A COMPLETE KNOT INVARIANT FROM CONTACT HOMOLOGY

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Abstract. We construct an enhanced version of knot contact homology, and show that we can deduce from it the group ring of the knot group together with the peripheral subgroup. In particular, it completely determines a knot up to smooth isotopy. The enhancement consists of the (fully noncommutative) Legendrian contact homology associated to the union of the conormal torus of the knot and a disjoint cotangent fiber sphere, along with a product on a filtered part of this homology. As a corollary, we obtain a new, holomorphic-curve proof of a result of the third author that the Legendrian isotopy class of the conormal torus is a complete knot invariant. Furthermore, we relate the holomorphic and sheaf approaches via calculations of partially wrapped Floer homology in the spirit of [BEE12].

1. Introduction

1.1. **Conormal tori and knot contact homology.** A significant thread in recent research in symplectic and contact topology has concerned the study of smooth manifolds through the symplectic structures on their cotangent bundles. In this setting, one can also study a pair of manifolds, one embedded in the other—in particular, a knot in a 3-manifold— via the conormal construction. If $K \subset \mathbb{R}^3$ is a knot, then its unit conormal bundle, the conormal torus Λ_K, is a Legendrian submanifold of the contact cosphere bundle $ST^*\mathbb{R}^3$. Isotopic knots produce conormal tori that are isotopic as Legendrian submanifolds, i.e., the Legendrian isotopy type of the conormal torus is a knot invariant. The fact that this invariant is nontrivial depends essentially on the contact geometry: the conormal tori of any two knots are smoothly isotopic, even if the knots themselves are not isotopic.

Symplectic field theory [EGH10] provides an algebraic knot invariant associated to this geometric invariant: the Legendrian contact homology of Λ_K, also known as the knot contact homology of K. This is the homology of a differential graded algebra generated by Reeb chords of Λ_K with differential given by counting holomorphic disks. In the past few years, there have been indications that knot contact homology and its higher genus generalizations are related via string theory to other knot invariants such as the A-polynomial, HOMFLY-PT polynomial, and possibly various knot homologies: the cotangent bundle equipped with Lagrangian branes along the conormal of the knot and the 0-section is the setting for an open topological string theory that has conjectural relations to all of these invariants. The physical account considers the holomorphic disks that go into knot contact homology, and also crucially takes into account higher genus information; this last part has not yet been fully developed in the mathematical literature, but some beginnings can be found in [AENV14].

In particular, it is explained there how certain quantum invariants should be conjecturally recovered from a quantization of knot contact homology arising from the consideration of non-exact Lagrangians. In any case, it appears that knot contact homology should be a very strong invariant, in the sense that it encodes a great deal of information about the underlying knot.

Recent work of the third author [She] shows that the Legendrian conormal torus Λ_K is in fact a complete invariant of K: two knots with Legendrian isotopic conormals must in fact be isotopic. Since Λ_K is the starting point for knot contact homology, this can be viewed as evidence for, or in any case is consistent with, the possibility that knot contact homology itself is a complete invariant. Other evidence in this direction is provided by the fact that knot contact homology recovers enough of the knot group (the fundamental group of the knot complement) to detect the unknot [Ng08] and torus knots, among others [GL]. These results use the ring structure on the "fully noncommutative" version of knot contact homology, where the algebra is generated by Reeb chords along with homology classes in $H_1(\Lambda_K)$ that do not commute with Reeb chords; see [Ng14, CELN]. However, the question of whether knot contact homology is a complete invariant remains open.

1.2. **Main results.** In this paper, we present an extension of knot contact homology by slightly enlarging the set of holomorphic disks that are counted. We will show that this enhancement contains the knot group along with the peripheral subgroup, and this in turn is enough to completely determine the knot [Wal68, GL89]. As a corollary, we have a new proof of the result from [She], using holomorphic curves rather than constructible sheaves.

For our purposes, we need the Legendrian contact homology of not just Λ_K but the union of Λ_K and a cotangent fiber Λ_p of $ST^*\mathbb{R}^3$; the inclusion of the latter is analogous to choosing a basepoint for the fundamental group. This new invariant, $\mathrm{LCH}_*(\Lambda_K \cup \Lambda_p)$, is a ring that contains the knot contact homology of K as a quotient. Using the "link grading" of Mishachev [Mis03], we can write:

$$\mathrm{LCH}_*(\Lambda_K \cup \Lambda_p) \cong (\mathrm{LCH}_*)_{\Lambda_K, \Lambda_K} \oplus (\mathrm{LCH}_*)_{\Lambda_K, \Lambda_p} \oplus (\mathrm{LCH}_*)_{\Lambda_p, \Lambda_K} \oplus (\mathrm{LCH}_*)_{\Lambda_p, \Lambda_p},$$

where $(\mathrm{LCH}_*)_{\Lambda_i, \Lambda_j}$ denotes the homology of the subcomplex generated by composable words of Reeb chords ending on Λ_i and beginning on Λ_j; see Section 2 for details.

From this set of data, we pick out what we call the *KCH-triple* (R_{KK}, R_{Kp}, R_{pK}) associated to $\Lambda_K \cup \Lambda_p$, defined by:

$$R_{KK} = (\mathrm{LCH}_0)_{\Lambda_K, \Lambda_K}, \qquad R_{Kp} = (\mathrm{LCH}_0)_{\Lambda_K, \Lambda_p}, \qquad R_{pK} = (\mathrm{LCH}_1)_{\Lambda_p, \Lambda_K}.$$

Of these, R_{KK} is precisely the degree 0 knot contact homology of K and contains a subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ once we equip K with an orientation and framing (which we choose to be the Seifert framing), where l, m denote the longitude and meridian of K; R_{Kp} and R_{pK} are left and right modules, respectively, over R_{KK}.

We need one further piece of data in addition to the KCH-triple: a product $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$. While the differential in $\mathrm{LCH}_*(\Lambda_K \cup \Lambda_p)$ counts holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $\mathbb{R} \times (\Lambda_K \cup \Lambda_p)$ and one positive puncture at a Reeb chord of $\Lambda_K \cup \Lambda_p$, the product μ counts holomorphic disks with *two* positive punctures, at mixed Reeb chords of $\Lambda_K \cup \Lambda_p$. Extending Legendrian contact homology to "Legendrian Rational Symplectic Field Theory" by counting disks with multiple positive punctures has not yet been successfully implemented in general; the difficulty comes from boundary breaking for holomorphic disks, which contributes to the codimension-1 strata of moduli spaces. However, partial results in this direction have been obtained by the first author [Ekh08] in the case of multiple-component Legendrian links when boundary breaking can be avoided for topological reasons, and (with less relevance for our purposes) by the second author [Ng10] in complete generality in the case of Legendrian knots in $\mathbb{R}^3$. In particular, the fact that μ is well-defined and invariant follows from [Ekh08].

Our main result is now as follows:

Theorem 1.1. *Let $K \subset \mathbb{R}^3$ be a knot and $p \in \mathbb{R}^3 \setminus K$ be a point, and let Λ_K, Λ_p denote the Legendrian submanifolds of $ST^*\mathbb{R}^3$ given by the unit conormal torus to K and the unit cotangent fiber over p. Then the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) constructed from the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$, equipped with the product $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$, is a complete invariant for K.*

More precisely, if there is an isomorphism between the KCH-triples for two knots K_0, K_1 that preserves μ, then:

(1) K_0 and K_1 are smoothly isotopic up to mirroring and orientation reversal;

(2) if the isomorphism from $R_{K_0 K_0}$ to $R_{K_1 K_1}$ restricts to the identity map on the subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, then K_0 and K_1 are smoothly isotopic as oriented knots.

A Legendrian isotopy between $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ induces an isomorphism between the KCH-triples that respects the product μ. Since $\mathbb{R}^3$ is noncompact, any Legendrian isotopy between the conormal tori Λ_{K_0} and Λ_{K_1} can be extended to an isotopy between $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ by pushing p away from the (compact) support of the isotopy. Thus we deduce from Theorem 1.1 a new proof of the following result.

Theorem 1.2 ([She]). *Let K_0, K_1 be smooth knots in $\mathbb{R}^3$ and let $\Lambda_{K_0}, \Lambda_{K_1}$ denote their conormal tori.*

(1) If Λ_{K_0} and Λ_{K_1} are Legendrian isotopic, then K_0 and K_1 are smoothly isotopic up to mirroring and orientation reversal.

(2) If Λ_{K_0} and Λ_{K_1} are parametrized Legendrian isotopic, then K_0 and K_1 are smoothly isotopic as oriented knots.

Here "parametrized Legendrian isotopic" means the following: each conormal torus Λ_K of an oriented knot K has two distinguished classes in $H_1(\Lambda_K)$ given by the meridian and

Seifert-framed longitude, and a parametrized Legendrian isotopy between conormal tori is an isotopy that sends meridian and longitude to meridian and longitude. We note that (1) in Theorem 1.2 is proved in [She], while (2) is implicit but not explicitly stated there; however, the techniques given in [She] extend to prove (2) as well as (1).

Our proof of Theorem 1.1 depends crucially on the results of [CELN], which relates knot contact homology to string topology. It is shown there that one can construct an isomorphism from degree 0 knot contact homology, $LCH_0(\Lambda_K)$, to a certain string homology constructed from paths ("broken strings") on the singular Lagrangian given by the union, inside the cotangent bundle, of the zero section and the conormal. This isomorphism is induced by mapping a Reeb chord to the chain of boundaries of all holomorphic disks asymptotic to the Reeb chord with boundary on the singular Lagrangian.

In this paper, we extend the isomorphism from [CELN] to show that the KCH-triple can also be computed using broken strings. Using this presentation, we prove an isomorphism

$$\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)] \cong \mathbb{Z} \oplus (R_{pK} \otimes_{R_{KK}} R_{Kp}).$$

Since knot groups are left-orderable, it follows that we can recover the knot group itself from the KCH-triple. A further consideration of the subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, which sits naturally in enhanced knot contact homology (more precisely, in R_{KK}), shows that we can also recover the longitude and meridian inside the knot group, and thus by [Wal68] we have a complete knot invariant.

We emphasize that the extra cosphere fiber is critical for our argument. It is shown in [CELN] that knot contact homology $LCH_0(\Lambda_K)$ is isomorphic to a certain subring of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$, and this can be used to prove that $LCH_0(\Lambda_K)$ detects the unknot and torus knots, as mentioned earlier. It is not clear whether this subring suffices to give a complete invariant. By contrast, the extra cosphere fiber allows the direct recovery of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ and thus $\pi_1(\mathbb{R}^3 \setminus K)$.

1.3. **Relation to sheaves.** We conclude this introduction with a discussion of the relation of the constructions in this paper to those of [She]; see Section 6 for more detail. Recall that for a smooth manifold M, there is an isomorphism between the wrapped Fukaya category of T^*M and the category of local systems on M: the former is generated by a single cotangent fiber F, and $\text{Hom}(F, F)$ is naturally identified with chains on the based loop space [Abo11]. This result has a cosheaf-theoretic interpretation (and proof): one can match the procedures of taking Hom in the wrapped Fukaya category of T^*M, and taking Hom in the space of global sections in the constant cosheaf of categories whose stalks is the category of chain complexes. The latter is chains on the based loop space by a formal argument.

More generally, Kontsevich [Kon09] has suggested the following picture. Given any Weinstein manifold W with a Legendrian Λ at infinity, the partially wrapped Fukaya category adapted to Λ (roughly, with wrapping stopped by Λ) should localize to a cosheaf over the

Lagrangian skeleton, i.e., the complement of all points which escape to $\partial W \setminus \Lambda$ under the Liouville flow. Moreover, by [Nad14] and forthcoming work [GPSa, GPSb], this cosheaf should be dual to the Kashiwara–Schapira stack which governs microlocal sheaves [KS94].

In [She], the category of interest was sheaves with microsupport in the conormal torus. By results from [KS94], this category is the global sections of a sheaf of categories over the skeleton formed by the union of the zero section and the conormal, which can be understood as modules over the global sections of a corresponding cosheaf. These global sections are generated by the images of generating elements of the stalk at a point on the zero section and a point on the conormal; these in turn correspond to Lagrangian disks F and C whose boundary is, respectively, a cosphere fiber and a sphere which links the torus.

We will sketch a calculation, in the spirit of the Legendrian surgery formulas in [BEE12], of the partially wrapped Floer homologies of these disks. In the notation above, we have:

$$HW_1(C, C) \cong R_{KK}$$
$$HW_1(C, F) \cong R_{pK}$$
$$HW_1(F, C) \cong R_{Kp}$$

and the product $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$ is just the ordinary pair-of-pants product in Floer homology. Then the KCH-triple and product determines a ring structure on

$$HW_1(F, F) \cong \mathbb{Z} \oplus (R_{pK} \otimes R_{Kp})$$

which, as mentioned previously, is precisely the group ring $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

In fact, we sketch an argument which shows that this ring isomorphism is induced from moduli spaces of holomorphic disks as follows. Applying Lagrange surgery to $\mathbb{R}^3 \cup L_K$, we obtain a Lagrangian M_K with the topology of $\mathbb{R}^3 \setminus K$. The disk F intersects M_K transversely in one point and the map above is induced from moduli spaces of holomorphic disks with one positive puncture at a Reeb chord of F, two Lagrangian intersection punctures at $M_K \cap F$, and boundary on $F \cup M_K$. This map is then directly analogous to the corresponding map in the cotangent bundle of a closed manifold and, as there, it gives an isomorphism of rings in homology. This picture using wrapped Floer homology helps to retroactively motivate the constructions we study in this paper, though we do not use it directly in the proof of our main result.

1.4. **Outline of the paper.** In Section 2, we introduce enhanced knot contact homology and the KCH-triple, along with the product map μ. We reformulate these structures in terms of string topology in Section 3 and then in terms of the knot group in Section 4, leading to a proof of Theorem 1.1 in Section 5. In Section 6, we discuss the relation to wrapped Floer homology and sheaves.

2. Enhanced Knot Contact Homology

In this section we present the ingredients of enhanced knot contact homology. In Section 2.1 we discuss the structure of the contact homology algebra of a two component Legendrian link, in Section 2.2 we specialize to the case of a link consisting of the conormal of a knot and the fiber sphere over a point. Finally, in Section 2.3 we introduce the product operation on enhanced knot contact homology.

2.1. **Legendrian contact homology for a link.** Let $V = J^1(M)$ be the 1-jet space of a compact manifold M with the standard contact structure, and let $\Lambda \subset V$ be a connected Legendrian submanifold. The Legendrian contact homology of Λ, which we will write as $\mathrm{LCH}_*(\Lambda)$, is the homology of a differential graded algebra $(\mathcal{A}_\Lambda, \partial)$, where $\mathcal{A}_\Lambda$ is a noncommutative unital algebra generated by Reeb chords of Λ and homology classes in $H_1(\Lambda)$, with the differential given by a count of certain holomorphic curves in the symplectization $\mathbb{R} \times V$ with boundary on $\mathbb{R} \times \Lambda$.

Remark 2.1. For a Legendrian submanifold Λ of a general contact manifold V the Legendrian algebra $\mathcal{A}_\Lambda$ is an algebra generated by both Reeb chords and closed Reeb orbits, where the orbits generate a commutative subalgebra. In the case of a 1-jet space there are no closed Reeb orbits and the algebra and its differential involves chords only.

Remark 2.2. Legendrian contact homology is often defined with coefficients in the group ring of $H_2(V, \Lambda)$ rather than $H_1(\Lambda)$, the difference being whether one associates to a holomorphic disk its relative homology class in $H_2(V, \Lambda)$ or the homology class of its boundary in $H_1(\Lambda)$. In the case of knot contact homology, our setup amounts to specializing to $U = 1$ in the language of [EENS13a, Ng11, Ng14] or $Q = 1$ in the language of [AENV14]. Also, as mentioned in the introduction, the version of the DGA that we consider here is the fully noncommutative DGA, in which homology classes in $H_1(\Lambda)$ do not commute with Reeb chords.

If $\Lambda \subset V$ is a disconnected Legendrian submanifold, then there is additional structure on the DGA of Λ first described by Mishachev [Mis03]; in modern language this is the "composable algebra", and we follow the treatment from [BEE12, EENS13b, NRS+]. For simplicity we restrict to the case $\Lambda = \Lambda_1 \cup \Lambda_2$. For $i, j = 1, 2$, let $\mathcal{R}^{ij}$ denote the set of Reeb chords that *end* on Λ_i and *begin* on Λ_j. The composable algebra $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is the noncommutative $\mathbb{Z}$-algebra generated by Reeb chords of $\Lambda_1 \cup \Lambda_2$, elements of $\mathbb{Z}[H_1(\Lambda_1)]$,
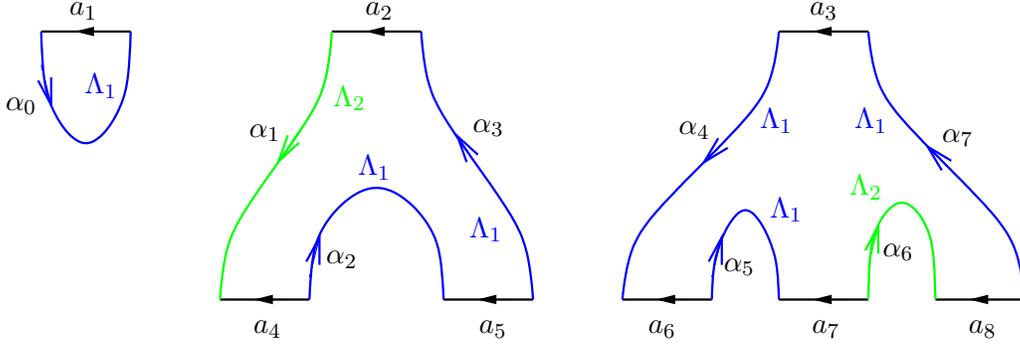
FIGURE 1. Terms contributing to the differential of $a_1 \in \mathcal{R}^{11}$, $a_2 \in \mathcal{R}^{21}$, $a_3 \in \mathcal{R}^{11}$: α_0, $\alpha_1 a_4 \alpha_2 a_5 \alpha_3$, $\alpha_4 a_6 \alpha_5 a_7 \alpha_6 a_8 \alpha_7$, respectively. Here $a_5, a_6 \in \mathcal{R}^{11}$, $a_4, a_8 \in \mathcal{R}^{21}$, $a_7 \in \mathcal{R}^{12}$, $\alpha_1, \alpha_6 \in H_1(\Lambda_1)$, and $\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7 \in H_1(\Lambda_2)$. The boundaries of the disks lie on $\mathbb{R} \times \Lambda_1$ and $\mathbb{R} \times \Lambda_2$ as shown. Small arrows denote orientations on Reeb chords.

elements of $\mathbb{Z}[H_1(\Lambda_2)]$, and two idempotents e_1, e_2, subject to the relations (where δ_{ij} is the Kronecker delta):

- $e_i e_j = \delta_{ij}$
- $e_{i'} a = \delta_{ii'} a$ and $a e_{j'} = \delta_{jj'} a$ for $a \in \mathcal{R}^{ij}$
- $e_j \alpha = \alpha e_j = \delta_{ij} \alpha$ for $\alpha \in \mathbb{Z}[H_1(\Lambda_i)]$.

Note that $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is unital with unit $e_1 + e_2$. For $i, j = 1, 2$, define $\mathcal{A}_{\Lambda_i, \Lambda_j} = e_i \mathcal{A}_{\Lambda_1 \cup \Lambda_2} e_j$; then

$$\mathcal{A}_{\Lambda_1 \cup \Lambda_2} = \bigoplus_{i,j \in \{1,2\}} \mathcal{A}_{\Lambda_i, \Lambda_j}.$$

In more concrete terms, $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is generated as a $\mathbb{Z}$-module by monomials of the form

$$\alpha_0 a_1 \alpha_1 a_2 \cdots a_n \alpha_n,$$

where there is some sequence $(i_0, \ldots, i_n)$ with $i_k \in \{1, 2\}$ such that $\alpha_k \in \mathbb{Z}[H_1(\Lambda_{i_k})]$ and $a_k \in \mathcal{R}^{i_{k-1} i_k}$ for all k. Monomials of this form are the "composable words". Generators of $\mathcal{A}_{\Lambda_i, \Lambda_j}$ are of the same form but specifically with $i_0 = i$ and $i_n = j$. Note that multiplication $\mathcal{A}_{\Lambda_i, \Lambda_j} \otimes \mathcal{A}_{\Lambda_{i'}, \Lambda_{j'}} \to \mathcal{A}_{\Lambda_i, \Lambda_{j'}}$ is concatenation if $j = i'$ and 0 otherwise.

The differential ∂ on $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is defined to be 0 on e_i and on elements of $\mathbb{Z}[H_1(\Lambda_i)]$ and is given by a holomorphic-disk count for Reeb chords of $\Lambda_1 \cup \Lambda_2$. For a Reeb chord $a \in \mathcal{R}^{ij}$ the disks counted in ∂a are maps into $\mathbb{R} \times V$ and have boundary on $(\mathbb{R} \times \Lambda_1) \cup (\mathbb{R} \times \Lambda_2)$, one positive puncture where it is asymptotic to $\mathbb{R} \times a$ and several negative punctures. The contribution to the differential is the composable word of homotopy classes and Reeb chords in the complement of the positive puncture along the boundary of the disk, see Figure 1. The differential thus respects the direct-sum decomposition $\bigoplus_{i,j \in \{1,2\}} \mathcal{A}_{\Lambda_i, \Lambda_j}$, and

this decomposition descends to the homology:

$$\mathrm{LCH}_*(\Lambda_1 \cup \Lambda_2) = H_*(\mathcal{A}_{\Lambda_1 \cup \Lambda_2}, \partial) = \bigoplus_{i,j \in \{1,2\}} (\mathrm{LCH}_*)_{\Lambda_i, \Lambda_j},$$

where $(\mathrm{LCH}_*)_{\Lambda_i, \Lambda_j} = H_*(\mathcal{A}_{\Lambda_i, \Lambda_j}, \partial)$. Recall that Legendrian isotopies induce isomorphisms on Legendrian contact homology via counts of holomorphic disks similar to the differential, see [Ekh08, EES07, EHK]. It follows that a Legendrian isotopy between 2-component Legendrian links induces a quasi-isomorphism between the DGAs that also respects the decomposition.

