Measurement of Generalized GDH Integrand for the Neutron and \(^3\text{He} \) at Low \(Q^2\)

by

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Weitao Yang

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics in the Graduate School of Duke University 2018
ABSTRACT

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Abstract

Despite the success of QCD at high energies where the perturbation calculations can be carried out because of the asymptotic freedom, many fundamental questions, regarding the confinement of quarks and gluons, the nuclear forces, and the nucleon mass and structure, still remain in the non-perturbative regime. Dispersive sum rules, based on universal principles, provide a data-driven approach to study the nucleon structure without model-dependencies. Among those sum rules, the well known Gerasimov-Drell-Hearn (GDH) sum rule relates the anomalous magnetic moment to a weighted integral over the photo-absorption cross section. Its generalized form is extended for the virtual photon absorption at an arbitrary four momentum transfer square (\(Q^2\)) and thus provides a unique relation to study the nucleon spin structure over an experimentally accessible range of \(Q^2\). The measured integrals can be compared with theoretical predictions for the spin dependent Compton amplitudes. Such experimental tests at intermediate and low \(Q^2\) deepen our knowledge of the transition from the asymptotic freedom regime to the color confinement regime in QCD.

Experiment E97-110 has been performed at the Thomas Jefferson National Accelerator Facility to precisely measure the generalized GDH sum rule and the moments of the neutron and \(^3\)He spin structure functions in the low energy region. During the experiment, a longitudinally-polarized electron beam with energies from 1.1 to 4.4 GeV was scattered from a \(^3\)He gas target which was polarized longitudinally or...
transversely at the Hall A center. Inclusive asymmetries and polarized cross-section differences, as well as the unpolarized cross sections, were measured in the quasielastic and resonance regions. In this work, the \(^3\)He spin dependent structure functions of \(g_1(\nu,Q^2)\) and \(g_2(\nu,Q^2)\) at \(Q^2 = 0.032-0.230\ \text{GeV}^2\) have been extracted from the experimental data, and the generalized GDH sum rule of \(^3\)He is firstly obtained for \(Q^2 < 0.1\ \text{GeV}^2\). The results exhibit a “turn-over” behavior at \(Q^2 = 0.1\ \text{GeV}^2\), which strongly indicates that the GDH sum rule for real photons will be recovered at \(Q^2 \rightarrow 0\).
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# List of Abbreviations and Symbols

## Abbreviations

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<th>Description</th>
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<tr>
<td>QCD</td>
<td>Quantum Chromodynamics.</td>
</tr>
<tr>
<td>GDH</td>
<td>Gerasimov-Drell-Hearn, who derived the GDH sum rule.</td>
</tr>
<tr>
<td>DIS</td>
<td>Deep inelastic scattering.</td>
</tr>
<tr>
<td>U(n)</td>
<td>Unitary group of degree ( n ).</td>
</tr>
<tr>
<td>SU(n)</td>
<td>Special Unitary group of degree ( n ).</td>
</tr>
<tr>
<td>TJNAF</td>
<td>Thomas Jefferson National Accelerator Facility, also known as Jefferson Lab or JLab.</td>
</tr>
<tr>
<td>PWIA</td>
<td>Plane wave impulse approximation.</td>
</tr>
<tr>
<td>FSI</td>
<td>Final state interactions.</td>
</tr>
<tr>
<td>MEC</td>
<td>Meson exchange currents.</td>
</tr>
<tr>
<td>QE</td>
<td>Quasielastic.</td>
</tr>
<tr>
<td>VVCS</td>
<td>Doubly virtual Compton scattering.</td>
</tr>
<tr>
<td>ChPT</td>
<td>Chiral perturbation theory.</td>
</tr>
<tr>
<td>BChPT</td>
<td>Baryonic Chiral perturbation theory.</td>
</tr>
<tr>
<td>IR</td>
<td>Infrared regularization.</td>
</tr>
<tr>
<td>LO</td>
<td>Leading-order.</td>
</tr>
<tr>
<td>NLO</td>
<td>Next-to-leading-order.</td>
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<tr>
<td>HRS</td>
<td>High Resolution Spectrometer.</td>
</tr>
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<td>SEOP</td>
<td>Spin exchange optical pumping.</td>
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</table>
PMT  Photomultiplier tube.
NMR  Nuclear magnetic resonance.
EPR  Electron Paramagnetic Resonance.
AFP  Adiabatic fast passage.
CEBAF Continuous Electron Beam Accelerator Facility.
BPM  Beam position monitor.
BCM  Beam current monitor.
PITA  Polarization Induced Transport Asymmetry.
RF   Radio frequency.
RMS  root-mean-square
DAQ  Data acquisition.
DC   Direct Current.
IHWP Insertable half-wave plate.
RHWP Rotatable half-wave plate.
VDC  Vertical drift chamber.
PID  Particle identification.
HLCS Hall A laboratory coordinate system.
TCS  Target coordinate system.
FCS  Focal plane coordinate system.
DCS  Detector coordinate system.
ERT  Elastic radiative tail.
Acknowledgements

Pursuing a PhD in Physics in the USA is probably the most important decision in my life. This seven and half years journey is not always smooth and happy: my original thesis project, got delayed due to the schedule conflict with the upgrade of the accelerator facility, so I had to switch to another project, which I was not so familiar with but need to finish it in a relatively short amount of time. Fortunately, the experiences in these two projects significantly expanded my knowledge in the nuclear physics, and I have learned a lot from the two collaborations that I have been working with. I believe this is the most valuable part I have gained throughout my graduate study.

Firstly, I would like to thank my advisor, Haiyan Gao, who introduced me to the field of experimental nuclear physics, and enlightened me not only in what is needed to become a physicist, but also in the methodology and the attitude one needs to accomplish a goal. I would also like to thank for what I have learned via the collaboration and the discussion with each of our group members, including W. Zheng, M. Huang, G. Laskaris, Q. Ye, Z. Yang, X. Yan, X. Li, W. Xiong, M. Meziane, P. Chu, S. Jawalker, Z. Ye, and C. Gu. Yan’s pursuit of excellence always encourages me to do better and better, and Xiong’s diligence always reminds me the key to be productive and successful.

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I will never forget all the nights and days we spent together to prepare for the experiment, to discuss and solve problems, and to take and analyze the data. I also want to acknowledge the saGDH collaboration, including J.-P. Chen, A. Deur, V. Sulkosky and N. Ton. The discussions with them are significantly helpful for me to understand the experiment and to finish the data analysis.

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The study of nucleon structure can be traced back to 1933, when Otto Stern and his collaborators [1, 2] discovered that the proton’s magnetic moment disagrees with Dirac’s prediction $\mu_N = \frac{e\hbar}{2M}$ for a point-like spin-$\frac{1}{2}$ particle. This deviation, called anomalous magnetic moment, was considered as the first evidence that the nucleon possesses a complicated internal structure.

The finite size of the nucleon was then probed by Hofstadter et al. through the elastic electron-proton scattering process [3] in 1955. A large disagreement was observed between the experimental results and the Mott formula, which assumed the proton as a structure-less particle and ignored the recoil of the target, and thus the spatial charge distribution of the proton was confirmed. In the next decade, extensive efforts of deep-inelastic scattering (DIS) experiments pioneered at Stanford Linear Accelerator Center (SLAC) [4] concluded that the nucleon is a composite of point-like particles named as partons, which are nowadays associated with quarks and gluons.

