Essays in Financial Economics

by

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Department of Economics
Duke University

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Approved:

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Ravi Bansal, Chair

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David Hsieh

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Adriano Rampini

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George Tauchen

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Economics in the Graduate School of Duke University 2009
Abstract

(Economics)

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Abstract

The central puzzles in financial economics commonly include violations of the expectations hypotheses, predictability of excess returns, and the levels and volatilities of nominal bond yields, in addition to well-known equity premium and the risk-free rate puzzles. Equally surprising is the recent evidence on large moves in asset prices, and the over-pricing of the out-of-the-money index put options relative to standard models. In this work, I argue that the long-run risks type model can successfully explain these features of financial markets. I present robust empirical evidence which supports the main economic channels in the model. Finally, I develop econometric methods to estimate and test the model, and find that it delivers plausible preference and model parameters and provides a good fit to the asset-price and macroeconomic data.

In the first chapter, which is co-authored with Ravi Bansal, we present a long-run risks based equilibrium model that can quantitatively explain the violations of expectations hypotheses and predictability of returns in bond and currency markets. The key ingredients of the model include a low-frequency predictable component in consumption, time-varying consumption volatility and investor’s preferences for early resolution of uncertainty. In this model, varying consumption volatility in the presence of the predictable consumption component leads to appropriate variation in bond yields and the risk premia to provide an explanation for the puzzling violations of the expectations hypothesis. Using domestic and foreign consumption and
asset markets data we provide direct empirical support for the economic channels highlighted in the paper.

In the second chapter, co-authored with Ravi Bansal, we develop a general equilibrium model in which income and dividends are smooth, but asset prices are subject to large moves (jumps). A prominent feature of the model is that the optimal decision of investors to learn the unobserved state triggers large asset-price jumps. We show that the learning choice is critically determined by preference parameters and the conditional volatility of income process. An important prediction of the model is that income volatility predicts future jumps, while the variation in the level of income does not. We find that indeed in the data large moves in returns are predicted by consumption volatility, but not by the changes in the consumption level. In numerical calibrations, we show that the model can quantitatively capture these novel features of the data.

In the third chapter, I present a long-run risks type model where consumption shocks are Gaussian, and the agent learns about unobserved expected growth from the cross-section of signals. The uncertainty about expected growth (confidence measure), as in the data, is time-varying and subject to jump-like risks. I show that the confidence jump risk channel can quantitatively account for the option price puzzles and large moves in asset prices, without hard-wiring jumps into consumption. Based on two estimation approaches, the model provides a good fit to the option price, confidence measure, returns and consumption data, at the plausible preference and model parameter values.
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A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets

1.1 Introduction

A prominent puzzle in financial economics is the violation of the expectations hypotheses and the ensuing predictability of returns in bond and currency markets. In this paper we show that a long-run risks framework (see Bansal and Yaron, 2004), which features low-frequency movements in consumption and time-varying consumption volatility, leads to significant time-variation in risk premia that can quantitatively explain the violations of expectations hypotheses in bond and currency markets. We also provide a direct empirical evidence that the covariation of measured consumption volatility in the data with the bond and currency prices across countries agrees with the theoretical predictions from our model – this evidence supports the economic channels highlighted in this paper.

The violations of the expectations hypotheses and the implied predictability of returns are one of the prominent features of bond and currency markets data. Empirical evidence in US and foreign countries typically shows a significant drop in long
rates following periods of high long-short yield differential, which stands in sharp contrast to the implications of the expectations hypothesis of the yield curve. Additional evidence on the predictability of bond returns is presented more recently in Cochrane and Piazzesi (2005), who show that a single bond factor constructed from a linear combination of three to five forward rates can sharply forecast future bond returns. In currency markets, Backus, Foresi, and Telmer (2001), Bansal (1997) and Fama (1984) show that the interest rate differential across countries forecasts future exchange rate changes — in particular, a rise in the domestic nominal rate forecasts an appreciation of the domestic currency. What economic mechanisms can account for all these bond and currency market puzzles? We argue that the economic channels of the long-run risks model can successfully account for the risk and returns in financial markets and explain the puzzling features of bond and currency markets.

The key ingredients of the long-run risks model include a low frequency component in consumption, time-varying aggregate volatility and Kreps and Porteus (1978) recursive utility of Epstein and Zin (1989). Time-varying consumption uncertainty provides a primary economic channel which generates predictability and time-variation of expected excess returns, while the expected growth component magnifies risk compensations which enables us to explain the violations of the expectations hypotheses in the data. In earlier work, Bansal and Yaron (2004) highlight the implications of this model for the slope of the real yield curve. Eraker (2006), Piazzesi and Schneider (2005) consider the long-run risks model as well, and show that the inflation risk induces an upward sloping nominal yield curve in this model. These papers do not attempt to explain the violations of the expectations hypothesis as in their specification, consumption volatility and hence all the risk premia are constant. A similar two-country version of the model considered in Colacito and Croce (2005) also has constant risk premia and consequently cannot explain the currency
market violations of the expectations hypothesis.

In terms of the model intuition, when agents prefer early resolution of uncertainty (i.e., when the risk aversion exceeds the reciprocal of the IES) a positive shock to consumption volatility moves the expected excess bond returns and the long-short yield spread in the same direction. Therefore, the slope of the term structure forecasts positively future excess returns on bonds, which can quantitatively account for the violations of expectations hypothesis in the data. Similarly, in foreign exchange markets, in response to a positive shock to domestic consumption uncertainty the agents demand higher expected excess returns in foreign bonds, forecast appreciation of the foreign currency and at the same time push the yield on domestic risk-free assets down. This can quantitatively account for the violations of the expectations hypothesis in currency markets.

There is considerable support for time-varying consumption volatility in the data—Kandel and Stambaugh (1990) present strong evidence of varying consumption volatility, while Stock and Watson (2002) highlight a time-variation and a decline in the volatility of several macro variables, such as real GDP and aggregate consumption. Further, in the data, the volatility of domestic consumption growth correlates negatively with the dollar price of foreign currencies and forward premia, and positively with the expected returns on foreign bonds for all the countries in our analysis. This evidence is consistent with the predictions of the model, and provides empirical support for the economic channels highlighted in the paper.


The rest of the paper is organized as follows. In the next section we document the violations of the expectations hypothesis in bond and currency markets. In Section 3 we setup the long-run risks model. We present the solution to the model and discuss its theoretical implications for financial markets in Section 4. Section 5 describes the data and calibration of the real and nominal economy and preference parameters. Model implications for bond and currency markets values are addressed in Section 6. In Section 7 we discuss an extension of the model to the two-volatility case which sharpens its quantitative results. Conclusion follows.

1.2 Predictability Puzzles and Evidence

A standard benchmark for the analysis of returns on bonds is provided by the expectations hypothesis. It states that in domestic bond markets, a high long-short yield spread today is offset by an anticipated loss on long maturity bonds in the fu-

---

ture, and therefore should forecast an increase in the long rates. In foreign exchange context, low risk-free rates at home are compensated by the future appreciation of dollar and therefore should predict expected depreciation of the foreign currency. These conclusions formally obtain in structural models when the expected excess returns are constant, e.g., when investors are risk-neutral or economic uncertainty is constant.

As discussed below, none of these implications of the expectations hypothesis are supported by the data; in fact, the signs in predictability regressions are exactly the opposite. Reduced-form empirical projections imply that a high yield spread forecasts a drop in future long rates, and the regression coefficients become more negative with maturity. Likewise, low forward premium predicts appreciation of the foreign currency, though, the violations are less severe in longer horizon. Therefore, the forecasts of the change in future bond and currency prices based on expectations hypothesis or empirical projections will be radically different, both in terms of their magnitude and sign. The violations of these predictions in the data pose a challenge to the economic understanding of the asset markets and seriously question the constant (zero) expected excess return assumptions used to justify the expectations hypothesis model.

The economic principle of no-arbitrage across bond, currency and equity markets implies that the expected return in all these markets should be explained by common economic risk channels. In the context of long-run risks model, we show that these channels can successfully account for the predictability puzzles in bond and currency markets.

In the next two sub-sections we establish the notations and document the key empirical findings on predictability of domestic and foreign bond returns.
1.2.1 Bond Market Puzzles

Denote $y_{t,n}$ the yield on the real discount bond with $n$ months to maturity. Then, we can write the excess log return on buying an $n$ months bond at time $t$ and selling it at time $t + m$ as an $n - m$ period bond as

$$r_{t+m,n} = ny_{t,n} - (n - m)y_{t+m,n-m} - my_{t,m}. \quad (1.1)$$

Variables with a dollar superscript will refer to nominal quantities, such as nominal risk-free rate $y_{t,1}$. To avoid clustering of superscripts, we lay out the discussion using real variables; same arguments apply for the nominal economy as well.

Under the expectations hypothesis, the expected excess bond returns are constant. This implies that the slope coefficient $\beta_{n,m}$ in bond regressions

$$y_{t+m,n-m} - y_{t,n} = const + \beta_{n,m} \frac{m}{n-m} (y_{t,n} - y_{t,m}) + \text{error} \quad (1.2)$$

should be equal to one at all maturities $n$ and time steps $m$. Indeed, with rational expectations, the population value for the slope coefficient is given by

$$\beta_{n,m} = 1 - \frac{\text{Cov}(E_t r_{t+m,n}, y_{t,n} - y_{t,m})}{m \text{Var}(y_{t,n} - y_{t,m})}. \quad (1.3)$$

If term-spread $y_{t,n} - y_{t,m}$ contains no information about expected excess bond returns $E_t r_{t+m,n}$, e.g. expected excess returns are constant under the expectations hypothesis, then the slope is equal to unity. Alternatively, high long-short spread should predict a proportional decline in future bond prices, which eliminates the yield advantage to long-term bonds by expected capital loss.

In the data, however, the regression coefficients in bond projections (1.2) are negative and increasing in absolute value with horizon (see Campbell and Shiller, 1991). In the second panel of Table 1.1, we tabulate the projection coefficients for
Table 1.1: **Bond Market Data**

<table>
<thead>
<tr>
<th></th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
</tr>
</thead>
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<tr>
<td><strong>Nominal Yield:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US Mean</td>
<td>5.56</td>
<td>5.77</td>
<td>5.94</td>
<td>6.07</td>
<td>6.16</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>2.91</td>
<td>2.86</td>
<td>2.79</td>
<td>2.75</td>
<td>2.72</td>
</tr>
<tr>
<td>UK Mean</td>
<td>7.33</td>
<td>7.35</td>
<td>7.39</td>
<td>7.43</td>
<td>7.46</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>2.82</td>
<td>2.57</td>
<td>2.45</td>
<td>2.40</td>
<td>2.37</td>
</tr>
<tr>
<td><strong>EH Projection:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US Slope</td>
<td>-0.70</td>
<td>-1.03</td>
<td>-1.41</td>
<td>-1.39</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.43)</td>
<td>(0.51)</td>
<td>(0.57)</td>
<td>(0.64)</td>
<td></td>
</tr>
<tr>
<td>UK Slope</td>
<td>-0.14</td>
<td>-0.12</td>
<td>-0.10</td>
<td>-0.09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.57)</td>
<td>(0.64)</td>
<td>(0.72)</td>
<td>(0.82)</td>
<td></td>
</tr>
<tr>
<td><strong>Single Latent Factor Projection:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US Slope</td>
<td>0.44</td>
<td>0.87</td>
<td>1.24</td>
<td>1.45</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.20</td>
<td>0.23</td>
<td>0.24</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>UK Slope</td>
<td>0.47</td>
<td>0.88</td>
<td>1.20</td>
<td>1.45</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.28</td>
<td>0.30</td>
<td>0.30</td>
<td>0.29</td>
<td></td>
</tr>
</tbody>
</table>

nominal bond yields in US and UK. Consistent with previous studies, all of the slope coefficients are negative, and they are increasing in absolute value with maturity for US. The standard errors of the estimates, however, are quite large. Similar evidence obtains for inflation-adjusted projections when we subtract the measure of expected inflation from nominal yields.

This empirical evidence suggests that, contrary to the expectations hypothesis, high long-short yield spread forecasts an increase in future prices. That is, expected excess bond returns are time-varying and predictable by the term-spread with a positive sign. Dai and Singleton (2002) provide further discussion of the violations of the expectations hypothesis and bond return risk premia in context of affine models of the term structure.

Forward rate projections provide additional evidence for the time-variation in
bond risk premia. We follow Cochrane and Piazzesi (2005) and regress the average of \( m \)–period excess returns on bonds of different maturities on the forward rates over equally spaced short, middle and long horizon. The fitted values \( \hat{r}_{x,t,m} \) from these regressions are then used as a single bond factor in projections

\[
rx_{t+m,n} = const + b_{m,n} \hat{r}_{x,t,m} + error. \tag{1.4}
\]

Cochrane and Piazzesi (2005) show that the estimates \( b_{m,n} \) are positive and increasing with horizon, and a single factor projection captures \( 20 - 30\% \) of the variation in bond returns. We demonstrate these results for US and UK bond markets in the last panel of Table 1.1.

### 1.2.2 Currency Market Puzzles

Let \( s_t \) stand for a real spot exchange rate, in logs, per unit of foreign currency (dollars spot price of one pound), and denote by \( f_{FX}^t \) the logarithm of the foreign exchange forward rate, i.e. current dollar price of a contract to deliver one pound tomorrow. Superscript \(^*\) will denote the corresponding variable in the second country, e.g. \( y^*_t,1 \) stands for the foreign risk-free rate. To avoid clustering of superscripts, we present the discussion in real terms.

A one-period excess dollar return in foreign bonds is given by

\[
rx_{t+1}^{FX} = s_{t+1} - s_t + y^*_{t,1} - y_{t,1}. \tag{1.5}
\]

This corresponds to an excess return on buying foreign currency today, investing the money into the foreign risk-free asset and converting the proceeds back using the spot rate next period.

Under the expectations hypothesis in currency markets, the excess returns are constant. Therefore, the slope coefficient in the projection

\[
s_{t+1} - s_t = const + \beta^{UIP}(y_{t,1} - y^*_{t,1}) + error. \tag{1.6}
\]
Table 1.2: Currency Market Data

<table>
<thead>
<tr>
<th></th>
<th>UK</th>
<th>Germany</th>
<th>Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Foreign Exchange Rate:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal Mean</td>
<td>-0.53</td>
<td>1.47</td>
<td>3.10</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>10.69</td>
<td>11.18</td>
<td>12.09</td>
</tr>
<tr>
<td>Real Mean</td>
<td>0.72</td>
<td>-0.25</td>
<td>0.65</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>10.80</td>
<td>11.17</td>
<td>12.18</td>
</tr>
<tr>
<td><strong>UIP Projection:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>-1.72</td>
<td>-0.85</td>
<td>-2.83</td>
</tr>
<tr>
<td>Nominal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.04</td>
<td>0.02</td>
<td>0.06</td>
</tr>
</tbody>
</table>

should be equal to one. Indeed, with rational expectations, the population value for the regression coefficient can be written as,

$$
\beta_{UIP} = 1 + \frac{\text{Cov}(E_t r_{t+1}^{FX}, y_{t+1} - y_{t+1}^*)}{\text{Var}(y_{t,1} - y_{t,1}^*)}.
$$

(1.7)

Therefore, if the forward premium $y_{t,1} - y_{t,1}^*$ contains no information about the foreign bond risk premium $E_t r_{t+1}^{FX}$, e.g. the latter is constant under the expectations hypothesis, the projection coefficient is unity. Alternatively, if the uncovered interest rate parity condition holds, high interest rate bearing countries are expected to experience a proportional depreciation of their currency.

Fama (1984), Hodrick (1987), Backus et al. (2001) and many other studies show that at short maturities, the regression coefficient in foreign exchange projection (1.6) is negative and statistically significant. In the second panel in Table 1.2, we document these findings for UK, Germany and Japan for an investment horizon of 1 month. To focus on the return dimension, we subtract the forward premium from the both sides of (1.6). As investment horizon gets larger, the violations of the UIP condition in the data are less severe. As shown by Chinn and Meredith (2004) and
Alexius (2001), at long maturities the slope coefficient turns positive but remains below one. We confirm these findings for nominal bonds in US and UK for 2 to 5 years to maturity: in monthly regressions from January 1988 to December 2005 (not shown), the slope coefficient is positive at 2 year horizon and is equal to 0.12 (0.98), and is very close to one for longer investment horizons.

To get additional insight into the violations of expectations hypothesis in currency markets, Fama (1984) decomposes the difference between the forward and spot price of the currency into the risk premium part and the expected depreciation of the exchange rate:

\[ f_t^{FX} - s_t = (f_t^{FX} - E_t s_{t+1}) + (E_t s_{t+1} - s_t) \]
\[ \equiv (-E_t r^{FX}_{t+1}) + (E_t s_{t+1} - s_t). \]  

(1.8)

As can be seen from the expression (1.7), to explain a negative slope in uncovered interest parity regressions, an asset-pricing model should deliver a negative covariance between the (negative of) foreign bond risk premium, \(-E_t r^{FX}_{t+1}\), and expected depreciation of the currency, \(E_t s_{t+1} - s_t\), and also a greater variance of the risk premium than that of the expected depreciation.

1.3 Long-Run Risks Model

1.3.1 Preferences and Real Economy

We consider a discrete-time real endowment economy developed in Bansal and Yaron (2004). The investors preferences over the uncertain consumption stream \(C_t\) can be described by the Kreps-Porteus, Epstein-Zin recursive utility function, (see Epstein and Zin, 1989; Kreps and Porteus, 1978):

\[ U_t = \left[ (1 - \delta)C_t^{1-\gamma} + \delta(E_t U_{t+1}^{1-\gamma})^{1-\gamma} \right]^{\frac{\theta}{1-\gamma}}. \]  

(1.9)
The time discount factor is $\delta$, $\gamma \geq 0$ is the risk aversion parameter and $\psi \geq 0$ is the intertemporal elasticity of substitution (IES). Parameter $\theta$ is defined as

$$\theta = \frac{1 - \gamma}{1 - \frac{1}{\psi}}.$$  

Its sign is determined by the magnitudes of the risk aversion and the elasticity of substitution, so that if $\psi > 1$ and $\gamma > 1$, then $\theta$ will be negative. Note that when $\theta = 1$, that is, $\gamma = 1/\psi$, the above recursive preferences collapse to the standard case of expected utility. As is pointed out in Epstein and Zin (1989), in this case the agent is indifferent to the timing of the resolution of uncertainty of the consumption path.

When risk aversion exceeds (is less than) the reciprocal of IES the agent prefers early (late) resolution of uncertainty of consumption path. Hence, these preferences allow for agent’s preference for the timing of the resolution of uncertainty. In the long-run risk model agents prefer early resolution of uncertainty of the consumption path.

As shown in Epstein and Zin (1989), the logarithm of the Intertemporal Marginal Rate of Substitution (IMRS) for these preferences is given by

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

(1.10)

where $\Delta c_{t+1} = \log(C_{t+1}/C_t)$ is the log growth rate of aggregate consumption and $r_{c,t+1}$ is the log of the return (i.e., continuous return) on an asset which delivers aggregate consumption as its dividends each time period. This return is not observable in the data. It is different from the observed return on the market portfolio as the levels of market dividends and consumption are not equal: aggregate consumption is much larger than aggregate dividends. Therefore, we assume exogenous process for consumption growth and use a standard asset-pricing restriction

$$E_t[\exp(m_{t+1} + r_{t+1})] = 1$$

(1.11)

which holds for any continuous return $r_{t+1} = \log(R_{t+1})$, including the one on the wealth portfolio, to solve for the unobserved wealth-to-consumption ratio in the model.
Following Bansal and Yaron (2004), we assume that the real consumption growth contains a small and persistent long-run expected growth component and the time-varying volatility (economic uncertainty). The consumption dynamics is thus the following:

\[
\Delta c_{t+1} = \mu_g + x_t + \sigma_g \eta_{t+1}, \tag{1.12}
\]

\[
x_{t+1} = \rho x_t + \varphi e \sigma_g e_{t+1}, \tag{1.13}
\]

\[
\sigma_{g,t+1}^2 = \sigma_g^2 + \nu_g (\sigma_{g,t}^2 - \sigma_g^2) + \sigma_{gw} w_{g,t+1}. \tag{1.14}
\]

The unconditional mean of the time-varying variance of consumption growth is \(\sigma_g^2\). The variance of the long-run component in expected growth and consumption volatility is determined by \(\varphi_e\) and \(\sigma_{gw}\), respectively. The parameters \(\rho\) and \(\nu_g\) control the persistence of shocks to expected growth and consumption volatility. For analytical tractability, we assume that all the innovations are Gaussian and independent from each other.

Hansen, Heaton, and Li (2008) consider a related long-run risks model and present empirical evidence in favor of the low-frequency expected growth factor \(x_t\). Related evidence regarding long-frequency predictability in consumption growth is also presented in Bansal, Kiku, and Yaron (2007b). In terms of time-varying volatility of consumption growth, Kandel and Stambaugh (1990) find strong evidence for the variation in conditional variances of consumption growth, which is associated with the business cycle. Stock and Watson (2002) provide evidence of time-varying and persistent volatility of consumption growth rate and other related macroeconomic series. Using financial markets data, Bansal, Khatchatrian, and Yaron (2005) and Lettau, Ludvigson, and Wachter (2006) document a significant co-movement between macroeconomic volatility and asset prices. In all, there is considerable support for time-varying consumption volatility.
1.3.2 Nominal Economy

The economic channels in the real economy suffice to explain the violations of the expectations hypotheses and predictability of returns in bond and currency markets in real terms. However, most of the asset markets data is in nominal terms, and data on real bonds is not observable. To make our model implications comparable to observed data, we model the inflation exogenously and derive asset prices in nominal terms and inflation adjusted terms. Our model implications for the violations of the expectations hypothesis are similar across these various measures—real or nominal.

Our approach to directly model inflation is similar to that pursued by Wachter (2006) and Piazzesi and Schneider (2005) in context of equilibrium model for bond yields. In particular, we assume that the inflation process follows

\[
\pi_{t+1} = \bar{\pi}_t + \varphi_{\pi g} \sigma_{gt} \eta_{t+1} + \varphi_{\pi x} \varphi_{e} \sigma_{gt} \epsilon_{t+1} + \sigma_{\pi} \xi_{t+1},
\]

(1.15)

where the expected inflation, \( \bar{\pi}_t \equiv E_t \pi_{t+1} \), is given by

\[
\bar{\pi}_{t+1} = \alpha_{\pi} (\bar{\pi}_t - \mu_{\pi}) + \alpha_{x} x_t + \varphi_{zg} \sigma_{gt} \eta_{t+1} + \varphi_{zx} \varphi_{e} \sigma_{gt} \epsilon_{t+1} + \sigma_{z} \xi_{t+1}.
\]

(1.16)

To maintain parsimony, we assume that the inflation shock \( \xi \) is homoscedastic and affects both the inflation rate and its conditional mean; extensions to time-varying volatility and separate shock structure are straightforward. Parameters \( \varphi_{\pi g}, \varphi_{zg} \) and \( \varphi_{\pi x}, \varphi_{zx} \) measure the sensitivity ("beta") of realized and expected inflation innovations to short and long-run consumption news. Thus, the conditional variance of realized and expected inflation is time-varying and proportional to that of consumption growth, \( \sigma_{gt}^2 \).

Our specification of the expected consumption and inflation growth rates is similar to that of Piazzesi and Schneider (2005). For parsimony, we assume that the inflation process has no effect on the real economy, while the real consumption growth,
in particular, its expected growth, affects future expectations of inflation rate. We
discuss the plausibility of this specification in the empirical section of the paper.

Given the real discount factor $m_{t+1}$ and inflation process $\pi_{t+1}$, the logarithm of
nominal pricing kernel is given by

$$m_{t+1} = m_{t+1} - \pi_{t+1}. \quad (1.17)$$

1.3.3 Two-Country Setup

We extend our model to two-country setup—a similar specification, without the
time-varying volatility in consumption, is also entertained in Colacito and Croce
(2005). In particular, we assume that the endowments are country specific, and the
agents derive utility only from consumption of domestic goods. Financial markets
are frictionless and open for domestic and foreign investment and equilibrium real
and nominal exchange rates adjust exactly to offset all the net payoffs and preclude
arbitrage. This is similar to setup considered by Backus et al. (2001).

For simplicity, we impose complete symmetry and equate model and preference
parameters across the two countries. Only the states and innovations are allowed to
be country-specific; those for the foreign country are indexed by a superscript $^*$. The
correlation structure of the innovations is summarized by

$$\text{Cov}(e, e^*) = \tau_e, \quad \text{Corr}(\eta, \eta^*) = \tau_\eta,$$

$$\text{Corr}(w_g, w_g^*) = \tau_{w_g}, \quad \text{Corr}(\xi, \xi^*) = \tau_\xi.$$

The discount factor used to price assets denominated in foreign currency is given
by

$$m_{t+1}^* = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1}^* + (\theta - 1)r_{c,t+1}^*, \quad (1.18)$$

where $g_{t+1}^*$ is the log growth rate of foreign endowment growth, $r_{c,t+1}^*$ is the log return
on foreign consumption portfolio, and $\delta, \gamma$ and $\psi$ are the preference parameters of
the representative agents at home and abroad.
1.4 Asset Markets

1.4.1 Real Marginal Rate of Substitution

The key ideas of the model rely on solutions which are derived using the standard log-linearization of returns. In particular, the log-linearized return on consumption claim is given by

\[ r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}, \]

where \( z_t \equiv \log \left( \frac{P_t}{C_t} \right) \) is the log price-to-consumption ratio. Parameters \( \kappa_0 \) and \( \kappa_1 \) are approximating constants which are based on the endogenous average price-to-consumption ratio in the economy.

The approximate solution for the price-consumption ratio is linear in states,

\[ z_t = A_0 + A_x x_t + A_{gs} (\sigma_g^2 - \sigma_y^2). \]

From the Euler condition (1.10) and the assumed dynamics of consumption growth, the solutions for \( A_x \) and \( A_{gs} \) satisfy

\[ A_x = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}, \]

\[ A_{gs} = \frac{1}{2} \frac{(1 - \gamma)(1 - \frac{1}{\psi})}{1 - \kappa_1 \nu_g} \left( 1 + \left[ \frac{\varphi \kappa_1}{1 - \kappa_1 \rho} \right]^2 \right). \]

It follows that \( A_x \) is positive if the IES, \( \psi \), is greater than one. In this case the intertemporal substitution effect dominates the wealth effect. In response to higher expected growth, agents buy more assets, and consequently the wealth to consumption ratio rises. In the standard power utility model with risk aversion larger than one, the IES is less than one, and hence \( A_x \) is negative — a rise in expected growth potentially lowers asset valuations. That is, the wealth effect dominates the substitution effect.

Coefficient \( A_{gs} \) measures the sensitivity of price-consumption ratio to volatility fluctuations. If the IES and risk aversion are larger than one, then \( A_{gs} \) is negative.
In this case a rise in consumption volatility lowers asset valuations and increases the risk premia on all assets. An increase in the permanence of volatility shocks, that is \( \nu_g \), magnifies the effects of volatility shocks on valuation ratios as changes in economic uncertainty are perceived by investors as being long lasting. Similarly, \( A_g \) increases in absolute value with the persistence of expected consumption growth \( \rho \), as the effects of volatility shocks are magnified as they feed in through the expected growth channel \( x_t \).

Using the approximate solutions for the price-consumption ratio, we can provide an analytical expression for the marginal rate of substitution in (1.10). The log of the IMRS \( m_{t+1} \) can always be decomposed to its conditional mean and innovation. The former is affine in expected mean and variance of consumption growth and can be expressed as

\[
E_t m_{t+1} = \mu_m - \frac{1}{\psi} x_t + \frac{1}{2} \left[ 1 + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] (\sigma^2_{gt} - \sigma^2_{g}) (1.22)
\]

for some constant \( \mu_m \).

The innovation in the IMRS is very important for thinking about risk compensation (risk premia) in various markets. Specifically, it is equal to

\[
m_{t+1} - E_t m_{t+1} = -\lambda_\eta \sigma_{gt} \eta_{t+1} - \lambda_e \varphi_e \sigma_{gt} \varphi_{t+1} - \lambda_{gw} \sigma_{gw} \varphi_{g,t+1}.
\]

Parameters \( \lambda_\eta \), \( \lambda_e \), and \( \lambda_{gw} \) are the market price of risks for the short-run, long-run, and volatility risks. The market prices of systematic risks, including the compensation for stochastic volatility risk in consumption, can be expressed in terms of the underlying preferences and parameters that govern the evolution of consumption.
growth:

\[ \lambda_\eta = \gamma \]

\[ \lambda_e = (\gamma - \frac{1}{\psi}) \left( \frac{\kappa_1}{1 - \kappa_1 \rho} \right) \]

\[ \lambda_{gw} = \frac{1}{2} (\gamma - \frac{1}{\psi})(1 - \gamma) \frac{\kappa_1}{1 - \kappa_1 \rho} \left( 1 + \left[ \frac{\kappa_1 \varphi e}{1 - \kappa_1 \rho} \right]^2 \right) \]

(1.24)

In the special case of power utility, \( \theta = 1 \) or more specifically, \( \gamma = \frac{1}{\psi} \), the risk compensation parameters \( \lambda_e \) and \( \lambda_{gw} \) are zero, and the IMRS collapses to the standard power utility specification

\[ m_{t+1}^{CRRA} = \log \delta - \gamma \Delta c_{t+1}. \]

With power utility there is no separate risk compensation for long-run growth rate risks and volatility risks — with generalized preferences both risks are priced. The pricing of long-run and volatility risks is an important feature of the long-run risks model. Specifically, at the calibrated parameter values, consumption volatility shocks, \( w_{g,t+1} \), together with the innovations into the expected consumption, \( e_{t+1} \), are the most important sources of risks in the economy, as measured by their contribution to the maximal Sharpe ratio in the economy (conditional variance of the discount factor).

The logarithm of nominal pricing kernel can be obtained from the condition (1.17). In particular, nominal prices of immediate and long-run consumption risks depend on the inflation betas to short and long-run consumption news. Real and nominal discount factors in foreign country have analogous solutions.
1.4.2 Bond Returns

The equilibrium real and nominal yields are affine in the state variables:

\[
y_{t,n} = \frac{1}{n} \left[ B_{0,n} B_{x,n} B_{gs,n} \right] \left[ 1 \ x_t \ \sigma_{gt}^2 - \sigma_g^2 \right]', \quad (1.26)
\]

\[
y_{t,n}^g = \frac{1}{n} \left[ B_{0,n}^g B_{x,n}^g B_{gs,n}^g B_{\pi,n}^g \right] \left[ 1 \ x_t \ \sigma_{gt}^2 - \sigma_g^2 \ \bar{\pi}_t - \mu_{\pi} \right]', \quad (1.27)
\]

The bond coefficients, which measure the sensitivity ("beta") of bonds to the fundamental risks in the economy, are pinned down by the preference and consumption dynamics parameters – the expressions for the loadings are presented in Appendix A.1.2.

For typical parameter values, real yields respond positively to expected growth shocks, \( B_x > 0 \), and negatively to consumption volatility \( B_{gs} < 0 \), – while this is also true with power utility (\( \gamma = \frac{1}{2} \)), the full model with recursive preferences separates the risk aversion from the IES which allows for greater flexibility in modeling and explaining the yield curve. The solutions for nominal bond yields are more complicated as they take into account inflation risks in the economy. For example, in our calibrations nominal yields load negatively on the expected growth factor, which allows us to match an upward sloping term-structure of nominal bonds.

As discussed in Section 1.2.1, the violations of expectations hypothesis in the data imply a positive covariation between the expected excess return on bonds and long-short yield spread. In the model, up to Jensen’s adjustment term the one period expected excess return on real bond with \( n \) months to maturity can be rewritten in the following form:

\[
E_t(r_{x_{t+1},n}) + \frac{1}{2} Var_t(r_{x_{t+1},n}) = -Cov_t(m_{t+1},r_{x_{t+1},n})
\]

\[
= -B_{gs,n-1}\lambda_{gw}\sigma_{gw}^2 - B_{x,n-1}\lambda_e\phi_e^2\sigma_{gt}^2,
\]

(1.28)
while the real term-spread is given by,

\[ y_{t,n} - y_{t,1} = \text{const} + \left( \frac{1}{n} B_{x,n} - B_{x,1} \right) x_t + \left( \frac{1}{n} B_{gs,n} - B_{gs,1} \right) \sigma^2_{gt}. \]  

(1.29)

The model implies that in a period of high consumption uncertainty, the expected excess return on real bonds is low. Additionally, both short and long real yields fall, and the long-short term-spread declines as well, as in calibrations long yields are more sensitive to consumption variance, that is, \( \frac{1}{n} |B_{gs,n}| > |B_{gs,1}| \). This leads to a positive correlation of the real term-spread and bond risk premia, as required to explain the violations of the expectations hypothesis in the data. The actual magnitudes of the slopes coefficients in bond regressions (1.2) depend on the amount of persistence and variation in bond risk premium generated by the model.