We can further refine the structure of $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ by considering the filtration

$$\mathcal{A}_{\Lambda_1 \cup \Lambda_2} = \mathcal{F}^0 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \mathcal{F}^1 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \cdots$$

where $\mathcal{F}^k \mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is the subalgebra generated as a $\mathbb{Z}$-module by words involving at least k mixed chords (Reeb chords either from Λ_1 to Λ_2 or from Λ_2 to Λ_1). This also gives a filtration on the summands of $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$:

$$\mathcal{A}_{\Lambda_1, \Lambda_1} = \mathcal{F}^0 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \mathcal{F}^4 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \cdots$$
$$\mathcal{A}_{\Lambda_1, \Lambda_2} = \mathcal{F}^1 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \mathcal{F}^3 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \mathcal{F}^5 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \cdots$$
$$\mathcal{A}_{\Lambda_2, \Lambda_1} = \mathcal{F}^1 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \mathcal{F}^3 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \mathcal{F}^5 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \cdots$$
$$\mathcal{A}_{\Lambda_2, \Lambda_2} = \mathcal{F}^0 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \mathcal{F}^4 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \cdots .$$

We note two properties of the filtration. First, it is compatible with multiplication: the product of elements of $\mathcal{F}^{k_1}$ and $\mathcal{F}^{k_2}$ is an element of $\mathcal{F}^{k_1+k_2}$. Second, the differential ∂ respects the filtration, since the differential of any mixed chord is a sum of words that each includes a mixed chord. As a consequence of this second property, there is an induced filtration on $\mathrm{LCH}_*(\Lambda_1 \cup \Lambda_2)$ as well as its summands $(\mathrm{LCH}_*)_{\Lambda_i, \Lambda_j}(\Lambda_1 \cup \Lambda_2)$.

We abbreviate successive filtered quotients as follows: for k even when $i = j$ and k odd when $i \neq j$, write

$$\mathcal{A}^{(k)}_{\Lambda_i, \Lambda_j} := \mathcal{F}^k \mathcal{A}_{\Lambda_i, \Lambda_j} / \mathcal{F}^{k+2} \mathcal{A}_{\Lambda_i, \Lambda_j}.$$

Then $\mathcal{A}^{(k)}_{\Lambda_i, \Lambda_j}$ is generated as a $\mathbb{Z}$-module by words with exactly k mixed chords. We will especially be interested in the following filtered quotients with their induced differentials:

- $\mathcal{A}^{(0)}_{\Lambda_i, \Lambda_i}$, which is the DGA of Λ_i itself;
- $\mathcal{A}^{(1)}_{\Lambda_i, \Lambda_j}$ with $i \neq j$, which is generated by words with exactly 1 mixed chord.

Note that for $i \neq j$, the DGAs of Λ_i and of Λ_j act on $\mathcal{A}^{(1)}_{\Lambda_i, \Lambda_j}$ on the left and right, respectively, by multiplication, and this gives $\mathcal{A}^{(1)}_{\Lambda_i, \Lambda_j}$ the structure of a differential bimodule.

2.2. Legendrian contact homology for the conormal and fiber.
We now restrict to the case where V is the contact manifold $ST^*\mathbb{R}^3 = J^1(S^2)$. If $K \subset \mathbb{R}^3$ is a knot and $\Lambda_K \subset V$

is the unit conormal bundle of K, then the knot contact homology of K is defined to be the Legendrian contact homology of Λ_K:

$$\mathrm{LCH}_*(\Lambda_K) = H_*(\mathcal{A}_{\Lambda_K}, \partial).$$

The conormal Λ_K is topologically a 2-torus and has trivial Maslov class. The triviality of the Maslov class gives a well-defined integer grading on $\mathcal{A}_{\Lambda_K}$, by the Conley–Zehnder index, see [EES07, EENS13b]. A choice of orientation for K gives a distinguished set of generators $l, m \in \pi_1(\Lambda_K) \cong \mathbb{Z}^2$, where m is the meridian and l is the Seifert-framed longitude. The group ring $\mathbb{Z}[H_1(\Lambda_K)] \cong \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is a subring of $\mathcal{A}_{\Lambda_K}$ in degree 0, and there is an induced map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}] \to HC_0(K)$ that is injective as long as K is not the unknot (see [CELN]). The algebra $\mathcal{A}_{\Lambda_K}$ is generated by Reeb chords of Λ_K.

Since the Reeb flow on $ST^*\mathbb{R}^3$ is the geodesic flow, Reeb chords correspond under the projection $ST^*\mathbb{R}^3 \to \mathbb{R}^3$ in a one to one fashion to oriented binormal chords of K: for the flat metric on $\mathbb{R}^3$ these are simply oriented line segments with endpoints on K that are perpendicular to K at both endpoints. Furthermore the Conley–Zehnder grading of such a chord agrees with the Morse index for the corresponding critical point of the distance function $K \times K \to \mathbb{R}$, and hence takes on only the values $0, 1, 2$.

Next suppose that in addition to the knot K, we choose a point $p \in \mathbb{R}^3 \setminus K$. Then we can form the Legendrian link $\Lambda = \Lambda_K \cup \Lambda_p \subset ST^*\mathbb{R}^3$, where Λ_K is the unit conormal to K as before and Λ_p is the unit cotangent fiber of $ST^*\mathbb{R}^3$ at p.

Let $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ be the DGA associated to the link $\Lambda_K \cup \Lambda_p$. Then $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ is generated by Reeb chords of $\Lambda_K \cup \Lambda_p$. There are no Reeb chords from Λ_p to itself, and so the Reeb chords of $\Lambda_K \cup \Lambda_p$ come in three types: from Λ_K to itself, to Λ_K from Λ_p, and to Λ_p from Λ_K. These all correspond to binormal chords of $K \cup \{p\}$, where the normality condition is trivial at p.

We now discuss the grading on $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$. The grading on pure Reeb chords from Λ_K to itself is as for $\mathcal{A}_{\Lambda_K}$. In order to define the grading of mixed Reeb chords of a two-component Legendrian submanifold of Maslov index 0 such as $\Lambda_K \cup \Lambda_p$, it is customary to choose a path connecting the two Legendrians, along with a continuous field of Legendrian tangent planes along this path (i.e., isotropic 2-planes in the contact hyperplanes along the path) interpolating between the tangent planes to the Legendrians at the two endpoints. There is a $\mathbb{Z}$'s worth of homotopy classes of such fields of tangent planes, and different choices affect the grading of mixed chords, shifting the grading of chords from Λ_K to Λ_p up by some uniform constant k and shifting the grading of chords from Λ_p to Λ_K down by k. Note here that the usual dimension formulas for holomorphic disks hold and are independent of the path chosen since for any actual disk the path is traversed algebraically zero times.

To assign a specific grading to mixed chords, it is convenient to place K and p in a specific configuration in $\mathbb{R}^3$. Let (x, y, z) be linear coordinates on $\mathbb{R}^3$. The unit circle in the xy plane is an unknot, and we can braid K around this unknot so that it lies in a small tubular

neighborhood of the circle; also, choose p to lie in the xy plane. If we view $\Lambda_K, \Lambda_p \subset J^1(S^2)$ as fronts in $J^0(S^2) = S^2 \times \mathbb{R}$, then the front of Λ_p is the graph of the function $v \mapsto p \cdot v$ for $v \in S^2$, and in particular the tangent planes to this front over the equator $\{z = 0\} \cap S^2$ are horizontal. On the other hand, if the braid for K has n strands, then the front of Λ_K has $2n$ sheets near the equator, n with positive $\mathbb{R}$-coordinate and n with negative, and the tangent planes to these sheets are nearly horizontal. We can now take the connecting path between Λ_p and Λ_K as follows: choose a point v in the equator with $p \cdot v > 0$, and over v join the unique point in the front of Λ_p to any of the n points in Λ_K with positive $\mathbb{R}$-coordinate. The tangent planes are horizontal at the Λ_p endpoint and nearly horizontal at the Λ_K endpoint; choose the path to consist of nearly horizontal planes over v joining these without rotation.

More explicitly, this choice of path results in the following. Reeb chords of $\Lambda_K \cup \Lambda_p$ are binormal chords of $K \cup \{p\}$; denote by "ind" the Morse index of the corresponding critical point for the distance function on $K \cup \{p\}$. Let γ be a binormal chord and let c be the corresponding Reeb chord. We then have:

- if $c \in \mathcal{R}^{KK}$ (c goes to Λ_K from Λ_K) then

$$|c| = \operatorname{ind}(\gamma);$$

- if $c \in \mathcal{R}^{Kp}$ (c goes to Λ_K from Λ_p) then

$$|c| = \operatorname{ind}(\gamma);$$

- if $c \in \mathcal{R}^{pK}$ (c goes to Λ_p from Λ_K) then

$$|c| = \operatorname{ind}(\gamma) + 1.$$

The formula for the degrees of self Reeb chords of Λ_K comes from [EENS13b, Lemma 3.7]. For mixed Reeb chords between Λ_K and Λ_p, we can use [EENS13b, Lemma 2.5] (cf. [EES05a, Lemma 3.4]), which writes the degree $|c|$ of a Reeb chord c between two sheets of a front projection in terms of the Morse index $\operatorname{ind}_{\text{loc}}$ of the difference between the functions corresponding to these two sheets, and the difference $D - U$ between the number of up and down cusps along a capping path for the chord, as $|c| = \operatorname{ind}_{\text{loc}} + (D - U) - 1$. In our case $\operatorname{ind}_{\text{loc}}$ is 2 for all mixed Reeb chords (and equal to $\operatorname{ind}(\gamma)$ for pure chords). Furthermore, moving from the part of the front for Λ_K with positive $\mathbb{R}$-coordinate to the part with negative $\mathbb{R}$-coordinate passes down one cusp near the north pole of S^2, and the reverse passes up one cusp. The stated formulas for degrees follow.

Since $\operatorname{ind}(\gamma)$ is in $\{0, 1, 2\}$ if γ joins K to itself and $\{0, 1\}$ otherwise, we find that Reeb chords in $\mathcal{R}^{KK}$, $\mathcal{R}^{Kp}$, and $\mathcal{R}^{pK}$ have degrees in $\{0, 1, 2\}$, $\{0, 1\}$, and $\{1, 2\}$, respectively. It follows that $\mathcal{A}_{\Lambda_K, \Lambda_K}$ (respectively $\mathcal{A}_{\Lambda_K, \Lambda_p}$, $\mathcal{A}_{\Lambda_p, \Lambda_K}$) is supported in degree ≥ 0 (respectively ≥ 0, ≥ 1), and in lowest degree is generated as a $\mathbb{Z}$-module by only words with the minimal possible number of mixed chords. In particular, $\mathcal{F}^2 \mathcal{A}_{\Lambda_K, \Lambda_K}$, $\mathcal{F}^3 \mathcal{A}_{\Lambda_K, \Lambda_p}$, and $\mathcal{F}^3 \mathcal{A}_{\Lambda_p, \Lambda_K}$ are all

zero in degree 0, 0, and 1 respectively, and so:

$$H_0(\mathcal{A}_{\Lambda_K,\Lambda_K}) \cong H_0(\mathcal{A}^{(0)}_{\Lambda_K,\Lambda_K})$$

$$H_0(\mathcal{A}_{\Lambda_K,\Lambda_p}) \cong H_0(\mathcal{A}^{(1)}_{\Lambda_K,\Lambda_p})$$

$$H_1(\mathcal{A}_{\Lambda_p,\Lambda_K}) \cong H_1(\mathcal{A}^{(1)}_{\Lambda_p,\Lambda_K}).$$

As noted in Section 2.1, the first of these, $H_0(\mathcal{A}^{(0)}_{\Lambda_K,\Lambda_K})$, is exactly the degree 0 Legendrian contact homology of Λ_K, that is, degree 0 knot contact homology. The homology coefficients $\mathbb{Z}[H_1(\Lambda_K)] \cong \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ form a degree 0 subalgebra of the DGA of Λ_K with zero differential, and so we have a map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ into the degree 0 knot contact homology of K. In addition, $\mathcal{A}_{\Lambda_K,\Lambda_K}$ acts on the left (respectively right) on $\mathcal{A}_{\Lambda_K,\Lambda_p}$ (respectively $\mathcal{A}_{\Lambda_p,\Lambda_K}$), with an induced action on homology.

Definition 2.3. The *KCH-triple* of $\Lambda_K \cup \Lambda_p$ is

$$(R_{KK}, R_{Kp}, R_{pK}) = (H_0(\mathcal{A}_{\Lambda_K,\Lambda_K}), H_0(\mathcal{A}_{\Lambda_K,\Lambda_p}), H_1(\mathcal{A}_{\Lambda_p,\Lambda_K})).$$

Here R_{KK} is viewed as a ring equipped with a map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}] \to R_{KK}$, and R_{Kp} and R_{pK} are an $(R_{KK}, \mathbb{Z})$-bimodule and a $(\mathbb{Z}, R_{KK})$-bimodule, respectively.

Note that although we have chosen a particular placement of K and p above, the KCH-triple of $\Lambda_K \cup \Lambda_p$ is unchanged by isotopy of Λ_K and Λ_p, since it can be defined strictly in terms of graded pieces of the homology of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$, which is invariant under Legendrian isotopy up to quasi-isomorphism. That is:

Proposition 2.4. *If $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ are Legendrian isotopic, then they have isomorphic KCH-triples $(R_{K_0K_0}, R_{K_0p}, R_{pK_0})$ and $(R_{K_1K_1}, R_{K_1p}, R_{pK_1})$, in the sense that there are isomorphisms*

$$\psi_{KK} : R_{K_0K_0} \xrightarrow{\cong} R_{K_1K_1}$$

$$\psi_{Kp} : R_{K_0p} \xrightarrow{\cong} R_{K_1p}$$

$$\psi_{pK} : R_{pK_0} \xrightarrow{\cong} R_{pK_1}$$

compatible with multiplications $R_{K_iK_i} \otimes R_{K_iK_i} \to R_{K_iK_i}$, $R_{K_iK_i} \otimes R_{K_ip} \to R_{K_ip}$, and $R_{pK_i} \otimes R_{K_iK_i} \to R_{pK_i}$. If furthermore the Legendrian isotopy is parametrized in the sense that it sends the basis l_0, m_0 of $H_1(\Lambda_{K_0})$ to the basis l_1, m_1 of $H_1(\Lambda_{K_1})$, then $\psi_{KK}(m_0) = m_1$, $\psi_{KK}(l_0) = l_1$.

Remark 2.5. As mentioned above, the gradings in $\mathcal{A}_{\Lambda_K,\Lambda_p}$ and $\mathcal{A}_{\Lambda_p,\Lambda_K}$ are not canonically defined but rather depend on a choice of homotopy class of a path connecting the tangent planes at base points in the components of the Legendrian link (possible choices are in one to one correspondence with $\mathbb{Z}$). In general, in Definition 2.3 we would want to set

$R_{Kp} = H_d(\mathcal{A}_{\Lambda_K, \Lambda_p})$ and $R_{pK} = H_{1-d}(\mathcal{A}_{\Lambda_p, \Lambda_K})$, where $d \in \mathbb{Z}$ corresponds to the choice of homotopy class of path. (In all cases we still have $R_{KK} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_K})$.)

This indeterminacy would seem to pose problems for Proposition 2.4. However, we can eliminate the ambiguity by stipulating that we have picked the unique choice of grading for which

$$\min\{d \,|\, H_d(\mathcal{A}_{\Lambda_K, \Lambda_p}) \neq 0\} = 0.$$

This is because with our preferred choice of grading, $H_d(\mathcal{A}_{\Lambda_K, \Lambda_p}) = 0$ for $d < 0$, while we will show that $R_{Kp} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_p})$ is nonzero (see for instance Proposition 4.13).

Remark 2.6. If we choose p sufficiently far away from K, then by action considerations (the action of the Reeb chord at the positive puncture of a holomorphic disk is greater than the sum of the actions at the negative punctures), the differential in $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ of any word containing exactly k mixed chords must only involve words again containing exactly k mixed chords. In this case, the DGA $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ is isomorphic to its associated graded DGA under the filtration $\mathcal{F}^k$, and the homology $\mathrm{LCH}_*(\Lambda_K \cup \Lambda_p)$ decomposes as a direct sum by number of mixed chords.

We will use the KCH-triple of $\Lambda_K \cup \Lambda_p$ to produce a complete knot invariant. More specifically, we have the following object created from the KCH-triple:

Definition 2.7. Let R_{pp} denote the $\mathbb{Z}$-module

$$R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}.$$

Alternatively, we can write R_{pp} in terms of the homology of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$:

Proposition 2.8. *We have*

$$R_{pp} \cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_p}) \cong H_1(\mathcal{A}^{(2)}_{\Lambda_p, \Lambda_p}).$$

Proof. Since there are no self Reeb chords of Λ_p, and since any mixed Reeb chord to Λ_p from Λ_K has degree ≥ 1, any degree 1 generator of $\mathcal{A}_{\Lambda_p, \Lambda_p}$ must consist of a mixed chord to Λ_p from Λ_K, followed by some number of Reeb chords of Λ_K, followed by a mixed chord to Λ_K from Λ_p. The result now follows from the definition of the KCH-triple. $\square$

In fact, we will show (Proposition 4.17) that there is a ring isomorphism

$$\mathbb{Z} \oplus R_{pp} \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$$

and this is the key to proving our main result, Theorem 1.1. To get this, we in particular need a multiplication operation on R_{pp}. In the next subsection, we will define a product map

$$\mu : R_{Kp} \otimes R_{pK} \to R_{KK}.$$

This will then induce a map

$$\mu : R_{pp} \otimes_{\mathbb{Z}} R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp} \otimes_{\mathbb{Z}} R_{pK} \otimes_{R_{KK}} R_{Kp} \to R_{pK} \otimes_{R_{KK}} R_{KK} \otimes_{R_{KK}} R_{Kp} = R_{pp},$$
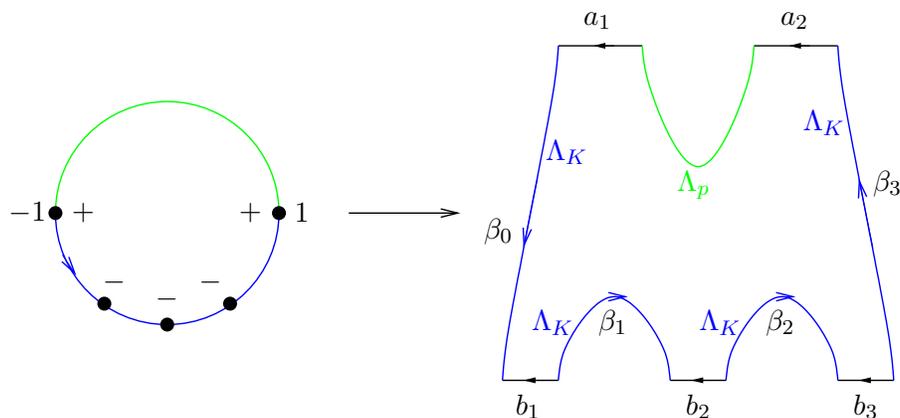
FIGURE 2. A disk in the moduli space $\mathcal{M}(a_1, a_2; \beta_0 b_1 \beta_1 b_2 \beta_2 b_3 \beta_3)$. Here the β_i coefficients record the homology classes of the depicted arcs in Λ_K. The arrows on Reeb chords denote their positive orientations.

which is the desired multiplication on R_{pp}.

2.3. **Product.** Recall that the differential in the contact homology DGA $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ that is used to define the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) counts holomorphic disks with one positive puncture in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $\mathbb{R} \times (\Lambda_p \cup \Lambda_K)$. As described in [Ekh08, BEE11], one can also produce invariants by counting holomorphic disks with two positive punctures at mixed Reeb chords, along with an arbitrary number of negative punctures at pure Reeb chords.

For general two-component Legendrian links, the resulting algebraic structure is a bit complicated to describe, but in our case it is simple because Λ_p has no self Reeb chords: reading along the boundary of any of these two-positive-punctured disks, we see a positive puncture from Λ_K to Λ_p, followed by a positive puncture from Λ_p to Λ_K, followed by some number of negative punctures from Λ_K to Λ_K. This allows us to define the product of a Reeb chord from Λ_K to Λ_p with a Reeb chord from Λ_p to Λ_K, or more generally the product of composable words in $\mathcal{A}_{\Lambda_K, \Lambda_p}$ and $\mathcal{A}_{\Lambda_p, \Lambda_K}$, each of which contains exactly one mixed Reeb chord. The result of the product in either case will be an alternating word of pure chords from Λ_K to Λ_K and homotopy classes of loops in Λ_K. (No mixed Reeb chords are involved.)

We now describe this construction in more detail. Let a_1 be a Reeb chord from Λ_K to Λ_p and a_2 a Reeb chord from Λ_p to Λ_K. Let $\mathbf{b} = \beta_0 b_0 \beta_1 \cdots b_m \beta_m$ be a word in Reeb chords b_i from Λ_K to itself and homology classes β_i in $H_1(\Lambda_K)$. Consider the moduli space of holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ of the following form. We take the domain of the disks to be the unit disk in the complex plane with punctures and boundary data as follows: there are two positive punctures at 1 and -1; the arc in the upper half plane connecting these two punctures maps to $\mathbb{R} \times \Lambda_p$; there are $m \geq 0$ negative punctures along the boundary arc in the lower half plane; and the boundary components in the lower half

plane all map to $\mathbb{R} \times \Lambda_K$ according to $\mathbf{b}$. See Figure 2. We write

$$\mathcal{M}(a_1, a_2; \mathbf{b})$$

for the moduli space of holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with punctures and boundary data as described.

The dimension of this moduli space is then the following, where $|c|$ denotes the grading of the Reeb chord c:

$$\dim(\mathcal{M}(a_1, a_2; \mathbf{b})) = |a_1| + |a_2| - |\mathbf{b}|.$$

Remark 2.9. This is a special case of a general dimension formula for holomorphic disks in the symplectization $\mathbb{R} \times ST^*Q$ of the cosphere bundle over an n-manifold, with boundary in $\mathbb{R} \times \Lambda$, where Λ is a Legendrian submanifold of Maslov class 0. If such a disk u has positive punctures at Reeb chords $a_1, \ldots, a_p$ and negative punctures at Reeb chords $b_1, \ldots, b_q$ then its formal dimension $\dim(u)$ is

$$\dim(u) = (n-3) + \sum_{j=1}^{p}(|a_j| - (n-3)) - \sum_{k=1}^{q}|b_k|.$$

As in the definition of the differential in Legendrian contact homology, we need to consider orientations of these moduli spaces induced by capping operators and the Fukaya orientation on the space of linearized Cauchy–Riemann operators on the disk with trivialized Lagrangian boundary condition.[1] There is basically only one point where the construction here differs from that used for the differential. The disks in the differential have a unique positive puncture and we write the capped-off linearized problem for a disk with positive puncture at a and negative punctures and boundary data according to $\mathbf{b} = \beta_0 b_1 \cdots b_m \beta_m$ as above as (with $C_c^{\pm}$ denoting the positive/negative capping operator at the Reeb chord c and L denoting the linearized Cauchy-Riemann operator at the holomorphic disk under consideration)

$$C_a^{+} \oplus L \oplus C_{b_1}^{-} \oplus \cdots \oplus C_{b_m}^{-} \approx F,$$

where F denotes a trivialized boundary condition on the closed disk and where "$\approx$" means "is related to via a linear gluing exact sequence", see [EES05b]. For the product, we have disks with two positive punctures and there is no natural way to order the punctures in general. However, in our case the two positive punctures are distinguished since both are mixed and have different endpoint configurations. We choose the following ordering:

$$C_{a_1}^{+} \oplus L \oplus C_{b_1}^{-} \oplus \cdots \oplus C_{b_m}^{-} \oplus C_{a_2}^{+} \approx F.$$

As usual this then induces a linear gluing sequence which in the transverse case orients the moduli space.

