SMOOTH INVARIANT DENSITIES FOR RANDOM SWITCHING ON THE TORUS.

YURI BAKHTIN, TOBIAS HURTH, SEAN D. LAWLEY, JONATHAN C. MATTINGLY

ABSTRACT. We consider a random dynamical system obtained by switching between the flows generated by two smooth vector fields on the 2d-torus, with the random switchings happening according to a Poisson process. Assuming that the driving vector fields are transversal to each other at all points of the torus and that each of them allows for a smooth invariant density and no periodic orbits, we prove that the switched system also has a smooth invariant density, for every switching rate. Our approach is based on an integration by parts formula inspired by techniques from Malliavin calculus.

1. Introduction

The main goal of this paper is to prove smoothness of invariant densities for a class of dynamical systems generated by random switching between two deterministic flows defined on the two dimensional torus $\mathbb{T}^2$. The individual flows are assumed to have everywhere positive smooth invariant densities with respect to Lebesgue measure and have no periodic orbits or fixed points.

Many authors have studied systems with random switchings (or, switching systems), and they are known independently under various titles: hybrid systems \[YZ10\], piecewise deterministic Markov processes (PDMP) (e.g. \[Dav93\], \[Mal15\]), random evolutions, see \[Her03\] for the history of the subject and extensive bibliography. Much of this work was inspired by \[Kar74\] (a reprint of an article published in 1956) where the first probabilistic representation of a second-order hyperbolic equation was obtained.

The random dynamics in question can be informally described as follows: given several smooth vector fields on a manifold, a point on the manifold follows one of them for a while and then, at a random time, switches to one of the other vector fields chosen at random, follows that vector field for a random time, switches again, and so on. If the switching times form a Poisson process with fixed intensity and the switches follow a Markov chain on the driving vector fields, then the two-component process composed of the point on the manifold and the driving vector field is also Markov. More general settings are possible, for instance it is often assumed in the literature that the rate at which switches occur depends on the point on the manifold.

Recently, several authors studied invariant measures for such two-component Markov processes, e.g. \[BLBMZ12\], \[FGRC09\], \[CH13\], \[LMR15\] and \[BCL16\]. Existence of an invariant measure holds true if the manifold is compact due to a Krylov–Bogolyubov type argument (see \[BLBMZ12\]), and can also be derived for some other systems from similar compactness arguments or via sufficient contractivity, see \[LMR15\].

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In [BH12], it was shown that uniqueness of an invariant distribution follows from
existence of a point \( x \) that (i) is accessible from every other point via orbits of the
driving vector fields (which can be viewed as admissible controls) and (ii) satisfies
a Hörmander-type hypoellipticity condition, i.e., the Lie algebra generated by the
driving vector fields at \( x \) coincides with the tangent space at \( x \). The same conditions
guarantee the absolute continuity of the invariant measure with respect to the vol-
ume on the manifold. Similar results were independently obtained in [BLBMZ12],
where it was also shown that, under these assumptions, the invariant distribution is
exponentially attracting in total variation for the action of the associated Markov
semigroup.

The results of [BH12] and [BLBMZ12] can be viewed as a simple way to look
at hypoellipticity from the probabilistic perspective. It is widely known that
Hörmander’s hypoellipticity conditions lead to smoothness of solutions of associated
parabolic equations, and these results have a probabilistic interpretation via Malli-
avin calculus and smoothness of transition (or invariant) densities for hypoelliptic
diffusions. However, the smoothness of invariant densities guaranteed by hypoellip-
ticity in the diffusion case does not hold in general for systems with switching. Even
in the simplest one-dimensional examples, invariant densities and their derivatives
may develop singularities at stable critical points of the driving vector fields. The
dynamical point of view of this phenomenon is based on mass accumulation near
criticalities due to the exponential contraction exhibited by the flow. In [BHMI15],
an exhaustive analysis of all kinds of singularities emerging in the one-dimensional
setting is carried out, and it is also shown that away from the critical points the
densities are smooth. The smoothness argument is based on the fact that time
averaging along an orbit of a vector field acts as a smoothing operator.

The situation becomes more involved in higher dimensions where singularities
of the density can be created by contraction towards attractors (potentially with
complicated structure) of one vector field and then propagated by other vector
fields along their orbits. On top of that, if one only requires hypoellipticity, the
smoothing properties of the integral operators involved are not as pronounced and
harder to exploit. In [LMRT15], sufficient contractivity of the system is leveraged to
prove existence and uniqueness of the invariant measure even in infinite dimensions.

In this paper, we introduce the simplest setting on the two-dimensional torus
that is devoid of the aforementioned difficulties. Namely, we will assume that there
are two driving vector fields that are transversal to each other everywhere on the
torus, which can be interpreted as a uniform ellipticity condition. In addition, we
will impose a requirement that precludes exponential contraction to avoid abnormal
mass accumulation. Our main result is that under these conditions, the invariant
density belongs to \( C^\infty \) (see Theorem 1 at the end of Section 2).

At the core of the proof is a study of regularizing properties of the transfer
operator associated to the switching system at the moment of time when a second
switch has just occurred (see 5). Our approach is based on integration by parts
with respect to times between consecutive switches to transfer the variation in
the initial point to a variation in the noise directions. This variation in the noise
directions can then be shifted by integrating by parts to the exponential density
generating the switching times.
This approach is inspired by the integration by parts at the heart of Malliavin calculus which was developed initially precisely to prove smoothness of the transition laws for stochastic differential equations driven by white noise. There are many conceptually related works. In [BC86, BBM07], Malliavin calculus and the associated integration by parts is developed for equations with jumps. While the setting is different, there are some conceptual similarities. Closer to the setting of this article [Loe16] uses integration by parts to study regularity of the one-dimensional marginals of the invariant density for a class of piecewise deterministic Markov processes with jumps.

It is possible to study two-dimensional and higher-dimensional systems based on vector fields that admit critical points, cycles, or hypoellipticity points. Our progress on those systems will be reported in another paper where we develop more delicate versions of the methods used in the present one. Furthermore, it is relatively straightforward to transfer the ideas here to the semi-Markov setting when the switching times are not exponentially distributed as long as they are given by a smooth density that decays sufficiently fast at infinity. The ease of transferring to the semi-Markovian setting stems largely from the fact that we work with the chain obtained after two successive jump times essentially as in [LMR15].

We close the introduction with an outline of the paper. In Section 2 we describe the class of switching systems we consider and state our main result, Theorem 1. The proof of Theorem 1 is a direct consequence of the smoothing result given in Theorem 4 obtained via integration by parts, all of which is proven in Section 6. In Sections 4 and 5 we record several auxiliary statements, notably a growth estimate on the flows generated by the two vector fields (Lemma 1) and an integral equation for the invariant density (Lemma 2). This integral equation is a prerequisite for the particular integration-by-parts argument in Section 6.

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2. The switching system

We consider a switching system intermittently driven by two smooth vector fields $u_0$ and $u_1$ on the two-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Throughout the paper, smoothness means $C^\infty$ smoothness although our results have versions involving lower regularity requirements and conclusions.

We usually identify $T^2$ with $[0,1)^2$ or work with the universal cover $\mathbb{R}^2$. In particular, this allows us to talk about the Lebesgue measure on the torus, and $\mathbb{R}^2$-vectors can serve as differences between points on $T^2$.

The smoothness of $u_0$ and $u_1$ implies that for $i \in \{0,1\}$ and $x \in T^2$, the initial-value problem

$$\dot{x}(t) = u_i(x(t)), \quad x(0) = x,$$

has a unique solution defined for all $t \in \mathbb{R}$. This lets us associate flows $(x,t) \mapsto \Phi^i_0(x)$ and $(x,t) \mapsto \Phi^i_1(x)$ to the vector fields $u_0$ and $u_1$. We define a stochastic process $X = (X_t)_{t \geq 0}$ on $T^2$ as follows. Given $i \in \{0,1\}$ and $x \in T^2$, the process $X$ follows the flow $t \mapsto \Phi^i_t(x)$, $t \geq 0$, for an exponentially distributed random time $\tau$. Then, a switch from $u_i$ to $u_{1-i}$ occurs and $X$ follows the flow $t \mapsto \Phi^i_{1-i}(y)$,
$t \geq \tau$, where $y = \Phi^t(x)$ is the point on $T^2$ where the switch happened. After another exponentially distributed time, we switch back to the vector field $u_1$, and so on. For simplicity, we assume that the exponential times between switches are i.i.d., so switching from $u_0$ to $u_1$ and from $u_1$ to $u_0$ happens with the same rate $\lambda > 0$. While the process $X$ by itself is not Markov, we obtain a Markov process when adjoining a second stochastic process $A = (A_i)_{t \geq 0}$ on the index set $\{0, 1\}$ that records the driving vector field at any given time. We denote the Markov semigroup of the two-component process $(X, A)$ with state space $T^2 \times \{0, 1\}$ by $(P^t)_{t \geq 0}$ and the corresponding transition probability measures by $P^t_{x,i}$. A probability measure $\mu$ on $T^2 \times \{0, 1\}$ is called an invariant measure of $(P^t)_{t \geq 0}$ if

$$\mu(E \times \{i\}) = \mu P^t(E \times \{i\}) := \sum_{j \in \{0, 1\}} \int_{T^2} P^t_{x,j}(E \times \{i\}) \mu(dx \times \{j\})$$

for any Borel set $E \subset T^2$, $i \in \{0, 1\}$ and $t \geq 0$.

