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CHAPTER 1

Holonomy and Special Geometries

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Introduction

According to the Oxford English Dictionary, it was Heinrich Hertz[1] who introduced the words *holonomic* and *nonholonomic* to describe a property of velocity constraints in mechanical systems.

Velocity constraints are *holonomic* if they force a curve in state space to stay in a proper subspace. As an example, the condition $\mathbf{p}\cdot d\mathbf{p} = 0$ for a vector particle $\mathbf{p} \in \mathbb{R}^n$ forces $\mathbf{p}$ to have constant length, while the constraint $\mathbf{p} \wedge d\mathbf{p} = 0$ forces $\mathbf{p}$ to move on a line.

Nonholonomic constraints, on the other hand, imply no such 'finite' constraints. A classical example is that of a ball rolling on a table without slipping or twisting. The state space is $B = \mathrm{SO}(3) \times \mathbb{R}^2$, where the $\mathrm{SO}(3)$ records the orientation of the ball and the $\mathbb{R}^2$ records its contact point on the plane. The rolling constraint is expressed as the set of differential equations

$$\alpha = a^{-1}\,da + a^{-1}\begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix} a = 0 \tag{1.1}$$

for a curve $\big(a(t); x(t), y(t)\big)$ in B. The curves in B satisfying this constraint are those tangent to the 2-plane field $D = \operatorname{Ker}\alpha$, which is transverse to the fibers of the projection $\mathrm{SO}(3)\times\mathbb{R}^2 \to \mathbb{R}^2$. It is not difficult to show that any two points of B can be joined by a curve tangent to D. I.e., one can roll a ball from any position to any other by choosing an appropriate rolling path $\big(x(t), y(t)\big)$.

Such constraints and their geometry have long been of considerable interest in the calculus of variations and control theory. For recent results, see the foundational work on Carnot-Carathéodory geometries by Gromov [**Gr96**].

A generalization of this idea, namely *holonomy*, was introduced into differential geometry by Schouten [**Sch18**] and Cartan [**Ca25, Ca26**] and has come to play an important role in differential geometry.

Of course, these notes can only serve as an overview and introduction to the subject. For details and further information, the reader is referred to the standard works on the subject, i.e., [**KN69, Hel78, Sa89, Jo00**].

1.1. Bundles and Connections

[1] Hertz died in 1984 and his book *Principien der Mechanik* was published in 1895. An English translation appeared in 1899 and it is this translation that the OED uses to date the terms under discussion.

1.1.1. Principal Bundles and Connections. Since I will only need to discuss bundles over manifolds, I will adopt a definition that is well-adapted to to this case.

Definition 1.1 (Principal H-bundles). Given a Lie group H and a smooth manifold M, a *principal right H-bundle* over M is a smooth manifold B endowed with both a smooth, surjective submersion $\pi : B \to M$ and a free, right H-action $R : B \times H \to B$ such that the fibers of π are the orbits of the H-action R.

Two principal right H-bundles (B_1, π_1, R_1) and (B_2, pi_2, R_2) over M are said to be *equivalent* if there exists a diffeomorphism $f : B_1 \to B_2$ such that $\pi_1 = \pi_2 \circ f$ and $f\bigl(R_1(u, h)\bigr) = R_2(u, h)$.

Remark 1.2 (Notation). For $u \in B$ and $h \in H$, is customary to write $u{\cdot}h$ for $R(u, h)$ and I will adopt this practice.

Remark 1.3 (Trivial Bundles). An obvious example is to let $B = M{\times}H$, with $\pi : B \to M$ being the projection on the first factor and $R : (M{\times}H) \times H \to M{\times}H$ being defined by group multiplication in the obvious way. This is said to be the *trivial* bundle over M.

More generally, a principal right H-bundle (B, π, R) over M is trivial (i.e., equivalent to the trivial bundle) if there is a smooth map $\sigma : M \to B$ that satisfies $\pi\bigl(\sigma(m)\bigr) = m$ for all $m \in M$. The mapping σ is said to be a *trivializing section*.

Remark 1.4 (Induced bundles and local triviality). If $\bigl(B, \pi, R\bigr)$ is a principal right H-bundle over M and $N \subset M$ is any submanifold, then, by the implicit function theorem, $B_N = \pi^{-1}(N)$ is a smooth submanifold of B that inherits the structure of a principal right H-bundle over N.

More generally, if $f : N \to M$ is any smooth mapping, then the subset $f^*(B) = \{(n, u) \mid f(n) = \pi(u)\} \subset N \times B$ defines a principal right H-bundle over N.

Note that the implicit function theorem implies that each point $m \in M$ has an open neighborhood $U \subset M$ such that B_U is a trivial H-bundle over U.

1.1.1.1. *Examples.* Of course, there are many examples of principal H-bundles. Here are a few of the most common:

Example 1.5 (Homogeneous bundles). If H is a closed subgroup of a Lie group G, then the left coset space G/H carries a unique smooth structure such that the coset projection $\pi : G \to G/H$ is a smooth submersion [**Hel78**]. Letting H act on G on the right via group multiplication makes G into a principal right H-bundle over $M = G/H$.

Example 1.6 (Coframe bundles). If $E \to M$ is a smooth (real) vector bundle of rank r, let $\mathsf{F}(E)$ denote the set of pairs (m, u) where $m \in M$ and where $u : E_m \to \mathbb{R}^r$ is a linear isomorphism. (One says that u is a *coframe* of E_m.)

One can regard $\mathsf{F}(E)$ as an open subset of the vector bundle $\mathrm{Hom}(E, \mathbb{R}^r) = E^* {\otimes} bR^r$, so it is naturally endowed with a smooth structure for which the basepoint projection $\pi : \mathsf{F}(E) \to M$ defined by $\pi(m, u) = m$ is a surjective submersion. Let $H = \mathrm{GL}(r, \mathbb{R})$ act on $\mathsf{F}(E)$ on the right via the action $(m, u){\cdot}h = \bigl(m, h^{-1} {\circ} u\bigr)$. Then the mappings π and R make $\mathsf{F}(E)$ into a principal right $\mathrm{GL}(r, \mathbb{R})$-bundle over M that is called the *coframe bundle* of E.

More generally, if $E \to M$ is a vector bundle over M that is endowed with some extra structure, one can often encode this extra structure into an associated principal coframe bundle. For example, if $E \to M$ is an Hermitian r-plane bundle, i.e., if E is endowed with a complex structure J and a compatible inner product $\langle , \rangle$, then one can consider the bundle $\mathsf{U}(E)$ consisting of the pairs (m, u) where m lies in M and $u : E_m \to \mathbb{C}^r$ is both complex linear and isometric. (Here, $\mathbb{C}^r$ is endowed with its standard inner product and complex structure.) Then $\mathsf{U}(E)$ is naturally a principal right $\mathrm{U}(r)$-bundle over M.

1.1.1.2. *Associated vector bundles.* The process of constructing principal H-bundles from vector bundles (possibly endowed with extra structure) can be 'inverted'. If (B, π, R) is a principal right H-bundle over M and $\rho : H \to \mathrm{GL}(V)$ is a representation of H as a set of linear transformations of some vector space V (where the ground 'field' can be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), then one can construct an *associated vector bundle* $E = B \times_\rho V = \big(B \times V\big)/\!\sim$, where one sets $(u, x) \sim \big(u{\cdot}h, \rho(h^{-1})x\big)$ and denotes the corresponding equivalence class of (u, x) by $[u, x]$. (To save notational clutter, the notation $[u, x]$ supresses the dependence on the representation ρ.)

The associated bundle inherits those properties from V that are invariant under the action of the elements $\rho(h)$ for $h \in H$. Thus, for example, if $\rho(H)$ preserves an inner product (resp., orientation, complex structure, etc.) on V, then E inherits a corresponding inner product (resp., orientation, complex structure, etc.)

If the representation ρ is faithful, one can then naturally interpret B as an H-subbundle of the $\mathrm{GL}(V)$-coframe bundle $\mathsf{F}(E)$.

This is useful because it is frequently important to be able to treat sections of the bundle E as functions in some sense. One can do this by interpreting a section $\sigma : M \to E$ as the mapping $s : B \to V$ that satisfies

$$\sigma\big(\pi(u)\big) = \big[u, s(u)\big]. \tag{1.2}$$

It is easy to see that

$$s(u{\cdot}h) = \rho(h^{-1})(u) \tag{1.3}$$

and, conversely, any function $s : B \to V$ that satisfies the equivariance (1.3) defines a section σ of E via (1.2).