Quantum Chromodynamics (QCD), a SU(3) Yang-Mills gauge theory of quarks and gluons emerged in 1970s, is nowadays viewed as the fundamental theory of strong interactions. The two main properties of QCD are the color confinement and
the asymptotic freedom. The former prohibits isolated color charged particles, i.e., quarks and gluons, while the latter is the result of the color antiscreening, and it in essence permits the use of the perturbation technique. In the high energy region, where the strong coupling constant $\alpha_s$ is small and the quarks in the hadrons are asymptotically free, the perturbative QCD has been verified by the experimental results. However, less is known regarding the hadron structure in the confinement regime, where the strong coupling constant is large due to the small energy scale, and the perturbative calculation is no longer valid. Therefore, chiral perturbation theory, an effective field theory based on one of the basic QCD symmetries and conserved currents, has been utilized to provide model-independent predictions in the low energy region.

The generalized Gerasimov-Drell-Hearn sum rule, derived from general principles such as causality, unitarity and gauge invariance, relates the nucleon spin-dependent structure to its static properties for all energy scales. It is an ideal tool for the experimental investigation of the nucleon spin structure in the non-perturbative regime. Experiment E97-110, performed in Hall A at the Thomas Jefferson National Accelerator Facility (TJNAF, also known as Jefferson Lab or JLab), studies the spin-dependent structure of the neutron and $^3$He by testing the sum rule predictions from chiral perturbation theory in the low energy region. In this thesis, we will describe the experimental approach and the theoretical background of this experiment. The detailed experimental setup and apparatus, as well as the data analysis and results will also be presented.
Inclusive Electron Scattering

The inclusive scattering of electrons off nucleons or nuclei, in which only the scattered electron is detected, has been a well-proved experimental technique for probing the internal structure of the targets. In this chapter, we present the details of this process as follows. Section 2.1 introduces the elastic and inelastic electron-nucleon scattering. The latter can be categorized into two different kinematic types: the resonance production and the deep inelastic scattering. Section 2.2 discusses the virtual photoabsorption process and its connection with the electron-nucleon scattering in the resonance region. The electron-nucleus scattering is presented in Section 2.3.

2.1 Electron-nucleon scattering

The inclusive lepton scattering off a nucleon target can be described as

\[ l(k) + N(p) \rightarrow l(k') + X. \]  

(2.1)

As shown in Figure 2.1, a charged lepton of four momentum \( k^\mu \) interacts with a nucleon target of four momentum \( p^\mu \) via the exchange of a single virtual photon with four-momentum transfer \( q^\mu = k^\mu - k'^\mu \). In an inclusive measurement, only
the scattered electron with four momentum \( k' \) is detected, while the final hadronic system \( X \) is not observed by the detection system.

\[
\begin{align*}
\nu &= \frac{p \cdot q}{M}, \\
Q^2 &= -q^2, \\
W^2 &= (p + q)^2,
\end{align*}
\]  

\[ (2.2) \]

where \( M \) represents the target mass. Sometimes \( \nu \) and \( Q^2 \) are replaced with two dimensionless variables

\[
\begin{align*}
x &= \frac{Q^2}{2M\nu}, \\
y &= \frac{p \cdot q}{p \cdot k}.
\end{align*}
\]  

\[ (2.3) \]

\textbf{Figure 2.1:} The lowest-order diagram for lepton-nucleon scattering in the target rest frame.

Several kinematic variables, including the electron energy transfer \( \nu \), the four momentum transfer square \( Q^2 \) and the invariant mass of the final hadronic system \( W \), are typically utilized for the inclusive scattering process. Their definitions are
In the laboratory frame, where the nucleon target is at rest and $p^\mu = (M, 0)$, $k^\mu = (E, \vec{k})$, $k'^\mu = (E', \vec{k}')$, these variables satisfy the following kinematic relations:

\[
\nu = E - E',
\]

\[
Q^2 = 2EE' - 2m^2 - 2\vec{k} \cdot \vec{k}' \approx 4EE' \sin^2 \frac{\theta}{2},
\]

\[
W^2 = M^2 + 2M\nu - Q^2,
\]

\[
x = \frac{Q^2}{2M\nu},
\]

\[
y = \frac{\nu}{E'},
\]

where $m$ represents the electron mass, and $\theta$ is the scattering angle between the incident electron momentum $\vec{k}$ and the scattered electron momentum $\vec{k}'$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{spectrum.png}
\caption{The spectrum of cross sections for inclusive electron-proton scattering over $W$ and $Q^2$. Inelastic cross sections are obtained from the fits by P. E. Bosted [5].}
\end{figure}
The differential cross section of the inclusive electron scattering is often studied in different kinematic regions. Figure 2.2 shows a spectrum of the cross sections for the electron-proton scattering with varied $W$ and $Q^2$. In this section, the elastic and inelastic electron-nucleon scattering will be discussed separately. The latter includes a resonance production region and a deep inelastic scattering (DIS) region.

### 2.1.1 Elastic electron scattering

In the elastic scattering process, the nucleon target remains in its ground state, so that the four momentum carried by the recoiling nucleon $p'\mu = p'^\mu + q'^\mu$, and the invariant mass of the final hadronic system $W = M$. One could easily derive that

$$\nu = \frac{Q^2}{2M}.$$  \hfill (2.5)

Thus there is only one independent kinematic variable, which is commonly chosen as $Q^2$.

Considering energy-momentum conservation, the energy of the scattered electron is determined by the incident electron energy and the scattering angle in the laboratory frame as

$$E' = \frac{E}{1 + \frac{2E}{M} \sin^2 \frac{\theta}{2}}.$$  \hfill (2.6)

For the differential cross section of the elastic electron-nucleon scattering, we shall begin with the well-known Mott formula including a recoil factor $\frac{E'}{E}$, written as

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{4E'^2} \cdot \frac{\cos^2 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} \cdot \frac{E'}{E},$$  \hfill (2.7)

where $\alpha \approx 1/137$ is the electromagnetic fine structure constant. This formula shows the differential cross section for an electron scattering from an unpolarized point-like particle with spin $\frac{1}{2}$ and mass $M$. The ultra-relativistic approximation (URA), in which the electron mass $m \to 0$, is assumed in its derivation [6].
Since the nucleon is an extended target, we need to include the effects from the internal structure of the target for the elastic electron-nucleon scattering process. The electromagnetic current for a nucleon can be written as [7]

\[ J^\mu = e \bar{u} (p') \Gamma^\mu u (p) e^{i(p'-p) \cdot x}, \]

where a general four-vector \( \Gamma^\mu \) is used to represent the structure related terms. Based on the requirement of being a Lorentz four-vector, \( \Gamma^\mu \) can be constructed from \( p, p', q \) and the Dirac-\( \gamma \) matrices. Only two independent terms are left due to the constraints from conservation of current and parity, so that \( \Gamma^\mu \) becomes

\[ \Gamma^\mu = F_1 (Q^2) \gamma^\mu + \frac{\kappa}{2M} F_2 (Q^2) i \sigma^{\mu \nu} q_\nu, \]

with \( \kappa \) the anomalous magnetic moment. \( F_1 \) and \( F_2 \), named as Dirac and Pauli form factors, respectively, parameterize the internal structure of a nucleon target. The differential cross section of the elastic electron-nucleon scattering with the Born approximation (one-photon exchange) can thus be expressed as

\[ \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( F_1^2 (Q^2) + \tau \kappa^2 F_2^2 (Q^2) + 2 \tau (F_1 (Q^2) + \kappa F_2 (Q^2))^2 \tan^2 \frac{\theta}{2} \right), \]

where \( \tau = \frac{Q^2}{4M^2} \). Equation (2.10) is often referred to as the Rosenbluth formula [8]. The form factors can be replaced with their linear combinations:

\[ G_E(Q^2) = F_1(Q^2) - \tau \kappa F_2(Q^2), \]
\[ G_M(Q^2) = F_1(Q^2) + \kappa F_2(Q^2). \]

Such a replacement leads to

\[ \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2 \tau G_M^2(Q^2) \tan^2 \frac{\theta}{2} \right), \]

\( G_E \) and \( G_M \), the so-called Sachs electromagnetic form factors [9], are closely related to the spatial distribution of a nucleon’s charge and magnetic moment in the Breit
frame, where $\vec{p} = -\vec{p}$ and $q^\mu = (0, \vec{q})$. In the non-relativistic limit, the Fourier transforms of electric form factor $G_E$ and magnetic form factor $G_M$ imply the charge density distribution and the current density distribution, respectively. More details about the nucleon form factors can be found in the review articles [10, 11].

