar{v}}{\partial t^2} \tag{3.24}
\]

\[- \bar{w} - \frac{\partial \bar{v}}{\partial \phi} - \nu \frac{\partial \bar{u}}{\partial x} - \frac{\bar{h}}{12} \left( \frac{\partial^4 \bar{w}}{\partial \phi^4} + 2 \bar{R}^2 \frac{\partial^4 \bar{w}}{\partial x^2 \partial \phi^2} + \bar{R}^4 \frac{\partial^4 \bar{w}}{\partial x^4} + \bar{w} + 2 \frac{\partial^2 \bar{w}}{\partial \phi^2} \right) - \ldots
\]

\[
\frac{\bar{h}}{12} \left( \frac{\bar{R} (1 - \nu)}{2} \frac{\partial^3 \bar{v}}{\partial \phi^2 \partial x} - \frac{\bar{R}^2 (3 - \nu)}{2} \frac{\partial^3 \bar{v}}{\partial x^2 \partial \phi} - \frac{\bar{R}^3 \partial^3 \bar{v}}{\partial x^3} \right) + \frac{\bar{p}_x \bar{R} (1 - \nu^2)}{\bar{h}} = \frac{\partial^2 \bar{w}}{\partial t^2} \tag{3.25}
\]

ND

\[
\bar{u}, \bar{v}, \bar{w} = \frac{u, v, w}{L}, \quad \bar{p}_x, \bar{p}_\phi, \bar{p}_r = \frac{p_x, p_\phi, p_r}{E}, \quad \nu, \quad \bar{h} = \frac{h}{R}, \quad \bar{R} = \frac{R}{L}
\]

\[
\bar{x} = \frac{x}{L}, \quad \bar{k} = kL, \quad \bar{t} = t \omega_n, \quad \bar{w} = \frac{\omega}{\omega_n} \text{ where } \omega_n = \sqrt{\frac{E}{\rho (1 - \nu^2) R^2}}
\]

The general solution form for a given azimuthal mode, \( n \), is substituted into the
above governing equations:

\[- R^2 \bar{k}^2 U - i \nu R \bar{k} W - \frac{(1 - \nu)}{2} n^2 U - \frac{(1 + \nu)}{2} \bar{R} \bar{k} n V + \bar{\omega}^2 U + \ldots\]

\[\frac{\hbar}{12} \left( - \frac{(1 - \nu)}{2} n^2 U - i R^3 \bar{k}^3 W + \frac{(1 - \nu)}{2} i \bar{R} \bar{k} n W \right) = 0 \quad (3.26)\]

\[- n^2 V - i n W - \frac{(1 - \nu)}{2} R^2 \bar{k}^2 V - \frac{(1 + \nu)}{2} \bar{R} \bar{k} n U + \bar{\omega}^2 V - \ldots\]

\[\frac{\hbar}{12} \left( \frac{3(1 - \nu)}{2} R^2 \bar{k}^2 V + \frac{(3 - \nu)}{2} i R^2 \bar{k}^2 n W \right) = 0 \quad (3.27)\]

\[- W + i n V + i \nu R \bar{k} U - \frac{\hbar}{12} \left( \left( n^2 + R^2 \bar{k}^2 \right)^2 + 1 - 2n^2 \right) W + \bar{\omega}^2 W - \ldots\]

\[\frac{\hbar}{12} \left( \frac{(1 - \nu)}{2} i \bar{R} n^2 U - \frac{(3 - \nu)}{2} i \bar{R}^2 \bar{k}^2 n V - i \bar{R} \bar{k}^3 U \right) = 0 \quad (3.28)\]

This system of equations can be rewritten as linear mapping, \([ L ]\), acting on its kernel, or eigenvector.

\[
[L] \begin{bmatrix} U \\ V \\ W \end{bmatrix} = [0]
\]
with \([ L ]\) defined by the shell governing equations as

\[
L_{11} = -\overline{R}^2\overline{K}^2 - \frac{(1 - \nu)}{2} \left( 1 + \frac{\overline{h}^2}{12} \right) n^2 + \overline{\omega}^2
\]

\[
L_{12} = -\frac{(1 + \nu)}{2} \overline{R} \overline{k} n
\]

\[
L_{13} = -i\nu \overline{R} \overline{k} + \frac{\overline{h}^2}{12} \left( \frac{(1 - \nu)}{2} i\overline{R}k n^2 - i\overline{R}^3 \overline{k}^3 \right)
\]

\[
L_{21} = -\frac{(1 + \nu)}{2} \overline{R} \overline{k} n
\]

\[
L_{22} = -n^2 - \frac{(1 - \nu)}{2} \overline{R}^2 \overline{K}^2 \left( 1 + \frac{\overline{h}^2}{4} \right) + \overline{\omega}^2
\]

\[
L_{23} = -i n - \frac{\overline{h}^2}{24} \left( (3 - \nu) i\overline{R}^2 \overline{k}^2 \overline{n} \right)
\]

\[
L_{31} = i\nu \overline{R} \overline{k} - \frac{\overline{h}^2}{12} \left( \frac{(1 - \nu)}{2} i\overline{R}k n^2 - i\overline{R}^3 \overline{k}^3 \right)
\]

\[
L_{32} = i n + \frac{\overline{h}^2}{24} \left( (3 - \nu) i\overline{R}^2 \overline{k}^2 \overline{n} \right)
\]

\[
L_{33} = -1 - \frac{\overline{h}^2}{12} \left( n^2 + \overline{R}^2 \overline{K}^2 \right)^2 + 1 - 2n^2 \right) + \overline{\omega}^2
\]

For a nontrivial eigenvector \([U, V, W]\) solution to exist, \([ L ]\) cannot be invertible and consequently must have a determinant of zero. Setting this condition formulates the dispersion relation which, for a given frequency and azimuthal mode, determines how waves are propagated down the axis of the shell. However, formulating the determinant of a 3 × 3 matrix and analytically solving for \(\overline{k}\), while not intractable, is not easy. Through observation, it is evident that the equation for the axial wavenumber will appear as an eight order polynomial with even powers that can be simplified to a quartic function. Modern computing allows for quick and accurate solutions to quartic polynomials, and as a result this thesis will computationally solve the dis-
persion relation for each instance when necessary. The final set of general solutions for the coupled deflections, with eight axial eigenvalues, is presented as:

\[
\begin{align*}
\bar{u} &= \sum_{n=0}^{\infty} \left[ \sum_{m=1}^{8} \overline{C}_{mn} U_{mn} e^{i(\overline{\omega} \tau - \overline{\kappa}_{mn} \tau - n\phi)} \right] \\
\bar{v} &= \sum_{n=0}^{\infty} \left[ \sum_{m=1}^{8} \overline{C}_{mn} V_{mn} e^{i(\overline{\omega} \tau - \overline{\kappa}_{mn} \tau - n\phi)} \right] \\
\bar{w} &= \sum_{n=0}^{\infty} \left[ \sum_{m=1}^{8} \overline{C}_{mn} W_{mn} e^{i(\overline{\omega} \tau - \overline{\kappa}_{mn} \tau - n\phi)} \right]
\end{align*}
\]

(3.29)  
(3.30)  
(3.31)

The free constants \( \overline{C}_{mn} \) will be determined through the end boundary conditions and forcing applied to the shell on a modal basis. Prior to this analysis, it is necessary to have a firm understanding of each azimuthal mode and how it effects the dispersion of propagated waves down the shell. For this discussion, the linear mapping will be simplified using the Donnell approximation allowing for a more succinct analytical examination [Flugge (1960)]. These approximations assume the shell is very thin thus removing many of the thickness dependent (\( h \)) terms with the exception of the radial bending resulting from transverse shear stresses.

\[
\begin{bmatrix} 
L_D \end{bmatrix} = \ldots
\]

\[
\begin{bmatrix}
-R^2 \overline{k}^2 - \frac{(1-\nu)}{2} n^2 + \overline{\omega}^2 & -\frac{(1+\nu)}{2} R \overline{k} n & -i \nu R \overline{k} \\
-\frac{(1+\nu)}{2} R \overline{k} n & -n^2 - \frac{(1-\nu)}{2} R^2 \overline{k}^2 + \overline{\omega}^2 & -in \\
i \nu R \overline{k} & in & -1 - \frac{n^2}{12} \left( n^2 + R^2 \overline{k}^2 \right)^2 + \overline{\omega}^2
\end{bmatrix}
\]
3.1.3 Monopole Dispersion ($n = 0$)

