Dynamics of the Disk-Pendulum Coupled System With Vertical Excitation

by

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Duke University

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Samuel C. Stanton

Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mechanical Engineering & Materials Science in the Graduate School of Duke University 2016
Abstract

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This thesis investigates the static and dynamic characteristics of the semi-elliptical rocking disk with an attached pendulum. This coupled system’s response is also analyzed analytically and numerically when a vertical harmonic excitation is applied to the bottom of the rocking disk. Lagrange’s Equation is used to derive the motion equations of the disk-pendulum coupled system. The second derivative test for the system’s potential energy shows how the location of the pendulum’s pivot point affects the number and stability of equilibria, and the change of location presents different bifurcation diagrams for different geometries of the rocking disk. For both vertically excited and unforced cases, the coupled system shows chaos, but properly chosen parameters enables the system display periodic motions. For the steady state of the vertically excited rocking disk without a pendulum, the variation of the excitation’s amplitude and frequency result in hysteresis for the response amplitude. When a pendulum is pinned on the rocking disk, three major categories of steady states are found numerically.
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\( m : m_E : m_R = 5 : 0 : 1, \quad \bar{\Omega} = 6\pi, \quad \mu = 0.4, \quad \mu_1 = 0.2 \).

5.2 System response (harmonic): \( a : b : l_1 : l_R = 50 : 45 : 60 : 80, \),
\( m : m_E : m_R = 5 : 0 : 1, \quad \bar{\Omega} = 6\pi, \quad \mu = 0.4, \quad \mu_1 = 0.2 \).

5.3 System response (beats): \( a : b : l_1 : l_R = 50 : 45 : -10 : 80, \),
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5.4 For the case in Fig. 5.3, the response of the rocking disk and pendulum in the frequency domain.
List of Abbreviations and Symbols

Symbols

\( m \) Mass of semi-elliptical disk
\( m_E \) Mass of pendulum end
\( m_R \) Mass of pendulum rod
\( \kappa \) Gyration radius of semi-elliptical disk
\( I_R \) Moment inertia around mass center of pendulum rod
\( a \) Major radius of the elliptical disk
\( b \) Minor radius of elliptical disk
\( l \) Length of line segment AO
\( l_1 \) Length of line segment OG
\( l_G \) Length of line segment AG
\( l_R \) Length of pendulum rod (line segment OE)
\( \theta \) Deviation of disk from vertical
\( \theta_1 \) Deviation of pendulum from vertical
\( A \) Velocity amplitude of base vibration
\( \Omega \) Angular velocity of base vibration
Due to their special dynamic characteristics, pendulums are widely used as a core object in energy harvesters. On one hand, the pendulum can easily respond to the horizontal or vertical excitation and thus transfer the energy from the external source to the kinetic energy of the pendulum’s rotation. On the other hand, the external oscillation can lead to a rotating pendulum, and the behavior “rotation” could be treated as an efficient way to harvest energy.

Many energy harvesters use an oscillating pendulum. Wang designed a novel weighted-pendulum-type device to harvest energy from a rotating wheel (Wang et al., 2013). Gu and Livermore presented a pendulum-excited harvester to realize self-tuning in the application of rotation (Gu and Livermore, 2010). Toh applied the method H-bridge power electronic interface to a pendulum-based device for marine energy harvesting (Toh et al., 2011). Wiercigroch determined how the external excitation can be converted to rotational motion (Wiercigroch et al., 2011). He also showed numerical, analytical and experimental evidence that the pendulum’s rotation can be maintained for a wide range of external excitation amplitude and frequencies; and even the desired response of the pendulum could be achieved by a
certain control strategy. With different types of excitation, pendulums show diverse bifurcations. Mann showed that even a subtle tile angle in the perfectly vertical or horizontal excitation would lead to symmetry breaking bifurcations (Mann and Koplow, 2006). In McLaughlin’s numerical investigations, period-doubling sequences of bifurcation occur both with and without damping (McLaughlin, 1981). For the double pendulum system, it is possible for its bifurcations to cause 2-D and 3-D tori, and double-period cascading bifurcations will result in chaos (Yu and Bi, 1998).

Apart from pendulums, the motion of rocking also plays an important role in the present study. In most cases, the pendulum cannot be used directly to convert the external excitation. It must be supported by or pinned to another rocking structure to oscillate. For the rocking motion of a ship, mounting a pendulum to the ship is one of the simplest methods to generate and harvest potential energy (Rastegar et al., 2006). Similar to the ship on the sea, the rocking disk is a proper choice of the supporting structure, which shows the typical motion of rocking as well. The rocking disk could easily convert the vertical excitation to rocking motions, thus helping drive the pendulum more effectively.

The rest of the thesis is organized as follows: Chapter 2 describes the physical model of the disk-pendulum coupled system, and uses Lagrange’s equation to derive the motion equations. Chapter 3 determines the static equilibria and stability for the non-excitation cases. Simplified basins of attraction are also presented in this chapter to show the occurrence of chaos and how they are affected by parameters. Chapter 4 investigates the dynamics of the rocking disk with vertical excitation. The pendulum is omitted in this chapter. Chapter 5 adds the pendulum to the rocking disk and uses numerical methods to investigate when the coupled system oscillates with periodic motion.
Fig. 2.1 shows a picture of the physical system, which consists of a harmonically excited base, a semi-elliptical rocking disk and a pendulum. The pendulum is pinned to the rocking disk. Its pivot point is on the axis of symmetry of the rocking disk. The vibrating base provides a vertical harmonic excitation to the disk-pendulum system from the bottom. We assume that the rocking disk does not slip horizontally on the base.