1.1.1.3. *Extension and Retraction.* If H is a subgroup of a Lie group G, one can 'extend' a principal right H-bundle B over M to a principal right G-bundle B' over M in a natural way:

Set $B' = \big(B \times G\big)/\!\sim$ where one sets $(u, g) \sim \big(u{\cdot}h, h^{-1}g\big)$ and denotes the equivalence class of (u, g) by $[u, g]$. It is easy to see that G acts naturally on the right on B' via $[u, g]{\cdot}g' = [u, gg']$ and that the quotient B' carries a unique smooth structure for which $B \times G \to B'$ is a smooth submersion. Then the natural projection $\pi' : B' \to M$ defined by $\pi'\big([u, g]\big) = \pi(u)$ completes the data needed to make B' into a principal right G-bundle over M. One says that B' is the *extension* of B to a G-bundle.

It is not generally true that every principal right G-bundle over M is the extension of a principal right H-bundle, but this is true when H contains a maximal compact subgroup of G (as follows from a partition of unity argument). In general, a principal right G-bundle B' over M is an extension of a principal right H-bundle B over M if and only if B can be found as a submanifold of B' that is invariant under the action of H. In this case, one says that the submanifold $B \subset B'$ is a *retraction* (or sometimes *reduction*) of the bundle B'.

1.1.1.4. *Connections.* Even though every principal right H-bundle $B \to M$ is locally trivial, it is not canonically trivial, i.e., there is no canonical way to identify the fibers B_x and B_y for points $x \neq y$ in M. Even in the case in which the bundle B is globally trivial, so that there exists a section $\sigma : M \to B$, this section is far from unique. Any other section σ' can be written in the form $\sigma'(m) = \sigma(m){\cdot}f(m)$ where $f : M \to H$ is any smooth function.

The notion of a *connection* was introduced to, at least partly, remedy this situation.

Definition 1.7. A *connection* on a principal right H-bundle B over M is a (linear) subbundle $L \subset TB$ that is invariant under the right action R, i.e., $(R_h)'(L_u) = L_{u\cdot h}$ for all $u \in B$ and $h \in H$ and that is complementary to the 'vertical subbundle' $\operatorname{Ker}\pi' \subset TB$, i.e., so that $L \cap \operatorname{Ker}\pi' = \{0\}$ and $TB = L \oplus \operatorname{Ker}\pi'$.

Every principal right H-bundle B has a connection [**KN69**], in fact, many of them.

Example 1.8 (Homogeneous connections). Consider the case where H is a closed subgroup of a Lie group G and let $\mathfrak{h} \subset \mathfrak{g} = T_eG$ denote the Lie algebras of the corresponding groups. The representation $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{g})$ then preserves the subspace $\mathfrak{h} \subset \mathfrak{g}$, but need not preserve a complementary subspace $\mathfrak{m} \subset \mathfrak{g}$ to $\mathfrak{h}$ and, if it does, such an $\mathrm{Ad}(H)$-invariant complement need not be unique. However, if there is such a complement $\mathfrak{m}$ (and one always exists if H is compact, or, more generally, reductive), then there is a unique left-invariant subbundle $L \subset TG$ that satisfies $L_e = \mathfrak{m}$ and this defines a connection on G (considered as a principal right H-bundle over $M = G/H$) that is invariant under the left action of G. Conversely, every such G-invariant connection on G over G/H comes from such an invariant complement $\mathfrak{m} \subset \mathfrak{g}$ to $\mathfrak{h}$.

1.1.1.5. *Connection* 1*-forms.* An equivalent formulation is in terms of *connection* 1*-forms*. Let $\mathfrak{h} = T_eH$ denote the Lie algebra of H. For any $v \in \mathfrak{h}$, there is a corresponding homomorphism $\mathbb{R} \to H$ denoted by $t \mapsto e^{tv}$. This homomorphism induces a unique vector field Y_v on B whose associated flow Φ_v satisfies $\Phi_v(t,u) = u{\cdot}e^{tv}$. The mapping $\iota_u : \mathfrak{h} \to T_uB$ defined by $\iota_u(v) = Y_v(u)$ is a linear isomorphism of $\mathfrak{h}$ with $\operatorname{Ker}\pi'(u) \subset T_uB$. Linear algebra now shows that, for each connection $L \subset TB$, there is a unique *connection* 1*-form* θ, i.e., an $\mathfrak{h}$-valued 1-form θ on B with the two properties that, first, $\theta(Y_v) = v$ for all $v \in \mathfrak{h}$, and, second, that $\theta(L) = 0$.

The invariance of L under the right action of H translates into the condition that θ have the equivariance ${R_h}^*(\theta) = \mathrm{Ad}(h^{-1})(\theta)$ for $h \in H$. Conversely, if one has an $\mathfrak{h}$-valued 1-form θ on B that satisfies $\theta(Y_v) = v$ for all $v \in \mathfrak{h}$ and ${R_h}^*(\theta) = \mathrm{Ad}(h^{-1})(\theta)$ for $h \in H$ defines a unique connection L by simply setting $L = \operatorname{Ker}\theta$.

1.1.1.6. *L-horizontal curves and vector fields.* Let $L \subset TB$ be a connection. By definition, $\pi'(u) : L_u \to M_{\pi(u)}$ is an isomorphism for all $u \in B$. A piecewise differentiable curve $u : [a,b] \to B$ is said to be *L-horizontal* if $u'(t)$ lies in $L_{u(t)}$ for all $t \in [a,b]$ for which $u'(t)$ exists. Because of the H-invariance of L, if $u : [a,b] \to B$ is L-horizonal, then so is $R_h{\circ}u : [a,b] \to B$ for all $h \in H$.

Standard results in ordinary differential equations then imply that, for every piecewise differentiable curve $c : [a,b] \to M$ and every $u_0 \in B_{c(0)}$, there is a unique L-horizontal curve $u : [a,b] \to B$ that satisfies $u(a) = u_0$ and that lifts c through π in the sense that $\pi\bigl(u(t)\bigr) = c(t)$ for all $t \in [a,b]$. Because of the H-invariance of L, the curve $c : [a,b] \to M$ induces a diffeomorphism $P_c(t) : B_{c(a)} \to B_{c(t)}$ for all $t \in [a,b]$ that satisfies $P_c(t)(u_0) = u(t)$ and $P_c(t)\bigl(u\cdot h\bigr) = P_c(t)\bigl(u\bigr)\cdot h$.

In general, the map $P_c(t)$ depends on the curve c and not just on the end-points $c(a)$ and $c(t)$.

One can also use L to lift vector fields on M in the following sense: For each vector field X on M, there is a unique vector field X^L on B that satisfies $\pi'(u)\bigl(X^L(u)\bigr) = X\bigl(\pi(u)\bigr)$ and $X^L(u) \in L_u$ for all $u \in B$. The H-invariance of L implies that X^L is invariant under the action of H.

This process of lifting vector fields allows one to define an important invariant of a connection: For vector fields X and Y on M, consider the vector field on B defined by the formula

$$Z = [X^L, Y^L] - [X,Y]^L. \tag{1.4}$$

Now, $Z(u)$ depends only on $X\bigl(\pi(u)\bigr)$ and $Y\bigl(\pi(u)\bigr)$, is linear in each factor, and takes values in $\operatorname{Ker}\pi'(u) \simeq \mathfrak{h}$. In particular, there is a well-defined map $K(u) : \Lambda^2(T_{\pi(u)}M) \to \mathfrak{h}$ that satisfies the equivariance

$$K(u\cdot h)(x,y) = \mathrm{Ad}(h^{-1})\bigl(K(u)(x,y)\bigr). \tag{1.5}$$

This is known as the *curvature* of the connection L.

1.1.2. Holonomy. It was Cartan who introduced the notion of the *holonomy group* of a connection and this has turned out to be a central notion in differential geometry.

Definition 1.9 (Holonomy bundle and group). Let (B,π,R) be a principal right H-bundle over a connected manifold M and let $L \subset TB$ be a connection on B.