The axisymmetric or monopole mode ($n=0$) represents a shell undergoing uniform deformation along the azimuthal cross section. For radial deflection, this appears as if the shell is ‘breathing’ or expanding/contracting. The approximate mapping becomes:

\[
\begin{bmatrix}
-R^2 k^2 + \bar{\omega}^2 & 0 & -i\nu R k \\
0 & -(1-\nu) R^2 k^2 + \bar{\omega}^2 & 0 \\
i\nu R k & 0 & -1 - \frac{R^2 \bar{\Omega}^4}{12} + \bar{\omega}^2
\end{bmatrix}
\]

The terms with angular dependence disappear and the tangential deflection of the shell becomes uncoupled from the longitudinal and radial deflection. More simply put, uniformly twisting the shell under this model will have no effect on the radial and
longitudinal displacements. Observing the simplified governing equations presents an interesting result:

**Tangential Forces**

\[
\frac{\partial^2 \tau}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 \tau}{\partial t^2} \quad \text{where} \quad a^2 = \frac{R^2 (1 - \nu)}{2}
\]  

and

**Longitudinal Forces**

\[
R^2 \frac{\partial^2 \bar{u}}{\partial x^2} + \nu R \frac{\partial \bar{w}}{\partial x} = \frac{\partial^2 \bar{u}}{\partial t^2}
\]  

**Radial Forces**

\[
- \bar{w} - \nu R \frac{\partial \bar{u}}{\partial x} - \frac{h^2}{12} R^4 \frac{\partial^4 \bar{w}}{\partial x^4} = \frac{\partial^2 \bar{w}}{\partial t^2}
\]

The tangential force balance equation appears in the form of the familiar wave equation. General solutions for the tangential deflection are propagated waves traveling in both axial directions.

\[
\bar{\nu} = C_1 e^{i (\omega t - \bar{k} x)} + C_2 e^{i (\omega t + \bar{k} x)} \quad \text{where} \quad \bar{k} = \frac{\bar{\nu}}{a}
\]

Meanwhile, the longitudinal and radial deflections remain coupled through the Poisson’s ratio. The dispersion relation for these coupled deflections becomes:

\[
\begin{bmatrix}
-\bar{R}^2 \bar{k}^2 + \bar{\omega}^2 & -i \nu \bar{R} \bar{k} \\
 i \nu \bar{R} \bar{k} & -1 - \frac{\bar{h}^2}{12} \bar{R}^4 \bar{k}^4 + \bar{\omega}^2
\end{bmatrix}
\begin{bmatrix}
U \\
W
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\Rightarrow \bar{k}^6 \left( \frac{\bar{h}^2}{12} \bar{R}^6 \right) + \bar{k}^4 \left( -\frac{\bar{h}^2}{12} \bar{\omega}^2 \bar{R}^4 \right) + \bar{k}^2 \left( \bar{R}^2 (1 - \nu^2 - \bar{\omega}^2) \right) + (\bar{\omega}^2 (\bar{\omega}^2 - 1)) = 0
\]  

A sixth order polynomial with even powers, the dispersion relation can be simplified to a cubic with positive/negative (left/right traveling) wavenumber symmetry. Real wavenumbers signify propagated waves while imaginary wavenumbers translate to evanescent waves that grow and decay axially. The eigenvector is also analyzed to examine which motion dominates over the frequency range. For a relatively thin and
long shell with an average Poisson ratio, the axial wavenumber (in pairs, magnitude of real and imaginary part), phase velocity and eigenvector are plotted versus frequency.

Figure 3.6: Monopole dispersion relation and eigenvectors

Observing the relative magnitudes of the wavenumbers, the radially dominant modes have a much lower wave speed than the longitudinal modes. As one could imagine, the thin wall of the shell allows for radial bending with relative ease, but remains stiff in compression/tension. The wavenumber plots also indicate a transition point near $\omega = 1$ that can be further examined through the force balance equations. If the radial and longitudinal deflection coupling is removed, $\nu = 0$, the radial equation
appears identical in form to the that of a beam on an elastic suspension:

Radial Shell Equation

\[ - \ddot{w} - \frac{h^2}{12R} \frac{\partial^4 w}{\partial x^4} = \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (3.37)

Beam on Elastic Suspension

\[ - \bar{K}_s \ddot{y} - \frac{\partial^4 y}{\partial x^4} = \frac{\partial^2 y}{\partial t^2} \]  \hspace{1cm} (3.38)

Examining the radial shell equation, the equivalent “elastic suspension” term originates from the azimuthal normal stress, \( N_\phi \), commonly referred to as the shell hoop stress. This structural component is necessary in maintaining curvature of the cylindrical shell.

![Figure 3.7: Shell and suspended beam analogy](image)

So like the suspended beam, the radial deflection has a cutoff frequency where no waves are propagated down the axis of the shell (\( \bar{k} = 0 \)). Below this frequency, the wavenumber is complex (evanescent) for all four solutions while above this frequency the solutions are strictly real (propagating) or imaginary. Without the longitudinal coupling, or Poisson ratio, this frequency is located at \( \bar{\omega} = 1 \). When the coupling is reintroduced, yielding the monopole governing shell equations, the cutoff frequency moves down to \( \bar{\omega}_{co} = \sqrt{1 - \nu^2} \) as indicated in the eigenvector plots of \( \bar{k}_3 \) and \( \bar{k}_4 \).
3.1.4 Dipole Dispersion \((n = 1)\)

As previously mentioned, the dipole distribution resembles the transverse beam deflection from the previous chapter. The angular derivatives are evaluated for \(n = 1\):
\[ [L_D] = \begin{bmatrix} -\overline{R^2 k^2} - \frac{1-\nu}{2} + \overline{\omega^2} & -\frac{1+\nu}{2} \overline{R k} & -i\nu \overline{R k} \\ -\frac{1+\nu}{2} \overline{R k} & -1 - \frac{1-\nu}{2} \overline{R^2 k^2} + \overline{\omega^2} & -i \\ i\nu \overline{R k} & i & -1 - \frac{\overline{\omega^2}}{12} \left(1 + \overline{R^2 k^2}\right)^2 + \overline{\omega^2} \end{bmatrix} \]

The determinant of the 3 x 3 matrix results in an even power eighth order polynomial equation for \( \overline{k} \) which can be simplified to a fourth order polynomial. Fortunately, an analytical closed form expression for finding the roots of a quartic polynomial exists and can be encoded with ease.

The monopole dispersion relation revealed modes which experienced cutoff behavior. Introducing the azimuthal terms in the dipole dispersion relation, this behavior reappears. If the wavenumber is set to zero, \( k = 0 \), the cutoff frequency can be determined in the linear mapping \([L_D]\).

\[ [L_{D,co}] = \begin{bmatrix} -\frac{(1-\nu)}{2} + \overline{\omega^2} & 0 & 0 \\ 0 & -1 + \overline{\omega^2} & -i \\ 0 & i & -1 - \frac{\overline{\omega^2}}{12} + \overline{\omega^2} \end{bmatrix} \]

\[ \det[L_{D,co}] = \left(-\frac{(1-\nu)}{2} + \overline{\omega^2}\right) \left(-1 - \frac{\overline{\omega^2}}{12} + \overline{\omega^2}\right) \left(-1 + \overline{\omega^2} - 1\right) = 0 \quad (3.39) \]

The positive roots of the above relation present the cutoff frequencies for the thin shell under beam-like motion. As evident in the Donnell linear mapping, there is now a longitudinal cutoff (first row of matrix) that did not appear in the monopole dispersion.

\[ \overline{\omega}_{co}^1 = \sqrt{\frac{1-\nu}{2}} \]

The radial and tangential modes, now coupled, experience a set of two cutoff frequencies that can be simplified by using a Taylor expansion about the thickness
\[
\omega_{co}^{2,3} = \sqrt{1 + \frac{h^2}{24} \pm \sqrt{1 + \frac{h^4}{96}}}
\]

where for a thin shell, \( \omega_{co}^{2} \approx \sqrt{2} \) and \( \omega_{co}^{3} \approx \frac{h\sqrt{6}}{12} \).