2.1.2 Inelastic electron scattering

In the inelastic scattering $eN \rightarrow eX$, the current $J^\mu$ in equation (2.8) cannot describe the hadronic vertex since the final hadronic system $X$ is not necessarily the recoiling nucleon. Instead, a more general form for the unpolarized electron-nucleon scattering is used:

$$\frac{d\sigma}{d\Omega dE} = \frac{\alpha^2}{Q^4} \cdot \frac{E'}{E} \cdot L_{\mu\nu} W^{\mu\nu},$$

(2.13)

with the leptonic tensor

$$L_{\mu\nu} = 2(k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k \cdot k'),$$

(2.14)

and the hadronic tensor

$$W^{\mu\nu} = W_1(\nu, Q^2) \left(-g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2}\right) + W_2(\nu, Q^2) \left(p^{\mu} - \frac{p \cdot q}{q^2} q^{\mu}\right) \left(p^{\nu} - \frac{p \cdot q}{q^2} q^{\nu}\right),$$

(2.15)

in which $W_1$ and $W_2$ parameterize the response from the nucleon. The antisymmetric part of the hadronic tensor is omitted because $L_{\mu\nu}$ is symmetric. Thus $W^{\mu\nu}$ only consists of two independent terms based on parity and current conservation. The relation in equation (2.6) is no longer valid in the inelastic scattering, so that both kinematic variables $\nu$ and $Q^2$ are necessary in the response functions $W_{1,2}$.

The differential cross section at the Born level thus results in the Rosenbluth form:

$$\frac{d^2\sigma}{d\Omega dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}^* \left(W_2(\nu, Q^2) + 2W_1(\nu, Q^2) \tan^2 \frac{\theta}{2}\right),$$

(2.16)
where the * symbol indicates the ignorance of the recoil factor \( \frac{E'}{E} \) in the Mott cross section.

As the energy transferred to the target increases, we start to observe a spectrum of excited states, namely the nucleon resonances, in the kinematic region of intermediate \( W \) and \( Q^2 \). Figure 2.3 shows a typical example of the differential cross section for the inelastic electron-nucleon scattering. Three prominent maxima are often observed in the resonance region. The first peak corresponds to the \( P_{33} (1232) \), also known as the \( \Delta (1232) \) resonance, which is a strong resonance that can be easily separated from other resonant states. The second one at \( W \approx 1500 \text{ MeV} \) includes the resonances \( D_{13} (1520) \) and \( S_{11} (1535) \). The third maximum at \( W \approx 1700 \text{ MeV} \) consists of multiple resonances including \( F_{15} (1680) \), which is the most pronounced state at low \( Q^2 \). The above resonances are denoted in the form of \( L_{2I2J} \), where \( L \) is the conventional nomenclature of the orbital angular momentum \( l \) (\( S, P, D, F \) for \( l = 0, 1, 2, 3 \), respectively), \( I \) is the isospin, and \( J = |l \pm \frac{1}{2}| \) represents the total angular momentum.

Separation of individual resonance peaks becomes extremely difficult when the \( Q^2 \) and \( W \) are both large. The differential cross section distributes smoothly in this region (usually \( Q^2 > 1-2 \text{ GeV}^2 \) and \( W > 2 \text{ GeV} \)), namely the deep inelastic regime. For the kinematic of DIS the exchanged virtual photon has a small wave length (\( \lambda \sim 1/\sqrt{Q^2} \)) and thus directly probes the constituents inside the nucleon.

One can replace the response functions \( W_{1,2} \) in equation (2.15) with two dimensionless structure functions:

\[
F_1(x, Q^2) = MW_1(v, Q^2),
\]

\[
F_2(x, Q^2) = \nu W_2(v, Q^2),
\]

where \( x \) is defined in equation (2.3). It is found that the structure functions depend

\* Here and thereafter the natural units are used, in which \( c = \hbar = 1 \)
mostly on the scaling variable $x$ when $Q^2$ is large:

$$F_1(x, Q^2) \xrightarrow{\text{large } Q^2} F_1(x),$$

$$F_2(x, Q^2) \xrightarrow{\text{large } Q^2} F_2(x).$$

The independence of $Q^2$ indicates that the electrons are scattered off a point charge. This scaling phenomenon was predicted by Bjorken [12], and then was explained in the parton model formulated by Feynman [13], Bjorken and Paschos [14]. In this model, the Bjorken scaling variable $x$ coincides with the fraction of the parent nucleon’s momentum carried by the probed parton. In an infinite momentum frame, where $\nu, Q^2 \to \infty$ but $x$ remains finite, the DIS process is an incoherent sum of electron scattering from structure-less partons with the impulse approximation. Therefore, the structure functions can be related to the unpolarized parton distribution functions $f_q$ as

$$F_1(x) = \frac{1}{2} \sum_q e_q^2 f_q(x) = \frac{1}{2} \sum_q e_q^2 [f_q^+(x) + f_q^-(x)],$$

Figure 2.3: The cross section of inelastic electron-proton scattering $ep \to eX$. Values are given by the fits from P. E. Bosted [5].
in which \( f_q^\uparrow \) (or \( f_q^\downarrow \)) denotes the distribution function of the partons with spin parallel (or anti-parallel) to the nucleon spin.

The well known Callan-Gross relation [15] predicted that the structure functions \( F_{1,2} \) can be connected by

\[
F_2(x) = 2xF_1(x),
\]

if the electromagnetic current only consisted of spin-\( \frac{1}{2} \) fields. The experimental verification of this relation suggests that partons have spin \( \frac{1}{2} \), which is expected for quarks.

Bjorken scaling only considers the lowest order of the electron-quark scattering, and ignores the contribution from gluon emissions. This approximation will become unsatisfied as the photon gains greater resolving power with larger \( Q^2 \) and hence observes more quark pairs and gluons surrounding the struck quark. Indeed, the experimental data shown in Figure 2.4 clearly indicates a \( Q^2 \) dependence of the proton structure function \( F_2^p \), which violates the scaling prediction. With the inclusion of radiative gluon effects, a logarithmic \( Q^2 \) dependence is developed for the structure functions [7]. The variation of the parton distribution functions (and hence the structure functions) with \( Q^2 \) can be evaluated by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [16, 17, 18] within the kinematic range where the perturbative QCD applies.