**Figure 2.1:** Schematic of a rocking semi-elliptical disk connected to a pendulum.
The equations of motion are derived with Lagrange’s equations:

\[
\frac{\partial L}{\partial \dot{\theta}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) = 0
\]

\[
\frac{\partial L}{\partial \dot{\theta}_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = 0
\]  

where the Lagrangian \( L = T - U \), \( T \) is the total kinetic energy of the system and \( U \) is the potential energy of the system. In order to derive the kinetic energy and potential energy expressions with respect to \( \theta, \theta_1, \dot{\theta} \) and \( \dot{\theta}_1 \), the position vectors of the disk’s and pendulum’s mass centers need to be identified. As shown in the Fig. 2.1, point P, A and G are the connect point between the disk and base, the center of the disk’s top surface, and the disk’s center of mass, respectively. The position vector of the mass center of disk \( \vec{r}_G \) is defined as the displacement from the connect point P in the upright position of the disk (the left disk in Fig. 2.1) to the mass center G in the displaced position of the disk (the right disk in Fig. 2.1). Therefore, this position vector could be written as:

\[
\vec{r}_G = \vec{r}_P + \vec{r}_{A/P} + \vec{r}_{G/A}
\]  

(2.2)

where \( \vec{r}_P = s\vec{I}, \vec{r}_{A/P} = -p\vec{I} + q\vec{J} \) and \( \vec{r}_{G/A} = -l\sin\theta\vec{I} - l\cos\theta\vec{J} \). Substitute these expressions into (2.2), and the position vector of the disk’s mass center becomes:

\[
\vec{r}_G = [s - p - (l + l_1)\sin\theta]\vec{I} + [q - (l + l_1)\cos\theta]\vec{J}
\]  

(2.3)

By applying the fact that the ground connect point (P) is always located at the minimum point on the disk’s periphery in the inertial reference frame (\( \vec{I} \) and \( \vec{J} \)), Mazzoleni derived the expressions of the contact point’s position with respect to \( \theta \).
(Mazzoleni et al., 2015). The undefined terms in (2.3) are given by:

\[ q = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \]  
(2.4a)

\[ p = \frac{(a^2 - b^2) \sin \theta \cos \theta}{q} \]  
(2.4b)

\[ w = \frac{(a^2 - b^2) \cos 2\theta - p^2}{q} \]  
(2.4c)

\[ s = \int_0^\theta \frac{a^2b^2}{q^3} d\theta \]  
(2.4d)

Note that \( p, q \) and \( s \) are shown in the Fig. 2.1 to express the position vector of the mass center, but \( w \) has no practical meaning. It is a new variable introduced by Mazzoleni for convenience to take derivatives of \( p, q \) and \( s \) with respect of time. He has also proved their mutual relationships:

\[ \dot{\dot{q}} = p\dot{\theta} \]  
(2.5a)

\[ \dot{p} = w\dot{\theta} \]  
(2.5b)

\[ \dot{w} = -\left(4p + \frac{3pw}{q}\right)\dot{\theta} \]  
(2.5c)

\[ \dot{s} = \frac{a^2b^2}{q^3}\dot{\theta} \]  
(2.5d)

Therefore the velocity of the disk’s mass center is:

\[ \vec{r}_G = \left[ \dot{s} - \dot{p} - (l + l_1) \dot{\theta} \cos \theta \right] \vec{I} + \left[ \dot{q} + (l + l_1) \dot{\theta} \sin \theta \right] \vec{J} \]  
(2.6)

Applying the vertical harmonic excitation to the semi-elliptical disk, the velocity of its mass center becomes:

\[ \vec{r}_G = \left[ \dot{s} - \dot{p} - (l + l_1) \dot{\theta} \cos \theta \right] \vec{I} + \left[ \dot{q} + (l + l_1) \dot{\theta} \sin \theta + A \cos (\Omega t) \right] \vec{J} \]  
(2.7)

where \( A \) and \( \Omega \) are the velocity amplitude and frequency of the base excitation. It is worth noting that \( l \) and \( l_1 \) do not strictly stand for “length.” Their modulus are the
Figure 2.2: Signs of $l$ and $l_1$ change with the pivot location.

lengths of line segments, but their signs are dependent on the position of the pivot point $O$ relative to point $A$ and $G$ (See Fig. 2.2).

The pendulum, whose mass is composed of the rod mass $m_R$ and the point mass at the rod end $m_E$, is pinned to the rocking disk. As shown in Fig. 2.1, the position vector to the free end could be written as $\vec{r}_E = \vec{r}_G + \vec{r}_{E/G}$, and for the mass center of rod, $\vec{r}_R = \vec{r}_G + \frac{1}{2}\vec{r}_{E/G}$, where $\vec{r}_{E/G} = l_R \sin \theta_1 \vec{I} - l_R \cos \theta_1 \vec{J}$ and $l_R$ is the length of the pendulum rod. Therefore, the velocities of the rod end and the rod mass center are:

$$\begin{align*}
\vec{v}_E &= \left(\dot{s} - \dot{p} - l\dot{\theta} \cos \theta + l_R \dot{\theta}_1 \cos \theta_1\right) \vec{I} + \left(\dot{q} + l \dot{\theta} \sin \theta + l_R \dot{\theta}_1 \sin \theta_1 + A \cos (\Omega t)\right) \vec{J} \\
\vec{v}_R &= \left(\dot{s} - \dot{p} - l\dot{\theta} \cos \theta + \frac{1}{2}l_R \dot{\theta}_1 \cos \theta_1\right) \vec{I} + \left(\dot{q} + l \dot{\theta} \sin \theta + \frac{1}{2}l_R \dot{\theta}_1 \sin \theta_1 + A \cos (\Omega t)\right) \vec{J}
\end{align*} (2.8a)$$

Given that the position vectors of the disk’s mass center, the pendulum’s mass center and end are all identified, the total translational kinetic energy of the system is obtained:

$$T_{trans} = \frac{1}{2} m_G \dot{\vec{r}}_G^2 + \frac{1}{2} m_R \dot{\vec{r}}_R^2 + \frac{1}{2} m_E \dot{\vec{r}}_E^2 \quad (2.9)$$
The total rotational kinetic energy of the system is:

\[ T_{\text{rot}} = \frac{1}{2} m (\kappa \dot{\theta})^2 + \frac{1}{2} I_R \dot{\theta}_1^2 \]  