For thin shells, the cutoff frequency \( \omega_{co}^{3} \) is much lower than the other two frequencies. Plotting the wavenumber solutions to the dispersion relation and respective eigenvectors reaffirms the cutoff behavior at the two positive frequencies. From an initial glance the motion appears highly coupled in comparison to the monopole dispersion, especially near the cutoff frequencies. The coupling between the longitudinal, tangential and radial modes is not predicted in simple beam theory and as will be shown, is essential to shell theory.
Figure 3.10: Dipole dispersion relation and eigenvectors

The eigenvalue plot for $k_3$ shows a propagating mode with relatively long wavelengths and comparable radial and tangential out-of-phase eigenvectors that represent the beam-like motion of the shell at low frequencies. At lower frequencies, the longitudinal deflection is not significant. However, after a cutoff frequency, $\omega_c^1$ evident in the eigenvalue plot for $k_4$, this motion becomes influential. At this frequency, the deformation of the shell in that mode is almost entirely longitudinal, with the top and bottom half of the shell cross section pulling in opposite directions to resemble the dipole. In this mode, frequencies below the cutoff in this mode translate to purely imaginary wavenumbers or evanescent behavior. Above this frequency, the
wavenumber becomes purely real or begins to propagate.

\[ \text{Dipole Deformation at Cutoff Frequency } \omega_{\text{co}}^1 \]

Figure 3.11: Dipole eigenvector at \( \omega_{\text{co}}^1 \) \( (k = 0) \)

At the cutoff frequency \( \omega_{\text{co}}^2 \), the radial and tangential components in the eigenvalue plot for \( \bar{k}_2 \) appear approximately equal in magnitude but out-of-phase. Since the zero eigenvalue removes the axial dependence, this eigenvector behaves as an almost rigid body translation in the transverse direction. For frequencies below this cutoff, the modal wavenumber is complex. Above this frequency, the wavenumber is purely real or propagating.
3.1.5 Higher Order Dispersion \((n > 1)\)

The other modes of vibration can be grouped as higher order modes, or those where \(n > 1\). While not as familiar as the monopole and dipole, these modes will affect the force transfer when they are excited, such as in the case for distributed pressure forcing. The cutoff linear mapping is generalized as:

\[
\begin{bmatrix}
L_{\text{co}}
\end{bmatrix} = \begin{bmatrix}
-\frac{(1-\nu) n^2 + \omega^2}{2} & 0 & 0 \\
0 & -n^2 + \omega^2 & -in \\
0 & in & -1 - \frac{n^2}{12} n^4 + \omega^2
\end{bmatrix}
\]

The three positive cutoff frequencies are plotted for a range of higher order modes:

- \(\omega_{\text{co}}^1\) longitudinal cutoff frequency
- \(\omega_{\text{co}}^2\) and \(\omega_{\text{co}}^3\) radial/tangential cutoff frequencies
3.1.6 Shell Boundary Conditions

Since there are eight constants in the general solution, there will need to be eight boundary conditions applied to the shell, or four on each axial end. The forcing on the cylindrical shell is established on a modal basis in combinations of stress resultants, moments and deflections. Motions are highly coupled, as evident in the governing equations, so a forcing in one direction could produce motion in the other orthogonal directions. The first shell configuration will resemble that of the cantilever thin beam as previously described.

Similar to the cantilever beam, the common boundary condition will be fixed on one end and forced on the other. For the fixed condition, the longitudinal, tangential and radial deflections along with the radial deflection slope will be set to zero. For the forced condition, the stress resultants on the circular cross section will need to be prescribed. Notice that there are five forces and moments ( \( N_x, N_{x\phi}, M_x, M_{x\phi} \) and \( Q_x \) ) but only four allowable conditions. The shear forces, \( N_{x\phi} \) and \( Q_x \), will need to be combined with the moment \( M_{x\phi} \) to form two effective shear forces, \( T_x \) and \( S_x \), that allow for the appropriate boundary conditions.
Figure 3.14: Effective shear force visual derivation [Flugge (1960)]

\( T_x = N_{x\phi} - \frac{1}{R} M_{x\phi} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial v}{\partial x} + \frac{1}{R} \frac{\partial u}{\partial \phi} \right) - \frac{Eh^3}{12(1+\nu)} \left( \frac{1}{R^2} \frac{\partial^2 w}{\partial \phi^2} \right) \)

\( S_x = Q_x + \frac{1}{R} \frac{\partial M_{x\phi}}{\partial \phi} = \frac{Eh^3}{12(1-\nu^2)} \left( \frac{\partial^3 w}{\partial x^3} + \frac{1}{R^2} \frac{\partial^3 w}{\partial x \partial \phi^2} \right) + \ldots \)

\[ \frac{Eh^3}{12(1+\nu)} \left( \frac{1}{R^2} \frac{\partial^3 w}{\partial \phi^2 \partial x} \right) \]  (3.40)

\[ \frac{h^2}{12(1-\nu^2)} \left( \frac{R^3}{\partial x} \frac{\partial^3 w}{\partial \phi^2} + R (2-\nu) \frac{\partial^3 w}{\partial x \partial \phi^2} \right) \]  (3.41)

where \( T_x = \frac{T_x}{EhL} \) and \( S_x = \frac{S_x}{EhL} \)

Both shear forces will be prescribed while the remaining resultants are set to zero on the forced end.

Fixed BCs: \( u|_{x=0} = 0 \), \( v|_{x=0} = 0 \), \( w|_{x=0} = 0 \) \( \frac{\partial w}{\partial x}|_{x=0} = 0 \)  (3.42)

Forced BCs: \( T_x|_{x=1} = \frac{F_1}{h} e^{-in\phi} \), \( S_x|_{x=1} = \frac{F_2}{h} e^{-in\phi} \), \( M_x|_{x=1} = 0 \) \( N_x|_{x=1} = 0 \)

(3.43)

where \( F_1, F_2 = \frac{F_1, F_2}{EL} \)

Similarly, the forced boundary condition can be made free by prescribing the homo-
geneous conditions.

Free BCs: \( T_x|_{\tau=1} = 0 \), \( S_x|_{\tau=1} = 0 \), \( M_x|_{\tau=1} = 0 \) and \( N_x|_{\tau=1} = 0 \) (3.44)

3.1.7 Modal Forcing

There are a variety of ways to excite the structure and observe its reaction. Similar to the beam excitation, the shell can be discretely forced using the previously defined shear resultants which appear in the following form, separated in the axial and azimuthal variables:

\[
\begin{align*}
\overline{T}_x &= \overline{T}(x) \ e^{-in\phi} \\
\overline{S}_x &= \overline{S}(x) \ e^{-in\phi}
\end{align*}
\]

The computed response will assess the force transfer function along the axis of the shell. For the dipole motion, modal distribution produces a net shear force transverse to the cross section, similar to the beam forcing.

![Azimuthal Forcing Distribution for n = 1](image)

**Figure 3.15**: Dipole shell forcing with net transverse resultant
Net Transverse Shear Force

\[
F_x = \int_0^{2\pi} (-\cos(\phi) T_x + \sin(\phi) S_x) \ Rd\phi
\]

\[
= EhL \int_0^{2\pi} (-\cos(\phi) \bar{T}(\bar{x}) + \sin(\phi) \bar{S}(\bar{x})) \ e^{-i\phi} \ d\phi
\]

\[
= \pi EhL \ (-\bar{T}(\bar{x}) - i\bar{S}(\bar{x})) \quad (3.45)
\]

\[
F_y = \int_0^{2\pi} (\sin(\phi) T_x + \cos(\phi) S_x) \ Rd\phi
\]

\[
= EhL \int_0^{2\pi} (\sin(\phi) \bar{T}(\bar{x}) + \cos(\phi) \bar{S}(\bar{x})) \ e^{-i\phi} \ d\phi
\]

\[
= \pi EhL \ (-i\bar{T}(\bar{x}) + \bar{S}(\bar{x})) \quad (3.46)
\]

If the shell is forced at \( \bar{x} = 1 \) by shear resultants \( \bar{T}_x|_{\bar{x}=1} = \frac{F_x}{h} \ e^{-in\phi} \) and \( \bar{S}_x|_{\bar{x}=1} = \frac{F_x}{h} \ e^{-in\phi} \) and fixed at \( \bar{x} = 0 \), then the transfer function can be defined as follows:

\[
TF_x = 20 \log_{10} \left( \frac{\text{Output Net } F_x}{\text{Input Net } F_x} \right) = 20 \log_{10} \left( \left| \frac{-\bar{T}(0) - i\bar{S}(0)}{-\bar{T}(1) - i\bar{S}(1)} \right| \right)
\]

\[
= 20 \log_{10} \left( \left| \frac{\bar{h} (-\bar{T}(0) - i\bar{S}(0))}{-F_1 - iF_2} \right| \right) \quad (3.47)
\]

\[
TF_y = 20 \log_{10} \left( \frac{\text{Output Net } F_y}{\text{Input Net } F_y} \right) = 20 \log_{10} \left( \left| \frac{-i\bar{T}(0) + \bar{S}(0)}{-i\bar{T}(1) + \bar{S}(1)} \right| \right)
\]

\[
= 20 \log_{10} \left( \left| \frac{\bar{h} (-i\bar{T}(0) + \bar{S}(0))}{-iF_1 + F_2} \right| \right) \quad (3.48)
\]