2.1.3 Doubly polarized electron scattering

So far we have only discussed the unpolarized electron-nucleon scattering, and thus the antisymmetric contributions in the tensor form has not been considered in equation (2.13). For a doubly polarized electron-nucleon scattering process as

\[
\bar{e}(k) + \bar{N}(p) \rightarrow l(k') + X,
\]

(2.21)
Figure 2.4: The proton structure function $F_2^p$ from Ref. [19].
its differential cross section can only be obtained with the inclusion of antisymmetric part in the leptonic and hadronic tensors:

\[
L_{\mu\nu} = L^S_{\mu\nu} + L^{A\pm}_{\mu\nu},
\]

\[
W^{\mu\nu} = W^S_{\mu\nu} + W^A_{\mu\nu}.
\]

The “±” sign in the leptonic tensor represents the helicity of the incident electron beam. A positive (negative) helicity is defined for the electron with the spin aligned parallel (anti-parallel) to its momentum direction. We have shown the expression for the symmetric part in equation (2.14) and (2.15). The antisymmetric contributions are written as

\[
L^{A\pm}_{\mu\nu} = \mp 2i\epsilon_{\mu\nu\alpha\beta}k^\alpha k^\beta,
\]

\[
W^A_{\mu\nu} = i\epsilon^{\mu\nu\alpha\beta}q_\alpha \left[ G_1(\nu, Q^2)S_\beta + \frac{G_2(\nu, Q^2)}{M^2}(p\cdot q S_\beta + S\cdot q p_\beta) \right],
\]

with the totally antisymmetric tensor \( \epsilon_{0123} = 1 \) and the nucleon spin four-vector

\[
S^\mu = \bar{u}(p)\gamma^\mu\gamma_5 u(p) \frac{1}{2M}.
\]

The two additional structure functions \( G_{1,2} \) account for the nucleon response regarding the polarization degree of freedom. Their dimensionless forms are called the spin-dependent structure functions, with the definitions of

\[
g_1(x, Q^2) = M_\nu G_1(\nu, Q^2),
\]

\[
g_2(x, Q^2) = \nu^2 G_2(\nu, Q^2).
\]

\( g_1 \) and \( g_2 \) can be extracted from the measured cross section differences between the parallel- and perpendicular-target-spin configurations. Given the incident electron beam with a positive (“\( + \)”) or negative (“\( - \)”) helicity, the parallel cross section difference, denoted as \( \Delta \sigma_{||} \), is obtained from the scattering off a nucleon target with
spin parallel ("\(\uparrow\)") to the direction of the incident electron’s momentum:

\[
\Delta \sigma_\parallel = \frac{d^2\sigma^{\uparrow\uparrow}}{d\Omega dE'} - \frac{d^2\sigma^{\uparrow\uparrow}}{d\Omega dE'} = \frac{4\alpha^2 E'}{M\nu Q^2 E} [(E + E' \cos \theta)g_1(x, Q^2) - 2Mxg_2(x, Q^2)].
\]  

(2.26)

Notice that the symmetric part is cancelled out in this difference, and thus only \(g_1\) and \(g_2\) related terms exist. Similarly, the perpendicular cross section difference is taken by aligning nucleon spin perpendicular ("\(\Rightarrow\)") with respect to the incident beam:

\[
\Delta \sigma_\perp = \frac{d^2\sigma^{\Rightarrow\Rightarrow}}{d\Omega dE'} - \frac{d^2\sigma^{\Rightarrow\Rightarrow}}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{M\nu Q^2 E} \sin \theta \cos \phi \left[ g_1(x, Q^2) + \frac{2E}{\nu} g_2(x, Q^2) \right].
\]  

(2.27)

where \(\phi\) is the azimuthal angle between the scattered electron and the polarization plane. This observable is maximized if the polarization plane and the scattering plane coincide, in which case \(\cos \phi = \pm 1\).

In practice, asymmetry measurements are often preferred in the experimental determination of \(g_1\) and \(g_2\) for better controlled systematic uncertainties. The parallel and perpendicular asymmetries can be formed by combining experimental data from the aforementioned polarization configurations:

\[
A_\parallel = \frac{\sigma^{\uparrow\uparrow} - \sigma^{\downarrow\downarrow}}{\sigma^{\uparrow\uparrow} + \sigma^{\downarrow\downarrow}},
\]

\[
A_\perp = \frac{\sigma^{\Rightarrow\Rightarrow} - \sigma^{\Rightarrow\Rightarrow}}{\sigma^{\Rightarrow\Rightarrow} + \sigma^{\Rightarrow\Rightarrow}}.
\]  

(2.28)

Analogous to the relation in equation (2.19), \(g_1\) can be expressed in terms of the polarized PDF \(\Delta f_1\) as

\[
g_1(x) = \frac{1}{2} \sum_q e_q^2 \Delta f_q(x) = \frac{1}{2} \sum_q e_q^2 \left[ f_q^+(x) - f_q^+(x) \right],
\]  

(2.29)
Unfortunately, it is difficult to interpret $g_2$ in the parton model [20]. We will revisit this structure function in Chapter 3.

2.2 Virtual photoabsorption cross sections

The previous section has shown that the most complicated part in the inclusive electron-nucleon scattering is the hadronic vertex, where a virtual photon interacts with the nucleon target. Four structure functions ($F_1$, $F_2$, $g_1$, $g_2$) have been introduced to describe the nucleon response due to its extended structure. If we viewed the scattered electron in this process as a source of the virtual photon $\gamma^*$ with four momentum $q^\mu = (\nu, \bar{q})$, then the $\gamma^*NX$ vertex can be interpreted as virtual photon-nucleon interactions.

2.2.1 Formalism

The differential cross section of the electron-nucleon scattering can be written equivalently as the cross sections of virtual photoabsorption [21]:

$$
\frac{d^2\sigma}{d\Omega dE'} = \Gamma \left[ \sigma_T(\nu, Q^2) + \epsilon \sigma_L(\nu, Q^2) - h \left( \cos \theta^* \sqrt{1 - \epsilon^2} \sigma_{TT}(\nu, Q^2) + \sin \theta^* \cos \phi^* \sqrt{2\epsilon(1 - \epsilon)} \sigma_{LT}(\nu, Q^2) \right) \right],
$$

(2.30)

with the virtual photon longitudinal polarization parameter

$$
\epsilon = \left[ 1 + 2(\nu^2/Q^2) + 1 \tan^2 \frac{\theta}{2} \right]^{-1},
$$

(2.31)

the kinematic factor

$$
\Gamma = \frac{\alpha}{2\pi^2 Q^2} \cdot \frac{E'}{E} \cdot \frac{K}{1 - \epsilon},
$$

(2.32)

and the beam helicity $h = \hat{\sigma} \cdot \hat{k}$. As shown in Figure 2.5, $\theta^*$ and $\phi^*$ represent the angles of the target spin direction with respect to the virtual photon three-momentum $\bar{q}$ and
the scattering plane, respectively. The factor $\Gamma$ depends on the virtual photon flux $K$, whose definition is with ambiguity. In this thesis we shall follow the convention originally proposed by Gilman [22]:

\[ K_G = |\vec{q}| = \sqrt{\nu^2 + Q^2}. \]  

(2.33)

in which the flux equals the three-momentum carried by the virtual photon in the laboratory frame. Another common convention is given by Hand [23], who defines the flux as the excitation energy from the virtual photon:

\[ K_H = \nu^* = \nu - \frac{Q^2}{2M}, \]  

(2.34)

The photoabsorption cross sections need to be adjusted accordingly in different conventions, i.e., $\sigma_i^G = K_H/K_G \cdot \sigma_i^H$. Notice that both conventions will be reduced to $K = \nu$ at the real photon point, where $Q^2 = 0$.

**Figure 2.5:** The schematic of target spin angles in the one photon-exchange diagram of doubly polarized electron-nucleon scattering.

Equation (2.30) consists of four different photon-nucleon amplitudes, in which the subscripts $L$ and $T$ denote the virtual photon’s transverse and longitudinal com-
ponents, respectively, with respect to the unit vector $\hat{q}$. The two spin-flip amplitudes $\sigma_{LT}$ (longitudinal-transverse) and $\sigma_{TT}$ (transverse-transverse) can only be accessed by a double polarization measurement, and thus are excluded from the cross section for the unpolarized electron-nucleon scattering. For real photons ($Q^2 = 0$), the contribution from longitudinal terms of $\sigma_L$ and $\sigma_{LT}$ vanishes, and the transverse term $\sigma_T(\nu, 0)$ reduces to the total photon-absorption cross section $\sigma_\gamma(\nu)$ in the unpolarized process.