(2.10)

where \( \kappa \) is the disk’s gyration radius, and \( I_R \) is the pendulum rod’s moment inertia around its mass center. Therefore, the expression for the total kinetic energy \( T = T_{\text{trans}} + T_{\text{rot}} \) and the total potential energy \( U \) are:

\[
\begin{align*}
T &= \frac{m}{2} \left\{ \left( \dot{s} - \dot{p} - (l + l_1) \dot{\theta} \cos \theta \right)^2 + \left( \dot{q} + (l + l_1) \dot{\theta} \sin \theta + A \cos(\Omega t) \right)^2 \right. \\
&\quad + \frac{m_E}{2} \left( \left( \dot{s} - \dot{p} - l \dot{\theta} \cos \theta + l_R \dot{\theta}_1 \cos \theta_1 \right)^2 + \left( \dot{q} + l \dot{\theta} \sin \theta + l_R \dot{\theta}_1 \sin \theta_1 + A \cos(\Omega t) \right)^2 \right) \\
&\quad + \frac{m_R}{2} \left( \left( \dot{s} - \dot{p} - l \dot{\theta} \cos \theta + \frac{1}{2} l_R \dot{\theta}_1 \cos \theta_1 \right)^2 + \left( \dot{q} + l \dot{\theta} \sin \theta + \frac{1}{2} l_R \dot{\theta}_1 \sin \theta_1 + A \cos(\Omega t) \right)^2 \right) \\
&\quad + \left. \frac{1}{2} I_R \dot{\theta}_1^2 \right\} 
\end{align*}
\]

(2.11a)

\[
U = mg \left[ q - (l + l_1) \cos \theta + \frac{A}{\Omega} \sin(\Omega t) \right] \\
+ m_E g \left( q - l \cos \theta - l_R \cos \theta_1 + \frac{A}{\Omega} \sin(\Omega t) \right) \\
+ m_R g \left( q - l \cos \theta - \frac{1}{2} l_R \cos \theta_1 + \frac{A}{\Omega} \sin(\Omega t) \right) 
\]

(2.11b)

After substituting (2.11a) and (2.11b) into Lagrange’s equations (2.1), the governing equations are obtained:

\[
m \left[ \kappa^2 \ddot{\theta} + K_7 \ddot{\theta} + K_8 \dot{\theta}^2 + l_1 \sin \theta (g - A\Omega \sin(\Omega t)) \right] + \frac{2m_E + m_R}{2} \left( K_3 \dot{\theta}_1 + K_4 \dot{\theta}_1^2 \right) \\
+ (m + m_E + m_R) \left[ K_1 \dot{\theta} + K_2 \dot{\theta}^2 + (p + l \sin \theta) (g - A\Omega \sin(\Omega t)) \right] = 0 
\]

(2.12a)

\[
\left( I_R + \frac{4m_E + m_R}{4} l_R^2 \right) \dddot{\theta}_1 + \frac{2m_E + m_R}{2} \left[ K_5 \dot{\theta}^2 + K_6 \ddot{\theta} + l_R \sin \theta_1 (g - A\Omega \sin(\Omega t)) \right] = 0 
\]

(2.12b)
where

\begin{align}
K_1 &= \left(\frac{a^2b^2}{q^3} - w - l \cos \theta \right)^2 + (p + l \sin \theta)^2 \tag{2.13a} \\
K_2 &= \left(\frac{a^2b^2}{q^3} - w - l \cos \theta \right) \left(4p + \frac{3pw}{q} + l \sin \theta - \frac{3a^2b^2p}{q^4}\right) + (p + l \sin \theta)(w + l \cos \theta) \tag{2.13b} \\
K_3 &= l_R \left[\sin \theta_1 (p + l \sin \theta) + \cos \theta_1 \left(\frac{a^2b^2}{q^3} - w - l \cos \theta \right)\right] \tag{2.13c} \\
K_4 &= l_R \left[\cos \theta_1 (p + l \sin \theta) - \sin \theta_1 \left(\frac{a^2b^2}{q^3} - w - l \cos \theta \right)\right] \tag{2.13d} \\
K_5 &= l_R \left[\cos \theta_1 \left(4p + \frac{3pw}{q} + l \sin \theta - \frac{3a^2b^2p}{q^4}\right) + \sin \theta_1 (w + l \cos \theta)\right] \tag{2.13e} \\
K_6 &= l_R \left[\cos \theta_1 \left(\frac{a^2b^2}{q^3} - w - l \cos \theta \right) + \sin \theta_1 (p + l \sin \theta)\right] \tag{2.13f} \\
K_7 &= l_1 \left[l_1 + 2l + 2p \sin \theta - 2\left(\frac{a^2b^2}{q^3} - w\right) \cos \theta\right] \tag{2.13g} \\
K_8 &= l_1 \left[\frac{a^2b^2}{q^3} \sin \theta + 3p \left(\frac{a^2b^2}{q^4} - \frac{w}{q} - 1\right) \cos \theta\right] \tag{2.13h}
\end{align}
3.1 Equilibria and stability trends

No excitation will be applied to the disk-pendulum coupled system in this section. In other words, the amplitude of the base excitation $A$ is zero.