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[1]In fact, for the purposes of this paper and in particular the proof of Theorem 1.1, one could ignore orientations and work over $\mathbb{Z}/2$ rather than $\mathbb{Z}$. However, for the purposes of the general theory, we will work over $\mathbb{Z}$ throughout.

With these orientations determined, we can now define μ. Suppose that we have $\mathbf{c}_1 a_1 \in \mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p}$ and $a_2 \mathbf{c}_2 \in \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K}$, where a_1, a_2 are mixed chords to Λ_K from Λ_p (respectively to Λ_p from Λ_K), and $\mathbf{c}_1$, $\mathbf{c}_2$ are words in pure Reeb chords on Λ_K and homology classes in Λ_K. Define:

$$\mu(\mathbf{c}_1 a_1, a_2 \mathbf{c}_2) = \sum_{|a_1| + |a_2| - |\mathbf{b}| = 1} |\mathcal{M}(a_1, a_2; \mathbf{b})/\mathbb{R}| \, \mathbf{c_1} \mathbf{b} \mathbf{c_2}.$$

This produces a map

$$\mu : \mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p} \otimes \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K} \to \mathcal{A}^{(0)}_{\Lambda_K, \Lambda_K}.$$

Proposition 2.10 ([Ekh08]). *The product map μ has degree -1 and satisfies the Leibniz rule:*

$$\mu \circ (\partial \otimes 1 + 1 \otimes \partial) = \partial \circ \mu.$$

Thus μ descends to a map on homology.

Here and in the rest of the paper, we use Koszul signs when defining the tensor product of maps: in particular, $(\partial \otimes 1)(a \otimes b) = (\partial a) \otimes b$ while $(1 \otimes \partial)(a \otimes b) = (-1)^{|a|} a \otimes (\partial b)$ if ∂ has odd degree. Although Proposition 2.10 is implicitly contained in [Ekh08], we give the proof for definiteness.

Proof of Proposition 2.10. Once we know that the moduli spaces are transversely cut out for generic data then the fact that μ has degree -1 follows from the dimension formula. The disks with two positive punctures considered here cannot be multiply covered for topological reasons (e.g. only one positive puncture is asymptotic to a chord from Λ_p to Λ_K). Thus the same argument as for disks with one positive puncture can be used to show transversality for generic almost complex structure where the formal dimension then equals the actual dimension, see e.g. [EES07, Proposition 2.3].

To see the displayed equation, we look at the boundary of moduli spaces $\mathcal{M}(a_1, a_2; \mathbf{b})$ of dimension 2. It follows by SFT compactness that the boundary consists of broken curves. We must check that there cannot be any boundary breaking. To see this note that any splitting arc in the domain that separates the positive punctures must connect boundary points that lie in distinct boundary components. Thus there is no boundary bubbling and several-level disks account for the whole boundary. The equation follows from identifying contributing terms with the boundary of an oriented 1-dimensional manifold. $\square$

We will also need the fact that μ is invariant under Legendrian isotopy. As in [Ekh08] this can be understood by looking at cobordism maps and homotopies of such. We will only need invariance on the level of homology, and this is slightly easier to prove: we need only the statement that the multiplication on homology induced by μ is invariant under Legendrian isotopy and this follows from properties of cobordism maps and analogues of these for the product. To see this first recall that a Legendrian isotopy $\Lambda_{K_t} \cup \Lambda_{p_t}$, $0 \leq t \leq 1$, from $\Lambda_{K_0} \cup \Lambda_{p_0}$ to $\Lambda_{K_1} \cup \Lambda_{p_1}$ gives an exact Lagrangian cobordism $L_K \cup L_p \subset \mathbb{R} \times ST^* \mathbb{R}^3$

that agrees with $(\mathbb{R} \times \Lambda_{K_0}) \cup (\mathbb{R} \times \Lambda_{p_0})$ in the positive end and $(\mathbb{R} \times \Lambda_{K_1}) \cup (\mathbb{R} \times \Lambda_{p_1})$ in the negative. Furthermore there is a cobordism map

$$\Phi \colon \mathcal{A}_{\Lambda_{K_0} \cup \Lambda_{p_0}} \to \mathcal{A}_{\Lambda_{K_1} \cup \Lambda_{p_1}}$$

that is a quasi-isomorphism respecting the filtration with respect to the number of mixed chords. This cobordism map counts holomorphic disks in the cobordism as follows. If a is a Reeb chord of $\Lambda_{K_0} \cup \Lambda_{p_0}$ and if $\mathbf{b}$ is an alternating word of Reeb chords and homotopy classes of paths in $\Lambda_{K_1} \cup \Lambda_{p_1}$ then let $\mathcal{M}^{\mathrm{co}}(a, \mathbf{b})$ denote the moduli space of holomorphic disks in $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $L_K \cup L_p$, with one positive puncture where the disk is asymptotic to the Reeb chord a and with several negative punctures which together with the boundary arcs give the word $\mathbf{b}$. The map Φ is then given by

$$\Phi(a) = \sum_{|a|-|\mathbf{b}|=0} |\mathcal{M}^{\mathrm{co}}(a, \mathbf{b})|\mathbf{b}.$$

In order to study invariance for the product, we look at moduli spaces in the cobordism analogous to the moduli spaces used in the definition of μ. If a_1 is a Reeb chord from Λ_{K_0} to Λ_{p_0}, a_2 a chord from Λ_{p_0} to Λ_{K_0}, and $\mathbf{b}$ a word of Reeb chords and homotopy classes of paths on Λ_{K_1} as above, then let $\mathcal{M}^{\mathrm{co}}(a_1, a_2; \mathbf{b})$ denote the moduli space of disks with boundary on $L_K \cup L_p$ with two positive punctures asymptotic to a_1 and a_2, and with negative punctures and boundary arcs mapping according to $\mathbf{b}$. Then define $\kappa \colon R_{K_0 p_0} \otimes R_{p_0 K_0} \to R_{K_1 K_1}$ by:

$$\kappa(\mathbf{c_1} a_1, a_2 \mathbf{c_2}) = \sum_{|a_1|+|a_2|-|\mathbf{b}|=0} |\mathcal{M}^{\mathrm{co}}(a_1, a_2; \mathbf{b})|\Phi(\mathbf{c_1})\mathbf{b}\Phi(\mathbf{c_2}).$$

Proposition 2.11 ([Ekh08])**.** *Given a Legendrian isotopy* $(\Lambda_{K_t}, \Lambda_{p_t})$, $t \le 0 \le 1$, *and product maps* μ_0 *and* μ_1 *for* $\Lambda_{K_0} \cup \Lambda_{p_0}$ *and* $\Lambda_{K_1} \cup \Lambda_{p_1}$ *as defined above, we have:*

(1) $$\Phi \circ \mu_0 - \mu_1 \circ \Phi - \kappa \circ (\partial_0 \otimes 1 + 1 \otimes \partial_0) + \partial_1 \circ \kappa = 0.$$

Thus, on the level of homology, Φ *sends* μ_0 *to* μ_1.

Proof. The proof follows from an analysis of the boundary of 1-dimensional moduli spaces of the form $\mathcal{M}^{\mathrm{co}}(a_1, a_2; \mathbf{b})$. By gluing and SFT compactness the boundary of such a moduli space consists of two-level buildings. Hence the terms contributing to (1) are in 1-to-1 correspondence with the boundary of an oriented 1-manifold. The homology statement follows from (1) together with the fact that Φ is a quasi-isomorphism respecting the filtration. $\square$

We now connect this general discussion of the product μ with the KCH-triple. Recall from Section 2.2 that we have:

$$R_{KK} \cong H_0(\mathcal{A}^{(0)}_{\Lambda_K, \Lambda_K}) \qquad R_{Kp} \cong H_0(\mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p}) \qquad R_{pK} \cong H_1(\mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K}).$$

Then the product gives a map

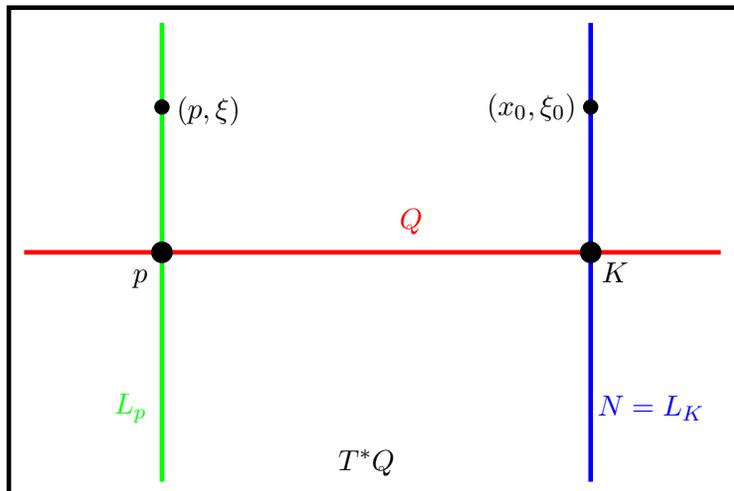$$\mu \colon R_{Kp} \otimes R_{pK} \to R_{KK}.$$

FIGURE 3. The cotangent bundle T^*Q with Lagrangians Q, N, L_p.

We can now write the invariance property as follows.

Proposition 2.12. *The R_{KK}-bimodule map $\mu : R_{Kp} \otimes_{\mathbb{Z}} R_{pK} \to R_{KK}$ is invariant under Legendrian isotopy of $\Lambda_K \cup \Lambda_p$: given isotopic $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ and isomorphisms $\psi_{KK}, \psi_{Kp}, \psi_{pK}$ of KCH-triples as in Proposition 2.4, we have*

$$\psi_{KK} \circ \mu_0 = \mu_1 \circ (\psi_{Kp} \otimes \psi_{pK}).$$

3. STRING TOPOLOGY

In this section we will describe how to extend the results from [CELN] to enhanced knot contact homology with the product μ. This will allow us to interpret $\mathrm{LCH}_*(\Lambda_K \cup \Lambda_p)$ in low degree in terms of a version of string topology and homotopy data; in particular, we will proceed in Section 4 to use string topology arguments to write the KCH-triple in terms of $\pi_1(\mathbb{R}^3 \setminus K)$.

The main result of [CELN] is an isomorphism in degree 0 between the Legendrian contact homology of Λ_K and a "string homology" defined using chains of broken strings, where a broken string is a loop in the union $\mathbb{R}^3 \cup L_K$ of the zero section and the conormal bundle of L_K in $T^*\mathbb{R}^3$. Here we will give a modification of this approach that produces an isomorphism between the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$ (in the appropriate degree) and string homology for broken strings in $\mathbb{R}^3 \cup L_K \cup L_p$, where L_p is the conormal of p in $T^*\mathbb{R}^3$, i.e. the fiber $T_p^*\mathbb{R}^3$. We will then prove that the product μ defined in Section 2.3 corresponds to the Pontryagin product on string homology under this isomorphism.

The discussion in this section closely parallels the treatment in [CELN], as our setup is nearly identical to the one there, differing only in the introduction of L_p. Where convenient, we adopt notation from [CELN] to make the correspondence clearer.

3.1. **Broken strings.** Here we recall the definition of broken strings from [CELN], suitably modified for our purposes. Let $K \subset \mathbb{R}^3$ be a knot and $p \in \mathbb{R}^3$ be a point in the complement of K. Write $Q = \mathbb{R}^3$ and view Q as the zero section in T^*Q, and let $N = L_K \subset T^*Q$ be the conormal bundle to K, while L_p is the cotangent fiber T_p^*Q. We then have three Lagrangians $Q, N, L_p \subset T^*Q$; N, L_p are disjoint, Q, L_p intersect transversely at p, and Q, N intersect cleanly along K. See Figure 3.

Fix base points $(x_0, \xi_0) \in N \setminus K$ and $(p, \xi) \in L_p \setminus \{p\}$. If we use a metric to identify T^*Q and TQ, then these points become (x_0, v_0) with $v_0 \in T_{x_0}N$ and (p, v) with $v \in T_pQ$. This metric also gives a diffeomorphism between N and a tubular neighborhood of K in Q, and we can view $Q \cup N$ as the disjoint union of Q and $N \subset Q$ glued along K. This allows us to identify T_xN with T_xQ for $x \in K$. Similarly we view $Q \cup N \cup L_p$ as the disjoint union of $Q \cup N$ and L_p with $p \in Q$ and $0 \in L_p$ identified, and the metric identifies T_pQ with $T_0L_p = T_p^*Q$.

Now consider a piecewise C^1 path in $Q \cup N \cup L_p$. This path can move between Q and N (in either direction) at a point on $K = Q \cap N$, and between Q and L_p at p; we call the points where the path changes components *switches*, either at K or at p.

Definition 3.1. A *broken string* is a piecewise C^1 path $s : [a, b] \to Q \cup N \cup L_p$ such that:

- the endpoints $s(a), s(b)$ are each at one of the two base points $(x_0, v_0) \in N$ or $(p, v) \in L_p$;
- if $s(t_0)$ is a switch at K from N to Q (i.e., for small $\epsilon > 0$, $s((t_0 - \epsilon, t_0]) \subset N$ and $s([t_0, t_0 + \epsilon)) \subset Q$), then:

$$\lim_{t \to t_0^-} (s'(t))^{\text{normal}} = \lim_{t \to t_0^+} (s'(t))^{\text{normal}},$$

  where we identify $T_{s(t_0)}N$ with $T_{s(t_0)}Q$ and v^{normal} denotes the component of v normal to K with respect to the metric on Q;
- if $s(t_0)$ is a switch at K from Q to N, then:

$$\lim_{t \to t_0^-} (s'(t))^{\text{normal}} = - \lim_{t \to t_0^+} (s'(t))^{\text{normal}};$$

- if $s(t_0)$ is a switch at p from L_p to Q, then:

$$\lim_{t \to t_0^-} s'(t) = \lim_{t \to t_0^+} s'(t);$$

- if $s(t_0)$ is a switch at p from Q to L_p, then:

$$\lim_{t \to t_0^-} s'(t) = - \lim_{t \to t_0^+} s'(t).$$

The portions of s in Q (respectively N, L_p) are called Q-strings (respectively N-strings, L_p-strings).

Remark 3.2. A broken string models the boundary of a holomorphic disk in T^*Q with boundary on $Q \cup N \cup L_p$ and one positive puncture at infinity at a Reeb chord for $\Lambda_K \cup \Lambda_p$.

The condition on the derivatives at a switch follows the behavior of the boundary of such a disk at a point where the boundary switches between Q and N, or between Q and L_p: if v_{in} and v_{out} denote the incoming and outgoing tangent vectors of a broken string at a switch then $v_{\text{out}} = Jv_{\text{in}}$, where J is the almost complex structure along the 0-section induced by the metric.

If we project from T^*Q to Q, then the endpoints of a broken string are each either at p or at the point on K that is the projection of x_0. With this in mind, we call a broken string s:

- a KK broken string if $s(a) = s(b) = (x_0, v_0)$
- a Kp broken string if $s(a) = (x_0, v_0)$ and $s(b) = (p, v)$
- a pK broken string if $s(a) = (p, v)$ and $s(b) = (x_0, v_0)$
- a pp broken string if $s(a) = s(b) = (p, v)$.

3.2. **String homology.** We now construct a complex from broken strings whose homology might be called "string homology"; in Section 3.4 below, we will describe an isomorphism between this homology and enhanced knot contact homology.

For $\ell \geq 0$, let Σ_ℓ denote the space of broken strings with ℓ switches at p (note that we do not count switches at K here). We write

$$\Sigma_\ell = \Sigma_\ell^{KK} \sqcup \Sigma_\ell^{Kp} \sqcup \Sigma_\ell^{pK} \sqcup \Sigma_\ell^{pp}$$

where Σ_ℓ^{ij} denotes the subset of Σ_ℓ corresponding to ij broken strings for $i, j \in \{K, p\}$, and then

$$C_k(\Sigma_\ell) = C_k^{KK}(\Sigma_\ell) \oplus C_k^{Kp}(\Sigma_\ell) \oplus C_k^{pK}(\Sigma_\ell) \oplus C_k^{pp}(\Sigma_\ell)$$

for the free $\mathbb{Z}$-module generated by generic k-dimensional singular simplices in Σ_ℓ (C_k^{ij} is the summand corresponding to ij broken strings). Here "generic" refers to simplices that satisfy the appropriate transversality conditions at switches and with respect to K and to p; compare [CELN, Definition 5.3].

In addition to the usual boundary operator $\partial : C_k(\Sigma_\ell) \to C_{k-1}(\Sigma_\ell)$ on singular simplices, there are two string operations

$$\delta_Q^K, \ \delta_N^K : C_k(\Sigma_\ell) \to C_{k-1}(\Sigma_\ell)$$

defined in [CELN] (where they are called δ_Q, δ_N). We refer to [CELN] for details, but qualitatively these operations take a generic k-dimensional family of broken strings, identify the subfamily consisting of broken strings where a Q-string or N-string has an interior point in K, and insert a "spike" in N or Q at this point; see Figure 4. We note that this interior intersection condition is codimension 1, and that adding a spike increases the number of switches at K by 2. In our setting, there are two more string operations

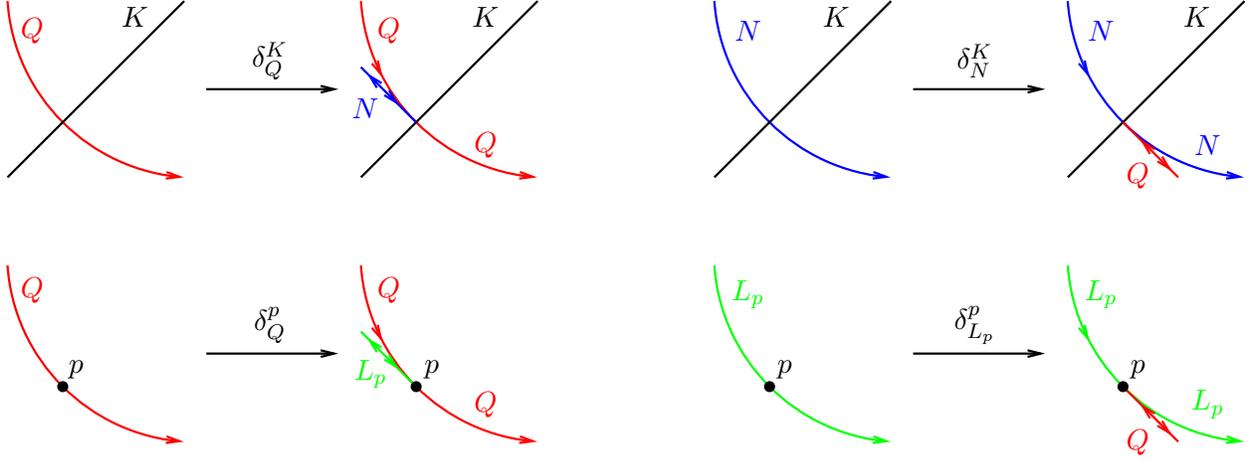$$\delta_Q^p, \ \delta_{L_p}^p : C_k(\Sigma_\ell) \to C_{k-2}(\Sigma_{\ell+2})$$

FIGURE 4. The maps δ_Q^K (respectively δ_N^K, δ_Q^p, $\delta_{L_p}^p$) insert an N-string (Q-string, L_p-string, Q-string) at an interior point of a Q-string (N-string, Q-string, L_p-string) that lies on K (K, p, p).

that are defined in the same way as δ_Q^K, δ_N^K, but inserting spikes in L_p or Q where a Q-string or L_p-string has an interior point at p; see Figure 4 again. Note now that the interior intersection condition is codimension 2, and that adding a spike increases the number of switches at p by 2.

We then have the following result, which is a direct analogue of Proposition 5.8 from [CELN] and is proved in the same way.

Lemma 3.3. *On generic 2-chains, the operations ∂, $\delta_Q^K + \delta_N^K$, and $\delta_Q^p + \delta_{L_p}^p$ each have square 0 and pairwise anticommute. In particular, we have*

$$(\partial + \delta_Q^K + \delta_N^K)^2 = 0,$$
$$(\partial + \delta_Q^K + \delta_N^K)(\delta_Q^p + \delta_{L_p}^p) + (\delta_Q^p + \delta_{L_p}^p)(\partial + \delta_Q^K + \delta_N^K) = 0,$$
$$(\delta_Q^p + \delta_{L_p}^p)^2 = 0.$$

Lemma 3.3 allows us to construct a complex out of broken strings in the following way. For $m \in \frac{1}{2}\mathbb{Z}$, define

$$C_m = \bigoplus_{k+\ell/2=m} C_k(\Sigma_\ell).$$

By consideration of the parity of the number of switches at p, we can write $C_m = C_m^{KK} \oplus C_m^{pp}$ when m is an integer and $C_m = C_m^{Kp} \oplus C_m^{pK}$ when m is a half-integer. We define a shifted complex $\tilde{C}_*$, $* \in \mathbb{Z}$, by:

$$\tilde{C}_m^{KK} = C_m^{KK}, \quad \tilde{C}_m^{Kp} = C_{m+1/2}^{Kp}, \quad \tilde{C}_m^{pK} = C_{m-1/2}^{pK}, \quad \tilde{C}_m^{pp} = C_m^{pp},$$
$$\tilde{C}_m = \tilde{C}_m^{KK} \oplus \tilde{C}_m^{Kp} \oplus \tilde{C}_m^{pK} \oplus \tilde{C}_m^{pp};$$

that is, we shift the grading up by $1/2$ if the beginning point is p and down by $1/2$ if the endpoint is p. By Lemma 3.3, $\partial + \delta_Q^K + \delta_N^K + \delta_Q^p + \delta_{L_p}^p$ is a differential on $\tilde{C}_*$ that lowers degree by 1.