For each $u \in B$, the *holonomy bundle* of L through u is the set $B^L(u)$ consisting of those points v for which there exists an L-horizontal curve $f : [a,b] \to B$ that satisfies $f(a) = u$ and $f(b) = v$. The *holonomy* of L at u is the set $\mathrm{Hol}^L(u) \subset H$ consisting of those $h \in H$ for which $u\cdot h$ lies in $B^L(u)$.

1.1.2.1. *Holonomy group structure.* The set $\mathrm{Hol}^L(u)$ is actually a subgroup of H. This can be seen as follows: First, since the constant path $f : [0,1] \to B$ defined by $f(t) = u$ is L-horizontal, it follows that the identity element e lies in $\mathrm{Hol}^L(u)$. Second, if h lies in $\mathrm{Hol}^L(u)$, then, by definition, there exists an L-horizontal curve $f : [0,1] \to B$ such that $f(0) = u$ while $f(1) = u\cdot h$. The mapping $g : [0,1] \to B$ defined by $g(t) = g(1-t)\cdot h^{-1}$ is then L-horizontal and joins u to $u\cdot h^{-1}$, so $\mathrm{Hol}^L(u)$ is closed under inverses. Finally, if h_1 and h_2 lie in $\mathrm{Hol}^L(u)$, then let $f_i : [0,1] \to B$ be L-horizontal curves that satisfy $f_i(0) = u$ and $f_i(1) = u\cdot h_i$. Define a new L-horizontal curve $g : [0,1] \to B$ by setting $g(t) = f_1(2t)$ for $t \in [0,\frac{1}{2}]$ while $g(t) = f_2(2t-1)\cdot h_1$ for $t \in [\frac{1}{2},1]$. Then $g(0) = u$ and $g(1) = (u\cdot h_2)\cdot h_1 = u\cdot(h_2h_1)$, showing that $\mathrm{Hol}^L(u)$ is closed under multiplication.

1.1.2.2. *Holonomy properties.* As the reader can verify, the group $\mathrm{Hol}^L(u)$ has the following properties:

$$\mathrm{Hol}^L(u\cdot h) = h^{-1}\mathrm{Hol}^L(u)h \tag{1.6}$$

and

$$\mathrm{Hol}^L(v) = \mathrm{Hol}^L(u) \qquad \text{for all } v \in B^L(u). \tag{1.7}$$

In particular, the connection L determines a conjugacy class of subgroups of H. Strictly speaking, the *holonomy* of L is this conjugacy class of subgroups, but it is common to speak of the holonomy of L as being a particular subgroup in this conjugacy class.

What is not so obvious is that $\mathrm{Hol}^L(u)$ is actually a (not necessarily closed) Lie subgroup of H. In fact, Cartan asserted this without giving what would now be considered to be an adequate proof. It was eventually proved (to modern standards) by Borel and Lichnerowicz:

Theorem 1.10 (Borel-Lichnerowicz [**BL52**]). *The group* $\mathrm{Hol}^L(u) \subset H$ *is a Lie subgroup, connected if* M *is simply connected. Moreover,* $B(u) \subset B$ *is a submanifold of* B *and a principal right* $\mathrm{Hol}^L(u)$*-bundle over* M*.*

For the proof, see [**KN69**]. The main idea is to reduce to the case in which M is simply connected, in which case one easily sees that $\mathrm{Hol}^L(u)$ is a path-connected subgroup of the Lie group H and this implies directly that it is a Lie subgroup of H.

A further important result relates the curvature of the connection L to its holonomy. To state this result, recall the definition eqrefeq:curvdef of the curvature map $K^L(u) : \Lambda^2\big(T_{\pi(u)}M\big) \to \mathfrak{h}$. Let $\mathfrak{k}^L(u) \subset \mathfrak{h}$ denote the image of this map.

Theorem 1.11 (Ambrose-Singer [**AS53**]). *The Lie algebra of* $\mathrm{Hol}^L(u) \subset H$ *is the smallest subalgebra of* $\mathfrak{h}$ *that contains the subspaces* $\mathfrak{k}^L(v) \subset \mathfrak{h}$ *for all* $v \in B^L(u)$*.*

Again, for the proof, see [**KN69**].

Note, however, a curious difficulty posed by Theorem 1.11: One cannot directly determine the smallest subalgebra of $\mathfrak{h}$ that contains the subspaces $\mathfrak{k}^L(v) \subset \mathfrak{h}$ for all $v \in B^L(u)$ without determining $B^L(u)$, which will, of course, determine $\mathrm{Hol}^L(u)$ directly. In fact, the main uses of Theorem 1.11 are theoretical, in which this result is used to provide a 'lower bound' for the holonomy group rather than to compute it directly, as will be seen.

1.2. Riemannian Holonomy

Undoubtedly, one of the most important uses, so far, of holonomy has been its application to the 'classification' of special geometries that can be compatible with Riemannian geometry.

1.2.1. Orthonormal coframe bundle. Let (M, g) be a Riemannian manifold. For simplicity, in these notes, I am going to assume that M is both connected and simply connected (though there are certainly interesting effects to consider when M has a non-trivial fundamental group). In this case, M is orientable, so I will go ahead and assume that M has been oriented as well.

Using the metric g and orientation, there is a natural $\mathrm{SO}(n)$-bundle defined over M. Namely, let $\pi : F \to M$ denote the set of orientation preserving, isometric isomorphisms $u : T_xM \to \mathbb{R}^n$, let $\pi(u) = x$, and let $\mathrm{SO}(n)$ act on F on the right by $u{\cdot}h = h^{-1}{\circ}u$. (The reader should recognize this as a special case of the construction in Example 1.6.) Then F is the bundle of oriented, orthonormal coframes of (M, g).

1.2.2. The Levi-Civita connection. Now, as is well-known, a Riemannian metric has a distinguished connection on its tangent bundle, i.e., the Levi-Civita. From now on, this will be understood to be the connection on F unless it is otherwise explicitly stated.

The resulting holonomy group through $u \in F$ is known as the *holonomy* at u of the Riemannian metric g and so will be denoted as $\mathrm{Hol}^g(u) \subset \mathrm{SO}(n)$. Because M has been assumed to be simply connected, $\mathrm{Hol}^g(u)$ is connected.

A special feature of F is that, for $u \in F$ with $\pi(u) = x$, the isomorphism $u : T_x \to \mathbb{R}^n$ allows one to interpret the curvature, as defined via (1.5), as a linear map

$$K_x : \Lambda^2(T_xM) \to \Lambda^2(T_xM) \simeq \mathfrak{so}(T_xM) \tag{1.8}$$

and the *first Bianchi identity* is expressed as the relation

$$K_x(u,v)w + K_x(v,w)u + K_x(w,u)v = 0 \qquad \text{for all } u,v,w \in T_xM. \tag{1.9}$$

Let $\mathfrak{h}(u) \subset \mathfrak{so}(n)$ denote the Lie algebra of $\mathrm{Hol}^g(u)$.

1.2.3. Reducibly acting holonomy. If $\mathrm{Hol}^g(u) \subset \mathrm{SO}(n)$ preserves a subspace $V \subset \mathbb{R}^n$ then it must also preserve its orthogonal $V^\perp$. In particular, the representation of $\mathrm{Hol}^g(u)$ on $\mathbb{R}^n$ is completely reducible, i.e., there is an orthogonal direct sum decomposition

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k \tag{1.10}$$

where each of the V_i is irreducible as a representation of $\mathrm{Hol}^g(u)$.

Now, a consequence of the first Bianchi identity (1.9) is that, if one sets $\mathfrak{h}_i(u) = \mathfrak{h}(u) \cap \mathfrak{so}(V_i)$, then one has a direct sum decomposition

$$\mathfrak{h}(u) = \mathfrak{h}_1(u) \oplus \mathfrak{h}_2(u) \oplus \cdots \oplus \mathfrak{h}_k(u). \tag{1.11}$$

This observation is due to Cartan [**Ca25**], who then went on to observe that, in such a case, the point $x = \pi(u) \in M$ has an neighborhood U that has a coordinate system

$$x = (x_1, \ldots, x_k) : U \to V_1 \oplus V_2 \oplus \cdots \oplus V_k \simeq \mathbb{R}^n \tag{1.12}$$

where each $x_i : U \to V_k$ is a submersion and for which there exist Riemannian metrics g_i on V_i such that, on U one has

$$g = (x_1)^*(g_1) + \cdots + (x_k)^*(g_k). \tag{1.13}$$

In other words, the metric g is locally a product metric.