The relations between the unpolarized structure functions $F_{1,2}$ and virtual photoabsorption cross sections are immediately extracted by comparing equation (2.30) and (2.16):

\[
\sigma_T = \frac{4\pi^2\alpha}{MK} F_1,
\]
\[
\sigma_L = \frac{4\pi^2\alpha}{K} \left( \frac{\nu^2 + Q^2 F_2}{\nu} - \frac{F_1}{M} \right). \tag{2.35}
\]

For the double polarization measurement, we can rewrite the cross section differences in equation (2.26) and (2.27) equivalently in terms of $\sigma_{LT}$ and $\sigma_{TT}$:

\[
\Delta \sigma_\parallel = 2\Gamma \left( 1 - \frac{E'}{E} \right) \left( \sigma_{TT} + \frac{\epsilon\sqrt{Q^2}}{E - E'\epsilon} \sigma_{LT} \right),
\]
\[
\Delta \sigma_\perp = 2\Gamma \left( 1 - \frac{E'}{E} \right) \sqrt{\frac{2\epsilon}{1 + \epsilon}} \left( \sigma_{LT} - \frac{\sqrt{Q^2}(1 + \epsilon)}{2(E - E'\epsilon)} \sigma_{TT} \right). \tag{2.36}
\]

Thus we retrieve the connections between the spin-dependent structure functions and the virtual photoabsorption cross sections as follows:

\[
\sigma_{LT} = \frac{4\pi^2\alpha}{MK} \sqrt{Q^2} \left( g_1 + g_2 \right),
\]
\[
\sigma_{TT} = \frac{4\pi^2\alpha}{MK} \left( g_1 - \frac{Q^2}{\nu^2} g_2 \right). \tag{2.37}
\]
2.2.2 Spin structure in the resonance region

Transitions from the nucleon ground state to a resonant excited state are commonly interpreted by the photon helicity amplitudes. We can write the virtual photon-nucleon cross sections for an isolated resonance in terms of the helicity amplitudes as

\[ \sigma_{T}(\nu_{R}, Q^2) = \frac{2M}{\Gamma_{R}W_{R}} \left( |A_{\frac{1}{2}}|^2 + |A_{\frac{3}{2}}|^2 \right), \]

\[ \sigma_{TT}(\nu_{R}, Q^2) = \frac{2M}{\Gamma_{R}W_{R}} \left( |A_{\frac{1}{2}}|^2 - |A_{\frac{3}{2}}|^2 \right), \]

\[ \sigma_{L}(\nu_{R}, Q^2) = \frac{4M}{\Gamma_{R}W_{R}} |S_{\frac{1}{2}}|^2, \]

\[ \sigma_{LT}(\nu_{R}, Q^2) = \frac{4M}{\Gamma_{R}W_{R}} S_{\frac{3}{2}} A_{\frac{1}{2}}, \] \hspace{1cm} (2.38)

with \( \nu_{R} = (W_{R}^2 - M^2 + Q^2) \), where \( W_{R} \) and \( \Gamma_{R} \) represent the invariant mass and the decay width of the corresponding resonant state, respectively. \( A_{\frac{1}{2}}, A_{\frac{3}{2}} \) and \( S_{\frac{1}{2}} \) are the three basic types of transition amplitude [6]. The symbol A (S) indicates the transverse (longitudinal) photon polarization, while the subscript \( \frac{1}{2} \) or \( \frac{3}{2} \) represents the total helicity projection between the photon and the nucleon.

The spin-dependent structure functions of \( g_1 \) and \( g_2 \) in the resonance region are often studied experimentally with the virtual photoabsorption asymmetries, which are expressed as

\[ A_{1} = \frac{\sigma_{TT}}{\sigma_{T}} = \frac{g_1(x, Q^2) - \gamma^2 g_2(x, Q^2)}{F_1(x, Q^2)}, \]

\[ A_{2} = \frac{\sigma_{LT}}{\sigma_{T}} = \frac{\gamma (g_1(x, Q^2) + g_2(x, Q^2))}{F_1(x, Q^2)}, \] \hspace{1cm} (2.39)

with \( \gamma^2 = Q^2/\nu^2 = 4M^2x^2/Q^2 \). The connections between the virtual photoabsorption asymmetries and the measured asymmetries in a doubly polarized electron-nucleon
scattering experiment are:

\[ A_\parallel = D(A_1 + \frac{\epsilon \sqrt{Q^2}}{E - E' \epsilon} A_2), \]
\[ A_\perp = D \sqrt{\frac{2\epsilon}{1 + \epsilon}} (A_2 - \frac{\sqrt{Q^2}(1 + \epsilon)}{2(E - E' \epsilon)} A_1), \]

(2.40)

with the photon depolarization factor

\[ D = \frac{E - E' \epsilon}{E(1 + \epsilon R(x, Q^2))}, \]

(2.41)

and \( R = \sigma_L/\sigma_T \). Positivity limits that \(|A_1| \leq 1\) and \(|A_2| \leq \sqrt{R}\). In the limit of \( \gamma \to 0 \), \( A_1 \) is solely dependent on \( g_1/F_1 \), while \( A_2 \) vanishes.

\( \Delta (1232) \) resonance of spin-3/2 includes both 1/2 and 3/2 projections. Its helicity amplitudes are linear combinations of the magnetic dipole (M1) and electric quadrupole (E2) transition amplitudes: [6]

\[ A_{\frac{1}{2}} = -\frac{1}{2} \left( M_{1+}^{(\frac{3}{2})} + 3E_{1+}^{(\frac{3}{2})} \right), \]
\[ A_{\frac{3}{2}} = -\frac{\sqrt{3}}{2} \left( M_{1+}^{(\frac{3}{2})} - E_{1+}^{(\frac{3}{2})} \right). \]

(2.42)

At low \( Q^2 \) the excitation of \( \Delta \) is dominated by the magnetic dipole (M1) transition, so that \( A_{\frac{3}{2}}/A_{\frac{1}{2}} \approx \sqrt{3} \) (the measured value for real photons is 1.889 \pm 0.092 [19]). Therefore, the asymmetry \( A_1 = (|A_{\frac{1}{2}}|^2 - |A_{\frac{3}{2}}|^2)/(|A_{\frac{1}{2}}|^2 + |A_{\frac{3}{2}}|^2) \) is expected to be approximately \(-\frac{1}{2}\). At large \( Q^2 \), perturbative QCD predicts that \( A_{\frac{3}{2}} \), which involves a change of helicity, should be suppressed as \( A_{\frac{3}{2}}/A_{\frac{1}{2}} \approx 1/Q^2 \), and hence \( A_1 \to 1 \) for \( Q^2 \to \infty \).

\( A_1 \) behaves differently for higher resonances. The excitation of the \( S_{11} (1535) \) resonance has no spin projection of \( \frac{3}{2} \), so the asymmetry \( A_1 \) stays unity at any \( Q^2 \).

For the resonances of \( D_{13} (1520) \) and \( F_{15} (1680) \) a dynamical constituent quark model
predicted that the asymmetry $A_1$ is dominated by $A_2$ at the real photon point ($Q^2 = 0$), and switches to the dominance of $A_1$ beyond $Q^2 \approx 1\text{GeV}^2$. This prediction is verified by experiments [25, 19], in which a rapid change from $A_1 \approx -1$ to $A_1 \approx 1$ with $0 < Q^2 < 3 \text{GeV}^2$ was observed.