To state our main result we need to introduce two conditions. We say that a smooth vector field $u$ on $T^2$ satisfies Condition A or the conjugacy condition if the flow generated by $u$ has an invariant measure with an everywhere positive, $C^\infty$ density with respect to Lebesgue measure and no periodic or fixed points. We will see that every such flow is smoothly conjugated to a flow with a simple structure. We will clarify the structure of this simple flow, the conjugacy, and the role of this assumption in Section 4. Here we only mention that presence of critical points or cycles may lead to invariant density singularities, which happens even in one-dimensional situations studied in [BHM15].

We say that a pair of two smooth vector fields $u$ and $v$ on $T^2$ satisfies Condition B if for every $x \in T^2$, the vectors $u(x)$ and $v(x)$ span the tangent space $T_x T^2 \cong \mathbb{R}^2$. We will also refer to Condition B as the ellipticity or transversality condition. Often in this paper, we use $(u, v)$ to denote the $2 \times 2$ matrix composed of two vector columns $u$ and $v$. The transversality condition may be rewritten as $\det(u(x), v(x)) \neq 0$ for all points $x \in T^2$.

Imposing the conjugacy conditions on individual vector fields $u_0$ and $u_1$ and the transversality condition on the pair $(u_0, u_1)$ defines a broad class of switched systems, see, e.g., Section 14.2 of [KH95]. For example, one can start with two linear flows on the torus, with distinct irrational slopes, and apply two conjugations to them separately, using transformations that are appropriately close to the identity map.

Since the torus is compact, the smoothness of driving vector fields guarantees, by a standard Krylov–Bogolyubov argument, that there is at least one invariant measure for the Markov semigroup $(P^t)_{t \geq 0}$. Theorem 1 in [BH12] ensures that $(P^t)_{t \geq 0}$ admits a unique invariant measure and that the invariant measure is absolutely continuous with respect to the product of Lebesgue measure on $T^2$ and counting measure on $\{0, 1\}$ if there is a point $x \in T^2$ that (i) satisfies a Hörmander hypoellipticity condition and (ii) is accessible from any other point of the torus.

In our setting, every point $x$ on the torus satisfies these requirements since (i) our ellipticity condition implies the Hörmander condition for all points $x \in T^2$, and (ii) our conjugacy condition guarantees that for any $x, y \in T^2$, every neighborhood of $x$ is visited by the orbit emitted from $y$. We will show the second part in Section 4.

We denote the unique invariant measure by $\mu$, the marginals of $\mu$ by $\mu_0$ and $\mu_1$, and
their respective density functions with respect to Lebesgue measure by \( \rho_0 \) and \( \rho_1 \). We call \( \rho_0 \) and \( \rho_1 \) invariant densities.

The main result of the present paper is the following:

**Theorem 1.** If smooth vector fields \( u_0 \) and \( u_1 \) each satisfy the conjugacy condition \( A \) and if the pair \((u_0,u_1)\) satisfies the transversality condition \( B \), then the invariant densities \( \rho_0 \) and \( \rho_1 \) admit \( C^\infty \) representatives for every switching rate \( \lambda > 0 \).

We prove Theorem 1 in Section 6.

3. The basic idea

The basic object of study will be the distribution of the process right after two switches. In this way, our smoothing results can also be applied to semi-Markov processes when the switching time distribution is no longer exponential but rather some other probability law on \((0,\infty)\) with smooth density \( \chi(t) \). It is natural to define the random map \( \Phi^{(S,T)}(x) = (\Phi^S \circ \Phi^T)(x) \) when \( T \) and \( S \) are independent, identically distributed random times with density \( \chi(t) \) each.

If \( Z_0 \) is distributed according to a law with density \( h_0 \) then the density of the law of \( Z_1 = \Phi^{(S,T)}(Z_0) \), denoted by \( h_1 \), is given by \( h_1(x) = (Qh_0)(x) \) where \( Q \) is the transfer operator defined by

\[
(Qh)(x) = \int_0^\infty \int_0^\infty \chi(s,t) J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \, ds \, dt.
\]

Here, \( \chi(s,t) = \chi(s)\chi(t) \) and \( \Psi^{(s,t)} = (\Phi^{(s,t)})^{-1} \) and \( J_{(s,t)}(x) \) is a Jacobian associated with the inverse flow. All of this will be defined precisely in Section 5.

Following this imbedded chain obtained by observing the system after jumps was the perspective taken in [LMR15]. Our goal is to study the smoothing properties of \( Q \).

Here we only want to outline the essence of the integration-by-parts estimate at the core of our results. Let us assume that for any direction \( \xi \in \mathbb{R}^2 \) we can find a corresponding direction \( \tau \in \mathbb{R}^2 \) so that

\[
(1) \quad \nabla_x \bigl( J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \bigr) \xi = \nabla_{(s,t)} \bigl( J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \bigr) \tau.
\]

Then, at least formally,

\[
\nabla_x ((Qh)(x)) \xi = \int_0^\infty \int_0^\infty \chi(s,t) \nabla_x \bigl( J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \bigr) \xi \, ds \, dt
\]

\[
= \int_0^\infty \int_0^\infty \chi(s,t) \nabla_{(s,t)} \bigl( J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \bigr) \tau \, ds \, dt.
\]

Assuming \( \chi(s,t) \) is smooth, then by integrating-by-parts the derivative \( \nabla_{(s,t)} \) can be moved onto the density \( \chi(s,t) \) at the price of generating a few boundary terms. However, none of the terms will have any derivatives of the function \( h \). Assuming that all of the terms are well defined, we obtain an expression for \( \nabla_x (Qh) \) which is well defined even if \( h \) is not smooth. This can then be parlayed into a proof that any invariant measure of the system must be smooth. The precise version needed to prove our main result is contained in Theorem 2 and its extensions Corollary 4 and Theorem 4. The latter two show how the above argument can be extended to \( Q^n \) to demonstrate that every successive application of \( Q \) further smoothenes the initial density.
4. Estimates on the deterministic flows.

Let us now clarify the conjugacy condition and show that it leads to at most polynomial growth of various derivatives in time. To emphasize the generality of Condition A, we momentarily consider flows in the more general setting of a compact manifold \( N \). For every smooth vector field \( u \) on \( N \), and for any \( x \in N \), the initial-value problem

\[
\dot{x}(t) = u(x(t)), \quad x(0) = x,
\]

has a unique solution \( \Phi^t_u(x) \) defined for all \( t \in \mathbb{R} \). The function \( \Phi_u : \mathbb{R} \times N \to N \), \( (t, x) \mapsto \Phi^t_u(x) \), is called the flow generated by \( u \). It is \( C^\infty \), jointly in \( t \in \mathbb{R} \) and \( x \in N \). We often treat the flow \( \Phi_u \) as a family of diffeomorphisms \( \Phi^t_u : N \to N \), \( t \in \mathbb{R} \).

Flows \( \Phi_u \) and \( \Phi_v \) generated by vector fields \( u \) and \( v \) on manifolds \( N \) and \( M \) are smoothly conjugated if there is a \( C^\infty \) diffeomorphism \( \sigma : N \to M \) such that for all \( t \in \mathbb{R} \) and \( x \in N \),

\[
\Phi^t_u(x) = \sigma^{-1} \circ \Phi^t_v \circ \sigma(x).
\]

One can prove (see Theorem 14.2.5 in [KH95]) that every smooth fixed-point-free flow on the torus conjugated to a flow preserving a smooth positive density is also conjugated to a special flow over a circle rotation under a smooth roof function. Let us describe this special flow.

Let \( S^1 = \mathbb{R}^1/\mathbb{Z}^1 \) be the unit circle, which we often view as the segment \([0, 1)\) with identified endpoints. Let \( H : S^1 \to (0, +\infty) \) be a smooth function and \( \omega \in [0, 1) \). Due to the smoothness of \( H \), the set

\[
M = \{(r, h) : r \in S^1, \ h \in [0, H(r))] \}/\sim,
\]

where \( \sim \) is the equivalence relation identifying points \((r, H(r))\) and \((r + \omega, 0)\) for all \( r \in S^1 \), is \( C^\infty \) diffeomorphic to the torus so that the flow \( \tilde{\Phi} = \Phi_{\partial_h} \) associated with the “vertical” vector field \((0, 1) = \partial_h\) is a smooth flow. Under this special flow, every point \((r, h) \in M\) moves in the vertical direction with constant speed 1, so that the \( h \) component keeps increasing until it reaches the value \( H(r) \). Upon reaching \((r, H(r))\), the point makes an instantaneous jump to \((r + \omega, 0)\) and from there continues moving upward with unit speed, etc.