Two further refinements can be made of this observation:

First, G. de Rham proved that if (M,g) is also complete (and, as usual, M is connected and simply connected), then the above product decomposition can be globalized:

THEOREM 1.12 (de Rham Splitting). *If (M,g) is a complete, simply-connected Riemannian manifold then (M,g) can be written as a Riemannian product*

$$(M,g) = (M_1,g_1) \times (M_2,g_2) \times \cdots \times (M_k,g_k) \tag{1.14}$$

where (M_i,g_i) is a simply-connected Riemannian n_i-manifold whose Riemannian holonomy is an irreducibly acting subgroup of $\mathrm{SO}(n_i)$.

Second, while the relation (1.11) shows that $\mathrm{Hol}^g(u)$ is a direct product of groups $H_i \subset \mathrm{SO}(V_i)$, it is not immediately clear (without the assumption that the metric g is complete) that each of the groups H_i is the holonomy of some Riemannian metric on a manifold of dimension $\dim V_i$. Nevertheless, this can be shown to be true. Consequently, in order to classify the possible holonomy groups of Riemannian metrics (of simply connected manifolds), it suffices to classify the possible holonomy groups of Riemannian metrics on n-manifolds whose holonomy groups act irreducibly on $\mathbb{R}^n$.

1.2.4. Berger's Classification. I now want to describe the classification of the possible Riemannian holonomy groups.

1.2.4.1. *Locally symmetric metrics.* There is a class of metrics for which the holonomy is not difficult to compute and that plays an important role in differential geometry: the locally symmetric metrics.

Definition 1.13 (Locally symmetric metrics). A connected Riemannian manifold (M, g) is said to be *locally symmetric* if each point $x \in M$ has an open neighborhood U on which there exists a g-isometric involution $\iota_x : U \to U$ that fixes x and satisfies $(\iota_x)'(v) = -v$ for all $v \in T_xM$.

It is not difficult to show that a locally symmetric (M, g) is locally homogeneous, i.e., that for any two points x and y in M there are open neighborhoods U of x and V of y and a g-isometry $\phi : U \to V$ that satisfies $\phi(x) = y$. Moreover, any locally symmetric (M, g) is locally isometric to a unique, simply-connected, complete locally symmetric space $(\overline{M}, \overline{g})$ and that, $(\overline{M}, \overline{g})$ is *globally symmetric* in the sense that, for each $x \in \overline{M}$, the involution ι_x can be taken to be defined on the entire manifold $\overline{M}$.

If (M, g) is a simply-connected, complete, locally symmetric Riemannian manifold, then, when one factors it as a product using the de Rham splitting theorem, each of the factors (M_i, g_i) is a simply-connected, complete, locally symmetric Riemannian manifold that cannot be further decomposed as a product. Thus, to classify the simply-connected, complete locally symmetric Riemannian manifolds, it suffices to classify the irreducible ones. Of course, if the dimension of (M, g) is 1, then (M, g) is isometric to the real line with its standard metric, so one can further assume that $\dim M > 1$.

Finally, in this case, one can write $M = G/K$ where G is the identity component of $\mathrm{Isom}(M, g)$ and $K \subset G$ is the subgroup of G that fixes a point, say $m_0 \in M$. It can then be shown that the holonomy of (M, g) is isomorphic to K as embedded into $\mathfrak{m} = \mathfrak{g}/gk$ via the adjoint homomorphism.

Thus, the local classification of the locally symmetric spaces is reduced to a classification problem involving Lie groups and Lie algebras. This is a long and beautiful story that I cannot do justice to here. The reader can consult [**Hel78**] or [**KN69**] for details. The main point is that the locally symmetric case is well-understood and that these metrics have been classified by (fairly sophisticated) algebraic means.

1.2.4.2. *Non locally symmetric metrics.* In Marcel Berger's 1954 thesis, he produced a list of possible conjugacy classes of subgroups $H \subset \mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$ and could be the holonomy of a Riemannian n-manifold (M, g) that was not locally symmetric. What he showed was that such a subgroup H would have to be conjugate to one of the groups on the following list: $\mathrm{SO}(n)$, $\mathrm{U}(\frac{1}{2}n)$, $\mathrm{SU}(\frac{1}{2}n)$, $\mathrm{Sp}(\frac{1}{4}n)\cdot\mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{1}{4}n)$, G_2 (if $n = 7$), $\mathrm{Spin}(7)$ (if $n = 8$), or $\mathrm{Spin}(9)$ (if $n = 16$).

1.2.4.3. *Berger's method.* Berger's method can be described as follows: If $H \subset \mathrm{SO}(n)$ is to be the holonomy of a Riemannian metric (M, g), then one can endow the bundle $\mathsf{F}(g)$ of oriented orthonormal coframes for the metric g with its Levi-Civita connection and find a holonomy subbundle $B(u) \subset \mathsf{F}(g)$ whose structure group is H. The associated Levi-Civita connection 1-form θ on F then pulls back to $B(u)$ to take values in the Lie algebra $\mathfrak{h} \subset \mathrm{SO}(n)$. Now, if one looks at the first Bianchi identity (1.9) applied to the curvature of θ pulled back to $B(u)$, one finds that it has to take values in a certain subspace $K(\mathfrak{h}) \subset \mathfrak{h}$ that, in fact, must be an ideal in $\mathfrak{h}$. One can describe this subspace as follows: For any Lie subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$, set

$$R(\mathfrak{h}) = \big\{k : \Lambda^2(\mathbb{R}^n) \to \mathfrak{h} \mid k(x,y)z + k(y,z)x + k(z,x)y = 0\big\} \tag{1.15}$$

and then

$$K(\mathfrak{h}) = \big\{k(x,y) \mid k \in R(\mathfrak{h}),\ x, y \in \mathbb{R}^n\big\}. \tag{1.16}$$

Now, the classification of the connected subgroups of $\mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$ is known (this is essentially the list of irreducible faithful representations of compact Lie groups), and, for each subgroup $H \subset \mathrm{SO}(n)$ on the list, one can check whether or not $K(\mathfrak{h})$ is a proper subalgebra of $\mathfrak{h}$ or not. If $K(\mathfrak{h})$ is a proper subalgebra of $\mathfrak{h}$, then by Theorem 1.11, the holonomy of the connection pulled back to $B(u)$ must be a proper subgroup of H, so H itself cannot be holonomy. In this way, one is reduced to considering the subgroups $H \subset \mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$ and for which $K(\mathfrak{h}) = \mathfrak{h}$. This latter list of subgroups is much, much smaller than the full list of subgroups $H \subset \mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$.

For the subgroups $H \subset \mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$ and satisfy $K(\mathfrak{h}) = \mathfrak{h}$, there is a further test one can apply: Using the second Bianchi identity, one can define a subspace $R^1(\mathfrak{h}) \subset R(\mathfrak{h}) \otimes \mathbb{R}^n$ with the property that the covariant derivative of the curvature tensor of the connection form on $B(u)$ must take values in this space $R^1(\mathfrak{h})$, which can also be computed via representation theory. If $H \subset \mathrm{SO}(n)$ satisfies $R^1(\mathfrak{h}) = (0)$, then any metric with holonomy H has to have covariant constant curvature, which, it turns out, is equivalent to being locally symmetric.

Thus, Berger went through the list of $H \subset \mathrm{SO}(n)$ that act irreducibly on $\mathbb{R}^n$ and that satisfy $K(\mathfrak{h}) = \mathfrak{h}$ and computed the corresponding $R^1(\mathfrak{h})$. He found that this latter vector space was nearly always zero and that, in fact, any subgroup on this list for which $R^1(\mathfrak{h})$ was not zero had to lie on the list as given above.

1.2.4.4. *Another approach and an elimination.* That was where matters stood for some time. Berger's proof was regarded by some as unsatisfactory because it relied so heavily on a case-by-case analysis of the irreducible representations of compact Lie groups.