Motion in the monopole and higher order modes creates no net shear forcing along the azimuth of the shell. The metric used for analyzing the transfer function in these cases will simply be the radial and azimuthal shear force amplitudes.
Figure 3.16: Forcing distributions with no net resultant

\[ TF_R = 20 \log_{10} \left( \frac{\text{Output } T_x \text{ Amplitude}}{\text{Input } T_x \text{ Amplitude}} \right) = 20 \log_{10} \left( \frac{T(0)}{T(1)} \right) \]

\[ = 20 \log_{10} \left( \frac{\bar{h} T(0)}{F_1} \right) \quad (3.49) \]

\[ TF_\phi = 20 \log_{10} \left( \frac{\text{Output } S_x \text{ Amplitude}}{\text{Input } S_x \text{ Amplitude}} \right) = 20 \log_{10} \left( \frac{\bar{S}(0)}{\bar{S}(1)} \right) \]

\[ = 20 \log_{10} \left( \frac{\bar{h} \bar{S}(0)}{F_2} \right) \quad (3.50) \]
The forced model is introduced with the cantilever shell. In order to emulate the beam motion, the shell is displaced in the dipole mode with effective shear forces equal in magnitude but out-of-phase. ($F_1 = i F_2$). The transverse force transfer function is plotted below for several Poisson ratios with added material damping.

$$TF_y = 20 \log_{10} \left( \frac{|T \left(-iT(0) + \bar{S}(0)\right)|}{2F_2} \right)$$
Figure 3.18: Cantilever single shell dipole transfer function for different Poisson ratios

Notice that at low frequencies the shape of the cantilever shell transfer function appears similar to that of the cantilever beam with smooth resonance peaks leveling off at 0 dB. The net shear force at the reaction (fixed) end of the shell is greater than or equal to the input force, hence the positive transfer function. The dispersion relation and eigenvector plot from Figure 3.10 reveals dominating radial and tangential deflections at low frequencies. However after the cutoff frequency $\omega_{\text{co}}^1$, the longitudi-
nal deflection and other higher frequency resonances become important. The transfer function beyond this point is clustered with resonances and interference, scattered with small attenuation bands at the higher frequencies due to the highly coupled nature of the shell equations.

Unlike with the beam analysis, the shell governing equations result in interference gaps for the single element cantilever configuration after a cutoff frequency which depends on the elastic properties of the shell material and its radius (notedly independent of its length). The beam equations used earlier in the thesis are rather simplified and do not account for longitudinal and tangential deflection of the beam cross section, thus cannot predict the interference. To visualize this discrepancy, the beam transfer function is plotted alongside the shell transfer function for the cantilever configuration with the same material and geometric properties. As an additional metric, the force response for the shell is computed using COMSOL, a finite element software.

Figure 3.19: COMSOL shell mesh
Figure 3.20: Cantilever single shell transfer function comparison between COMSOL, shell theory and beam theory

While the COMSOL and shell theory show close correlation, including the location of the cutoff frequency, the beam theory significantly deviates from the results after only a few resonances. This comparison further solidifies the necessity for shell theory in describing the dynamics of shell like structures. Table 3.1 includes the dimensional longitudinal (dipole) cutoff frequencies for common materials.
Table 3.1: Dipole cutoff frequency

<table>
<thead>
<tr>
<th>Material</th>
<th>Dipole Cutoff Frequency in Unit Radius $\omega_{co}^1 \cdot R$ (Hz m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass Fibre</td>
<td>≈ 560 Hz m</td>
</tr>
<tr>
<td>6061 Aluminum and 1020 Steel</td>
<td>500 Hz m</td>
</tr>
<tr>
<td>Polyvinyl Chloride (PVC)</td>
<td>150 Hz m</td>
</tr>
<tr>
<td>Lead</td>
<td>110 Hz m</td>
</tr>
</tbody>
</table>

3.1.9 Cantilever Single Shell: Monopole and Higher Order Forcing

If the shell is forced axisymmetrically or in a higher order azimuthal mode ($n > 1$) then there is no net force applied to the shell as mentioned before. Instead, the amplitude of the radial force transfer that arises from forcing the shell at $\bar{x} = 1$ with a radial shear resultant ($F_1 = 1$, $F_2 = 0$) can be analyzed.

For the monopole deformation, a radial cutoff frequency is evident in the dispersion relation at $\omega_{co} = \sqrt{1 - \nu^2}$. This frequency indicates the transition between evanescent and propagating radial waves along the axial direction. The force transfer function is plotted for several Poisson ratios under the cantilever boundary conditions.
As expected, a transition point appears at the cutoff frequency in which the shell initiates axial wave propagation. For frequencies lower than this, the radial waves quickly decay.

The cutoff frequencies appear in the higher order modes similar to that of the dipole. However, the low frequency radial/tangential cutoff $\omega_{co}^3$, not prominent in the dipole forcing, increases in frequency value for the higher azimuthal modes and
becomes clearly visible. Plotting the transfer function for the higher order modes reveals similar but more profound cutoff behavior.

![Force Transfer Function for Single Cantilever Shell](image)

**Figure 3.22**: Cantilever single shell higher order transfer function

Past the cutoff frequency, the behavior resembles the simple non-interfering wave propagation with distributed resonance peaks. Both the monopole and dipole cantilever responses indicate that for the set of cantilever boundary conditions with end forcing, the transfer functions yield highly evanescent modes below a specified cutoff
frequency. Or in an applicable sense, the force transmission can be minimized by designing the single shell such that the forcing frequency is below the cutoff point.

3.1.10 Radial Pressure Distribution

The inhomogeneous forcing discussed in this thesis has been a discrete loading applied at the ends of the shell. While this is a useful metric in analyzing the force transfer across the full length of the shell, there is interest in modeling force distributions along the cylindrical face. In structural applications, disturbances in the interior or exterior of the shell can propagate towards the boundary ends. This is modeled by applying an axial and azimuthal distribution of radial pressure, $p_r$, to the face of the shell.

![Figure 3.23: Single shell radial pressure forcing](image)

The azimuthal distribution is in the form of the previously derived harmonic eigenfunctions while the axial distribution will be a quadratic polynomial. Nondimensionalized by the Young’s modulus, this forcing appears as:

$$ p_r(x, \phi, t) = e^{i\omega t} \sum_{n=0}^{\infty} P_n(x) e^{-in\phi} $$

(3.51)

The general solution will be a linear combination of the previously derived homogeneous solution and the inhomogeneous solution to the pressure forcing $(\mathcal{U}_{pn}, \mathcal{V}_{pn})$.

$$ \mathcal{P}_n(x) = P_{0n} + P_{1n}x + P_{2n}x^2 $$

(3.52)
\[ u = e^{i\omega t} \sum_{n=0}^{\infty} \left[ e^{-\text{i}k_{mn} \bar{x}} \left( U_{pn}(\bar{x}) + \sum_{m=1}^{8} C_{mn} U_{mn} e^{-\text{i}k_{mn} \bar{x}} \right) \right] \] (3.53)

\[ v = e^{i\omega t} \sum_{n=0}^{\infty} \left[ e^{-\text{i}k_{mn} \bar{x}} \left( V_{pn}(\bar{x}) + \sum_{m=1}^{8} C_{mn} V_{mn} e^{-\text{i}k_{mn} \bar{x}} \right) \right] \] (3.54)

\[ w = e^{i\omega t} \sum_{n=0}^{\infty} \left[ e^{-\text{i}k_{mn} \bar{x}} \left( W_{pn}(\bar{x}) + \sum_{m=1}^{8} C_{mn} W_{mn} e^{-\text{i}k_{mn} \bar{x}} \right) \right] \] (3.55)

The inhomogeneous deflection is determined by first assuming a form of the general solution to match that of the pressure distribution.

\[ U_{pn}(\bar{x}) = U_{0n} + U_{1n} \bar{x} + U_{2n} \bar{x}^2 \] (3.56)

\[ V_{pn}(\bar{x}) = V_{0n} + V_{1n} \bar{x} + V_{2n} \bar{x}^2 \] (3.57)

\[ W_{pn}(\bar{x}) = W_{0n} + W_{1n} \bar{x} + W_{2n} \bar{x}^2 \] (3.58)

The solution is substituted into the shell governing equations. Gathering the coefficients yields nine total equations for the nine unknown constants, a linear mapping matrix \([ L_P ]\). This matrix is inverted and multiplied by the pressure matrix, \([ P ]\), to yield the inhomogeneous solution coefficients.