The equilibria can be derived from (2.12a) and (2.12b) by setting $\dot{\theta} = \ddot{\theta} = \ddot{\theta}_1 = \dot{\theta}_1 = 0$. There is no surprise that the equilibria of the pendulum is same with that of the classic single degree-of-freedom pendulum (3.1b). It is also well known that the pendulum has multiple equilibrium points. One is the stable equilibria at the bottom $(\tilde{\theta}_1 = 2n\pi)$, and the other one is the unstable equilibria at the top $(\tilde{\theta}_1 = 2n\pi + \pi)$, where $n = 1, 2, 3, 4$...

$$
sin \theta \left[ \frac{(a^2 - b^2) \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} + l + m' l_1 \right] = 0 \quad (3.1a)
$$

$$
sin \theta_1 = 0 \quad (3.1b)
$$

where $m' = m / (m + m_E + m_R)$

Comparatively, the equilibria of the disk are more complicated. From (3.1a), a trivial solution is obtained, $\sin \tilde{\theta} = 0$. Unlike the pendulum which may spin, the physical model limits the disk angle between $-\pi/2$ and $\pi/2$. Therefore the trivial
solution $\tilde{\theta} = 0$. For the non-trivial solutions (3.2), the number of real roots depend on $a$, $b$, and $l_1$. In this equation, we use $l_G$ ($l_G = l_1 + l$) instead of $l$ because $l_G$ stands for the position of the disk’s center of mass, which is only dependent on the vertical radius $b$ and always positive. The disk equilibria are given by:

$$\tilde{\theta} = \begin{cases} 
0 & l_1 \leq R_m l_G \\
0, \pm \tan^{-1} \left[ \frac{1}{a} \sqrt{\left( \frac{b^2 - a^2}{l_G - l_1/R_m} \right)^2 - b^2} \right] & \begin{align*} 
R_m l_G < l_1 & \leq R_m \left( l_G + \frac{a^2 - b^2}{b} \right) & a < b \\
l_1 & > R_m \left( l_G + \frac{a^2 - b^2}{b} \right) & a = b \\
0 & R_m \left( l_G + \frac{a^2 - b^2}{b} \right) \leq l_1 < R_m l_G & a > b \\
0 & l_1 \geq R_m l_G 
\end{align*} 
\end{cases}$$

where $R_m = (m + m_E + m_R)/(m_E + m_R)$.

Given that the disk-pendulum coupled system is conservative in absence of the base excitation and we have derived the potential energy function (2.11b), the Lagrange-Dirichlet method can be used to determine the stability of the equilibrium points. Since the stability of the pendulum is known, we only need to evaluate the second derivative of the potential energy with respect to the angle of the disk. When substituting $\theta$ with $\tilde{\theta}$, $\partial^2 U/\partial \theta^2 > 0$ demonstrates a stable equilibrium and $\partial^2 U/\partial \theta^2 < 0$ demonstrates an unstable equilibrium. This term is given by:

$$\frac{\partial^2 U}{\partial \theta^2} = g (m + m_E + m_R) \left[ \left( l_G - \frac{l_1}{R_m} \right) \cos \theta + \frac{(a^2 - b^2) \left( b^2 \cos^4 \theta - a^2 \sin^4 \theta \right)}{\left( a^2 \sin^2 \theta + b^2 \cos^2 \theta \right)^{3/2}} \right]$$

(3.3)

For the disk-pendulum coupled system, the “type of bifurcation” changes at $a/b = 1$ (Fig. 3.1), but this is not a conventional critical point because the “stability”, not the “bifurcation type”, changes at a critical point. It is worth noting
Figure 3.1: Two cases (a < b and a > b) show the different types of bifurcation for the rocking disk. These two bifurcation diagrams demonstrate how the number and the stability of equilibria evolve with the position of pendulum’s pivotal point. The solid blue line indicates the stable equilibria and the dashed orange line indicates the unstable equilibria.
that for the case of rocking semi-elliptical disk without a pendulum, a super-critical pitchfork bifurcation occurs at \( a/b = \sqrt{1 - 4/3\pi} \) (Mazzoleni et al., 2015). When \( a/b > \sqrt{1 - 4/3\pi} \), the rocking disk (without a pendulum) owns one stable equilibrium angle at \( \tilde{\theta} = 0 \); and when \( a/b < \sqrt{1 - 4/3\pi} \), the equilibrium angle \( \tilde{\theta} = 0 \) becomes unstable and two stable equilibria appear symmetrically to \( \tilde{\theta} = 0 \) (Fig. 3.2).

Therefore, the gap between \( a/b = 1 \) and \( a/b = \sqrt{1 - 4/3\pi} \) enable the pendulum to change the stability of the rocking disk. As shown in the Fig. 3.3, on one hand, if we have a bistable semi-elliptical rocking disk \( (a/b < \sqrt{1 - 4/3\pi}) \), we are able to make it monostable by placing a pendulum pinned on the disk’s axis of symmetry. On the other hand, if we have a monostable on which \( \sqrt{1 - 4/3\pi} < a/b < 1 \), we are able to make it bistable in the same way. However, if \( a/b > 1 \), the pendulum will not affect the stability of the rocking disk.
3.2 Basins of attraction

The disk-pendulum coupled system is a 2 degree-of-freedom dynamic system. Considering that its motion equation consists of second order differential equations, a 4 dimensional initial condition \((\theta (0), \theta_1 (0), \dot{\theta} (0), \dot{\theta}_1 (0))\) is required to determine its motion. Because it is impractical to visualize a 4 dimensional basins of attraction diagram, the basins of attraction shown in Fig. 3.4–3.7 are simplified by setting \(\dot{\theta} (0) = \dot{\theta}_1 (0) = 0\). Although the simplified basins of attraction cannot fully demonstrate all the dynamical features of the disk-pendulum coupled system, they still show the strong sensitivity to initial conditions and how the geometry affects the predictability of the motion.