Theorem 14.2.5 in [KH95] implies that any flow associated to a smooth vector field \( u \) on \( \mathbb{T}^2 \) satisfying Condition A is smoothly conjugated to a special flow with appropriately chosen \( \omega = \omega_u \) and \( H = H_u \). Moreover, since Condition A requires that \( u \) does not admit any periodic orbits, the number \( \omega \) in the above construction has to be irrational. Therefore, all orbits are dense in \( M \) for the special flow and in \( \mathbb{T}^2 \) for \( \Phi_u \). The conjugacy of the flows \( \Phi_u \) and \( \Phi \) by the diffeomorphism \( \sigma \) can be rewritten as \( \nabla_x \sigma(x) u(x) = (0, 1) \) for all \( x \in \mathbb{T}^2 \). Here, \( \nabla_x \sigma(x) \) is the Jacobian matrix of the map \( \sigma \) at point \( x \). For fixed \( t \in \mathbb{R} \), we denote the Jacobian matrix of \( x \mapsto \Phi^t(x) \) by \( \nabla_x \Phi^t(x) \). For nonnegative integers \( n_1 \) and \( n_2 \), we write \( \partial_1^{n_1} \partial_2^{n_2} \Phi^t(x) \) for the coordinatewise partial derivative of \( x \mapsto \Phi^t(x) \), where each coordinate of \( \Phi^t(x) \) is differentiated \( n_1 \) times with respect to the first coordinate of \( x \) and \( n_2 \) times with respect to the second coordinate of \( x \). Finally, for any \( n \in \mathbb{N} \), we denote the Euclidean norm on \( \mathbb{R}^n \) by \(|-|\).

The following polynomial estimate on the Jacobian of the flow is crucial for our analysis. It is based on conjugacy to a special flow described above. This
estimate implies that the Lyapunov exponents of the flows we consider are equal to zero. If the Lyapunov exponents are non-zero, one must be positive and one must be negative. Excluding negative Lyapunov exponents is natural as the associated contraction often leads to invariant densities with singularities.

**Lemma 1.** For any smooth vector field $u$ on $\mathbb{T}^2$ satisfying Condition A, there is a constant $c > 1$ and a family of constants $c_n > 0$, $n \geq 0$, such that for all $t > 0$ and for all $x \in \mathbb{T}^2$, the flow $\Phi = \Phi_u$ satisfies the following estimates:

$$|\partial_{n_1} \partial_{n_2}^2 \Phi^t(x)| \leq c_{n_1 + n_2} (1 + t)^{n_1 + n_2}$$

for $n_1, n_2 \geq 0$ and $n_1 + n_2 \geq 1$, and

$$c^{-1} \leq \det \nabla_x \Phi^t(x) \leq c.$$

The proof of Lemma 1 is given in Section 7. It relies heavily on smooth conjugacy of $\Phi_u$ to a special flow described above.

5. An integral equation for invariant densities

We return to the setting from Section 2. For notational convenience, we define the inverse flows

$$\Psi^t_i(x) = (\Phi^t_i)^{-1}(x) = \Phi^{-t}_i(x), \quad i \in \{0, 1\}, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}^2,$$

and the composition

$$\Psi^{(s,t)}(x) = (\Psi^s_1 \circ \Psi^t_0)(x), \quad (s, t) \in \mathbb{R}^2, \quad x \in \mathbb{T}^2.$$

Furthermore, we define $F^t_i(x) = \nabla_x \Psi^t_i(x)$ and the Jacobian

$$J_{(s,t)}(x) = \det (F^s_1(\Psi^t_0(x))F^t_0(x)).$$

Finally, let

$$U(x) = (u_1(x), u_0(x))$$

be the matrix with columns $u_1(x)$ and $u_0(x)$.

Now, we extend the integral equation from Lemma 2 in [BHM15] to the case of the 2D-switching system introduced in Section 2. Instead of considering just the latest switch, we consider the latest 2 switches leading to the current state. To that end, we define the transfer operator

$$Qh(x) = \int_{\mathbb{R}^2} \lambda^2 e^{-\lambda(s+t)} J_{(s,t)}(x) h(\Psi^{(s,t)}(x)) \, ds \, dt, \quad x \in \mathbb{T}^2$$

for real-valued integrable functions $h$ on $\mathbb{T}^2$. Observe that if $S$ and $T$ are independent exponentially distributed random variables with parameter $\lambda$ then

$$Qh(x) = \mathbb{E} \left[ J_{(S,T)}(x) h(\Psi^{(S,T)}(x)) \right].$$

According to the following lemma, the invariant density $\rho_0$ is a fixed point of $Q$.

**Lemma 2.** We have $\rho_0 = Q\rho_0$.

**Remark 1.** For $x \in \mathbb{T}^2$, the term $Q\rho_0(x)$ can be interpreted as an average over possible histories of the previous two switches leading up to point $x$ and driving vector field $u_0$. 
Proof of Lemma 2: As in Lemma 2 in [BHM15], one can show that
\[ \rho_i(x) = \int_{\mathbb{R}^+} \lambda e^{-\lambda t} \det F_t^i(x) \rho_{1-i}(\Psi_t^i(x)) \, dt, \quad i \in \{0, 1\}. \]

The lemma follows from plugging the instances of this identity for \(i = 0\) and \(i = 1\) into one another and using the fact that the pushforward of a function under the cumulative flow \(\Phi_0^t \circ \Phi_1^s\) is the composition of pushforwards under the individual flows \(\Phi_0^t\) and \(\Phi_1^s\). □

6. Smoothness through integration by parts

In this section, we prove the main result on the smoothness of the invariant density (Theorem 1) using integration by parts with respect to the times between switches. We begin by defining a collection of “Good” functions \(G\) for which integration by parts can be performed.

We define \(G\) to be the set of all \(C^\infty\) functions \(G: \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}\) such that the following conditions hold.

(1) There is a polynomial \(p: \mathbb{R}^2 \to \mathbb{R}\) such that
\[ |G(x, s, t)| \leq p(s, t), \quad x \in \mathbb{T}^2, \ (s, t) \in \mathbb{R}_+^2. \]

(2) For all \(n \in \mathbb{N}\) and \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(\alpha_l\) equal to \((s, t)\) or \(x\), there is a polynomial \(q: \mathbb{R}^2 \to \mathbb{R}\) such that
\[ |\nabla^n_\alpha G(x, s, t)| \leq q(s, t) \prod_{l=1}^n |\xi_l| \]
for all \(x \in \mathbb{T}^2, \ (s, t) \in \mathbb{R}_+^2\) and \(\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{2n}\) with \(\xi_l \in \mathbb{R}^2\) for \(1 \leq i \leq n\). Here, \(\nabla^n_\alpha G(x, s, t)\) denotes an \(n\)-fold differential of \(G\) at the point \((x, s, t)\), which can be thought of as a multilinear form on the \(n\)-fold Cartesian product of \(\mathbb{R}^2\) with itself.

These conditions are equivalent to saying that \(G\) and all higher-order partial derivatives of \(G\) are bounded by polynomials in \(s\) and \(t\). Observe that if \(P\) is a polynomial in \(n\) variables and if \(G^{(1)}, \ldots, G^{(n)}\) are in \(G\), then \(P(G^{(1)}, \ldots, G^{(n)})\) is in \(G\) as well. Furthermore, if \(G\) and \(H^{(3)}, \ldots, H^{(4)}\) are in \(G\), so is \(G(H^{(1)}, \ldots, H^{(4)})\). Finally, if \(G \in G\), then the partial derivatives of \(G\) of any order are in \(G\) as well.

The following lemma, which will be proven in Section 7, shows that most objects of interest are in \(G\).

Lemma 3. The components of \((x, s, t) \mapsto U(x)^{-1}\) and \((x, s, t) \mapsto \Psi_t^i(x), \ i \in \{0, 1\}\), (both defined in Section 5) are in \(G\).

As an immediate corollary of Lemma 3, the components of \((x, s, t) \mapsto F_t^i(x), \ i \in \{0, 1\}\), and the Jacobian \(J_{(s,t)}(x)\) (which were also defined in Section 5) are in \(G\).
6.1. Integration by parts. We begin with a remark on our notation for derivatives. If the differential operator precedes a term in parentheses, e.g. \( \nabla_x h(\Psi^{(s,t)}x) \), we apply the operator to the entire term, so in the given example we would differentiate the function \( x \mapsto h(\Psi^{(s,t)}x) \). If we wish to differentiate only the function \( h \) and then evaluate the derivative at \( \Psi^{(s,t)}(x) \), we write \((\nabla_x h)(\Psi^{(s,t)}x)\).