\[
\begin{bmatrix}
U_{0n} \\
U_{1n} \\
U_{2n} \\
V_{0n} \\
V_{1n} \\
V_{2n} \\
W_{0n} \\
W_{1n} \\
W_{2n}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\overline{P}_{0n} \\
\overline{P}_{1n} \\
\overline{P}_{2n}
\end{bmatrix}
\]

The free constants in the homogeneous solution, \( C_{mn} \), are determined on a modal basis by applying the end boundary conditions. Note that in the boundary conditions,
the displacement derivatives evaluated at each end of the shell will include both the homogeneous and inhomogeneous solutions. The prescribed boundary conditions for the single shell forced through a radial pressure distribution are:

**Fixed BCs:**
\[ u|_{\vec{x}=0} = 0 , \quad \bar{v}|_{\vec{x}=0} = 0 , \quad \bar{w}|_{\vec{x}=0} = 0 \quad \text{and} \quad \frac{\partial \bar{w}}{\partial \vec{x}}|_{\vec{x}=0} = 0 \]
\[ u|_{\vec{x}=0} = 0 , \quad \bar{v}|_{\vec{x}=1} = 0 , \quad \bar{w}|_{\vec{x}=1} = 0 \quad \text{and} \quad \frac{\partial \bar{w}}{\partial \vec{x}}|_{\vec{x}=1} = 0 \]

The transfer function for the distributed forcing with fixed-fixed boundary conditions examines the shear forcing at the two end boundary locations in comparison to the integrated pressure forcing. For the dipole mode, the net forcing is in the transverse direction and the pressure integration is as follows:

\[
F_x = \int_0^L \int_0^{2\pi} \cos(\phi) \ p_r \ R \ d\phi \ dx
\]
\[
= ERL \int_0^1 \bar{P}_1(\vec{x}) \left( \int_0^{2\pi} \cos(\phi) \ e^{-i\phi} \ d\phi \right) \ d\vec{x}
\]
\[
= \pi ERL \int_0^1 \bar{P}_1(\vec{x}) \ d\vec{x}
\]

The transfer function becomes:

\[
TF_x = 20 \log_{10} \left( \frac{\text{Output Net } F_x}{\text{Input Net } F_x} \right)
\]
\[
= 20 \log_{10} \left( \frac{\pi EhL | - \bar{T}(0) - i \bar{S}(0)| + | - \bar{T}(1) - i \bar{S}(1)|}{\pi ERL \int_0^1 \bar{P}_1(\vec{x}) \ d\vec{x}} \right)
\]
\[
= 20 \log_{10} \left( \frac{\bar{h} | - \bar{T}(0) - i \bar{S}(0)| + | - \bar{T}(1) - i \bar{S}(1)|}{\int_0^1 \bar{P}_1(\vec{x}) \ d\vec{x}} \right)
\]

It is important to only add the magnitudes of the shear reaction forces for the numerator of the transfer function.

As an example, the shell is analyzed under the dipole mode pressure forcing. The axial pressure distribution can be defined as a parabola such that the maximum
forcing is in the middle of the shell ($\bar{x} = .5$):

$$\bar{p}_r(x, \phi, t) = \left(-\bar{x} + x^2\right) e^{i(\pi x - \phi)}$$

Figure 3.24: Single shell radial pressure forcing with parabolic axial distribution

Rather than observing the transfer of force from one shell end to the other, the response now encapsulates forcing at all lengths of the shell. The transfer function is plotted below:
As evident in the plot, the cutoff frequency behavior is present at $\omega_{co}^1$. Unlike with the end forcing results though, the transfer function does not level off at 0 dB and instead is affected by interference gaps caused by the distributed forcing.

### 3.2 Double Thin Cylindrical Shell

The behavior of the single shell was evaluated through various forcing methods at different azimuthal modes. Due to the highly coupled longitudinal, tangential and radial deflections, the transfer function response can be intricate and contain cutoff frequencies. The MEMP two shell configuration will now be formulated and analyzed in comparison to the single shell. The mixed cantilever boundary condition that was found to be most effective in the beam results will be the center of discussion.
3.2.1 Axially Discrete Elastic Coupling Formulation

The MEMP configuration consists of two concentric shells, an inner and outer shell, of equal length. The two shells are coupled together by axially discrete, azimuthally continuous springs with stiffness $K_s$ that resist motion in the radial direction. The radial governing equations for the coupled shell system with springs located at $x = 0$
and a pressure distribution on only one of the shells are:

\[
\frac{E_1 h_1}{(1 - \nu_1^2)} [-w_1 - \frac{\partial v_1}{\partial \phi} - R_1 \nu_1 \frac{\partial u_1}{\partial x} - \ldots
\]

\[
\frac{h_1^2}{12 R_1^2} \left( \frac{\partial^4 w_1}{\partial \phi^4} + 2 R_1^2 \frac{\partial^4 w_1}{\partial x^2 \partial \phi^2} + R_1^4 \frac{\partial^4 w_1}{\partial x^4} + w_1 + 2 \frac{\partial^2 w_1}{\partial \phi^2} \right) - \ldots
\]

\[
\frac{h_1^2}{12 R_1^2} \left( \frac{R_1 (1 - \nu_1)}{2} \frac{\partial^3 u_1}{\partial \phi^2 \partial x} - \frac{R_1^2 (3 - \nu_1)}{2} \frac{\partial^3 v_1}{\partial x^2 \partial \phi} - R_1^4 \frac{\partial^3 u_1}{\partial x^3} \right) ] + \ldots
\]

\[
R_1^2 p_{r1} - R_1^2 K_s (w_1 - w_2) \delta(x) = R_1^2 \rho_1 h_1 \frac{\partial^2 w_1}{\partial t^2} \quad (3.59)
\]

\[
\frac{E_2 h_2}{(1 - \nu_2^2)} [-w_2 - \frac{\partial v_2}{\partial \phi} - R_2 \nu_2 \frac{\partial u_2}{\partial x} - \ldots
\]

\[
\frac{h_2^2}{12 R_2^2} \left( \frac{\partial^4 w_2}{\partial \phi^4} + 2 R_2^2 \frac{\partial^4 w_2}{\partial x^2 \partial \phi^2} + R_2^4 \frac{\partial^4 w_2}{\partial x^4} + w_2 + 2 \frac{\partial^2 w_2}{\partial \phi^2} \right) - \ldots
\]

\[
\frac{h_2^2}{12 R_2^2} \left( \frac{R_2 (1 - \nu_2)}{2} \frac{\partial^3 u_2}{\partial \phi^2 \partial x} - \frac{R_2^2 (3 - \nu_1)}{2} \frac{\partial^3 v_1}{\partial x^2 \partial \phi} - R_2^4 \frac{\partial^3 u_2}{\partial x^3} \right) ] - \ldots
\]

\[
R_2^2 K_s (w_2 - w_1) \delta(x) = R_2^2 \rho_2 h_2 \frac{\partial^2 w_2}{\partial t^2} \quad (3.60)
\]

The two shell nondimensionalizations are made in accordance with the algorithm
used in the MEMP beams:

\[
\nu_{1,2}, \nu_{1,2}, w_{1,2} = \frac{u_{1,2}, v_{1,2}, w_{1,2}}{L} , \quad p_x, p_\phi, p_r = \frac{p_x, p_\phi, p_r}{E_1} , \quad \nu_1, \nu_2
\]

\[
\overline{R}_1 = \frac{R_1}{L} , \quad \overline{R}_2 = \frac{R_2}{L} , \quad \overline{h}_1 = \frac{h_1}{R_1} , \quad \overline{h}_2 = \frac{h_2}{R_2} , \quad \overline{\omega} = \frac{\omega}{\omega_n}
\]

\[
\overline{K}_s = \frac{K_s (1 - \nu_1^2)}{E_1} , \quad \overline{\beta}^2 = \frac{E_1 (1 - \nu_2^2)}{E_2 (1 - \nu_1^2)} , \quad \gamma^2 = \frac{\rho_2 R_2^2}{\rho_1 R_1^2} \quad \text{where} \quad \omega_n = \sqrt{\frac{E_1}{\rho_1 (1 - \nu_1^2) R_1^2}}
\]