Multiple basins of attraction occur only when the rocking disk is bistable. As shown in Fig. 3.3, there are two types of bistable rocking disks with a pendulum. The first one originates from the bistable semi-elliptical rocking disk \((a/b < \sqrt{1 - 4/3\pi})\); adding a pendulum on it with a proper position of the pivotal point will keep its bistability. The second one originates from the monostable semi-elliptical rocking disk \((\sqrt{1 - 4/3\pi} < a/b < 1)\); a proper position of the pivotal point will turn its monostability to bistability. For both cases, the “proper position” is \(R_m \left(l_G + \frac{a^2 - b^2}{b}\right) < l_1 < R_m l_G\) (Fig. 3.1a). In Table 3.1, 8 groups of parameters are listed: 2 different
Table 3.1: Different combinations of parameters for basins of attraction

<table>
<thead>
<tr>
<th>No.</th>
<th>(a) (mm)</th>
<th>(b) (mm)</th>
<th>(m : m_E : m_R)</th>
<th>(l_1) (mm)</th>
</tr>
</thead>
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<tr>
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<td>90</td>
<td>5:2:3</td>
<td>-3.6</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>90</td>
<td>5:2:3</td>
<td>16.4</td>
</tr>
<tr>
<td>3</td>
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<td>90</td>
<td>10:2:3</td>
<td>-5.4</td>
</tr>
<tr>
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<td>90</td>
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<td>60.7</td>
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<td>8</td>
<td>85</td>
<td>100</td>
<td>10:2:3</td>
<td>77.4</td>
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</tbody>
</table>

\(l_1\) = \(-3.6\) (mm), \(l_1\) = 16.4 (mm)

**Figure 3.4**: \(a = 60\) (mm), \(b = 90\) (mm), \(m : m_E : m_R = 5 : 2 : 3\). The lower pivot point (plot a) causes a more sensitive dependence on initial conditions. Although the higher pivot point (plot b) could enhance the system’s predictability, the easier toppling failure may limit the system’s working range (larger black area).

geometries \((a/b < \sqrt{1 - 4/3\pi} \text{ and } \sqrt{1 - 4/3\pi} < a/b < 1)\), 2 different mass ratios \((m : m_E : m_R = 5 : 2 : 3 \text{ and } 10 : 2 : 3)\), and two different positions of the pivotal points \((l_1 = R_m \left(l_G + \frac{3\pi}{a^2-b^2}\right) \text{ and } R_m \left(l_G + \frac{4\pi}{a^2-b^2}\right))\). This particular choice of \(l_1\) guarantees not only its belonging to the “proper position”, but also the equilibrium angles far away from 90° (conceptually, the closer to 90° the equilibria are, the more easily the rocking disk topples). It should be noted that \(l_G = 4b/3\pi\) for a uniform-density semi-elliptical disk (Myers and LAKE, 1962).

In Fig. 3.4–3.7, the blue dots symbolize the case where the disk-pendulum coupled system will stop at the negative equilibrium angle, while the yellow dots symbolize
Figure 3.5: $a = 60 \text{ (mm)}, b = 90 \text{ (mm)}, m : m_E : m_R = 10 : 2 : 3$. The lower pivot point (plot a) causes a more sensitive dependence on initial conditions. Although the higher pivot point (plot b) could enhance the system’s predictability, the easier toppling failure may limit the system’s working range. However, the working range becomes wider than that in Fig. 3.4 (smaller black area).

Figure 3.6: $a = 85 \text{ (mm)}, b = 100 \text{ (mm)}, m : m_E : m_R = 5 : 2 : 3$. This basin of attraction diagram illustrates more clearly borderline between two equilibria. Compared with the system where $a = 60 \text{ (mm)}, b = 90 \text{ (mm)}$, this system is more predictable despite its low pivot point. The higher pivot point could filter out the system’s unpredictability partly, but it may make the system more easily to topple as well.
Figure 3.7: \( a = 85 \ (mm), b = 100 \ (mm), m : m_E : m_R = 10 : 2 : 3 \). What is different with the case shown in Fig. 3.6 lies in that while the higher pivot point provides a better predictability and narrower working range in both cases, the shrunk working range remains wide for the system in this case.

stopping at the positive equilibrium angle. Due to the special physical feature of the disk-pendulum model, there appears a third destination the coupled system will reach: the black dots represent the cases where the rocking disk will finally topple.

In general, when the pendulum-pinned disk originates from the bistable rocking disk \((a/b < \sqrt{1-4/3\pi}, \text{Fig } 3.4 \text{ and } 3.5)\), it shows more unpredictability than the one originating from the monostable rocking disk \((\sqrt{1-4/3\pi} < a/b < 1, \text{Fig. } 3.6 \text{ and } 3.7)\). Furthermore, comparing Fig. 3.4(a) to 3.7(a) with Fig. 3.4(b) to 3.7(b), the higher pivotal point (greater \(l_1\)) may help enhance the system’s predictability, but it may also reduce its working range and make the rocking disk easier to topple.
Vertical Excitation of Only the Semi-Elliptical Disk

This chapter investigates applying a vertical excitation to the semi-elliptical rocking disk. By using numerical investigations, the excitation frequency and the amplitude were varied to study their effect on the behavior of the rocking disk. Furthermore, in all the cases with vertical excitation, the proper combinations of the frequency and the amplitude are chosen to prevent the rocking disk from jumping off the platform; and we assume that there is no horizontal slip between the rocking disk and the platform.