Integration by parts applied to an integral over \([0, \infty) \times (0, \infty)\) results in five terms: an integral over the interior of \([0, \infty)^2\), two boundary terms corresponding to the coordinate axes \( s = 0 \) and \( t = 0 \), and two boundary terms corresponding to \( s = \infty \) and \( t = \infty \). In our setting, the terms corresponding to \( s = \infty \) and \( t = \infty \) vanish. To deal with the remaining three terms, we introduce the projections \( \pi_0(s, t) = (s, t) \), \( \pi_1(s, t) = (s, 0) \) and \( \pi_2(s, t) = (0, t) \) for \((s, t) \in \mathbb{R}^2\).

**Theorem 2.** Fix \( G \in \mathcal{G} \). For any \( \xi \in \mathbb{R}^2 \), there exist \( G^{(0)}_\xi, G^{(1)}_\xi, G^{(2)}_\xi \in \mathcal{G} \) such that

\[
\mathbb{E} \left[ G(x, S, T) \nabla_x (h(\Psi^{(S,T)}x)) \right] = \sum_{i=0}^2 \mathbb{E} \left[ G^{(i)}_\xi(x, \pi_i(S, T)) h(\Psi^{\pi_i(S,T)}x) \right]
\]

for all \( C^1 \) functions \( h : \mathbb{T}^2 \to \mathbb{R} \). In addition, there exists \( K > 0 \) (depending only on \( G \)) such that \( \mathbb{E} |G^{(i)}_\xi(x, \pi_i(S, T))| \leq K|\xi| \) for all \( i \in \{0, 1, 2\} \) and \( x \in \mathbb{T}^2, \xi \in \mathbb{R}^2 \).

**Proof:** Let \( \xi \in \mathbb{R}^2 \) and \((s, t) \in \mathbb{R}^2_+ \). In the notation of Section 5 we have that

\[
\nabla_x \Psi^{(s,t)}(x) \xi = F^s_1(\Psi^0_0 x) F^t_0(x) \xi,
\]

\[
\nabla_{(s,t)} \Psi^{(s,t)}(x) = -F^s_0(\Psi^0_0 x) U(\Psi^1_0 x).
\]

The first equation is a straightforward application of the chain rule. The second is obtained by using the forward derivative defined by \( \frac{1}{\delta} [\Psi^s_0 \circ \Psi^t_0 - \Psi^s_0] \) as \( \delta \to 0 \) for the \( t \) derivative and the backward derivative defined by \( \frac{1}{\delta} [\Psi^t_1 \circ \Psi^t_1 - \Psi^s_1] \) as \( \delta \to 0 \) for the \( s \) derivative. By the uniform ellipticity condition, \( U(x) \) is invertible for all \( x \). Setting

\[
\tau_t(x) = -U(\Psi^0_0 x)^{-1} F^t_0(x),
\]

and combining the equations in (7) produces

\[
\nabla_x \Psi^{(s,t)}(x) \xi = \nabla_{(s,t)} \Psi^{(s,t)}(x) \tau_t(x) \xi.
\]

Hence, this choice of \( \tau \) realizes the relationship promised in (11) which transfers a variation in \( x \) to one in \( (s, t) \).

For any function \( h : \mathbb{T}^2 \to \mathbb{R} \) which is \( C^1 \), a direct calculation yields

\[
\nabla_x (h(\Psi^{(s,t)}x)) = (\nabla_x h)(\Psi^{(s,t)}x) \nabla_x \Psi^{(s,t)}(x) \xi,
\]

which when combined with (8) produces

\[
\nabla_x (h(\Psi^{(s,t)}x)) \xi = \nabla_{(s,t)} (h(\Psi^{(s,t)}(x))) \tau_t(x) \xi.
\]

Since \( \nabla_x h \) is bounded and \( G \in \mathcal{G} \), equation (9) implies that

\[
\mathbb{E} \left[ G(x, S, T) \nabla_x (h(\Psi^{(S,T)}x)) \right] = \mathbb{E} \left[ G(x, S, T) \nabla_{(s,t)} (h(\Psi^{(S,T)}x)) \tau_T(x) \xi \right]
\]

\[
= \int_{\mathbb{R}^2_+} \lambda^2 e^{-\lambda(s+t)} G(x, s, t) \nabla_{(s,t)} (h(\Psi^{(s,t)}x)) \tau_t(x) \xi \, ds \, dt.
\]
After observing that the components of \((x, s, t) \mapsto \tau_t(x)\) are in \(\mathcal{G}\) by Lemma 3 we apply integration by parts to (10). The divergence of the two-dimensional vector \(\lambda^2 e^{-\lambda(s+t)}G(x, s, t)\tau_t(x)\xi\) with respect to \((s, t)\) equals \(-\lambda^2 e^{-\lambda(s+t)}G^{(0)}_\xi(x, s, t)\), where

\[
G^{(0)}_\xi(x, s, t) := G(x, s, t) \left( \lambda \left( \mathbb{1} \cdot \tau_t(x) \xi \right) - (e_2 \cdot \partial_t \tau_t(x) \xi) \right) - \nabla_{(s, t)} G(x, s, t) \tau_t(x) \xi.
\]

Here, \(\cdot\) denotes the Euclidean inner product, \(\mathbb{1} = (1, 1)^T\) and \(e_i\) is the \(i\)th standard unit vector in \(\mathbb{R}^2\) for \(i \in \{1, 2\}\). Since \(h\) is bounded and since \(G\) and the components of \(\tau_t(x)\) are in \(\mathcal{G}\), there is a polynomial \(p\) such that

\[
\left| G(x, s, t) h(\Psi^{(s,t)}x)(\tau_t(x) \xi \cdot e_i) \right| \leq p(s, t), \quad x \in \mathbb{T}^2, \quad (s, t) \in \mathbb{R}^2_+, \quad i \in \{1, 2\}.
\]

Thus, for \(t \in \mathbb{R}_+\),

\[
\left| \int_0^\infty \lambda^2 e^{-\lambda(s+t)}G(x, s, t) h(\Psi^{(s,t)}x)(\tau_t(x) \xi \cdot e_2) \, ds \right| \leq \int_0^\infty \lambda^2 e^{-\lambda(s+t)} p(s, t) \, ds,
\]

and the integral on the right tends to 0 as \(t \to \infty\). Similarly,

\[
\lim_{s \to \infty} \int_0^\infty \lambda^2 e^{-\lambda(s+t)}G(x, s, t) h(\Psi^{(s,t)}x)(\tau_t(x) \xi \cdot e_1) \, dt = 0.
\]

The integration-by-parts formula implies that the integral in (10) equals

\[
\int_{\mathbb{R}^2_+} \lambda^2 e^{-\lambda(s+t)}G^{(0)}_\xi(x, s, t) h(\Psi^{(s,t)}x) \, ds \, dt
\]

\[
- \int_0^\infty \lambda^2 e^{-\lambda s}G(x, 0, t) h(\Psi^{(0,t)}x)(\tau_t(x) \xi \cdot e_1) \, dt
\]

\[
- \int_0^\infty \lambda^2 e^{-\lambda s}G(x, s, 0) h(\Psi^{(s,0)}x)(\tau_0(x) \xi \cdot e_2) \, ds.
\]

The single integrals converge because \(G \in \mathcal{G}\) and the double integral converges because all other integrals do. Defining

\[
G^{(1)}_\xi(x, s, 0) = -\lambda G(x, s, 0)(\tau_0(x) \xi \cdot e_2),
\]

\[
G^{(2)}_\xi(x, 0, t) = -\lambda G(x, 0, t)(\tau_0(x) \xi \cdot e_1),
\]

we have

\[
E \left[ G(x, S, T) \nabla_x (h(\Psi^{(S,T)}x)) \xi \right] = \sum_{i=0}^2 E \left[ G^{(i)}_\xi(x, \pi_i(S, T)) h(\Psi^{\pi_i(S,T)}x) \right],
\]

and from Lemma 3 it follows that \(G^{(i)}_\xi \in \mathcal{G}\) for \(i \in \{0, 1, 2\}\). The second part of Theorem 2 is a consequence of the fact that for \(i \in \{0, 1, 2\}\), \(G^{(i)}_\xi\) can be written as the dot product of \(\xi\) and a vector-valued function whose components are in \(\mathcal{G}\). □
6.2. \textbf{$L^1$ smoothing estimates.} In this subsection, building on the integration by parts formula of the last section, we derive a formula for the derivative of $Qh(x)$ which does not involve the derivative of $h$. This lets us bound the $L^1$ norm of $\nabla(Qh)$ in terms of the $L^1$ norm of $h$. We denote the $L^1$ norm on $T^2$ by $\| \cdot \|_{L^1(T^2)}$ or just by $\| \cdot \|_{L^1}$. For $n \in \mathbb{N}$ and a $C^n$ function $h : T^2 \to \mathbb{R}$, let

$$\|\nabla^n x h\|_{L^1} := \int_{\mathbb{T}^2} \sup_{\xi \in \mathbb{R}^n} |\nabla^n x h(x)| \, dx,$$

where the supremum is taken over all $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathbb{R}^2$ and $|\xi_i| = 1$ for $1 \leq i \leq n$. We begin with a simple estimate which lets us bound various expectations with respect to the $L^1$ norm.