Notably, all the three ingredients of the long-run risks model—preference for early resolution of uncertainty and time-variation in expected growth and volatility of consumption—are critical to explain the predictability of bond returns and violations of expectations hypothesis in the data. Indeed, as can be seen from expression (1.28), the risk premium on holding period bond returns reflects the compensation for long-run and volatility risks. With power utility, these two sources of risks are not priced, as \( \lambda_{gw} \) and \( \lambda_e \) are all zero, so up to Jensen’s adjustment term, the risk premium is zero. Further, all the time-variation in bond risk premium comes from the uncertainty about expected growth rate, so that if the expected consumption growth is constant (\( \varphi_e = 0 \)) or the aggregate volatility is constant (\( \sigma^2_{gt} = \sigma^2_{g}, \)) the bond risk premium is constant as well, and expectation hypothesis holds.

A similar discussion of the bond risk premia holds in nominal terms. Market prices of risks and bond return sensitivities to sources of uncertainty should now account for inflation risks, so the responses of bond yields and excess bond returns to consumption volatility depend on the calibration of consumption and inflation.
rates. For example, in our calibration, both nominal bond risk premium and nominal long-short yield spread actually increase with the variance of consumption growth. Still, same as for the real bonds, as the expected bond returns and the term-spread move in the same direction in response to consumption volatility shock, the slope coefficient in expectations hypothesis regressions for nominal bonds is also below one and negative.

1.4.3 Expected Depreciation and the Forward premium

As discussed in Backus et al. (2001), the Euler equation implies that the change in exchange rate is equal to the difference between the logarithms of the discount factors in the two countries:

$$ s_{t+1} - s_t = m_{t+1}^* - m_{t+1}, $$

(1.30)

and similar expression holds for nominal exchange rates.

Therefore, given the equilibrium solution to the pricing kernel, one-period expected depreciation rate of the domestic currency can be calculated in the following way:

$$ E_t s_{t+1} - s_t = const + \frac{1}{\psi}(x_t - x_t^*) + \frac{(\gamma - \frac{1}{\psi})(\gamma - 1)}{2} \left[ 1 + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] (\sigma_{gt}^2 - \sigma_{gt}^*)^2, $$

(1.31)

while the expression for the forward premium satisfies

$$ y_{t,1} - y_{t,1}^* = const + \frac{1}{\psi}(x_t - x_t^*) $$

$$ - \frac{1}{2} \left( \lambda^2 + \varphi_e^2 \lambda^2 + \frac{1}{\psi} - \gamma \right)(\gamma - 1) \left[ 1 + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] (\sigma_{gt}^2 - \sigma_{gt}^*)^2, $$

(1.32)

Therefore, the solution to expected excess return in foreign bond is linear in
consumption volatilities at home and abroad:

\[ E_t r_{t+1}^{FX} = E_t \left( s_{t+1} - s_t + y_{t,1}^* - y_{t,1} \right) \]
\[ = \frac{1}{2} \left( \lambda^2 \eta + \varphi e^{2 \lambda} \right) \left( \sigma_{gt}^2 - \sigma_{gt}^{*2} \right). \] (1.33)

As can be seen from the above expressions, in the full long-run risks specification with time-varying economic uncertainty and preference for early resolution of uncertainty, the expected excess return on foreign bonds unambiguously increases when the consumption uncertainty at home is high. Indeed, with a positive shock to domestic uncertainty, the equilibrium dollar price of the foreign currency \( s_t \) drops immediately, so that relative to the new level today, the foreign currency is expected to \textit{appreciate} tomorrow (expression (1.31)), and the dollar return on investments abroad is expected to be high (expression (1.33)). At the same time, when \( \gamma > 1 \) and \( \psi > 1 \), increase in domestic consumption uncertainty lowers the yields on domestic bonds and the yield spread across countries (equation (1.32)). All together, in response to a positive shock to consumption uncertainty, the agents demand higher expected excess returns in foreign bonds, forecast appreciation of the foreign currency and at the same time push the yield on domestic risk-free assets down. This can qualitatively account for the violations of the uncovered interest rate parity condition in the data. The actual magnitude of the model-implied slope coefficients in foreign exchange projections depend on the calibration of preference and consumption growth parameters.

For example, if the consumption uncertainty \( \sigma_{gt}^2 \) is constant, the expected excess returns on foreign bonds are constant as well, so that the expectations hypothesis holds and the projection coefficient in foreign exchange regression should be centered at one. At the other extreme, when agent has power utility, the slope coefficient is
given by
\[ \beta_{CRRA}^{\text{UIP}} = \frac{\text{Var}(x_t - x_t^*)}{\gamma^2 \text{Var}(\sigma_{gt}^2 - \sigma_{gt}^*) + \text{Var}(x_t - x_t^*)}, \]
so that it is less than one, but bigger than zero.

In terms of Fama (1984) conditions, while the power utility model can generate a negative covariation of the expected excess returns in foreign bonds and forward premium, so that the slope coefficient is below one, it fails to produce enough variation in the risk premium \( E_t r_{FX,t+1} \). In the long-run risks model, the risk premium is magnified by the compensation for the uncertainty in expected consumption growth, see expression (1.33). In particular, for the right persistence and variance of the long-run and volatility risks, we can match the empirical findings that the projection coefficients are negative at short horizon but become positive and closer to one at longer maturities.

The discussion for nominal variables is completely analogous, and is omitted for the interests of space.

1.5 Data and Calibration

1.5.1 Real Economy

We choose four countries for our empirical analysis, such as United Kingdom, Germany, Japan and United States (domestic country). The financial data for the foreign countries is taken from Datastream. This includes spot and forward rates, Euro-Currency Middle Rates of 1 month to maturity and returns on Morgan Stanley International Index for the period of January 1976 (July 1978 for Japan) to November 2005. Additional data on 1 to 5 year nominal discount bonds in US and UK come from CRSP and Bank of England, respectively. Market returns in US are calculated for a broad value-weighted portfolio from CRSP. The consumption
and CPI measures for foreign countries are taken from the IMF’s International Financial Statistics,\(^2\) while the US consumption data come from BEA tables of real expenditures on non-durable goods and services.

The second column of Table 1.5 shows summary statistics for the US quarterly consumption series for the period from 1976Q2 to 2005Q2. The real consumption growth rate is mildly persistent, has an annualized volatility just below 1% and correlates positively with household expenditure rates in UK. To view the long-horizon properties of the series we compute autoregressive coefficients at different lags as well as the variance ratios which are themselves determined by the autocorrelations. In the data the variance ratios first rise significantly and at about 3 years start to decline, while the autocorrelations fall uniformly with number of lags. The standard errors on these statistics, not surprisingly, are quite substantial. As the constructed series in other countries are more noisy proxies for the true consumption process of the agent, they are more volatile and less persistent than the US series. For the interests of space, we do not report their statistics in the paper.

The first panel in Table 1.3 tabulates summary statistics for excess market returns and inflation-adjusted rates across the four countries\(^3\). Notably, one-month interest rates are very persistent and vary from 1.9% in Japan to 3.61% in UK, while their standard deviations range between 2.1% and 3.1% for the sample period. For a post-war sample in US, the interest rate is equal to 0.8% with a standard deviation of 2.8%. Realized excess equity returns are several times more volatile and average 4.9% – 5.5% for UK, Germany and US (7.4% for a post-war US sample). High mean and variance of the market return relative to the interest rate are well-known puzzles in financial literature (see Mehra and Prescott, 1985).

\(^2\) We thank John Campbell for providing us the dataset.

\(^3\) Inflation adjustment of interest rates is based on AR(2) filtered inflation, while realized inflation is used to adjust changes in foreign exchange rates.
Table 1.3: Interest Rate and Equity Return Data

<table>
<thead>
<tr>
<th></th>
<th>UK</th>
<th>Germany</th>
<th>Japan</th>
<th>US</th>
<th>US47</th>
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<tr>
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<tr>
<td><strong>Interest Rate:</strong></td>
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<td></td>
</tr>
<tr>
<td>Mean</td>
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<td>2.73</td>
<td>1.91</td>
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<tr>
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<td>2.06</td>
<td>2.80</td>
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</tr>
<tr>
<td>AR(1)</td>
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<td>0.91</td>
<td>0.84</td>
<td>0.78</td>
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<td>0.05</td>
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<td>0.37</td>
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<tr>
<td><strong>Interest Rate:</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
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</table>

The first panel of Table 1.2 reports summary statistics for the nominal and real foreign exchange rates across the countries. Consistent with previous findings, foreign exchange rates have a zero autoregressive coefficient (not reported), and the range for their volatilities is 10.7% – 12.1%.

1.5.2 Nominal Economy

The second panel of Table 1.3 contains summary statistics for seasonally-adjusted monthly inflation rates and 1 month nominal yields across countries, while a more detailed description of the quarterly inflation in US is given in the second column of Table 1.6. The inflation rates are fairly persistent, as evidenced by the first and tenth-order autoregressive coefficients of 0.64 and 0.36, respectively, in the US sample. In fact, the variance ratios rise substantially and start to decline only at 8.5 year horizon. The series co-move positively across countries, with a correlation of US
and UK inflation rates of 0.73, and negatively with consumption growth rates, −0.21 being the correlation coefficient in US sample. The output for the other countries is similar, safe for a low predictability of inflation rates in Japan, and is omitted for the interest of space.

To capture the sensitivity of inflation news to consumption uncertainty, we set up and estimate a bivariate VAR(1) for these two series. We use the estimated model to compute analytically a $k$-period inflation beta defined as

$$\hat{\text{Cov}}_t\left(\frac{1}{k} \sum_{i=1}^{k} \pi_{t+i}, \frac{1}{k} \sum_{i=1}^{k} \Delta c_{t+i}\right) / \hat{\text{Var}}_t\left(\frac{1}{k} \sum_{i=1}^{k} \Delta c_{t+i}\right).$$

A solid line in Figure 1.1 draws the inflation beta as a function of horizon. Consistent with negative correlation of consumption and inflation rates in the data, the inflation beta is negative and stabilizes at −1.3 at long horizons. Notably, if the conditional expectations of consumption growth and inflation rates were constant, the inflation beta computed using the unconditional moments in Tables 1.5 and 1.6 would amount to −0.44 at all maturities. This evidence suggests that the expected consumption and inflation rates are time-varying and negatively correlated.
1.5.3 Calibration of Consumption and Inflation

We calibrate the model outlined in (1.12) - (1.14) and (1.15) - (1.16) at monthly frequency and time-aggregate the output from monthly simulations to match the key aspects of the 1976Q2 - 2005Q2 sample of quarterly consumption growth and inflation rate in US. We use the solutions provided in the theoretical section of the paper to derive our model implications for the asset prices. In particular, we use the numerical method discussed in Bansal et al. (2007b). They develop a procedure to solve for the endogenous constants $\kappa_0$ and $\kappa_1$ in equation (1.19) associated with each return and document that the numerical solution to the model is accurate. We provide some details on this method in Appendix A.1.1.

The baseline calibration parameter values are reported in Table 1.4. Specifically, we set the persistence in the expected consumption growth $\rho$ at 0.991. Our choice of $\phi_e$ and $\sigma$ ensures that the model matches the moments of consumption growth in the data. In particular, the annualized volatility of monthly consumption growth is set to 1.1%, while the long-run risks volatility parameter is $\phi_e = 0.055$. The persistence of the variance shocks is set at $\nu_g = 0.996$.

To capture the international dimensions of the data, similar to Colacito and Croce (2005) we set the correlation of long-run news $\tau_e$ to be nearly perfect, 0.999, and additionally impose a high correlation of the volatility news across the countries, $\tau_{gw} = 0.99$. In the extension of the model discussed subsequently in Section 1.7, we allow the volatilities of short-run and long-run consumption news to be different from each other, and calibrate the correlation between the short-run volatilities across countries to be zero, while the correlation of the long-run volatilities is set to one. This captures the intuition that in the long run, the distribution of consumption processes across all the countries are nearly identical (the long-run means and volatilities are the same), while in the short-run, they can be quite different due
<table>
<thead>
<tr>
<th>Parameter Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preference Parameters:</td>
<td></td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>$\delta$ 0.9987</td>
</tr>
<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\psi$ 1.5</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>$\gamma$ 8</td>
</tr>
<tr>
<td>Consumption Growth Parameters:</td>
<td></td>
</tr>
<tr>
<td>Mean of consumption growth</td>
<td>$\mu_c$ 0.0016</td>
</tr>
<tr>
<td>Long-run risks persistence</td>
<td>$\rho$ 0.991</td>
</tr>
<tr>
<td>Long-run risks volatility</td>
<td>$\varphi_e$ 0.055</td>
</tr>
<tr>
<td>Volatility level</td>
<td>$\sigma_g$ 0.0032</td>
</tr>
<tr>
<td>Volatility persistence</td>
<td>$\nu_g$ 0.996</td>
</tr>
<tr>
<td>Volatility of volatility</td>
<td>$\sigma_{gw}$ 1.15e-06</td>
</tr>
<tr>
<td>Dividend Growth Parameters:</td>
<td></td>
</tr>
<tr>
<td>Dividend leverage</td>
<td>$\phi$ 1.5</td>
</tr>
<tr>
<td>Volatility loading of dividend growths</td>
<td>$\varphi_d$ 6.0</td>
</tr>
<tr>
<td>Correlation of consumption and dividend news</td>
<td>$\tau_{gd}$ 0.1</td>
</tr>
<tr>
<td>Inflation Parameters:</td>
<td></td>
</tr>
<tr>
<td>Mean of inflation rate</td>
<td>$\mu_\pi$ 0.0032</td>
</tr>
<tr>
<td>Inflation leverage on long-run news</td>
<td>$\varphi_{\pi x}$ -2.0</td>
</tr>
<tr>
<td>Inflation shock volatility</td>
<td>$\sigma_\pi$ 0.0035</td>
</tr>
<tr>
<td>Expected inflation AR coefficient</td>
<td>$\alpha_{\pi}$ 0.83</td>
</tr>
<tr>
<td>Expected inflation loading on long-run risks</td>
<td>$\alpha_x$ -0.35</td>
</tr>
<tr>
<td>Expected inflation leverage on long-run news</td>
<td>$\varphi_{xx}$ -1.0</td>
</tr>
<tr>
<td>Expected inflation shock volatility</td>
<td>$\sigma_z$ 4.0e-06</td>
</tr>
<tr>
<td>Cross-Country Parameters:</td>
<td></td>
</tr>
<tr>
<td>Correlation of long-run news</td>
<td>$\tau_e$ 0.999</td>
</tr>
<tr>
<td>Correlation of short-run news</td>
<td>$\tau_\eta$ 0.0</td>
</tr>
<tr>
<td>Correlation of volatility news</td>
<td>$\tau_{gw}$ 0.99</td>
</tr>
<tr>
<td>Correlation of inflation news</td>
<td>$\tau_\xi$ 0.0</td>
</tr>
</tbody>
</table>

to uncorrelated immediate consumption and short-run volatility news at home and abroad.

In Table 1.5 we report the calibration output of our model, which is based on 1,000 simulations of 360 months of consumption series aggregated to quarterly horizon. The model implications for the volatility, persistence and multi-horizon properties of consumption growth rates are close to their empirical counterparts. Additionally,
Table 1.5: Consumption Growth Dynamics: Data and Model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>0.78 (0.09)</td>
<td>1.38</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.36 (0.07)</td>
<td>0.38</td>
</tr>
<tr>
<td>AR(2)</td>
<td>0.20 (0.08)</td>
<td>0.21</td>
</tr>
<tr>
<td>AR(5)</td>
<td>0.08 (0.08)</td>
<td>0.17</td>
</tr>
<tr>
<td>VR(2)</td>
<td>1.36 (0.07)</td>
<td>1.38</td>
</tr>
<tr>
<td>VR(5)</td>
<td>2.15 (0.27)</td>
<td>2.03</td>
</tr>
<tr>
<td>VR(10)</td>
<td>3.01 (0.65)</td>
<td>2.78</td>
</tr>
<tr>
<td>$\text{Corr}(\Delta c, \Delta c^*)$</td>
<td>0.25 (0.09)</td>
<td>0.24</td>
</tr>
</tbody>
</table>

The model also delivers low correlation coefficient of consumption growth rates across countries, which matches well the historical estimates (0.24 in the model versus 0.25 in the US and UK data).

The inflation rate process is calibrated as follows. To maintain parsimony, we zero out inflation and expected inflation betas to immediate consumption news, $\varphi_{\pi g} = \varphi_{zg} = 0$, and set their sensitivity to long-run risks to be negative, $\varphi_{x} = -2$ and $\varphi_{z} = -1$. This is consistent with Piazzesi and Schneider (2005), who show that the negative correlation of inflation innovations with future long-horizon consumption growth helps explain the term structure of nominal bonds. We calibrate the parameters of the expected inflation to match the key properties of the data. In particular, the expected inflation loads negatively on the expected consumption growth, $\alpha_{x} = -0.35$, and its own autoregressive coefficient is $\alpha_{\pi}$ is 0.83. For simplicity, we set to zero the correlation of independent inflation shocks across countries and capture the co-movements in the series through covariation of long-run risks shocks.

Table 1.6 shows the calibration output for the inflation process, while Figure 1.1 depicts the model-implied inflation beta to consumption news based on the bivariate VAR(1) specification. The model quite successfully matches the univariate properties of the inflation series, as well as the joint behavior of inflation and consumption.
Table 1.6: Inflation Dynamics: Data and Model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
</tr>
<tr>
<td>(\sigma(\pi))</td>
<td>1.62 (0.32)</td>
<td>1.68</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.64 (0.16)</td>
<td>0.70</td>
</tr>
<tr>
<td>AR(2)</td>
<td>0.67 (0.14)</td>
<td>0.60</td>
</tr>
<tr>
<td>AR(5)</td>
<td>0.57 (0.17)</td>
<td>0.48</td>
</tr>
<tr>
<td>AR(10)</td>
<td>0.36 (0.17)</td>
<td>0.32</td>
</tr>
<tr>
<td>VR(2)</td>
<td>1.61 (0.17)</td>
<td>1.69</td>
</tr>
<tr>
<td>VR(5)</td>
<td>3.56 (0.60)</td>
<td>3.41</td>
</tr>
<tr>
<td>VR(10)</td>
<td>6.44 (1.81)</td>
<td>5.69</td>
</tr>
</tbody>
</table>

\(\text{Corr}(\pi, \Delta c)\) -0.21 (0.14) -0.39 -0.13 -0.63
\(\text{Corr}(\pi, \pi^*)\) 0.73 0.07 0.63 0.86 0.28

growth rate and the correlation of the inflation rates across countries.

1.5.4 Preference Parameters

We calibrate the subjective discount factor \(\delta = 0.9987\). The risk-aversion coefficient is set at \(\gamma = 8\). Mehra and Prescott (1985) and Bansal and Yaron (2004) do not entertain risk aversion values larger than 10.

There is a debate in the literature about the magnitude of the IES. As in Bansal and Yaron (2004), we focus on an IES of 1.5 — an IES value larger than one is important for our quantitative results. Bansal et al. (2005) document that the asset valuations fall when consumption volatility is high; this is consistent only with \(\psi > 1\). Further, as we show in the next section of the paper, in the data domestic consumption volatility co-moves negatively with dollar prices of foreign currency and forward premia and positively with expected returns on foreign bonds. This evidence is consistent with model predictions only when \(\gamma > \frac{1}{\psi}\) and \(\psi > 1\), which further supports our calibration of preference parameters.
1.6 Model Implications

1.6.1 Bond Markets

As shown in the first panel of Table 1.7, at the calibrated parameter values the model-implied term structure of nominal bond yields is upward sloping. The one-year nominal yield is 5.44%, and it increases to 6.22% at 5 years. The volatilities of the yields fall uniformly from 2.25% at 1 year to 1.91% at 5 year horizon. The population values for levels and volatilities of nominal yields are thus consistent with US historical estimates reported in Table 1.1. The term-structure of real rates is downward sloping: the model-implied real rate is 1.3% at one year horizon and 0.52% at 5 years. At the calibrated parameter values, the inflation risk premium is increasing with the maturity, which enables us to match the upward slope of the nominal term structure.

The second panel of Table 1.7 shows model-implied slope coefficients in bond projections (1.2). These regressions are done using the annual time step and bond
maturities of 2 to 5 years, so they are directly comparable to the projections in the data reported in Table 1.1. The theoretical slope coefficients are all negative and decreasing with horizon: the slopes fall from $-0.17$ to $-0.34$. These values are consistent with the estimates based on historical data in US and UK bond markets in Table 1.1, taking into account the standard error of the estimates. As in the data, these violations are more severe at longer horizons, as model-implied slope coefficients in bond regressions increase in absolute value with horizon. In the model we also verify that the slope coefficients for real bond regressions are also negative and increasing in absolute value with horizon, and their values are similar to the ones based on nominal regressions. As we do not observe the counterparts for real yields in the data, we do not report the model-based regression results in the paper.

Additional evidence on the time-variation of bond risk premia comes from forward-rate projections considered in Cochrane and Piazzesi (2005). The preferred regression model of Cochrane and Piazzesi (2005) includes five forward rates, but we have to limit ourselves to three regressors to avoid perfect multicollinearity in the model. Indeed, with three states — expected consumption growth, expected inflation rate and consumption volatility — the three forward rates summarize all the information in the economy. Last panel of Table 1.7 and second panel of Table 1.1 compare the magnitudes of the coefficients and $R^2$ in common bond factor regressions (1.4) in the data and in the model. The loadings on a single bond factor are very similar across the countries: for US, they increase from 0.44 at 2 year horizon to 1.45 at 5 years. These values are well captured by the model: the slope coefficients increase from 0.39 to 1.58 for 2 and 5 year maturities, respectively. The $R^2$ in the data are in $20\%-30\%$ range. The population $R^2$s are about $10\%-12\%$, however, the estimates in small samples (not reported) often reach magnitudes found by Cochrane and Piazzesi (2005).
Table 1.8: Model Implications: Currency Markets

<table>
<thead>
<tr>
<th></th>
<th>1m</th>
<th>3m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>UIP Projection</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>-1.23</td>
<td>-1.18</td>
<td>-0.71</td>
<td>0.20</td>
<td>1.25</td>
<td>2.30</td>
<td>3.19</td>
</tr>
<tr>
<td><strong>Nominal FX Rate</strong></td>
<td>19.86</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Real FX Rate</strong></td>
<td>19.80</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.6.2 Currency Markets

Table 1.8 shows that the nominal slope coefficient in foreign exchange projections is equal to $-1.23$ at one month horizon, and it increases to $-0.71$ at 1 year and to 3.19 at 5 year horizon. The value of the nominal projection coefficient at short horizon matches well the empirical estimates shown in Table 1.2. The model-implied nominal slope becomes positive at maturities above 2 years, so that the violations of the expectations hypothesis are less pronounced at longer maturities, which is consistent with the evidence reported in Chinn and Meredith (2004) and Alexius (2001). Further, consistent with the data, our model delivers that the foreign exchanges rates are virtually unpredictable – the $R^2$s in foreign exchange projections (not shown) and the persistence in changes in spot prices of foreign currencies are all very close to zero. In the model we also verify that the slope coefficients in real regressions are negative as well – as there is no direct counterpart for real yields in the data, we do not report the model output in the paper. These findings are broadly consistent with Hollifield and Yaron (2003), who argue that risks from the real side of the economy are potentially important to capture the violations of the uncovered interest rate parity condition.

As shown in Table 1.2, the model-implied volatility of the foreign exchange rates is 19.86%, both in real and nominal terms, which is somewhat higher than the usual
estimates of 11 – 12% in the data.

We can provide a direct evidence for the volatility channel in the model using consumption and asset prices data across countries. In Figures 1.2 - 1.4 we plot inflation-adjusted spot prices, forward premia and expected foreign bond returns (solid line) against the difference in consumption volatility across countries (dashed line). We follow Bansal et al. (2005) and construct consumption volatility measures non-parametrically as a 4.5 year sum of absolute residuals from AR(3) projections of consumption growth rates. Inflation adjustment for the interest rates is based on the fitted values from AR(2) model for the inflation rate. Consistent with the theoretical predictions in Section 1.4.3 for $\gamma > \frac{1}{\psi}$ and $\psi > 1$, consumption volatility differential co-moves negatively with dollar prices of foreign currency and forward premia, and positively with the expected dollar returns in foreign bonds. Indeed, the correlation coefficients for the spot exchange prices range between -0.1 and -0.5, and are equal to about -0.3 and 0.3 for the forward premia and expected excess returns, respectively, for all the countries in the sample.

The magnitudes of the preference parameters and in particular, the value of the IES relative to one, have the first-order implications for the dynamics of asset markets in the model. Indeed, if $\psi > 1$, a rise in consumption volatility lowers asset valuations (see equation (1.21)), while the opposite happens if the IES is less than one. This has a direct effect on the co-movements of the consumption variance with asset and currency valuations and forward premium in the economy. Ultimately, it determines the ability of the model to explain the violations of the expectations hypothesis. In Figure 1.5 we plot the model-implied nominal slope coefficient against the IES when all the other parameters are fixed at their benchmark values. When IES is less than one, the theoretical slope coefficient is positive, so calibrating the IES the at the value above one is important to explain the foreign exchange puzzle in the data.
Figure 1.2: Real Exchange Rate and Consumption Volatility Differential,
Figure 1.3: Forward Premium and Consumption Volatility Differential.
Figure 1.4: Expected Excess Foreign Bond Returns and Consumption Volatility Differential.
1.6.3 Equity Return

We note that the model, at the calibrated values for preferences and consumption above, also matches the equity data. This is not surprising, given the earlier work on equity markets in Bansal and Yaron (2004). Following this work, we consider a dividend process of the form,

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_g \eta_{d,t+1}. \quad (1.34)$$

and calibrate it similar to that in Bansal and Yaron (2004). The third panel of Table 1.7 computes summary statistics for the real excess return on market portfolio and the real one-month interest rate. The model generates a sizable equity premium of 6.6%, which matches well the historical estimates in US and foreign countries reported in Table 1.3. The model-implied population value of the volatility of market returns is

---

4 We calibrate the dividend process and set the exposure of the corporate sector to long-run risks to $\phi = 1.5$, and choose $\varphi_d = 6$ to match a high volatility of dividend stream relative to consumption. The correlation of consumption and dividend news is set to $\tau_{gd} = 0.1$. The model-implied dynamics of the dividend growth series matches the data very well, and is omitted for the interest of space.
about 11.11%, which is broadly comparable to the historical estimates of 14% – 15% in US data (see Table 1.3). Model-implied real interest rate is 1.5%.

1.7 Robustness and Model Extension: Two Volatility Model

One interpretation of the evidence regarding consumption volatility in Stock and Watson (2002) on one hand and Kandel and Stambaugh (1990) on the other is that there are two volatility processes that drive macroeconomic volatility. Indeed, the evidence in Stock and Watson (2002) highlights a long-run decline in aggregate uncertainty, which can be modeled as a very slow mean-reverting process, and another that is featured in Kandel and Stambaugh (1990) is a shorter run volatility which mean-reverts more quickly after rising during recessions. This two volatility process view is consistent with the evidence on macroeconomic volatility, and at the same time, when incorporated in the long-run risks model, has sharp implications for the violations of the expectations hypothesis – in particular, it brings many of the quantitative model implications closer to the magnitudes in the data. We feature this model in this section and also point out the quantitative magnitudes upon which this two-volatility model improves.

Specifically, we extend the benchmark model specification and allow the volatilities of long-run and short-run consumption news to be driven by individual shocks—the structure is,

\[
\begin{align*}
\Delta c_{t+1} &= \mu_g + x_t + \sigma_{gt} \eta_{t+1}, \\
x_{t+1} &= \rho x_t + \sigma_{xt} e_{t+1}, \\
\sigma^2_{g,t+1} &= \sigma^2_g + \nu_g (\sigma^2_{gt} - \sigma^2_g) + \sigma_{gw} w_{g,t+1}, \\
\sigma^2_{x,t+1} &= \sigma^2_x + \nu_x (\sigma^2_{xt} - \sigma^2_x) + \sigma_{wx} w_{x,t+1}.
\end{align*}
\]

The immediate news in consumption growth have a stochastic volatility \( \sigma^2_{gt} \), while
\( \sigma^2_{zt} \) is equal to the conditional variance of the low-run component \( x_t \). We view the volatility of \( x_t \) as corresponding to the long-run volatility, and the volatility for the short-run shock to be associated with the rapidly mean reverting short-run volatility. We show that with a minimal change in the calibration of the model, we can provide a sharper explanation of the predictability of bond and currency returns while being consistent with the consumption and inflation data. Indeed, as we show in Tables in Appendix A.2, the coefficients in the expectations hypothesis regressions in bond markets become more negative and are quite close to those the data. We also obtain higher population \( R^2 \) in the single factor projections. Relative to the benchmark case with one volatility, it increases from from 9\% to 11 − 13\% (see Tables 1.7 and A.2).

In currency markets, a relatively high mean-reverting short-run volatility which is uncorrelated across countries help better match the slope coefficients in foreign exchange projections at long maturities and the volatility of foreign exchange rate. Indeed, nominal slope coefficient in foreign exchange regressions is -1.2 at one month horizon, turns positive within 2 to 3 years and remains below one at the considered maturities (it is equal to 0.86 at 5 year horizon). The volatility of the exchange rate is about 15\%, which is quite close to its data counterpart.

This augmented long-run risks model provides a sharper match to the data, however, the main economic insights are quite similar to the main model presented in the paper. For this reason, the details of the solution to the this model and its quantitative implications are discussed in Appendix A.2.

Conclusion

We show that the long-run risks model (see Bansal and Yaron, 2004) can explain the violations of expectation hypothesis and predictability puzzles in bond and currency
markets. The key ingredients of the model include long-run growth fluctuations, time-varying consumption uncertainty and preference for early resolution of uncertainty. These channels generate a significant variation in risk premia, driven by the consumption volatility, which can quantitatively account for the negative coefficients in the tests of expectations hypothesis and for the level of return predictability in bond and currency markets. Using consumption and asset markets data, we provide direct empirical evidence to support the key economic channels highlighted in the paper.

The model captures the intuition that a positive shock to consumption volatility moves the expected excess bond returns and the yield-spread in the same direction, which explains negative slope coefficients in the expectations hypothesis regressions in bond markets. At the same time, forward premium decreases and the domestic currency is expected to depreciate, which accounts for the violations of the uncovered interest rate parity condition in currency markets. In numerical calibrations, we show that our model can match the key dimensions of bond and currency markets in the data.
Learning and Asset-Price Jumps

2.1 Introduction

A prominent feature of financial markets is infrequent but large price movements (jumps). In this paper, we develop a model in which income and dividends have smooth Gaussian dynamics, however, asset prices are subject to large infrequent jumps. In our model, large moves in asset prices obtain from the actions of the representative agent to acquire more information about the unobserved state of the economy for a cost. We show that the optimal decision to incur a cost and learn the true economic state is directly related to the level of uncertainty in the economy. This implies that aggregate economic volatility, as well as market volatility, should predict jumps in returns. We show that indeed in the data, consistent with the model, return jumps are predicted by consumption volatility (market volatility). Further, the implied asset-price implications from our model are consistent with the

key findings from parametric models about frequency and predictability of jumps as discussed in Singleton (2006) as well as nonparametric jump-detection analysis of Barndorff-Nielsen and Shephard (2006). Based on our evidence, we argue that our structural model provides an economic basis for realistic reduced-form models of stock price dynamics with time-varying volatility and jumps.

We rely on the long-run risks model of Bansal and Yaron (2004), which key ingredients are a small and persistent low-frequency expected growth component, time-varying income volatility, and recursive utility of Epstein and Zin (1989) and Weil (1989). The expected growth is unobserved and has to be estimated from the history of the data; in addition, the representative agent also has an option to incur a cost and learn the true economic state. This setup is designed to capture the intuition that some of the key aspects of the economy are not directly observable, but the agents can learn more about them through additional costly exploration. We show that the optimal decision to pay a cost and observe the true state endogenously depends on the aggregate volatility, the variance of the filtering error and agent’s preferences. In particular, with preference for early resolution of uncertainty, the optimal frequency of learning about the true state after incurring a cost increases when consumption volatility rises. On the other hand, with expected utility, the agent has no incentive to learn the true state even if costs are zero. Learning about the true state may lead to large revisions in expectations about future income, which translate into large moves in asset prices. These large moves in asset prices obtain even though the underlying income in the economy is smooth and has no jumps. Such asset-price moves, we show, do not occur in economies where an option to learn about the true expected growth for a cost is absent.

Earlier studies, such as Gennette (1986), David (1997), Veronesi (1999) and Ai (2007) feature Kalman filter learning about the unobserved states to derive implica-
tions about the asset valuations in the economy. Veldkamp (2006) and Van Nieuwerburgh and Veldkamp (2006) specify a learning model where the endogenous information flow varies with the level of economic activity. Hansen and Sargent (2006) consider alternative approach to learning, which captures investors’ concerns about robustness and potential model misspecification. In our approach, we modify standard Kalman filtering problem to account for endogenous learning about the true state after paying a cost. The endogeneity of agent’s information set changes the sources and prices of risks relative to standard models; in particular, the actions of investors to learn about the true state alter asset valuations and can lead to asset-price jumps.