Give the pendulum is not pinned to the disk in this chapter, the motion equations (2.12a) could be simplified by setting $m_E = m_R = 0$ and setting $l_1 = 0$ in (2.13g) and (2.13h) for convenience, the equation of motion for only the semi-elliptical rocking disk becomes:

$$\left(\kappa^2 + K_1\right) \ddot{\theta} + K_2 \dot{\theta} + (l_G \sin \theta + p) \left[ g - A \Omega \sin(\Omega t) \right] = 0$$

(4.1)
where

\[ K_1 = \left( \frac{a^2 b^2}{q^3} - w - l_G \cos \theta \right)^2 + (p + l_G \sin \theta)^2 \] (4.2a)

\[ K_2 = \left( \frac{a^2 b^2}{q^3} - w - l_G \cos \theta \right) \left( 4p + \frac{3pw}{q} + l_G \sin \theta - \frac{3a^2 b^2 p}{q^4} \right) + (p + l_G \sin \theta) (w + l_G \cos \theta) \] (4.2b)

A new time variable \( \tau = t\sqrt{g/a} \) is now applied to non-dimensionalize the motion equation:

\[ \ddot{K}_1 \theta'' + \ddot{K}_2 \theta'^2 + \frac{l_G \sin \theta + p}{a} \left[ 1 - A \dot{\Omega} \sin \left( \Omega \tau \right) \right] = 0 \] (4.3)

where \( \ddot{K}_1 = (\kappa^2 + K_1)/a^2 \), \( \ddot{K}_2 = K_2/a^2 \), \( A = A/\sqrt{ga} \) and \( \dot{\Omega} = \Omega/\sqrt{ga} \). It is worth noting that the equation of motion for this excited semi-elliptical rocking disk is independent of its mass, while the mass ratio does matter in the disk-pendulum coupled system.

The command \texttt{ode45} in Matlab is used to find the numerical solutions to the rocking disk’s governing equation (4.3). To obtain the periodic solutions numerically, a damping term \( \mu \theta' \) should be firstly introduced (4.4). The damping \( \mu \) will be carefully chosen in the following cases. It should be reasonably large to help the rocking disk go through the initial transient quickly and also be moderately small to prevent the rocking disk from stopping at the equilibrium points.

\[ \theta'' + \mu \theta' + \frac{\ddot{K}_2}{K_1} \theta'^2 + \frac{l_G \sin \theta + p}{K_1 a} \left[ 1 - A \dot{\Omega} \sin \left( \Omega \tau \right) \right] = 0 \] (4.4)

4.1 Frequency Sweep

The amplitude \( \tilde{A} \) and the frequency \( \tilde{\Omega} \) of the vertical excitation both affect the rocking disk’s dynamic stability. With the different geometries of the rocking disk, these two parameters affect the system in different ways. Firstly, the amplitude of the excitation is fixed and the frequency varies in a certain range.
\( \tau \times 10^4 \)  

\( \theta \quad \pi/2 \quad \pi/4 \quad 0 \quad \pi/4 \quad \pi/2 \)

\( \tilde{\Omega} \)

\( 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \)

**Figure 4.1:** \( a = 50(mm), \ b = 40(mm), \ \tilde{A} = 7.14 \times 10^{-3}, \ \mu = 0.7 \). Plot (a) shows the excitation frequency change and disk response in time series. The blue squares in the Plot (b) represent the amplitude of the system response \( \theta_{Amp} \) while \( \tilde{\Omega} \) is increasing, and the orange triangles represent the amplitude while \( \tilde{\Omega} \) is decreasing.

The simulation for a monostable rocking disk is presented in the Fig. 4.1. Fig. 4.1(a) shows the time series data of the angle \( \theta \) from the simulation, demonstrating how the amplitude of response \( \theta_{Amp} \) varies with the excitation’s frequency where \( \tilde{\Omega} \) increases from 1.5 to 3.5, then decrease to 1.5. Retrieving the data from Fig. 4.1(a), Fig. 4.1(b) shows how \( \theta_{Amp} \) evolves with \( \tilde{\Omega} \) in a scatter plot. From time series data, the rocking disk is statically stable at its equilibria \( \tilde{\theta} = 0 \) when \( \tilde{\Omega} \) is small. When \( \tilde{\Omega} \) increases to around 2.35, \( \tilde{\theta} = 0 \) is no longer stable and \( \theta \) has to leave from this equilibrium point, therefore the rocking disk suddenly starts to oscillate with a large amplitude. As \( \tilde{\Omega} \) increases, the response of the rocking disk gets smaller, and when \( \tilde{\Omega} > 2.8 \), \( \tilde{\theta} = 0 \) becomes stable again, thus the rocking disk appears stationary at \( \tilde{\theta} = 0 \). After reaching the maximum of frequency, \( \tilde{\Omega} \) starts to decrease. When it becomes smaller than 2.8, \( \tilde{\theta} = 0 \) turns unstable. Different from the bifurcation occurring at \( \tilde{\Omega} = 2.35 \) with the increasing \( \tilde{\Omega} \), the disk’s response changes smoothly as \( \tilde{\Omega} \) decreases. However, as \( \tilde{\Omega} \) reaches 1.57, the increasing response suddenly vanishes and the rocking disk goes back to its stable equilibrium \( \tilde{\theta} = 0 \). For the increasing and decreasing \( \tilde{\Omega} \), the bifurcation takes place at the obviously different \( \tilde{\Omega} \). It makes
hysteresis occur in the Fig. 4.1(b).

Comparatively, the simulation for a bistable rocking disk is much more complex. As shown in the Fig. 4.2, the rocking disk may oscillate around its stable equilibria with a small amplitude (Fig. 4.2(b)), or move across the unstable equilibrium at $\tilde{\theta} = 0$ and oscillate with a large amplitude. It should be noted that the rocking disk oscillates with $\frac{1}{2}\tilde{\Omega}$, the half of the excitation frequency. However, in some special cases, the frequency of the rocking disk could go down to $\frac{1}{4}\tilde{\Omega}$.

In the Fig. 4.3, when a frequency varying excitation is applied to the bistable rocking disk which slightly oscillates around its stable equilibria ($\tilde{\theta} \approx 0.903 = 51.72^\circ$), the disk’s response resembles the monostable disk oscillating around $\tilde{\theta} = 0$ (Fig. 4.1). The only difference lies in that the hysteresis occurs twice: one at low frequency and the other one at high frequency.