Remark 3.4. The $\frac{1}{2}$-grading for strings broken at p has the following geometric counterpart for holomorphic disks with switching Lagrangian boundary conditions on $L_p \cup Q$ and a puncture at the intersection point $p = L_p \cap Q$. The contribution to the dimension formula of such a puncture is the Maslov index of a $-\frac{\pi}{2}$ rotation in each complex coordinate line and 1 for the conformal variation corresponding to moving the puncture along the boundary. This gives a total contribution of $-\frac{1}{2}$. In order to have each puncture contributing with an integer we can for example deform L_p by rotating the first coordinate line towards Q. This way the Maslov contribution for one of the punctures at p becomes -1 and the other -2 giving total dimension contributions 0 and -1. We choose this rotation and the corresponding grading shift to match our choice of capping path connecting Λ_K to Λ_p: that is, so that both Reeb chords that start at L_p gets shifted up by 1 compared to the Morse grading and so that chains of broken strings starting at p are also shifted up.

3.3. Switches at a point in an example.
On the complex of broken strings, there are four string operations, δ_Q^K, δ_N^K, δ_Q^p, and $\delta_{L_p}^p$. The two that introduce switches on the knot, δ_Q^K and δ_N^K, have appeared before and are studied at length in [CELN]. For the other two, δ_Q^p and $\delta_{L_p}^p$, which introduce switches at a point, the only property we need for our main argument is their codimension; in the following section, Section 3.4, we use this to prove an isomorphism to knot contact homology. Here we examine δ_Q^p and $\delta_{L_p}^p$ more closely in a model case. This is a digression from the main argument and can be skipped without loss of continuity, but provides some context for these operations within contact geometry.

Consider $Q = S^n$ and as usual let $L_p \subset T^*S^n$ be the cotangent fiber over p, with $\Lambda_p \subset ST^*S^n$ the Legendrian sphere given by the unit cotangent fiber. By the surgery result from [BEE12, §5.5], we can compute the DGA of Λ_p in ST^*S^n via the DGA for the Legendrian unknot $U \subset S^{2n-1}$, where S^{2n-1} is the standard contact $(2n-1)$-sphere, i.e., the contact boundary of the standard symplectic $2n$-ball, with the differential in the latter DGA twisted by a point condition at p.

There is (effectively) only one Reeb chord a of U of grading $|a| = n-1$, see [BEE12], and the differential is $\partial a = p$. To see this one can use the flow tree description of holomorphic disks: it is easy to see that for the standard front of the unknot there is exactly one rigid point constrained Morse flow tree with positive puncture at a. Thus the DGA of Λ_p is generated by chords a^r, $r \geq 1$, of grading $r(n-1) + (n-2)$, and the differential is

$$\partial(a^r) = \sum_{j=2}^{r} a^{j-1} \cdot a^{r-j}.$$

We claim that this DGA is chain isomorphic to the complex of broken strings in $Q \cup L_p$ with differential given by $\partial + \delta_Q^p + \delta_{L_p}^p$. For the latter, note that L_p is contractible so we simply forget the N-strings and think of the chains of broken strings as the tensor algebra of chains on the based loop space of S^n with differential $\partial + \delta_p$, where δ_p splits a chain over the locus where its evaluation map hits p. By Morse theory, the space of non-constant based loops in S^n is a cell complex with a cell in dimensions

$$(n-1),\ 2(n-1),\ 3(n-1),\ 4(n-1),\ \ldots.$$

For degree reasons there are only quadratic terms in the differential $\partial + \delta_p$ and in order to compute δ_p we need to see the unstable manifolds of the cells that correspond to Morse flow in the Bott-manifolds followed by shrinking the loops over half-disks. It is not hard to see that δ_p acts on the Morse cells by splitting the cell of dimension $r(n-1)$ into two cells of dimensions $j(n-1)$ and $k(n-1)$, where $j + k = r - 1$, in all possible ways.

We thus conclude that the complex of broken strings in $Q \cup L_p$ is indeed isomorphic to the DGA of the cosphere Λ_p. Furthermore, one can check that this isomorphism is induced by the map that associates to a Reeb chord c of ∂L_p the chain carried by the moduli-space of disks with positive puncture at c and switching boundary condition on $Q \cup L_p$. Note that each pair of switches in the boundary of such a disk contributes $-(n-2)$ to the dimension of the moduli space, see Remark 3.4, which explains the difference in grading between the generators of the DGA of Λ_p and generators of the complex of chains of broken strings ($r(n-1) + (n-2)$ versus $r(n-1)$ for $r \geq 1$).

3.4. String homology and enhanced knot contact homology.

In [CELN] the DGA $\mathcal{A}_{\Lambda_K}$ was related to string homology via a chain map defined through a count of holomorphic disks with switching boundary condition. Here we similarly relate $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ to string homology. More precisely, if a is a Reeb chord of $\Lambda_K \cup \Lambda_p$ then we let $\mathcal{M}^{\mathrm{sw}}(a)$ denote the moduli space of holomorphic disks in $T^*\mathbb{R}^3$ with one positive puncture asymptotic to the Reeb chord a at infinity, and such that the disk has switching boundary on $Q \cup N \cup L_p$: that is, the boundary of the disk lies on $Q \cup N \cup L_p$, and there are several punctures where the boundary switches between the Lagrangians L_p and Q or between L_K and Q, in either direction.

The boundary of a disk in $\mathcal{M}^{\mathrm{sw}}(a)$, oriented counterclockwise, is a broken string in $Q \cup N \cup L_p$; see Figure 5. More precisely, each endpoint of a Reeb chord of $\Lambda_K \cup \Lambda_p$ is a point in $\Lambda_K \cup \Lambda_p$; fix paths in L_p or N that connect these points to the base points (p, ξ) or (x_0, ξ_0). Then the union of the boundary of a disk in $\mathcal{M}^{\mathrm{sw}}(a)$ and the paths for the endpoints of a is a broken string.

We can stratify $\mathcal{M}^{\mathrm{sw}}(a)$ by the number of switches at p: for $\ell \geq 0$, let $\mathcal{M}_\ell^{\mathrm{sw}}(a)$ denote the subset of $\mathcal{M}^{\mathrm{sw}}(a)$ of disks with ℓ switches at p. The moduli space $\mathcal{M}_\ell^{\mathrm{sw}}(a)$ is an oriented C^1-manifold and we let $[\mathcal{M}_\ell^{\mathrm{sw}}(a)]$ denote the chain of broken strings in Σ_ℓ carried by this
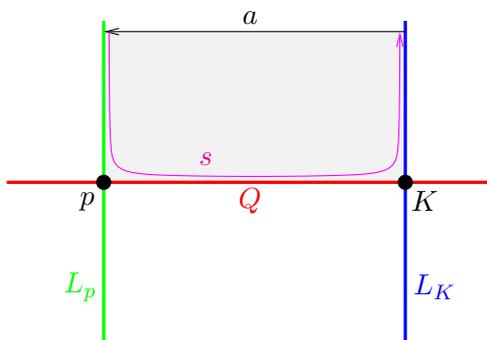
FIGURE 5. A Reeb chord a to Λ_p from Λ_K and a holomorphic disk in $\mathcal{M}^{\mathrm{sw}}(a)$ whose boundary is the depicted broken string s.

moduli space (that is, the chain given by the boundaries of disks in the moduli space). Now define $\Phi\colon \mathcal{A}_{\Lambda_K \cup \Lambda_p} \to \tilde{C}_*$ by

$$\Phi(a) = \sum_{\ell} [\mathcal{M}^{\mathrm{sw}}_{\ell}(a)].$$

Proposition 3.5. *The map*

$$\Phi \colon (\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial) \to (\tilde{C}_*, \partial + \delta_Q^K + \delta_N^K + \delta_Q^p + \delta_{L_p}^p)$$

is a degree zero chain map of differential graded algebras, where multiplication on $\tilde{C}_*$ is given by chain-level concatenation of broken strings.

Proof. The proof is very similar to [CELN, Proposition 5.8]. We first check that the map Φ has degree 0. The dimension of $\mathcal{M}^{\mathrm{sw}}_{\ell}(a)$ is

$$\dim(\mathcal{M}^{\mathrm{sw}}_{\ell}(a)) = |a| - \ell',$$

where ℓ' is the number of arcs in the boundary of the disk that *end* at a switch at p. To see this, note that the dimension input from a pair of Lagrangian intersection punctures at p is -1 (2 for conformal structures and -3 from the Maslov index), and from Remark 3.4, this contribution is distributed between the punctures as 0 for the puncture beginning at a switch at p and -1 for the puncture ending at a switch at p.

We now have three cases. If a joins Λ_K to itself, then $\ell = 2\ell'$ and

$$[\mathcal{M}^{\mathrm{sw}}_{\ell}(a)] \in C_{|a|-\ell/2}(\Sigma^{KK}_{\ell}) \subset \tilde{C}^{KK}_{|a|}.$$

If a goes to Λ_K from Λ_p, then if we traverse the boundary of a disk in $\mathcal{M}^{\mathrm{sw}}_{\ell}(a)$ beginning at the positive puncture, we begin on N, then alternately switch to and from L_p, and end on L_p; thus $\ell = 2\ell' + 1$ and

$$[\mathcal{M}^{\mathrm{sw}}_{\ell}(a)] \in C_{|a|-(\ell-1)/2}(\Sigma^{Kp}_{\ell}) \subset \tilde{C}^{Kp}_{|a|}.$$

Finally, if a goes to Λ_p from Λ_K, then the same argument gives $\ell = 2\ell' - 1$ and

$$[\mathcal{M}^{\mathrm{sw}}_{\ell}(a)] \in C_{|a|-(\ell+1)/2}(\Sigma^{pK}_{\ell}) \subset \tilde{C}^{pK}_{|a|}.$$

In all cases we find that Φ preserves degree.

We next study the chain map equation. To this end we must understand the codimension 1 boundary of $\mathcal{M}^{\mathrm{sw}}(a)$ which contributes the singular boundary $\partial\Phi(a)$. This boundary consists of three parts:

(i) 2-level disks with one level of dimension $\dim(\mathcal{M}^{\mathrm{sw}}(a)) - 1$ and a level of dimension 1 in the symplectization end;

(ii) 1-level disks in which a boundary arc in Q or in L_K shrinks to a point in K, or equivalently a disk with one boundary arc that hits K in an interior point;

(iii) 1-level disks in which a boundary arc in Q or in L_p shrinks to a point at p, or equivalently a disk with one boundary arc that hits p in an interior point.

Configurations of type (i) are counted by $\Phi(\partial a)$, configurations of type (ii) by $(\delta_Q^K + \delta_N^K)\Phi(a)$, and configurations of type (iii) by $(\delta_Q^p + \delta_N^p)\Phi(a)$. The chain map equation follows. $\qquad\square$

We will be especially interested in the subcomplexes $\tilde{C}_*^{KK}$, $\tilde{C}_*^{Kp}$, $\tilde{C}_*^{pK}$ in the lowest degree. These are given as follows, where the differential is $d = \partial + \delta_Q^K + \delta_N^K$ (the coproducts δ_Q^p, $\delta_{L_p}^p$ do not appear for degree reasons):

$$\tilde{C}_1^{KK} = C_1^{KK}(\Sigma_0) \oplus C_0^{KK}(\Sigma_2) \xrightarrow{d} \tilde{C}_0^{KK} = C_0^{KK}(\Sigma_0)$$

$$\tilde{C}_1^{Kp} = C_1^{Kp}(\Sigma_1) \oplus C_0^{Kp}(\Sigma_3) \xrightarrow{d} \tilde{C}_0^{Kp} = C_0^{Kp}(\Sigma_1)$$

$$\tilde{C}_2^{pK} = C_1^{pK}(\Sigma_1) \oplus C_0^{pK}(\Sigma_3) \xrightarrow{d} \tilde{C}_1^{pK} = C_0^{pK}(\Sigma_1).$$

Note that d acts on the first summand in each case, and is 0 on the second.

A main result from [CELN] is that Φ induces an isomorphism in degree 0 homology. In our setting, this becomes the following:

Proposition 3.6. *The map Φ induces isomorphisms*

$$R_{KK} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}) \xrightarrow{\cong} H_0(\tilde{C}_*^{KK}, d) = \mathrm{coker}\left(\partial + \delta_Q^K + \delta_N^K : C_1^{KK}(\Sigma_0) \to C_0^{KK}(\Sigma_0)\right)$$

$$R_{Kp} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}) \xrightarrow{\cong} H_0(\tilde{C}_*^{Kp}, d) = \mathrm{coker}\left(\partial + \delta_Q^K + \delta_N^K : C_1^{Kp}(\Sigma_1) \to C_0^{Kp}(\Sigma_1)\right)$$

$$R_{pK} = H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}) \xrightarrow{\cong} H_1(\tilde{C}_*^{pK}, d) = \mathrm{coker}\left(\partial + \delta_Q^K + \delta_N^K : C_1^{pK}(\Sigma_1) \to C_0^{pK}(\Sigma_1)\right).$$

Proof. The isomorphism for R_{KK} is proven in [CELN, §7] via an action/length filtration argument, and the other isomorphisms use exactly the same argument. A short description of the argument is as follows. A length filtration on chains of broken strings given by the supremum norm of the sum of the lengths of the Q-strings is introduced. On the DGA there is the action filtration and for a suitable choice of almost complex structure on $T^*\mathbb{R}^3$ the chain map Φ respects this filtration. A standard approximation argument shows that the string homology complex is quasi-isomorphic to the string homology complex of piecewise linear broken strings. On the complex of piecewise linear broken strings, a length-decreasing flow (with splittings when the segments cross the knot) then deforms the

complex to a complex generated by certain chains associated to binormal chords and using basic holomorphic strips over binormal chords and the action/length filtrations then shows that Φ is a quasi-isomorphism. $\qquad\square$

Remark 3.7. It is likely that Φ is in fact an isomorphism in all degrees. The reason that we restrict to the lowest degree (0 for $\mathcal{A}_{\Lambda_K,\Lambda_K}$ and $\mathcal{A}_{\Lambda_K,\Lambda_p}$ and 1 for $\mathcal{A}_{\Lambda_p,\Lambda_K}$) here, following the same restriction in [CELN], is that the proof of the isomorphism in [CELN] involves an explicit examination of moduli spaces of holomorphic disks with switching boundary conditions of dimensions ≤ 2. To extend the isomorphism to higher degrees would require one to work out the relevant string homology in degree $d+2$, imposing conditions at endpoints of the strings that match degenerations in higher dimensional moduli spaces of holomorphic disks, and this has not been worked out for moduli spaces of dimensions ≥ 3.

Remark 3.8. In [CELN], $\operatorname{coker}\left(\partial + \delta_Q^K + \delta_N^K : C_1^{KK}(\Sigma_0) \to C_0^{KK}(\Sigma_0)\right)$ is written as "string homology" $H_0^{\text{string}}(K)$, and the first isomorphism in Proposition 3.6 states that $H_0^{\text{string}}(K)$ is isomorphic to knot contact homology in degree 0. A variant of this construction, modified string homology $\tilde{H}_0^{\text{string}}(K)$, is also considered in [CELN, §2], and it is observed there that $\tilde{H}_0^{\text{string}}(K) \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. In our language, modified string homology is defined by

$$\tilde{H}_0^{\text{string}}(K) = \operatorname{coker}\left(\partial + \delta_Q^K + \delta_N^K : C_1^{pp}(\Sigma_2) \to C_0^{pp}(\Sigma_2)\right).$$

Unfortunately this is not directly one of the terms in the homology of $\tilde{C}_*$, or else we could use this to deduce $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ directly from $\operatorname{LCH}_*(\Lambda_K \cup \Lambda_p)$; but we can recover it by invoking the Pontryagin product, which we discuss next.

3.5. **String topology and the product.** Having established isomorphisms Φ in low degree between enhanced knot contact homology and string homology, we now examine the behavior of the product map μ under this isomorphism. Recall from Section 2.3 that μ is a map

$$\mu : \mathcal{A}_{\Lambda_K,\Lambda_p}^{(1)} \otimes \mathcal{A}_{\Lambda_p,\Lambda_K}^{(1)} \to \mathcal{A}_{\Lambda_K,\Lambda_K}^{(0)}.$$

We will show that under the isomorphism Φ, μ maps to the Pontryagin product at the point p, which we now define.

Consider two chains of broken strings in $C_{k_1}^{Kp}(\Sigma_{\ell_1})$ and $C_{k_2}^{pK}(\Sigma_{\ell_2})$. We define their Pontryagin product at p as the concatenation at p followed by removing the path between the switches at p that precede and follow this concatenation. This gives a map

$$C_{k_1}^{Kp}(\Sigma_{\ell_1}) \otimes C_{k_2}^{pK}(\Sigma_{\ell_2}) \to C_{k_1+k_2}^{KK}(\Sigma_{\ell_1+\ell_2-2}).$$

Summing over integers k_1, k_2 and half-integers ℓ_1, ℓ_2, we get the Pontryagin product at p

$$P : \tilde{C}_*^{Kp} \otimes \tilde{C}_*^{pK} \to \tilde{C}_*^{KK}$$

which has degree -1.

We now treat the relation between P and μ. Note that $\Phi \colon \mathcal{A}_{\Lambda_K \cup \Lambda_p} \to \tilde{C}_*$ induces maps

$$\mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p} \to \tilde{C}^{Kp}_*, \qquad\qquad\qquad \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K} \to \tilde{C}^{pK}_*.$$

We claim that these Φ maps intertwine P and μ on the level of homology. This is not true on the chain level, but the difference can be measured by a map Ψ that we now define.

If a_1 and a_2 are Reeb chords to Λ_K from Λ_p and to Λ_p from Λ_K, respectively, then we write $\mathcal{M}^{\mathrm{sw}}(a_1, a_2)$ for the moduli space of holomorphic disks $u \colon D \to T^*Q$ that send positive punctures at 1 and -1 to a_1 and a_2 at infinity, the arc in the upper half plane connecting these punctures to L_p, and the arc in the lower half plane to $Q \cup L_K \cup L_p$. We can stratify $\mathcal{M}^{\mathrm{sw}}(a_1, a_2)$ by the number of switches that the boundary of a holomorphic disk has at p; for $\ell \geq 0$ even, write $\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)$ for the subset of $\mathcal{M}^{\mathrm{sw}}(a_1, a_2)$ corresponding to disks with ℓ switches at p. The formal dimension of this moduli space is, see Remark 2.9,

$$\dim(\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)) = |a_1| + |a_2| - \ell/2.$$

With notation as in Section 2.3, define a map

$$\Psi \colon \mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p} \otimes \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K} \to C^{KK}_*$$

as follows:

$$\Psi(\mathbf{b}_1 a_1 \otimes a_2 \mathbf{b}_2) = \sum_{|a_1| + |a_2| - \ell/2 = 0} \Phi(\mathbf{b}_1) \cdot [\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)] \cdot \Phi(\mathbf{b}_2),$$

where $[\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)]$ denotes the chain in $C^{KK}_0(\Sigma_\ell) \subset C^{KK}_{\ell/2} = \tilde{C}^{KK}_{\ell/2}$ carried by the moduli space (i.e., the chain of broken strings corresponding to the disks in $\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)$) and where $\cdot$ denotes the concatenation product: given broken strings in $\Phi(\mathbf{b}_1)$, $\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)$, and $\Phi(\mathbf{b}_2)$, we concatenate the three to obtain another broken string.

Remark 3.9. To see that $[\mathcal{M}^{\mathrm{sw}}_\ell(a_1, a_2)]$ is a chain of broken strings we use [CEL10, Theorem 1.2] which implies that there is a uniform bound of the number of switches on the boundary of a disk in any moduli space with two positive punctures.

Proposition 3.10. *On* $\mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p} \otimes \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K}$ *we have the following:*

$$\Phi \circ \mu - P \circ \Phi \otimes \Phi + \Psi \circ (1 \otimes \partial + \partial \otimes 1) - (\partial + \delta^K_Q + \delta^K_N + \delta^p_Q + \delta^p_{L_p}) \circ \Psi = 0.$$

Proof. To see this we note that the codimension one boundary $\partial \mathcal{M}^{\mathrm{sw}}(a_1, a_2)$ consists of the following breakings:

- Two-level disks with one level in the symplectization of dimension one and one level in the cotangent bundle. These are accounted for by the first, third, and fourth terms.
- Lagrangian intersection breaking at K, accounted for by the operations $\delta^K_Q + \delta^K_N$ in the last term.
- Lagrangian intersection breaking at p in the upper half disk, accounted for by the second term.

- Lagrangian intersection breaking at p in the lower half disk, accounted for by the operations $\delta_Q^p + \delta_{L_p}^p$ in the last term.

The formula follows. □

We can now assemble our results in low degree into the following result. Recall that $R_{KK} \cong H_0(\mathcal{A}_{\Lambda_K,\Lambda_K}^{(0)})$, $R_{Kp} \cong H_0(\mathcal{A}_{\Lambda_K,\Lambda_p}^{(1)})$, and $R_{pK} \cong H_1(\mathcal{A}_{\Lambda_p,\Lambda_K}^{(1)})$. From Proposition 3.6, we have an isomorphism $\Phi : R_{KK} \to H_0(\tilde{C}_*^{KK}, d)$. The following is now an immediate consequence of Proposition 3.10.

Proposition 3.11. *The following diagram commutes:*

$$
\begin{array}{ccc}
R_{Kp} \otimes R_{pK} & \xrightarrow{\mu} & R_{KK} \\
\downarrow{\scriptstyle \Phi\otimes\Phi} & & \downarrow{\scriptstyle \Phi} \\
H_0(\tilde{C}_*^{Kp}, d) \otimes H_1(\tilde{C}_*^{pK}, d) & \xrightarrow{P} & H_0(\tilde{C}_*^{KK}, d).
\end{array}
$$

4. Legendrian Contact Homology and the Knot Group

In this section, we use the isomorphism from Section 3 between Legendrian contact homology and string topology to write the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) defined in Section 2.2 in terms of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$. This will allow us to recover the knot group from the KCH-triple along with the product $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$. Along the way, we present the KCH-triple in terms of the cord algebra and deduce that enhanced knot contact homology encodes the Alexander module.

4.1. **String homology and the cord algebra.** From Proposition 3.6, we have isomorphisms between the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) and parts of the homology of the string complex $(\tilde{C}_*, d)$. As in [CELN], we can interpret this string homology in terms of the "cord algebra" of K, essentially by considering only the Q-strings. Here we give this cord algebra interpretation of string homology, which will allow us in Section 4.3 to rewrite the KCH-triple in terms of the knot group. The cord-algebra approach has the added benefit of readily yielding the Alexander module of the knot as the homology of a certain linearization of enhanced knot contact homology, as we will see.

We first review the cord algebra as presented in [CELN, §2.2], adapted to our purposes. Let $K \subset Q$ be an oriented knot and $p \in Q$ be a point in the knot complement, where $Q = \mathbb{R}^3$ as before. Let K' be a parallel copy of K in the Seifert framing, and choose a base point $*$ on K'.