It was eventually noticed that the groups listed above are subgroups that act transitively on the unit sphere in $\mathbb{R}^n$. In a ground-breaking 1962 paper, Simons [**Si62**] gave a direct proof that any subgroup $H \subset \mathrm{SO}(n)$ that acts irreducibly on $\mathbb{R}^n$ and that can be the holonomy of a Riemannian n-manifold (M, g) that is not locally symmetric must act transitively on the unit sphere in $\mathbb{R}^n$.

However a few years later, Alekseevski [**Al68**] and Brown and Gray [**BG72**] (independently) noticed that, for $H = \mathrm{Spin}(9) \subset \mathrm{SO}(16)$, one actually has $R^1(\mathfrak{h}) =$

(0), so that the group $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ never belonged on Berger's list in the first place[2].

1.2.4.5. *Existence.* Berger's and Simons' arguments do not show that there *are* metrics with these holonomies, just that these were the only possibilities for locally irreducible metrics that were not locally symmetric.

Some of these holonomies were known to exist already when the classification was proved. I will just give some examples now, saving the more extensive discussion for the next section.

For example, it is easy to show that the generic metric on an (orientable) n-manifold has holonomy $\mathrm{SO}(n)$.

The condition of (M^n, g) having holonomy in $H = \mathrm{U}(\frac{1}{2}n)$ is equivalent to the manifold supporting a complex structure J that is orthogonal and parallel with respect to g, i.e., that (M, g, J) is a Kähler manifold. Of course, many of these had been known for quite some time because of their appearance in the study of complex manifolds and algebraic geometry.

Calabi constructed an example of a metric on the canonical bundle of $\mathbb{CP}^{n-1}$ whose holonomy is $H = \mathrm{SU}(\frac{1}{2}n)$ and an example of a metric on the cotangent bundle of $\mathbb{CP}^{n/2}$ whose holonomy is $H = \mathrm{Sp}(\frac{1}{4}n)$.

Alekseevski constructed examples of left-invariant metrics on solvmanifolds of dimension $n = 4m$ whose holonomy is $H = \mathrm{Sp}(\frac{1}{4}n){\cdot}\mathrm{Sp}(1)$.

Finally, in 1987, Bryant constructed examples of metrics in dimension 7 (resp. 8) with holonomy G_2 (resp. $\mathrm{Spin}(7)$. (Abstract existence proofs had been completed in 1984, but explicit examples were not available until 3 years later.)

Thus, all of the groups that remained on Berger's list are known to occur as holonomy groups of Riemannian manifolds. These holonomies and their commonly accepted names are summarized in Table 1.1.

TABLE 1.1. The irreducible nonsymmetric Riemannian holonomies

Manifold Dimension	Holonomy Group	Geometry
n	$\mathrm{SO}(n)$	Generic
$n = 2m$	$\mathrm{U}(m)$	Kähler
$n = 2m$	$\mathrm{SU}(m)$	Calabi-Yau
$n = 4m$	$\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$	quaternion-Kähler
$n = 4m$	$\mathrm{Sp}(m)$	HyperKähler
$n = 7$	G_2	*Associative*
$n = 8$	$\mathrm{Spin}(7)$	*Cayley*

1.2.5. The holonomy principle. The reader may be wondering just why the holonomy group of a metric says anything important about it. The answer is that the holonomy of a Riemannian manifold (M, g) governs what sorts of algebraic structures can be defined on M that are as compatible as possible with the metric g.

[2]In fact, it is interesting to look at Berger's original paper: He had done the calculations needed to show that $\mathrm{Spin}(9)$ satisfies $K^1(\mathfrak{h}) = (0)$, but had not realized it.

For example, suppose that $\sigma \in \Gamma\big(\otimes^k(T^*M)\big)$ is a covariant tensor field of degree k on M. Its covariant derivative with respect to the Levi-Civita connection is denoted by $\nabla\sigma \in \Gamma\big(\otimes^{k+1}(T^*M)\big)$ and, by definition, vanishes if and only if σ is parallel with respect to eh Levi-Civita connection.

Since the bundle $\otimes^p(T^*M)$ is the vector bundle associated to F via the p-th tensor power of the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, it follows that σ is represented by a function $s : F \to \otimes^k(\mathbb{R}^n)$ that is equivariant with respect to the action of $\mathrm{SO}(n)$. The condition of being covariant constant is then seen to be equivalent to the condition that s be constant on each holonomy bundle $F(u) \subset F$. By equivariance, it then follows that $s(u) \in \otimes^k(\mathbb{R}^n)$ must be fixed by $H = \mathrm{Hol}^g(u)$.

Conversely, if $s_0 \in \otimes^k(\mathbb{R}^n)$ is fixed by a subgroup $H \subset \mathrm{SO}(n)$ and (M, g) is a Riemannian manifold whose holonomy is conjugate to a subgroup of H, then, by choosing a $u \in F$ such that $\mathrm{Hol}^g(u) \subset H$, one can identify $\otimes^k(T^*M)$ as the associated bundle of $F(u)$ defined by the k-th tensor power of the induced representation of $\mathrm{Hol}^g(u)$ on $\mathbb{R}^n$ and hence construct the section $\sigma \in \Gamma\big(\otimes^k(T^*M)\big)$ that corresponds to the constant function $s_0 : F(u) \to \otimes^k(\mathbb{R}^n)$. This resulting section σ will then be covariant constant with respect to the metric g.

This correspondence between the elements in the representations of $\mathrm{SO}(n)$ that are fixed by a subgroup $H \subset \mathrm{SO}(n)$ and the covariant constant tensor fields on Riemannian manifolds whose holonomy is conjugate to a subgroup of H is known as the *holonomy principle*. In particular, the holonomy of a metric determines what sorts of parallel tensor fields the metric supports.

Given the classification of the possible holonomy groups of metrics that are not locally symmetric, there are three particularly applications of the holonomy principle:

1.2.5.1. (*Almost*) *complex structures.* Three of the possible holonomy groups, $\mathrm{U}(\frac{1}{2}n)$, $\mathrm{SU}(\frac{1}{2}n)$, and $\mathrm{Sp}(\frac{1}{4}n)$ fix (at least one) (orthogonal) complex structure on $\mathbb{R}^n$. Consequently, a Riemannian manifold (M, g) with such a holonomy group supports a g-parallel, orthogonal almost complex structure J. Since such a J is g-parallel, i.e., $\nabla^g J = 0$, it follows that, in geodesic normal coordinates about a point $x \in M$, the coefficients of J are constant to first order, i.e., that, centered on any point of M, there are coordinates $x = (x^i)$ such that, when J is written in the form

$$J = a^i_j(x)\,\frac{\partial}{\partial x^i} \otimes dx^j, \tag{1.17}$$

the coefficients satisfy

$$a^i_{j,k}(0) = \frac{\partial a^i_j}{\partial x^k}(0) = 0. \tag{1.18}$$

In particular, since the Nijnhuis tensor of J is linear in the functions $a^i_{j,k}$, it follows that the Nijnhuis tensor of J must vanish, i.e., that the almost complex structure J is actually complex.

1.2.5.2. *Differential forms.* Similar considerations apply when one considers the space of parallel exterior forms that a Riemannian manifold (M, g) supports. If the holonomy of g is conjugate to a subgroup of $H \subset \mathrm{SO}(n)$, the each of the elements of $\Lambda^*(\mathbb{R}^n)^H$, the ring of H-invariant differential forms on $\mathbb{R}^n$, gives rise to a g-parallel differential form.

By an argument similar to the one above showing that every g-parallel almost complex structure must be integrable, one can show that any g-parallel differential form on M is necessarily both closed and co-closed.

It is a remarkable fact that, for each of the subgroups $H \subset \mathrm{SO}(n)$ listed in Table 1.1, the ring $\Lambda^*(\mathbb{R}^n)^H$ actually characterizes H, i.e., H is actually equal to the subgroup of $\mathrm{SO}(n)$ that fixes the elements of $\Lambda^*(\mathbb{R}^n)^H$. For each of the groups listed in Table 1.1, generators (and their degrees) of $\Lambda^*(\mathbb{R}^n)^H$ are listed in Table 1.2.