One of the key implications of the model is that the income volatility predicts future large moves in returns. We provide empirical support that large moves in the stock market can be predicted by volatility measures in the economy. Specifically, we document a positive correlation of return jump-indicator with lags of conditional variance of consumption. On annual frequency, the volatility of annual consumption significantly predicts next year large moves in market returns with an $R^2$ of 9%, which we show using two alternative measures of consumption volatility, including the usual GARCH model. Further, in the data there is no evidence for predictability of large moves in returns by the levels of the real aggregate variables. We show that the model can match both of these novel and important data features. Earlier evidence in Bates (2000), Pan (2002) and Eraker (2004a) documents that market volatility also predicts jumps. In our structural model, market variance is related to aggregate income volatility, which consequently enables us to match this data feature as well and provide an economic motivation for this empirical finding.

Our target is to match the key evidence on frequency, magnitude and predictability of jumps in the data. In the data we identify 24 years with at least one significant
price move (i.e. jump) in daily return for the 80 year period from 1926 to 2006; hence, the frequency of jump-years is one every 3.4 years\(^2\). In our sample, we find that the relative contribution of jumps to the total return variance is 7.5%, which is consistent with the evidence in Huang and Tauchen (2005) and other studies. We calibrate the model so we can match these dimensions along with other key asset-market facts. We use standard calibrations of income and preference parameters, while our calibration of learning costs is similar to observation and transactions costs in Abel, Eberly, and Panageas (2007a, b).\(^3\) We show that at the calibrated value of learning cost parameters, investors optimally choose to observe the true state about once every one and a half year, and the per annum expenditure on costly learning is merely 0.02% of the aggregate income. The model with constant aggregate volatility delivers the average frequency of jump-years once every 4.5 years, and the contribution of jumps to return variance of 7.7%. When we allow for time-varying aggregate volatility, the average frequency of jump-years increases to once every 4 years, while the relative contribution of jumps — to 9.5%. In standard models with no option to learn the true state for a cost, asset prices do no exhibit jumps. Further, we show that the model with costly learning delivers positive and significant correlation of large return move indicator with endowment and return variances and zero correlation with endowment growth. The magnitudes of the correlation coefficients are comparable to the data.

The rest of the paper is organized as follows. In the next section we review the empirical evidence on large infrequent movements in asset valuations in the data. In Section 3 we set up a model and describe preference, information structure and

\(^2\) This provides a conservative estimate for the frequency of return jumps in the data, as there can be more than 1 jump in daily returns on a given year.

\(^3\) Rational inattention channel is also used to explain infrequent adjustments of stock portfolio (Duffie and Sun 1990) or the consumption and saving plans of investors (Reis, 2006).
income dynamics in the economy. In Section 4 we characterize solutions to the optimal learning policy and equilibrium asset valuations. Finally, in Section 2.5 we use numerical calibrations to quantify model implications for asset-price jumps. Conclusion follows.

2.2 Evidence on Asset Price Jumps

Empirical evidence suggests that asset prices display infrequent large movements which are too big to be Gaussian shocks. In the first panel of Figure 2.1 we plot the time-series of daily inflation-adjusted returns on a broad market index for the period of 1926-2006\(^4\). Occasional large spikes in the series suggest presence of large moves (jumps). Indeed, the empirical quantiles plot on Figure 2.2 indicates that there are substantial deviations in the distribution of market returns from Normality, with a number of observations falling far in the tails. Consistent with this evidence, the kurtosis of market returns is 21, relative to 3 for Normal distribution, as shown in the first panel of Table 2.1.

\(^4\) We prorate monthly inflation rate to daily frequency to obtain inflation-adjusted returns from nominal ones. The results for the nominal returns are very similar.
For further evidence on large movements in asset prices, we apply non-parametric jump-detection methods (see Barndorff-Nielsen and Shephard, 2006), used in a stream of papers in financial econometrics. This approach allows us to identify years with one or more large price moves in daily returns.

Let $r_T$ stand for a total return from time $T - 1$ to $T$, and denote $r_{T,j}$ the $j$th intra-period return from $T - 1 + (j - 1)/M$ to $T - 1 + j/M$, for $j = 1, 2, \ldots, M$. The two common measures which capture the variation in returns over the period are the Realized Variation, given by the sum of squared intra-period returns,

$$RV_T = \sum_{j=1}^{M} r_{T,j}^2$$

and the Bipower Variation, which is defined as the sum of the cross-products of current absolute return and its lag:

$$BV_T = \frac{\pi}{2} \left( \frac{M}{M - 1} \right) \sum_{j=2}^{M} |r_{T,j-1}| |r_{T,j}|.$$  \hspace{1cm} (2.2)

When the underlying asset-price dynamics is a general jump-diffusion process, for finely sampled intra-period returns the Realized Variation $RV_T$ measures the total variation.
Table 2.1: **Summary Statistics: Data and Model**

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>Kurt</th>
<th>Jump-year Freq</th>
<th>Jump Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Data:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return</td>
<td>7.98</td>
<td>16.61</td>
<td>21.19</td>
<td>3.38</td>
<td>7.46</td>
</tr>
<tr>
<td><strong>Model Output:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant Volatility:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return with costly learning</td>
<td>7.92</td>
<td>15.28</td>
<td>18.37</td>
<td>4.47</td>
<td>7.68</td>
</tr>
<tr>
<td>Return, no learning</td>
<td>8.10</td>
<td>14.10</td>
<td>3.01</td>
<td>47.93</td>
<td>1.94</td>
</tr>
<tr>
<td>Time-Varying Volatility:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return with costly learning</td>
<td>7.82</td>
<td>15.22</td>
<td>35.14</td>
<td>3.97</td>
<td>9.47</td>
</tr>
<tr>
<td>Return, no learning</td>
<td>7.92</td>
<td>14.12</td>
<td>3.21</td>
<td>48.82</td>
<td>2.00</td>
</tr>
</tbody>
</table>

variation coming both from Gaussian and jump components of the price, while the Bipower Variation $BV_T$ captures the contribution of a smooth Gaussian component only (see, e.g. Barndorff-Nielsen and Shephard, 2006).\(^5\) Hence, these two measures reveal the magnitudes of smooth and jump components in the total variation of returns. A scaled difference between these two measures (Relative Jump statistics) provides a direct estimate of the percentage contribution of jumps to the total price variance:

$$RJ_T = \frac{RV_T - BV_T}{RV_T}. \quad (2.3)$$

Under the assumption of no jump and some regularity conditions, Barndorff-Nielsen and Shephard (2006) show that the joint asymptotic distribution of the two variation measures is conditionally Normal. This allows us compute a t-type statistics

\(^5\) More precisely, under some technical conditions,

$$\lim_{M \to \infty} RV_T = \int_{T-1}^T \sigma_p^2(s)ds + \sum_{j=1}^{N_T} k_{T,j}^2, \quad \lim_{M \to \infty} BV_T = \int_{T-1}^T \sigma_p^2(s)ds,$$

where $\sigma_p(s)$ is the instantaneous volatility of the Brownian motion component of the price, $k_{T,j}$ is the jump size and $N_T$ is the number of jumps within the period $T$. 47
to test for abnormally large price movements, which are indicative of jumps. A popular version of this statistics is

$$z_T = \frac{RV_T - BV_T}{\sqrt{\left(\left(\frac{\pi}{2}\right)^2 + \pi - 5\right) \frac{1}{M} TP_t}},$$

(2.4)

where the jump-robust Tri-Power Quarticity measure $TP_t$ estimates the scale of the variation measures and is defined as

$$TP_T = \left(\frac{M^2}{M - 2}\right) \left(E(|N(0,1)|^{4/3})\right)^{-3} \sum_{j=3}^{M} |r_{T,j-2}|^{4/3} |r_{T,j-1}|^{4/3} |r_{T,j}|^{4/3}.$$  

(2.5)

Under the null hypothesis of no jumps and conditional on the sample path, the jump-detection statistics $z_T$ is asymptotically standard Normal. Thus, if the value of $z_T$ is higher than the cut-off corresponding to the chosen significance level, then the test detects at least one large price move during the period $T$.

To calculate the jump-detection statistics over a year, we use the data on 266 daily returns, on average$^6$. $M = 266$ is a typical number in high-frequency studies, where it roughly corresponds to using 5-minutes returns to compute daily (24 hours) statistics. Huang and Tauchen (2005) discuss the performance of the tests in finite samples.

On Figure 2.1 we plot daily inflation-adjusted market returns and the corresponding years detected by jump-detection statistic for the period of 1927 - 2006. At 1% significance level, we flag 24 years with at least one significant move in daily asset prices. The relative contribution of large movements to the total return variation, as measured by the average relative jump measure $RJ$, is 7.5%. This estimate is consistent with other studies.

$^6$ For predictability regressions, we construct similar measures on monthly and quarterly frequencies.
Naturally, the detection of jump-years depends on the chosen significance level of the test. When the significance level drops, the cut-off value for the jump statistics in equation (2.4) increases, so that only larger jumps get flagged. On Figure 2.9 we plot the average frequency of detected jump-years in the data for a range of significance levels from 0.5% to 5%. As the significance level increases, the detected frequency of jump periods increases from one every 4 years to one every 2.5 years. In subsequent analysis, we fix the significance level to 1% – the results for other values are qualitatively very similar.

2.2.1 Predictability of Large Price Moves

In this section, we provide empirical evidence that fundamental macroeconomic volatility and variance of market returns can predict future large moves in asset prices in the data. On the other hand, there is no consistent evidence in the data for the link between large moves in returns and the levels of aggregate macro variables. That is, at the considered frequencies of large moves in returns, jumps in asset prices are not predicted by the variations in the level of real economy.

On the top panel of Figure 2.3 we plot the correlations of jump year indicator with annual consumption growth rate, its conditional variance and the conditional variance of market returns up to 5 year leads and lags (conditional variance calculations are based on AR(1)-GARCH(1) fit). The correlations of large move indicator with lagged aggregate volatility are all positive and are within 0.2-0.3 range. Similarly, positive market variance positively predicts future jump years 1 to 4 years ahead, with the correlation coefficient of about 0.1. As for the level of consumption growth, while the correlation coefficients are negative for 1 and 2 year lags and around -0.1, they turn positive at 3 year lag.

The predictability patterns are stronger at quarterly and monthly frequencies, as the persistence of variance measure and the frequency of detected jump periods
Figure 2.3: Jump Year Correlations with Aggregate Output, Aggregate Volatility and Market Volatility in the Data
increase. As consumption data is not available at such frequencies for a long historical sample, we use the industrial production index growth, whose monthly and quarterly observations are available from 1930s.\footnote{On annual frequency, the correlation of growth rates in consumption and industrial production is 0.55, while the correlation of their conditional variances is 0.84.} On the bottom panels of Figure 2.3 we plot the lead-lag correlations of the jump indicator with levels and conditional volatilities of industrial production growth rate and variance of market return at quarterly and monthly frequency. The results present a robust evidence for positive correlations of future large move indicators with variance measures and no consistent link with the level of the real economy.

To sharpen quantitative results, we construct a measure of macroeconomic volatility based on the financial markets data. We regress annual consumption growth on its own lag, the lags of market price-dividend ratio and junk bond spread and extract consumption innovation. The square of this innovation is further projected on the price-dividend ratio and junk bond spread, so that the fitted value $\hat{\sigma}_T^2$ captures the level of ex-ante aggregate volatility in the economy. The results of the two projections are summarized in the top panel Table 2.2. The $R^2$s are in excess of 20\%, and the signs of the slope coefficients are economically intuitive: low asset valuations and high bond spreads predict low expected growth and high uncertainty.

We use the extracted factor $\hat{\sigma}^2$ to forecast next year jump indicator statistic. The probit regression of the next-period jump indicator on current measure of macroeconomic volatility yields a statistically significant coefficient on $\hat{\sigma}^2_T$ with a t-statistics in excess of 3, and $R^2$ of 9\%. Specifically,

$$Pr(JumpIndicator_{T+1}) = \Phi \left( \frac{-0.95 + 1411.35\hat{\sigma}_T^2}{(0.22)} \right),$$

where $JumpIndicator_T$ is equal to 1 if year $T$ is flagged as a jump-year and 0 otherwise. On Figure 2.4 we plot the jump-detection statistics $z_T$ itself and the
Table 2.2: Estimation of Consumption Volatility

<table>
<thead>
<tr>
<th>Projection:</th>
<th>$\Delta c$</th>
<th>$pd$</th>
<th>$spread$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta c$</td>
<td>0.347</td>
<td>0.007</td>
<td>-0.004</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>(0.065)</td>
<td>(0.006)</td>
<td>(0.004)</td>
<td></td>
</tr>
<tr>
<td>$\hat{\sigma}^2 \times 10^4$</td>
<td>-0.546</td>
<td>4.602</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.156)</td>
<td>(2.056)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GARCH:</th>
<th>$AR(1)$</th>
<th>$Vol$</th>
<th>$GARCH$</th>
<th>$ARCH$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta c$</td>
<td>0.32</td>
<td>2.81e-06</td>
<td>0.82</td>
<td>0.14</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(7.5e-06)</td>
<td>(0.07)</td>
<td>(0.06)</td>
<td></td>
</tr>
</tbody>
</table>

The spikes in fitted probabilities broadly agree with large values of the jump statistics, even for the 1955-1980 period when no significant price moves were detected.

For robustness, we also check the results using GARCH measure of annual consumption volatility in the data. The bottom panel of Table 2.2 shows that the estimated aggregate consumption volatility is very persistent in the data. The probit estimation of predictability of future jump-year indicator is given by,

$$
\hat{Pr}(JumpIndicator_{T+1}) = \Phi \left( -0.80 + 899.23 \hat{\sigma}^2_T \right),
$$

so that the consumption volatility is statistically significant predictor of future jump years with $t-$statistics of 2.3, and the $R^2$ of 6%.

Predictability of future jumps by the consumption variance is a novel dimension of this paper. Predictability of future return jumps by market variance is consistent with the evidence in earlier studies which estimate parametric models of asset-price dynamics, see Bakshi et al. (1997), Bates (2000), Pan (2002), Eraker (2004a) and Singleton (2006).
2.3 Model Setup

Our model builds on the long-run risks framework developed in Bansal and Yaron (2004), where the investor has a full information about the economy. In contrast, we assume that investors do not observe all the relevant state variables, and hence there is an important role for learning about the true underlying state of the economy. In the model we show that the actions of the agents to learn the unobserved states can lead to asset-price dynamics which exhibits jumps.

2.3.1 Preferences and Information

Denote $\mathcal{I}_t$ the beginning-of-period information set of the agent, which includes current and past observed variables. The information set by the end of the period is endogenous and depends on the decision of investors to learn about the true state. Let us introduce a binary choice indicator $s_t \in \{0, 1\}$, which is equal to one if the agent learns about the true state for a cost in period $t$, and zero otherwise. Let $\mathcal{I}_t(s_t)$ be the time-$t$ (end-of-period) information set following a choice $s_t$. With no learning about the true state ($s_t = 0$), the end-of-period information set coincides
with that in the beginning of the period: \( I_t(0) \equiv I_t \). On the other hand, when \( s_t = 1 \), investors acquire new information during the day which enriches their information set: \( I_t(1) \supset I_t \). Further, let \( E_t \) denote the conditional expectation with respect to the information set \( I_t \), while denote \( E^{st}_t(\cdot) \) the conditional expectation based on the information following a binary choice \( s_t : E^{st}_t(\cdot) \equiv E[\cdot|I_t(s_t)] \).

We consider recursive preferences of Epstein and Zin (1989) over the uncertain consumption stream, with the intertemporal elasticity of substitution parameter set to one:

\[
U_t = Ct^{1-\beta} (J^{st}_t(U_{t+1}))^\beta, \tag{2.6}
\]

\[
J^{st}_t(U_{t+1}) = (E^{st}_t U_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}. \tag{2.7}
\]

\( C_t \) denotes consumption of the agent and \( J^{st}_t(U_{t+1}) \) is the certainty equivalent function which formalizes how the agent evaluates uncertainty across the states. Parameter \( \beta \) is the subjective discount factor, and \( \gamma \) is the risk-aversion coefficient of the agent. Note that certainty equivalent function depends on the choice indicator \( s_t \in \{0, 1\} \), as the information set of the agent is different whether the investors learn about the true state \( (s_t = 1) \) or not \( (s_t = 0) \).

To solve the model, we consider an optimization problem of the social planner, who wants to optimally allocate the exogenous output stream between consumption and learning expenditures. The solution to the social planner problem can then be decentralized in the competitive markets, which we verify by solving the representative agent problem directly.

### 2.3.2 Social Planner Problem

Consider the life-time utility of the agent \( U_t(s_t) \) for a given learning choice of the social planner \( s_t \in \{0, 1\} \):

\[
U_t(s_t) = C_t(s_t)^{1-\beta} (J^{st}_t(U_{t+1}))^\beta, \tag{2.8}
\]
where $U_{t+1}$ is the optimal utility tomorrow, and $C_t(s_t)$ denotes a choice specific consumption of the agent. The risk-sensitive certainty equivalent operator $J_t^*(U_{t+1})$ is specified in equation (2.7).

The objective of the social planner is to maximize the certainty equivalent of the life-time utility of the agent $U_t(s_t)$ with respect to the beginning-of-period information set $\mathcal{I}_t$ by choosing whether or not to learn about the true state for a cost:

$$s^*_t = \arg\max_{s_t} \{ J_t(U_t(s_t)) \} .$$  \hspace{1cm} (2.9)

The true value of the state is not known to the planner in the beginning of the period. As the agents are risk-sensitive to the new information about the state, the planner chooses to learn about the state for a cost if the certainty equivalent of agent’s life-time utility with learning is bigger than the life-time utility without learning. Following a decision to learn, the social planner then uses part of the endowment to pay the learning cost.

Define $Y_t$ the aggregate income process. Then, the budget constraint of the social planner states that the aggregate income is equal to consumption and learning cost expenditures:

$$Y_t = C_t(s_t) + s_t \xi_t .$$ \hspace{1cm} (2.10)

The learning cost $\xi_t$ represents the resources required to acquire and process the new information about the underlying economic state. It is similar to costs of observing the value of wealth and costs of transferring assets and rebalancing the portfolio featured in rational inattention literature, see Abel, Eberly, and Panageas (2007a, b). For analytical tractability, we make $\xi_t$ proportional to the aggregate income:

$$\xi_t = \chi Y_t ,$$ \hspace{1cm} (2.11)

for $0 \leq \chi < 1$. This specification preserves the homogeneity of the problem and simplifies the solution of the model.
In Appendix B.1 we show that in equilibrium, the life-time utility of investors following learning choice \( s_t \) are proportional to the level of income,

\[
U_t(s_t) = \phi_t(s_t)Y_t, \quad \text{for } s_t \in \{0, 1\}.
\]  

(2.12)

where the utility per income ratio \( \phi_t(s_t) \) satisfies the following recursive equation:

\[
\phi_t(s_t) = (1 - s_t\chi)^{1-\beta} \left( E_t^{s_t} \left[ \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right]^{1-\gamma} \right)^{\frac{\beta}{1-\gamma}}.
\]  

(2.13)

Learning about the true state has two opposite effects on the utility of investors. First, the agent’s consumption drops as part of the aggregate endowment is sacrificed to cover the learning costs. This decreases the agent’s utility, as evident from examining the first bracket in the expression above. On the other hand, learning enriches the information set of investors, and the ensuing reduction in the uncertainty about future economy may increase their utility (second part of the expression (2.13)). The net effect depends on the attitude of investors to the timing of resolution of uncertainty and the magnitude of learning costs. For example, in Appendix B.2 we show that with expected utility, the agent never learns about the true state even at zero costs, as there is no preference for early resolution of uncertainty. On the other hand, if investors have preference for early resolution of uncertainty, they will choose to learn for a cost if the cost is small enough.

The decentralization of the social planner problem leads to the usual equilibrium Euler equation

\[
E_t^{s_t} [M_{t+1}R_{i,t+1}] = 1,
\]  

(2.14)

where \( R_{i,t+1} \) is the return on any asset traded in the economy. The expression for the discount factor \( M_{t+1} \) is also standard, safe for an endogenous information set which
depends on the optimal choice indicator \( s^*_t \):

\[
M_{t+1} = \beta \left( \frac{Y_{t+1}}{Y_t} \right)^{-1} \frac{U^{1-\gamma}_{t+1}}{E_t(U^{1-\gamma}_{t+1})}.
\]

\[\text{(2.15)}\]

2.3.3 Income Dynamics

The log income growth rate process incorporates a time-varying mean \( x_t \) and stochastic volatility \( \sigma^2_t \):

\[
\Delta y_{t+1} = \mu + x_t + \sigma_t \eta_{t+1},
\]

\[\text{(2.16)}\]

\[
x_{t+1} = \rho x_t + \varphi_e \sigma_t \epsilon_{t+1},
\]

\[\text{(2.17)}\]

\[
\sigma^2_{t+1} = \sigma^2_0 + \nu(\sigma^2_t - \sigma^2_0) + \sigma_w \sigma_t w_{t+1}.
\]

\[\text{(2.18)}\]

where \( \eta_t, \epsilon_t \) and \( w_t \) are independent standard Normal innovations. Parameters \( \rho \) and \( \nu \) determine the persistence of the mean and variance of the income growth rate, respectively, while \( \varphi_e \) and \( \sigma_w \) govern their scale. The empirical motivation for the time-variation in the conditional moments of the income process comes from the long-run risks literature, see e.g. Bansal and Yaron (2004), Hansen et al. (2008).

We assume that the volatility \( \sigma^2_t \) is known to the agent at time \( t \), which can be justified as the availability of high-frequency data allows for an accurate estimation of the conditional volatility in the economy. On the other hand, the true expected income state \( x_t \) is not directly observable to the investors. The investors can learn about the state from the observed data using standard filtering techniques, and they also have an additional option to pay a cost to learn its true value. This setup is designed to capture the intuition that some of the key aspects of the economy are not directly observable, but the agents can learn more about them through additional costly exploration. The learning costs in this paper are similar to the observation and information costs in rational inattention literature; see Abel, Eberly, and Panageas (2007b), Reis (2006) and Duffie and Sun (1990).
We apply a separation principle and solve a filtering problem before the optimization problem of the agent. Given the setup of the economy, the beginning-of-period information set of the agent consists of the history of income growth, income volatility and observed true states up to time $t$: 

$$I_t = \{y_\tau, \sigma^2_\tau, s_{\tau-1}^x \}_{\tau=1}^t.$$  

If the agent does not learn the true state in period $t$, the end-of-period information set is the same as in the beginning of the period: $I_t(0) = I_t$. On the other hand, if the agent learns the true value of the expected income state, the information set immediately adjusts to include $x_t$: $I_t(1) = I_t \cup x_t$. Define a filtered state $\hat{x}_t(s_t)$, which gives the expectation of the true state $x_t$ given the information set of the agent and the costly learning decision $s_t$:

$$\hat{x}_t(s_t) = E_{s_t}^s(x_t),$$  

and denote $\omega^2_t(s_t)$ the variance of the filtering error which corresponds to the estimate $\hat{x}_t(s_t)$:

$$\omega^2_t(s_t) = E_{s_t}^s(x_t - \hat{x}_t(s_t))^2.$$  

If the agent chooses to learn about the true state, we obtain, naturally, that $\hat{x}_t(1) = x_t$ and $\omega^2_t(1) = 0$.

Given the history of income, income volatility and past observed expected growth states, the agent updates the beliefs about unobserved expected income state in a Kalman filter manner. Indeed, as the income volatility is observable, evolution of the system is conditionally Gaussian, so that the expected mean and variance of the filtering error are the sufficient statistics to track the beliefs of the agent about the economy next period. Specifically, for a given choice indicator $s_t$ today, the evolution of the states in the beginning of the next period follows from the one-step-ahead
innovation representation of the system (2.16)-(2.18):

\[\Delta y_{t+1} = \mu + \hat{x}_t(s_t) + u_{t+1}(s_t), \quad (2.21)\]

\[\hat{x}_{t+1}(0) = \rho \hat{x}_t(s_t) + K_t(s_t)u_{t+1}(s_t), \quad (2.22)\]

\[\omega_{t+1}^2(0) = \sigma_t^2 \left( \phi^2 + \rho^2 \frac{\omega_t^2(s_t)}{\omega_t^2(s_t) + \sigma_t^2} \right), \quad (2.23)\]

where the gain of the filter is equal to

\[K_t(s_t) = \frac{\rho \omega_t(s_t)^2}{\omega_t(s_t)^2 + \sigma_t^2}. \quad (2.24)\]

The filtered consumption innovation \(u_{t+1}(s_t) = \sigma_t \eta_{t+1} + x_t - \hat{x}_t(s_t)\) is learning choice specific, and contains short-run consumption shock and filtering error. The two cannot be separately identified unless the agent learns the true \(x_t\), in which case the filtered consumption innovation is equal to the true consumption shock, \(u_{t+1}(1) = \sigma_t \eta_{t+1}\). The filtered consumption innovation is used to update the estimate of the expected growth \(\hat{x}_{t+1}(0)\), as shown in (2.22). This Kalman filter estimate is known to the agent at the beginning of next period. If investors decide to pay the cost to learn the true \(x_{t+1}\), the expected income and variance of the filtering error are immediately adjusted to reflect the new information. We can then express the values of the states tomorrow in the following way:

\[\hat{x}_{t+1}(s_{t+1}) = s_{t+1}x_{t+1} + (1 - s_{t+1})\hat{x}_{t+1}(0), \quad (2.25)\]

\[\omega_{t+1}^2(s_{t+1}) = (1 - s_{t+1})\omega_{t+1}^2(0). \quad (2.26)\]

Recall that the variance shocks \(w_{t+1}\) are assumed to be independent from the income innovations at all leads and lags. That is, future volatility shocks do not help predict tomorrow’s expected income, and neither can learning about \(x_t\) affect the agent’s beliefs about future volatility. Therefore the dynamics of the income
volatility is independent of the learning choice of the agent and follows (2.18). If income volatility is constant, we obtain a standard Kalman Filter result that the variance of the filtering error $\omega^2_0(0)$ increases in a deterministic fashion since the last costly learning. On the other hand, when income volatility is stochastic, the variance of the filtering error fluctuates and typically increases faster at times of heightened aggregate volatility. Learning models considered by David (1997) and Veronesi (1999) use regime-shift specification for expected growth component and feature alternative time-varying dynamics of the filtering uncertainty.

We specified the evolution of the economy in the beginning of the next period in (2.21)-(2.23), and instantaneous adjustments of the expected income and variance of the filtering error when the agent chooses to learn the true state for a cost in (2.25)-(2.26). The decision to learn is endogenous and is determined as a part of the equilibrium solution of the model, which we discuss in the next section.

2.4 Model Solution

2.4.1 Optimal Costly Learning

The life-time utility of the agent depends on the beginning-of-period information and, at times when the agent chooses to learn about the true state for a cost, on the true value of the expected income growth. As the volatility and consumption shocks are uncorrelated, we can separate the expected growth and volatility components in the equilibrium utility per income ratio, which simplifies the solution to the model. In Appendix B.3 we show that the life-time utility per income ratio can be written in the following way:

$$
\phi_t(s_t) = e^{B\hat{x}_t(s_t)+f(s_t,\sigma^2_t,\omega^2_0(0))},
$$

(2.27)
where the sensitivity of the utility to expected income growth is independent of the costly learning choice and is given by

\[ B = \frac{\beta}{1 - \beta \rho}. \]  

(2.28)

The volatility function \( f(s_t, \sigma_t^2, \omega_t^2(0)) \) depends on the learning choice \( s_t \), the volatility states \( \sigma_t^2 \) and \( \omega_t^2(0) \), as well as risk aversion of the agent \( \gamma \) and learning cost parameter \( \chi \). It satisfies the recursive equation given in the Appendix (B.18).

The agent chooses to observe the true state if the ex-ante life-time utility with learning exceeds the utility with no learning about the true state. Given the equilibrium solution to the life-time utility per income ratio in (2.27), investor’s life-time utility with no learning is

\[ \phi_t(0) = e^{B\hat{x}_t(0) + f(0, \sigma_t^2, \omega_t^2(0))}, \]  

(2.29)

while the certainty equivalent of the life-time utility (per income) with costly learning is

\[ J_t(\phi_t(1)) = e^{B\hat{x}_t(0) + \frac{1}{2}(1-\gamma)B^2\omega_t^2(0) + f_t(1, \sigma_t^2, \omega_t^2(0))}. \]  

(2.30)

In our setup, the agent has recursive preferences when the risk-aversion coefficient \( \gamma \) is different from 1; when \( \gamma = 1 \) preferences collapse to a standard expected log utility case. The incentive to learn the unobserved state for a cost critically depends on the recursive preferences of the agent. Indeed, when \( \gamma = 1 \), the volatility functions \( f(s_t, \sigma_t^2, \omega_t^2(0)) \) are constant and depend only on the learning choice \( s_t \); moreover, the level of life-time utility following a learning choice is smaller than the level of utility without learning for any positive learning cost parameter \( \chi \). Hence, in expected utility case, the agent never learns for a cost. On the other hand, with recursive utility, the volatility functions \( f(s_t, \sigma_t^2, \omega_t^2(0)) \) are time-varying, so that the ex-ante life-time utilities of the agent with and without learning depend on the income volatility and
variance of the filtering error. Learning choice is optimally determined by the relative difference between these utilities. Given our distributional assumptions on the economy, this difference depends only on volatility states $w_t^2(0)$ and $\sigma_t^2$, but not on the expected growth:

$$s_t^* = 1[J_t^0(\phi_t(1)) > \phi_t(0)]$$

$$= 1 \left[ \frac{1}{2}(1 - \gamma)B^2\omega_t^2(0) + f_t(1, \sigma_t^2, \omega_t^2(0)) > f_t(0, \sigma_t^2, \omega_t^2(0)) \right].$$

Hence, the optimal learning choice of the agent is governed by the income volatility, variance of the filtering error and preference and learning cost parameters. In general, solutions for the volatility function $f$ and the optimal learning choice $s_t^*$ are not available in a closed form, so we have to solve for them numerically. In the case when income shocks are homoscedastic, the variance of the filtering error is a deterministic function of time since the last costly learning, so investors optimally learn about the true state at constant time intervals. On the other hand, when income volatility is time-varying, we show that the agent chooses to learn the true state when the filtering variance gets too high relative to the underlying volatility of the income growth, so that the utility benefits from reducing the uncertainty outweigh the learning costs. Further, the frequency of costly learning increases at times of heightened income volatility, as the filtering uncertainty accumulates faster when income volatility rises.

2.4.2 Risk Compensation and Asset Prices

Using the solution to the equilibrium discount factor in (2.15), we can express the equilibrium discount factor in terms of the underlying variables in the economy. The conditional mean of the discount factor is equal to the negative of the expected
income growth plus the contribution of the income and filtering variance:

\[ E_t^{s_t} m_{t+1}(s_t) = \log \beta - \mu - \hat{x}_t(s_t) - \frac{1}{2}(1 - \gamma)^2(BK_t(s_t) + 1)^2(\omega_t^2(s_t) + \sigma_t^2) \]

\[ - (\gamma - 1)E_t^{s_t} f_{t+1} - \ln E_t^{s_t} e^{(1-\gamma)[f_{t+1} + \frac{1}{2}(1 - \gamma)B^2 \omega_{t+1}(0)s_{t+1}^*]} ] . \]  

(2.32)

The innovation into the log discount factor satisfies,

\[ m_{t+1}(s_t) - E_t^{s_t} m_{t+1}(s_t) = -(1 + (\gamma - 1)(1 + BK_t(s_t))) u_{t+1}(s_t) \]

\[ -(\gamma - 1)Bs_{t+1}^*(x_{t+1} - \hat{x}_{t+1}(0)) - (\gamma - 1)(f_{t+1} - E_t^{s_t} f_{t+1})] . \]

(2.33)

If the expected endowment growth factor is observable by the agent, (e.g., the information costs are zero), we obtain a standard long-run risks setup. The price of a short-run consumption risk is then \( \gamma \), and prices of long-run and volatility risks are constant and provided in the above studies.

The option to learning about the true state for a cost changes the sources and prices of risks relative to standard models. In our model, the price of immediate consumption risk \( u_{t+1} \) is time-varying because the expected growth state is unobservable and has to be learned from the data. As the decision to learn about the true state depends on income volatility, the pricing of income volatility shocks is also more complicated, which is reflected in the non-linear volatility function \( f_{t+1} \).

Finally, while in the standard long-run risks model the agent fears the true innovations into the expected growth, \( x_{t+1} - E_t x_{t+1} \), in our model its counterpart is the revision of the state, \( s_{t+1}^*(x_{t+1} - \hat{x}_{t+1}(0)) \). Although both risks have the same market price \( (\gamma - 1)B \), the amount of risk can be much higher in our case as the variance of the filtering error can significantly exceed the conditional volatility of long-run risks shocks. Therefore, the option to learn about the true state can have a considerable effect on the asset valuations in the economy. In an alternative specification with robust control, Hansen and Sargent (2006) show that the concerns of the agents
Table 2.3: **Costly Learning Model Calibration**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>7.27e-05</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.9963</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>8.53e-04</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_w$</td>
<td>5.07e-05</td>
</tr>
<tr>
<td>$\varphi_e$</td>
<td>8.95e-03</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>5</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.9999</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>10</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.080087</td>
</tr>
</tbody>
</table>

about model misspecification can also generate substantial magnification of the risk premia.