Definition 4.1. A *cord* is a continuous map $\gamma : [0, 1] \to Q$ with $\gamma(0), \gamma(1) \in (K' \setminus \{*\}) \cup \{p\}$ and $\gamma([0, 1]) \cap K = \emptyset$. A cord is a KK (respectively Kp; pK; pp) cord if $\gamma(0), \gamma(1) \in K'$ (respectively $\gamma(0) \in K'$, $\gamma(1) = p$; $\gamma(0) = p$, $\gamma(1) \in K'$; $\gamma(0) = \gamma(1) = p$).

Definition 4.2 ([CELN])**.** The *cord algebra* of K, Cord_{KK}, is the noncommutative unital ring freely generated by homotopy classes of KK cords and $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, modulo the following skein relations, where the cord is drawn in red, K in black, and K' in gray:

(1) $= 1 - m$

(2) $= m \cdot$ and $=$ $\cdot m$

(3) $= l \cdot$ and $=$ $\cdot l$

(4) $-$ $=$ $\cdot$ $.$

On the Kp and pK cords, we can also impose the same skein relations: for example, (4) gives a relation between two Kp cords (the left hand side) and a product of a KK cord and a Kp cord (the right hand side).

Definition 4.3. The *Kp cord module* of K, Cord_{Kp}, is the right Cord_{KK}-module freely generated by Kp cords, modulo the skein relations (2), (3), and (4) from Definition 4.2. Similarly, the *pK cord module* of K, Cord_{pK}, is the left Cord_{KK}-module freely generated by pK, modulo the same skein relations.

Now a broken string in $C_0^{KK}(\Sigma_0)$ (respectively $C_0^{Kp}(\Sigma_1)$, $C_0^{pK}(\Sigma_1)$) produces an element of Cord_{KK} (respectively Cord_{Kp}, Cord_{pK}) given by the product of the Q-strings taken in order. This map induces maps from the cord algebra and modules to string homology, and as in [CELN] we can show that these maps are isomorphisms. Combined with Proposition 3.6, this shows that the cord algebra and modules are isomorphic to the KCH-triple (R_{KK}, R_{Kp}, R_{pK}), and this is the fact that we will exploit in this section to prove Theorem 1.1.

Proposition 4.4. *There are isomorphisms*

$$\mathrm{Cord}_{KK} \cong H_0(\tilde{C}_*^{KK}, d) \cong R_{KK}$$

$$\mathrm{Cord}_{Kp} \cong H_0(\tilde{C}_*^{Kp}, d) \cong R_{Kp}$$

$$\mathrm{Cord}_{pK} \cong H_1(\tilde{C}_*^{pK}, d) \cong R_{pK}$$

where the first line is a ring isomorphism, and the second and third lines send the left and right actions of Cord_{KK} to the left and right actions of R_{KK}. Under these isomorphisms, the map $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$ is the concatenation map

$$\mathrm{Cord}_{Kp} \otimes \mathrm{Cord}_{pK} \to \mathrm{Cord}_{KK}.$$

Proof. The first line is proved in Proposition 2.9 of [CELN], and the other two lines have the same proof. The fact that these isomorphisms preserve multiplication follows formally from

the construction of the cord algebra and modules. The description of μ as a concatenation product is a direct consequence of Proposition 3.11. $\qquad\square$

4.2. Enhanced knot contact homology and the Alexander module.

Here we digress from the main argument to observe that we can use the cord modules to recover the Alexander module $H_1(\tilde{X}_K)$ of K, where $\tilde{X}_K$ is the infinite cyclic cover of $\mathbb{R}^3 \setminus K$ and $H_1(\tilde{X}_K)$ is viewed as a $\mathbb{Z}[m^{\pm 1}]$-module as usual by deck transformations. As a consequence, we show that a certain canonical linearization of enhanced knot contact homology contains the Alexander module and thus the Alexander polynomial.

It was previously known [Ng08] that the Alexander module can be extracted from the same linearization of usual knot contact homology $\mathrm{LCH}_*(\Lambda_K)$, but in a somewhat obscure way— essentially, the degree 1 linearized homology is the second tensor product of $H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}]$, with the proof involving an examination of the combinatorial form of the DGA of Λ_K in terms of a braid representative for K, and a relation to the Burau representation. Here we will see that with the introduction of the fiber Λ_p alongside Λ_K, we can instead deduce the Alexander module in a significantly simpler way.

We first present a variant of the cord algebra and modules, following [Ng08] and especially the discussion in [CELN, §2.2]. Choose a base point $*$ on K corresponding to the base point $*$ on K'. Let an *unframed cord* of K be a path whose endpoints are in $(K \setminus \{*\}) \cup \{p\}$ and which is disjoint from K in its interior; we can divide these into KK, Kp, pK, pp cords depending on where the endpoints lie.

Definition 4.5 ([Ng08])**.** The *unframed cord algebra* of K, Cord'_{KK}, is the noncommutative algebra over $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ generated by homotopy classes of unframed KK cords, modulo the following skein relations:
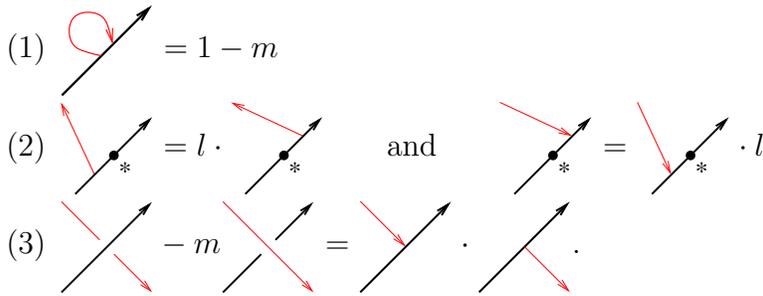
The *unframed Kp (respectively pK) cord module* of K, Cord'_{Kp} (respectively Cord'_{pK}), is the right (respectively left) Cord'_{KK}-module generated by unframed Kp (respectively pK) cords, modulo the skein relations (2) and (3).

Note that Cord'_{KK}, Cord'_{Kp}, and Cord'_{pK} are all $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$-modules, unlike their framed counterparts Cord_{KK}, Cord_{Kp}, Cord_{pK}, where elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ do not necessarily commute with cords. However, we have the following.

Proposition 4.6. *The unframed cord algebra and modules* Cord'_{KK}, Cord'_{Kp}, Cord'_{pK} *are isomorphic to the quotients of the cord algebra and modules* Cord_{KK}, Cord_{Kp}, Cord_{pK} *obtained by imposing the relations that elements of* $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ *commute with cords.*

Proof. This is essentially laid out in [CELN, §2.2]. Fix a cord γ_0 from p to a point $x_0 \in K' \setminus \{*\}$. Given any cord γ, we can produce a loop $\tilde{\gamma}$ in $\mathbb{R}^3 \setminus K$ based at p, by joining any endpoint of γ on K' to x_0 along K', and appending γ_0 or $-\gamma_0$ as necessary. Let γ' be the unframed cord obtained from γ by joining any endpoint of γ on K' to the corresponding point on K by a straight line segment normal to K. Then the map

$$\gamma \mapsto m^{-\operatorname{lk}(\tilde{\gamma}, K)} \gamma'$$

gives the desired isomorphisms from the quotients of Cord_{KK}, Cord_{Kp}, Cord_{pK} to Cord'_{KK}, Cord'_{Kp}, Cord'_{pK}. Note that this map sends the skein relations (1), (3), (4) in Definition 4.2 to (1), (2), (3) in Definition 4.5, and the normalization by powers of m means that (2) from Definition 4.2 becomes trivial under this map. $\square$

Now from [Ng08], there is a canonical augmentation of the DGA for K,

$$\epsilon : (\mathcal{A}_{\Lambda_K}, \partial) \to (\mathbb{Z}[m^{\pm 1}], 0),$$

which can be defined simply by its action on Cord'_{KK}, which by Proposition 4.6 is the quotient of $H_0(\mathcal{A}_{\Lambda_K}, \partial)$ by setting l, m to commute with everything:

$$\epsilon(l) = 1$$
$$\epsilon(m) = m$$
$$\epsilon(\gamma) = 1 - m$$

for any unframed KK cord γ. (Note that ϵ preserves the skein relations for Cord'_{KK} and is thus well-defined.) We can extend ϵ to an augmentation of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ by setting ϵ to be 0 for any mixed chord between Λ_K and Λ_p.

Remark 4.7. Applying [AENV14, Theorem 6.15] to the holomorphic strips over binormal chords shows that the augmentation ϵ is induced by an exact Lagrangian filling M_K diffeomorphic to the knot complement, obtained by joining the conormal L_K and the zero-section Q via Lagrange surgery along the knot K.

Linearizing with respect to this augmentation gives the linearized contact homology

$$\mathrm{LCH}^\epsilon_*(\Lambda_K \cup \Lambda_p) = (\mathrm{LCH}^\epsilon_*)_{\Lambda_K, \Lambda_K} \oplus (\mathrm{LCH}^\epsilon_*)_{\Lambda_K, \Lambda_p} \oplus (\mathrm{LCH}^\epsilon_*)_{\Lambda_p, \Lambda_K} \oplus (\mathrm{LCH}^\epsilon_*)_{\Lambda_p, \Lambda_p}.$$

As discussed previously, in [Ng08] it is shown that $(\mathrm{LCH}^\epsilon_1)_{\Lambda_K, \Lambda_K}$ recovers the Alexander module $H_1(\tilde{X}_K)$. Here instead we have the following.

Proposition 4.8. *We have isomorphisms of* $\mathbb{Z}[m^{\pm 1}]$-*modules*

$$(\mathrm{LCH}^\epsilon_0)_{\Lambda_K, \Lambda_p} \cong (\mathrm{LCH}^\epsilon_1)_{\Lambda_p, \Lambda_K} \cong H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}].$$
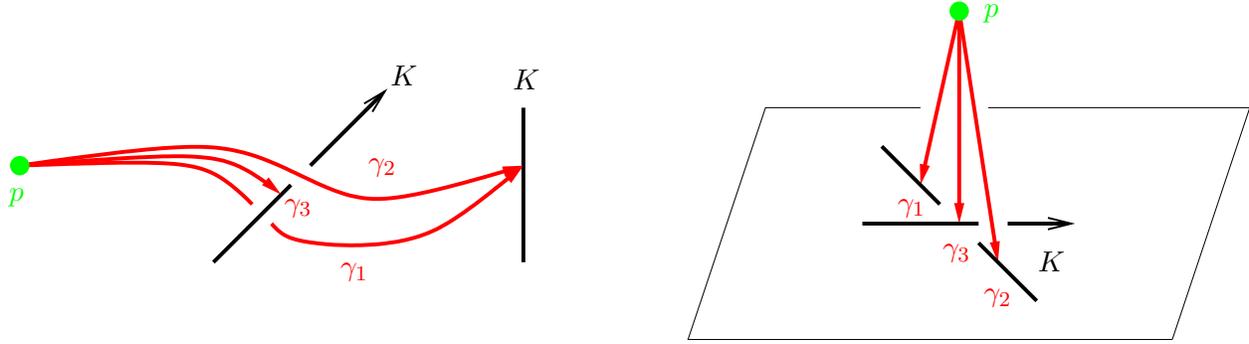
FIGURE 6. Three pK cords $\gamma_1, \gamma_2, \gamma_3$ related by $\gamma_1 - m\gamma_3 = (1-m)\gamma_2$.

Proof. We will prove the isomorphism for $(\mathrm{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$; the isomorphism for $(\mathrm{LCH}_0^\epsilon)_{\Lambda_K, \Lambda_p}$ follows by symmetry between Cord'_{Kp} and Cord'_{pK}. The complex whose homology computes $(\mathrm{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$ is the free $\mathbb{Z}[m^{\pm 1}]$-module generated by Reeb chords to Λ_p from Λ_K, with the differential given by applying the augmentation ϵ to all pure Reeb chords from Λ_K to itself to the usual differential ∂. Thus we have:

$$(\mathrm{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K} \cong R_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}] \cong \mathrm{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}].$$

Here by "$\otimes_\epsilon$" we mean $\otimes_{R_{KK}}$ (or $\otimes_{\mathrm{Cord}'_{KK}}$) where we use ϵ to give $\mathbb{Z}[m^{\pm 1}]$ the structure of an R_{KK}-module (or Cord'_{KK}-module), and implicitly we are setting l, m to commute with everything in $(\mathrm{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$ and R_{pK}.

Now $\mathrm{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}]$ is the quotient of the free $\mathbb{Z}[m^{\pm 1}]$-module generated by unframed pK cords by the skein relations (2) and (3) from Definition 4.5, where l is sent to 1 and all KK cords are sent to $1 - m$. Relation (2) then says that pK cords are unchanged if we move their K endpoint over $*$, while relation (3) becomes:

That is, if $\gamma_1, \gamma_2, \gamma_3$ are unframed pK cords that are related as shown in the left side of Figure 6, then we impose the relation:

$$\gamma_1 - m\gamma_2 = (1-m)\gamma_3.$$

Thus we can describe $\mathrm{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}]$ in terms of a knot diagram for K as follows. Use the diagram to place K in a neighborhood of the xy plane in $\mathbb{R}^3$, and place p high above the xy plane along the z axis. If the diagram has n crossings, then it divides K into n strands from undercrossing to undercrossing. Then $\mathrm{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}]$ is generated by n unframed pK cords, namely straight line segments from p to any point on each of these strands, and each crossing gives a relation $\gamma_1 - m\gamma_2 = (1-m)\gamma_3$ if $\gamma_1, \gamma_2, \gamma_3$ are as shown in the right side of Figure 6. But this is the well-known presentation of $H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}]$ from knot

colorings. In particular, what we have just described is the Alexander quandle of K, see [Joy82]. □

Remark 4.9. The description we have given in this section for the unframed cord modules is highly reminiscent of the construction of the knot quandle from [Joy82], which is known to be a complete invariant. However, we do not know how to extract the entire knot quandle, rather than just the Alexander quandle (which is a quotient), from the unframed cord module.

4.3. **String homology in terms of the knot group.** Having expressed the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) in terms of cords in Section 4.1, our next step en route to proving Theorem 1.1 is to rewrite the KCH-triple further, in terms of the knot group and the peripheral subgroup of the knot K. For R_{KK}, which is the degree 0 knot contact homology of K, this was done in [CELN, §2.3–2.4], and we follow the treatment there.

Write $\pi = \pi_1(\mathbb{R}^3 \setminus K)$ for the knot group and $\hat{\pi} = \pi_1(\Lambda_K)$ for the peripheral subgroup. A framing and orientation on K gives meridian and longitude classes $m, l \in \hat{\pi}$, which we can then view as classes in π as well. In what follows, we place square brackets around elements of π and curly brackets around elements of $\hat{\pi}$.

Define S to be the $\mathbb{Z}$-module freely generated by words that are formal products of nontrivial words whose letters are alternately in π and $\hat{\pi}$, divided by the following string relations:

(1) $\cdots_1 [x\alpha_1]\{\alpha_2\}\cdots_2 = \cdots_1 [x]\{\alpha_1\alpha_2\}\cdots_2$
(2) $\cdots_1 \{\alpha_1\}[\alpha_2 x]\cdots_2 = \cdots_1 \{\alpha_1\alpha_2\}[x]\cdots_2$
(3) $(\cdots_1 [x_1 x_2]\cdots_2) - (\cdots_1 [x_1 m x_2]\cdots_2) = \cdots_1 [x_1]\{1\}[x_2]\cdots_2$
(4) $(\cdots_1 \{\alpha_1\alpha_2\}\cdots_2) - (\cdots_1 \{\alpha_1 m\alpha_2\}\cdots_2) = \cdots_1 \{\alpha_1\}[1]\{\alpha_2\}\cdots_2.$

Note that there is no restriction on generators of S as to whether the first or last letters are in π or $\hat{\pi}$. We can define a product on S as follows: multiplication of two words w_1, w_2 generating S is zero unless the last letter of w_1 and the first letter of w_2 are both in π or both in $\hat{\pi}$, in which case it is concatenation combined with the product in π or $\hat{\pi}$; that is,

$$(\cdots_1 \{\alpha_1\}) \cdot (\{\alpha_2\}\cdots_2) = \cdots_1 \{\alpha_1\alpha_2\}\cdots_2$$
$$(\cdots_1 [x_1]) \cdot ([x_2]\cdots_2) = \cdots_1 [x_1 x_2]\cdots_2.$$

We now have the following result identifying R_{KK}, R_{Kp}, R_{pK} from Section 3.1 with summands of S.

Proposition 4.10. *R_{KK}, R_{Kp}, and R_{pK} are isomorphic to the $\mathbb{Z}$-submodules of S generated by the following sets:*

- *for R_{KK}, words beginning and ending in $\hat{\pi}$;*
- *for R_{Kp}, words beginning in $\hat{\pi}$ and ending in π;*
- *for R_{pK}, words beginning in π and ending in $\hat{\pi}$.*

Multiplication in S induces maps $R_{KK} \otimes R_{KK} \to R_{KK}$, $R_{KK} \otimes R_{Kp} \to R_{Kp}$, $R_{pK} \otimes R_{KK} \to R_{pK}$ that agree with, respectively, the ring structure on R_{KK} and the R_{KK}-module structure on R_{Kp} and R_{pK}.

Proof. Same as the proof of [CELN, Proposition 2.14]. Briefly, by Proposition 4.4, the KCH-triple is isomorphic to $(\mathrm{Cord}_{KK}, \mathrm{Cord}_{Kp}, \mathrm{Cord}_{pK})$. Given a cord, we can produce a closed loop in $\mathbb{R}^3 \setminus K$ based at p, and hence an element of π, as in the discussion at the beginning of Section 4.2. Thus products of cords, with elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ in between, correspond to alternating products of elements of π and $\hat{\pi}$. The string relations on S come from the skein relations on cords. □

To clarify:

- R_{KK} is generated by $\{\alpha_1\}$, $\{\alpha_1\}[x_1]\{\alpha_2\}$, $\{\alpha_1\}[x_1]\{\alpha_2\}[x_2]\{\alpha_3\}$, . . .
- R_{Kp} is generated by $\{\alpha_1\}[x_1]$, $\{\alpha_1\}[x_1]\{\alpha_2\}[x_2]$, . . .
- R_{pK} is generated by $[x_1]\{\alpha_1\}$, $[x_1]\{\alpha_2\}[x_2]\{\alpha_2\}$, . . .

where $\alpha_i \in \hat{\pi}$ and $x_i \in \pi$. To this, we can then add:

- $R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}$ is generated by $[x_1]\{\alpha_1\}[x_2]$, $[x_1]\{\alpha_1\}[x_2]\{\alpha_2\}[x_3]$, ...

Finally, the product $\mu : R_{Kp} \otimes R_{pK} \to R_{KK}$ has a simple interpretation in terms of S, since by Proposition 4.4 it is the concatenation product:

$$\mu(\{\alpha_0\} \cdots \{\alpha_1\}[x_1], [x_2]\{\alpha_2\} \cdots \{\alpha_3\}) = \{\alpha_0\} \cdots \{\alpha_1\}[x_1 x_2]\{\alpha_2\} \cdots \{\alpha_3\}.$$

The product on $R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}$ induced by μ is then also given by concatenation.

4.4. The KCH-triple within $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

Although the notation from Section 4.3 using square and curly brackets is natural from the viewpoint of broken strings, it will be convenient for our purposes to reinterpret the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) directly in terms of the group ring of the knot group, which we henceforth denote by

$$R := \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)].$$

This is the content of Proposition 4.13 below.

To prepare for this result, extend the notation $\cdots [x_1]\{\alpha_1\}[x_2] \cdots \in S$, where up to now we have $x_i \in \pi$ and $\alpha_i \in \hat{\pi}$, by linearity to allow for arbitrary $x_i \in R = \mathbb{Z}\pi$. Given any element of S of the form $\cdots [x_1]\{\alpha_1\}[x_2] \cdots$ with $\alpha_i \in \hat{\pi}$ and $x_i \in \mathbb{Z}\pi$, the string relations on S allow us to get rid of any internal part in curly braces, where "internal" means not at the far left or far right. More precisely, by (1) and (3) from the defining relations for S in Section 4.3, we can write:

$$\cdots [x_1]\{\alpha_1\}[x_2] \cdots = \cdots [x_1 \alpha_1 x_2] \cdots - \cdots [x_1 \alpha_1 m x_2] \cdots .$$

This allows us to inductively reduce the number of internal curly braces until none are left.

Thus for instance we can write any element of R_{Kp} as a linear combination of elements of the form $\{\alpha_1\}[x_1]$, where $x_1 \in \mathbb{Z}\pi$, and this in turn is equal to $\{1\}[\alpha_1 x_1]$ by string relation (2). Similar results hold for R_{pK} and R_{KK}, as well as for R_{pp}, and we conclude the following:

Proposition 4.11. *As $\mathbb{Z}$-submodules of S, we have:*

- *R_{KK} is generated by the elements of the form $\{\alpha\}$ and $\{1\}[x]\{1\}$ for $\alpha \in \hat{\pi}$ and $x \in \pi$;*
- *R_{Kp} is generated by $\{1\}[x]$ for $x \in \pi$;*
- *R_{pK} is generated by $[x]\{1\}$ for $x \in \pi$;*
- *R_{pp} is generated by $[x_1]\{1\}[x_2]$ for $x_1, x_2 \in \pi$.*

Write $\hat{R} = \mathbb{Z}[\hat{\pi}] = \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, and view $\hat{R}$ as a subring of R. In [CELN, Proposition 2.20], it is shown that the map $\{\alpha\} \mapsto \alpha$, $\{1\}[x]\{1\} \mapsto x(1 - m)$ induces an isomorphism from R_{KK} to $\hat{R} + R(1 - m)$, where the latter is viewed as a subring of R (and $R(1 - m)$ is the left ideal generated by $1 - m$).

Remark 4.12. To be precise, the map in [CELN, Proposition 2.20] is from R_{KK} to $\hat{R} + (1 - m)R$ rather than $\hat{R} + R(1 - m)$, and sends $\{1\}[x]\{1\}$ to $(1 - m)x$ rather than $x(1 - m)$. This however is just a choice of where to place the $(1 - m)$ factors. Our convention can be derived from the convention in [CELN] by the symmetry that reverses the order of words in S.

We can now generalize this isomorphism to the entire KCH-triple.