For the case where the rocking disk oscillates with a large amplitude like Fig. 4.2(a), the system could reach a steady state with a constant excitation frequency (Fig. 4.5). However, it is hard to reach and maintain another steady state when the excitation frequency changes, no matter how slightly and slowly the frequency is varying. The system often shows chaos with a time varying excitation frequency (Fig. 4.4). As mentioned before, when the rocking disk oscillates periodically with a large amplitude, the frequency of the response could become $\frac{1}{4}\tilde{\Omega}$ occasionally (Fig. 4.4).
Figure 4.3: $a = 40(mm)$, $b = 60(mm)$, $\tilde{A} = 1.60 \times 10^{-2}$, $\mu = 0.7$. Plot (a) shows the excitation frequency change and disk response in time series. The blue squares in the Plot (b) represent the amplitude of the system response $\theta_{Amp}$ while $\tilde{\Omega}$ is increasing, and the orange triangles represent the amplitude while $\tilde{\Omega}$ is decreasing.

4.5(b)). It is a rare steady state resulting from a group of strictly chosen parameters: disk geometry, excitation frequency and amplitude, damping, and initial conditions. Any small perturbation of those parameters will reduce the response frequency to $\frac{1}{2} \tilde{\Omega}$ (Fig. 4.5(a)).

Figure 4.4: $a = 60(mm)$, $b = 82(mm)$, $\tilde{A} = 5.21 \times 10^{-2}$, $\mu = 0.4$. The disk response shows chaos in time series with the time varying excitation frequency.
4.2 Amplitude Sweep

After sweeping frequencies of the vertical excitation, the frequency is fixed in the following simulations, and we will see how the varying excitation amplitude affects the response of the semi-elliptical rocking disk. Again, the maximum of the amplitude is set to be reasonably small to prevent the rocking disk from jumping off the platform.

Figure 4.5: $a = 60\,(mm)$, $b = 82\,(mm)$, $\tilde{A} = 5.21 \times 10^{-2}$, $\tilde{\Omega} = 0.71$. The disk in the Plot (a) oscillates with $\frac{1}{2} \tilde{\Omega}$ and the disk in the Plot (b) oscillates with $\frac{1}{4} \tilde{\Omega}$.

Figure 4.6: $a = 40\,(mm)$, $b = 40\,(mm)$, $\tilde{\Omega} = 15.74$, $\mu = 1.5$. Plot (a) shows the excitation amplitude change and disk response in time series. The blue squares in the Plot (b) represent the amplitude of the system response $\theta_{Amp}$ while $\tilde{A}$ is increasing, and the orange triangles represent the amplitude while $\tilde{A}$ is decreasing.
Figure 4.7: $a = 40\, (mm), b = 65\, (mm), \tilde{\Omega} = 15.74, \mu = 1.5$. Plot (a) shows the excitation amplitude change and disk response in time series. The blue squares in the Plot (b) represent the amplitude of the system response $\theta_{Amp}$ while $\tilde{A}$ is increasing, and the orange triangles represent the amplitude while $\tilde{A}$ is decreasing.

In the Fig. 4.6, the numerical solution for a monostable rocking disk is provided. Fig. 4.6(a) shows the time series data of the angle $\theta$ from the simulation, demonstrating how the amplitude of response $\theta_{Amp}$ varies with the excitation’s amplitude where $\tilde{A}$ increases from 0 to 570, then decrease to 0. Reforming the data from Fig. 4.6(a), Fig. 4.6(b) shows how $\theta_{Amp}$ evolves with $\tilde{A}$ in a scatter plot. From time series data, the rocking disk is statically stable at its equilibria $\theta = 0$ when $\tilde{A}$ is small. When $\tilde{\Omega}$ increases to around 210, $\theta = 0$ is no longer stable and the system suddenly oscillates with a comparatively large amplitude. As $\tilde{A}$ increases, the response of the rocking disk gets greater until $\tilde{A}$ reaches the maximum. While $\tilde{A}$ is decreasing, the rocking disk’s amplitude becomes correspondingly smaller. Unlike the bifurcation and the sudden change of response which occur at $\tilde{A} = 210$ with the increasing $\tilde{A}$, the response amplitude for the different steady states changes slowly and smoothly as $\tilde{A}$ decreases. It makes hysteresis occur in the Fig. 4.6(b).

In the Fig. 4.7, a amplitude varying excitation is applied to the bistable rocking disk. Similar to the case for the monostable disk, the bistable disk keeps stationary at its equilibrium point ($\tilde{\theta} \approx 1.049 \approx 60.10^\circ$ in Fig. 4.7(a)) until $\tilde{A}$ increases to
Figure 4.8: $a = 60(mm)$, $b = 82(mm)$, $\tilde{\Omega} = 6.99$, $\mu = 0.3$. The disk in Plot (a) dissipates quickly and keeps stationary at its equilibrium $\tilde{\theta} \approx -0.541 \approx -30.98^\circ$. The disk in Plot (b) reaches the steady state and its response amplitude is hardly affected by excitation amplitude.

around 265. The rocking disk’s amplitude gains a sudden increment and then varies with $\tilde{A}$ smoothly.

Like the cases for frequency sweeping, the steady state where the bistable rocking disk goes across its unstable equilibrium ($\tilde{\theta} = 0$) and oscillates with a large amplitude is treated as a rare event (Fig. 4.2(a)). It is uneasy for a bistable rocking disk to “reach” and “maintain” this special steady state with a varying frequency, however, it is merely hard to “reach” that with a varying amplitude. In other words, “maintaining” this special steady states is comparatively easy for the cases with a varying excitation amplitude.