Consider an asset with a dividend stream proportional to income growth,

$$
\Delta d_t = \mu + \varphi_d (\Delta y_t - \mu).
$$

(2.34)

Bansal and Yaron (2004) specify dividend dynamics which includes idiosyncratic dividend shock. The specification above is simpler as it does not require extension of the model to multivariate Kalman filter, but preserves model results and intuition.

Using the equilibrium solution to the discount factor in (2.32)-(2.33) and the Euler condition (2.14), we can solve for the equilibrium log price-dividend ratio,

$$
v_t(s_t) = H\hat{x}_t(s_t) + h(s_t, \sigma_t^2, \omega_t^2(0)),
$$

(2.35)

where the solutions for $H$ and $h(s_t, \sigma_t^2, \omega_t^2(0))$ are given in Appendix B.4.

The asset valuations depend on filtered or, if $s_t = 1$, true expected income growth. When investors learn about the true state, the equilibrium price-dividend ratio responds to the the revision in the expected income state magnified by $H$. For example, as $H$ is positive, when the true $x_t$ is lower than what the agent expected, asset prices
Table 2.4: **Endowment Dynamics: Data and Model Calibration**

|              | Data          |          |           |           |           |
|--------------|--------------|----------|-----------|-----------|
|              | Estimate     | S.E.     | Median    | 5%        | 95%       |
| Mean         | 1.95         | (0.32)   | 1.94      | 1.18      | 2.57      |
| Vol          | 2.13         | (0.52)   | 2.17      | 1.81      | 2.60      |
| AR(1)        | 0.44         | (0.13)   | 0.52      | 0.37      | 0.64      |
| AR(2)        | 0.16         | (0.18)   | 0.15      | -0.06     | 0.36      |
| AR(5)        | -0.01        | (0.10)   | -0.02     | -0.27     | 0.17      |

can fall sharply. The probability of costly learning, and consequently, large asset-price moves depends on the volatility states in the economy. In particular, when aggregate volatility is high, investors learn for a cost more often, which triggers more frequent large moves in returns. As the volatility of equilibrium returns also increases in aggregate volatility, the model can thus explain the predictability of future asset-price jumps by macroeconomic and return variance. In Section 2.5 we calibrate the economy and show that the model-implied jump implications are quantitatively consistent with the data.

### 2.5 Model Output

#### 2.5.1 Model Calibration

The model is calibrated on daily frequency. The baseline calibration parameter values, which are reported in Table 2.3, are similar to the ones used in standard long-run risks literature (see e.g. Bansal and Yaron, 2004), safe for an adjustment to daily frequency. Specifically, we set the persistence in the expected income growth \( \rho \) at 0.9963. The choice of \( \varphi_e \) and \( \sigma_0 \) ensures that the model matches the annualized aggregate volatility of about 2%, while the volatility persistence is set to \( \nu = 0.99 \). To calibrate dividend dynamics, we set the leverage parameter of the corporate sector \( \varphi_d \) to 5. We calibrate the model on a daily frequency and then time-aggregate to
annual horizon. Table 2.4 shows that we can successfully match the unconditional mean, volatility, and auto-correlations of the endowment dynamics in the data.

As for the preference parameters, we let the subjective discount factor $\delta$ equal 0.9999 and set the risk aversion parameter at 10. The learning expenditure includes the resources that the investors spend to acquire and process the information about true value of the underlying economic state, which includes opportunity costs of time and effort. We calibrate the cost parameter similar to observation and information costs in rational inattention literature, see Abel, Eberly, and Panageas (2007a, b). At this level of learning costs, investors are willing to optimally learn the true state about once every one and a half year, so that expenditure on costly learning accounts merely for 0.02% of annual aggregate income. The calibration of the learning cost parameter is sensitive to the assumed values of other model parameters, such as risk aversion and level of the volatility shocks. We discuss these issues later in the section.

2.5.2 Constant Volatility Case

First we consider a special case when income volatility is constant, that is, $\sigma_t = \sigma_0$. This provides a useful benchmark case as the numerical solution to the model is greatly simplified. Indeed, in a standard homoscedastic Kalman Filter setup the variance of the filtering error $\omega_t^2$ is a deterministic function of time since the last learning about the true state. Therefore, the optimal learning policy is purely time-dependent, so that the agent chooses to learn every $N$th period, where $N$ depends on the preference and income dynamics parameters of the model. We provide the details of the solution to the model in Appendix B.5.

On Figure 2.5 we show the optimal length of the filtering period $N$ as a function of the cost parameter $\chi$ for risk aversion levels of 5, 10 and 15. When cost of learning increases, the agent chooses to learn the true expected income state less frequently. This decision is also very sensitive to the risk attitude of the investors, so that more
risk-averse investors learn about the true state more often for any value of the learning cost parameter. On the other hand, when the risk-aversion coefficient is less than or equal to 1, the agent will never choose to learn the true state for a cost, as there is no preference for early resolution of uncertainty.

Table 2.1 reports asset-pricing implications of the option to learn for a cost in the model with constant income volatility. Simulated mean and volatility of returns are 7.9% and 15.3%, respectively, which match statistics in the data. On Figure 2.6 we plot a typical simulation of the economy for 80 years. The log income growth is conditionally Normal, and the filtered expected income state closely tracks the true state with a correlation coefficient in excess of 0.7. About every 2 years the agent pays the cost and learns the true state. The revision in expectations about future income growth triggers proportional adjustments to the equilibrium asset prices, as can be seen from equation (2.35). In presence of highly persistent long-run risks shocks, asset prices are very sensitive changes in expected income state. Therefore, even small deviations in the filtered state from the truth, when uncovered, can lead
to large changes in valuations that look like large price moves. Notably, although the number of periods between successive days with costly learning is constant, the years with flagged jumps do not have to occur at regular intervals, as shown on Figure 2.6. Indeed, the jump-detection statistics is designed to pick out only large jumps, hence the significance level of 1%, so that some of the smaller price adjustments remain undetected.

For comparison, the last graph on Figure 2.6 depicts the equilibrium market returns which would obtain in this economy if agent could not learn the true state for a cost and have to exclusively rely on standard Kalman filtering. As can be seen in the second panel of Table 2.1, the specification with no learning cannot deliver large price movements observable before. The jump-detection statistics typically do
not find more than 2 or 3 instances of large price moves in 80 years of simulated
daily data; the detected jumps represent pure-chance large random draws in the
simulation. On the other hand, when agent can learn about the true state for a cost,
the detected jump frequency is about 4.5 years, and the contribution of jumps to the
total return variation is 7.7%. These numbers are consistent with the data (see first
panel of Table 2.1).

Similar conclusions obtain from comparison of the unconditional distributions
of returns. When the agent has no option to learn the true state, the kurtosis of
return distribution is equal to 3, relative to 18 for the returns in the economy with
learning, and 21 for the data. As income volatility is kept constant, the heavy tails
in the return distribution are driven solely by the discrete adjustments to the asset
valuations.

The constant-volatility case can deliver the key result that the equilibrium asset
prices can display infrequent large movements which cannot be explained by standard
Gaussian shocks. However, when income shocks are homoscedastic, the decision
to learn is purely time-dependent, and volatilities of macroeconomic and financial
variables are constant. We can address these issues by opening up stochastic volatility
channel, which we discuss in the next section.

2.5.3 Time-Varying Volatility Case

The asset-pricing implications of costly learning in a time-varying volatility setup
take into account state-dependence and time-variation of the optimal costly learning
rule. Indeed, as both aggregate volatility and filtering variance are now time-varying,
the optimal decision to learn about the true state is stochastic, as shown in income
dynamics simulation on Figure 2.7. We further characterize the dependence of opti-
mal costly learning on filtering and income volatilities on Figure 2.8, which depicts
the expected number of periods till next costly learning given current filtering vari-
For high, medium and low values of aggregate volatility. Investors choose to learn for a cost if the variance of the filtering error grows too high in the economy, so that it is optimal to sacrifice part of the current endowment, pay learning cost and reduce the amount of uncertainty. Costly learning is more frequent in high income volatility states, as in those states the uncertainty about the filtered estimate accumulates faster and is expected to reach the costly learning cut-off point sooner. These actions of investors to learn about the underlying state can lead to large adjustments in daily asset prices, detected as jumps by annual jump-detection statistics, as shown in return simulation on Figure 2.10. Relative to constant volatility case, the detected jump-years are more frequent, averaging one every 4 years, and contribute more to the total variation in returns, 9.5% versus 7.5% in a constant-volatility case and in

Figure 2.7: Income Dynamics in Time-Varying Volatility Case
the data (see Table 2.1). For robustness, on Figure 2.9 we also show the model-implied jump-year frequency for a range of significance levels for the jump-detection test. As the significance level increases, the null of no jumps is rejected more often, so that the frequency of detected jump-years increases. As the Figure shows, the model can broadly match the evidence on the average frequency of jump-years in the data, as all the values are well within the 5% – 95% confidence band. These large moves in returns cannot be obtained in economy without costly learning, as can be visually seen on the time-series plot of returns on Figure 2.10. Without costly learning, the average frequency of detected jump-years is less than 1 in 80 years, and the detected “jumps” are merely pure-chance large random draws. The comparison of the fourth moments of return distribution is revealing: without an option to learn, the kurtosis of market returns is 3, and it reaches 35 when the agent can learn the true state for a cost.

On Figure 2.11 we plot the unconditional distribution of the number of periods between the detected jump-years based on long simulation from the full model with
time-varying consumption volatility. In jump-diffusion models of asset prices with constant arrival intensity of jumps, the number of periods between successive jumps is exponential, so for comparison, we also provide an exponential fit to the jump-year duration distribution in the model. The mean of the fitted exponential distribution is 3.6 years, which agrees with the estimate of the jump-year frequency reported in Table 2.1. While the exponential distribution generally fits the distribution of jump duration, there is evidence for clustering of jumps – the unconditional distribution has heavier left tail than exponential, so a jump-year is likely to follow another. Clustering and predictability of jumps is an important aspect of our model, which we discuss in the next section.

2.5.4 Predictability of Jumps

In the model with time-varying consumption volatility, the frequency of learning and consequently, the likelihood of price jumps, is increasing with aggregate volatility, so that returns jumps are more frequent at times of high aggregate volatility. As
discussed before in Section 2.2, the predictability of return jumps by the aggregate
volatility is an important feature of the data, and our model can capture this ef-
flect. Furthermore, as the aggregate volatility also drives the variation in equilibrium
market returns, our model can provide an economic explanation for the predictabil-
ity of large asset-price moves by the variance of returns in the data. Finally, as in
the data, the levels of income does not predict future return jumps, as the optimal
learning choice depends only on the income volatility and variance of filtering error.
This highlights an important aspect of the model and the data that the second mo-
ments are critical to forecast future jumps, while the movements in the level are not
informative about future jumps in returns.

The model can quantitatively reproduce the key features of predictability of re-

Figure 2.10: Return Dynamics in Time-Varying Volatility Case
turn jumps by consumption and market variance, and absence of predictability of future jumps by the level of consumption. On Figure 2.12 we show model-implied lead-lag correlations of jump indicator with endowment growth and conditional variance of endowment growth and returns at monthly frequency, constructed in the same way as the empirical counterparts on Figure 2.3. The model with costly learning delivers positive and significant correlation of large return move indicator with endowment and return variances and zero correlation with endowment growth. The magnitudes of the correlation coefficients are comparable to the data. As shown on the bottom panel of Figure 2.12, the model with no costly learning cannot account for the predictability of return jumps in the data, as all the correlation coefficients are zero. Results on quarterly and annual frequencies are very similar, and are omitted in the interest of space.

The predictability of large moves in returns that our model is able to capture is consistent with the evidence from parametric models for asset prices, which feature
stochastic volatility and jumps in returns whose arrival intensity is increasing in market variance; see examples in Bates (2000), Pan (2002), Eraker (2004a) and Singleton (2006). To further compare our model implications to the results from the parametric studies of return dynamics, we fit a discrete-time GARCH-jump specification for returns, which feature autoregressive stochastic volatility and time-varying arrival intensity of jumps in returns. In Appendix B.6 we discuss the estimation results in the data and full model. The model can match quite well the dynamics of the time-varying volatility of returns, as well as the key findings in the literature regarding the frequency and predictability of jumps. In particular, the frequency of jumps is
positive and highly significant in the data and the model. The jumps explain 10% of the variance of returns in the model and in the data, and the estimated frequency of jumps is one every one and two years, respectively. Thus, the model can account for the key features of the conditional distribution of returns in the data, so it can serve as an economic basis for realistic reduced-form models of asset prices which incorporate time-varying volatility and jump components.

2.6 Conclusion

We present a general equilibrium model which features smooth Gaussian dynamics of income and dividends and large infrequent movements in asset prices (jumps). The large moves in asset prices are triggered by the optimal actions of investors to learn the unobserved expected growth. We show that the optimal decision to learn the true state is stochastic and depends on the time-varying volatility of income growth and the variance of the filtering error, as well as the preference parameters. The revisions in the expected income due to costly learning lead to large moves in asset valuations which look like jumps. These large price moves cannot be obtained in the economy without costly learning of the true state, or in the economy with standard expected utility.

A prominent feature of the model is that the frequency of costly learning, and consequently, the likelihood of asset price jumps, increases in the income volatility in the economy, so that returns jumps are more frequent at times of high aggregate volatility. We show that predictability of returns jumps by consumption variance is an important and novel aspect of the data. Furthermore, the model can provide an economic explanation for the predictability of large asset-price moves by the variance of returns, and lack of return jump predictability by the levels of income in the data. This highlights an important aspect of the model and the data that the second
moments are critical to forecast future jumps, while the movements in the level are not informative about future jumps in returns.

Using calibrations, we find that the model can quantitatively reproduce the key features of predictability of return jumps by consumption and market variance, and absence of predictability of future jumps by the level of consumption. In addition, the model can account for the frequency and magnitude of price jumps in the data, fat-tail distribution of market returns, equity premium, and other asset-pricing features. We argue that our structural model can serve as an economic basis for realistic reduced-form models of asset prices which incorporate time-varying volatility and jump components.
Learning, Confidence and Option Prices

3.1 Introduction

Option markets are important, as they can provide significant information about the risks that investors perceive in financial markets. One of the central issues in option data is that deep out-of-the-money index put options appear overpriced, so that the insurance for large downward movements in the stock market is too expensive relative to standard models (see e.g. Rubinstein, 1994). Equally puzzling are the substantial variation and large moves in the option prices. These findings indicate that investors are concerned with large, negative moves in underlying prices, which occur approximately once every 18 months in the data. However, there is no persuasive evidence in the data for large contemporaneous moves in the real economy at the considered frequencies, which presents a challenge for an economic explanation of option markets. In this paper, I show that fluctuating confidence of investors about unobserved expected growth can quantitatively explain asset-price

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anomalies in derivative markets and account for the observed large moves in returns, while keeping fundamental consumption dynamics smooth Gaussian. Based on two estimation approaches, I find that the model with learning and confidence jump risks delivers plausible preference and model parameters and provides a good fit to option prices, investors’ confidence, returns, and consumption in the data.

Earlier structural works which address issues in option markets typically introduce jumps into the fundamental consumption process: see Eraker and Shaliastovich (2008), Drechsler and Yaron (2008), Santa-Clara and Yan (2008), Gabaix (2007), Bates (2006), Benzoni, Collin-Dufresne, and Goldstein (2005), Liu et al. (2005). In this paper, I do not entertain the possibility of jumps in consumption, and instead show that learning and fluctuating confidence about expected growth can account for the key features of option and equity data. The economy setup closely follows Bansal and Shaliastovich (2008) and, as in standard long-run risks model of Bansal and Yaron (2004), features Gaussian dynamics of true consumption growth with a persistent expected growth component, time-varying consumption volatility, and the recursive utility of Epstein and Zin (1989) and Weil (1989). However, unlike the standard long-run risks model, expected growth is not directly observable, and investors learn about it using a cross-section of signals. The quality of signals determines the uncertainty of investors about their estimate of expected growth. This uncertainty, referred to as ”confidence measure,” is time-varying and contains large positive shocks. Due to imperfect information and learning, the confidence measure affects the beliefs of investors about future consumption and impacts equilibrium asset prices in the economy.

As in a standard long-run risks specification, investors in the model demand compensation for short-run, long-run and consumption volatility risks. The novel contribution of the model is that the confidence risks are also priced in equilibrium,
so that when agents have a preference for early resolution of uncertainty, states with higher uncertainty about expected growth are discounted more heavily. Notably, the confidence jump shocks receive risk compensation although there are no jump risks in consumption. Learning and confidence jump risk channels can explain the key features of option price data. Out-of-the-money put options hedge jump risks in the confidence measure and thus appear expensive relative to models with no jump risks. This can account for the smirk pattern in option prices, where Black-Scholes implied volatilities are decreasing in the strike price of the contract. Further, endogenous jumps in equilibrium prices due to positive jumps in uncertainty about future growth can account for large downward moves in asset prices, and a negatively skewed and heavy-tailed unconditional distribution of returns.

The key economic mechanism in this paper, such as learning about expected growth, is featured in a number of asset-pricing models. In the class of learning models considered by David (1997), Veronesi (1999) and Ai (2007), the unobserved drift is modeled as a regime-shift process, so that investor’s uncertainty about the estimate is stochastic and is related to fundamental shocks in the economy. David and Veronesi (2002) show that this channel endogenously generates a correlation between equilibrium returns and return volatility which can explain time-variation in option-implied volatility and the skewness and kurtosis premium in option prices. The model of Buraschi and Jitsov (2006) features heterogeneous agents and learning about the dividend growth rate and can explain option prices and the dynamics of option volume. Alternative learning models are presented in Hansen and Sargent (2006), who specify model-selection rules which capture investors’ concerns about robustness and potential model misspecification, and Piazzesi and Schneider (2007), who use survey data to characterize and study the subjective beliefs of agents in the

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economy. Relative to the models in the literature, the novel dimension of this paper is the time-variation in the quality of signals available to the investors and the ensuing confidence jump risks in asset markets. Fluctuations in the confidence measure are consistent with theoretical model of Veldkamp (2006) and Van Nieuwerburgh and Veldkamp (2006), where information flow about the unobserved economic state endogenously varies with the level of economic activity.

The main target in this paper is to quantitatively explain option pricing puzzles and at the same time account for the key dimensions of consumption, returns and the confidence measure in the data. I use the cross-section of forecasts of next-quarter GDP from the Survey of Professional Forecasters and construct the empirical confidence measure as the variance of the average forecast, consistent with the theoretical specification. I show that in the data, the confidence measure contains significant information about Black-Scholes volatilities in the option market. The option volatilities across all strikes and maturities are about 7% higher in quarters when uncertainty is high, relative to quarters when uncertainty about future growth is low. Further, in projections of option-implied volatilities 2 and 3 quarters ahead, the slope coefficient on the confidence measure is large and significant at all strikes, while the slope coefficient on the current value of option volatility is small and is typically decreasing with the horizon. In addition, the empirical confidence measure exhibits large positive moves, whose frequencies and magnitudes are plausible to account for the jump features of option and asset market data. Indeed, using formal econometric analysis, Bansal and Shaliastovich (2008) find significant evidence for jump-like shocks in the confidence measure. The large moves in the confidence measure are related to large moves in returns and in the variance of returns. On the other hand, there is no persuasive evidence in the data for the link between large moves in returns and large moves in the real economy at the considered frequencies.
I use two econometric approaches to estimate and test the model. For GMM estimation, I consider moments of the confidence measure and equity returns, which characterize non-Gaussian features of the distribution, as well as the information in interest rates and option-implied volatilities in the data. I also employ the latent-factor MLE approach, where I treat the confidence measure as well as consumption volatility and expected growth state as latent factors and back them out from the option, return and consumption data, similar to Duffie and Singleton (1997), Pan (2002) and Santa-Clara and Yan (2008). The quantitative implications from the two estimation approaches are very similar and provide empirical support for the long-run risks model with learning, fluctuating investors’ confidence and jump-like confidence risks. I obtain plausible preference parameters, which indicate that investors have a preference for early resolution of uncertainty. The estimated model parameters suggest that the confidence measure significantly fluctuates over time; moreover, nearly all the variation in the series is driven by Poisson jumps. Large moves in uncertainty about future growth can quantitatively explain over-pricing of out-of-the-money put options and produce an implied volatility curve comparable to the data. Using the backed out confidence measure and consumption volatility states from the MLE estimation, I show that these states account for more than 95% of the total variation in option volatilities. Due to jumps, shocks in the confidence measure are more important for out-of-the-money and longer maturities contracts.

Based on GMM estimation, the estimated frequency of jumps in asset prices, driven endogenously by jumps in the confidence measure, is one every 5 months, and the average jump in returns is $-3.3\%$, monthly. Using MLE estimates, the frequency of large moves in returns is about once every 9 months, while average jump in return is $-7.5\%$, monthly. The frequency of moves in returns of such magnitude in the data is consistent with the model; for example, in my sample monthly returns fall
below the cutoff of $-3.3\%$ once every 6 months. Confidence jump risks contribute about 2% to the total equity premium of 6%, while expected growth shocks account for 3%. The estimates of the jump risk premium in the economy is consistent with Pan (2002) and Broadie et al. (2007), who find that jump risks account for about one-third of the total equity premium in the economy.

Based on GMM estimation, the model with confidence jumps is not rejected in the data, with a $p-$value of 0.3. The in-sample and out-of-sample tests suggest that the model can account for the cross-section of option prices and distribution of the confidence measure and returns in the data. The dynamics of consumption and confidence measure from the two estimations are consistent with features of the data based on a long historical sample. On the other hand, the restricted model with no jump risks in the confidence measure is rejected both in sample and out of sample, as it fails to capture the over-pricing of out-of-the-money put options and non-Gaussian features of returns and confidence measure in the data. Overall, the empirical results strongly indicate that the confidence jumps risk is a key channel to empirically explain option and equity prices in the data without introducing jumps into the fundamental consumption process.

Earlier structural models which aim to explain option prices and large moves in returns typically hardwire jumps into consumption fundamentals. Eraker and Shaliastovich (2008) show that when investors have preference for the timing of resolution of uncertainty, jumps in consumption fundamentals are priced in equilibrium and affect asset valuations and returns. In particular, positive jumps in consumption volatility endogenously translate into negative jumps in equilibrium prices, which can capture the shape of the implied volatility curve in option prices. Benzoni et al. (2005) consider jumps in expected consumption, which they show can also rationalize the volatility smirk observed in the data. Eraker (2007) and Drechsler and
Yaron (2008) further argue that jumps in conditional moments of consumption can account for some key features of the risk-neutral variance of returns implied by the cross-section of option prices in the data. In a related literature, Liu et al. (2005) introduce rare jumps into the endowment dynamics and argue that concerns for model uncertainty can explain the over-pricing of out-of-the money puts and the smirk pattern of option prices in the data. This implied volatility pattern can also be generated in a rare disaster model with a time-varying probability of a crash, as discussed by Gabaix (2007). In a similar vein, Santa-Clara and Yan (2008) estimate risks of investors implied from the option markets and argue for substantial Peso issues in measuring jumps from the stock market data alone. Bates (2006) studies the equilibrium implications of the model which features jump news in dividends and crash-averse investors with heterogeneous attitudes towards crash risk. In an earlier study, Naik and Lee (1990) analyze general-equilibrium option pricing when the underlying dividend follows a jump-diffusion process. Relative to the above literature, I do not entertain the possibility of jumps in consumption; rather, I show that learning and fluctuating confidence of investors about expected growth can account for the empirical jump evidence in option and equity data.

Other approaches which incorporate learning and option prices include Campbell and Li (1999), who consider learning about volatility regimes, and Guidolin and Timmermann (2003), who study Bayesian learning implications for option pricing in context of the equilibrium model. A number of papers highlight the importance of information in option prices to learn about the risks in financial markets. The empirical evidence presented in Bollerslev, Tauchen, and Zhou (2008), Todorov (2007), Buraschi and Jackwerth (2001), Bakshi and Kapadia (2003), as well as from parametric models of asset prices, suggest that the risk premia in options cannot be explained by compensation for diffusive stock market risk alone. A number of pa-
pers also use option market data to study the characteristics of investor preferences; these works include Brown and Jackwerth (2004), Bondarenko (2003), Garcia, Luger, and Renault (2003), Ait-Sahalia and Lo (2000), Jackwerth (2000).

The rest of the paper is organized as follows. In the next section I set up the model and discuss preferences of the representative agent and dynamics of the economy given the information set of investors. Solutions to the discount factor and asset and option prices are shown in Section 3.3. Section 3.4 describes the data and empirical evidence on the option pricing puzzles. I present GMM estimation results and implications to option prices and equity premium in Section 3.5, while the MLE estimation is discussed in Section 3.6, followed by the Conclusion.

3.2 Model Setup

3.2.1 Preferences

I consider a discrete-time real endowment economy. Investor’s preferences over the uncertain consumption stream \( C_t \) are described by the Kreps and Porteus (1978) recursive utility function of Epstein and Zin (1989):

\[
U_t = \left\{ (1 - \delta)C_t^{\frac{1-\gamma}{\psi}} + \delta(E_t[U_{t+1}^{1-\gamma}])^{1/\theta} \right\}^{\frac{\theta}{1-\gamma}}, \tag{3.1}
\]

where \( \gamma \) is a measure of a local risk aversion of the agent, \( \psi \) is the intertemporal elasticity of substitution and \( \delta \in (0, 1) \) is the subjective discount factor. The conditional expectation is taken with respect to date-\( t \) information set of the agent, which is discussed later in the paper. For notational simplicity, parameter \( \theta \) is defined as

\[
\theta = \frac{1 - \gamma}{1 - \frac{1}{\psi}}. \tag{3.2}
\]

When \( \theta = 1 \), that is, \( \gamma = 1/\psi \), the above recursive preferences collapse to standard expected utility. As is pointed out in Epstein and Zin (1989), in this case the agent is
indifferent to the timing of resolution of uncertainty in the consumption path. When risk aversion exceeds the reciprocal of the intertemporal elasticity of substitution, investors prefer early resolution of uncertainty; otherwise they prefer late resolution of uncertainty. Preference for the timing of the resolution of uncertainty has important implications for risk channels and equilibrium asset-prices in the economy. In the long-run risk model agents prefer early resolution of uncertainty in the consumption path.

As shown in Epstein and Zin (1989), the logarithm of the intertemporal marginal rate of substitution for these preferences is given by

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

(3.3)

where $\Delta c_{t+1} = \log(C_{t+1}/C_t)$ is the log growth rate of aggregate consumption and $r_{c,t+1}$ is the log of the return (i.e., continuous return) on an asset which delivers aggregate consumption as its dividends. This return is not observable in the data. It is different from the observed return on the market portfolio as the levels of market dividends and consumption are not equal; aggregate consumption is much larger than aggregate dividends. To solve the model, I assume an exogenous process for consumption growth and use a standard asset pricing restriction

$$E_t[\exp(m_{t+1} + r_{t+1})] = 1,$$

(3.4)

which holds for any log return $r_{t+1} = \log(R_{t+1})$ to calculate asset prices in the economy.

The dynamics of the real economy and agent’s information set is described in the next sections.
3.2.2 Real Economy

Following Bansal and Yaron (2004), the true dynamics for log consumption growth $\Delta c_{t+1}$ incorporates a time-varying mean $x_t$ and stochastic volatility $\sigma_t^2$:

\[
\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_t, \\
x_{t+1} = \rho x_t + \varphi_e \sigma_t \epsilon_{t+1}, \\
\sigma_{t+1}^2 = \sigma_t^2 + \nu_c (\sigma_t^2 - \sigma^2) + \varphi_w \sigma_t \nu_{c,t+1},
\]

where $\eta_t$, $\epsilon_t$, and $w_{c,t+1}$ are independent standard Normal shocks which capture short-run, long-run and volatility risks in consumption, respectively. Parameters $\rho$ and $\nu_c$ determine the persistence of the conditional mean and variance of the consumption growth rate, while $\varphi_e$ and $\varphi_w$ govern their scale. The empirical motivation for the time-variation in the conditional moments of the consumption process comes from the long-run risks literature, see e.g. Bansal and Yaron (2004), Hansen et al. (2008) and Bansal et al. (2007b).

As in Bansal and Shaliastovich (2008), I assume that the agent knows the structure and parameters of the model and can observe consumption volatility $\sigma_t^2$. However, the true expected growth factor $x_t$ is not directly observable and has to be inferred from the data, which includes the history of consumption and a cross-section of signals about future growth. These signals, together with consumption data, provide all the information about the expected growth state in the economy.

Specifically, I assume that agents receive $n$ signals about the expected growth $x_{i,t}$, for $i = 1, 2, \ldots n$. Each signal deviates from the true state $x_t$ by a random noise $\xi_{i,t}$,

\[
x_{i,t} = x_t + \xi_{i,t},
\]

where the errors $\xi_{i,t}$ are randomly drawn from a Normal distribution and are uncorrelated with fundamental shocks in the economy.
The date-\(t\) imprecision in signal \(i\) is captured by \(V_{i,t}\):

\[\xi_{i,t} \sim N(0, V_{i,t}).\]  

(3.9)

In general, the imprecision in the signal can be different across signals and can vary across time, hence the subscripts \(i\) and \(t\). For simplicity, I further assume that all the signals are ex-ante identical, so that at each date \(t\) the uncertainty in each signal is the same and denote \(V_{0,t} \equiv V_{i,t}\) for all \(i\).

As all the signals come from the same distribution and are ex-ante equally informative, the investor assigns the same weight to each of them. That is, in the end the average signal is a sufficient statistic for the cross-section of all the individual ones. Define the average signal \(\bar{x}_t\), which corresponds to the sample average of the individual signals. Then, using (3.8),

\[\bar{x}_t \equiv \frac{1}{n} \sum_{i=1}^{n} x_{i,t} = x_t + \xi_t,\]  

(3.10)

where the cross-sectional uncertainty in the average signal \(V_t\) and the average signal error are given by

\[V_t = \frac{1}{n} V_{0,t}, \quad \xi_t = \frac{1}{n} \sum_{i=1}^{n} \xi_{i,t},\]  

(3.11)

so that

\[\xi_t \sim N(0, V_t).\]  

(3.12)

The uncertainty \(V_t\) determines the confidence of investors about their estimate of expected growth; as in Bansal and Shaliastovich (2008), I also refer to it as the confidence measure. In the model, the confidence measure is assumed to be observable to investors. It can be estimated in the data from the cross-section of individual signals. Indeed, the signal equation (3.8) implies that

\[E \left( \frac{1}{n-1} \sum_{i=1}^{n} (x_{i,t} - \bar{x}_t)^2 \right) = E \left( \frac{1}{n-1} \sum_{i=1}^{n} (\xi_{i,t} - \xi_t)^2 \right) = V_{0,t},\]  

(3.13)
so that the cross-sectional variance of the signals adjusted by the number of signals $n$ can provide an estimate of the confidence measure $V_t$ in the data.

The confidence measure in the model captures the uncertainty that the agents have about their estimate of future growth. The variation in the confidence measure across time reflects the fluctuations in the quality of information in the economy, so that at times when information is poor, signals are less precise and the uncertainty is high ($V_t$ increases). The time-variation in the confidence measure and ensuing confidence risks are the novel contribution of the model.


3.2.3 Confidence Measure Dynamics

As discussed in the previous section, the confidence measure $V_t$ reflects the uncertainty of investors about future expected growth. The specification of the confidence measure is a key ingredient of our model. The key features of the confidence measure, such as fluctuations and large positive moves, are motivated by the theoretical literature on this issue and the empirical work. In terms of theoretical work, Veldkamp (2006) and Van Nieuwerburgh and Veldkamp (2006) present a model with endogenous learning, which features large discrete moves in the information about future economy. These moves are broadly consistent with the model specification of the confidence dynamics. The large, discrete moves in investors’ uncertainty about future economy also obtain in the costly learning models due to lumpy information. Finally, Bansal and Shaliastovich (2008) discuss the empirical support for fluctuations and large moves in the confidence measure in the data; further details are provided in Section 3.4.2.
Based on these considerations, I follow Bansal and Shaliastovich (2008) and I set-up a discrete-time jump-diffusion model for the confidence measure, which features persistence and both Gaussian and jump-like innovations:

$$V_{t+1} = \sigma_v^2 + \nu(V_t - \sigma_v^2) + \sigma_w \sqrt{V_t} w_{t+1} + Q_{t+1}. \quad (3.14)$$

The parameters $\sigma_v^2$ is the mean value of $V_t$, $\nu$ captures its persistence while $\sigma_w$ determines the volatility of the smooth Gaussian shock $w_{t+1}$. The non-Gaussian innovation in the confidence process is denoted by $Q_{t+1}$. I model it as a compound Poisson jump,

$$Q_{t+1} = \sum_{i=1}^{N_{t+1}} J_{i,t+1} - \mu_j \lambda_t, \quad (3.15)$$

where $N_{t+1}$ is the Poisson process, whose intensity $\lambda_t \equiv E_t N_{t+1}$ corresponds to the probability of having one jump in the continuous-time model, while $J_{i,t+1}$ determines the distribution of the size of the jump. Parameter $\mu_j$ is the unconditional mean of jump size, so subtracting $\mu_j \lambda_t$ on the right-hand side of the above equation ensures that the jump innovation $Q_{t+1}$ is conditionally mean zero\(^3\). In estimation of the model, I consider an exponential distribution for jumps, which is convenient as it is fully described by a single parameter $\mu_j$.