Proposition 4.13. *We have $\mathbb{Z}$-module isomorphisms between the KCH-triple and the following $\mathbb{Z}$-submodules of $R = \mathbb{Z}[\pi]$:*

$$R_{KK} \xrightarrow{\cong} \hat{R} + R(1 - m) \qquad \{\alpha\} \mapsto \alpha\,, \ \{1\}[x]\{1\} \mapsto x(1 - m);$$

$$R_{Kp} \xrightarrow{\cong} R \qquad \{1\}[x] \mapsto x;$$

$$R_{pK} \xrightarrow{\cong} R(1 - m) \qquad [x]\{1\} \mapsto x(1 - m),$$

where the second and third isomorphisms hold for any knot K and the first isomorphism holds as long as K is not the unknot. We use ϕ to denote all of these isomorphisms; then it is furthermore the case that ϕ sends all multiplications $R_{KK} \otimes R_{KK} \to R_{KK}$, $R_{KK} \otimes R_{Kp} \to R_{Kp}$, $R_{pK} \otimes R_{KK} \to R_{pK}$, as well as the product $R_{Kp} \otimes R_{pK} \to R_{KK}$, to multiplication in R.

Proof. This follows the proof of [CELN, Proposition 2.20]. To see that ϕ is well-defined, extend the definition of ϕ to all generators of S (ignoring Proposition 4.11 for the moment)

by:

$$\{\alpha_1\}[x_1]\{\alpha_2\}\cdots\{\alpha_{k-1}\}[x_k]\{\alpha_k\} \mapsto \alpha_1 x_1 \alpha_2 (1-m)\cdots\alpha_{k-1}(1-m)x_k\alpha_k$$

$$\{\alpha_1\}[x_1]\{\alpha_2\}\cdots\{\alpha_{k-1}\}[x_k] \mapsto \alpha_1 x_1 \alpha_2 (1-m)\cdots\alpha_{k-1}(1-m)x_k$$

$$[x_1]\{\alpha_1\}[x_2]\cdots[x_k]\{\alpha_k\} \mapsto x_1 \alpha_1 (1-m)x_2\cdots x_k\alpha_k(1-m)$$

$$[x_1]\{\alpha_1\}[x_2]\cdots[x_k] \mapsto x_1 \alpha_1 (1-m)x_2\cdots x_k;$$

that is, replace each term $\{\alpha\}$ in curly braces by $\alpha(1-m)$ unless $\{\alpha\}$ is at the end of a word, in which case replace it by α. It is easily checked that this map preserves the string relations on S, and so it gives a well-defined map $\phi : S \to R$. Restricted to generators of the form $\{\alpha\}$, $\{1\}[x]\{1\}$, $\{1\}[x]$, $[x]\{1\}$, ϕ is as given in the statement of the proposition; note now by Proposition 4.11 that these suffice to determine ϕ.

We next check bijectivity. The maps ϕ are clearly surjective. It is proved in [CELN, Proposition 2.20] that ϕ on R_{KK} is injective as long as K is knotted. The fact that ϕ is injective on R_{Kp} is trivial: by Proposition 4.11, any element of R_{Kp} can be written as $\{1\}[x]$ for some $x \in R$, and then $\phi(\{1\}[x]) = 0$ implies $x = 0$. To prove that ϕ is injective on R_{pK}, note that if $\phi([x]\{1\}) = 0$, then $x(1-m) = 0$ in R; then since knot groups are left orderable, $R = \mathbb{Z}\pi$ has no zero divisors, and so $x = 0$.

Finally, the fact that ϕ respects multiplication and μ follows readily from the definition of ϕ: for example,

$$\phi(\mu(\{\alpha_0\}\cdots\{\alpha_1\}[x_1], [x_2]\{\alpha_2\}\cdots\{\alpha_3\})) = \phi(\{\alpha_0\}\cdots\{\alpha_1\}[x_1 x_2]\{\alpha_2\}\cdots\{\alpha_3\})$$

$$= \alpha_0(1-m)\cdots\alpha_1(1-m)x_1 x_2 \alpha_2(1-m)\cdots\alpha_3$$

$$= \phi(\{\alpha_0\}\cdots\{\alpha_1\}[x_1]) \cdot \phi([x_2]\{\alpha_2\}\cdots\{\alpha_3\}).$$

$\square$

Remark 4.14. In Section 6.5 we give an interpretation of the maps in Proposition 4.13 in terms of moduli spaces of holomorphic disks inducing maps from partially wrapped Floer homology into chains on spaces of paths and loops in $\mathbb{R}^3 \setminus K$.

Recall that the product μ gives a ring structure on $R_{pp} = R_{pK} \otimes R_{Kp}$. Similarly to Proposition 4.13, we then have the following.

Proposition 4.15. *The isomorphisms $\phi : R_{pK} \xrightarrow{\cong} R$ and $R_{Kp} \xrightarrow{\cong} R(1-m)$ induce a ring isomorphism*

$$\phi : R_{pp} \xrightarrow{\cong} R(1-m)R$$

where $R(1-m)R$ denotes the two-sided ideal of R generated by $1-m$.

Proof. Same as the proof of Proposition 4.13, but now use the fact that ϕ is defined on R_{pp} by

$$[x_1]\{\alpha_1\}[x_2]\cdots\{\alpha_{k-1}\}[x_k] \mapsto x_1 \alpha_1 (1-m)x_2\cdots\alpha_{k-1}(1-m)x_k$$

and that any element of R_{pp} can be written as $[x]$ for some $x \in R$. $\qquad\square$

Although $R(1-m)R$ is not all of R, we observe the following.

Proposition 4.16. *The submodules $\mathbb{Z} = \mathbb{Z} \cdot 1$ and $R(1-m)R$ of R are complementary: $R \cong \mathbb{Z} \oplus R(1-m)R$ as $\mathbb{Z}$-modules.*

Proof. The homomorphism from π to the trivial group induces a map $R = \mathbb{Z}[\pi] \to \mathbb{Z}$, which restricts to the identity on $\mathbb{Z}$ and to the zero map on $R(1-m)R$; thus $\mathbb{Z} \cap R(1-m)R = 0$. On the other hand, $\pi = \pi_1(\mathbb{R}^3 \setminus K)$ is generated by a finite collection of meridians, each of which is an element of π of the form $\gamma m \gamma^{-1}$ for some $\gamma \in \pi$, and $\gamma m \gamma^{-1} = 1 - \gamma(1-m)\gamma^{-1} \in \mathbb{Z} + R(1-m)R$; thus $\mathbb{Z} + R(1-m)R = R$. $\qquad\square$

We can restate the combination of Propositions 4.15 and 4.16 as follows. Consider the direct sum $\mathbb{Z} \oplus R_{pp}$, and give this a ring structure by defining the generator 1 of $\mathbb{Z}$ to be the multiplicative identity and setting multiplication on the factor R_{pp} to be as usual. That is:

$$(n_1, r_1) \cdot (n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2).$$

Then:

Proposition 4.17. *We have a ring isomorphism $\phi: \mathbb{Z} \oplus R_{pp} \overset{\cong}{\to} R$ defined by*

$$\phi(n, [x_1]\{\alpha_1\}[x_2]) = n + x_1 \alpha_1 (1-m) x_2.$$

Remark 4.18. The ring $\mathbb{Z} \oplus R_{pp}$ may seem like an odd candidate to be isomorphic to $R = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. In fact, it is the correct object to consider from at least two perspectives. One is through the cord algebra and modules: we have $R_{pp} \cong \mathrm{Cord}_{pK} \otimes \mathrm{Cord}_{Kp}$, and we can use the skein relation for cords (Definition 4.2 (4)) to rewrite the product of a pK cord with a Kp cord as the difference of two pp cords. This gives a map

$$\mathrm{Cord}_{pK} \otimes \mathrm{Cord}_{Kp} \to \mathrm{Cord}_{pp}.$$

This map is not surjective, since it maps to the ideal generated by differences of cords, but it becomes an isomorphism if we add the $\mathbb{Z}$-module generated by the trivial pp cord to the left hand side to obtain $\mathbb{Z} \oplus R_{pp}$. On the other hand, any pp cord is a loop in $\mathbb{R}^3 \setminus K$, and this induces an isomorphism between Cord_{pp} and $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

The other perspective is through partially wrapped Floer homology; see Section 6.4.

5. Proof of Theorem 1.1

Here we present the proof of our main result. Briefly, by Proposition 4.17, we can recover $R = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ from the ring structure on R_{pp}, which itself is determined by the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$ along with the product μ. This allows us to recover the knot K itself. The details of the proof are broken into two subsections corresponding to statements (1) and (2) from Theorem 1.1, and at the end of this section we make some remarks about the difference between the two.

5.1. **Proof of Theorem 1.1 (2).** Suppose that K_0, K_1 are knots such that there are isomorphisms between the KCH-triples of K_0 and K_1; these isomorphisms are compatible with the products $\mu : R_{K_ip} \otimes R_{pK_i} \to R_{K_iK_i}$; and the isomorphism $R_{K_0K_0} \overset{\cong}{\to} R_{K_1K_1}$ sends the meridian and longitude m_0, l_0 of K_0 to the meridian and longitude m_1, l_1 of K_1, respectively. Under these assumptions, we want to conclude that K_0, K_1 are isotopic in $\mathbb{R}^3$ as oriented knots. Since R_{KK} detects the unknot (see e.g. [CELN, Corollary 1.5]), we will assume that K_0, K_1 are both knotted.

For $i = 0, 1$, write $R_i = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K_i)]$ and $\hat{R}_i = \mathbb{Z}[m_i^{\pm 1}, l_i^{\pm 1}] \subset R_i$. By Proposition 4.13, we can write the isomorphisms between the KCH-triples $(R_{K_0K_0}, R_{K_0p}, R_{pK_0})$ and $(R_{K_1K_1}, R_{K_1p}, R_{pK_1})$ as a triple of maps

$$(\psi_{KK}, \psi_{Kp}, \psi_{pK}) : (\hat{R}_0 + R_0(1-m_0), R_0, R_0(1-m_0)) \overset{\cong}{\to} (\hat{R}_1 + R_1(1-m_1), R_1, R_1(1-m_1)).$$

These maps are compatible with multiplication and the product μ in the way described in Proposition 4.13.

Write $R_{pp}^i = R_{pK_i} \otimes_{R_{K_iK_i}} R_{K_ip}$; then the map $\psi_{pK} \otimes \psi_{Kp}$ is an isomorphism from R_{pp}^0 to R_{pp}^1, and it preserves the ring structure on R_{pp}^i determined by μ. By Proposition 4.17, we can then view

$$\psi_{pp} := \text{id} \oplus (\psi_{pK} \otimes \psi_{Kp}) : \mathbb{Z} \oplus R_{pp}^0 \overset{\cong}{\to} \mathbb{Z} \oplus R_{pp}^1$$

as a ring isomorphism $\psi_{pp} : R_0 \overset{\cong}{\to} R_1$.

Now $R_i = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K_i)]$ and knot groups are left-orderable, and so any isomorphism between R_0 and R_1 must come from a group isomorphism between the knot groups. More precisely, the set of units in the group ring $\mathbb{Z}[G]$ of a left-orderable group G are exactly the elements $\pm g$ for $g \in G$, and so there exists an isomorphism

$$\psi : \pi_1(\mathbb{R}^3 \setminus K_0) \overset{\cong}{\to} \pi_1(\mathbb{R}^3 \setminus K_1)$$

such that $\psi_{pp}(\gamma) = \pm\psi(\gamma)$ for all $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_0)$.

Now that we know that the knot groups of K_0 and K_1 are isomorphic, it remains to show that the isomorphism preserves the peripheral structure. We will show that there is some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$ such that $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} l_1 \gamma$, whence the composition of ψ and conjugation by γ gives an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \overset{\cong}{\to} \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0, l_0 to m_1, l_1 as desired.

For this, we use the assumption that $\psi_{KK}(m_0) = m_1$ and $\psi_{KK}(l_0) = l_1$. The elements $1 \in R_0 \cong R_{K_0p}$ and $1 - m_0 \in R_0(1 - m_0) \cong R_{pK_0}$ have images under ψ_{Kp} and ψ_{pK}

$$\psi_{Kp}(1) = x \qquad\qquad \psi_{pK}(1 - m_0) = x'(1 - m_1)$$

for some $x, x' \in R_1$. In R_1, we have

$$1 - m_1 = \psi_{KK}(1 - m_0) = \psi_{KK}\mu(1, 1 - m_0)$$
$$= \mu(\psi_{Kp}(1), \psi_{pK}(1 - m_0)) = \mu(x, x'(1 - m_1)) = xx'(1 - m_1).$$

Now the group ring of a left-orderable group has no zero divisors, and so it follows that $xx' = 1 \in R_1$ and hence there exists some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$ such that $x = \pm\gamma$, $x' = \pm\gamma^{-1}$.

Next, for $\alpha \in \mathbb{Z}$, view $l_0^\alpha(1-m_0) = (1-m_0) \cdot l_0^\alpha \cdot 1$ as an element of $R_{pK_0} \otimes_{R_{K_0 K_0}} R_{K_0 K_0} \otimes_{R_{K_0 K_0}} R_{K_0 p} = R_{pp}^0$. Then we have

$$\psi_{pp}(l_0^\alpha(1-m_0)) = \psi_{pK}(1-m_0) \cdot \psi_{KK}(l_0^\alpha) \cdot \psi_{Kp}(1) = \gamma^{-1}(1-m_1) \cdot l_1^\alpha \cdot \gamma.$$

Now since $\psi_{pp} = \pm\psi$ on elements of $\pi_1(\mathbb{R}^3 \setminus K_0)$, both of $\psi_{pp}(l_0^\alpha(1-m_0))$ and $\gamma^{-1}(1-m_1) \cdot l_1^\alpha \cdot \gamma$ are binomials, and equating terms gives

$$\{\psi(l_0^\alpha), \psi(m_0 l_0^\alpha)\} = \{\gamma^{-1} l_1^\alpha \gamma, \gamma^{-1} m_1 l_1^\alpha \gamma\}.$$

Plugging in $\alpha = 0$ and $\alpha = 1$ in succession gives $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} l_1 \gamma$, as desired.

5.2. Proof of Theorem 1.1 (1).

We now consider the case where there is an isomorphism between the KCH-triples of K_0 and K_1 preserving the product μ, but without the assumption that longitude and meridian classes are mapped to themselves. We will prove that there is an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \cong \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0 to $m_1^{\pm 1}$ and l_0 to $l_1^{\pm 1}$. It then follows from Waldhausen [Wal68] that K_0 is smoothly isotopic to either K_1 or the mirror of K_1, as unoriented knots. (Note that our argument identifies not only the peripheral subgroups but also the meridians in each, and so we do not need to appeal to Gordon and Luecke [GL89].) As in the previous proof, we can assume that K_0 and K_1 are both knotted.

The setup is as in Section 5.1, except that $\psi_{KK}(m_0) = m_1^{n_1} l_1^{n_2}$ and $\psi_{KK}(l_0) = m_1^{n_3} l_1^{n_4}$ for some $\left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ not necessarily the identity matrix. As in the previous proof, ψ_{pp} is induced by a group isomorphism $\psi : \pi_1(\mathbb{R}^3 \setminus K_0) \to \pi_1(\mathbb{R}^3 \setminus K_1)$. Now however when we write $\psi_{Kp}(1) = x$, $\psi_{pK}(1-m_0) = x'(1-m_1)$ for $x, x' \in R_1$, we have

$$(2) \qquad\qquad\qquad xx'(1-m_1) = 1 - m_1^{n_1} l_1^{n_2}.$$

We can then appeal to the following algebraic result.

Lemma 5.1. *Suppose G is left orderable and $m, g \in G$, $z \in \mathbb{Z}[G]$ satisfy $z(1-m) = 1-g$ in $\mathbb{Z}[G]$. Then $g = m^n$ for some $n \in \mathbb{Z}$.*

Proof. Let $<$ be the left-invariant ordering on G. Without loss of generality, we may assume $m > 1$; the lemma is trivial if $m = 1$, and we can replace m by m^{-1} if $m < 1$. Write $z = \sum_{i=1}^k a_i g_i$ for $a_i \in \mathbb{Z}$ and $g_i \in G$; we may assume that $a_i \neq 0$ for all i and $g_1 < g_2 < \cdots < g_k$. Then $g_i < g_i m$ for all i, and so among the subset $S = \{g_1, \ldots, g_k, g_1 m, \ldots, g_k m\}$ of G (which may contain repeated elements), g_1 is strictly lowest. Furthermore, let $g_j m$ be the largest among the k distinct elements $g_1 m, \ldots, g_k m$; then $g_j m$ is strictly largest among the elements of S. It follows that the expansion of $z(1-m) = \sum_{i=1}^k (a_i g_i - a_i g_i m)$ involves g_1 and $g_j m$ at least. Since $1-g$ is a binomial, it follows that $\{g_1, g_j m\} = \{1, g\}$, and furthermore that all terms in the expansion not involving g_1 or $g_j m$ must cancel. Thus $g_1 m$ must be canceled by

g_{i_1} for some i_1, whence $g_{i_1}m$ must be canceled by g_{i_2} for some i_2, and so forth. We conclude that there is a sequence $i_0 = 1, i_1, \ldots, i_\ell = j$ such that $g_{i_r} = g_{i_{r-1}}m$ for all $r = 1, \ldots, \ell$, and so $g_j = g_1 m^{r-1}$. The lemma follows. $\qquad\square$

We now continue with the proof of Theorem 1.1 (1). By Equation (2) and Lemma 5.1, $m_1^{n_1} l_1^{n_2} = m_1^n$ for some n. By the Loop Theorem, since K_2 is knotted, l_1 is not a power of m_1, and so $n_2 = 0$. Since $\left(\begin{smallmatrix} n_1 & n_2 \\ n_3 & n_4 \end{smallmatrix}\right)$ is invertible, it follows that $n_1 = \pm 1$ and $n_4 = \pm 1$.

We treat the cases $n_1 = 1$ and $n_1 = -1$ separately. If $n_1 = 1$, then $\psi_{KK}(m_0) = m_1$ and $\psi_{KK}(l_0) = m_1^{a_3} l_1^{\pm 1}$. As in Section 5.1, we have $xx'(1 - m_1) = 1 - m_1$ and so $x = \pm\gamma$, $x' = \pm\gamma^{-1}$ for some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$. Now for any $\alpha \in \mathbb{Z}$, we compute

$$\psi_{pp}(l_0^\alpha(1 - m_0)) = \psi_{pK}(1 - m_0) \cdot \psi_{KK}(l_0^\alpha) \cdot \psi_{Kp}(1) = \gamma^{-1} m_1^{\alpha a_3} l_1^{\pm\alpha}(1 - m_1)\gamma.$$

Identifying terms as in Section 5.1 gives

$$\{\psi(l_0)^\alpha, \psi(m_0)\psi(l_0)^\alpha\} = \{\gamma^{-1} m_1^{\alpha n_3} l_1^{\pm\alpha}\gamma, \gamma^{-1} m_1^{\alpha n_3 + 1} l_1^{\pm\alpha}\gamma\}.$$

Plugging in $\alpha = 0$ and $\alpha = 1$ in succession gives $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} m_1^{n_3} l_1^{\pm 1}\gamma$. Conjugating ψ by γ gives a group isomorphism $\pi_1(\mathbb{R}^3 \setminus K_1) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_2)$ sending m_0 to m_1 and l_0 to $m_1^{n_3} l_1^{\pm 1}$. Since the longitude is the identity in (and the meridian generates) the abelianization of π_1, we must have $a_3 = 0$: thus m_0 is sent to m_1 and l_0 to $l_1^{\pm 1}$.

If instead $n_1 = -1$, we can run the same argument to conclude $x = \pm\gamma$, $x' = \mp\gamma^{-1} m_1^{-1}$; $\psi_{pp}(l_0^\alpha(1 - m_0)) = -\gamma^{-1} m_1^{-1}(1 - m_1)m_1^{\alpha n_3} l_1^{\pm\alpha}\gamma$; $\psi(m_0) = \gamma^{-1} m_1^{-1}\gamma$ and $\psi(l_0) = \gamma^{-1} m_1^{n_3} l_1^{\pm 1}\gamma$; and finally there is an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0 to m_1^{-1} and l_0 to $l_1^{\pm 1}$.

This completes the proof of Theorem 1.1 (1).

5.3. **A note on different types of Legendrian isotopy.** The difference between statements (1) and (2) in Theorem 1.1 is in the strength of the assumption about the Legendrian isotopy relating two conormal tori Λ_{K_0} and Λ_{K_1}. One might ask if the weaker assumption—an unparametrized Legendrian isotopy between Λ_{K_0} and Λ_{K_1}—might still imply the stronger result—a smooth isotopy between K_0 and K_1 as oriented knots. This appears to be possible, but our invariants do not show this.

The issue is a symmetry of $ST^*\mathbb{R}^3$: the diffeomorphism of $\mathbb{R}^3$ given by $(x, y, z) \mapsto (x, y, -z)$ induces a coorientation-preserving contactomorphism of $ST^*\mathbb{R}^3$ that preserves cotangent fibers and sends the conormal torus of K to the conormal torus of the mirror $m(K)$ of K. It follows that there is an isomorphism

$$\mathrm{LCH}_*(\Lambda_p \cup \Lambda_K) \cong \mathrm{LCH}_*(\Lambda_p \cup \Lambda_{m(K)}),$$

and indeed between the DGAs for $\Lambda_p \cup \Lambda_K$ and $\Lambda_p \cup \Lambda_{m(K)}$. Further, this isomorphism preserves the product μ. (On the homology of the conormal torus, the mirroring map preserves l but sends m to m^{-1}, and so this symmetry does not contradict Theorem 1.1 (2).)

There is a result that is somewhere in between statements (1) and (2) from Theorem 1.2: if $\Lambda_{K_0}, \Lambda_{K_1}$ are Legendrian isotopic in an orientation-preserving manner, then K_0, K_1 are smoothly isotopic as unoriented knots. Indeed, from the proof of Theorem 1.1 (1) in Section 5.2, the Legendrian isotopy must send (m_0, l_0) to $(m_1^{\pm 1}, l_1^{\pm 1})$, and the orientation-preserving condition implies that the two signs agree. It follows that there is an isomorphism between $\pi_1(\mathbb{R}^3 \setminus K_0)$ and $\pi_1(\mathbb{R}^3 \setminus K_1)$ that preserves the peripheral subgroup, possibly after changing the orientation of K_1.

6. Lagrangian Skeleta, Wrapped Floer Homology, and Microlocal Sheaves

In this section we describe a geometric setup where the enhanced knot contact homology appears as the partially wrapped Floer homology of certain Lagrangian disks. This Floer homology is closely related to the microlocal sheaf approach to these questions, see e.g. [She]. We calculate the partially wrapped Floer homology via a version of the Legendrian surgery isomorphism [BEE12].