\textbf{Lemma 4.} Let $G_0$ be a, possibly uncountable, subset of $G$ such that there exists a single polynomial $p(s, t)$ with $|G(x, s, t)| \leq p(s, t)$ for all $G \in G_0$, $x \in \mathbb{T}^2$, and $s, t \geq 0$. Then there exists a constant $K$ so that for any $i \in \{0, 1, 2\}$ and any $h \in L^1(T^2)$,

$$\int_{T^2} \sup_{G \in G_0} E[(G(x, S, T)||h(\Psi_{i}(S, T)(x))|)] \, dx \leq K \|h\|_{L^1}.$$

\textbf{Proof:} Fix $i \in \{0, 1, 2\}$ and $h \in L^1(T^2)$. Since $|G(x, s, t)| \leq p(s, t)$ for all $G \in G_0$, we have

$$\int_{T^2} \sup_{G \in G_0} E[(G(x, S, T)||h(\Psi_{i}(S, T)x)|)] \, dx \leq E \left[ p(S, T) \int_{T^2} |h(\Psi_{i}(S, T)x)| \, dx \right].$$

For fixed $s, t \geq 0$, we make the change of variables $y = \Psi_{i}(s, t)(x)$ and let $\Phi_{i}(s, t)$ denote the inverse of $\Psi_{i}(s, t)$. Since $\Psi_{i}(s, t) (T^2) = T^2$, the bound $|\det \nabla_x \Phi_{i}(s, t)(y)| \leq c$ from Lemma 4 implies that the term on the right-hand side of (11) is less than or equal to

$$cE[p(S, T)] \int_{T^2} |h(y)| \, dy. \tag{12}$$

We now show the announced $L^1$ estimate on $\nabla(Qh)$ for $C^1$ functions $h$.

\textbf{Theorem 3.} For any $\xi \in \mathbb{R}^2$, there exist $G^{(0)}_{\xi}, G^{(1)}_{\xi}, G^{(2)}_{\xi} \in G$ such that

$$\nabla_x(Qh(x)) = \sum_{i=0}^{2} E \left[ G^{(i)}_{\xi}(x, \pi(S, T))h(\Psi_{i}(S, T)x) \right] \tag{13}$$

for all $C^1$ functions $h$. Furthermore,

$$\|\nabla_x(Qh)\|_{L^1} \leq K \|h\|_{L^1},$$

for some $K > 0$ independent of $h$.

\textbf{Remark 2.} One can extend the above theorem to show that for any $h \in L^1(T^2)$, $Qh(x)$ is in the Sobolev space $W^{1,1}$, functions whose weak derivatives belong to $L^1(T^2)$, with the weak derivative given by the right-hand side of (13). The argument is given in the proof of Theorem 4 where more is proven.
Proof of Theorem 3: Let us fix a $C^1$ function $h$. Since $h$ is in $C^1(\mathbb{T}^2)$ and since $J_{i(s,t)}(x)$ is in $\mathcal{G}$, (11) implies that for any $\xi \in \mathbb{R}^2$: 

$$
\nabla_x (Qh(x)) \xi = E \left[ \nabla_x J_{i(s,t)}(x) \xi h(\Psi^{(S,T)}(x)) + J_{i(s,t)}(x) \nabla_x (h(\Psi^{(S,T)}(x))) \right].
$$

Invoking again that $J_{i(s,t)}(x)$ is in $\mathcal{G}$, we deduce from Theorem 2 that there exist $G^{(0)}_\xi, G^{(1)}_\xi, G^{(2)}_\xi \in \mathcal{G}$, not depending on $h$, such that (12) holds. Moreover, each function $G^{(i)}_\xi$ can be written as the dot product of $\xi$ and a vector-valued function whose components are in $\mathcal{G}$ and do not depend on $\xi$. Therefore, there exists a single polynomial $p(s,t)$ such that 

$$
|G^{(i)}_\xi(x,s,t)| \leq p(s,t)
$$

for all $i \in \{0,1,2\}$, $x \in \mathbb{T}^2$, $(s,t) \in \mathbb{R}_+^2$, and $\xi \in \mathbb{R}^2$ such that $|\xi| = 1$. By Lemma 4 there exists $K > 0$ independent of $h$ such that 

$$
\|\nabla_x (Qh)\|_{L^1} \leq \sum_{i=0}^{2} \int_{\mathbb{T}^2} \sup_{|\xi| = 1} E \left[ |G^{(i)}_\xi(x,\pi_i(S,T))| |h(\Psi^{(S,T)}(x))| \right] dx \leq K \|h\|_{L^1}.
$$

2

6.3. Smoothness. We will now generalize the approach from the previous subsections in order to show that the invariant density $\rho_0$ is $C^\infty$ smooth. In particular, we will show that for any positive integer $n$, the derivative $\nabla^n_x (Q^n h)$ is bounded in $L^1$ by the $L^1$-norm of $h$. We begin with a generalization of Theorem 2.

Corollary 1. Let $n \geq 2$ and $G \in \mathcal{G}$. There exists $K > 0$ such that for any $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathbb{R}^2$ and for any $C^n$ function $h : \mathbb{T}^2 \to \mathbb{R}$, the term 

$$
E \left[ G(x,S,T) \nabla^n_x (h(\Psi^{(S,T)}x)) \xi \right]
$$

can be written as a linear combination of integrals of the form 

$$
E \left[ H^{(j)}(x,\pi_j(S,T)) \nabla_x^{n-1-k} (h(\Psi^{\pi_j(S,T)}x)) \eta \right],
$$

where $j \in \{0,1,2\}$, $0 \leq k \leq n-1$, $\eta \in \mathbb{R}^{2(n-1-k)}$ equal to a subset of $\xi$ with complement $\zeta \in \mathbb{R}^{2(k+1)}$, and $H^{(j)}_\xi \in \mathcal{G}$ such that 

$$
E[H^{(j)}_\xi(x,\pi_j(S,T))| \leq K \prod_{l=1}^{k+1} |\zeta_l|.
$$

Neither the functions $H^{(j)}_\xi$ from $\mathcal{G}$ nor the coefficients of the linear combination depend on $h$.

Proof: Let $h$ be a $C^n$ function and let $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathbb{R}^2$ for $1 \leq i \leq n$. Set $\tilde{\xi} = (\xi_1, \ldots, \xi_n)$. Then, 

$$
\nabla^n_x (h(\Psi^{(s,t)}x)) \xi = \nabla_x^{n-1} (\nabla_x (h(\Psi^{(s,t)}x)) \xi_1) \tilde{\xi}.
$$