To capture the dependence of jump probability on the level of the confidence measure, I assume that the arrival intensity $\lambda_t$ is linear in $V_t$,

$$\lambda = \lambda_0 + \lambda_1 V_t. \quad (3.16)$$

When $\lambda_1 > 0$, the probability of jumps increases in the level of the confidence measure, so jumps are more frequent when the uncertainty about expected growth is high.

\(^3\) Indeed,

$$E_t(Q_{t+1}) = E_t(E_t(Q_{t+1} | N_{t+1})) = E_t(\mu_j N_{t+1}) - \mu_j \lambda_t = 0.$$
This specification of the time-series evolution of the uncertainty about future growth is very similar to the models for the variance process in continuous time considered in Eraker (2004b), Broadie et al. (2007) and Eraker and Shaliastovich (2008).

### 3.2.4 Filtering Dynamics

At each point in time, the agent estimates expected consumption growth given the information set $\mathcal{I}_t$, which includes the history of consumption, consumption volatility, signals and confidence measure:

$$\mathcal{I}_t = \left\{ \Delta c_{t-j}, \sigma^2_{t-j}, \{x_{i,t-j}\}_{i=1,2,...}, V_{t-j}\right\}_{j=0,1,...}. \quad (3.17)$$

Let $\hat{x}_t$ stand for investors’ estimate of the expected growth,

$$\hat{x}_t = E(x_t | \mathcal{I}_t), \quad (3.18)$$

and denote $\omega^2_t$ the variance of the filtering error which corresponds to the estimate $\hat{x}_t$:

$$\omega^2_t = E \left( (x_t - \hat{x}_t)^2 | \mathcal{I}_t \right). \quad (3.19)$$

Appendix C.1 shows that the filtering problem of the agent has a one-step ahead innovation representation, where the expectations about future growth are updated using the observed consumption and average signal data. The optimal weights given to consumption and signal news are time-varying and reflect the relative quality of consumption and signal information, that is, $\sigma^2_t$ versus $V_t$. In general, solutions to the optimal signal $\hat{x}_t$ and filtering uncertainty $\omega^2_t$ are complicated non-linear functions of the whole history of consumption and signal data. To simplify the solution to the model, I follow Bansal and Shaliastovich (2008) and consider an approximate specification where the Kalman Filter weight on consumption news is 0, and that on
the signal news is set to a constant steady-state value: a positive number slightly less than 1. This approximation is exact in a complete information case when the average signal perfectly reveals the true state, that is, when $V_t = 0$. The approximation is very accurate when the uncertainty in the average signal is much smaller than the consumption variance. I verify that at the considered model parameter values the time-series correlation of the filtered expected growth states from the approximate and exact Kalman Filter specification is in excess of 0.99, and utility losses from the considered approximate setup are small.

The approximate solution to the agents’ filtering problem implies that the evolution of the economy given the information of the agent is given by,

$$
\Delta c_{t+1} = \mu + \hat{x}_t + a_{c,t+1},
$$

$$
\bar{x}_{t+1} = \rho \hat{x}_t + a_{x,t+1},
$$

$$
\hat{x}_{t+1} = \rho \hat{x}_t + K_2 a_{x,t+1}.
$$

(3.20) (3.21) (3.22)

The immediate filtered consumption innovations are given by $a_{c,t+1}$, while $a_{x,t+1}$ denotes the filtered news about the average signal. As shown in (3.22), the agents update their expectations about the true expected growth based on the filtered news about the average signal $a_{x,t+1}$, so that the estimate of the expected state can also be written as a weighted average of the expected value of the state as of last period and current average signal:

$$
\hat{x}_{t+1} = (1 - K^2) \rho \hat{x}_t + K_2 \bar{x}_{t+1}.
$$

(3.23)

The weight on the average signal news $K_2$ is constant and is given by the steady-state solution to the Kalman Filter problem of the agent (see Appendix C.1).

Investor’s uncertainty about the estimate of expected growth $\omega_t^2$ is directly related to the confidence measure from the cross-section of signals:

$$
\omega_t^2 = K_2 V_t.
$$

(3.24)
If uncertainty about future growth is constant, a standard Kalman Filter result obtains that the steady-state variance of the filtering error is constant. On the other hand, when investors’ confidence measure is stochastic, the variance of the filtering error fluctuates one-to-one with the uncertainty about future growth. Learning models considered by David (1997) and Veronesi (1999) use regime-shift specification for expected growth component and feature alternative time-varying dynamics of the filtering uncertainty.

The innovations into consumption and average signal contain fundamental short and long-run consumption shocks and filtering errors; in general, the three cannot be separately identified based on the information set of the agent:

\[ a_{c,t+1} = x_t - \hat{x}_t + \sigma_t \eta_{t+1}, \]
\[ a_{x,t+1} = \rho(x_t - \hat{x}_t) + \varphi \sigma_t \epsilon_{t+1} + (\bar{x}_{t+1} - x_t). \]  

(3.25)

In a complete information setting, investors observe the true expected growth, so the two innovations above collapse to standard short-run and long-run consumption shocks. On the other hand, with imperfect information, the confidence of investors about their estimate of expected growth affects their beliefs about the distribution of future consumption. Even if the fundamental consumption volatility is constant, the variance of consumption growth tomorrow given the available information of investors is time-varying due to the variation in the precision of the signals, and lower confidence of investors (high \( V_t \)) implies higher uncertainty about future consumption.

The equations (3.20) - (3.22), together with the time-series model for the confidence measure in (3.14) and aggregate consumption volatility in (3.7) fully describe the evolution of the economy given agent’s period-\( t \) information. In the next section, I incorporate preferences and solve the equilibrium asset prices in the economy.
3.3 Model Solution

3.3.1 Discount Factor

To solve the model, I first use the dynamics of the economy given the information set of the agent and Euler equation (3.4) to calculate the price of the consumption claim. The equilibrium price-consumption ratio is linear in the expected growth state, aggregate consumption volatility, and the confidence level of the investors:

\[ pc_t = B_0 + B_x \hat{x}_t + B_v V_t + B_\sigma \sigma_t^2, \]  

(3.26)

where the expressions for the loadings are provided in Appendix C.

The loading \( B_x \) measures the sensitivity of the price-consumption ratio to expected growth. It is positive for \( \psi > 1 \), so that when the substitution effect dominates the income effect, prices rise following positive news about expected consumption, similar to a standard long-run risks model. The loadings \( B_v \) and \( B_\sigma \) capture the effects of the confidence measure and consumption volatility on asset valuations. When the agent has a preference for early resolution of uncertainty (\( \gamma > 1/\psi \)), these loadings are negative. In this case, lack of confidence about the expected growth state and high aggregate uncertainty decrease equilibrium asset valuations and the utility of the agent.

The relative magnitudes of the loadings of the price-consumption ratio on the aggregate volatility and confidence measure depend on the quality of signal information about expected growth. In the complete information case, the true expected state is known and the consumption volatility factor \( \sigma_t^2 \) alone determines the conditional variation of short-run and long-run consumption shocks. On the other hand, with learning, the volatilities of these shocks are now driven by two factors, \( \sigma_t^2 \) and \( V_t \) (see equation (3.25)), so that the volatility channel is now represented by consumption volatility and confidence measure states. This reduces the price of consumption
volatility risks and the risk compensation for consumption volatility shocks relative to the complete information case.

Using the equilibrium solution to the consumption asset, I can express the discount factor in (3.3) in terms of the underlying states and shocks in the economy. The equilibrium solution to the discount factor and the Euler equation (3.4) can then be used to directly obtain equity, bond and option prices in the economy. In equilibrium, the log discount factor is equal to,

\[ m_{t+1} = m_0 + m_x \hat{x}_t + m_v V_t + m_{\sigma} \sigma_t \]

\[ - \gamma a_{c, t+1} - \lambda_x K_2 a_{x, t+1} - \lambda_v \left( \sigma_w \sqrt{V_t} w_{t+1} + Q_{t+1} \right) - \lambda_{\sigma} \varphi_w \sigma_t w_{c, t+1}, \]

where the expressions for the discount factor loadings and prices of risks are pinned down by the model and preference parameters of the investors. Their expressions are provided in Appendix C.

Innovations in the discount factor determine the risks that investors face in the economy. As in standard long-run risks model with complete information, short-run, long-run and consumption volatility risks are priced. The novel dimension of the model is that the confidence shocks also receive risk compensation; in particular, the confidence jump risks \( Q_{t+1} \) are priced even though there are no jumps in fundamental consumption. Due to learning, the magnitudes of risk prices change relative to a standard model. As investors cannot observe the true long-run risks shocks, the price of long-run risk decreases, while the price of short-run consumption risk increases relative to complete information; this is consistent with Croce et al. (2006). In addition, the risk compensation for consumption volatility shocks also decreases relative to a standard long-run risks model.

Using the solution for the discount factor, I can derive the expressions for the equilibrium risk-free rates in the economy. Real interest rates with \( n \) periods to maturity are linear in the expected growth state, investors’ confidence and consumption
variance:

\[ r_{ft,n} = -F_{0,n} - F_{x,n} \hat{x}_t - F_{v,n} V_t - F_{\sigma,n} \sigma_t^2. \]  

(3.28)

where the bond coefficients are given in the Appendix C. In particular, real yields increase in the expected growth state, and decrease with positive shocks to the confidence measure.

### 3.3.2 Risk-Neutral Probability

The evolution of the consumption process in (3.20)-(3.22), the confidence measure in (3.14) and the consumption volatility in (3.7) is specified under the objective probability measure in the data. The economy dynamics can also be written under the risk-neutral probability, which is characterized by the condition that the price of any payoff \( R_{t+1} \) can be computed by taking the expectation of its payoff under the risk-neutral measure discounted by the risk-free rate:

\[ E_t(M_{t+1}R_{t+1}) = e^{-rf_t} E^q_t R_{t+1}, \]  

(3.29)

where \( E^q_t \) refers to the expectation of the payoff tomorrow under the risk-neutral measure.

Given the solution to the discount factor, the dynamics of the states under the risk-neutral measure is given by,

\[ \Delta c_{t+1} = \mu + \hat{x}_t - \gamma \text{Var}_{t+1}(a_{c,t+1}) - \lambda_x \text{Cov}_{t+1}(a_{c,t+1}, K_2 a_{x,t+1}) + a^q_{c,t+1}, \]  

(3.30)

\[ \hat{x}_{t+1} = \rho \hat{x}_t - \lambda_x \text{Var}_{t+1}(K_2 a_{x,t+1}) - \gamma \text{Cov}_{t+1}(a_{c,t+1}, K_2 a_{x,t+1}) + K_2 a^q_{x,t+1}, \]  

(3.31)

\[ \sigma_{t+1}^2 = \sigma_v^2 + \nu_c (\sigma_t^2 - \sigma_v^2) - \lambda_\sigma \varphi_w^2 \sigma_t^2 + \varphi_w \sigma_t w^q_{c,t+1}, \]  

(3.32)

\[ V_{t+1} = \sigma_v^2 + \nu (V_t - \sigma_v^2) - \lambda_v \sigma_w^2 V_t + \sigma_w \sqrt{V_t} w^q_{v,t+1} + Q^q_{t+1}. \]  

(3.33)

The risk-neutral transformation of the probability measure is standard and reflects risk compensation for the underlying shocks in the economy. The drifts of
consumption growth and expected consumption in (3.30) and (3.31) are adjusted by the risk prices multiplied by the variance-covariance of the corresponding shocks \( a_{c,t+1}^q \) and \( a_{x,t+1}^q \), while the conditionally Gaussian distributions of these shocks are unchanged. Further, under the objective measure, expected growth shocks \( a_{x,t+1}^q \) are Gaussian given current volatility states and next-period confidence measure \( V_{t+1} \):

\[
Var_{t+1}(a_{x,t+1}^q | \mathcal{I}_t, V_{t+1}) = \rho^2 K_2 V_t + \varphi^2 \sigma_t^2 + V_{t+1},
\]

so \( Var_{t+1}(a_{x,t+1}^q) \) depends on \( V_{t+1} \) (see Appendix C.1). Then, as is evident from expression (3.31), the total innovation into expected growth under the risk-neutral measure incorporates confidence shocks in \( V_{t+1} \). Hence, under the risk-neutral measure investors’ estimate of expected growth state exhibits large moves, as positive jumps in the confidence measure (high uncertainty) cause negative jumps in the expected growth magnified by the price of risk parameter \( \lambda_x \). In contrast, under the objective measure, shocks in expected growth and investors’ confidence measure are uncorrelated. Due to the negative correlation of shocks into the expected growth state and the confidence measure, the risk adjustment of confidence shocks, \( \lambda_{vx} = \lambda_v - \frac{1}{2} \lambda_x^2 K_2^2 \), depends on the price of confidence risks \( \lambda_v \) and the risk compensation for shocks to expected growth \( \lambda_x \).

Confidence jump shocks are compound Poisson both under the objective and risk-neutral measures,

\[
Q_{t+1}^q = \sum_{i=1}^{N_{t+1}^q} \tilde{J}_{i,t+1} - \mu_j \lambda_t,
\]

but the frequency and distribution of jumps are different under the two measures. When investors prefer an early resolution of uncertainty, they dislike positive shocks to \( V \) (\( \lambda_{vx} < 0 \)), so that the jump component in the confidence measure is magnified under the risk-neutral measure. Indeed, relative to objective measure, jumps are
expected to arrive more frequently,

\[ \lambda^q_t \equiv E_t^Q N_{t+1}^q = \frac{\lambda_t}{1 + \mu_j \lambda_{\text{ex}}} > \lambda_t, \]  

(3.36)

and their size is larger,

\[ \mu^q_j = \frac{\mu_j}{1 + \mu_j \lambda_{\text{ex}}} > \mu_j, \]  

(3.37)

under the risk neutral measure.

3.3.3 Equity Prices

To obtain implications for equity prices, I consider a dividend process of the form

\[ \Delta d_{t+1} = \mu_d + \phi (\Delta c_{t+1} - \mu) + \varphi_d \sigma_t \eta_{d,t+1}, \]  

(3.38)

where \( \eta_{d,t+1} \) is a dividend shock independent from all other innovations in the economy. I continue to maintain the assumption that the average signal data is much more informative about the expected growth than consumption or dividend data, so investors learn about the expected state only from the average signals (see specification (3.20)-(3.22)).

The equilibrium price-dividend ratio is linear in the expected growth state and the level of the confidence measure of the investors:

\[ pd_t = H_0 + H_x \hat{x}_t + H_v V_t + H_\sigma \sigma^2_t, \]  

(3.39)

where solutions for the loadings are provided in Appendix C. Similar to the valuation of consumption asset, equity prices increase in expected growth factor and decrease when the confidence of investors is low or the aggregate volatility is high. In particular, large positive moves in \( V_t \) endogenously translate into large jumps in asset valuations and returns. Indeed, the equilibrium log return on the dividend asset
satisfies
\[ r_{d,t+1} = \mu_r + b_x \hat{x}_t + b_v V_t + b_{\sigma} \sigma_t^2 + \phi a_{c,t+1} + \kappa_{d,1} H_x K_2 a_{x,t+1} \]
\[ + \kappa_{d,1} H_v \left( \sigma_w \sqrt{V_{i,t+1} + Q_{t+1}} \right) + \kappa_{d,1} H_d \varphi_w \sigma_t \varphi_{c,t+1} + \varphi_d \sigma_t \eta_{d,t+1}, \]
(3.40)

for certain loadings \( b_x, b_v \) and \( b_{\sigma} \). As the return beta to the confidence measure is negative \( (H_v < 0) \), large positive shocks in the confidence measure translate into negative moves in returns, magnified by the loading \( H_v \). This channel plays an important role to empirically explain large moves in asset prices and over-pricing of out-of-the-money put options, keeping the consumption dynamics smooth as in the data.

The dynamics of returns under the objective measure in (3.40) and the evolution of the states under the risk-neutral measure in (3.30)-(3.33) can be also used to characterize the variation in returns under both probability measures. The conditional variance of returns under the two measures is linear in the confidence measure and consumption variance. Hence, positive jumps in the confidence measure endogenously translate into simultaneous positive jumps in conditional variance of returns and negative jumps in prices. In a related model, Eraker and Shaliastovich (2008) show that positive jumps in aggregate volatility of consumption \( \sigma_t^2 \) can also lead to negative jumps in equilibrium returns and positive jumps in the conditional variance of returns.

3.3.4 Option Prices

The equilibrium asset-pricing framework can be used to compute prices of options written on the dividend claim. In Appendix C.4 I show that price \( C_t(K/P_t,n) \) of a put option contract with moneyness \( K/P_t \) and maturity \( n \) depends on the underlying
expected growth, confidence measure and aggregate volatility states:

\[
\frac{C_t(K/P_t, n)}{P_t} = \frac{1}{2\pi} \frac{K}{P_t} \int_{iz_1 - \infty}^{iz_1 + \infty} \frac{e^{G_{0,n} + G_{x,n} \hat{x}_t + G_{v,n} V_t + G_{\sigma,n} \sigma_t^2 + iz \log(K/P_t)}}{iz - z^2} dz,
\]

(3.41)

where \(z_i \equiv \text{Im}(z) < 0\), and complex-valued loadings \(G\) depend on the model and preference parameters. The option price can be easily computed numerically for given states and parameters of the economy.

I convert theoretical option prices \(C_t(K/P_t, n)\) into Black-Scholes implied volatility units \(\sigma_{BS,t}^2\) using model-implied interest rate \(r_{f,t,n}\) and log price-dividend ratio \(pd_t\) (see expressions (3.28) and (3.39), respectively). This transformation is convenient, as the implied volatilities are easier to interpret than the original option prices. Indeed, implied volatilities are directly comparable across strikes and maturities; in fact, the observed differences in implied volatilities constitute major puzzles in the option pricing literature. In addition, while the price of the option in (3.41) can in principle depend on all the expected growth, confidence and aggregate volatility states, in numerical simulations I verify that Black-Scholes implied volatilities are driven nearly entirely by the confidence measure and aggregate volatility alone. This is not surprising, as the variance of market returns under physical and risk-neutral measure depends linearly on \(V_t\) and \(\sigma_t^2\), so the expected growth state is expected to have an insignificant effect on the volatilities implied in the option contracts. This insight proves very useful in the MLE estimation of the model, as it allows me to back out the confidence measure and aggregate volatility states directly from the implied volatilities, while inferring about the expected growth state using macroeconomic and asset price data.

In addition, as option-implied volatilities in the model are driven nearly entirely by the confidence measure and consumption volatility, positive jumps in the confidence measure endogenously translate into positive jumps in the option-implied
3.4 Empirical Evidence

3.4.1 Data

I collect monthly data on European S&P 500 index option prices for the period of January 1996 to June 2007 from the OptionMetrics database. The dataset also includes index price level, zero coupon yields at different maturities and dividend yield implied from the put-call parity relationship in the option market. The option contracts typically expire at the end of every third week of the month. As the theoretical model is specified on a monthly frequency, I use Wednesday prices every third week of the month to ensure that the time to expiration is an integer. Specifically, I use options with maturities of 1 and 2 months and moneyness closest to 0.9, 0.95, 1.00, 1.05 and 1.10, which are among the most actively traded contracts on the exchange. To mitigate microstructure problems, I exclude all observations with option prices less than one eights of a dollar, as well as those with no trading volume or with open interest less than 100 contracts. In the last step, I check for basic arbitrage violations in the option markets. For estimation, I consider put option prices only, as they are more actively traded than call options and the latter would be redundant given the put-call parity relationship.

Using the interpolated zero coupon rates and price-dividend ratios, I convert option prices into Black-Scholes implied volatility units. That is, I solve for the implied Black-Scholes volatility of the put contract given the observed option price, its strike price, time to maturity, current index level and the interest rate and log price-dividend ratio in the data. As discussed in the previous section, implied volatilities are easier to interpret than the original option prices; further, focusing on implied volatilities forces the estimation to directly address the key option pricing puzzles.
Table 3.1: **Summary Statistics for Key Macroeconomic and Financial Series**

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>AR(1)</th>
<th>Std. Dev.</th>
<th>Skew</th>
<th>Kurt</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Implied volatility</strong></td>
<td>17.66</td>
<td>0.78</td>
<td>5.85</td>
<td>0.78</td>
<td>3.41</td>
</tr>
<tr>
<td></td>
<td>(0.97)</td>
<td>(0.04)</td>
<td>(0.52)</td>
<td>(0.23)</td>
<td>(0.59)</td>
</tr>
<tr>
<td><strong>Smirk</strong></td>
<td>3.72</td>
<td>0.46</td>
<td>0.86</td>
<td>0.50</td>
<td>3.95</td>
</tr>
<tr>
<td></td>
<td>(0.12)</td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.31)</td>
<td>(0.71)</td>
</tr>
<tr>
<td><strong>Consumption growth</strong></td>
<td>2.00</td>
<td>-0.37</td>
<td>0.93</td>
<td>-0.11</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>(0.20)</td>
<td>(0.07)</td>
<td>(0.09)</td>
<td>(0.21)</td>
<td>(0.45)</td>
</tr>
<tr>
<td><strong>Interest rate</strong></td>
<td>1.56</td>
<td>0.97</td>
<td>0.51</td>
<td>-0.29</td>
<td>1.71</td>
</tr>
<tr>
<td></td>
<td>(0.33)</td>
<td>(0.04)</td>
<td>(0.02)</td>
<td>(0.29)</td>
<td>(0.25)</td>
</tr>
<tr>
<td><strong>Log return</strong></td>
<td>6.99</td>
<td>-0.06</td>
<td>15.17</td>
<td>-0.69</td>
<td>4.51</td>
</tr>
<tr>
<td></td>
<td>(4.36)</td>
<td>(0.08)</td>
<td>(1.41)</td>
<td>(0.25)</td>
<td>(0.71)</td>
</tr>
<tr>
<td><strong>Confidence Measure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1996-2007</td>
<td>0.07</td>
<td>0.45</td>
<td>0.02</td>
<td>1.55</td>
<td>5.77</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.11)</td>
<td>(0.002)</td>
<td>(0.27)</td>
<td>(1.42)</td>
</tr>
<tr>
<td>1968-2007</td>
<td>0.11</td>
<td>0.53</td>
<td>0.06</td>
<td>4.36</td>
<td>30.80</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.07)</td>
<td>(0.01)</td>
<td>(0.69)</td>
<td>(9.67)</td>
</tr>
</tbody>
</table>

I obtain the data on the real consumption growth rate, monthly, for the same period of 1996 to 2007 from the BEA Tables. Additionally, I construct an empirical measure of the confidence of investors as an estimate of the cross-sectional variance of the average forecast of real GDP from the Survey of Professional Forecasts. The calculations follow Bansal and Shaliastovich (2008), and the details are provided in Appendix C.5.

The key features of consumption and return data, shown in Table 3.1, are comparable to the standard estimates in the literature. Mean log return is 7% and mean inflation-adjusted interest rate is 1.6%, so the average excess return in the sample is 5.4%. Interest rates are quite persistent, with autocorrelation coefficient of 0.97 and annualized volatility of 0.5%. Consumption growth averages 2% and has a standard deviation of just below 1%. A well-known feature of consumption growth data at monthly frequency is negative autocorrelation. In my sample, the estimated persis-
The persistence coefficient is $-0.37$, while the persistence of consumption growth is reliably positive at lower frequencies and longer historical sample; see Table 3.5. To deal with the data issues in monthly consumption, I introduce a measurement noise in log consumption level in the MLE estimation of the model.

I discuss the option-price evidence and related dimensions of return and macroeconomic data in the next section.

### 3.4.2 Option Pricing Puzzles

One of the key puzzles in option markets is that out-of-the-money put options appear overpriced, so that the insurance for large downward movements in asset prices is too expensive relative to standard models (see e.g. Rubinstein, 1994). According to the Black-Scholes model, the option-implied volatilities across all strikes and maturities should be equal to the volatility of the underlying asset. Table 3.1 reports that in the data, the average volatility of out-of-the-money options of 21.4% exceeds the at-the-money volatility of 17.7% by nearly 4%. In fact, this difference (‘volatility smirk’) is always positive in the sample and ranges between 2% and 7%, as shown.
Table 3.2: Option-Implied Volatility

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 month to maturity:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average implied vol</td>
<td>25.79</td>
<td>21.38</td>
<td>17.66</td>
<td>15.68</td>
<td>15.74</td>
</tr>
<tr>
<td>(0.94)</td>
<td>(0.52)</td>
<td>(0.97)</td>
<td>(0.88)</td>
<td>(0.76)</td>
<td></td>
</tr>
<tr>
<td>Low uncertainty V</td>
<td>20.56</td>
<td>15.70</td>
<td>11.85</td>
<td>10.29</td>
<td>11.45</td>
</tr>
<tr>
<td>High uncertainty V</td>
<td>28.01</td>
<td>23.61</td>
<td>19.71</td>
<td>17.40</td>
<td>17.19</td>
</tr>
<tr>
<td><strong>2 months to maturity:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average implied vol</td>
<td>24.15</td>
<td>21.14</td>
<td>18.42</td>
<td>16.97</td>
<td>16.64</td>
</tr>
<tr>
<td>(0.96)</td>
<td>(0.96)</td>
<td>(0.95)</td>
<td>(0.89)</td>
<td>(0.80)</td>
<td></td>
</tr>
<tr>
<td>Low uncertainty V</td>
<td>18.62</td>
<td>15.54</td>
<td>12.66</td>
<td>11.39</td>
<td>11.67</td>
</tr>
<tr>
<td>High uncertainty V</td>
<td>26.30</td>
<td>23.22</td>
<td>20.42</td>
<td>18.71</td>
<td>18.37</td>
</tr>
</tbody>
</table>

in the second panel of Figure 3.1. Similar results obtain for a broader range of put option strikes and for longer maturities (see Table 3.2). The empirical evidence of over-pricing of out-of-the-money put options suggests that the cross-section of option prices cannot be explained by standard Gaussian models and points to the jump risk factors in the economy⁴.

Consistent with this evidence, option and asset prices exhibit large moves (jumps) in the data. The unconditional distribution of returns is characterized by negative skewness of −0.7 and high kurtosis of 4.5 — for Normal distribution, these statistics are 0 and 3, respectively. Excess kurtosis and negative skewness are indicative of large negative moves in returns. Similarly, positive skewness in implied variance indicates the presence of large positive movements in the series. Sizeable variation across time and occasional large positive spikes are apparent on the plot of option-implied volatility on Figure 3.1. Direct evidence for large moves can be obtained by isolating abnormal movements in prices. Specifically, I identify large move as a

⁴ Statistical evidence on the importance of jumps for option prices is discussed in Bakshi et al. (1997), Bates (2000), Pan (2002), Broadie et al. (2007), Santa-Clara and Yan (2008).
two standard deviation or higher innovation based on the AR(1)-GARCH(1,1) fit. In the data, the frequency of identified large moves in returns and implied variance is the same, once every 17 months. 75% of the identified large moves in returns are negative, while all of the large moves in implied variance are positive. The timing of large moves in implied variance and returns is highly related, as 5 out of 8 identified large moves in the two series occur at the same time. These findings on large moves in asset prices are broadly consistent with jump evidence from the parametric models of asset-prices discussed in Singleton (2006), and with empirical results in Tauchen and Todorov (2008), who present strong evidence for common jumps in stock price and implied volatility from the option markets based on the high-frequency data.

While there is strong support for large common moves in asset and option prices, there is no evidence for large moves in the real economy that can economically account for the jump features of financial data at the considered frequencies. In my sample, none of the large moves in financial prices can be explained by a simultaneous large jump in real consumption. The estimated conditional mean and variance of consumption growth are even smoother than the underlying series and also show no large moves that could explain jumps in prices. Similar evidence is presented in Bansal and Shaliastovich (2008), who document that there is no link in the data between large moves in equity returns and moves in a variety of macroeconomic variables, while in Chapter 2 we argued that years with daily jumps in returns are not predictable by the level of the real economy.

While there is no direct empirical evidence for jump risks in consumption, measures of investors’ uncertainty about future growth exhibit substantial fluctuations and large moves in the data, which can potentially explain the cross-section of option prices and the time-series dynamics of asset and option prices. Indeed, the confidence measure has significant information about the option price volatilities in
Table 3.3: Option Price Predictability by Confidence Measure

<table>
<thead>
<tr>
<th></th>
<th>Option Variance</th>
<th>Confidence Measure</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Out-of-the-Money</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 quarter ahead</td>
<td>0.52**</td>
<td>0.11</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.11)</td>
<td></td>
</tr>
<tr>
<td>2 quarters ahead</td>
<td>0.27</td>
<td>0.43*</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.18)</td>
<td></td>
</tr>
<tr>
<td>3 quarters ahead</td>
<td>0.31</td>
<td>0.51**</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>(0.17)</td>
<td>(0.18)</td>
<td></td>
</tr>
<tr>
<td><strong>At-the-Money</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 quarter ahead</td>
<td>0.42**</td>
<td>0.09</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.07)</td>
<td></td>
</tr>
<tr>
<td>2 quarters ahead</td>
<td>0.33**</td>
<td>0.24**</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.11)</td>
<td></td>
</tr>
<tr>
<td>3 quarters ahead</td>
<td>0.13</td>
<td>0.30**</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.10)</td>
<td></td>
</tr>
<tr>
<td><strong>In-the-Money</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 quarter ahead</td>
<td>0.27**</td>
<td>0.12</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>(0.09)</td>
<td>(0.08)</td>
<td></td>
</tr>
<tr>
<td>2 quarters ahead</td>
<td>0.30**</td>
<td>0.19**</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.08)</td>
<td></td>
</tr>
<tr>
<td>3 quarters ahead</td>
<td>0.08</td>
<td>0.24**</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.10)</td>
<td></td>
</tr>
</tbody>
</table>

the data. As shown in Table 3.2, option volatilities across all strikes and maturities are about 7% higher in quarters where the uncertainty is high, relative to quarters where the uncertainty about future growth is low. Further, the confidence measure in the data has significant information about future option-implied volatilities, even controlling for the current implied volatility of the contract. Table 3.3 documents that, in projections of option-implied volatilities 2 and 3 quarters ahead, the slope coefficient on the confidence measure is large and significant at all strikes, while the slope coefficient on the current value of option volatility is small and is typically de-
creasing with horizon. (Beyond 3 quarters, both slopes become insignificant.) This evidence is consistent with Burasi and Jitsov (2006), who show that the cross-sectional dispersion of forecasts from the Survey of Professional Forecasts and the Consumer Confidence Survey has information about the level and slope of the option smile and future realized volatility of returns.

The empirical confidence measure exhibits large positive moves, whose frequencies and magnitudes are plausible to account for the jump features of option and asset market data. As shown in the last panel of Table 3.1, the unconditional distribution of the confidence measure in the data is very heavy-tailed and positively skewed, especially for the full period from 1968 to 2008. The large positive spikes in the series depicted on Figure 3.2 indicate the possibility of large positive shocks to the uncertainty about future growth. Indeed, using formal econometric analysis, Bansal and Shaliastovich (2008) find significant evidence for a jump-like component in the confidence measure and document that large moves in the confidence measure in the data are related to large moves in returns and in variance of returns implied from the option markets.

**Figure 3.2: Time-Series of Confidence Measure**
Table 3.4: Calibration of Consumption Dynamics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.0017</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.978</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0058</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>0.987</td>
</tr>
<tr>
<td>$\varphi_w$</td>
<td>5.72e-04</td>
</tr>
<tr>
<td>$\varphi_e$</td>
<td>0.044</td>
</tr>
</tbody>
</table>

Hence, based on the empirical evidence, a measure of investors’ uncertainty about future growth provides a channel to explain option prices puzzles and the jump features of financial market data, without relying on jumps in consumption. I formally assess the ability of the model with time-varying confidence jump risks to explain option, return and macroeconomic data in the next section.

3.5 GMM Model Estimation

3.5.1 Consumption Calibration

I calibrate the parameters of consumption dynamics and estimate preference, confidence measure and dividend parameters using a GMM approach.

The consumption dynamics parameters are calibrated on monthly frequency. The baseline values for the parameters, which are reported in Table 3.4, are very similar to those used in the long-run risks literature (see e.g. Bansal and Yaron, 2004). Specifically, the annualized consumption growth rate is set at 2%. The persistence in the expected growth $\rho$ of 0.978 and the magnitudes of the scale parameters $\varphi_e$ and $\sigma$ ensure that the model can match historic volatility of consumption growth of 2% and its persistence of about 0.45. Similar to Bansal and Yaron (2004), I set the persistence of the consumption variance to $\nu_c = 0.987$, and calibrate the volatility of
Table 3.5: Consumption Dynamics: Data and Model Calibration

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
</tr>
<tr>
<td>Mean</td>
<td>1.95</td>
<td>(0.32)</td>
</tr>
<tr>
<td>Vol</td>
<td>2.13</td>
<td>(0.52)</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.44</td>
<td>(0.13)</td>
</tr>
<tr>
<td>AR(2)</td>
<td>0.16</td>
<td>(0.18)</td>
</tr>
<tr>
<td>AR(5)</td>
<td>-0.01</td>
<td>(0.10)</td>
</tr>
<tr>
<td>VR(2)</td>
<td>1.58</td>
<td>(0.18)</td>
</tr>
<tr>
<td>VR(5)</td>
<td>2.23</td>
<td>(0.86)</td>
</tr>
</tbody>
</table>

volatility parameter to \( \varphi_w = 5.72 \times 10^{-4} \). I calibrate the model on monthly frequency and then time-aggregate to an annual horizon. Table 3.5 shows that the model can successfully match mean, volatility, auto-correlations and variance ratios of the consumption dynamics over the long period in the data.