An even more remarkable fact emerges when the individual entries are examined: Except for the cases $H = \mathrm{SO}(n)$ and $H = \mathrm{U}(\frac{1}{2}n)$ and $H = \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$, (i.e., the first two lines in the table and the case $m = 1$ in the quaternion-Kähler case), the elements of $\Lambda^*(\mathbb{R}^n)^H$ serve to characterize H as a subgroup of $\mathrm{GL}(n, \mathbb{R})$. In fact, this leads to a remarkable result, which is a combination of several different results:

THEOREM 1.14. *Let $H \subset \mathrm{SO}(n)$ be one of the groups $\mathrm{SU}(\frac{1}{2}n)$, $\mathrm{Sp}(\frac{1}{4}n)\cdot\mathrm{Sp}(1)$ $(n > 8)$, $\mathrm{Sp}(\frac{1}{4}n)$, G_2 $(n = 7)$, or $\mathrm{Spin}(7)$ $(n = 8)$ and let $R = \Lambda^*(\mathbb{R}^n)^H$ be the ring of H-invariant differential forms. Suppose that M^n is an n-manifold endowed with a ring $\mathcal{R} \subset \Omega^*(M)$ of closed differential forms with the property that, for each $x \in M$, there exists an isomorphism $u : T_xM \to \mathbb{R}^n$ such that $\mathcal{R}_x = u^*(R)$. Then there exists a Riemannian metric g on M whose holonomy is conjugate to a subgroup of H and with respect to which each of the forms in $\mathcal{R}$ is parallel.*

REMARK 1.15. As already remarked, this theorem is a combination of results individually proved for each of the groups H mentioned in the theorem. For example, the results for $H = \mathrm{G}_2$ or $\mathrm{Spin}(7)$ are due to a combination of results of Bryant and Fernandez and Gray (see [**Bry87**] for an account). The result for $H = \mathrm{SU}(\frac{1}{2}n)$ or $\mathrm{Sp}(\frac{1}{4}n)$ is due to Hitchin. For the history of the result for $H = \mathrm{Sp}(\frac{1}{4}n)\cdot\mathrm{Sp}(1)$ $(n > 8)$, see [**Sa89**]. (The fact that this latter result requires $n > 8$, i.e., that the theorem fails in this case for $n = 8$, has been known for quite some time, but I am not sure to whom it should be attributed.)

The main use of Theorem 1.14 is that it allows one to construct examples of metrics with these holonomies via exterior differential systems techniques and to analyze their local generality. For more details on this method, see [**Bry96**].

TABLE 1.2. Invariant Exterior Forms

Manifold Dimension	Holonomy Group	Exterior forms (generators)
n	$\mathrm{SO}(n)$	$1 \in \Lambda^0,\ *1 \in \Lambda^n$
$n = 2m$	$\mathrm{U}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2$
$n = 2m$	$\mathrm{SU}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2,\ \phi, \psi \in \Lambda^m$
$n = 4m$	$\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$
$n = 4m$	$\mathrm{Sp}(m)$	$1 \in \Lambda^0,\ \omega_1, \omega_2, \omega_3 \in \Lambda^2$
$n = 7$	G_2	$1 \in \Lambda^0,\ \phi \in \Lambda^3,\ *\phi \in \Lambda^4$
$n = 8$	$\mathrm{Spin}(7)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$

1.2.5.3. *Spinors.* Finally, there is the relation with parallel spinor fields. Each of the groups $\mathrm{SU}(\frac{1}{2}n)$, $\mathrm{Sp}(\frac{1}{4}n)$, G_2 $(n = 7)$, or $\mathrm{Spin}(7)$ $(n = 8)$ is characterized as the stabilizer of one or more spinors in a spin (or semi-spin) representation of $\mathrm{SO}(n)$ and hence metrics with these holonomies are characterized by the parallel spinor fields that they admit.

Because such a manifold admits parallel spinor fields, it follows from Lichnerowicz' formula that these metrics must have vanishing scalar curvature. In fact, more is true: Riemannian manifolds with these holonomies must have vanishing Ricci tensor.

1.3. Special Riemannian Geometries

In this final section, I will make some short remarks about interesting features characteristic of each of these geometries. These remarks are intended merely to give the reader some sense of the overall geometric features, not to even begin an exhaustive treatment. After all, these geometries are currently the focus of very active research and an extensive treatment of even one of these cases would be very long indeed. The references mentioned in the introduction constitute a very good place to start to read more about these geometries.

1.3.1. Kähler and Calabi-Yau geometries. As has already been mentioned, a Riemannian manifold (M, g) has holonomy conjugate to a subgroup of $\mathrm{U}(n)$ if and only if it admits a g-parallel, orthogonal almost complex structure J that must, in fact, be integrable. The combined structure (M, g, J) defines what is known as a *Kähler* manifold.

Given g, specifying J is equivalent to specifying the 2-form ω defined by

$$\omega(x, y) = g(Jx, y) \tag{1.19}$$

This 2-form is known as the *Kähler form* of the structure. It is evident that any two of (g, J, ω) determine the third via (1.19).

1.3.1.1. *Local analysis for $H = \mathrm{U}(\frac{1}{2}n)$.* In fact, this relationship gives one a way locally to generate all Riemannian metrics whose holonomy is conjugate to a subgroup of $\mathrm{U}(\frac{1}{2}n)$: Since J is integrable, setting $n = 2m$, one can find J-holomorphic local coordinate charts $z = (z^a)$ and it is not difficult to show that, in such coordinates, one can always find a local function (called the *Kähler potential*) h such that

$$\omega = \frac{\sqrt{-1}}{2}\frac{\partial^2 h}{\partial z^a \partial \overline{z}^b}\ dz^a \wedge d\overline{z}^b \qquad \text{and} \qquad g = \frac{\partial^2 h}{\partial z^a \partial \overline{z}^b}\ dz^a \circ d\overline{z}^b \tag{1.20}$$

Conversely, if h is a function on a domain in $\mathbb{C}^m$ such that the quadratic-form g defined by (1.20) is positive definite, then g is a metric for which the standard complex structure on $\mathbb{C}^m$ is orthogonal and parallel. (Of course, the 2-form ω is also g-parallel.)

It follows that, in a certain sense, the local Kähler metrics depend on one 'arbitrary' function of $n = 2m$ real variables.

1.3.1.2. *Global examples.* Global examples are easy to come by: Any complex submanifold of a Kähler manifold is Kähler when it is given the induced metric. In particular, since the (locally symmetric) $\mathrm{SU}(n{+}1)$-invariant metric on $\mathbb{CP}^n = \mathrm{SU}(n{+}1)/S\big(\mathrm{U}(1)\mathrm{U}(n)\big)$ is Kähler, every nonsingular algebraic variety inherits a Kähler metric.

Kähler geometry is of enormous importance in algebraic geometry and complex analysis, and these examples only scratch the surface of a vast and beautiful theory.

1.3.1.3. *Local analysis for* $H = \mathrm{SU}(\frac{1}{2}n)$. If the holonomy of (M, g) is a subgroup of $\mathrm{SU}(\frac{1}{2}n)$, then, in addition to a parallel complex structure J and $(1,1)$-form ω, the Riemannian manifold supports a nonzero parallel $(n,0)$-form, i.e., a complex volume form $\Psi = \phi + \sqrt{-1}\,\psi$. Because this form must be closed, it follows that this volume form is actually holomorphic. Moreover, one can then choose local J-holomorphic coordinates so that

$$\Psi = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m. \tag{1.21}$$

The fact that Ψ is parallel with respect to g then is equivalent to having the Kähler potential h satisfy the complex Monge-Ampère equation

$$\det\left(\frac{\partial^2 h}{\partial z^a \partial \overline{z}^b}\right) = C \tag{1.22}$$

for some constant c.

This gives a complete local description of the metrics on n-dimensional domains whose holonomy lies in $\mathrm{SU}(\frac{1}{2}n)$. One finds that, in Cartan's sense of 'generality', the 'general' local metric with holonomy in $\mathrm{SU}(\frac{1}{2}n)$ depends on two functions of $n-1$ variables. For more details on this, see [**Bry96**].

1.3.1.4. *Global examples.* On the global front, it is a famous theorem of Yau that a conjecture of Calabi is true:

THEOREM 1.16. *If (M, J) is a compact complex manifold that has a nonvanishing holomorphic volume form Ψ and that admits a Kähler metric g_0 with associated Kähler 2-form ω_0, then there exists a unique metric g_1 on M so that (M, g_1, J) is Kähler, the associated Kähler form ω_1 belongs to the same cohomology class as ω_0, and Ψ is g_1-parallel.*

It is for this reason that metrics with holonomy in $\mathrm{SU}(\frac{1}{2}n)$ are nowadays referred to as *Calabi-Yau metrics*.