(3.61)
and the governing equations become:

\[-w_1 - \frac{\partial \nu_1}{\partial \phi} - \nu_1 \bar{R}_1 \frac{\partial u_1}{\partial x} - \frac{\bar{h}_1^2}{12} \left( \frac{\partial^4 w_1}{\partial \phi^4} + 2\bar{R}_1^2 \frac{\partial^4 w_1}{\partial x^2 \partial \phi^2} + \bar{R}_1 \frac{\partial^4 w_1}{\partial x^4} + w_1 + 2 \frac{\partial^2 w_1}{\partial \phi^2} \right) - \ldots \]

\[= \frac{\bar{h}_1^2}{12} \left( \frac{\bar{R}_1 (1 - \nu_1)}{2} \frac{\partial^3 u_1}{\partial \phi^3 \partial x} - \frac{\bar{R}_1^2 (3 - \nu_1)}{2} \frac{\partial^3 v_1}{\partial x^3 \partial \phi} - \bar{R}_1 \frac{\partial^3 u_1}{\partial x^3} \right) + \ldots \]

\[-w_2 - \frac{\partial v_2}{\partial \phi} - \nu_2 \bar{R}_2 \frac{\partial u_2}{\partial x} - \frac{\bar{h}_2^2}{12} \left( \frac{\partial^4 w_2}{\partial \phi^4} + 2\bar{R}_2^2 \frac{\partial^4 w_2}{\partial x^2 \partial \phi^2} + \bar{R}_2 \frac{\partial^4 w_2}{\partial x^4} + w_2 + 2 \frac{\partial^2 w_2}{\partial \phi^2} \right) - \ldots \]

\[= \frac{\bar{h}_2^2}{12} \left( \frac{\bar{R}_2 (1 - \nu_2)}{2} \frac{\partial^3 u_2}{\partial \phi^3 \partial x} - \frac{\bar{R}_2^2 (3 - \nu_2)}{2} \frac{\partial^3 v_2}{\partial x^3 \partial \phi} - \bar{R}_2 \frac{\partial^3 u_2}{\partial x^3} \right) + \ldots \]

\[= \frac{\beta^2 K}{\bar{h}_2} (\bar{w}_2 - \bar{w}_1) \frac{\delta(x)}{\bar{h}_1} = \frac{\beta^2}{\bar{h}_2^2} \frac{\partial^2 \bar{w}_2}{\partial \phi^2} \]

(3.62)

Solutions for each individual shell appear in the same form as the single shell with their respective eigenvalues/eigenvectors:

\[\bar{u}_{1,2} = \sum_{n=0}^{\infty} \sum_{m=1}^{8} C_{(1,2)mn} U_{(1,2)mn} e^{i(\bar{\sigma}t - \bar{\tau}(1,2)mn\bar{x} - n\phi)} \]

(3.64)

\[\bar{v}_{1,2} = \sum_{n=0}^{\infty} \sum_{m=1}^{8} C_{(1,2)mn} V_{(1,2)mn} e^{i(\bar{\sigma}t - \bar{\tau}(1,2)mn\bar{x} - n\phi)} \]

(3.65)

\[\bar{w}_{1,2} = \sum_{n=0}^{\infty} \sum_{m=1}^{8} C_{(1,2)mn} W_{(1,2)mn} e^{i(\bar{\sigma}t - \bar{\tau}(1,2)mn\bar{x} - n\phi)} \]

(3.66)

The same process that was used for the MEMP beams is now used for the shells in determining what the boundary conditions are at the spring junction, presented for shell one. In order to maintain mass conservation of the material, the deflections are
made continuous across the jump.

\[ \begin{align*}
\frac{\partial w_1}{\partial x}^{0+}_{0-} &= 0 \\
\frac{\partial^2 w_1}{\partial x^2}^{0+}_{0-} &= 0 \\
\frac{\partial^3 w_1}{\partial x^3}^{0+}_{0-} &= 0
\end{align*} \] (3.67)

The shell equations (longitudinal, tangential and radial) are integrated at a perturbation distance \( \epsilon \) around the spring location and the limit as \( \epsilon \) approaches zero is evaluated. The longitudinal governing equation can be integrated twice to yield:

\[ \left[ \frac{R_1^2 \partial^2 u_1}{\partial x^2} + \frac{\hat{h}_1}{12} \left( -\frac{R_1^3 \partial^2 w_1}{\partial x^2} \right) \right]^{0+}_{0-} = 0 \] (3.70)

\[ \left[ \frac{\hat{h}_1}{12} \left( -\frac{R_1^3 \partial^2 w_1}{\partial x^2} \right) \right]^{0+}_{0-} = 0 \] (3.71)

It can be concluded that the first axial derivative of the radial deflection must be continuous. Using this condition and integrating the tangential governing equation:

\[ \left[ \left[ \frac{R_1^2 (1 - \nu_1)}{2} + \frac{\hat{h}_1}{12} \left( \frac{3R_1^2 (1 - \nu_1)}{2} \right) \right] \frac{\partial^2 \tau_1}{\partial x} \right]^{0+}_{0-} = 0 \] (3.72)

Thus, the axial derivative of the tangential deflection is also continuous. These conclusions are finally inserted into the integration of the radial governing equation. The sifting property is again used in integrating the delta dirac function.

\[ \left[ -\frac{\hat{h}_1}{12} \left( R_1^4 \frac{\partial^3 w_1}{\partial x^3} \right) + \frac{\hat{h}_1}{12} \left( R_1^4 \frac{\partial^2 w_1}{\partial x^2} \right) \right]^{0+}_{0-} - \frac{K_s R_1 (\bar{w}_1(0) - \bar{w}_2(0))}{\hat{h}_1} = 0 \] (3.73)

The radial governing equation is integrated again and compared to previous conditions to yield that the first derivative of the longitudinal deflection and the second axial derivative of the radial deflection must be continuous. Collecting these conti-
nuity conditions:

\[
\frac{\partial \bar{u}_1}{\partial x} \bigg|_{0^+} = 0 \quad (3.74)
\]

\[
\frac{\partial \bar{v}_1}{\partial x} \bigg|_{0^+} = 0 \quad (3.75)
\]

\[
\frac{\partial \bar{w}_1}{\partial x} \bigg|_{0^+} = 0 \quad (3.76)
\]

\[
\frac{\partial^2 \bar{w}_1}{\partial x^2} \bigg|_{0^+} = 0 \quad (3.77)
\]

\[
- \frac{h_1^2}{12} \left( \bar{R}_1 \frac{\partial^3 \bar{w}_1}{\partial x^3} \right) + \frac{h_1^2}{12} \left( \bar{R}_1 \frac{\partial^3 \bar{u}_1}{\partial x^3} \right) \bigg|_{0^+} = \frac{K_s}{h_1} \bar{R}_1 (\bar{w}_1(0) - \bar{w}_2(0)) = 0 \quad (3.78)
\]

Similar conditions can be derived for the second shell. With the connectivity boundary conditions formulated and the general shell solutions known, the free constants can be determined from the forcing conditions on each shell.

### 3.2.2 Computational Modeling

The algorithm used for the MEMP beams is applied to the MEMP shells in solving for the constants. The amendments to the code are highlighted below:

- single modal structure (transverse)
- azimuthal mode iteration
- 4 eigenvalue solutions for each beam
- 8 eigenvalue solutions for each beam
- 4 boundary/connectivity conditions
- 8 boundary/connectivity conditions
- invert \(8 \times (1 + s)\) size matrix for \(s\) springs
- invert \(16 \times (1 + s)\) size matrix for \(s\) springs
A GUI was developed in MATLAB to facilitate the modeling and organize results.

Figure 3.27: MEMP shell graphic user interface in MATLAB

The MEMP shell can now be analyzed under the various forcing configurations and boundary conditions.
3.2.3 Mixed Cantilever Double Shell: Discrete End Forcing

The shells can be configured in the mixed cantilever boundary setup: the outer shell is free at \( x = 0 \) and forced at \( x = 1 \) by shear resultants \( T_x|_{x=1} = \frac{T_1}{h_1} e^{-i\phi} \) and \( S_x|_{x=1} = \frac{F_2}{h_1} e^{-i\phi} \) while the inner shell is fixed at \( x = 0 \) and free at \( x = 1 \). For the dipole mode, the force transfer function is defined as the ratio of the net transverse shear force on the inner shell at the fixed location to the net input force on the outer shell.

\[
TF_x = 20 \log_{10} \left( \frac{\text{Output Net } F_x}{\text{Input Net } F_x} \right)
\]

\[
= 20 \log_{10} \left( \frac{\frac{h_2 R_2}{\beta} (1 - \nu_2^2)}{\frac{h_1 R_1}{\beta^2} (1 - \nu_1^2)} \cdot \left| \frac{-T_2(0) - iS_2(0)}{T_1(1) - iS_1(1)} \right| \right)
\]

\[
= 20 \log_{10} \left( \frac{\frac{h_2 R_2}{\beta} (1 - \nu_2^2)}{\frac{h_1 R_1}{\beta^2} (1 - \nu_1^2)} \cdot \left| \frac{-T_2(0) - iS_2(0)}{-T_1 - iS_1} \right| \right)
\]

In comparing the theoretical MEMP beam results to the single beam results, the reference single beam was designed by creating a continuous distribution of rigid
springs using the MEMP algorithm. This method was chosen since the exact geometry of the cross section was not prescribed in the nondimensional analysis. However, for the current case the geometry is chosen to be the cylindrical shell. Therefore, setting aside certain constraints allows for an appropriate reference single shell to the MEMP shell configurations. If the density, Poisson ratio and length are constant and the radius of the single shell is taken as the average of the two MEMP shells, then the thickness can be solved for to conserve mass between the MEMP and single shell configurations. In the cases presented, the radius is chosen to be constant so that the single shell thickness is double that of the MEMP shells. In cases where the stiffness ratio $\beta$ is not one, then the single shell elasticity is identical to the less stiff of the two MEMP shells.