3.5.2 GMM Estimation

I estimate preference, confidence measure and dividend parameters using standard GMM approach. In particular, I bring in quarterly observations on the confidence measure, based on real GDP forecasts from the Survey of Professional Forecasts, and corresponding observations on monthly returns, interest rates and volatilities implied from option contracts with moneyness closest to 0.95, 1.00 and 1.05 and maturities of 1 and 2 months, for the period from 1996 to mid 2007. As the confidence measure in the data is based on quarterly forecasts of real GDP, I adjust the confidence measure data to account for scale and time-aggregation issues, as discussed in Appendix C.5.

In estimation, I set up four blocks of orthogonality conditions which correspond to the moments of the confidence measure, real excess return, interest rate and option prices. For the confidence measure, I choose to match its first four unconditional moments and quarter-ahead auto-covariance, which characterize the level, variation,
persistence, and non-Gaussian properties of the series. For the stock market data, I also use the first four moments of excess equity returns, its auto-covariance as well as the covariance of excess returns with the confidence measure. Further, I construct orthogonality conditions based on the level, variation and persistence of interest rates, and the levels of 6 option volatilities which span three moneyness categories (0.95, 1.00 and 1.05) and two maturities (1 and 2 months). Further details of moment conditions are provided in the Appendix C.6.

Let \( g_t(\Theta) \) stand for the overall vector of orthogonality conditions based on the confidence measure, equity return, interest rate and option price data, \( \Theta \) is the vector of estimated parameter and \( \bar{g}(\Theta) \) denotes the sample average of moment conditions across time. Then, the GMM objective function is given by,

\[
Q_{GMM}(\Theta) = T \bar{g}(\Theta)' \Sigma_T^{-1} \bar{g}(\Theta).
\]

(3.42)

As usual, the estimation proceeds in two steps, and the optimal weighting matrix \( \Sigma_T^{-1} \) corresponds to the inverse of the Newey-West estimate of the variance of the moment conditions based on the first-step parameter estimates. The computation of standard errors and hypothesis testing follows standard asymptotic results for GMM, as discussed in Hansen (1982).

3.5.3 GMM Parameter Estimates

The estimation uses 20 moment conditions to estimate 10 parameters of investor preferences, confidence measure, and dividend growth rate. The consumption dynamics parameters are calibrated as discussed in the Section 3.5.1. For identification reasons, I also fix the subjective discount factor at 0.999. Bansal, Gallant, and Tauchen (2007a) discuss that it is difficult to separately identify the subjective discount factor and the inter-temporal elasticity of substitution.

The estimated parameters of the full model, which features time-varying con-
Table 3.6: GMM Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>Full Model Estimate</th>
<th>S.E.</th>
<th>No Jumps Estimate</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preference:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>δ</td>
<td>0.999</td>
<td></td>
<td>0.999</td>
<td></td>
</tr>
<tr>
<td>ψ</td>
<td>1.36 (0.11)</td>
<td></td>
<td>1.43 (0.15)</td>
<td></td>
</tr>
<tr>
<td>γ</td>
<td>11.03 (1.30)</td>
<td></td>
<td>10.56 (5.09)</td>
<td></td>
</tr>
<tr>
<td><strong>Dividend:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>φ</td>
<td>3.68 (0.85)</td>
<td></td>
<td>4.01 (2.00)</td>
<td></td>
</tr>
<tr>
<td>φ_d</td>
<td>4.88 (1.49)</td>
<td></td>
<td>5.52 (3.03)</td>
<td></td>
</tr>
<tr>
<td><strong>Confidence Measure:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ_v × 10^4</td>
<td>3.73 (0.11)</td>
<td></td>
<td>3.07 (0.11)</td>
<td></td>
</tr>
<tr>
<td>ν</td>
<td>0.91 (0.01)</td>
<td></td>
<td>0.94 (0.03)</td>
<td></td>
</tr>
<tr>
<td>σ_w × 10^5</td>
<td>0.34 (1.15)</td>
<td></td>
<td>10.34 (0.60)</td>
<td></td>
</tr>
<tr>
<td>λ_0 × 10^2</td>
<td>6.37 (2.76)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ_1 × 10^{-6}</td>
<td>1.00 (0.51)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>μ_j × 10^7</td>
<td>1.95 (0.25)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Consumption volatility and fluctuating confidence driven by Normal and jump-like innovations, are presented in the left column of Table 3.6. The estimated preference parameters are quite standard in the long-run risk literature: the inter-temporal elasticity of substitution is 1.36, while the risk aversion is 11.03. This parameter configuration implies that agents prefer early resolution of uncertainty, so that they dislike negative shocks to expected consumption and positive shocks to consumption volatility and confidence measure.

The leverage of dividend growth to expected consumption is estimated at φ = 3.7, and the dividend volatility scale is φ_d = 4.9. These values are consistent with the literature (see e.g. Bansal and Yaron, 2004), and imply the annualized dividend volatility of 11% and the persistence of 0.4, which agree well with the historical data.

The estimated parameters of the confidence dynamics are presented in the lower panel of Table 3.6. The unconditional level of the confidence measure is 15 times
smaller than the level of consumption volatility, which is consistent with the evidence in the data. The level of the confidence measure is directly related to the uncertainty that investors face about their estimate of expected growth. Indeed, as shown in equation (3.24), the variance of the filtering error \( \omega_t^2 \) due to Kalman Filter learning about the state is proportional to the confidence measure, where the proportionality coefficient \( K_2 \) (Kalman Filter weight on the forecast innovations) is equal to 0.5, based on the estimates of the model. Hence, the model-implied two standard deviations band around the investors’ estimate of expected consumption growth is on average ±0.25%, annualized. The average uncertainty about expected growth is quite small; for comparison, the calibrated standard deviation of the consumption growth is 2% on annual basis.

The estimated model parameters indicate that the confidence measure significantly fluctuates over time. Indeed, the persistence of the confidence shocks is 0.91, which translates into a half-life of 7 months. The model for the confidence measure in (3.14) allows both for Gaussian and jump-like confidence shocks, however, the estimation results suggest that nearly all the variation in the series is driven by Poisson jumps. The Gaussian volatility parameter \( \sigma_w \) is very small and insignificant, while all the jump parameters \( \mu_j, \lambda_0, \lambda_1 \) are highly significant individually. Further, the Wald test for their joint significance overwhelmingly rejects the null that jump parameters are zero with \( p \)-value well below 1%. The average frequency of jumps in the confidence measure is one every 5 months, which agrees with the estimates in the data. Finally, the results indicate that the probability of confidence jump is increasing when the confidence measure is high (high uncertainty): the intensity parameter \( \lambda_1 \) is estimated positive and statistically significant.

The right column of Table 3.6 presents the estimation results for the restricted model with no jumps in the confidence measure. To generate enough variation in the
confidence measure, the volatility of Gaussian shocks increases thirtyfold relative to
the model with jumps, while the persistence of shocks goes up to 0.94. The estimation
results suggest that the fit of the model with no confidence jumps substantially
worsens, as the Likelihood Ratio test based on the difference in GMM objective
functions of full and restricted models overwhelmingly rejects the restrictions that
jump parameters are zero, consistent with an earlier result on the joint significance
of the jump parameters based on the Wald Test.

3.5.4 Option Pricing Implication

Confidence jump risks play an important role to explain the cross-section of option
prices and jump-like features of asset market data. Indeed, as shown on Figure 3.3,
the model with confidence jump risks can quantitatively explain overpricing of the
out-of-the-money put options in the data. On the other hand, in the restricted model
with no confidence jumps, the implied volatility curve is virtually flat. This result
is consistent with Pan (2002), Bates (2000), Bakshi et al. (1997), who show that
standard models with Gaussian shocks cannot account for the cross-section of option
prices in the data.

The full model with confidence jumps can deliver an implied volatility smile simi-
lar to the data. While the model somewhat under-predicts the volatilities of the deep
out-of-the-money options, the magnitudes of option pricing errors are comparable to
other studies. On average, the difference between option volatility in the data and in
the model are less than 2% in Black-Scholes volatility units for put contract used in
the estimation. For comparison, Santa-Clara and Yan (2008) document root-square
errors of 2%, while Pan (2002) reports absolute pricing errors in the range from 1%
to 3%. The underlying tension in the estimation is between fitting the tails of re-
turn and confidence measure distributions and the slope of the option smile. Indeed,
higher jump contribution can substantially increase the deep out-of-the-money op-
Figure 3.3: Implied Volatility Smile

tion volatilities; however, it will adversely impact the fit of return and confidence measure by making their distribution more heavy-tailed.

In the model with confidence jump risks, out-of-the-money put options hedge large positive moves in the uncertainty about future growth, which explains the over-pricing of these contracts relative to standard models. As discussed in Section 3.3.3, large positive jumps in the confidence measure endogenously translate into negative moves in returns and positive contemporaneous moves in the conditional variance of returns. This can account for the heavy tails of the unconditional distribution of
returns and implied option volatility, as well as the evidence on common large moves in these series in the data. The model with confidence jumps delivers the kurtosis of the unconditional distribution of excess returns equal to 6 and negative skewness of -0.2. These values are broadly consistent with statistics in the data, shown in Table 3.1. For comparison, in the model with no confidence jumps, kurtosis of return distribution is 3.8 and skewness is 0. The plots of the unconditional distributions of excess returns from the two model specifications and from the data are shown on Figure 3.4. These plots visually indicate that the model with jumps provides a better fit to the return distribution in the data than the model with no confidence jumps. To focus on tail properties of returns, on Figure 3.5 I show the QQ plot of quantiles of return distribution in the model versus the data. In a model with confidence jumps, the points cluster along the 45-degree line, which indicates a close fit of the model-implied distribution of returns to the data. On the other hand, the model with no jumps cannot account for the left tail of the distribution of return in the data, as in absence of jump risks the model cannot generate negative skewness
In the model, jumps in the asset prices are endogenously driven by large moves in investors confidence measure. At the estimated parameters, the average frequency of jumps is about one every 5 months. An average jump in returns implied from the model is $-3.3\%$, monthly. The moves of such magnitudes are quite common in the data; indeed, in my sample monthly returns fall below the cutoff of $-3.3\%$ once every 6 month. As shown in Section 3.3.2, preference for early resolution of uncertainty implies that the contribution of the confidence jump risk is magnified under the risk-neutral measure, with jumps being larger and more frequent relative to the objective measure. Indeed, the mean jump in return is $-3.6\%$, and the average frequency of jumps is 4 months under the risk-neutral measure.

I discuss the implications of the confidence jump risks for the total premium in the economy in the next section.
Table 3.7: **Equity Premium Decomposition**

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Growth</th>
<th>Short-Run Growth</th>
<th>Confidence Measure</th>
<th>Consumption Volatility</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Learning</td>
<td>3.32</td>
<td>1.35</td>
<td>0.00</td>
<td>2.07</td>
<td>6.27</td>
</tr>
<tr>
<td>GMM - Full Model</td>
<td>3.19</td>
<td>1.36</td>
<td>1.73</td>
<td>0.17</td>
<td>6.11</td>
</tr>
<tr>
<td>GMM - No Jumps</td>
<td>3.52</td>
<td>1.37</td>
<td>0.28</td>
<td>0.21</td>
<td>4.87</td>
</tr>
<tr>
<td>Latent Factor MLE</td>
<td>2.36</td>
<td>0.95</td>
<td>2.44</td>
<td>0.02</td>
<td>5.18</td>
</tr>
</tbody>
</table>

3.5.5 **Equity Premium**

Table 3.7 shows the magnitude of the model-implied equity premium and its decomposition due to sources of risks in the economy. For a benchmark specification, I consider a standard, complete information long-run risks model where the uncertainty about expected growth is zero (see Bansal and Yaron, 2004). Based on the calibration of consumption dynamics and the estimated dividend and preference parameters from the full model, the implied equity premium in the model with no learning is 6.3%. Most of the compensation (3.3%) is due to investors’ exposure to the long-run risks; consumption volatility risks demand 2.1%, while immediate consumption growth shocks — about 1.4%. Learning and fluctuating confidence channels change the compensation for the fundamental risks in the economy relative to the benchmark case. As investors cannot separate true short-run and long-run consumption innovations, the compensation for the expected growth risks decreases to 3.2% while the compensation for immediate consumption shocks increases slightly; this is consistent with Croce et al. (2006). In addition, as discussed in Section 3.3.1, fluctuating confidence of investors diminishes the importance of consumption volatility channel and reduces risk compensation for consumption volatility risks, compared to the complete information setup. Based on the parameter estimates of the full model, consumption volatility risks now contribute less than 0.2% to the total equity.
premium of 6.1%. On the other hand, confidence risk is the second most important risk channel in the economy, adding 1.7% to the equity premium. As nearly all of the confidence risks are jump risks, the compensation for the confidence risks of almost 2% determines the overall contribution of jump risks to the equity premium in the economy. This magnitude is consistent with other studies. For example, Broadie et al. (2007) and Pan (2002) estimate the jump risk premium between 2% – 3% and 3.5%, respectively, or one third of the total equity premium in the sample, which agrees with the estimate provided in this paper. Finally, in the model with no jumps in the confidence measure, the estimated equity premium drops to 4.9%, as the compensation for purely Gaussian confidence shocks decreases to 0.3%.

I also verify that the model with the confidence jump risks delivers sensible implications for the bond markets. The model-implied 1 month interest rate is 2%. Its volatility is 0.4%, annualized, and the persistence is 0.91. These values broadly agree with the statistics for interest rate reported in Table 3.1. The model-implied real term-structure is nearly flat, and the real interest rate at 5 year maturity is 1.3.5.6 GMM Model Fit

As shown in Hansen (1982), under the null hypothesis that the model is correctly specified the GMM criterion function in (3.42) has a $\chi^2$ distribution, which can be used for an overall goodness-of-fit test of the model. The test does not reject the null that the orthogonality conditions based on the full model with confidence jump risks are equal to zero: the $p$-value for the test is 0.30. In addition, I follow Eichenbaum, Hansen, and Singleton (1988) to test the blocks of moment equations corresponding to the confidence measure, equity returns, interest rate and option-price data, respectively; see Appendix C.6 for details. These moment restrictions are not rejected in the data: the p-value for the test that confidence measure moments are zero is 0.60, it is 0.8 for the return moments, 0.5 for the moments of interest rate
and 0.3 for the orthogonality conditions based on the option price data.

In the estimation, I do not incorporate deep out- and in-the-money options with moneyness closest to 0.9 and 1.1, respectively, at 1 and 2 months to maturity; these options can be used now for an out-of-sample test of the model. The orthogonality conditions for the levels of these option volatilities are not rejected at 5% significance level; in Section 3.5.4 I showed that the model somewhat under-predicts the implied volatility for deep out-of-the-money options, especially for longer maturity. Similarly, I can use the available data from 1968 to 1996 for an out-of-sample test of the orthogonality restrictions on the confidence measure. The average uncertainty is higher in the earlier period, so that the moment condition for the level of the series is rejected at 1% significance level. However, the joint test of all the higher moment restrictions which characterize the variation and non-Gaussian features of the confidence measure cannot be rejected at 5%.

The goodness of fit of the model substantially deteriorates when the confidence shocks do not include a jump-like component. The blocks of moment restrictions corresponding to the confidence measure, return and option volatility data are rejected in sample at a 1% significance level. Further, the out-of-sample tests of moment conditions based on the additional option price data for moneyness 0.9 and 1.1 and the confidence data from 1968 to 1996 are also rejected with p-value below 1%.

3.6 Latent State MLE Model Estimation

3.6.1 Econometric Method

In the second approach, I treat the confidence measure, as well as consumption volatility and expected growth state, as latent factors and estimate the model using the maximum likelihood method. The estimation framework allows me to use monthly observations on real consumption growth, equity returns, interest rates and
the cross-section of 6 option prices to recover preference, consumption and confidence dynamics parameters and back out the unobserved states.

The main idea in the estimation is that in the model, option-implied volatilities are driven almost entirely by the confidence measure and aggregate volatility states. This allows me to back out volatility states from the implied volatilities alone and to estimate expected growth using the data on consumption, risk-free rate and equity return. The method is similar to Pan (2002), who inverts the option price to solve for the latent market volatility, and Santa-Clara and Yan (2008), who use two option prices to solve for the unobserved variance and stochastic intensity states. More generally, this approach is motivated by the literature on estimation of affine dynamic term structure models, see for example Duffie and Singleton (1997) and Duffee (2002). I outline the estimation approach below, and provide further details in Appendix C.7.

Denote Θ the parameters of the model, and let a vector

$$Z_t = \begin{bmatrix} r_t^{\text{data}} & r_{d,t} & \Delta c_{t+1}^{\text{data}} \end{bmatrix}'$$

contain period-t observations on interest rate, log real return and log consumption growth. I allow for i.i.d. Normal measurement errors in the log consumption level and interest rate to deal with measurement issues of these series in the data.

Each period, given the parameters Θ, I solve for the unobserved confidence measure and consumption volatility states $V_t(Θ)$ and $\sigma_t^2(Θ)$ to directly match Black-Scholes volatilities of out- and at-the-money short-term put options in the model with moneyness of 0.95 and 1, respectively, and maturity of 1 month to their counterparts in the data. Using the model for the confidence measure and consumption volatility specified in (3.14) and (3.8), I can compute the conditional likelihood of the two implied option volatilities $l_t(\sigma_{BS,t}^2)$ which are used to invert the states. Further, given the history of observed macro and return data and the current implied confidence and consumption volatility states, the conditional distribution of asset market and consumption data $Z_t$ is Normal. Therefore, I can apply standard Kalman Filter
methods to write down the likelihood of the observed data \( l_t(Z_t) \) and the evolution of the estimate of the expected state \( \tilde{x}_t(\Theta) \). To bring into the estimation the implied volatilities of the remaining put options, I assume that the pricing errors \( \xi_{bs,t} \), that is, the difference between the model predicted and observed implied volatility for these contracts, are pure measurement errors which are Normal and independent from each other and fundamental shocks in the economy. Hence, their conditional likelihood \( l_t(\xi_{bs,t}) \) is Normal.

I can combine the information from the consumption and asset-price data and the cross-section of option volatilities into the period-\( t \) log likelihood:

\[
l_t(Z_t, \sigma_{BS,t}^2, \xi_{bs,t}) = l_t(Z_t) + l_t(\sigma_{BS,t}^2) + l_t(\xi_{bs}). \tag{3.43}
\]

The total log likelihood function of the sample is given by,

\[
L(\Theta) = \sum_{t=1}^{T} l_t(Z_t, \sigma_{BS,t}^2, \xi_{bs,t}). \tag{3.44}
\]

The optimal parameter value maximizes the sample likelihood function \( L(\Theta) \) given that the solution to the model and the implied states exist, and that the implied confidence measure \( V_t(\Theta) \) and consumption variance \( \sigma_t^2(\Theta) \) are greater than zero. I use parametric bootstrap method to compute standard errors on the estimated parameters.

3.6.2 MLE Estimation Results

Table 3.8 presents parameter estimates for the full model with fluctuating confidence and time-varying consumption volatility. As before, I fix the subjective discount factor at 0.999. In addition, I set the mean of consumption and dividend growth to be equal to 2%. This is a standard estimate in the data; indeed, Tables 3.1 and Table 3.5 report the average consumption growth of 2%, both for the recent and long
Table 3.8: MLE Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>Full Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
</tr>
<tr>
<td><strong>Preference:</strong></td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.999</td>
</tr>
<tr>
<td>$\psi$</td>
<td>2.44</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>9.66</td>
</tr>
<tr>
<td><strong>Consumption:</strong></td>
<td></td>
</tr>
<tr>
<td>$\mu \times 10^3$</td>
<td>1.66</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.976</td>
</tr>
<tr>
<td>$\varphi_e \times 10^2$</td>
<td>4.50</td>
</tr>
<tr>
<td><strong>Dividend:</strong></td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>3.5</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>6.22</td>
</tr>
<tr>
<td><strong>Confidence Measure:</strong></td>
<td></td>
</tr>
<tr>
<td>$\sigma_v \times 10^4$</td>
<td>2.01</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.92</td>
</tr>
<tr>
<td>$\sigma_w \times 10^5$</td>
<td>9.81</td>
</tr>
<tr>
<td>$\lambda_0 \times 10^2$</td>
<td>3.47</td>
</tr>
<tr>
<td>$\lambda_1 \times 10^{-6}$</td>
<td>1.96</td>
</tr>
<tr>
<td>$\mu_j \times 10^7$</td>
<td>1.06</td>
</tr>
<tr>
<td><strong>Consumption Volatility:</strong></td>
<td></td>
</tr>
<tr>
<td>$\sigma \times 10^3$</td>
<td>5.95</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>0.86</td>
</tr>
<tr>
<td>$\varphi_w \times 10^3$</td>
<td>2.07</td>
</tr>
</tbody>
</table>

historical samples. Finally, to stabilize estimation I also fix the dividend leverage parameter $\phi$ to 3.5, which is a common value in the literature (see e.g. Bansal and Yaron, 2004).

The estimation results for the model parameters are comparable to the ones from unconditional GMM. The estimated risk aversion coefficient is almost 10 and the intertemporal elasticity of substitution is 2.4; this indicates preference for early resolution of uncertainty. Expected consumption growth is very persistent, $\rho = 0.976$, so that the half-life of expected growth shocks is almost 2.5 years. At the same time,
the variation in expected growth is very small and accounts for less than 5% of the total variation in consumption. Estimated consumption volatility is on average 2%, and it is moderately persistent, with an autoregression coefficient of 0.85. As for the estimates of the confidence dynamics, its average level, in volatility units, is about 30 times lower than the unconditional volatility of consumption growth. Confidence shocks are quite persistent, with an autocorrelation coefficient of 0.92. Most of the fluctuation in the confidence measure is driven by non-Gaussian shocks, as jumps account for three quarters of the conditional variance of the series. Though the consumption and confidence measure parameters are estimated using a recent sample from 1996 to 2007, the implied dynamics of the series is consistent with a long historical sample from 1930-2006 for consumption and from 1968 to 2008 for the confidence measure. I simulate the model and verify that the key features of their distribution, such as volatility, skewness and kurtosis of the confidence measure and the ratio of the confidence measure over the time-series volatility, match very well
their counterparts in the data. The details are omitted in the interest of space.

The confidence measure and consumption volatility states implied from option prices are plotted in Figure 3.6. The confidence measure exhibits substantial variation over time, with occasional large positive spikes, while nearly hitting a zero boundary in the late period of the sample. At the quarterly frequency, the correlation of the confidence measure implied from option prices and the confidence measure in the data constructed from the professional forecasts of future GDP is 0.4. The estimated expected growth state is depicted in Figure 3.7. The extracted drift component significantly predicts next-period consumption with $R^2$ of 2%. 

3.6.3 Option and Asset Prices

The model with the confidence jump risks can quantitatively explain the cross-section of option prices and the variation in option-implied volatilities. As shown in Table 3.9, the absolute pricing errors for implied volatilities are 1.2% for 1 month in-the-money options and less than 0.8% for those with 2 months to maturity, so that the
Table 3.9: **Implied Volatilities Fit from MLE**

<table>
<thead>
<tr>
<th>$K/P$</th>
<th>Pricing error</th>
<th>Explained Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RMSE</td>
<td>Abs. Mean</td>
</tr>
<tr>
<td><strong>1 month</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.05</td>
<td>1.33</td>
<td>1.15</td>
</tr>
<tr>
<td><strong>2 months</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>0.90</td>
<td>0.75</td>
</tr>
<tr>
<td>1.00</td>
<td>0.86</td>
<td>0.68</td>
</tr>
<tr>
<td>1.05</td>
<td>0.89</td>
<td>0.75</td>
</tr>
</tbody>
</table>

The model implications for the distribution of return and implied volatility and jumps are similar to the ones based on GMM fit. Relative to GMM, the frequency of large moves in returns decreases to one every 9 months, while average jump in return is higher, $-7.5\%$, monthly. The jump dimensions of return distribution are consistent with the data.

The model-implied equity premium is 5.2% (see Table 3.7) which is close to
Figure 3.8: Option-Implied Volatility Smile
the estimate in the data of 5.5%. Expected growth shocks and confidence shocks contribute about 2.4% to the total equity premium, so that the relative contribution of confidence jump risks somewhat increases relative to the unconditional GMM parameter fit. The time-series of the fitted equity premium, as well as the premia due to expected growth and confidence shocks, are shown on Figure 3.9. The implied equity premium in the economy shows substantial variation over time, driven by the compensation for the expected growth and confidence jump risks.

A model without confidence jumps cannot explain the cross-section of option prices in the data. For robustness checks, I also estimate the model without time-varying consumption volatility. Most of the results are qualitatively similar; however, the cross-sectional fit of option prices deteriorates.

3.7 Conclusion

I present a long-run risks type model which features learning and fluctuating investors’ confidence about their estimate of unobserved expected growth. Uncertainty
about expected growth (confidence measure) is time-varying and subject to jump-like
risks. This confidence jump risk channel can quantitatively account for the cross-
section of option prices and large moves in asset prices, without hard-wiring jumps
into consumption. Out-of-the-money put options hedge jump risks in the confidence
measure and thus appear expensive relative to models with no jump risks. Positive
jumps in the confidence measure endogenously translate into negative jumps in equi-
librium prices, which can account for large downward moves and negatively skewed
and heavy-tailed unconditional distribution of returns.

I provide empirical evidence that the confidence measure in the data contains
significant information about current and future implied volatilities. Further, I use
two econometric approaches to formally estimate and test the model. The empirical
results provide a strong support for a long-run risks model with learning, fluctuating
confidence of investors and jump-like confidence risks. The model is not rejected in
the data and provides a good fit to the option price, confidence measure, returns, and
consumption data. Overall, empirical results strongly indicate that the confidence
jump risk plays an important role to explain option and equity prices in the data
without introducing jumps into fundamental consumption.
Appendix A

Model Solution

A.1 Baseline Model

A.1.1 Solution to \( \kappa_1 \)

From the log-linearization of returns, we obtain that the mean price-to-consumption ratio \( E(z_t) = A_0 \) and a constant \( \kappa_0 \) are equal to

\[
A_0 = \log \frac{\kappa_1}{1 - \kappa_1}, \quad k_0 = -\log \left[ (1 - \kappa_1)^{1 - k_1} k_1 \right]. \tag{A.1}
\]

We can substitute the above expressions for \( A_0 \) and \( \kappa_0 \) into the solution for \( A_0 \) from the Euler equation, and after some algebra obtain

\[
\log \kappa_1 = \log \delta + (1 - \frac{1}{\psi}) \mu_g + A_{gs} (1 - \kappa_1 \nu_g) \sigma_g^2 + \frac{1}{2} \theta \kappa_1^2 A_{gs}^2 \sigma_{gw}^2. \tag{A.2}
\]

To find the solution for \( \kappa_1 \) at the calibrated parameter values, we iterate on the equation above from an initial value \( \delta \). Similar expression can be derived for the approximating coefficients in the log-linearization of the return on the stock market portfolio. This approach closely follows (Bansal et al., 2005).
A.1.2 Solution to Bond Yields

The log prices of real and nominal discount bonds \( q_{t,n} \), \( q^s_{t,n} \) satisfy the Euler conditions

\[
e^{q_{t,n}} = E_t e^{m_{t+1} + q_{t+1,n-1}}, \quad e^{q^s_{t,n}} = E_t e^{m^s_{t+1} + q^s_{t+1,n-1}}, \quad (A.3)
\]

for \( q_{t,0} = q^s_{t,0} = 0 \). Using the solutions to the real and nominal pricing kernels in (1.22) and (1.17), we obtain that the bond prices and therefore, bond yields, are affine in the states, as shown in expression (1.26) and (1.27), and the bond yield loadings satisfy

\[
B_{x,n} = \rho B_{x,n-1} + \frac{1}{\psi}, \\
B_{gs,n} = \nu_g B_{gs,n-1} - \frac{(1 - \gamma)(\gamma - 1)}{2} \left[ 1 + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] - \frac{1}{2} \left( \lambda^2_e + \varphi^2_e (\lambda_e + B_{x,n-1})^2 \right), \\
B_{0,n} = B_{0,n-1} - \mu_m - \frac{1}{2} \left( \sigma^2_g [\lambda^2_\eta + \varphi^2_e (\lambda_e + B_{x,n-1})^2] + \sigma^2_{gw} [\lambda_{gw} + B_{gs,n-1}]^2 \right)
\]

for real bonds, and

\[
B^s_{x,n} = \rho B^s_{x,n-1} + \alpha_x B^s_{x,n-1} + \frac{1}{\psi}, \\
B^s_{\pi,n} = \alpha_\pi B^s_{\pi,n-1} + 1, \\
B^s_{gs,n} = \nu_g B^s_{gs,n-1} - \frac{(1 - \gamma)(\gamma - 1)}{2} \left[ 1 + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] \\
- \frac{1}{2} \left( [\lambda_\eta + \varphi_{\pi g} + \varphi_{z g} B^s_{\pi,n-1}]^2 + \varphi^2_e [\lambda_e + \varphi_{\pi x} + \varphi_{z x} B^s_{\pi,n-1} + B^s_{x,n-1}]^2 \right), \\
B^s_{0,n} = B^s_{0,n-1} - \mu_m + \mu_\pi - \frac{1}{2} \left( [\sigma_\pi + B^s_{\pi,n-1} \sigma_x]^2 + \sigma^2_{gw} [\lambda_{gw} + B^s_{gs,n-1}]^2 \right. \\
+ \left. \sigma^2_g ([\varphi_{\pi g} + \lambda_\eta + \varphi_{z g} B^s_{\pi,n-1}]^2 + \varphi^2_e (\varphi_{\pi x} + \lambda_e + B^s_{x,n-1} + \varphi_{z x} B^s_{\pi,n-1})^2) \right)
\]

for nominal ones.
A.2 Two Volatilities Model

A.2.1 Model Solution

The equilibrium price-to-consumption ratio is affine in the expected consumption and inflation rates and the two volatilities:

\[ v_t = \mu_v + A_x x_t + A_{gs} (\sigma_{gt}^2 - \sigma_g^2) + A_{xs} (\sigma_{xt}^2 - \sigma_x^2), \quad (A.4) \]

where the loadings satisfy

\[ A_x = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}, \quad A_{gs} = \frac{1}{2} \frac{(1 - \gamma)(1 - \frac{1}{\psi})}{1 - \kappa_1 \nu_g}, \quad A_{xs} = \frac{1}{2} \kappa_1^2 \left( \frac{1 - \gamma}{1 - \kappa_1 \nu_g} \right)^2 \quad (A.5) \]

The log-linearization parameter \( \kappa_1 \) is given implicitly by

\[ \log \kappa_1 = \log \delta + (1 - \frac{1}{\psi}) \mu_g + A_{gs} (1 - \kappa_1 \nu_g) \sigma_g^2 + A_{xs} (1 - \kappa_1 \nu_x) \sigma_x^2 + \frac{1}{2} \theta \kappa_1^2 \left( A_{gs}^2 \sigma_g^2 + A_{xs}^2 \sigma_x^2 \right) \quad (A.6) \]

The real discount rate now takes the following form:

\[ m_{t+1} = \mu_m + m_x x_t + m_{gs} (\sigma_{gt}^2 - \sigma_g^2) + m_{xs} (\sigma_{xt}^2 - \sigma_x^2) \]

\[ - \lambda_\eta \sigma_{gt} \eta_{t+1} - \lambda_e \sigma_{xt} e_{t+1} - \lambda_g \sigma_g w_{g,t+1} - \lambda_x \sigma_x w_{x,t+1}, \quad (A.7) \]

for

\[ \mu_m = \theta \log \delta - (\theta - 1) \log \kappa_1 - \gamma \mu_g, \quad m_x = -\frac{1}{\psi}, \quad m_{gs} = \frac{1}{2} (1 - \gamma)(\frac{1}{\psi} - \gamma), \quad (A.8) \]

\[ m_{xs} = \frac{1}{2} (\gamma - 1)(\frac{1}{\psi} - \gamma) \left( \frac{\kappa_1}{1 - \kappa_1 \rho} \right)^2, \]
The equilibrium real and nominal bond prices are affine in the state variables:

\[
q_{t,n} = -B_{0,n} - B_{x,n} x_t - B_{gs,n} (\sigma_{gt}^2 - \sigma_g^2) - B_{xs,n} (\sigma_{xt}^2 - \sigma_x^2), \tag{A.10}
\]

\[
q^\$_{t,n} = -B^\$_{0,n} - B^\$_{x,n} x_t - B^\$_{gs,n} (\sigma_{gt}^2 - \sigma_g^2) - B^\$_{xs,n} (\sigma_{xt}^2 - \sigma_x^2) - B^\$_{\pi,n} (\bar{\pi}_t - \mu_\pi), \tag{A.11}
\]

where the loadings satisfy the recursions

\[
B_{x,n} = \rho B_{x,n-1} - m_x,
\]

\[
B_{gs,n} = \nu_g B_{gs,n-1} - (m_{gs} + \frac{1}{2}\lambda_\eta^2),
\]

\[
B_{xs,n} = \nu_x B_{xs,n-1} - m_{xs} - \frac{1}{2}(\lambda_e + B_{x,n-1})^2,
\]

\[
B_{0,n} = B_{0,n-1} - \mu_m - \frac{1}{2}(\lambda_\eta^2 \sigma_g^2 + (\lambda_e + B_{x,n-1})^2 \sigma_x^2 + (\lambda_{gw} + B_{gs,n-1})^2 \sigma_{gw}^2 + (\lambda_{xw} + B_{xs,n-1})^2 \sigma_{xw}^2).
\]

and

\[
B^\$_{x,n} = \rho B^\$_{x,n-1} + \alpha_x B^\$_{\pi,n-1} - m_x,
\]

\[
B^\$_{\pi,n} = \alpha_\pi B^\$_{\pi,n-1} + 1,
\]

\[
B^\$_{gs,n} = \nu_g B^\$_{gs,n-1} - m_{gs} - \frac{1}{2}(\varphi_{\pi g} + \lambda_\eta + \varphi_{xg} B^\$_{\pi,n-1})^2,
\]

\[
B^\$_{xs,n} = \nu_x B^\$_{xs,n-1} - m_{xs} - \frac{1}{2}(\varphi_{\pi x} + \lambda_e + B^\$_{x,n-1} + \varphi_{zx} B^\$_{\pi,n-1})^2,
\]

\[
B^\$_{0,n} = B^\$_{0,n-1} - \mu_m + \mu_\pi - \frac{1}{2}((\sigma_{\pi} + B^\$_{\pi,n-1} \sigma_z)^2 + (\lambda_{gw} + B^\$_{gs,n-1})^2 \sigma_{gw}^2 + (\lambda_{xw} + B^\$_{xs,n-1})^2 \sigma_{xw}^2 + (\varphi_{\pi g} + \lambda_\eta + \varphi_{xg} B^\$_{\pi,n-1})^2 \sigma_g^2 + (\varphi_{\pi x} + \lambda_e + B^\$_{x,n-1} + \varphi_{zx} B^\$_{\pi,n-1})^2 \sigma_x^2).
\]

A.2.2 Model Implications

Table A.1 reports the adjustments to the baseline calibration values for the variance parameters and correlations of news across the countries. We keep the persistence of
Table A.1: Model Parameter Values, Two-Volatility Specification

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Consumption Growth Parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>Short-run volatility level</td>
<td>$\sigma_g$ 0.004</td>
</tr>
<tr>
<td>Short-run volatility persistence</td>
<td>$\nu_g$ 0.85</td>
</tr>
<tr>
<td>Short-run volatility of volatility</td>
<td>$\sigma_{gw}$ 1.15e-06</td>
</tr>
<tr>
<td>Long-run volatility level</td>
<td>$\sigma_x$ 0.04 $\times$ $\sigma_{g0}$</td>
</tr>
<tr>
<td>Long-run volatility persistence</td>
<td>$\nu_x$ 0.996</td>
</tr>
<tr>
<td>Long-run volatility of volatility</td>
<td>$\sigma_{xw}$ 0.06$^2$ $\times$ $\sigma_{gw}$</td>
</tr>
<tr>
<td><strong>Cross-Country Parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>Correlation of long-run news</td>
<td>$\tau_e$ 0.99995</td>
</tr>
<tr>
<td>Correlation of short-run news</td>
<td>$\tau_\eta$ 0.0</td>
</tr>
<tr>
<td>Correlation of short-run volatility news</td>
<td>$\tau_{gw}$ 0.00</td>
</tr>
<tr>
<td>Correlation of long-run volatility news</td>
<td>$\tau_{xw}$ 1.00</td>
</tr>
</tbody>
</table>

$s_2^{xt}$ at 0.996, and decrease the short-run volatility autoregressive coefficient to 0.85.