2.3 Electron-nucleus scattering

The electron-nucleus scattering has a few more reaction channels than that of the electron-nucleon scattering. As the general spectrum shown in Figure 2.6, electron-nucleon interactions can be characterized by the energy transfer $\nu$ at a fixed $Q^2$. The first peak from elastic electron-nucleus scattering occurs at $\nu = \frac{Q^2}{2M_T}$, where $M_T$ is the mass of the nuclear target. As the energy transfer increases, the nucleus is excited to its resonant states with several isolated maxima and a quasi-bound giant resonances peak observed. At energy above the nuclear resonance region, a broad peak locates approximately at $\nu = \frac{Q^2}{2M_N}$, with $M_N$ the nucleon mass. This peak, corresponding to the elastic electron scattering from a quasi-free nucleon, is slightly shifted from $\nu = \frac{Q^2}{2M_N}$ due to the nuclear binding energy and is broadened because of the Fermi motion. With higher energy transfer, the electron effectively “penetrates” the nucleus, and hence a spectrum analogous to the inelastic spectrum of the electron-nucleon scattering is obtained.

In this thesis, we are interested in the polarized electron scattering from a polarized $^3\text{He}$ nucleus. Since the nuclear resonance region does not exist for $^3\text{He}$, only the elastic and quasielastic (QE) channels will be discussed.

2.3.1 Effective polarized neutron target

Experimental study of the neutron structure is difficult due to the lack of free neutron target as its mean life time is only $880.3 \pm 1.1$ seconds [19]. Several light nuclei, such as deuteron and $^3\text{He}$, are hence used as effective neutron targets. Polarized $^3\text{He}$ often
Figure 2.6: A typical cross section spectrum of electron-nucleus scattering with constant $Q^2$ over the energy transfer $\nu$.

serves as an effective polarized neutron target since its spin at the ground state is mostly carried by the unpaired neutron. The average nucleon polarization in a $^3$He nucleus is written as \[26\].

\[
\langle \sigma_z \rangle^p = -\frac{2}{3}(P_D - P_{S'}) ,
\]

\[
\langle \sigma_z \rangle^n = P_S - \frac{1}{3}(P_D - P_{S'}) ,
\]

where $P_k$ denotes the probability of the nucleons to be in the $k$ state. The space-symmetric $S$ state predominantly contributes to the ground state of the polarized $^3$He, with $P_S \approx 90\%$.

In the effective polarization approximation, the nuclear binding is neglected and $x^2/Q^2 \to 0$, the polarized $^3$He structure functions can be formed by a weighted sum of the nucleon structure functions as

\[
g_i^{^3He}(x, Q^2) = 2P_i^p g_i^p(x, Q^2) + P_i^n g_i^n(x, Q^2) ,
\]

where $i = 1, 2$, and the superscript denotes the nucleus or the nucleon. The weighting factor $P_i^N$ is the effective polarization, which is dominated by the lowest-order
contribution $\langle \sigma_+ \rangle^N$ and can be calculated from models of $^3$He wave functions, such as SS [27] and KPSV [28] spectral functions. However, this approach is not reliable in the nucleon resonance region or at large $x$, a detailed analysis with various nuclear corrections can be found in Ref. [29].

2.3.2 Differential cross section

The differential cross section of unpolarized elastic electron-$^3$He scattering follows the Rosenbluth separation. Thus equation (2.12) applies for this process as long as the Mott cross section includes a charge factor of $Z^2$, and the nucleon electromagnetic form factors are replaced with those of $^3$He.

A general formula of the differential cross section for the electron scattering from a spin-$\frac{1}{2}$ nucleus is written as [30]

$$\frac{d\sigma}{d\Omega dE'} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^* \left[ \nu_T R_T (\nu, Q^2) + \nu_L R_L (\nu, Q^2) \right.
- \nu_T' \cos \theta^* R_T' (\nu, Q^2) + 2 \nu_{TL}' \sin \theta^* \cos \phi^* R_{TL'} (\nu, Q^2) \left. \right],$$

(2.45)

with the kinematic factors

$$\nu_L = \left( \frac{Q^2}{\nu^2 + Q^2} \right)^2,$$

$$\nu_T = \frac{Q^2}{2(\nu^2 + Q^2)} + \tan^2 \frac{\theta}{2} = \frac{Q^2}{2 \epsilon (\nu^2 + Q^2)},$$

$$\nu_T' = \tan \frac{\theta}{2} \sqrt{\frac{Q^2}{\nu^2 + Q^2}} + \tan^2 \frac{\theta}{2} = \nu_T \sqrt{1 - \epsilon^2},$$

$$\nu_{TL}' = -\frac{Q^2}{\sqrt{2}(\nu^2 + Q^2)} \tan \frac{\theta}{2} = -\sqrt{\frac{Q^2}{2(\nu^2 + Q^2)}} \cdot \nu_T \sqrt{2 \epsilon (1 - \epsilon)}.$$

(2.46)

We note that this formula takes an equivalent form of equation (2.30). The definitions of the target spin angles $\theta^*$, $\phi^*$, the beam helicity $h$, and the virtual photon
longitudinal polarization parameter $\epsilon$ remain the same. Four nuclear response functions, including two symmetric terms of $R_L$ and $R_T$ and two antisymmetric pieces of $R_T'$ and $R_{TT'}$, have been introduced to parameterize the nuclear response to the virtual photon.

2.3.3 Theoretical calculations in the quasielastic region

The quasielastic process can be simply interpreted in the Plane Wave Impulse Approximation (PWIA). In the impulse approximation, the virtual photon interacts incoherently with individual nucleons inside the target nucleus. The struck nucleon is then knocked out freely without any further interactions with the residual baryon system. The PWIA approximates the final states of the target nucleus with products of plane waves for a single nucleon and correlated states of the remaining nucleons (for $^3$He the states are the deuteron or the two-nucleon scattering) [31]. Within the framework of PWIA, the nuclear current tensor is related to the current tensor for single nucleon. Therefore, the response functions of $^3$He can be calculated with the spectral functions constructed from realistic wave functions of this three-nucleon system [27, 28, 32]. Unfortunately, as pointed out in Ref. [28], the PWIA calculations for the polarized response functions suffer from the corrections for final state interactions (FSI), meson exchange currents (MEC), relativistic effects, and the non-nucleonic degrees of freedom (such as the $\Delta$-isobar) in the bound state. Most importantly, large FSI effects limit the application of PWIA at low $Q^2$. In experiment E94-010 [33] at JLab, the PWIA calculations [32, 34] successfully described the experimental data in the QE region at large $Q^2$, while a significant discrepancy between the calculations and the results had been observed at $Q^2 = 0.05$ GeV$^2$.

The unsatisfied descriptions from PWIA at low $Q^2$ emphasized the importance of FSI effects in this kinematic region. An exact treatment of FSI had been incorporated in the non-relativistic Faddeev calculation by Golak et al. [35]. The Faddeev
equation, formulated by L. D. Faddeev [36, 37], reduces the Schrödinger equation of the three-nucleon system to a set of three coupled equations for two-body interactions, and thus achieves a significant simplification in the solutions of the 3N bound state and 3N scattering states. The calculations by the same group [38, 39] then included the corrections for MEC, and exhibited a good agreement on the transverse asymmetry with the experimental data at $Q^2 = 0.1 \text{ GeV}^2$ [40, 41]. The E94-010 [33] results at $Q^2 = 0.05 \text{ GeV}^2$ had also been well reproduced by the non-relativistic Faddeev calculation [42], which utilized the AV18 [43] nucleon-nucleon potentials and the Höhler parameterization [44] for the single nucleon current. Furthermore, the $\Delta$-degree of freedom was explicitly included in the Faddeev calculations by Deltuva et al. [45, 46, 47, 48]. The non-relativistic Faddeev calculations, though proved very accurate at low $Q^2$, is not expected to be precise at large $Q^2$ when the relativistic effects become significant.
Dispersive Sum Rules and Theoretical Predictions

The famous Gerasimov-Drell-Hearn (GDH) sum rule [49, 50, 51] connects a particle’s spectrum of photoabsorption cross sections with its anomalous magnetic moment $\kappa$. Recent experimental and theoretical efforts on the GDH sum rule for real photons have expanded our knowledge about the nucleons and the nuclear effects in light nuclei. Taking the underlying principles of the GDH sum rule, several dispersive sum rules generalize this relation for virtual photons at an arbitrary $Q^2$, and relate the moments of spin-dependent structure functions to the real or virtual Compton amplitudes. By testing the theoretical calculations of the Compton amplitudes, these generalized sum rules offer great opportunities to study the nucleon spin structure and its evolution with $Q^2$.