By (9), the right-hand side of (15) can be written as 

$$
\nabla_x^{n-1} \left( \sum_{i=1}^{2} \left[ (\nabla_{(s,t)} (h(\Psi^{(s,t)}x)) e_i) \right] \left[ \tau_i(x) \xi_1 \cdot e_i \right] \right) \tilde{\xi}
$$
where \( e_i \) the standard basis in \( \mathbb{R}^2 \). Using the product rule, this derivative is a linear combination of terms of the form
\[
\nabla_x^{n-1-k} (\nabla_{s,t} (h(\Psi^{(s,t)} x)) e_i) \eta] [\nabla_x^k (\tau_i(x) \xi_1 \cdot e_i) \tilde{\zeta}],
\]
where \( 0 \leq k \leq n - 1 \), \( \eta \in \mathbb{R}^{2(n-1-k)} \) equal to a subset of \( \tilde{\zeta} \), and \( \tilde{\zeta} \in \mathbb{R}^{2k} \) the complement of \( \eta \) in \( \tilde{\zeta} \). Fixing \( k, i \) and \( \eta \) and interchanging the order of differentiation in the first term in the preceding product gives
\[
\nabla_x^{n-1-k} (\nabla_{s,t} (h(\Psi^{(s,t)} x)) e_i) \eta = \nabla_{s,t} (\nabla_x^{n-1-k} (h(\Psi^{(s,t)} x)) \eta) e_i.
\]
Hence, if we set \( \zeta = (\zeta_1, \ldots, \zeta_{k+1}) := (\xi_1, \tilde{\zeta}) \) and
\[
H_\zeta(x, s, t) = G(x, s, t) \nabla_x^k (\tau_i(x) \xi_1 \cdot e_i) \tilde{\zeta},
\]
we can write
\[
\text{(16)} \quad \mathbb{E} \left[ G(x, S, T) \nabla_x^{n-1-k} \left( \nabla_{s,t} (h(\Psi^{(s,t)} x)) e_i) \eta] [\nabla_x^k (\tau_i(x) \xi_1 \cdot e_i) \tilde{\zeta}] \right]
\]
\[
= \mathbb{E} \left[ H_\zeta(x, S, T) \nabla_x^{n-1-k} (h(\Psi^{(s,t)} x)) \eta) e_i \right].
\]
The divergence of the two-dimensional vector \( \lambda^2 e^{-\lambda (s+t)} H_\zeta(x, s, t) e_i \) with respect to \( (s, t) \) is \( -\lambda^2 e^{-\lambda (s+t)} H_\zeta(x, s, t) e_i \), where
\[
H_\zeta(x, s, t) = \lambda H_\zeta(x, s, t) - \nabla_{s,t} H_\zeta(x, s, t) e_i.
\]
Similarly to the proof of Theorem 3 we obtain from integration by parts that the integral on the right side of (16) equals
\[
\sum_{j=0}^2 \mathbb{E} \left[ H_\zeta^{(j)}(x, S, T) \nabla_x^{n-1-k} (h(\Psi^{(s,t)} x)) \eta \right],
\]
where
\[
H_\zeta^{(1)}(x, s, t) = -\lambda H_\zeta(x, s, t) e_i \cdot e_2),
\]
\[
H_\zeta^{(2)}(x, s, t) = -\lambda H_\zeta(x, t) e_i \cdot e_1).
\]
Since \( H_\zeta \in \mathcal{G} \), we also have \( H_\zeta^{(j)} \in \mathcal{G} \) for \( 0 \leq j \leq 2 \). In addition to \( k, i \) and \( \eta \), fix \( j \in \{0, 1, 2\} \). We show that there is a constant \( K > 0 \), independent of \( \xi \) and \( h \), such that (16) holds. This will complete the proof of Corollary 4 because for given \( n \) and \( \xi \), there are only finitely many ways of choosing \( k, i, j \) and \( \eta \). As \( G \) and the components of \( \tau_i \) are in \( \mathcal{G} \), \( H_\zeta^{(j)}(x, s, t) \) can be written as
\[
\sum_{i_1=1}^2 \cdots \sum_{i_{k+1}=1}^2 \sum_{i_1 \cdots i_{k+1} \in \{1, 2\}} g_{i_1 \cdots i_{k+1}}(x, s, t) \prod_{l=1}^{k+1} (\zeta_i)_{i_l},
\]
where \((\zeta_i)_{i_l}\) is the \( i_l \)-th component of \( \zeta_l \) and \( g_{i_1 \cdots i_{k+1}} \) are functions in \( \mathcal{G} \) that do not depend on \( \xi \) or \( h \). Thus,
\[
\mathbb{E}\left| H_\zeta^{(j)}(x, S, T) \right| \leq \max_{i_1 \cdots i_{k+1} \in \{1, 2\}} \mathbb{E}|g_{i_1 \cdots i_{k+1}}(x, S, T)| 2^{k+1} \prod_{l=1}^{k+1} |\zeta_l|.
\]

The next corollary generalizes the smoothing estimate in Theorem 3.
Theorem 4. We will see that Theorem 1 is essentially a corollary of this result.

Corollary 2. Let $n$ be a positive integer. There exists $K_n > 0$ such that for any $C^n$ function $h : \mathbb{T}^2 \to \mathbb{R}$, we have

$$\|\nabla^j_x(Q^n h)\|_{L^1} \leq K_n \max\{\|h\|_{L^1}, \|\nabla_x h\|_{L^1}, \ldots, \|\nabla^{n-1}_x h\|_{L^1}\}.$$ 

This corollary in turn implies the following result which captures the smoothing effects of $Q^n$ and the intuition that each application of $Q$ leads to another round of averaging; and hence, another degree of smoothness.

Corollary 3. For any $n \in \mathbb{N}$ there exists $K_n > 0$ such that for any $C^n$ function $h : \mathbb{T}^2 \to \mathbb{R}$, we have

$$\|\nabla^j_x(Q^n h)\|_{L^1} \leq K_n \|h\|_{L^1}, \quad 0 \leq j \leq n. \quad (18)$$

Proof of Corollary 3. Applying Corollary 2 to the function $f = Q^n h$ produces

$$\|\nabla^j_x(Q^n h)\|_{L^1} \leq C \max\{\|Q^n h\|_{L^1}, \|\nabla_x(Q^n h)\|_{L^1}, \ldots, \|\nabla^{n-1}_x(Q^n h)\|_{L^1}\}$$

for $1 \leq j \leq n$. Repeatedly applying this type of estimate to each of the terms of the form $\|\nabla^j_x(Q^n h)\|_{L^1}$ on the right-hand side shows that there exists $C > 1$ so that

$$\|\nabla^j_x(Q^n h)\|_{L^1} \leq C \|Q^n h\|_{L^1}.$$ 

Since $Q$ is a bounded operator on $L^1(\mathbb{T}^2)$, there exists $C_k$ so that $\|Q^k h\|_{L^1} \leq C_k \|h\|_{L^1}$ for $0 \leq k \leq n$. Thus, (18) holds with $K_n = C \max\{C_0, \ldots, C_n\}$. \qed

From Corollary 3, we can now deduce Theorem 4 via an approximation argument. We will see that Theorem 4 is essentially a corollary of this result.

Theorem 4. For any $h \in L^1(\mathbb{T}^2)$ and $n \in \mathbb{N}$, $Q^n h$ is in the Sobolev space $W^{n,1}$ which consists of functions whose weak derivatives up to and including order $n$ exist and are in $L^1(\mathbb{T}^2)$. Additionally $Q^{n+3} h$ is in $C^n(\mathbb{T}^2)$ which is the space of $n$-times continuously differentiable functions.

Proof: Since $h \in L^1(\mathbb{T}^2)$ and since $C^\infty(\mathbb{T}^2)$ is dense in $L^1(\mathbb{T}^2)$, there is a sequence $(h_k)_{k \geq 1}$ of $C^\infty$ functions that converges to $h$ in $L^1(\mathbb{T}^2)$. We can choose the approximating sequence in such a way that $\|h_k\|_{L^1} \leq \|h\|_{L^1}$ for all $k \geq 1$. Fix a positive integer $n$. By Corollary 3, we have

$$\|\nabla^j_x(Q^n h_k)\|_{L^1} \leq K_n \|h_k\|_{L^1} \leq K_n \|h\|_{L^1}$$

for $0 \leq j \leq n$ and $k \geq 1$. The sequence $(Q^n h_k)_{k \geq 1}$ is therefore a bounded sequence in the Sobolev space $W^{n,1}(\mathbb{T}^2)$ of $L^1$-functions whose weak derivatives up to order $n$ are also in $L^1(\mathbb{T}^2)$. (As we have seen, the derivatives of $Q^n h_k$ even exist in the classical sense.)

Now, the Rellich–Kondrachov theorem (see e.g. [Ada75, Theorem 6.2]) implies that $W^{n+3,1}(\mathbb{T}^2)$ is compactly embedded in $C^n(\mathbb{T}^2)$. Thus, there is a subsequence $(Q^{n+3} h_{k_i})_{i \geq 1}$ that converges to a limit in $C^n(\mathbb{T}^2)$. On the other hand, $(Q^{n+3} h_{k_i})_{i \geq 1}$ also converges to $Q^{n+3} h$ in $L^1(\mathbb{T}^2)$ because $Q$ is bounded. This implies that $Q^{n+3} h$ has a representative in $C^n(\mathbb{T}^2)$. \qed

We now turn to the proof of the main result Theorem 4 which is an immediate consequence of Theorem 4 and the invariance of $\rho_0$. 


Proof of Theorem 1. For any $n \in \mathbb{N}$, Theorem 1 implies that $Q^{n+3}\rho_0 \in C^n(T^2)$. Since the invariance of $\rho_0$ implies that $\rho_0 = Q^{n+3}\rho_0$, the proof is complete. □

We now return to the proof of Corollary 2 which will require the following lemma.