While we increase the variance of consumption growth and the volatility of the long-run variance, the variance long-run risks was decreased to keep the predictability of consumption growth low. In international dimension, we increase the correlation of long-run risks shocks across the countries and shift all the co-movements of volatilities to low frequency. For the interest of space, we do not report the calibration output for consumption, inflation and dividend processes as they are virtually the same as the in baseline case.

In Table A.2 we show the output for the nominal term structure and the violations of expectations hypothesis in the two-volatility setup. The first panel documents that the levels and variances of yields are very similar to the baseline calibration and are consistent with historical evidence. On the other hand, the violations of the expectations hypothesis in bond markets are more pronounced than in the baseline setup: for example, the slope coefficients in nominal bond projection are now equal to -0.26 and -0.45 at 2 and 5 year frequencies, respectively. The predictability of bond returns in single bond factor regressions also increases relative to the benchmark.
Table A.2: Two-Volatility Model: Bond Markets

<table>
<thead>
<tr>
<th></th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nominal Term Structure:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(y^s)$</td>
<td>5.23</td>
<td>5.35</td>
<td>5.52</td>
<td>5.73</td>
<td>5.98</td>
</tr>
<tr>
<td>$\sigma(y^s)$</td>
<td>2.04</td>
<td>1.92</td>
<td>1.84</td>
<td>1.82</td>
<td>1.84</td>
</tr>
<tr>
<td><strong>EH Projection:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>-0.26</td>
<td>-0.39</td>
<td>-0.43</td>
<td>-0.45</td>
<td></td>
</tr>
<tr>
<td><strong>Single Factor Projection:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>0.39</td>
<td>0.82</td>
<td>1.21</td>
<td>1.58</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.11</td>
<td>0.12</td>
<td>0.12</td>
<td>0.13</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Excess Market Return:</strong></td>
<td>7.51</td>
<td>12.11</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>Real Risk Free Rate:</strong></td>
<td>1.32</td>
<td>1.21</td>
<td>0.99</td>
</tr>
</tbody>
</table>

The population $R^2$ increase to $11 - 13\%$, while the loadings on the single bond factor remain the same.

Table A.3 shows that the nominal slope coefficients in foreign exchange regressions is $-1.2$ at 1 month horizon. It turn positive within 1 year and remains below one at the considered maturities (nominal one is equal to 0.86 at 5 year horizon), which is consistent with historical evidence. The decrease in persistence of short-run consumption volatility also helps to match the volatility of foreign exchange rate, which now becomes $15.78\%$ and is closer to the historical estimates of $11 - 12\%$ in the data.

Finally, the third panel of Table A.2 shows that model generates the equity risk premium of $7.5\%$ and the standard deviation of market returns of $12.11\%$, which is again closer to the estimates in the data. The mean of the real risk-free rate ($1.32\%$) and its volatility ($1.21\%$) are quite similar to the benchmark values. Therefore, the extension of the model to the two volatilities case can improve the quantitative predictions of the benchmark model.
Table A.3: **Two-Volatility Model: Currency Markets**

<table>
<thead>
<tr>
<th></th>
<th>1m</th>
<th>3m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>UIP Projection:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>-1.18</td>
<td>-0.81</td>
<td>0.25</td>
<td>0.66</td>
<td>0.81</td>
<td>0.82</td>
<td>0.86</td>
</tr>
<tr>
<td><strong>Std. Dev. AR(1)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal FX Rate:</td>
<td>15.77</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Real FX Rate:</td>
<td>15.68</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix B

Costly Learning Model Solution

B.1 Social Planner’s Problem

The learning decision of the social planner maximizes the ex-ante utility of the agent:

\[ s_t^* = \arg\max_s \{ J_t(U_t(s)) \}, \quad (B.1) \]

subject to the resource constraint (2.10):

\[ Y_t = C_t(s_t) + s_t \xi_t, \quad (B.2) \]

where the learning cost \( \xi_t \) is proportional to the aggregate income \( Y_t \):

\[ \xi_t = \chi Y_t, \quad (B.3) \]

for \( 0 \leq \chi < 1 \). From the resource constraint, it immediately follows that

\[ C_t(s_t) = Y_t(1 - \chi s_t). \quad (B.4) \]

Therefore, when the planner does not learn about the true state \( (s_t = 0) \), the agent’s consumption is equal to the aggregate income. On the other hand, when the planer
learns about the true state, \((s_t = 1)\), part of the endowment is sacrificed to cover the learning cost.

Conjecture that the life-time utility functions are proportional to income:

\[ U_t(s_t) = \phi_t(s_t)Y_t, \tag{B.5} \]

for \(s_t \in \{0, 1\}\). The optimal utility of the agent then is given by the learning choice specific counterpart evaluated at the optimal indicator \(s^*_t : U_t = U_t(s^*_t)\). The optimal utility next period takes into account the optimal learning choice tomorrow and can be written as \(U_{t+1} = \phi_{t+1}Y_{t+1}\), where to simplify the notations, we denote \(\phi_{t+1} \equiv \phi_{t+1}(s^*_{t+1})\).

Substitute the conjecture for \(U_t(s_t)\) and \(U_{t+1}\), and the consumption rule (B.4) into the definition of the life-time utility of the agent in (2.8) to obtain the following recursive formula for the utility per income \(\phi_t(s_t)\):

\[ \phi_t(s_t) = (1 - s_t\chi)^{1-\beta} \left( E_t^{s_t} \left( \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} \right)^{\frac{\beta}{1-\gamma}}. \tag{B.6} \]

As aggregate income \(Y_t\) is known in the beginning-of-the-period, it can be factored out from the the optimal condition for learning (B.1). We can then rewrite it in the following way:

\[ s^*_t = 1 \quad \text{if} \quad J_t(\phi_t(1)) > J_t(\phi_t(0)) \]

\[ = 0 \quad \text{if} \quad J_t(\phi_t(1)) \leq J_t(\phi_t(0)). \tag{B.7} \]

### B.2 Timing of Resolution of Uncertainty

The key aspect of our model is that the agent has a preference for a timing of the resolution of uncertainty. With standard expected utility preferences, the agent is indifferent to the timing of the resolution of uncertainty, and as a consequence, has
no incentive to learn for a cost. Indeed, consider a case when learning costs are zero, that is $\chi = 0$. Then, the utility of the agent corresponding to the indicator variable $s_t \in \{0, 1\}$ satisfies

$$U_t(s_t) = E_t^{s_t} \sum_{j=0}^{\infty} \beta^j u(Y_{t+j}).$$

The optimal learning policy in the expected utility case is based on the ex-ante expected utility given the beginning of period information. However, applying the law of iterated expectations, we obtain that

$$E_t U_t(1) = E_t \left( E^1_t \sum_{j=0}^{\infty} \beta^j u(Y_{t+j}) \right) = E_t^0 \sum_{j=0}^{\infty} \beta^j u(Y_{t+j}) = U_t(0).$$

In expectation, new information does not increase the utility of the agent. Therefore, even though the new information is costless, the agent has no incentive to learn it.

The results are very different when the agent has a preference for a timing of the resolution of uncertainty. Using the recursive solution for the utility per income ratio in (B.6), the solution to the optimal choice indicator above can be expanded in the following way:

$$s^*_t = 1 \left[ (1 - \chi)^{1-\beta} \left( E_t \left[ E^1_t \left( \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} \right]^{\beta} \right)^{\frac{1}{1-\gamma}} > E_t \left( \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right)^{\frac{\beta}{1-\gamma}} \right].$$

In general, the equilibrium utility per income ratio $\phi_t$ depends on the whole history of choice indicators $(s_t, s_{t-1}, \ldots)$. Let us make a simplifying assumption, which holds in the model setup of this paper, that if the agent learns the state today, then all the past learned states become irrelevant for the life-time utility of the agent. That is, $\phi_t(1)$ does not depend on the variables observed before time $t$. Now consider a case when the agent has a preference for early resolution of uncertainty, i.e. when the
risk aversion parameter $\gamma$ is greater than one. Conjecture that it is optimal to learn tomorrow, so that by assumption above, $\phi_{t+1}$ does not depend on the current choice indicator $s_t$. Then, using Jensen’s inequality type argument, it is easy to show that the agent learns the true state for a cost today if cost parameter $\chi$ is 0. Therefore, for small information costs, it is always optimal to learn about the underlying states if the agent prefers early resolution of uncertainty. On the other hand, if information costs are large, $(\chi \to 1)$, it is never optimal learn, so that $s_t$ is fixed at 0 in all time periods. For medium costs, the optimal solution to the choice indicator depends on the underlying state variables in the economy.

B.3 Utility and Learning Choice

As the volatility and consumption shocks are uncorrelated, we can separate the expected growth and volatility components in the equilibrium utility per income ratio, which simplifies the solution to the fixed-point recursion in (2.13). In this section we consider a general case with time-varying volatility, while in Appendix B.5 we show that the solution can be simplified even further when the volatility is constant.

Conjecture that for each choice indicator $s_t$ and corresponding states $\hat{x}_t(s_t), \omega_t^2(0)$ and $\sigma_t^2$ today, the life-time utility per income ratio satisfies,

$$\phi(s_t, \hat{x}_t(s_t), \omega_t^2(0), \sigma_t^2) = e^{B\hat{x}_t(s_t) + f(s_t, \sigma_t^2, \omega_t^2(0))},$$

(B.11)

for some utility loading $B$ and volatility function $f(s_t, \sigma_t^2, \omega_t^2(0))$. Note that the variance of the filtering error used in the value function is based on the beginning of period information; the actual value $\omega_t^2(s_t)$ depends deterministically on the beginning-of-period estimate $\omega_t^2(0)$ and learning choice $s_t$, see equation (2.26).

Let us fix the optimal choice $s_{t+1}^*$ tomorrow. We conjecture that $s_{t+1}^*$ depends only on the income volatility and beginning-of-period variance of the filtering error,
and not on the expected income and dividend factors, i.e. \( s_{t+1}^* = s^*(\sigma_{t+1}^2, \omega_{t+1}^2(0)) \).

Consider the equilibrium life-time utility from next period onward:

\[
\phi_{t+1} = \phi(s_{t+1}^*, \hat{x}_{t+1}(s_{t+1}^*), \omega_{t+1}^2(0), \sigma_{t+1}^2) = e^{B\hat{x}_{t+1}(s_{t+1}^*) + f_{t+1}},
\]

where for notational simplicity, we define \( f_{t+1} = f(s_{t+1}^*, \sigma_{t+1}^2, \omega_{t+1}^2(0)) \). Now, using (2.25),

\[
\log \left( \frac{\phi_{t+1} Y_{t+1}}{Y_t} \right) = B \left( \hat{x}_{t+1}(0) + s_{t+1}^*(x_{t+1} - \hat{x}_{t+1}(0)) \right) + f_{t+1} + \Delta y_{t+1}.
\]

Consider a recursive equation for the optimal utility per income ratio (B.6) for a given choice indicator \( s_t \) today. To evaluate \( E_t^{s_t} \left( \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} \), we use the law of iterated expectations where we first condition on \( I_{t+1} \). Then, \( \Delta y_{t+1}, \sigma_{t+1}^2 \) and therefore \( \hat{x}_{t+1}(0), s_{t+1}^* \) and \( f_{t+1} \) are known, while the only random component is the true state \( x_{t+1} \). Due to the Kalman filter procedure,

\[
x_{t+1} | I_{t+1} \sim N(\hat{x}_{t+1}(0), \omega_{t+1}^2(0)),
\]

where \( \hat{x}_{t+1}(0) \) and \( \omega_{t+1}^2(0) \) satisfy (2.25) and (2.26). Therefore the right-hand side expectation in the utility recursion (2.13) is equal to,

\[
E_t^{s_t} \left( \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} = E_t^{s_t} e^{(1-\gamma)[B\hat{x}_{t+1}(0) + f_{t+1} + \Delta y_{t+1} + \frac{1}{2}(1-\gamma)B\omega_{t+1}^2(0)s_{t+1}^*]}.
\]

Now by conjecture, \( s_{t+1}^* \) and thus \( f_{t+1} \) and \( \omega_{t+1}(s_{t+1}^*) \) are driven by income volatility shocks, which are independent from income innovations and therefore, from the filtered shock \( u_{t+1}(s_t) \). Thus,

\[
E_t^{s_t} \left( \phi_{t+1} \frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} = e^{(1-\gamma)(\mu + (B\rho + 1)\hat{x}_t(s_t) + \frac{1}{2}(1-\gamma)(BK_t(s_t)+1)^2(\omega_t^2(s_t) + \sigma_t^2))}
\times E_t^{s_t} e^{(1-\gamma)[f_{t+1} + \frac{1}{2}(1-\gamma)B^2\omega_{t+1}^2(0)s_{t+1}^*]}.
\]

(B.16)
Therefore, using the equilibrium utility recursion (B.6) and the conjectured solution for the life-time utility of the agent (B.11) and matching the coefficients, we obtain that loading on expected growth is equal to

\[ B = \frac{\beta}{1 - \beta \rho}, \]  

(B.17)

while the volatility function satisfies

\[
\begin{align*}
    f(s_t, \sigma_t^2, \omega_t^2(0)) &= (1 - \beta) \ln (1 - s_t \chi) + \beta \mu \\
    &+ \beta \frac{1}{2} (1 - \gamma)(BK_t(s_t) + 1)^2 (\omega_t^2(s_t) + \sigma_t^2) + \frac{\beta}{1 - \gamma} \ln E_t s_{t+1} e^{(1-\gamma)\left[f_{t+1} + \frac{1}{2} (1-\gamma)B^2 \omega_{t+1}^2(0)s_{t+1}^2\right]}.
\end{align*}
\]

(B.18)

Solution to $B$ and $f$ verifies the conjecture for the life-time utility of the agent.

Now, given the utility equation (B.11) and the dynamics of the factors, we can rewrite the optimal condition for a learning choice (2.9). Notably, the expected growth component drops out, so that the optimal choice indicator depends only on the learning and aggregate variance:

\[
    s_t^* = 1 \left[ \frac{1}{2} (1 - \gamma) B^2 \omega_t^2(0) + f_t(1, \sigma_t^2, \omega_t^2(0)) > f_t(0, \sigma_t^2, \omega_t^2(0)) \right].
\]

(B.19)

Using the optimal condition for $s_{t+1}^*$ tomorrow to rewrite the recursive equation of the volatility function (B.18) in the following way:

\[
\begin{align*}
    f(s_t, \sigma_t^2, \omega_t^2(0)) &= (1 - \beta) \ln (1 - s_t \chi) + \beta \mu + \beta \frac{1}{2} (1 - \gamma)(BK_t(s_t) + 1)^2 (\omega_t^2(s_t) + \sigma_t^2) \\
    &+ \frac{\beta}{1 - \gamma} \ln E_t s_{t+1} e^{(1-\gamma)\max\left[\frac{1}{2} (1-\gamma)B^2 \omega_{t+1}^2(0) + f_{t+1}(1, \sigma_{t+1}^2, \omega_{t+1}^2(0)) : f_{t+1}(0, \sigma_{t+1}^2, \omega_{t+1}^2(0))\right]}.
\end{align*}
\]

(B.20)

That is, the volatility function $f$ can be obtained as fixed-point solution to the equation above, given the evolution of the variance of the filtering error in (2.23) and (2.26).
B.4 Dividend Asset

Consider an asset with dividend stream \( \Delta d_t = \mu + \varphi_d(\Delta y_t - \mu) \).

The equilibrium price-dividend ratio solves,

\[
\nu_t(s_t) = H \hat{x}_t(s_t) + h(s_t, \sigma_t^2, \omega_t^2(0)),
\]

(B.21)

The log-linearized returns satisfy

\[
\begin{align*}
    r_{d,t+1} &= \kappa_0 + \varphi_d \mu + (H(H_1 - 1) + \varphi_d) \hat{x}_t(s_t) - h(s_t, \sigma_t^2, \omega_t^2(0)) \\
    &+ (\kappa_1 H K_t(s_t) + \varphi_d) u_{t+1}(s_t) + \kappa_1 h(s_{t+1}^*, \sigma_{t+1}^2, \omega_{t+1}^2(0)) + \kappa_1 H s_{t+1}^*(x_{t+1} - \hat{x}_{t+1}(0)).
\end{align*}
\]

(B.22)

Using Euler conditions and the equilibrium solution for discount factor for the log-linearized dividend return, we obtain that the loading \( H \) satisfies

\[
H = \frac{\varphi_d - 1}{1 - \kappa_1 \rho}.
\]

(B.23)

The price-dividend levels are given recursively by

\[
\begin{align*}
    h_t(s_t, \sigma_t^2, \omega_t^2(0)) &= \ln \beta + \kappa_0 \\
    &+ \frac{1}{2} (\varphi_d - 1 + \kappa_1 H K_t(s_t))(\varphi_d - 1 + \kappa_1 H K_t(s_t) - 2(\gamma - 1)(1 + B K_t(s_t))(\sigma_t^2 + \omega_t^2(s_t)) \\
    &+ \ln E_t e^{\kappa_1 h_{t+1} + \frac{1}{2}(\kappa_1 H - (\gamma - 1) B )^2 s_{t+1}^2 \omega_{t+1}^2(0) - (\gamma - 1) f_{t+1}} \\
    &- \ln E_t e^{(1-\gamma)(f_{t+1} + \frac{1}{2}(1-\gamma) B^2 s_{t+1}^2 \omega_{t+1}^2(0)),}
\end{align*}
\]

(B.24)

To solve for the approximating constants \( \kappa_0 \) and \( \kappa_1 \), we use the numerical procedure discussed in Bansal et al. (2007b), who develop a method to solve for the endogenous constants associated with each return and document that the numerical solution to the model is accurate.
B.5 Constant Volatility Case

In the general case with time-varying volatility, we first solve the model numerically by discretizing income and filtering volatility states and applying fixed-point iterations to the volatility functions \( f(s_t, \sigma_t^2, \omega_t^2(0)) \) and \( h(s_t, \sigma_t^2, \omega_t^2(0)) \). We verify that the numerical solutions to these functions are very close to being linear, so in calibrations we approximate these function to be affine in the two volatilities, and solve for the loadings numerically.

When the income volatility is constant \( \sigma_t = \sigma_0^2 \), the variance of the filtering error becomes a deterministic function of time since the last learning about the true state. In this case, the optimal learning decision is purely time-dependent, so that the investors choose to learn about the underlying state if the last time they did so was \( N \) or more periods ago.

Assume we know the optimal \( N \), and consider the time interval from 1 to \( N \). In equilibrium, the agent starts filtering in period 1 and learns about the true state for a cost in period \( N \), afterwards the solution repeats itself.

The equilibrium volatility functions are non-random functions of time, so to simplify the notations, denote them \( f_i : \)

\[
f_i = f(0, \sigma_0^2, \omega_i^2(0)), \quad 1 \leq i < N,
\]

\[
f_N = f(1, \sigma_0^2, \omega_N^2(0)).
\]

Now we can rewrite the recursions in (B.18) as a system of linear equations (to
simplify the exposition, we consider the case $N > 2$:

$$f_1 - \beta f_2 = \beta \mu + \frac{1}{2} \beta (1 - \gamma) (BK_1(0) + 1)^2(\omega_1^2(0) + \sigma_0^2),$$

\[ \ldots \]

$$f_i - \beta f_{i+1} = \beta \mu + \frac{1}{2} \beta (1 - \gamma) (BK_i(0) + 1)^2(\omega_i^2(0) + \sigma_0^2), \quad 2 \leq i < N - 1$$

$$f_{N-1} - \beta f_N = \beta \mu + \frac{1}{2} \beta (1 - \gamma) ((BK_{N-1}(0) + 1)^2(\omega_{N-1}^2(0) + \sigma_0^2) + B^2 \omega_N^2(0)),$$

$$f_N - \beta f_1 = (1 - \beta) \ln(1 - \chi) + \beta \mu + \frac{1}{2} \beta (1 - \gamma) (BK_N(1) + 1)^2(\omega_N^2(1) + \sigma_0^2).$$

(B.25)

This system can be easily solved for equilibrium volatility functions $f_i, i = 1, 2, \ldots, N$.

Now we need to make sure that the chosen $N$ is indeed optimal, that is, the agent is not better off deviating from the conjectured learning rule. If investors were to learn about the state earlier than in the $N$th period, their utility would be $f_N$. To preclude this deviation, we need to have that (see condition (B.19))

$$\frac{1}{2} (1 - \gamma) B^2 \omega_i^2(0) + f_N < f_i,$$

for $1 \leq i < N$.

On the other hand, consider a scenario when investors fail to learn about the true state at time $N$. By conjecture, the optimal behavior in period $N + 1$ is to learn, therefore, from the expression (B.18), the utility that the investors would get by deviating is given by,

$$\tilde{f}_N = \beta \mu + \frac{1}{2} \beta (1 - \gamma) ((BK_N(0) + 1)^2(\omega_N^2(0) + \sigma_0^2) + B^2 \omega_{N+1}^2(0)) + \beta f_N.$$

Following optimality condition for choice indicator (B.19), we then need to have that

$$\frac{1}{2} (1 - \gamma) B^2 \omega_N^2(0) + f_N > \tilde{f}_N.$$

(B.27)
In practice, we loop from a low value of \( N \) until we satisfy both optimality conditions (B.26)-(B.27), where the volatility functions \( f_i \) solve the linear system (B.25). In numerical calibrations, the optimal \( N \) is always unique: when \( N \) is lower than optimum, we violate the last condition (B.27), so that the agent can increase the utility by estimating, rather than learning about the state for a cost; for \( N \) higher than optimum, (B.26) is not satisfied, and investors would want to learn sooner.

We follow the same approach to find the volatility functions in the price-dividend ratio. As \( h_s \) are no longer random, we can rewrite their recursion in (B.24) much in the same way as (B.25), as we already know the optimal choice indicator and utility functions \( f_i \). To solve for the approximating constants \( \kappa_0 \) and \( \kappa_1 \), we use the numerical procedure discussed in Bansal et al. (2007b).

### B.6 Parametric Jump Model

To compare our model implications to the results from the above studies, we fit a discrete-time GARCH-jump specification for returns, which feature autoregressive stochastic volatility and time-varying arrival intensity of jumps in returns\(^1\). Specifically, the return dynamics is given by

\[
    r_t = \mu_r + a_{1,t} + a_{2,t}. \tag{B.28}
\]

The first component \( a_{1,t} \) represents a smooth Gaussian component of returns, whose conditional volatility is time-varying and follows GARCH(1,1) process:

\[
a_{1,t} = \sqrt{v_{t-1}} z_t, \quad z_t \sim N(0,1), \tag{B.29}
\]

\[
v_t = \sigma_v^2 + \beta_v v_{t-1} + \alpha_v (r_t - \mu_r)^2. \tag{B.30}
\]

The second shock $a_{2,t}$ is driven by Poisson jumps:

$$a_{2,t} = \sum_{k=1}^{n_t} \xi_{t,k} - \mu_j \lambda_{t-1}. \quad (B.31)$$

The jump size distribution is Normal:

$$\xi_{t,k} \sim N(\mu_j, \sigma_j^2), \quad (B.32)$$

and the arrival of number of jumps $n_t = 0, 1, 2, \ldots$ is described by a conditional Poisson distribution with intensity $\lambda_t$, so that

$$P_{r_{t-1}}(n_t = j) = \frac{\exp(-\lambda_{t-1}) \lambda_{t-1}^j}{j!}. \quad (B.33)$$

As we are interested in the predictability of jumps by market variance, we follow the literature and model the jump intensity to be linear in the variance of returns,

$$\lambda_t = \lambda_0 + \lambda_t \nu_t. \quad (B.34)$$

The above specification of return dynamics can be readily estimated by MLE using the sample and simulated data. In estimations, as we want to capture large, infrequent moves in asset prices, we restrict the unconditional jump intensity not to exceed 1 jump per year.

We perform a Monte-Carlo study where we estimate specification (B.28)-(B.34) for 100 simulations of 80 years of daily returns from a time-varying volatility model, and we compare the results to the estimates based on the sample data. As can be seen in Table B.1, the model matches quite well the dynamics of the time-varying volatility of smooth component of the returns: the overall persistence is 0.96 in the model, compared to 0.98 in the data, and the intercept and ARCH and GARCH parameter coefficients are close as well. The model can also capture the key findings
Table B.1: **Estimation of GARCH-jump dynamics: Data and Model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Data</th>
<th></th>
<th>Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
<td>Mean</td>
<td>S.E.</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.08</td>
<td>(0.004)</td>
<td>0.12 (0.021)</td>
<td></td>
</tr>
<tr>
<td>$\beta_v$</td>
<td>0.91</td>
<td>(0.004)</td>
<td>0.94 (0.018)</td>
<td></td>
</tr>
<tr>
<td>$\alpha_v$</td>
<td>0.07</td>
<td>(0.003)</td>
<td>0.03 (0.005)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>0.00</td>
<td>(0.001)</td>
<td>0.00 (0.002)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_l$</td>
<td>25.78</td>
<td>(16.21)</td>
<td>152.00 (60.06)</td>
<td></td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>-1.6</td>
<td>(0.46)</td>
<td>0.60 (0.78)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_j$</td>
<td>2.9</td>
<td>(0.33)</td>
<td>4.30 (0.77)</td>
<td></td>
</tr>
</tbody>
</table>

In the literature regarding the frequency and predictability of jumps. The estimated jump arrival intensity loading on market variance is positive and highly significant both in the model and in the data, though, it is estimated with a large standard error. The mean jump size is $-1.6\%$ in the data, and is slightly positive but insignificant in the model. The standard deviations of the jump distribution are 3% and 4% in the data and model, respectively. The jumps explain about 10% of the variation of the returns in the data and the model, and their average frequency is once in one and two years, respectively.
Appendix C

Confidence Model Solution

C.1 Kalman Filter

Given the dynamics of the underlying economy in (3.5)-(3.6) and the specification of signals in (3.8), the distribution of the states given the current information set and next-period confidence measure is conditionally Normal:

\[
\begin{bmatrix}
        x_{t+1} \\
        \Delta c_{t+1} \\
        \bar{x}_{t+1}
\end{bmatrix} \mid \mathcal{I}_t, V_{t+1} \sim N\left( \begin{bmatrix} \hat{x}_t \\
        \mu + \hat{x}_t \\
        \rho \hat{x}_t \end{bmatrix}, \Sigma_{t+1} \right),
\]

(C.1)

where the variance-covariance matrix is given by,

\[
\Sigma_{t+1} = \begin{bmatrix}
        \rho^2 \omega_t^2 + \varphi^2 \sigma_t^2 & \rho \omega_t^2 & \rho^2 \omega_t^2 + \varphi^2 \sigma_t^2 \\
        \rho \omega_t^2 & \omega_t^2 + \sigma_t^2 & \rho \omega_t^2 \\
        \rho^2 \omega_t^2 + \varphi^2 \sigma_t^2 & \rho \omega_t^2 & \rho^2 \omega_t^2 + \varphi^2 \sigma_t^2 + V_{t+1}
\end{bmatrix}.
\]

(C.2)

The innovation representation of the system can then be written in the following
way:

\[
\Delta c_{t+1} = \mu + \hat{x}_t + a_{c,t+1}, \quad (C.3)
\]
\[
\bar{x}_{t+1} = \rho \hat{x}_t + a_{x,t+1}, \quad (C.4)
\]
\[
\hat{x}_{t+1} = \rho \hat{x}_t + K_{1,t+1} a_{c,t+1} + K_{2,t+1} a_{x,t+1}, \quad (C.5)
\]

where the Kalman Filter weights and the update for the filtering variance \(\omega_t^2\) satisfy standard equations

\[
K_{t+1} = \Sigma_{t+1}^{12} (\Sigma_{t+1}^{22})^{-1},
\]
\[
\omega_{t+1}^2 = \Sigma_{t+1}^{11} - \Sigma_{t+1}^{12} (\Sigma_{t+1}^{22})^{-1} \Sigma_{t+1}^{21}, \quad (C.6)
\]

where the superscripts refer to the partitioning of \(\Sigma_{t+1}\) into four blocks, such that \(\Sigma_{t+1}^{11}\) is the \((1, 1)\) element of the matrix, \(\Sigma_{t+1}^{12}\) contain the elements from the first row and second and third columns, etc. The explicit solutions for the Kalman Filter weights satisfy

\[
K_{1,t+1} = \frac{\rho \omega_t^2 V_{t+1}}{(\omega_t^2 + \sigma_t^2)V_{t+1} + (\varphi_e^2 \sigma_t^2 + (\varphi_e^2 + \rho^2)\omega_t^2)\sigma_t^2}, \quad (C.7)
\]
\[
K_{2,t+1} = \frac{(\varphi_e^2 \sigma_t^2 + (\varphi_e^2 + \rho^2)\omega_t^2)\sigma_t^2}{(\omega_t^2 + \sigma_t^2)V_{t+1} + (\varphi_e^2 \sigma_t^2 + (\varphi_e^2 + \rho^2)\omega_t^2)\sigma_t^2}, \quad (C.8)
\]

while the evolution of the variance of the filtering error is given by

\[
\omega_{t+1}^2 = V_{t+1} K_{2,t+1}. \quad (C.9)
\]

In the preferred specification, the Kalman Filter weights in the innovations representation of the system are constant. When investors do not look at consumption data and only update based on the average forecast, \(K_1 = 0\) and \(K_2\) is a steady-state solution to

\[
K_{2,t+1} = \frac{\rho^2 \omega_t^2 + \varphi_e^2 \sigma_t^2}{\rho^2 \omega_t^2 + \varphi_e^2 \sigma_t^2 + V_{t+1}}. \quad (C.10)
\]
To solve for the steady state of the system, I plug the solution for filtering uncertainty in \( w_t^2 = K_2V_t \) into the above equation and solve a quadratic equation for the constant value of \( K_2 \) when the volatility processes \( V_t \) and \( \sigma_t^2 \) are set to their unconditional means.