In this chapter, we begin with an introduction to the GDH sum rule, and then extend the discussion to dispersive sum rules for all $Q^2$. The following sections present common theoretical tools and phenomenological models that provide predictions for these sum rules in various energy regions. In the last section, the comparison between existing experimental data and theoretical calculations is briefly reviewed.
3.1 The GDH sum rule

Being derived from the dispersion relation and the low energy theorem [52, 53], the GDH sum rule for spin-$\frac{1}{2}$ nucleon takes the form of

$$I_{N}^{GDH} = \int_{\nu_{th}}^{\infty} \frac{d\nu}{\nu} \left( \sigma_{\frac{1}{2}}(\nu) - \sigma_{\frac{3}{2}}(\nu) \right) = 2\pi^{2} \alpha \frac{K^{2}}{M^{2}},$$  \hspace{1cm} (3.1)

where $\nu_{th}$ is the pion production threshold for the target nucleon. In this equation, $\sigma_{\frac{1}{2}}$ and $\sigma_{\frac{3}{2}}$ are the helicity dependent photoabsorption cross sections with total helicity projection of $\frac{1}{2}$ and $\frac{3}{2}$, respectively. Figure 3.1 illustrates these two processes and the corresponding helicity projections between the photon and the nucleon. In the parton model, $\sigma_{\frac{1}{2}}$ and $\sigma_{\frac{3}{2}}$ are related to the parton density $f_{q}$ in the photon-parton Breit frame. In the case of total helicity of $\frac{1}{2}$, the photon can only be absorbed by a quark with the same helicity as the nucleon because the quark spin should remain $\frac{1}{2}$, so $\sigma_{\frac{1}{2}}$ only depends on $f_{q}^{1}$. For the same constraint only a quark with the opposite helicity can be involved in $\sigma_{\frac{3}{2}}$, which thus relates to $f_{q}^{1}$. Consequently, the average over these two helicity dependent cross sections leads to the transverse term of the total photoabsorption cross section$^*$:

$$\sigma_{\frac{1}{2}} + \sigma_{\frac{3}{2}} = 2\sigma_{T},$$  \hspace{1cm} (3.2)

while the helicity difference is related to the transverse-transverse interference term:

$$\sigma_{\frac{1}{2}} - \sigma_{\frac{3}{2}} = 2\sigma_{TT}.$$  \hspace{1cm} (3.3)

Based on the GDH sum rule, precise knowledge of a particle’s anomalous magnetic moment serves as an experimentally testable prediction for the weighted integral of its excitation spectrum, which relates to the particle’s intrinsic degrees of freedom.

$^*$Equation (2.38) has already revealed these relations.
Intensive experimental efforts have been devoted to testify the GDH sum rule for nucleons and light nuclei.

Predictions of full GDH integrals for the proton and the neutron can be determined from equation (3.1):

\[
I_p^{GDH} = 204.8 \, \mu b \, (\kappa_p = 1.793), \\
I_n^{GDH} = 233.2 \, \mu b \, (\kappa_n = -1.916).
\] (3.4)

The GDH sum rule for the proton was firstly investigated by the joint effort of GDH and A2 collaborations [54, 55] at Mainz Microtron (MAMI) with the photon energy \(200 \, \text{MeV} < E_\gamma < 800 \, \text{MeV}\). The energy range was then extended to \(680 \, \text{MeV} < E_\gamma < 2900 \, \text{MeV}\) in a subsequent experiment by GDH collaboration [56, 57] at the accelerator facility ELSA in Bonn. These experiments measured the helicity dependence of the total photoabsorption cross section on the proton, and together yielded \(254 \pm 5(\text{stat}) \pm 12(\text{sys}) \, \mu b\) for the proton GDH integral over the interval of \(200 \, \text{MeV} < \nu < 2900 \, \text{MeV}\). The unmeasured part can be evaluated based on models or phenomenological parameterizations. The integral from the pion production threshold to 200 MeV is given as \(-28 \, \mu b\) in SAID model [58] or \(-27 \, \mu b\).
in MAID model [59]. The combined result deviated from the prediction by approximately $-22 \, \mu b$. This discrepancy was initially thought from the unmeasured high energy region, but the $\pi^0$ contribution was found to be a probable cause for the discrepancy in a later experiment by LSC collaboration [60] at the Laser Electron Gamma Source (LEGS) at Brookhaven National Laboratory (BNL). The experiment measured inclusive pion production from both hydrogen and deuteron with polarized photon beams covering the $\Delta$ (1232) resonance region, resulting in a correction of $-17.5 \pm 5.7 \, \mu b$ to the Mainz and Bonn results. Consequently, these results lead to $208 \pm 6(\text{stat}) \pm 14(\text{sys}) \, \mu b$ [60], which is consistent with the prediction for the proton GDH integral.

The first experimental investigation of the neutron GDH sum rule was conducted by the GDH collaboration [61] at ELSA. The experiment measured the helicity dependent total photoabsorption cross sections from a longitudinally polarized LiD target with the photon energy from 815 MeV to 1825 MeV, and extracted a value of $33.9 \pm 5.5(\text{stat}) \pm 4.5(\text{sys}) \, \mu b$ for the neutron GDH integral in the corresponding photon energy range. For the unmeasured part below 815 MeV, theoretical estimates of $125.7 \, \mu b$ and $-2.5 \, \mu b$ for the single $\pi$ and $\eta$-meson contributions are given by the MAID [59] and ETA-MAID [62] models, respectively. In addition, the contribution from $\pi\pi N$ channel in this range has been roughly estimated utilizing the proton model from [63], and results in $40 \, \mu b$. The non-resonance contribution at high photon energy can be evaluated via the Regge approach [64], which obtains $30 \, \mu b$ for the neutron GDH integral above 1825 MeV. Combining all the above experimental and theoretical results, the full GDH integral on neutron is found to be approximately $227 \, \mu b$, and agrees with the prediction of $233 \, \mu b$.

For a nuclear target with spin-$S$ equation (3.1) is generalized to

$$I^{GDH} = \int_{\nu_{th}}^{\nu} \frac{d\nu}{\nu} (\sigma_P(\nu) - \sigma_A(\nu)) = 4\pi^2 \alpha \frac{\kappa^2}{M^2} S, \quad (3.5)$$

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where $\nu_{th}$ becomes the two-body breakup threshold for the nuclear target (2.2 MeV for deuteron and 5.5 MeV for $^3$He [65]). Here $\sigma_P$ ($\sigma_A$) represents the helicity dependent photoabsorption cross section, and its subscript $P$ ($A$) denotes the beam helicity parallel (anti-parallel) to the target spin. This equation immediately leads to the GDH sum rule predictions for deuteron of spin-1 and $^3$He of spin-$\frac{1}{2}$:

$$I_{GDH}^D = 0.65 \, \mu b \ (\kappa_D = -0.143),$$

$$I_{GDH}^{^3He} = 497.9 \, \mu b \ (\kappa_{^3He} = -8.371).$$

(3.6)

Following the experiment [61], a measurement [66] dedicated to the test of deuteron GDH integrand was performed by the GDH and A2 collaborations at MAMI. It had measured the GDH integral of deuteron between 200 MeV and 800 MeV, and yielded a value of $407 \pm 20$(stat) $\pm 24$(sys) $\mu b$. The statistical uncertainty was then significantly improved in a repeated measurement [67], which extracted a value of $388 \pm 7$(stat) $\pm 21$(sys) $\mu b$ for the same range. The combination of experimental results from [61] and [67] allowed an estimate for the deuteron GDH sum rule in the range of 200 MeV to 1800 MeV, amounted to $452 \pm 9$(stat) $\pm 24$(sys) $\mu b$ [67]. Therefore, the deuteron GDH sum rule, which predicted the full integral to be $0.65 \, \mu b$, suggests a significant contribution from the photo-disintegration region in order to cancel out the measured GDH integral in the resonance region. This suggestion was supported by an analysis [68] of world experimental data, which obtained a value of $-603 \pm 43 \, \mu b$ for the deuteron GDH integral from the photo-disintegration threshold to 6 MeV.