Lemma 5. Let $n \in \mathbb{Z}_+$. There exists a polynomial $p_n(s,t)$ such that for any $C^n$ function $h : T^2 \to \mathbb{R}$, we have

$$||\nabla^n_x (h \circ \Psi(s,t))||_{L^1} \leq p_n(s,t) \max\{||h||_{L^1}, ||\nabla_x h||_{L^1}, \ldots, ||\nabla^n_x h||_{L^1}\}.$$  

Proof: We prove the lemma by induction. The base case $n = 0$ follows from Lemma 4 after the change of variables $y = (s,t)(x)$. In the induction step, assume that the inequality holds for some $n \in \mathbb{Z}_+$. For a fixed $C^{n+1}$ function $h$, $x \in T^2$, $\xi_1 \in \mathbb{R}^2$ and $\xi = (\xi_2, \ldots, \xi_{n+1}) \in \mathbb{R}^{2n}$ with $|\xi_1| = \ldots = |\xi_{n+1}| = 1$, we have

$$\nabla_x^{n+1} (h(\Psi(s,t)x))(\xi_1, \xi) = \nabla_x^n (\nabla_x (h(\Psi(s,t)x))\xi_1)\xi,$$

by the same reasoning as in the proof of Corollary 1, the derivative on the right side of (19) is a linear combination of terms of the form

$$\nabla_x^{n-k}((\nabla_x h)\Psi(s,t)x)\eta \nabla_x^k \Psi(s,t)x(\xi_1 \cdot e_i)\xi,$$

where $0 \leq k \leq n$, $i \in \{1, 2\}$, $\eta \in \mathbb{R}^{2(n-k)}$ a subset of $\xi$ and $\zeta \in \mathbb{R}^{2k}$ the complement of $\eta$ with respect to $\zeta$. For $i \in \{1, 2\}$, let $g_i(y) := (\nabla_x h)(y)e_i$. Since $g_i$ is in $C^n$, the induction hypothesis implies that for any $k \in \{0, \ldots, n\}$ and $\eta \in \mathbb{R}^{2(n-k)}$,

$$||\nabla_x^{n-k}(g_i \circ \Psi(s,t))||_{L^1} \leq p_n-k(s,t) \max\{||g_i||_{L^1}, \ldots, ||\nabla_x^{n-k}g_i||_{L^1}\} \leq p_n-k(s,t) \max\{||h||_{L^1}, \ldots, ||\nabla_x^{n+1}h||_{L^1}\}.$$  

Recall that the components of $\Psi(s,t)(x)$ are in $\mathcal{G}$. This implies that for every $k \in \{0, \ldots, n\}$, there is a polynomial $q_k$ such that

$$||\nabla_x^k (\nabla_x \Psi(s,t)x)\xi_1 \cdot e_i ||_{L^1} \leq q_k(s,t).$$

Here, it is important to note that the term on the right depends neither on $x$ nor on $\zeta$. Applying the estimates in (21) and (22) to the term in (20) yields the desired result. □

Proof of Corollary 2. The case $n = 1$ was treated in Theorem 1 so we may assume without loss of generality that $n \geq 2$. Let $h$ be a $C^n$ function and let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{2n}$ with $|\xi_1| = \ldots = |\xi_n| = 1$. Since $h$ is assumed to be in $C^n$ and since $J(s,t)(x)$ is in $\mathcal{G}$, (6) implies that we can write $\nabla_x^k(Qh)(x)\xi$ as a linear combination of terms of the form

$$E \left[ (\nabla_x^{n-k}J(s,T)(x)\eta) (\nabla_x^k (h(\Psi(s,T)x))\zeta) \right],$$

where $0 \leq k \leq n$, $\eta \in \mathbb{R}^{2(n-k)}$ equal to a subset of $\xi$ and $\zeta \in \mathbb{R}^{2k}$ equal to the complement of $\eta$ in $\xi$. Again because of $J(s,t)(x) \in \mathcal{G}$, there are polynomials $q_1, \ldots, q_n : \mathbb{R}^2 \to \mathbb{R}$, independent of $\xi$, such that

$$||\nabla_x^m J(s,t)(x)\eta|| \leq q_m(s,t)$$

for all $m \leq n$. □
for all \( x \in T^2 \), \( (s, t) \in \mathbb{R}_+^2 \), \( 1 \leq m \leq n \), and \( h \in \mathbb{R}^{2m} \) a subset of \( \xi \). By Lemma 5 there are also polynomials \( p_0, \ldots, p_n \), independent of \( h \), such that

\[
\|\nabla^m_x (h \circ \Psi(s, t))\|_{L^1} \leq p_m(s, t) \max\{\|h\|_{L^1}, \ldots, \|\nabla^m_x h\|_{L^1}\}
\]

for \( 0 \leq m \leq n \). Thus, for \( k < n \), we have

\[
\int_{T^2} \sup_{\xi \in \mathbb{R}^m, |\xi|_1 = \ldots = |\xi|_n = 1} \left| \mathbb{E} \left[ \left( \nabla^{n-k}_x J_{(S, T)}(x) h(\Psi(S, T)x) \right) \nabla^k_x (h(\Psi(S, T)x)) \xi \right] \right| \, dx
\]

\[
\leq \mathbb{E} \left[ q_{n-k}(S, T) \int_{T^2} \sup_{\xi \in \mathbb{R}^m, |\xi|_1 = \ldots = |\xi|_n = 1} \left| \nabla^k_x (h(\Psi(S, T)x)) \xi \right| \, dx \right]
\]

\[
\leq \mathbb{E} \left[ q_{n-k}(S, T) p_k(S, T) \right] \max\{\|h\|_{L^1}, \ldots, \|\nabla^m_x h\|_{L^1}\}.
\]

Moreover, we can deduce from Corollary 1 that

\[
\mathbb{E} \left[ J_{(S, T)}(x) \nabla^n_x (h(\Psi(S, T)x)) \xi \right]
\]

can be written as a linear combination of integrals of the form

\[
(23) \quad \mathbb{E} \left[ H^{(j,k)}_\xi(x, \pi_j(S, T)) \nabla^{n-1-k}_x (h(\Psi(S, T)x)) \eta \right],
\]

where \( j \in \{0, 1, 2\} \), \( 0 \leq k \leq n-1 \), \( \eta \in \mathbb{R}^{2(n-1-k)} \) a subset of \( \xi \) with complement \( \zeta \) and \( H^{(j,k)}_\xi \in \mathcal{G} \). Recall from the proof of Corollary 1 that for fixed \( j \), \( k \) and \( \eta \), \( H^{(j,k)}_\xi(x, \pi_j(s, t)) \) can be written in the form of (17). Since the functions \( g_1, \ldots, g_{k+1} \) in (17) are in \( \mathcal{G} \), there is a polynomial \( q_k \), independent of \( x \) and \( \zeta \), such that

\[
\left| H^{(j,k)}_\xi(x, \pi_j(s, t)) \right| \leq q_k(\pi_j(s, t)).
\]

Therefore,

\[
\int_{T^2} \sup_{\xi \in \mathbb{R}^m, |\xi|_1 = \ldots = |\xi|_n = 1} \left| \mathbb{E} \left[ H^{(j,k)}_\xi(x, \pi_j(S, T)) \nabla^{n-1-k}_x (h(\Psi(S, T)x)) \eta \right] \right| \, dx
\]

\[
\leq \max\{\|h\|_{L^1}, \ldots, \|\nabla^m_x h\|_{L^1}\} \mathbb{E} \left[ q_k(S, T) p_{n-1-k}(\pi_j(S, T)) \right].
\]

Combining the estimates above and keeping in mind that the coefficients in the linear combinations do not depend on \( h \) or \( \xi \), we obtain the desired estimate on \( \|\nabla^m_x (Qh)\|_{L^1} \). \( \square \)