C.2 Discount Factor

The aggregate consumption volatility \( \sigma_t^2 \) follows a square-root process specified in (3.7), while the dynamics of the confidence measure is given by a discrete-time jump-diffusion specification outlined in (3.14). The distribution of jump size \( J_{t,t+1} \) is defined by its moment generating function,

\[
l(y) \equiv Ee^{yJ_t}.
\]  
(C.11)

For example, when jump size follows exponential distribution with mean jump \( \mu_j \),

\[
l(y) = (1 - \mu_jy)^{-1}.
\]  
(C.12)

The conditional variance-covariance of consumption and expected growth shocks is given by,

\[
\Sigma_{cx,t+1} = Var \begin{bmatrix} \sigma_{c,t+1} \\ \sigma_{x,t+1} \end{bmatrix} = \begin{bmatrix} K_2V_t + \sigma_t^2 & \rho K_2V_t \\ \rho K_2V_t & \rho^2 K_2V_t + \varphi_\sigma^2 \sigma_t^2 + V_{t+1} \end{bmatrix}.
\]  
(C.13)

The log price-to-consumption ratio \( p_c \) is linear in the states of the economy:

\[
p_c = B_0 + B_x \tilde{x}_t + B_vV_t + B_\sigma \sigma_t^2.
\]  
(C.14)

Using Euler equation (1.11), I can directly solve for the loading \( B_x \):

\[
B_x = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}.
\]  
(C.15)
The loading on the confidence measure $B_v$ satisfies non-linear equation

$$\frac{1}{2} \theta_t^2 \kappa_1^2 \sigma_\omega^2 z^2 - (1 - \kappa_1 (\nu - \lambda_1 \mu_j)) z + \frac{1}{2} \theta B^2 K_2 ((1 - (1 - K) \kappa_1 \rho) + \kappa_1 K_2) + \frac{\lambda_1}{\theta} (l(\theta \kappa_1 z) - 1) = 0,$$

(C.16)

for $z = B_v + \frac{1}{2} \theta \kappa_1 B^2 K_2$, while $B_\sigma$ solves a quadratic equation

$$\frac{1}{2} \theta_t^2 \kappa_1^2 \varphi_w^2 B^2_\sigma - (1 - \nu c \kappa_1) B_\sigma + \frac{1}{2} \theta \left( (1 - \frac{1}{\psi})^2 + \kappa_1^2 B^2_\sigma K^2_2 \varphi^2_e \right) = 0. \tag{C.17}$$

Finally, the log-linearization parameter, which is pinned down by the equilibrium level of the price-consumption ratio, satisfies the following non-linear equation:

$$\log \kappa_1 = \log \delta + (1 - \frac{1}{\psi}) \mu + B_\sigma (1 - \kappa_1 \nu c) \sigma^2$$

$$+ (B_v (1 - \kappa_1) + \kappa_1 (1 - \nu) z) \sigma^2_v + \frac{\lambda_0}{\theta} (l(\theta \kappa_1 z) - \theta \kappa_1 z \mu_j - 1). \tag{C.18}$$

As in Eraker and Shaliastovich (2008), in case of multiple roots for $B_\sigma$ and $B_v$ I choose the solution which is non-explosive as the variation in $V_t$ or $\sigma^2_t$ is approaching zero.

Using the equilibrium solution to the price-consumption ratio, I can write down the expression for the discount factor in the following way:

$$m_{t+1} = m_0 + m_x x_t + m_v V_t + m_\sigma \sigma^2_t$$

$$- \lambda_c a_\zeta t + \lambda_z K_2 a_{x,t-1} - \lambda_v \left( \sigma_w \sqrt{V_t w_{t+1}} + Q_{t+1} \right) - \lambda_\sigma \varphi \sigma_t w_{c,t+1}, \tag{C.19}$$

where the discount factor loadings and the prices of risks are pinned down by the dynamics of factors and preference parameters of the investors. Their solutions are given by,

$$m_x = -\frac{1}{\psi}, \quad m_v = (1 - \theta) B_v (1 - \kappa_1 \nu), \quad m_\sigma = (1 - \theta) B_\sigma (1 - \kappa_1 \nu c),$$

$$m_0 = \theta \log \delta + (1 - \theta) \log \kappa_1 - \gamma \mu - m_v \sigma^2_v - m_\sigma \sigma^2, \tag{C.20}$$

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and
\[ \lambda_x = (1 - \theta) \kappa_1 B_x, \quad \lambda_\sigma = (1 - \theta) \kappa_1 B_\sigma, \quad \lambda_v = (1 - \theta) \kappa_1 B_v. \] (C.21)

C.3 Asset Prices

Consider a log payoff tomorrow expressed as,
\[ p_{n-1,t+1} = F_0 n-1 + F_{x,n-1} x_{t+1} + F_{v,n-1} V_{t+1} + F_{\sigma,n-1} \sigma_{t+1}^2 + F_{g,n-1} \Delta c_{t+1} + F_{d,n-1} \eta_{d,t+1}. \] (C.22)

Then, the solution for the coefficients in its log price today \( p_{n,t} \) satisfies
\[ F_{g,n} = F_{d,n} = 0, \]
\[ F_{x,n} = m_x + F_{x,n-1} \rho + F_{g,n-1}, \]
\[ F_{\sigma,n} = m_\sigma + F_{\sigma,n-1} \nu_c + \frac{1}{2} ((F_{g,n-1} - \lambda_c)^2 + \varphi_c^2 (F_{x,n-1} - \lambda_x)^2 K_2^2 + \varphi_v^2 (F_{\sigma,n-1} - \lambda_\sigma)^2 + F_{d,n-1}^2), \]
\[ F_{v,n} = m_v + \frac{1}{2} (F_{g,n-1} - \lambda_c + \rho (F_{x,n-1} - \lambda_x) K_2) K_2 + (q_{v_c} + \lambda_v) \nu + \frac{1}{2} q_{v_{12}}^2 \sigma_w^2 + \lambda_1 (l(q_{v_c}) - q_{v_c} \mu_j - 1), \]
\[ F_{0,n} = m_0 + F_{0,n-1} + F_{g,n-1} \mu + F_{\sigma,n-1} \sigma^2 (1 - \nu_c) + (q_{v_c} + \lambda_v) \sigma_v^2 (1 - \nu) + \lambda_0 (l(q_{v_c}) - q_{v_c} \mu_j - 1) \] (C.23)

for \( q_{v_c} = F_{v,n-1} - \lambda_v + \frac{1}{2} (F_{x,n-1} - \lambda_x)^2 K_2^2. \)

Setting \( F_{0,n-1} = F_{x,n-1} = F_{v,n-1} = F_{\sigma,n-1} = F_{g,n-1} = F_{d,n-1} = 0 \) in the above recursion, I can obtain the solution to \( n \)-period real risk-free rate.

On the other hand, the price-dividend ratio is given by,
\[ pd_t = H_0 + H_x x_t + H_v V_t + H_\sigma \sigma_t^2, \] (C.24)
where the loadings satisfy the following equations:
\[ H_x = m_x + \kappa_{d,1} \rho H_x + \phi, \]
\[ H_\sigma = m_\sigma + \kappa_{d,1} H_\sigma \nu_c + \frac{1}{2} ((\phi - \lambda_c)^2 + \varphi_c^2 (\kappa_{d,1} H_x - \lambda_x)^2 K_2^2 + \varphi_v^2 (\kappa_{d,1} H_\sigma - \lambda_\sigma)^2 + \varphi_2^2), \]
\[ H_v = m_v + \frac{1}{2} (\phi - \lambda_c + \rho (\kappa_{d,1} H_x - \lambda_x) K_2) K_2 + (q_{v_c} + \lambda_v) \nu + \frac{1}{2} q_{v_{12}}^2 \sigma_w^2 + \lambda_1 (l(q_{v_c}) - q_{v_c} \mu_j - 1), \] (C.25)
for \( q_{vx} = \kappa_{d,1}H_v - \lambda_v + \frac{1}{2}(\kappa_{d,1}H_x - \lambda_x)^2K_2^2 \), and the log-linearization parameter

\[
\log \kappa_{d,1} = m_0 + \mu_d + \left( H_v(1 - \kappa_{d,1}\nu) + \frac{1}{2}(\kappa_{d,1}H_x - \lambda_x)^2K_2^2(1 - \nu) \right) \sigma_v^2 \tag{C.26}
\]

\[ + H_\sigma(1 - \kappa_{d,1}\nu_c)\sigma^2 + \lambda_0(l(q_{vx}) - q_{vx}\mu_j - 1). \]

C.4 Option Prices

The option prices are computed using the approach in Lewis (2000). Unlike other methods in the literature, it relies on a single integration along the complex line, which reduces computational burden (see Eraker and Shaliastovich, 2008).

The option price with strike \( K \) and maturity \( n \) is given by,

\[
C_t(K/P_t, n) = E_t \left[ M_{t,t+n} \max(e^{pt+n} - K, 0) \right]
= \frac{1}{2\pi} \left[ \int_{iz_1}^{iz_2} E_t \left( M_{t,t+n}e^{-izp_t+n} \right) \hat{w}(z)dz \right], \tag{C.27}
\]

where \( M_{t,t+n} \) is the discount factor which can be used to price \( n \)-period ahead payoffs, \( p_t \) is the log equity price and \( \hat{w}(z) \) is the generalized Fourier transform of the payoff function of the option equal to,

\[
\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx} (e^x - K)^+ dx
= -\frac{K^{iz+1}}{z^2 - iz}. \tag{C.28}
\]

The integration region is parallel to the real line in the complex plane, and \( z_i \equiv \text{Im}(z) > 1 \) for call options and \( z_i < 0 \) for put options.

Using the equilibrium solution to the discount factor and asset valuations, the expectation inside the integral in (C.27) is given by

\[
\log E_t e^{mt+n-izp_t+n} = G_{0,n} + G_{x,n}\hat{x}_t + G_{v,n}V_t + G_{\sigma,n}\sigma_t^2 - izp_t, \tag{C.29}
\]

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where complex-valued loadings $G_{0,n}, G_{x,n}, G_{v,n}$ and $G_{\sigma,n}$ satisfy recursive equations similar to those computed in Appendix C.3.

Hence, the equilibrium put option price normalized by the equity price satisfies

$$
\frac{C_t(K/P_t, n)}{P_t} = \frac{1}{2\pi P_t} \left[ e^{G_{0,n} + G_{x,n} \hat{x}_t + G_{v,n} V_t + G_{\sigma,n} \sigma_t^2 + iz \log(K/P_t)} \int_{iz - \infty}^{iz + \infty} \frac{dz}{iz - z^2} \right] \quad \text{(C.30)}
$$

C.5 Confidence Measure

I use the cross-section of individual forecasts from the Survey of Professional Forecasts to calculate the average (consensus) forecast and its standard error for the next-quarter GDP for the period of 1996 to 2007. The survey started in the last quarter of 1968 as a joint project of the American Statistical Association and the National Bureau of Economic Research; in 1990 it was taken by the Federal Reserve Bank of Philadelphia. The data set contains quarterly forecasts on a variety of macroeconomic and financial variables made by the professional forecasters who largely come from the business world and Wall Street, see Croushore (1993) for details and Zarnowitz and Braun (1993) for a comprehensive study of the survey.

I use forecasts of nominal GDP and price index and to back out the average forecast and uncertainty in average forecast for real GDP. Specifically, for each quarter $t$ let $NGDP_{i,t}$ and $P_{i,t}$ denote the next quarter forecasts of nominal GDP and price level of forecaster $i$. If $n_t$ is the number of available forecasts, then the average forecast for the log real GDP ($RGDP$) growth rate is

$$
\Delta \log(RGDP)_t = \frac{1}{n_t} \sum_{i=1}^{n_t} \left( \log \frac{NGDP_{i,t}}{NGDP_t} - \log \frac{P_{i,t}}{P_t} \right), \quad \text{(C.31)}
$$

where $NGDP_t$ and $P_t$ are the current values of the series known to the forecasters. For my sample, the average number of forecasts each period is about 40.
The cross-sectional variance of forecasts divided by the number of forecasts then determines the uncertainty (variance) in the average forecast:

\[
V_t = \frac{1}{n_t} \text{Var} \left( \log \frac{RGDP_{i,t}}{RGDP_t} \right)
\]

\[
= \frac{1}{n_t} \left( \frac{1}{n_t - 1} \sum_{i=1}^{n_t} \left( \log \frac{NGDP_{i,t}}{NGDP_t} - \log \frac{P_{i,t}}{P_t} - \Delta \log(RGDP)_t \right)^2 \right). \tag{C.32}
\]

The cross-sectional variance of the forecasts adjusted by the number of forecasts provides an unbiased estimate for the confidence measure in the model. To make the inference robust to possible outliers and errors, I delete observations which are more than two standard deviations from the sample mean. The above calculations of the confidence measure in the data follow Bansal and Shaliastovich (2008); David and Veronesi (2008) and Buraschi and Jitsov (2006) use similar approach to compute the uncertainty measures which rely on the cross-sectional dispersion in signals.

The confidence measure in the data is based on quarterly forecasts of real GDP, however, in the model the signals are for the next month expected consumption growth. To deal with the difference in scale, I adjust the confidence measure in the data by the ratio of the calibrated unconditional consumption variance \( \sigma^2 \) over the variance of realized GDP over the sample period (its standard deviation is 1%, annualized). Indeed, the properties of the scaled confidence measure are very similar across different macroeconomic series in the data; Bansal and Shaliastovich (2008) document that the ratio of the confidence measure over conditional variance behaves very similarly for real GDP and industrial production forecasts.

I next deal with the time-aggregation in the uncertainty in forecasts using the approximation of multi-period consumption growth dynamics developed in Bansal et al. (2007b). Using Taylor expansion, log quarterly consumption is approximately
equal to,

$$\log(C_{t+1} + C_{t+2} + C_{t+3}) \approx \log(3C_t) + \Delta c_{t+1} + \frac{2}{3} \Delta c_{t+2} + \frac{1}{3} \Delta c_{t+3}. \quad (C.33)$$

Using the specification of the consumption dynamics, the true expected quarterly consumption can be written as

$$E_t^{true} \log(C_{t+1} + C_{t+2} + C_{t+3}) \approx \log(3C_t) + 2\mu + (1 + \frac{2}{3} \rho + \frac{1}{3} \rho^2)x_t. \quad (C.34)$$

The true expected quarterly consumption growth is proportional to the true expected monthly consumption growth, up to known constants and consumption levels. Hence, the uncertainty in the estimate of expected quarterly consumption is proportional to the uncertainty in the estimate of expected monthly consumption, with the proportionality coefficient of $(1 + \frac{2}{3} \rho + \frac{1}{3} \rho^2)^2$.

I use the above scaling factor to further adjust the confidence measure based on quarter-ahead forecasts to obtain its counterpart for the next month expected growth.
C.6 GMM Estimation

The unconditional moments of $V$ are given by,

$$E(V_t) = \sigma_v^2,$$

$$E(V_t - \sigma_v^2)^2 = \frac{\sigma_w^2 \sigma_v^2 + 2 \mu_j^2 (\lambda_0 + \lambda_1 \sigma_v^2)}{1 - \nu^2},$$

$$E(V_t - \sigma_v^2)^3 = \frac{3}{1 - \nu^3} (\nu (\sigma_w^2 + 2 \mu_j^2 \lambda_1) E(V_t - \sigma_v^2)^2 + \mu_j^3 (\lambda_0 + \lambda_1 \sigma_v^2)), $$

$$E(V_t - \sigma_v^2)^4 = \frac{1}{1 - \nu^4} (6 \nu^2 (\sigma_w^2 + 2 \mu_j^2 \lambda_1) E(V_t - \sigma_v^2)^3$$

$$+ (6 \nu^2 (\sigma_w^2 \sigma_v^2 + 2 \mu_j^2 (\lambda_0 + \lambda_1 \sigma_v^2))) + 3 \sigma_w^4 + 10 \mu_j^2 \sigma_w^2 \lambda_1 + 12 \mu_j^3 \nu \lambda_1) E(V_t - \sigma_v^2)^2$$

$$+ (3 \sigma_w^4 + 10 \mu_j^2 \sigma_w^2 \lambda_1) \sigma_v^4 + (10 \mu_j^2 \sigma_w^2 \lambda_0 + 16 \mu_j^4 \lambda_1) \sigma_v^2 + 16 \mu_j^4 \lambda_0),$$

$$E(V_{t+m} V_t) = \sigma_v^4 + \nu^m E(V_t - \sigma_v^2)^2.$$  \hspace{1cm} (C.35)

Moments of consumption volatility follow directly from above by setting jump parameters to zero. The moments of excess returns can be computed in a similar way using the model dynamics for returns; the results are available upon request.

Denote $g_v^\nu(\Theta)$ the vector of orthogonality condition based on the confidence data:

$$g_v^\nu(\Theta) = [V_t \ V_t^2 \ V_t^3 \ V_t^4 \ V_t V_{t-3}]' - m_v(\Theta),$$

where $\Theta$ are the estimated parameters and $m_v(\Theta)$ is the corresponding vector of unconditional moments implied by the model.

Denote $er_{t+1} = r_{d,t+1} - r_{f,t}$ the excess return in the data. Then, the orthogonality conditions based on the return data can be expressed in the following way:

$$g^{er}_t(\Theta) = [er_t \ er_t^2 \ er_t^3 \ er_t^4 \ er_{t} er_{t-3} \ er_t V_t]' - m_r(\Theta),$$ \hspace{1cm} (C.36)

where $m_r(\Theta)$ is the vector of unconditional moments calculated based on the return dynamics in Section 3.3.3. Further, I use the information in the level, variation and
persistent of the interest rate, so I construct the orthogonality conditions based on the interest rate data,

\[ g_{t}^{rf}(\Theta) = \begin{bmatrix} r_{t} & r_{t}^2 & r_{t}r_{t-3} \end{bmatrix}' - m_{rf}(\Theta), \tag{C.37} \]

and I use the level of 6 option price volatilities, stacked in vector \( \sigma_{BS,t} \), which span three moneyness categories (0.95, 1.00 and 1.05) and two maturities (1 and 2 months). The orthogonality conditions based on the option price data are then,

\[ g_{t}^{BS}(\Theta) = \sigma_{BS,t} - m_{BS}(\Theta), \tag{C.38} \]

where \( m_{BS}(\Theta) \) are the corresponding implied volatilities from the model.

Let \( g_{t}(\Theta) \) stand for the overall vector of orthogonality conditions based on the confidence measure, equity return, interest rate and option price data:

\[ g_{t}(\Theta) = \begin{bmatrix} g_{t}^{v}(\Theta) & g_{t}^{cr}(\Theta) & g_{t}^{rf}(\Theta) & g_{t}^{BS}(\Theta) \end{bmatrix}, \tag{C.39} \]

and \( \bar{g} \) denote its sample average across time. Then, the GMM objective function is given by,

\[ Q_{GMM}(\Theta) = T\bar{g}(\Theta)'\Sigma_{T}^{-1}\bar{g}(\Theta). \tag{C.40} \]

As usual, the estimation is proceeded in two steps, and the optimal weighting matrix \( \Sigma_{T}^{-1} \) corresponds to the inverse of the Newey-West estimate of the variance of the moment conditions based on the first-step parameter estimates. The computation of standard errors and hypothesis testing follows from the standard asymptotic results for GMM.

Hansen (1982) shows that, under the null hypothesis that the model is correctly specified the GMM criterion function in (C.40) has a \( \chi^2 \) distribution with 10 degrees of freedom, which can be used for an overall goodness-of-fit test of the model. I follow Eichenbaum et al. (1988) to test individual blocks of moment conditions (see also
Singleton, 2006). For instance, under the null hypothesis that moment conditions for the confidence measure are satisfied, a statistics

\[ T \min_{\Theta} \bar{g}(\Theta)' \Sigma_T^{-1} \bar{g}(\Theta) - T \min_{\Theta} \left[ \begin{array}{ccc} g_{\tau}^{e}(\Theta) & g_{\tau}^{f}(\Theta) & g_{\tau}^{i}(\Theta) \end{array} \right]' \Sigma_T^{-1} \left[ \begin{array}{ccc} g_{\tau}^{e}(\Theta) & g_{\tau}^{f}(\Theta) & g_{\tau}^{i}(\Theta) \end{array} \right] \]

follows a \( \chi^2 \) distribution with degrees of freedom equal to the number of moment restrictions in the confidence measure block, while the weighting matrix \( \Sigma_T^{-1} \) can be constructed using the corresponding lower-right block of the estimate of the variance of the vector of moment conditions under the null. Then LR test that jump parameters are zero is conducted in a similar way by comparing the GMM objective functions of the restricted and unrestricted specifications.

For an out-of-the-sample test of moment conditions corresponding to the deep out- and in-the-money put options, I construct moment conditions for the levels of these volatilities \( \bar{g}^{IV} \) as in (C.38). Then, the statistics

\[ T \bar{g}^{IV}(\Theta)' \Sigma_{T,IV}^{-1} \bar{g}^{IV}(\Theta) \sim \chi^2(4), \]

where \( \Sigma_{T,IV}^{-1} \) refers to the appropriate partition of the variance-covariance matrix \( \Sigma_T^{-1} \), is distributed \( \chi^2 \) with 4 degrees of freedom under the null that orthogonality conditions are satisfied. Similar test procedure is used for the orthogonality conditions for the confidence data from 1968 to 1996.

C.7 Latent Factor MLE Estimation

Given the dynamics of the economy, the conditional log-likelihood for the implied confidence measure and consumption volatility states is given by,

\[ l_t(V_t(\Theta), \sigma^2_t(\Theta)) = l(V_t(\Theta), \sigma^2_t(\Theta) \mid \{ Z_{t-j}, V_{t-j}(\Theta), \sigma^2_{t-j}(\Theta) \}_{j=1,2,...}) = l_t(V_t(\Theta)|V_{t-1}(\Theta)) + l_t(\sigma^2_t(\Theta)|\sigma^2_{t-1}(\Theta)). \]
This decomposition reflects the assumption that the two processes are independent. As the consumption variance follows a square-root process, its conditional distribution is Gaussian. The confidence measure follows a mixture of Normal-Gamma, and the details of the computation of its likelihood are provided below in Appendix C.7.1.

Given the likelihood of these states, I can write down the conditional likelihood of the two implied volatilities that are used to invert the confidence measure and consumption variance states:

\[
l_t(\sigma^2_{BS,t}) = l_t(\sigma^2_{BS,t} \mid \{Z_{t-j}, V_{t-j}(\Theta), \sigma^2_{t-j}(\Theta)\}_{j=1,2,...}, V_t(\Theta), \sigma^2_t(\Theta)) \\
= l_t(V_t(\Theta), \sigma^2_t(\Theta)) - \log(\text{abs}(J_t)),
\]

where \(J_t\) is the Jacobian of the transformation of the confidence measure and consumption variance into the two implied variance states. As the option variances are nearly linear in the confidence measure and consumption variance, the Jacobian is computed numerically from the projection of implied option variances on the two variance states.

As for the observed consumption and interest rate data, I assume that one-period risk-free rate and log consumption level are observed with a measurement error:

\[
r_f^{data}_t = r_f_t + \xi_{r_f,t}, \quad (C.43)
\]

\[
= - F_0 - F_x \hat{x}_t - F_v V_t - F_\sigma \sigma^2_t + \xi_{r_f,t}.
\]

and

\[c^{data}_t = c_t + \xi_{c,t}, \quad (C.44)\]

so that

\[
\Delta c^{data}_{t+1} = \mu + \hat{x}_t + a_{c,t+1} + \xi_{c,t+1} - \xi_{c,t}. \quad (C.45)
\]

Indeed, real interest rates are not observed in the data, so \(\xi_{r_f,t}\) captures measurement errors due to inflation adjustment of nominal yields, interpolation and
other data issues. The measurement error in observed consumption $\xi_{c,t}$ can account for a negative autocorrelation of monthly consumption growth rate. For simplicity, the measurement errors $\xi_{r,t}$ and $\xi_{c,t}$ are assumed to be Normal, homoscedastic and independent from each other and all the other shocks in the economy.

In Appendix C.7.2 I show that the conditional distribution of observed asset market and consumption data $Z_t$ given the history of $Z_t$ and past and current values of implied states $\sigma^2_t(\Theta)$ and $V_t(\Theta)$ is Normal. Therefore, I can apply standard Kalman Filter methods to write down the conditionally Normal likelihood of the observed data,

$$ l_t(Z_t) = l(Z_t | \{Z_{t-j}, V_{t-j}(\Theta), \sigma^2_{t-j}(\Theta)\}_{j=1,2,...}, V_t(\Theta), \sigma^2_t(\Theta)) \quad (C.46) $$

and the evolution of the estimate of the expected state $\tilde{x}_t(\Theta)$ in a recursive way as a function of the observed macro and asset-price data and the implied confidence and volatility states.

**C.7.1 Likelihood for Confidence Measure**

To simplify the exposition, I drop the dependence of the implied state $V_t(\Theta)$ on model parameters $\Theta$.

The dynamics for the confidence measure is given by,

$$ V_{t+1} = \sigma^2_v + \nu(V_t - \sigma^2_v) + \sigma_w \sqrt{V_t} w_{t+1} + \sum_{i=1}^{N_{t+1}} J_{i,t+1} - \mu_j \lambda_t, \quad (C.47) $$

where $w_{t+1}$ is Gaussian shock, $N_{t+1}$ is the Poisson process with stochastic intensity $\lambda_t = \lambda_0 + \lambda_1 V_t$, and $J_{i,t+1}$ is jump size, whose distribution is i.i.d exponential with mean $\mu_j$.

To simplify the notations, denote the total variance shock $\xi_{t+1}$,

$$ \xi_{t+1} = V_{t+1} - (\sigma^2_v + \nu(V_t - \sigma^2_v) - \mu_j \lambda_t), $$

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which is equal to the sum of conditionally Normal shock \( \sigma_w \sqrt{V_t} w_{t+1} \) and jump shocks \( \sum_{i=1}^{N_{t+1}} J_{i,t+1} \).

To find the conditional likelihood of variance shocks \( f(\xi_{t+1}|V_t) \) I can first condition on the number of jumps \( N_{t+1} \), so that

\[
f(\xi_{t+1}|V_t) = \sum_{N=0}^{\infty} f(\xi_{t+1}|V_t, N_{t+1} = N) Pr(N_{t+1} = N),
\]

where the conditional probability function of the Poisson distribution is given by

\[
Pr(N_{t+1} = N) = \lambda^N_t \exp(-\lambda_t) / N!.
\]

As the sum of \( N \) exponentially distributed variables follows Gamma distribution, given previous state \( V_t \) and the number of jumps \( N_{t+1} \) jump shock has Gamma distribution \( \text{Gamma}(N, \mu_j) \), so that the distribution of the total variance shock \( \xi_{t+1} \) is Normal-Gamma. For \( N = 0 \), its conditional likelihood is Normal; for \( N > 1 \) it can be written in the following way:

\[
f(\xi_{t+1}|V_t, N) = \frac{(\sigma_w \sqrt{V_t})^{N-1}}{\sqrt{2\pi} \Gamma(N) \mu_j^N} \exp \left( -\frac{\xi_{t+1}^2}{\mu_j^2} + \frac{\sigma_w^2 V_t^2}{2\mu_j^2} \right) \int_{b_t}^{\infty} (t - b_t)^{N-1} \exp \left( -\frac{t^2}{2} \right) dt,
\]

for

\[
b_t = -\frac{\xi_{t+1}}{\sigma_w \sqrt{V_t}} + \frac{\sigma_w \sqrt{V_t}}{\mu_j}.
\]

To calculate the integral on the right-hand side, I use the fact the representation

\[
\int_{b_t}^{\infty} t^{N-1} \exp \left( -\frac{t^2}{2} \right) dt = 2^{\frac{N-2}{2}} \Gamma(N/2) (1 - \text{sgn}(b_t)^N F_{\chi^2}(b_t^2|N)),
\]

where \( F_{\chi^2}(.,|N) \) is the cdf of \( \chi^2 \)–square distribution with \( N \) degrees of freedom, and \( \text{sgn}(b_t) \) gives the sign of \( b_t \).
C.7.2 Likelihood for Consumption and Asset-Price Data

To simplify the exposition, I drop the dependence of the implied states $V_t(\Theta)$ and $\sigma^2_t(\Theta)$ on model parameters $\Theta$.

Denote by $\mathcal{H}_t$ the period-$t$ history of data and implied volatility states observed by econometrician:

$$\mathcal{H}_t = \{Z_{t-j}, V_{t-j}, \sigma^2_{t-j}\}_{j=0,1,...},$$

(C.48)

and let $\tilde{x}_t$ and $\tilde{\xi}_{c,t}$ denote the filtered value of unobserved expected growth state and measurement error in consumption, respectively, while $\omega^2_{xc,t}$ stand for the variance of the filtering error of the econometrician,

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{\xi}_{c,t} \end{bmatrix} = E\left( \begin{bmatrix} \hat{x}_t \\ \xi_{c,t} \end{bmatrix} \mid \mathcal{H}_t \right), \quad \omega^2_{xc,t} = E\left( \begin{bmatrix} \tilde{x}_t - \hat{x}_t \\ \tilde{\xi}_{c,t} - \xi_{c,t} \end{bmatrix}^2 \mid \mathcal{H}_t \right).$$

(C.49)

The one-step ahead conditional distribution of unobserved states and consumption and asset-price data is Normal,

$$\begin{bmatrix} \tilde{x}_{t+1} \\ \xi_{c,t+1} \\ Z_{t+1} \end{bmatrix} \mid \mathcal{H}_t, V_{t+1}, \sigma^2_{t+1} \sim N\left( \begin{bmatrix} \mu_{Zt} \\ 0 \\ 0 \end{bmatrix}, \Omega_{t+1} \right),$$

(C.50)

where the drift is given by

$$\mu_{Zt} = \begin{bmatrix} -F_0 - F_x \rho \tilde{x}_t - F_x V_{t+1} - F_x \sigma^2_{t+1} \\ r_0 + \frac{1}{\psi} \tilde{x}_t + H_v (\kappa_{d,1} V_{t+1} - V_t) + H_{\sigma} (\kappa_{d,1} \sigma^2_{t+1} - \sigma^2_t) \\ \mu + \tilde{x}_t - \tilde{\xi}_{c,t} \end{bmatrix},$$

(C.51)

$$r_0 = \mu - \log \kappa_{d,1} + (1 - \kappa_{d,1}) (H_v \sigma^2_v + H_{\sigma} \sigma^2),$$

and the variance-covariance matrix satisfies

$$\Omega_{t+1} = N \times \text{diag} \begin{bmatrix} \Sigma_{cx,t+1} \\ \omega^2_{xc,t} \\ \sigma^2_f \\ \sigma^2_{\xi,c} \\ \sigma^2_t \end{bmatrix} \times N', \quad N = \begin{bmatrix} 0 & K_2 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -F_x K_2 & -F_x \rho & 0 & 1 & 0 & 0 \\ \phi & \kappa_1 H_x K_2 & \frac{1}{\psi} & 0 & 0 & 0 & \varphi_d \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$
The updates for the filtered states and the variance of the filtering error satisfy standard Kalman Filter recursions

\[
\begin{bmatrix}
\tilde{x}_{t+1} \\
\tilde{\xi}_{c,t+1}
\end{bmatrix}
= \begin{bmatrix}
\rho \tilde{x}_t \\
0
\end{bmatrix} + K_{t+1} (Z_{t+1} - \mu_{Z_t})
\]

\[
\omega_{xc,t+1}^2 = \Omega_{t+1}^{11} - \Omega_{t+1}^{12} \left( \Omega_{t+1}^{22} \right)^{-1} \Omega_{t+1}^{21},
\]

where the Kalman Filter Gain is given by

\[
K_{t+1} = \Omega_{t+1}^{12} \left( \Omega_{t+1}^{22} \right)^{-1}.
\]

The superscripts refer to the partitioning of \( \Omega_{t+1} \) into four blocks, such that \( \Omega_{t+1}^{11} \) is the two by two upper corner of the matrix, \( \Omega_{t+1}^{12} \) contain the elements from first two rows and third to fourth columns, etc. The conditional distribution of the observed asset-price and macro data in \( Z_{t+1} \) is Normal with mean \( \mu_{Z_t} \) and variance-covariance matrix given by \( \Omega_{t+1}^{22} \) which allows me write down the conditional likelihood \( l_t(Z_t) \).
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Biography

Ivan Shaliastovich was born in Minsk, Belarus on August 9, 1982. He earned his B.A. degree with honors in Economics and Mathematics from the American University in Bulgaria in May 2003, and MA degree in Economics from Duke University in January 2006. Ivan has accepted an assistant professor position at the Wharton School of the University of Pennsylvania.