As with the MEMP mixed cantilever double beam, the force attenuation is directly correlated to the strength of the coupling springs: a spring stiffness of $K_s = 0$ uncouples the shells and yields complete attenuation. Additionally, the beam results concluded that for mismatched beam properties, or individual beam wavespeeds, there were more opportunities for destructive interference. The results for the two shell configuration with soft springs ($K_s = .01$) along with mismatched and identical shell properties ($\beta = .5, 1$) are plotted below. For all MEMP shell results, as with the beam results, there will be ten linearly spaced springs along the shell length. The dipole cutoff frequency is labeled for both the outer (1) shell and inner (2) shell.
As predicted, similar MEMP behaviors that appeared in the beams are evident in the shells, especially peak splitting for out-of-phase modes. Overall, the case with mismatched shell properties ($\beta = .5$) experiences higher attenuations than the case with identical shell properties ($\beta = 1$). Also, the low in-phase resonances are present and signify the lower frequency limit for attenuation just as in the beam analysis. The shell behavior becomes interesting with the inclusion of the cutoff frequencies ($\omega_{1,co}^1$ and $\omega_{2,co}^1$) as indicated by the dashed red and black lines on the
plots. Taking the identical shell case as an example, the cutoff frequencies for each shell coincide. The eigenvector analysis revealed that at this frequency, there is a mode in which the motion of the shell is predominately longitudinal while the radial and tangential deflections are not excited. The force transfer function, which measures the transfer of the radial and tangential shear forces along the shell, thus experiences an attenuation dip at each of the shell’s cutoff frequencies as noted in each of the plots. In comparison to the single shell case, drastic improvements are seen when the MEMP shells have mismatched properties even after the cutoff frequency.

If the shell is forced in a higher order azimuthal mode, the influence of the cutoff frequency becomes more apparent. The single shell results for the cantilever configuration deforming in the $n = 4$ mode reveal distinct radial cutoff behavior at $\omega_3^c$. The MEMP results for the mixed cantilever double shell are plotted below for various stiffness ratios.
The azimuthal mode plotted in the figure is for $n = 4$. As previously discussed, the higher order modes experience cutoff frequencies which are strongly dependent on the nondimensional thickness ratio. This phenomena is why the reference single shell ($\bar{h} = 2 \bar{h}_1$) experiences cutoff at a higher frequency than the mass conserved MEMP shells. Examining the MEMP identical shell case ($\bar{\beta} = 1$), the cutoff for both the inner shell and outer shell coincide. Interference gaps appear past this frequency not evident in the single shell result. In the mismatched case ($\bar{\beta} = 2$), there are
two different and distinct cutoff locations, one for each shell. Past the highest cutoff point, the MEMP shell in this configuration generally attenuates more vibration transmission in comparison to the identical shell case, a conclusion in agreement with the similar MEMP beam analysis. The frequency region between the inner and outer shell cutoff frequencies provides a new opportunity for reduction. In this region, radial vibrations propagate axially along the excited outer shell while those on the inner shell quickly decay in an evanescent behavior.

The monopole transfer function leads to similar conclusions with improvements shown in cases with mismatched inner and outer shell stiffnesses. Results are plotted below.
The MEMP configurations are able to drastically reduce the vibration transmission through the shells when forced discretely on the boundary end of a shell. Comparisons are made with a reference single shell that weighs the same as the two MEMP shells combined, thus conserving mass. This weight relation is set by prescribed certain material properties of the shells (identical radii, poisson ratio, etc). However, there are other various methods to reach this constraint that involve altering the geometric and material properties of the MEMP shells. These free vari-
ables provide opportunities to maximize the static strength of the MEMP shell while maintaining the dynamic force attenuation.

3.2.4 Double Shell Vibration Shield: Radial Pressure Distribution

![Diagram of Vibration Absorber Double Shell Configuration]

**Figure 3.32:** Vibration absorber shell configuration with boundary conditions on inner and outer shell

The MEMP design can also be examined in response to a distributed forcing, specifically in the form of a radial pressure. The boundary conditions are chosen as follows: the inner shell is fixed both at \( \bar{x} = 0 \) and at \( \bar{x} = 1 \) while the outer shell subject to the radial pressure distribution and is free both at \( \bar{x} = 0 \) and at \( \bar{x} = 1 \). Since the forcing is applied to the outer shell, it acts as a sort of vibrational shield to the inner shell. The transmission of force from the outer shell towards the fixed boundaries of
the inner shell is calculated:

\[ TF_x = 20 \log_{10} \left( \frac{\text{Output Net } F_x}{\text{Input Net } F_x} \right) \]

\[ = 20 \log_{10} \left( \frac{\pi E_2 h_2 L \left[ \left| -\overline{T}_2(0) - i\overline{S}_2(0) \right| + \left| -\overline{T}_2(1) - i\overline{S}_2(1) \right| \right]}{\pi E_1 R_1 L \int_0^1 \overline{P}_1(\varphi) \, d\varphi} \right) \]

\[ = 20 \log_{10} \left( \frac{(1 - \nu_2^2) \overline{h}_2 \overline{R}_2 \left[ \left| -\overline{T}(0) - i\overline{S}(0) \right| + \left| -\overline{T}(1) - i\overline{S}(1) \right| \right]}{(1 - \nu_1^2) \overline{\beta}^2 \overline{R}_1 \int_0^1 \overline{P}_1(\varphi) \, d\varphi} \right) \quad (3.79) \]

A parabolic axial distribution is applied for the radial pressure in the dipole mode. The results are shown in comparison to the appropriate single shell. In the case for \( \overline{\beta} = .5 \), the nondimensional frequency is plotted in reference to the stiffer shell.
The attenuation trends are similar to what was determined in the end loading configuration: the MEMP shells with mismatched properties again generally performs better than the case with identical shells with reduction strongly dependent on the spring coupling strength. While only the dipole mode is presented, the end forcing results predict that shells forced in the monopole and higher order modes would experience more prominent cutoff behavior than the dipole when subject to...
distributed pressure loadings.

A statement must be said on axial location of the forcing function. The attenuation achieved in the MEMP shells and beams is reliant on the highly coupled structural impedances. The calculated transfer function measures the impedance’s effect on the force transmission between the excitation and receiver (fixed end). If the excitation is placed directly on the receiver, there is of course no force attenuation. Thus as the pressure loading becomes concentrated near the receiver end, the MEMP effect is less prominent. The above case presents a centralized and symmetric parabolic distribution along the outer shell, however there could be applications where the forcing is located closer to the inner shell fixtures.
4.1 MEMP Design Conclusions on Thin Beams and Shells

Multi-element multi-path design is introduced in thin beams as an exploration into vibration transmission reduction on lightweight flexible structures with discrete elastic couplings. Theoretical simulation results reveal wideband and robust attenuation through the mixed cantilever double beam configuration in which the excitation (source) and fixture (receiver) are placed on different beams such that the wave paths must travel through the couplings in order to reach the fixture. In this configuration, the lower coupling stiffness translates to higher attenuations. Similarly, mismatched properties on the substructures leads to interference between waves propagating at different speeds and creates more opportunities for reduction. Results show low frequency attenuation beginning after the in-phase resonant frequency of the system, after which the interference is initiated. Viscoelastic damping is included into the elastic couplings, serving to level out the transfer function and bring down the resonant peaks. The wideband transmission loss evident in the MEMP beams are not present in the single element structures but are instead inherent to the physical phe-
nomena of multi-layered design, including cutoff behavior, peak splitting and wave path interactions. Algorithms are developed for quick and efficient solutions to the linear system of equations of the MEMP coupled substructures, visualized with an elegantly designed graphic user interface (GUI).