Let $K \subset \mathbb{R}^3$ be a knot. We will associate a Weinstein manifold W_K to K with Lagrangian skeleton $\mathbb{R}^3 \cup L_K$. Here a Lagrangian skeleton of a Weinstein domain X is a Lagrangian subvariety $L \subset X$ (possibly singular but looking like the symplectization of a Legendrian Λ at infinity) such that X is a regular neighborhood of L. The space W_K will be topologically identical to $T^*\mathbb{R}^3$, but the Liouville vector field on W_K is somewhat different than the one on $T^*\mathbb{R}^3$. We use this Liouville field to define partially wrapped Floer homology adapted to the Legendrian Λ.

Before we describe W_K, we first discuss our version of partially wrapped Floer homology in more generality. We then explain the analogue of the construction of W_K in one dimension down ($\mathbb{R}^2$ rather than $\mathbb{R}^3$) before proceeding to W_K itself and calculating partially wrapped Floer homology in this case. We caution that the discussion in this section lacks full details at times; rather than giving a detailed and rigorous treatment, our goal is to use wrapped Floer homology to provide motivation for the constructions in the previous sections and connect our argument to the sheaf-theoretic argument from [She].

6.1. **Partially wrapped Floer homology.** First we give an informal description of what we call partially wrapped Floer homology adapted to a Legendrian; see Figure 7 for an illustration. Let W be a Weinstein manifold and let Λ be a Legendrian submanifold in the ideal contact boundary V of W. Consider the end $\approx [0, \infty) \times V$ of W and the Lagrangian cylinder $[0, \infty) \times \Lambda \subset [0, \infty) \times V$. We will consider a Liouville vector field Z on W (the Liouville condition says that $\mathcal{L}_Z \omega = \omega$, where $\mathcal{L}$ is the Lie derivative and ω the symplectic form) adapted to L. The condition of being adapted to L means that Z agrees with the standard Liouville vector field $p \cdot \partial_p$ along fibers in a cotangent neighborhood of $[0, \infty) \times \Lambda$. The contact boundary V' of W equipped with such a Liouville vector field Z is then non-compact but in a controlled way as follows. It consists of a compact piece V'_c which is a
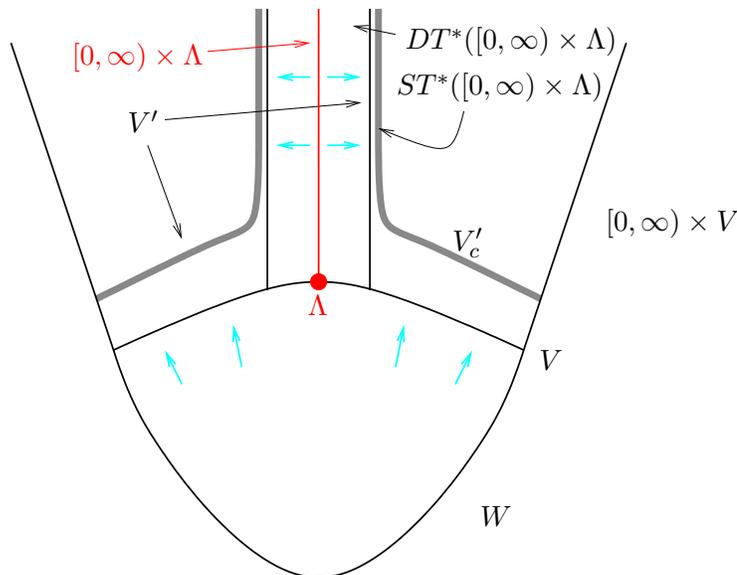
FIGURE 7. The setup for partially wrapped Floer homology with wrapping stopped by Λ. The small arrows indicate the Liouville field Z.

manifold with boundary and the boundary $\partial V_{\mathrm{c}}'$ equals $ST^*([0,\infty) \times \Lambda|_{\{0\}\times\Lambda})$. The noncompact part equals

$$ST^*([0,\infty) \times \Lambda) \approx \partial V_{\mathrm{c}}' \times [0,\infty).$$

The flow of the Liouville vector field Z starting at V' then gives an end of W that looks like $[1,\infty) \times V'$. As in the definition of ordinary symplectic homology or wrapped Floer homology we consider a Hamiltonian on W that is approximately equal to 0 outside the end and is a function $h(r)$ of only the first coordinate in the end $[1,\infty) \times V'$, where the first two derivatives of h are non-negative and $h(r) = ar + b$ for constants a and b for $r > 2$. We next consider Lagrangian submanifolds in W that agree with cylinders over compact Legendrian submanifolds in V' in the end. For such Lagrangians we define a Floer homology just as wrapped Floer homology is defined but using the Hamiltonians described above. We call the resulting Floer homology *partially wrapped Floer homology adapted to* Λ and write $HW(L_1, L_2)$ for the partially wrapped Floer homology of L_1 and L_2.

Remark 6.1. One can rephrase the above construction starting from a compact Weinstein domain $\overline{W}$ with a compact Lagrangian $\overline{L}$ with Legendrian boundary Λ. Then attach a neighborhood of the zero-section in $T^*\Lambda \times [0,\infty)$ to $\partial\overline{W}$ along Λ and use a standard interpolation between the Liouville vector fields $\overline{Z}$ pointing outwards along $\partial\overline{W}$ and standard Liouville vector field $p \cdot \partial_p$ along fibers in the cotangent bundle attached. This results in a non-compact Weinstein domain with non-compact contact boundary V' as above. Our Weinstein manifold is then obtained by adding the positive end of the symplectization of V'. Note that this is very similar to Lagrangian handle attachment. We will use this in calculations below.

Remark 6.2. In our setting the wrapping is "stopped" by the Legendrian Λ, in the sense that the Hamiltonian flow does not cross the cylinder over Λ. The term "partially wrapped" is due to Auroux [Aur10a, Aur10b], but in the setting considered there, the wrapping is stopped by a codimension-1 submanifold of V rather than a Legendrian. The constructions there and here are of a similar spirit but we do not intend to study the exact relation here. Our notion of partially wrapped Floer homology perhaps aligns more closely with [Nad14].

Remark 6.3. For a prototypical example of partial wrapping consider $\mathbb{R}^{2n} \approx T^*\mathbb{R}^n$ with coordinates $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$. We equip it with the Liouville vector field $p \cdot \partial_p$ and Hamiltonian $H(p, q) = \frac{1}{2}p^2$. We use this Hamiltonian to define partially wrapped Floer homology adapted to the boundary sphere in the 0-section; that is, in this case $W = D^{2n}$ and $\Lambda \subset S^{2n-1}$ is the Legendrian sphere $\{p = 0\}$. Unlike the usual Fukaya category for D^{2n}, which is trivial, the partially wrapped Fukaya category is nontrivial: for instance, if F denotes the fiber then $HW(F, F)$ has rank 1.

Below we will give a gluing operation similar to Lagrangian handle attachment/Legendrian surgery that allows us to construct Liouville manifolds with wrapping stopped by a Legendrian and to compute the resulting partially wrapped Floer homology in the spirit of the surgery formula of [BEE12]. Although this approach works rather generally we will restrict to the case of W_K mentioned above. Here the relevant Legendrian surgery operation that we perform can be thought of as an S^1-family of punctured disk attachments. We start by describing an individual punctured disk attachment.

6.2. Attaching a punctured handle in dimension 2.

Before proceeding to W_K, we first consider a model case in 1 dimension down. Our goal here is to describe partially wrapped Floer homology for a Lagrangian fiber disk as well as its pair-of-pants product. See Figure 8 for a picture of the construction.

Consider $T^*\mathbb{R}^2$ as above and fix the cotangent fiber L_0 thought of as the conormal of the point at the origin. We will construct a Weinstein manifold $W_0(\epsilon)$ with Lagrangian skeleton $\mathbb{R}^2 \cup L_0$ by attaching a punctured Lagrangian handle to the Legendrian ideal boundary Λ_0 of L_0. The manifold $W_0(\epsilon)$ with its Liouville vector field is then the ambient space for partially wrapped Floer homology adapted to ∂L_0 and the boundary circle of the 0-section $\mathbb{R}^2$. Let D' denote a punctured disk equipped with the complete hyperbolic metric. To get a concrete model, consider the upper half plane with the hyperbolic metric:

$$H = \{(x, y) \colon y > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y}.$$

Let $H_{y \geq 1} = H \cap \{y \geq 1\}$ and let $\mathbb{Z}$ act on $H_{\geq 1}$ by $n(x, y) = (x + n, y)$. Then

$$D' = H_{\geq 1}/\mathbb{Z}.$$

FIGURE 8. The construction of $W_0(\epsilon)$ from $T^*\mathbb{R}^2$.

The metric induces a Hamiltonian $\frac{1}{2}p^2$ on T^*D' for which the Liouville vector field $p \cdot \partial_p$ is gradient like. We now perform Lagrangian handle attachment of the ϵ-disk cotangent bundle $D_\epsilon T^*D'$, gluing $\partial D'$ to Λ. This gives a Weinstein domain $W_0(\epsilon)$ of the desired type.

We next consider Reeb chords of the Legendrian boundaries of basic Lagrangian disks in $W_0(\epsilon)$. Let ξ denote a point in $\mathbb{R}^2$, $\xi \neq 0$, and let η denote a point in D'. Let L_ξ and C_η denote the Lagrangian disk fibers at ξ and η. Let $\mathcal{C}_{\xi\xi}$ denote the set of Reeb chords from ∂L_ξ to itself in $\partial W_0(\epsilon)$, $\mathcal{C}_{\xi\eta}$ denote the chords from ∂L_ξ to ∂C_η, and interpret $\mathcal{C}_{\eta\xi}$ and $\mathcal{C}_{\eta\eta}$ similarly. In $ST^*\mathbb{R}^2$, there is a unique Reeb chord $c_{\xi 0}$ connecting Λ_ξ to Λ_0 (following the straight line segment from ξ to 0) and a unique Reeb chord $c_{0\xi}$ connecting Λ_0 to Λ_ξ. Let $c^+, c^- \in \Lambda_0$ denote the endpoint of $c_{\xi 0}$ and the beginning point of $c_{0\xi}$, respectively.

Lemma 6.4. *For $\epsilon > 0$ sufficiently small, the Reeb chords in $\partial W_0(\epsilon)$ are as follows:*

- *The set $\mathcal{C}_{\eta\eta}$ is in natural 1-1 correspondence with homotopy classes of loops in D' connecting η to η.*
- *The set $\mathcal{C}_{\eta\xi}$ is in natural 1-1 correspondence with homotopy classes of paths in D' connecting p to c^-.*
- *The set $\mathcal{C}_{\xi\eta}$ is in natural 1-1 correspondence with homotopy classes of paths in D' connecting c^+ to η.*
- *The set $\mathcal{C}_{\xi\eta}$ is in natural 1-1 correspondence with homotopy classes of loops in D' connecting c^+ to c^-.*

Proof. We first note that there are no Reeb chords connecting Λ_ξ to itself before the handle attachment, and as stated above there are unique Reeb chords between Λ_ξ to Λ_0. Furthermore, since the Reeb flow on the boundary $S_\epsilon T^* D'$ of $D_\epsilon T^* D'$ is geodesic flow with respect to the hyperbolic metric on D', there is a unique Reeb chord in $S_\epsilon T^* D'$ connecting η to c^- in any homotopy class of paths, and similarly there is a unique Reeb chord connecting c^+ to η in any homotopy class. A Reeb chord in $\partial W_0(\epsilon)$ converges as $\epsilon \to 0$ to an alternating word of Reeb chords inside and outside the handle. A straightforward fixed point argument as in [BEE12] shows that any such word corresponds to a chord after the surgery. $\square$

We can now compute the partially wrapped Floer homology $HW(L_\xi, L_\xi)$. Fixing a path connecting c^+ to c^-, we have as in Lemma 6.4 a 1-1 correspondence between Reeb chords in $ST^* D'$ connecting the fibers of these points and homotopy classes of loops in D'. We write m for the generator of $\pi_1(D')$ and $c_{\xi 0} m^k c_{0\xi}$ for the Reeb chord in ∂V_0 corresponding to this word of Reeb chords. To compute $HW(L_\xi, L_\xi)$ we shift L_ξ in the positive Reeb direction to obtain a shifted Lagrangian L'_ξ and then consider chords from L'_ξ to L_ξ. Generators for the model of $HW(L'_\xi, L_\xi)$ are then the following:

- an intersection point between L_ξ and L'_ξ in the middle of the disk L_ξ;
- Reeb chords of the form $c_{\xi 0} m^k c_{0\xi}$.

As in [BEE12], there is a chain map from $HW(L'_\xi, L_\xi)$ to this complex with the differential induced by the DGA differential before surgery, which is a chain isomorphism by an action filtration argument. Since the DGA differential is trivial we find that the wrapped Floer homology is generated by the generators of the model complex.

Finally, we describe the pair-of-pants product on $HW(L_\xi, L_\xi)$ using the above isomorphism. To this end, we must first describe that product on the model for wrapped Floer homology without Hamiltonian. The description is a consequence of arguments used in [EHK] and [EO] as follows. Let S denote a horizontal strip in $\mathbb{C}$ with a horizontal slit pointing rightwards. We can view Floer holomorphic disks with two positive punctures as maps whose domain is S. For $\tau \in \mathbb{R}$, consider maps $u: (S, \partial S) \to (W_0(\epsilon), L_\xi)$ that solve the equation

$$(du + \beta_\tau \otimes X_H)^{0,1} = 0,$$

where X_H is the Hamiltonian vector field and where β is a non-positive 1-form on S that in standard coordinates $s + it$ on S interpolates in a region of width 1 around $s = \tau$ between dt to the left of the region and 0 to the right. We consider the solution space of dimension 0 as τ ranges over $\mathbb{R}$. For $\tau = +\infty$ we find the isomorphism map from linearized contact homology to wrapped Floer homology followed by the standard pair-of-pants product. For $\tau = -\infty$ we find the product on linearized contact homology that counts disks with two positive and one negative puncture followed by the isomorphism. Split curves at other values of τ count rigid curves with interpolations with a curve contributing to boundary attached either above or below. It follows that these configurations do not contribute on the level of homology. We

conclude that the pair-of-pants product on wrapped Floer homology corresponds to a count of disks with two positive punctures. (These disks may also involve punctures at Lagrangian intersection points.)

With this established we use a similar argument to transport the pair-of-pants product to the description of the wrapped Floer homology before the surgery: compose the product with the isomorphism map and study splittings. Here the other end of moduli space corresponds to two isomorphism disks joined by a disk with two mixed positive punctures (and several negative punctures at pure Reeb chords). We find that on words the product counts disks with two positive mixed punctures. Drawing the Lagrangian projection of Λ_ξ and Λ_0 before the surgery, one can check that there are two such disks and they lie in distinct homotopy classes 1 and m. Thus if $\cdot$ denotes the product we have that the empty word 1 in the DGA acts as the identity and

$$(c_{\xi 0} m^{\alpha_1} c_{0\xi}) \cdot (c_{\xi 0} m^{\alpha_2} c_{0\xi}) = c_{\xi 0} m^{\alpha_1 + \alpha_2} c_{0\xi} - c_{\xi 0} m^{\alpha_1 + \alpha_2 + 1} c_{0\xi}.$$

Remark 6.5. The above calculations show that the wrapped Floer homology of $HW(L_\xi, L_\xi)$ with its pair-of-pants product is ring isomorphic to $\mathbb{Z}[m^{\pm 1}]$ via the map

$$1 \mapsto 1, \quad c_{\xi 0} m^k c_{0\xi} \mapsto m^k (1 - m).$$

Not coincidentally, $\mathbb{Z}[m^{\pm 1}]$ is the group ring of $\pi_1(\mathbb{R}^2 \setminus \{0\})$, cf. Remark 6.11 below.

6.3. **The construction of W_K.** We next turn to the construction of W_K. Consider $T^* \mathbb{R}^3$ and let L_K denote the conormal Lagrangian as usual. In analogy with the construction of W_0 and the surgery calculation above, we want to construct a Weinstein domain W_K so that if we want to compute partially wrapped Floer homology in $T^* \mathbb{R}^3$ adapted to $\partial \mathbb{R}^3 \cup \Lambda_K$, we can write the partially wrapped Floer homology of fibers over the top-dimensional strata of the Lagrangian skeleton in terms of the contact homology DGA of the Legendrian attaching locus.

We will construct W_K by attaching the cotangent bundle of $[0, \infty) \times T^2$ to $T^* \mathbb{R}^3$ as follows; see Figure 9. Note first that the conormal torus of the knot Λ_K can be viewed as an S^1-family of conormal circles over points in K; now consider $S^1 \times D'$ with the product metric (the metric on D' is the complete hyperbolic metric as in Section 6.2) and its cotangent bundle $T^*(S^1 \times D')$. We perform Lagrangian attachment of the ϵ-disk cotangent bundle $D_\epsilon T^*(S^1 \times D')$ to the disk cotangent bundle $DT^* \mathbb{R}^3$ by gluing $S^1 \times \partial D'$ to Λ_K. This is our desired Weinstein domain $W_K(\epsilon)$, which depends on the size ϵ of the disk bundle attached. In analogy with the above we calculate the wrapped Floer homology for two basic Lagrangian disks in W_K, the *outer* disk F which is the fiber of a point in $\mathbb{R}^3$ not on K and the *inner* disk C, the fiber over a point in $S^1 \times D'$.

We begin by a description of the Reeb chords after surgery in terms of data before. We write $\mathcal{C}(\Lambda_1, \Lambda_2)$ for the set of Reeb chords in ∂W_K starting on Λ_1 and ending on Λ_2. As in Section 2.1, write $\mathcal{R}^{KK}$ for Reeb chords of Λ_K in $ST^* \mathbb{R}^3$, and $\mathcal{R}^{Kp}$ and $\mathcal{R}^{pK}$ for Reeb
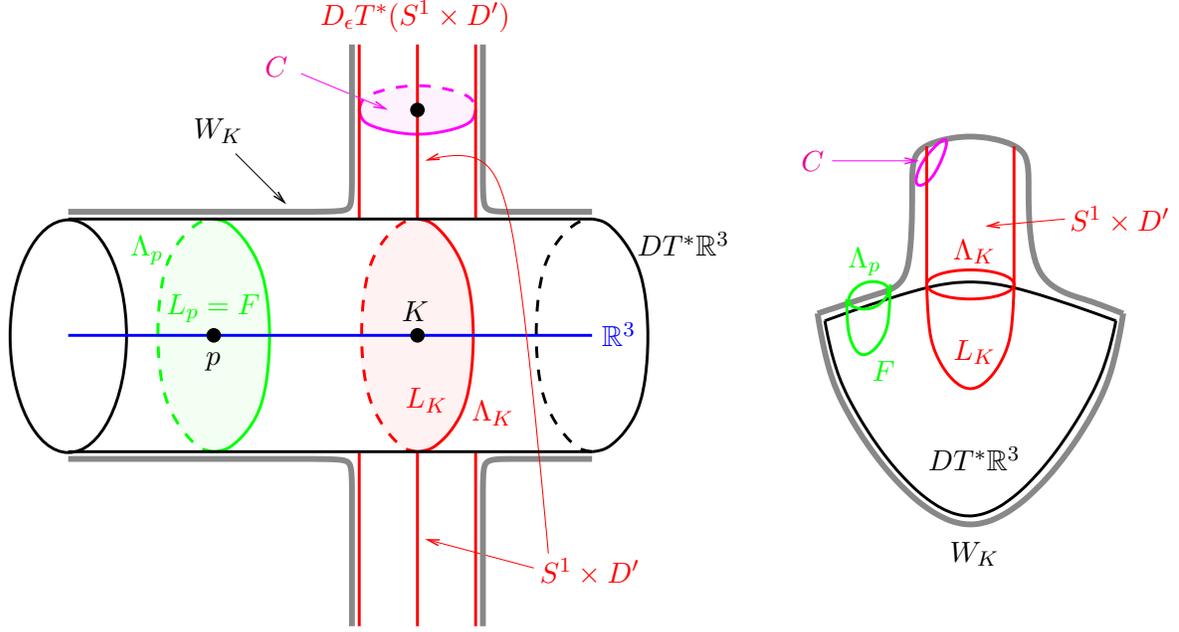
FIGURE 9. The construction of W_K from $T^*\mathbb{R}^3$ (left), and an alternate schematic picture (right).

chords in $ST^*\mathbb{R}^3$ to Λ_K from $\Lambda_p = \partial B$, and to Λ_p from Λ_K, respectively. Also, write $\pi_1(S^1 \times D') = \mathbb{Z}^2 = \langle l, m \rangle$.

Lemma 6.6. *For $\epsilon > 0$ sufficiently small we have the following.*

- *There is a natural 1-1 correspondence between the elements of $\mathcal{C}(\partial C, \partial C)$ and words of the form*

$$l^{\alpha_1} m^{\beta_1} \, c_1 \, l^{\alpha_2} m^{\beta_2} \, c_2 \, \cdots c_r \, l^{\alpha_{r+1}} m^{\beta_{r+1}},$$

  where $c_j \in \mathcal{R}^{KK}$ and either $r \geq 1$ or $r = 0$ and $(\alpha_1, \beta_1) \neq (0,0)$.

- *There is a natural 1-1 correspondence between the elements of $\mathcal{C}(\partial C, \partial F)$ and words of the form*

$$al^{\alpha_1} m^{\beta_1} \, c_1 \, l^{\alpha_2} m^{\beta_2} \, c_2 \cdots c_r \, l^{\alpha_{r+1}} m^{\beta_{r+1}},$$

  where $r \geq 0$, $c_j \in \mathcal{R}^{KK}$, and $a \in \mathcal{R}^{pK}$.

- *There is a natural 1-1 correspondence between the elements of $\mathcal{C}(\partial F, \partial C)$ and words of the form*

$$l^{\alpha_1} m^{\beta_1} \, c_1 \, l^{\alpha_2} m^{\beta_2} \, c_2 \, \cdots c_r \, l^{\alpha_{r+1}} m^{\beta_{r+1}} \, b,$$

  where $r \geq 0$, $c_j \in \mathcal{R}^{KK}$, and $b \in \mathcal{R}^{Kp}$.

- *There is a natural 1-1 correspondence between the elements of $\mathcal{C}(\partial F, \partial F)$ and words of the form*

$$a \, l^{\alpha_1} m^{\beta_1} \, c_1 \, l^{\alpha_2} m^{\beta_2} \, c_2 \cdots c_r \, l^{\alpha_{r+1}} m^{\beta_{r+1}} \, b,$$

  where $r \geq 0$, $c_j \in \mathcal{R}^{KK}$, $a \in \mathcal{R}_{pK}$, and $b \in \mathcal{R}_{Kp}$.