For Calabi-Yau metrics, there are, in addition to the complex submanifolds that play a role in their modern analysis, other special types of submanifolds. For example, a very important class are the m-dimensional submanifolds $L^m \subset M^{2m}$ that are Lagrangian with respect to the 2-form ω and on which Ψ pulls back to be a real m-form (i.e., the imaginary part of Ψ pulls back to zero on L). These submanifolds are, by an argument due to Harvey and Lawson, calibrated submanifolds and hence are homologically mass-minimizing. These submanifolds are now known as *special Lagrangian submanifolds* and currently play a very important role in string theory.

1.3.2. Hyperkähler and Quaternion Kähler.

Recall that an almost quaternion structure on a manifold M^n is a triple (J_1, J_2, J_3) where each J_i is an almost complex structure on M and the three almost complex structure anticommute ($J_1J_2 = -J_2J_1$, etc.) and satisfy $J_1J_2 = J_3$ (and hence, $J_2J_3 = J_1$, etc.). In particular, it follows that $J_\lambda = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3$ is an almost complex structure whenever ${\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 = 1$. Such an almost quaternion structure is said to be *integrable* if each J_λ is integrable, which is equivalent to the condition that any two of the complex structures in $\{J_1, J_2, J_3\}$ be integrable. The reader should note that this notion of integrability does *not* imply that there are local coordinates in which each of the J_i has constant coefficients.

A Riemannian manifold (M^n, g) where $n = 4m$ has holonomy conjugate to a subgroup of $\mathrm{Sp}(m) \subset \mathrm{SO}(n)$ if and only if it admits a g-parallel, orthogonal almost quaternion structure (J_1, J_2, J_3). Of course, since each J_i is g-parallel, it follows that each J_i is integrable and hence the almost quaternion structure (J_1, J_2, J_3) is integrable and, moreover, that the nondegenerate 2-form ω_i associated to the Kähler structure (M, g, J_i) is g-parallel and hence closed and co-closed for each i.

1.3.2.1. *Definition via 2-forms.* The three 2-forms $(\omega_1, \omega_2, \omega_3)$ alone are sufficient to recover metric g and the almost quaternion structure (J_1, J_2, J_3). Indeed, one has

$$\begin{aligned} \omega_1(J_2x, y) &= \omega_3(x, y), \\ \omega_2(J_3x, y) &= \omega_1(x, y), \\ \omega_3(J_1x, y) &= \omega_2(x, y), \end{aligned} \tag{1.23}$$

and these equations serve to determine (J_1, J_2, J_3) while g is then defined by

$$g(x, y) = -\omega_1(J_1x, y) = -\omega_2(J_2x, y) = -\omega_3(J_3x, y). \tag{1.24}$$

Moreover, given a triple $(\omega_1, \omega_2, \omega_3)$ of closed nondegenerate 2-forms on M for which the maps J_i defined by (1.23) form an almost quaternion structure and the quadratic form g then defined by (1.24) is a positive definite quadratic form on M, one finds that each J_i is g-parallel. (As already mentioned, this observation is due to Hitchin.)

1.3.2.2. *Local analysis for $H = \mathrm{Sp}(\frac{1}{4}n)$.* Although the analysis is somewhat more complicated, it is not difficult to show that, locally, these metrics depend on $\frac{1}{2}n$ functions of $\frac{1}{2}n + 1$ variables. Again, see [**Bry96**] for details. There is also a 'twistor' description of these local metrics that is due to Hitchin. See [**Bes87**] for this description.

1.3.2.3. *Compact construction for $H = \mathrm{Sp}(\frac{1}{4}n)$.* The construction of compact examples can be accomplished with the aid of complex geometry, Berger's classification of holonomy, and Theorem 1.16 because of the following observation: Using the definitions, one sees that

$$\omega_2(J_1x, y) + \sqrt{-1}\,\omega_3(J_1x, y) = \sqrt{-1}\big(\omega_2(x, y) + \sqrt{-1}\,\omega_3(x, y)\big), \tag{1.25}$$

so it follows that the $\mathbb{C}$-valued 2-form $\Omega = \omega_2 + \sqrt{-1}\omega_3$ is actually of type $(2, 0)$ with respect to the complex structure J_1. Moreover, it is visibly nondegenerate and hence defines a holomorphic symplectic structure on the complex manifold (M, J_1). Since Ω^m is then a non-vanishing holomorphic volume form, it follows that (M, g, J_1) is a Calabi-Yau manifold.

Conversely, suppose that one starts with a compact complex manifold (M, J_1) that admits both a holomorphic symplectic structure $\Omega_0 \in \Omega^{2,0}(M)$ and a Kähler 2-form $\omega_0 \in \Omega^{1,1}_+(M)$. (Several methods of constructing simply connected examples of such manifolds are now known, building on the original constructions of Futaki and Beauville.) Then, by Theorem 1.16, there exists a unique Kähler 2-form $\omega_1 \in \Omega^{1,1}_+(M)$ that lies in the same cohomology class as ω_0 and for which ${\Omega_0}^m$ is g-parallel for the metric g associated to (M, J_1, ω_1). Now, it is not difficult to show that Ω_0 itself must be parallel with respect to g. (This uses the fact that g must be Ricci-flat since its holonomy lies in $\mathrm{SU}(2m)$ and then a Weitzenböck formula that shows that, because Ω_0 is closed and holomorphic, it must be parallel.) It then easily follows, from the classification of holonomies that the holonomy of g must be conjugate to a subgroup of the form

$$\mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) \times \cdots \times \mathrm{Sp}(m_k) \subset \mathrm{SU}(2m) \tag{1.26}$$

where $m_1 + \cdots + m_k \leq m$. If one now chooses M so that it is not a product (again, this extra condition can be satisfied by examples constructed using algebro-geometric techniques), then the holonomy is forced to be $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$, as desired.

1.3.2.4. $\mathrm{Sp}(\frac{1}{4}n)\cdot\mathrm{Sp}(1)$*-structures.* The three 2-forms $(\omega_1, \omega_2, \omega_3)$ suffice to determine the underlying $\mathrm{Sp}(\frac{1}{4}n)$-structure on an n-manifold (where $n = 4m$). On the other hand, the 4-form Ω defined by

$$6\,\Omega = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2 \tag{1.27}$$

has a significantly larger stabilizer group, since it is stabilized not only by $\mathrm{Sp}(\frac{1}{4}n)$, the orthogonal quaternion linear transformations, but by scalar multiplication by unit quaternions as well, i.e., the commutator of $\mathrm{Sp}(\frac{1}{4}n)$ in $\mathrm{SO}(n)$.

In fact, it is not difficult to prove that, when $m > 1$, the group $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$ is defined (as a subgroup of $\mathrm{GL}(4m, \mathbb{R})$) as the stabilizer of a 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^{4m})$. Thus, one can define a $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$-structure on a $4m$-manifold by specifying a 4-form Ω of the same algebraic type at each point as Ω_0. One could say that the pair (M, Ω) is an *almost quaternion-Kähler* manifold. As was mentioned in Theorem 1.14, when $m > 2$, the condition that this $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$-structure underly a metric g with respect to which Ω is parallel is simply that $d\Omega = 0$. (When $m = 2$, this is not adequate and further conditions must be imposed.)

There are some very well-known examples of such metrics: The *Wolf* spaces. These are the quaternion-Kähler symmetric spaces of compact type and there is one such symmetric space associated to each of the simple Lie groups. The best known are, of course, $\mathbb{HP}^n = \mathrm{Sp}(m{+}1)/\big(\mathrm{Sp}(m){\times}\mathrm{Sp}(1)\big)$, i.e., the quaternionic projective spaces, but the other classical ones are $\mathrm{SO}(n{+}4)/\big(\mathrm{SO}(n){\times}\mathrm{SO}(4)\big)$, the Grassmannian of oriented 4-planes in $\mathbb{R}^{n+4}$ and $\mathrm{SU}(n{+}2)/S\big(\mathrm{U}(n){\times}\mathrm{U}(2)\big)$, the Grassmannian of complex 2-planes in $\mathbb{C}^{n+2}$. The first exceptional case $\mathrm{G}_2/\mathrm{SO}(4)$ plays an important role in calibrated geometry in dimension 7.