An experimental setup is arranged involving two aluminum MEMP beams and a shaker apparatus with binder clip springs used as the elastic couplings. The properties of the materials are measured and the transfer function is recorded for different numbers of couplings, then compared to the simulation results. The theoretical and experimental transfer functions correspond well together, both predicting the wideband attenuation for soft spring couplings. When a stiffener is attached to one of the beams creating mismatched beam properties, there is greater reduction as predicted by theoretical analysis.

The design approach is further explored with thin cylindrical shells coupled by discrete radial springs subject to radial and tangential loads. The theoretical model consists of the highly coupled shell equations (longitudinal, tangential and radial deflection components) that are solved for monopole, dipole and higher order azimuthal modes. The single shell dispersion relation yields interesting results, revealing cutoff frequencies and intersecting eigenvectors components. When a single element shell structure is forced through beam-like motion in the dipole mode, the shell results diverge from those of the beam analysis at higher frequencies, emphasizing the importance of the deflection coupling. End forcing results examined on each azimuthal mode for the mixed cantilever double shell configuration again reveal wideband attenuation for soft couplings and mismatched properties. However, there is an additional cutoff component of the shell behavior that allows for tuning of individual shells in the MEMP design. The MEMP shells are also studied when forced through radial pressure functions with parabolic axial distributions, resulting in similar wideband attenuation gaps.
4.2 Future Work

The presented research serves as an introduction into the reduction opportunities available through MEMP design on lightweight, flexible structures. The results from the simulations and experimental studies are promising and indicate the need for further examination into the subject, especially when applied to larger systems. Additionally, modifications can be made to the spring couplings to augment the wave interference and improve the vibration reduction. Examples include various degree of freedom spring motions and nonperiodic distributions.

As with other passive approaches, optimization is an intricate problem that requires vigorous computational methods. This methodology consists of searching for parameters that result in maximum attenuation bands while adhering to constraints in static stiffness and mass, complicated by the matrix inversion used in the MEMP algorithms. While techniques in control engineering are certainly available, this problem may become intractable as the number of substructures and consequently the size of the inverted matrix become large.

The attenuations in vibration transmission suggests that MEMP design can be applied to reducing the underwater acoustic scattering and sound transmission through simple structures. For this analysis, the cylindrical shell would be most applicable due to its ubiquity in frameworks of submersibles (submarine hulls as an example). Currently, research is being conducted on developing an impedance matching algorithm for two-dimensional acoustic shell cloaking at low frequencies.
Appendix A

Coupled Mass Transfer Function

As a simplification for the MEMP mixed cantilever two beam (Configuration C), the elasticity can be removed the beams such that they behave as lumped masses. The two masses ($m_1$ and $m_2$) are coupled together by an extensional spring with strength $K_s$ and move with displacements $x_1$ and $x_2$. A forcing is applied onto the first mass through a harmonic excitation. The governing equations are:

$$F_0 e^{i\omega t} - K_s (x_1 - x_2) = m_1 \frac{\partial^2 x_1}{\partial t^2}$$

$$-K_s (x_2 - x_1) = m_2 \frac{\partial^2 x_2}{\partial t^2}$$

Figure A.1: Two coupled masses diagram
The steady state displacement solutions can be written with unknown amplitudes as:

\[ x_1 = X_1 \ e^{i\omega t} \]
\[ x_2 = X_2 \ e^{i\omega t} \]

Substituting the general solutions into the governing equations and nondimensionalizing the parameters yields:

\[ 1 - (X_1 - X_2) = -\omega^2 \ X_1 \]
\[ - (X_2 - X_1) = -m \ \omega^2 \ X_2 \]

where \[ X_1 = \frac{X_1 \ K_s}{F_0} \quad X_2 = \frac{X_2 \ K_s}{F_0} \quad \omega = \omega \sqrt{\frac{m_1}{K_s}} \quad m = \frac{m_2}{m_1} \]

The displacement amplitudes can be solved through linear algebra:

\[ X_1 = \frac{1 - m \ \omega^2}{\omega^2 \ (m \ \omega^2 - m - 1)} \]
\[ X_2 = \frac{1}{\omega^2 \ (m \ \omega^2 - m - 1)} \]

And the force transfer from the excitation towards the second mass is:

\[ TF = \frac{\text{Force on } m_2}{\text{Input Force}} = \frac{m_2 \ \frac{d^2 x_2}{dt^2}}{F_0 \ e^{i\omega t}} = \frac{\bar{m}}{m + 1 - \bar{m} \ \omega^2} \tag{A.1} \]

The transfer function and individual mass responses are plotted below. After the two mass resonance, the transfer function reaches below the quasistatic limit and decays to zero as the frequency approaches infinity.
Figure A.2: Force transfer function and mass responses for two coupled masses
The single element reference structure for the MEMP beams is established by fixing both beams at $\overline{x} = 0$ and equally forcing both beams at $\overline{x} = 1$. The spring couplings are set to rigid ($K_s \to \infty$ limit) and the number of couplings, $s$, is increased to reach the continuous distribution. In this manner, the two beams only behave in-phase as a single structure.

The MEMP top beam and reference top beam are forced with with excitation $F_1 e^{i\omega t}$ while the reference bottom beam is forced with $F_2 e^{i\omega t}$. The quasistatic deflection on the forcing end of the top beam is $y_{\text{MEMP}}$. The deflections on the forcing ends of both beams of the reference single beam are identical since the beams are
assumed to be “bolted” and noted as $y_s$. After the nondimensionalization, the forcing amplitude on each beam appear as:

$$
\overline{F}_1 = \frac{F_1 L^2}{E_1 I_1} \quad \overline{F}_2 = \frac{F_2 L^2}{E_2 I_2}
$$

If $\overline{F}_1 = 1$ and $\overline{F}_2 = \beta^2$, then:

$$
\frac{F_2}{F_1} = \frac{E_2 I_2 L^2}{E_1 I_1 L^2} = 1
$$

and the static strengths as:

- Reference Single Beam Static Strength $= K_{\text{Rigid}} = \frac{F_2 + F_1}{y_s}$

- MEMP Mixed Cantilever Double Beam Static Strength $= K_{\text{MEMP}} = \frac{F_1}{y_{\text{MEMP}}}$

The relative static strength is then defined as the MEMP static strength divided by the single beam static strength where deflections $\overline{y}_s$ and $\overline{y}_{\text{MEMP}}$ are both determined through computational methods:

$$
\text{Relative Static Strength} = \frac{K_{\text{MEMP}}}{K_{\text{Rigid}}} = \frac{\frac{F_1}{y_{\text{MEMP}}}}{\frac{F_2 + F_1}{y_s}} = \frac{y_s}{2} \frac{\overline{y}_s}{y_{\text{MEMP}}}
$$
Appendix C

Experimental Dimensional Parameters

To compare the theoretical and experimental MEMP beam results, the dimensional parameters needed to be measured. The elasticity of the couplings and beam were determined through a universal testing machine. A custom built apparatus was used to convert translational force to torsion for measuring the torsional spring constant. The results are plotted.

![Load vs Deflection for Cantilever Experimental Beam](image)

**Figure C.1**: Cantilever beam experimental deflection curves

\[ K = 278.6 \text{ N/m}, \quad R^2 = 0.962 \]
\[ K = 299.9 \text{ N/m}, \quad R^2 = 0.969 \]
\[ K = 304.9 \text{ N/m}, \quad R^2 = 0.971 \]

Mean Value \( K = 294.5 \text{ N/m} \) and \( E_{\text{beam}} = 5.89 \times 10^{10} \text{ N/m}^2 \)
Figure C.2: Coupling stiffness experimental deflection and rotation curves
Table C.1: Dimensional parameters for experimental apparatus

<table>
<thead>
<tr>
<th>Dimensional Parameters for Experimental MEMP Study</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Young’s modulus ($E$)</td>
<td>58.9 MPa</td>
</tr>
<tr>
<td>Beam density ($\rho$)</td>
<td>2700 kg/m$^3$</td>
</tr>
<tr>
<td>Beam thickness ($h$)</td>
<td>$\frac{1}{8}$ inch</td>
</tr>
<tr>
<td>Beam width ($w$)</td>
<td>1$\frac{1}{2}$ inch</td>
</tr>
<tr>
<td>Beam total length ($L$)</td>
<td>16 inch</td>
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<tr>
<td>Coupling extensional stiffness ($K_s$)</td>
<td>4710 N/m</td>
</tr>
<tr>
<td>Coupling torsional stiffness ($K_t$)</td>
<td>0.19 N m/rad</td>
</tr>
<tr>
<td>Coupling weight</td>
<td>0.0018 kg</td>
</tr>
<tr>
<td>Bolt weight</td>
<td>0.0035 kg</td>
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</tbody>
</table>
Bibliography