Proof. As $\epsilon \to 0$ Reeb chords in W_K, traversed from bottom to top, have limits as the words described, where each c_i, a, b is also traversed backwards. (We follow the Reeb chords in W_K backwards in order to obtain composable words; note that composable words in the spirit of Section 2.1 consist of a concatenation of Reeb chords whose consecutive endpoints line up *if* the chords are traversed backwards.) As in [BEE12] a fixed point argument shows that every word glues to a unique Reeb chord for small $\epsilon > 0$. $\qquad\square$

Remark 6.7. Note that in the construction of W_K above we could have attached any model of $T^*S^1 \times (D^2 \setminus \{0\})$ with Reeb flow agreeing with that of a complete metric at infinity. The reason for the particular choice we made was to get a model suitable for direct calculations. This can be compared to the concrete handle model used in [BEE12] which is not the only possible choice but well suited for calculations.

6.4. **Partially wrapped Floer homology in W_K.** We now turn to partially wrapped Floer homology in $T^*\mathbb{R}^3$ via W_K. In particular, we study $HW_*(L_1, L_2)$ where each of L_1, L_2 is one of the Lagrangians F, C in W_K. The complexes $CW(C, F)$ and $CW(F, C)$ are generated by elements of $\mathcal{C}(\partial C, \partial F)$ and $\mathcal{C}(\partial F, \partial C)$, respectively, while $CW(C, C)$ and $CW(F, F)$ are generated by elements of $\mathcal{C}(\partial C, \partial C)$ and $\mathcal{C}(\partial F, \partial F)$, along with an additional generator corresponding to an intersection point between the Lagrangian and its pushoff.

Now note that the Reeb chords in Lemma 6.6 can be viewed as composable elements in the DGA $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$. For $CW(C, F)$ and $CW(F, C)$, these are precisely the composable words with exactly 1 mixed Reeb chord. For $CW(C, C)$, the Reeb chords in $\mathcal{C}(\partial C, \partial C)$ are all possible words involving only pure Reeb chords of Λ_K, with the exception of $l^0 m^0 = 1$; but $CW(C, C)$ has an extra generator corresponding to 1. That is, we have:

$$CW(C, C) = \mathcal{A}^{(0)}_{\Lambda_K, \Lambda_K}$$
$$CW(C, F) = \mathcal{A}^{(1)}_{\Lambda_p, \Lambda_K}$$
$$CW(F, C) = \mathcal{A}^{(1)}_{\Lambda_K, \Lambda_p}.$$

For $CW(F, F)$, the Reeb chords in $\mathcal{C}(\partial F, \partial F)$ are the composable words beginning and ending on Λ_p with exactly 2 mixed Reeb chords, and $CW(F, F)$ is generated by these along with the extra generator. Thus the generating set of $CW(F, F)$ is naturally the same as the generating set of $\mathcal{F}^0 \mathcal{A}_{\Lambda_p, \Lambda_p} / \mathcal{F}^4 \mathcal{A}_{\Lambda_p, \Lambda_p}$ (the extra generator plays the role of 1), and we have:

$$CW(F, F) = \mathbb{Z} \oplus \mathcal{A}^{(2)}_{\Lambda_p, \Lambda_p}.$$

We next claim that the differentials on the Floer complexes agree with the differentials from the DGA. Indeed, there are chain maps Φ from CW to the corresponding filtered quotients of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ that count the following holomorphic disks:

- For $CW(C, C)$, disks with one positive puncture, two Lagrangian intersection punctures at $C \cap (S^1 \times D')$, and negative punctures at chords in $\mathcal{R}^{KK}$.
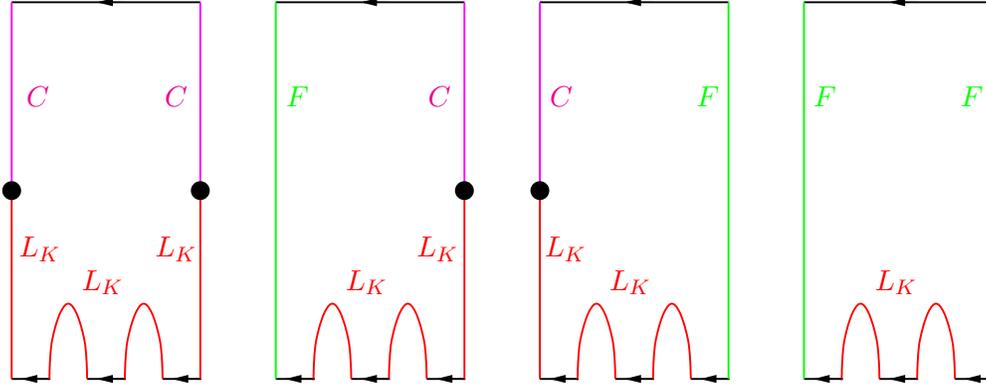
FIGURE 10.   The holomorphic disks defining the maps Φ on, from left to right: $CW(C,C)$; $CW(C,F)$; $CW(F,C)$; $CW(F,F)$. In each case, a Reeb chord with endpoints on $\partial F \cup \partial C$ is the input at the top, and the output at the bottom is a product of Reeb chords and homology classes for $\Lambda_K \cup \Lambda_p$. For the L_K boundary conditions, we extend L_K into W_K by taking its union with $S^1 \times D'$, see Figure 9. Arrows represent orientations of Reeb chords.

- For $CW(C,F)$, disks with one positive puncture, a first negative puncture at a chord in $\mathcal{R}^{pK}$, several negative punctures at chords in $\mathcal{R}^{KK}$, and a Lagrangian intersection puncture at $C \cap (S^1 \times D')$.
- For $CW(F,C)$, disks with one positive puncture, a Lagrangian intersection at $C \cap (S^1 \times D')$, several negative punctures at chords in $\mathcal{R}^{KK}$, and a last negative puncture at a chord in $\mathcal{R}^{Kp}$.
- For $CW(F,F)$, disks with one positive puncture, a first negative puncture at a chord in $\mathcal{R}^{pK}$, several negative punctures at chords in $\mathcal{R}^{KK}$, and a last negative puncture at a chord in $\mathcal{R}^{pK}$.

See Figure 10.

Lemma 6.8. *The chain maps Φ are chain isomorphisms. The degrees of the chain maps on $CW(F,F)$ and $CW(C,C)$ are 0 and 1, respectively, and there is a choice of reference path connecting ∂C to ∂F so that the grading shifts on $CW(C,F)$ and $CW(F,C)$ are 0 and 1, respectively.*

Proof. This follows from constructions of curves over almost trivial strips and an energy filtration argument as in [BEE12]. For the degree statement note that two Lagrangian corners of codimension 3 gives a contribution to the dimension by $-3+2 = -1$ which means that the grading of the Reeb chord at the positive puncture is one above the sums of the gradings at the negative end. (Compare the grading shift by $(n-2)$ in the wrapped homology surgery isomorphism from [BEE12].) $\square$

Lemma 6.8 allows us to relate the wrapped Floer picture to the KCH-triple. The maps Φ descend in homology to isomorphisms

$$HW_1(C,C) \overset{\cong}{\to} H_0(\mathcal{A}^{(0)}_{\Lambda_K,\Lambda_K}) \cong R_{KK}$$

$$HW_1(C,F) \overset{\cong}{\to} H_1(\mathcal{A}^{(1)}_{\Lambda_p,\Lambda_K}) \cong R_{pK}$$

$$HW_1(F,C) \overset{\cong}{\to} H_0(\mathcal{A}^{(1)}_{\Lambda_K,\Lambda_p}) \cong R_{Kp}.$$

That is, the KCH-triple is precisely given by partially wrapped Floer homology in lowest degree. In addition, Φ gives an isomorphism

$$HW_1(F,F) \overset{\cong}{\to} \mathbb{Z} \oplus H_1(\mathcal{A}^{(2)}_{\Lambda_p,\Lambda_p}) \cong \mathbb{Z} \oplus R_{pp},$$

where the second isomorphism comes from Proposition 2.8.

We examine this last isomorphism $HW_1(F,F) \cong \mathbb{Z} \oplus R_{pp}$ more carefully. The right hand side has a product given by μ. The left hand side also has a product: on $CW(F,F)$, there is a pair-of-pants product

$$\Pi \colon CW(F,F) \otimes CW(F,F) \to CW(F,F)$$

which here has degree -1, see Remark 2.9. We claim that this agrees in homology with μ under the isomorphism Φ.

Lemma 6.9. *There is a degree 0 pairing $P \colon CW(F,F) \otimes CW(F,F) \to CW(F,F)$ such that*

$$\Phi \circ \Pi - \mu \circ (\Phi \otimes \Phi) + P \circ (1 \otimes d + d \otimes 1) - \partial \circ P = 0$$

where d is the differential on $CW(F,F)$. In particular, on homology Φ sends Π to μ.

Proof. Consider the 1-dimensional moduli space of disks in the cobordism with two positive punctures at chords connecting ∂F to ∂F and with negative punctures at a mixed chord $\partial F \to \Lambda_K$ followed by an alternating word of chords and homotopy classes on Λ_K and then a mixed chord $\Lambda_K \to \partial F$. If P counts the corresponding rigid curves in the cobordism then terms in the left hand side of the equation count ends of a 1-dimensional compact oriented moduli space. The lemma follows. $\square$

Thus the isomorphism between $HW_1(F,F)$ and $\mathbb{Z} \oplus R_{pp}$ is a ring isomorphism. By Proposition 4.17, we conclude that:

$$HW_1(F,F) \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)].$$

This is as expected by the general theory, see Remark 6.11 below.

Remark 6.10. We can use the isomorphism Φ to describe other pair-of-pants products of the form $HW(L_2, L_3) \otimes HW(L_1, L_2) \to HW(L_1, L_3)$, as follows:

- Exactly as for $HW(F,F)$ above, the product

$$HW_1(F,C) \otimes HW_1(C,F) \to HW_1(C,C)$$

  corresponds to the product μ counting rigid disks with two positive mixed punctures and several negative pure punctures.

- Arguing as in [BEE12, Theorem 5.8], we find that the products

$$HW_1(C,C) \otimes HW_1(C,C) \to HW_1(C,C),$$
$$HW_1(C,F) \otimes HW_1(C,C) \to HW_1(C,F),$$
$$HW_1(C,C) \otimes HW_1(F,C) \to HW_1(F,C),$$
$$HW_1(C,F) \otimes HW_1(F,C) \to HW_1(F,F)$$

  correspond to concatenation in the DGA's.

Remark 6.11. We have seen that $HW_1(C,C)$ is knot contact homology in degree 0, while $HW_1(F,F)$ is the group ring of $\pi_1(\mathbb{R}^3 \setminus K)$. Both of these are consistent with general considerations in wrapped Floer homology. The picture for the partially wrapped Floer homology of C is analogous to the calculation of the wrapped Floer homology of a cocore disk in Legendrian handle attachment, which via the surgery isomorphism [BEE12] is isomorphic to the Legendrian contact homology of the attaching sphere. In our setting, C plays the role of the cocore disk, while Λ_K plays the role of the Legendrian attaching sphere. Compared to the situation in [BEE12], we are attaching a different handle, and the Reeb flow inside the handle is more complicated but easy to control from the point of view of holomorphic disks since all Reeb chords inside the punctured handle have index 0.

Our calculation of the partially wrapped Floer homology of F relates it to the DGA of $\Lambda_K \cup \partial F = \Lambda_K \cup \Lambda_p$ again as in [BEE12], but this time we compute the wrapped Floer homology of F in the manifold with the punctured solid torus handle attached in terms of data before the attachment. Via string homology of broken strings on the Lagrangian skeleton we find that in the lowest degree the partially wrapped Floer homology of F is given by $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. This should be compared to the result from [AS10] (cf. [Abo12]) that the wrapped Floer cohomology of a cotangent fiber in T^*Q is isomorphic to the homology of the based loop space, with the pair-of-pants product in wrapped Floer cohomology mapping to the Pontryagin product on chains of loops, see Section 6.5 below for further discussion of this relation.

6.5. **Wrapped Floer homology and the knot complement.** Here we describe a direct connection between partially wrapped Floer homology of F and C in W_K and the topology of the loop space of the knot complement, which explains the geometry underlying the isomorphisms in Proposition 4.13 and the ring isomorphism $HW_1(F,F) \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

As a first step, following [AENV14], one can construct a Lagrangian submanifold $M_K \subset W_K$ that is diffeomorphic to the knot complement $M_K \approx \mathbb{R}^3 \setminus K$, see Figure 11. This is the
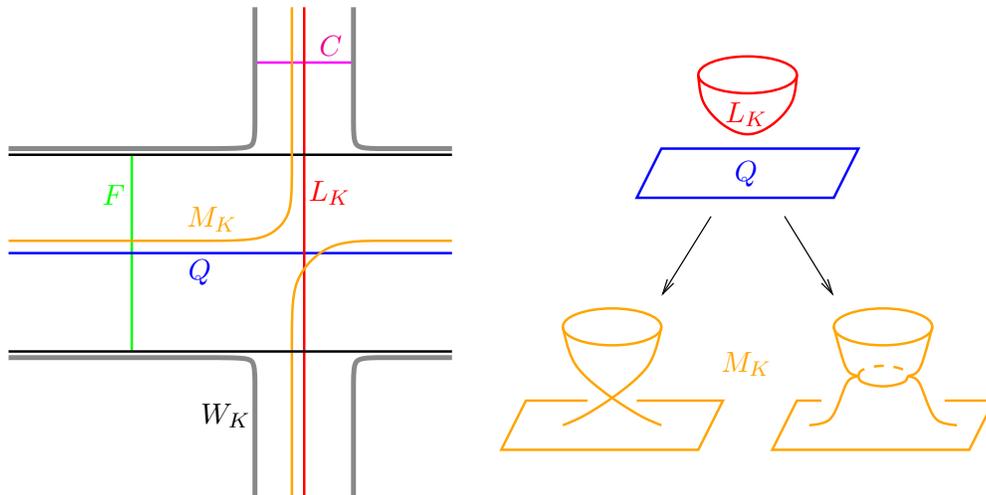
FIGURE 11. Constructing the Lagrangian M_K from the Lagrangians Q and L_K: on the left, depicting M_K within W_K; on the right, the two possible Lagrange surgeries.
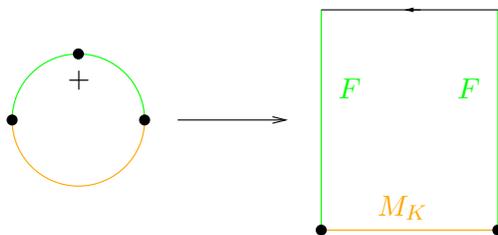

FIGURE 12. A holomorphic disk with boundary on $F \cup M_K$, used to define a map from Reeb chords of ∂F to chains in the loop space of M_K.

result of applying Lagrange surgery to $Q \cup L_K$ along the clean intersection $Q \cap L_K = K$. In fact there are two distinct Lagrange surgeries: in the fiber over a point in K we see two transversely meeting Lagrangian 2-disks and we can smooth their intersection in two ways, as shown in the right diagram of Figure 11. (If the two disks are a local model for the double point of the Whitney sphere in $\mathbb{R}^4$, then one smoothing gives the Clifford torus and the other the Chekanov torus.) Note that M_K intersects each of F and C transversely in one point.

In Section 6.4, we described chain isomorphisms Φ from the partially wrapped Floer homologies of C and F to the DGA $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$. Here we outline geometric counterparts of Φ, mapping into chains of paths on M_K rather than to $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$.

We can construct a map from $CW(F, F)$ into chains of based loops in M_K by associating to a Reeb chord of F the chain of based loops in M_K (based at the point $F \cap M_K$) carried by the moduli space of holomorphic disks with one positive puncture at the Reeb chord, two Lagrangian intersection punctures at $F \cap M_K$, and boundary on $F \cup M_K$, see Figure 12. Similarly we have maps from $CW(C, C)$, $CW(C, F)$, and $CW(F, C)$ to chains of loops in M_K

(for the first one) or paths in M_K (for the last two) by replacing the F boundary conditions in Figure 12 by C as needed. All four maps are chain maps and induce maps on homology.

Now Theorem 6.15 in [AENV14] relates holomorphic disks with positive puncture at a Reeb chord of Λ_K and switching boundary conditions on $Q \cup L_K$, via a smoothing procedure, to disks with the same positive puncture but now with boundary on M_K. We can combine this with the surgery isomorphism relating Reeb chords of $\partial C \cup \partial F$ (the generators of the partially wrapped complexes) to words of Reeb chords of $\Lambda_K \cup \Lambda_p$. The geometric maps from $CW(C, C)$, $CW(C, F)$, and $CW(F, C)$ to chains in loop/path spaces then descend in homology to maps from $R_{KK} = HW_1(C, C)$, $R_{Kp} = HW_1(C, F)$, and $R_{pK} = HW_1(F, C)$ to $\mathbb{Z}[\pi_1(M_K)]$. These are precisely the isomorphisms in Proposition 4.13 from the KCH-triple to subrings of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)] = \mathbb{Z}[\pi_1(M_K)]$; furthermore, the map on $CW(F, F)$ induces the isomorphism from $\mathbb{Z} \oplus R_{pp} \cong HW_1(F, F)$ to the entire group ring from Proposition 4.17.

Indeed, the algebraic maps constructed in the proof of Proposition 4.13 correspond to the maps from chains of broken strings to chains of paths on M_K determined by [AENV14, Theorem 6.5]. More precisely, the algebraic maps correspond to the disks for which the length functional on the image chain is almost equal to the action of the Reeb chord at the positive puncture. Using the action/length filtration as in [CELN] we then find that the algebraic maps are homotopic to the geometric maps.

The factors of $1 - m$ in Proposition 4.13 come from the fact that a holomorphic disk with switching boundary at $Q \cap L_K$ can be smoothed to have boundary on M_K in two different ways, differing in homotopy by a meridian in M_K. In this context, we note that the choice of where to place the $1-m$ factors in the isomorphisms from Proposition 4.13, see Remark 4.12, corresponds to the choice of Lagrange surgery (Clifford or Chekanov) along K.

The geometric chain maps that we have described here have direct analogues for the wrapped Floer homology of a cotangent fiber in a closed manifold [AS10, Abo12]. In particular, the identification of the pair-of-pants product in wrapped Floer cohomology of the fiber in a closed manifold with the Pontryagin product on chains of loops is directly analogous to our identification of the products in $HW_1(F, F)$ and $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

6.6. **Relation to sheaves and cosheaves.** This paper has presented a holomorphic-curve approach to the result that the conormal torus is a complete knot invariant. This result was previously established by the methods of microlocal sheaf theory in [She]. We now explain how these two approaches are related.

Consider a closed manifold M. Abouzaid has shown on the one hand that the wrapped Floer homology of a cotangent fiber in T^*M is naturally identified with chains on the based loop space of M [Abo12], and on the other hand that this cotangent fiber generates the wrapped Fukaya category [Abo11]. One way to express these facts simultaneously is to say that the wrapped Fukaya category of T^*M localizes to a cosheaf of categories over M.

To generalize this notion to the cotangent bundle of an open manifold, and in particular to make sense of what the sections of the above cosheaf should be over a smaller open set in symplectic-geometric terms, requires considerations of ambient symplectic spaces more general than standard Liouville domains because of the following. For a manifold M with non-empty boundary ∂M, the standard Liouville structure on the cotangent bundle (with Liouville vector field $p \cdot \partial_p$ pointing radially outwards along fibers) does not have contact type boundary. This problem can be resolved in more or less equivalent ways by either allowing such "Liouville manifolds with boundary" as ambient symplectic manifolds (essentially the approach we have taken here), or by taking the more conventional Liouville structure on T^*M (obtained by rounding the corners over $T^*_{\partial M}M$) but remembering ∂M as a Legendrian submanifold in the contact boundary and construct the wrapped Floer theory in such a way that this Legendrian serves to stop the wrapping. We point out that it does not suffice to take the conventional Liouville structure on T^*M alone and forget the Legendrian: with this structure T^*M is subcritical, the wrapped Fukaya category is trivial, and the Floer homology of a cotangent fiber vanishes and can in particular not be identified with chains on the based loop space as in the closed case, cf. Remark 6.3.

The setup for cotangent bundles of manifolds with boundary applies in more general situations. Consider a Weinstein manifold W with ideal contact boundary V that contains a possibly singular Legendrian $\Lambda \subset V$. We write $\mathbb{L} \subset W$ for the Lagrangian skeleton, by which in this setting we mean the complement of all points in W which escape to $V \setminus \Lambda$ under the Liouville flow. In this context, it has long been expected [Kon09] that the Fukaya category defined by stopping wrapping at Λ will localize on $\mathbb{L}$. In the special case that $\mathbb{L}$ is locally a conormal variety (results of Nadler [Nad] suggest that this can always be achieved after perturbing the Liouville structure), it is further expected that the resulting cosheaf is dual to the Kashiwara–Schapira stack which governs microlocal sheaves [Nad14]. Results along these lines will soon appear [GPSa, GPSb], where it will also be shown that in good circumstances, the wrapped Fukaya category is again generated by fibers, one in each connected component of the smooth locus of the skeleton.

We now return to the specific case studied here. In [She], the conormal torus Λ_K of a knot $K \subset \mathbb{R}^3$ was studied via the category of sheaves microsupported in the union of the zero section and the cone over Λ_K. This sheaf of categories is known by [KS94] to be the global sections of a sheaf of categories over the skeleton formed by the union of the zero section and the conormal. These in turn are modules over the global sections of a corresponding cosheaf that is locally constant away from the singular locus. It then follows formally that the global sections are generated by the stalk in each connected component of the smooth locus and the stalk at the singular locus. The stalk at the smooth locus is just the category of perfect complexes and a calculation at a singular point of the skeleton shows that in fact the stalks at the smooth points suffice.

By the above discussion, the sheaf category studied in [She] is the category of modules over the wrapped Fukaya category of $T^*\mathbb{R}^3$ with wrapping stopped by $\partial\mathbb{R}^3 \cup \Lambda_K$, which is generated by two cotangent fibers, one at a point on the zero section and one at a point on the of the Lagrangian conormal of K. These generators are the Lagrangian disks F and C discussed above.

The surgery formula calculates the partially wrapped Floer homology (along with the pair-of-pants product) for these Lagrangians F and C, which generate the Fukaya category of the neighborhood of the Lagrangian skeleton, from holomorphic-curve considerations on the Legendrian attaching locus. In light of the above discussion, the constructions of the previous sections then have a natural interpretation as calculating morphisms in a category equivalent to that studied in [She]. Note however that this connection is not explicitly used in the proof presented here that these morphism spaces determine a complete knot invariant. Rather, in the proof of our main theorem we map the holomorphic curve theories to the string topology of the Lagrangian skeleton. Hence, in a sense, the holomorphic curve theory of the Legendrian conormal allows us to see rather concretely how the Fukaya category of W_K localizes to its Lagrangian skeleton in the case of conormals of knots.

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