Assume from now on that (M^{4m}, g) is a quaternion-Kähler manifold whose holonomy acts irreducibly. One can show without difficulty that g must be Einstein. If the scalar curvature of g is zero, then, in fact, the holonomy lies in $\mathrm{Sp}(m)$ and so the metric is actually hyperKähler. If the scalar curvature is nonzero, then either g is locally symmetric (and hence is locally isometric to some Wolf space or its noncompact dual) or else, by Berger's classification, it must have holonomy conjugate to $\mathrm{Sp}(m){\times}\mathrm{Sp}(1)$.

1.3.2.5. *Local generality.* By using methods similar to those for studying local hyperKähler structures, one can show that the 'general' quaternion-Kähler metric depends on $2m$ functions of $2m{+}1$ variables (the same generality as for local hyperKähler metrics). Thus, there are many such metrics, locally.

1.3.2.6. *Global constructions.* Aside from the Wolf spaces (and their products), no complete examples of quaternion-Kähler manifolds with positive curvature are known and there is a conjecture that all such manifolds are products of symmetric spaces. This conjecture has been verified in the low dimensions.

On the other hand, quaternion-Kähler orbifolds of positive scalar curvature have been constructed by a sort of reduction process and many examples are now known.

For negative scalar curvature, Alekseevski has constructed examples on certain solvmanifolds that are homogeneous under the action of a Lie group.

1.3.3. G_2 **geometry.** This is the first of the two possible *exceptional* holonomy groups. For many years, this was just a curiosity (in fact metrics with this holonomy were not known to exist, even locally, until 1984), but it became more important after Havey and Lawson did their groundbreaking work on calibrated geometries. Even then, interest in such manifolds languished for a while until the work of Joyce (see below) and the appearance of G_2-geometry in certain extensions of string theory, which stimulates interest in this geometry among theoretical physicists these days.

One way of defining G_2-geometry is in terms of a metric in dimension 7 that admits a parallel spinor field. However, in this brief discussion, I will concentrate on its definition via 3-forms.

Definition 1.17. The group $\mathrm{G}_2 \subset \mathrm{GL}(7,\mathbb{R})$ is the group of linear transformations that preserve the 3-form ϕ_0 on $\mathbb{R}^7$ defined by

$$\phi_0 = dx^{123} - dx^{145} - dx^{167} - dx^{246} - dx^{275} - dx^{347} - dx^{356} \tag{1.28}$$

where dx^{ijk} is shorthand for $dx^i \wedge dx^j \wedge dx^k$.

It can be shown that G_2 is a connected and simply-connected Lie group of dimension 14 that lies in $\mathrm{SO}(7)$. It follows that the orbit of ϕ_0 under the action of $\mathrm{GL}(7,\mathbb{R})$ has dimension $49 - 14 = 35 = \dim \Lambda^3(\mathbb{R}^7)$. In particular, this orbit is open in $\Lambda^3(\mathbb{R}^7)$, so that ϕ_0 is what is nowadays called 'stable', i.e., it is linearly equivalent to any small perturbation of itself (just as a positive definite quadratic form is linearly equivalent to any small perturbation of itself).

A G_2-structure on a 7-manifold M can now be defined by specifying a 3-form ϕ on M such that, at every point $x \in M$, there exists a linear isomorphism $u : T_xM \to \mathbb{R}^7$ satisfying $\phi_x = u^*(\phi_0)$. By the above remark, if ϕ is such a form on M, then any sufficiently small perturbation of ϕ also defines a G_2-structure on M.

Moreover, because G_2 is a subgroup of $\mathrm{SO}(7)$, it follows that a G_2-structure on M defines an underlying metric g_ϕ and orientation, which in turn, determines a Hodge duality operator $*_\phi$.

The fundamental theorem in the subject is a combination of work by Fernandez and Gray with that of Bryant:

Theorem 1.18. *If ϕ is a G_2-defining 3-form on M^7, then ϕ is parallel with respect to the metric g_ϕ if and only if $d\phi = d(*_\phi\phi) = 0$.*

Consequently, in this case, the holonomy of g_ϕ will be a subgroup of G_2. By the classification of holonomy, if the holonomy of g_ϕ acts irreducibly (i.e., if g_ϕ is not locally a product metric), then the holonomy must be all of G_2.

1.3.3.1. *Local generality.* This result provides the essential step needed to determine the generality of such metrics locally. By an exterior differential systems analysis [**Bry87**], one shows that the local metrics with holonomy G_2 depend on six functions of six variables up to local diffeomorphism.

1.3.3.2. *Global existence.* In a stunning development, Joyce proved that compact manifolds with holonomy G_2 exist and, in fact, constructed many series of such manifolds. This is a long and beautiful story that I do not have time to go into here, but the reader can learn all about it (and the beautiful geometry of such manifolds) by consulting the fundamental reference [**Jo00**].

1.3.4. Spin(7)**.** This is the second of the two possible *exceptional* holonomy groups. Again, metrics of with this holonomy remained mysterious until their existence was proved and then became more interesting when Joyce proved that there were compact examples. The significance of these metrics for mathematical physics still remains unclear.

One way of defining Spin(7)-geometry is in terms of a metric in dimension 8 that admits a parallel spinor field. However, in this brief discussion, I will concentrate on its definition via 4-forms.

DEFINITION 1.19. The group $\mathrm{Spin}(7) \subset \mathrm{GL}(8, \mathbb{R})$ is the group of linear transformations that preserve the 3-form Φ_0 on $\mathbb{R}^8 = \mathbb{R}\oplus\mathbb{R}^8$ defined by

$$\Phi_0 = dx^0 \wedge \phi_0 + *\big(\phi_0\big) \tag{1.29}$$

where ϕ_0 is as defined in (1.28).

It can be shown that Spin(7) is a connected and simply-connected Lie group of dimension 21 that lies in SO(8) and is the double cover of SO(7). The orbit of Φ_0 under the action of $\mathrm{GL}(8, \mathbb{R})$ has dimension $64 - 21 = 43 < \dim \Lambda^4(\mathbb{R}^8)$, so this orbit is not open in $\Lambda^4(\mathbb{R}^8)$. (In fact, for dimension reasons, $\Lambda^4(\mathbb{R}^8)$ clearly does not have any open $\mathrm{GL}(8, \mathbb{R})$-orbits.) It is not difficult to show that Φ_0 and its multiples are the only nontrivial forms (other than the ones of degrees 0 or 8) on $\mathbb{R}^8$ that are left invariant by Spin(7).

A Spin(7)-structure on n 8-manifold M can now be defined by specifying a 4-form Φ on M such that, at every point $x \in M$, there exists a linear isomorphism $u : T_xM \to \mathbb{R}^8$ satisfying $\Phi_x = u^*(\Phi_0)$.

Again, because Spin(7) is a subgroup of SO(8), it follows that a Spin(7)-structure on M defines an underlying metric g_Φ.

The fundamental theorem in the subject is essentially due to Fernandez:

THEOREM 1.20. *If Φ is a* Spin(7)*-defining* 4*-form on M^8, then Φ is parallel with respect to the metric g_Φ if and only if $d\Phi = 0$.*

Consequently, in this case, the holonomy of g_ϕ will be a subgroup of Spin(7). By the classification of holonomy, if the holonomy of g_ϕ acts irreducibly (i.e., if g_ϕ is not locally a product metric), then the holonomy must be one of Sp(2), SU(4), or Spin(7). The former two are distinguished by the fact that metrics with these holonomies admit parallel 2-forms. Thus, such a closed form Φ whose underlying metric g_Φ is not locally a produce and does not admit nontrivial parallel 2-forms necessarily has Spin(7) as its holonomy group.

1.3.4.1. *Local generality.* This result provides the essential step needed to determine the generality of such metrics locally. By an exterior differential systems analysis [**Bry87**], one shows that the local metrics with holonomy Spin(7) depend on twelve functions of seven variables, up to local diffeomorphism.

1.3.4.2. *Global existence.* Joyce has also proved that compact manifolds with holonomy Spin(7) exist and, in fact, has constructed many series of such manifolds. Again, the reader is referred to [**Jo00**] for further information.

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