In the Fig. 4.8, these two cases share the same disk geometry, initial conditions, excitation frequency and the maximum of the excitation amplitude. The only difference lies in that the excitation amplitude $\tilde{A}$ increases from 307.6 in Fig. 4.8(a) and the other one in Fig. 4.5(b) increases from 316.5. The two $\tau - \tilde{A}$ plots look almost same while disk’s responses in time series are distinct. On one hand, the subtle difference in $\tilde{A}$ makes the rocking disk in Fig. 4.8(a) lose the opportunity
forever to reach the special steady state. The system dissipates gradually and stops
at its equilibrium point no matter how greatly $\tilde{A}$ changes. Therefore the parameters should be extremely precise and chosen carefully to help the rocking disk reach
the special steady state. On the other hand, in Fig. 4.5(b), once the rocking disk
has reached the special steady states, the system keeps oscillating with stable large
amplitude. Even though $\tilde{A}$ is varying, the system stays in the special steady state
easily. Compared with the frequent chaos for the bistable rocking disk with the
varying excitation frequency (Fig. 4.4), the varying excitation amplitude makes the
system more predicable and more easily to maintain the steady states.
Vertical Excitation of the Coupled Disk-Pendulum System

When a pendulum is pinned to the semi-elliptical rocking disk, the motion of the whole coupled system is complicated and diverse. Distinct system responses may result from any subtle change of the parameters: geometry, masses, damping, initial conditions and vertical excitation. In this chapter, this coupled system’s motion falls into two main categories: steady state and chaos, each of whom includes several typical motion patterns.

For convenience, the equations of motion (2.12a) and (2.12b) could be written as a simplified version, and then the two damping terms of the rocking disk and pendulum, $\mu$ and $\mu_1$, are introduced as below:

$$\ddot{\theta} + \mu \dot{\theta} = F \left( \theta, \theta_1, \dot{\theta}_1^2, \sin(\Omega t) \right)$$

(5.1a)

$$\ddot{\theta}_1 + \mu_1 \dot{\theta}_1 = F_1 \left( \theta, \theta_1, \dot{\theta}_1^2, \dot{\theta}_2^2, \sin(\Omega t) \right)$$

(5.1b)

where $\sin(\Omega t)$ represents the vertical excitation with the frequency $\Omega$.

The most common case for steady state is shown in Fig. 5.1. Both the rocking disk and pendulum oscillate around their own stable equilibria with the same frequency.
0.5Ω. It should be noted that the response’s frequency is half of the excitation frequency, and the rocking disk and pendulum oscillate in phase.

For a monostable rocking disk, an excessively high pivotal point will make it lose the stability around $\theta = 0$ and topple (Fig. 3.1(a)). However, a moderately high pivotal point can keep the disk’s monostability and help reach the steady state with a frequency $\Omega$ (Fig. 5.2). The amplitude of response shrinks obviously for the higher frequency, and the oscillation of the rocking disk and pendulum turns out-of-phase.

When the pivotal point is moved below the disk’s center of mass, the system runs in another different way: beat oscillation. For the cases in Fig. 5.1 and 5.2, the system sometimes oscillates like beats in the initial transient but these beats vanish soon when the system is reaching the steady state. In contrast, the beats in Fig. 5.3 is constantly stable and should be treated as one of the steady states of the disk-pendulum coupled system. By using Fourier transformation, the system’s response in the frequency domain is presented in Fig. 5.4. The rocking disk and pendulum have different frequency compositions, but both of them have a major
When the pivot point is moderately high, the system oscillates with small amplitudes but large frequency, which is identical to the excitation frequency. The disk and pendulum are oscillating out of phase.

Beat oscillation occurs when the pivot point is moderately low.

response around $\frac{1}{2}\Omega$. 

Figure 5.2: $a : b : l_1 : l_R = 50 : 45 : 60 : 80$, $m : m_E : m_R = 5 : 0 : 1$, $\Omega = 6\pi$, $\mu = 0.4$, $\mu_1 = 0.2$. When the pivot point is moderately high, the system oscillates with small amplitudes but large frequency, which is identical to the excitation frequency.

Figure 5.3: $a : b : l_1 : l_R = 50 : 45 : -10 : 80$, $m : m_E : m_R = 5 : 0 : 1$, $\Omega = 8\pi$, $\mu = 0.8$, $\mu_1 = 0.5$. Beat oscillation occurs when the pivot point is moderately low.
Figure 5.4: For the case in Fig. 5.3, the response of the rocking disk and pendulum in the frequency domain.
This thesis investigates the static and dynamic response of a coupled disk-pendulum system using both analytical and numerical methods.

For the case of unforced behavior, when the pendulum is not pinned on the rocking disk, the disk’s geometry determines whether the bistability or monostability occurs. When adding a pendulum on the disk, the number and the stability of equilibria will change with the location of the pivot point. The proper choice of the location can make a bistable disk monostable and vice versa, or alter its original stability. The disk-pendulum coupled system is sensitive to initial conditions. Therefore, the system easily becomes chaotic and highly unpredictable. The higher pivot point may help enhance the system’s predictability, but it may also diminish the working range and make the system easier to topple.

When the rocking disk without a pendulum is vertically excited, the variation of the excitation’s amplitude and frequency results in the hysteresis for the amplitude of the response. The frequency of response may not be identical to the frequency of excitation. In most cases, the response frequency is half of the excitation frequency. In another cases investigated, the response frequency can even be a quarter of the
excitation frequency. Additionally, there are two major patterns of the steady state: one is the small perturbation around the stable equilibrium, and the other one is oscillating across the unstable equilibrium with a large amplitude. For the steady state with large amplitude, both the time varying frequency and time varying amplitude of the excitation make the rocking disk difficult to reach the steady state. However, once the rocking disk has reached steady state, the rocking disk can maintain the steady state easily with excitation of time varying amplitude, while the excitation with the time varying frequency brings about chaos for the rocking disk.

When a pendulum is pinned on the rocking disk, three major categories of steady states were observed with numerical investigations. With the moderately high pivot point, the system might obtain a periodic response where the response frequency is identical to the excitation frequency and the disk and the pendulum oscillate out of phase. With a lower pivotal point, the system could oscillate with the half of the excitation frequency and the oscillations of disk and the pendulum are in phase. As the pivot point move downwards below the disk’s center of mass, the system’s response shows “beats” oscillation.
Bibliography