7. Proof of estimates on the deterministic flows

7.1. Proof of Lemma 1 Let us first study the conjugated special flow \( \hat{\Phi} \) where the conjugation is realized via a diffeomorphism \( \sigma \). Let \( t > 0 \), \( x \in T^2 \) and \( y = (r, h) \in M \) such that \( y = \sigma(x) \). We define \( S = \{ s \in (0, t) : h(\hat{\Phi}^s(y)) = 0 \} \), and introduce an ordering on \( S \) by \( S = \{ t_1, \ldots, t_{n(x,t)} \} \) with \( t_1 < \ldots < t_{n(x,t)} \). We also set \( t_0 = 0 \) and \( t_{n(x,t)+1} = t \). One can cover the trajectory \( \{ \hat{\Phi}^s(y) \}_{s \in [0, t]} \) by a family of \( n(x, t) + 1 \) charts such that for \( 1 \leq k \leq n(x, t) + 1 \), the \( k \)th chart contains the vertical line segment connecting \( \hat{\Phi}^{t_{k-1}+0}(y) \) to \( \hat{\Phi}^{t_0}(y) \). We can define these charts in such a way that the flow within each chart is a parallel translation, so in the canonical coordinates \( (r, h) \) on \( M \), \( \nabla_y \hat{\Phi}^s(y) \) is the product of \( n(x, t) \) Jacobian matrices of coordinate changes between the charts. The linear map associated with such a Jacobian matrix maps vectors \( (1, H'(r_k)) \) and \( (0, 1) \) to \( (0, 0) \) and \( (0, 1) \), respectively,
where \( r_k = r(\hat{\Phi}^t_{k-1} y) \). Therefore, these matrices are given by \( J_{-H^t(r_k)} \), where a shear matrix \( J_a \) is defined by
\[
J_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad a \in \mathbb{R}.
\]
Since \( J_a J_b = J_{a+b} \) for \( a, b \in \mathbb{R} \), we obtain that
\[
\nabla_y \hat{\Phi}^t(y) = J_{-\sum_{k=1}^{n(x,t)} H^t(r_k)} = \begin{pmatrix} 1 & 0 \\ -\sum_{k=1}^{n(x,t)} H^t(r_k) & 1 \end{pmatrix}.
\]
We immediately conclude that for all \( t \),
\[
det \nabla_y \hat{\Phi}^t(y) = 1.
\]
Since
\[
\nabla_x \Phi^t(x) = \nabla_x [\sigma^{-1} \circ \hat{\Phi}^t \circ \sigma](x) = \nabla_y \sigma^{-1}(\tilde{\Phi}^t y) \nabla_y \hat{\Phi}^t(y) \nabla_x \sigma(x),
\]
we obtain due to (26):}
\[
det \nabla_x \Phi^t(x) = \det \nabla_y \sigma^{-1}(\tilde{\Phi}^t y) \det \nabla_x \sigma(x).
\]
The last identity together with compactness of \( T^2 \) and smoothness of \( \sigma \) imply (3).
Using (24) and the identity \((\partial_x r_k, \partial_h r_k) = (1, 0)\), we obtain that
\[
|\partial_{r}^{n_1} \partial_{h}^{n_2} \tilde{\Phi}^t(y)| \leq \left| \sum_{k=1}^{n(x,t)} H^{(n_1)}(r_k) \right|.
\]
Since there is \( c_0 > 0 \) such that \( n(x, t) \leq c_0 (1 + t) \) for all \( t > 0 \), we can use smoothness of \( H \) and compactness of its domain to write
\[
(28) \quad |\partial_{r}^{p_1} \partial_{h}^{p_2} \tilde{\Phi}^t(y)| \leq c_1 (1 + t)
\]
for some \( c_1 > 0 \) that only depends on \( n_1 \), and for all \( t > 0 \).
For the remainder of the proof, we introduce the notation \((\Phi^t_1(x), \Phi^t_2(x))\) for the coordinates of \( \Phi^t(x) \) on \( T^2 \) and \((\tilde{\Phi}^t_1(y), \tilde{\Phi}^t_2(y))\) for the coordinates of \( \tilde{\Phi}^t(y) \) on \( M \). With (25) in hand, to prove (2), it remains to see that for \( l \in \{1, 2\}, \partial_{r}^{p_1} \partial_{h}^{p_2} \Phi^t_l(x) \) can be represented as a finite sum of terms of the form
\[
f(\hat{\Phi}^t(\sigma(x))) g(x) \prod_{i=1}^{p} \partial_{e_k(i)} \partial_{i}^{m(i)} \tilde{\Phi}^t_{j(i)} (\sigma(x)),
\]
where \( f : M \to \mathbb{R} \) and \( g : T^2 \to \mathbb{R} \) are smooth functions, \( p \leq n_1 + n_2, k(i), m(i) \in \mathbb{Z}_+ \) and \( j(i) \in \{r, h\} \) for all \( i \in \{1, \ldots, p\} \). This can be checked by induction, starting with (26) as the induction basis.

### 7.2. Proof of Lemma 3
Since \( U(x)^{-1} \) does not depend on \((s, t)\), we only need to verify that \( x \mapsto U(x)^{-1} \) has derivatives of all orders and that these derivatives are bounded on \( T^2 \). This follows from smoothness of the vector fields and from the uniform ellipticity condition.

We will now show that the components of \( \Phi^t_1(x) \) are in \( \mathcal{G} \). In this proof, we will write the \( k \)-th coordinate of a point \( y \in T^2 \) as \( e_k \cdot y \), where \( e_1, e_2 \) are the standard basis vectors inherited from \( \mathbb{R}^2 \).
Let us fix \( i \in \{0, 1\} \) and \( k \in \{1, 2\} \). As \( (e_k \cdot \Phi^t_1(x)) \) is bounded, it only remains to check that its derivatives are bounded by polynomials in \( t \). For any finite sequence \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of elements from \( \{1, 2, 3\} \), let \( \partial_\alpha = \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_n} \), where \( \partial_1 = \partial_{x_1}, \partial_2 = \partial_{x_2} \) and \( \partial_3 = \partial_t \). We first consider the case where all indices in \( \alpha \) are
from \( \{1, 2\} \), i.e., where we only take spatial derivatives. In this situation, Lemma 1 implies that
\[
|\partial_\alpha(e_k \cdot \Psi_t^l(x))| \leq c_n(1 + t)^n, \quad x \in \mathbb{T}^2, \ t \geq 0,
\]
where \( c_n > 0 \) is some constant. The general case where \( \partial_\alpha \) includes a mixture of spatial and temporal derivatives can then be reduced to the special case we have just discussed. Namely, we will show that for any \( n \in \mathbb{Z}_+ \) and \( \alpha \in \{1, 2, 3\}^n \), \( \partial_\alpha(e_k \cdot \Psi_t^l(x)) \) can be written as a polynomial in variables of the form \( \partial_\beta(e_l \cdot \Psi_t^j(x)) \) for \( \beta \in \bigcup_{j=0}^n \{1, 2\}^j \) and \( l \in \{1, 2\} \). Here, \( \partial_\alpha \) should be interpreted as the identity operator if \( \alpha \in \{1, 2, 3\}^0 \). This statement will follow via a standard induction argument once we show that for \( n \in \mathbb{Z}_+ \), \( \alpha \in \{1, 2\}^n \), and \( m \in \{1, 2, 3\} \), \( \partial_m \partial_\alpha(e_k \cdot \Psi_t^l(x)) \) can each be written as a polynomial in variables of the form \( \partial_\beta(e_l \cdot \Psi_t^j(x)) \) for \( \beta \in \bigcup_{j=0}^n \{1, 2\}^j \) and \( l \in \{1, 2\} \). If \( m \in \{1, 2\} \), we have
\[
\partial_m \partial_\alpha(e_k \cdot \Psi_t^l(x)) = \partial_\beta(e_k \cdot \Psi_t^l(x)),
\]
where \( \beta = (\alpha, m) \in \{1, 2\}^{n+1} \) is the concatenation of \( \alpha \) and \( m \). In addition,
\[
\partial_m((\partial_\alpha(e_k \cdot u_i))(\Psi_t^l(x))) = (\nabla_x \partial_\alpha(e_k \cdot u_i))(\Psi_t^l(x)) \cdot (\partial_m \Psi_t^l(x))
= \sum_{l=1}^2 (\partial_{(\alpha, l)}(e_k \cdot u_i))(\Psi_t^l(x)) \partial_m(e_l \cdot \Psi_t^l(x)),
\]
and the right-hand side is in the desired form. If \( m = 3 \), interchanging the order of differentiation yields
\[
\partial_m \partial_\alpha(e_k \cdot \Psi_t^l(x)) = -\partial_\alpha(e_k \cdot u_i)(\Psi_t^l(x)).
\]
By the chain rule for higher-order derivatives (see for instance Theorem 2.1 in [CS96]), the term on the right can be written as a polynomial in variables of the form \( \partial_\beta(e_k \cdot u_i)(\Psi_t^l(x)) \) and \( \partial_\beta(e_l \cdot \Psi_t^j(x)) \) for \( \beta \in \bigcup_{j=0}^n \{1, 2\}^j \) and \( l \in \{1, 2\} \). Finally,
\[
\partial_3((\partial_\alpha(e_k \cdot u_i))(\Psi_t^l(x))) = -\sum_{l=1}^2 (\partial_{(\alpha, l)}(e_k \cdot u_i))(\Psi_t^l(x))(e_l \cdot u_i(\Psi_t^l(x))).
\]
Since for any \( \beta \in \bigcup_{j=0}^n \{1, 2\}^j \) and \( l \in \{1, 2\} \),
\[
\sup_{x \in \mathbb{R}^2, t \geq 0} (\partial_\beta(e_l \cdot u_i))(\Psi_t^l(x)) < \infty,
\]
we infer that \( (e_k \cdot \Psi_t^l(x)) \in \mathcal{G} \).

**References**


SMOOTH INVARIANT DENSITIES FOR RANDOM SWITCHING


Courant Institute of Mathematical Sciences, New York University, 251 Mercer St, New York, NY 10012 USA

Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St, Toronto, ON M5S 2E4 Canada

Department of Mathematics, University of Utah, Salt Lake City, UT 84112 USA

Mathematics Department, Duke University, Durham, NC 27708 USA