The experimental study of the $^3$He GDH sum rule has commenced recently with several accomplished helicity dependence measurements. On one hand, the GDH integral of $^3$He in the resonance region was firstly measured by the experiment [69] performed at MAMI with the photon energy from 200 MeV to 500 MeV. The integral in this range was found to be $135 \pm 20$(stat) $\pm 12$(sys) $\mu b$. On the other hand, several experiments at the High Intensity Gamma-Ray Source (HIGS) facility [70, 71, 72]
studied the $^3$He GDH integrand below the pion production threshold. The first measurement [70, 71] determined the contribution from the three-body break up channel to the GDH integrand for $^3$He at the photon energies of 12.8 MeV and 14.7 MeV. The results were found to be consistent with the theoretical predictions from Deltuva et al. [45, 46, 47, 48]. However, the measurement [72] with photon energy of 16.5 MeV showed a discrepancy larger than one standard deviation between the experimental data and theoretical predictions. More experimental efforts with the photon energy above 16.5 MeV are required for a better understanding of the $^3$He GDH integrand below the pion production threshold.

The aforementioned measurements of the GDH integrals beyond the pion production threshold are summarized in table 3.1.

### Table 3.1: GDH sum rules and the measured integral for nucleons and light nuclei [54, 55, 56, 57, 60, 61, 66, 67, 69].

<table>
<thead>
<tr>
<th>Target</th>
<th>$I^{GDH}$ (µb)</th>
<th>Measured Range (MeV)</th>
<th>$I^{GDH}_{exp}$ (µb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton</td>
<td>204.8</td>
<td>200-2900</td>
<td>254 ± 5 ± 12 - 17.5 ± 5.6</td>
</tr>
<tr>
<td>Neutron</td>
<td>233.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deuteron</td>
<td>0.65</td>
<td>200-1800</td>
<td>452 ± 9 ± 24</td>
</tr>
<tr>
<td>$^3$He</td>
<td>497.9</td>
<td>200-500</td>
<td>135 ± 20 ± 12</td>
</tr>
</tbody>
</table>

3.2 Dispersive sum rules

The GDH sum rule can be generalized for virtual photons at an arbitrary $Q^2$ via the forward dispersion relations in the doubly virtual Compton scattering (VVCS). In this section, we introduce the generalization procedure following the Refs. [73, 74].

Figure 3.2 shows the lowest-order diagram of the VVCS process. The virtual photon is characterized by its momentum $q^\mu = (q_0, \vec{q})$ and polarization $\epsilon^\mu = (\epsilon_0, \vec{\epsilon})$. In the laboratory frame, if the z-axis is chosen to coincide with the direction of photon propagation and hence $q^\mu = (\nu, 0, 0, |\vec{q}|)$, the longitudinal polarization of a
virtual photon can be defined as

$$\varepsilon_0^\mu = \frac{1}{Q}(|q|, 0, 0, \nu), \quad (3.7)$$

while the transverse part can be described by the circular polarization for photons with helicities of \( \pm 1 \):

$$\varepsilon_\pm^\mu = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0). \quad (3.8)$$

Therefore, the general form of the VVCS amplitude for forward scattering of virtual photons is written as [73]

$$T(\nu, Q^2, \theta = 0) = \varepsilon^{*\mu} \cdot \bar{\varepsilon}_f T(\nu, Q^2) + f_L(\nu, Q^2)$$

$$+ i \bar{\sigma} \cdot (\varepsilon^{*\mu} \times \bar{\varepsilon}) g_{TT}(\nu, Q^2)$$

$$+ i (\varepsilon^{*\mu} - \bar{\varepsilon}) \cdot (\bar{\sigma} \times \bar{q}) g_{LT}(\nu, Q^2). \quad (3.9)$$

---

**Figure 3.2**: The schematic of lowest-order diagrams for forward doubly virtual Compton scattering.

The Compton amplitude is symmetric under photon crossing, i.e., \( \varepsilon^{*\mu} \leftrightarrow \varepsilon_\mu \) and
$q_\mu \leftrightarrow -q_\mu$, so that the four amplitudes satisfy the following transformations:

\[ f_T(\nu, Q^2) = f_T(-\nu, Q^2), \]
\[ f_L(\nu, Q^2) = f_L(-\nu, Q^2), \]
\[ g_{TT}(\nu, Q^2) = -g_{TT}(-\nu, Q^2), \]
\[ g_{LT}(\nu, Q^2) = g_{LT}(-\nu, Q^2). \]

(3.10)

The unitarity in the optical theorem relates the imaginary part of these terms to the photoabsorption cross sections:

\[ \text{Im} f_T(\nu, Q^2) = \frac{K}{4\pi} \sigma_T(\nu, Q^2), \]
\[ \text{Im} f_L(\nu, Q^2) = \frac{K}{4\pi} \sigma_L(\nu, Q^2), \]
\[ \text{Im} g_{TT}(\nu, Q^2) = \frac{K}{4\pi} \sigma_{TT}(\nu, Q^2), \]
\[ \text{Im} g_{LT}(\nu, Q^2) = \frac{K}{4\pi} \sigma_{LT}(\nu, Q^2), \]

(3.11)

Here $K$ is the virtual photon flux introduced in Section 2.2, and we have chosen the convention in equation (2.33).

Considering only the inelastic contribution beyond the pion production threshold of $\nu_{th}$, the unsubtracted dispersion relation for the spin-flip amplitude $g_{TT}$ is written as

\[ \text{Re} \left[ g_{TT}(\nu, Q^2) - g_{TT}^{\text{pole}}(\nu, Q^2) \right] = \frac{2\nu}{\pi} \mathcal{P} \int_{\nu_{th}}^{\infty} \frac{\text{Im} g_{TT}(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu', \]
\[ = \frac{\nu}{2\pi^2} \mathcal{P} \int_{\nu_{th}}^{\infty} \frac{K(\nu', Q^2)\sigma_{TT}(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu', \]

(3.12)

where $\mathcal{P}$ is the principal value integral, and $g_{TT}^{\text{pole}}$ represents the elastic contribution. An appropriate high energy behavior is assumed so that this dispersion integral converges without a subtraction. Performing a low energy expansion on the left-
The leading term yields the generalized GDH sum rule $I_{TT}(Q^2)$, whose expression can be obtained by comparing the Taylor expansion of the right-hand side (RHS) in equation (3.12), and thus reads

$$I_{TT}(Q^2) = \frac{M^2}{4\pi^2\alpha} \int_{\nu_{th}}^{\infty} \frac{K(\nu, Q^2)\sigma_{TT}(\nu, Q^2)}{\nu^2} d\nu$$

$$= \frac{2M^2}{Q^2} \int_{0}^{x_{th}} \left[ g_1(x, Q^2) - \frac{4M^2}{Q^2} x^2 g_2(x, Q^2) \right] dx.$$  

(3.14)

The second term, denoted as $\gamma_{TT}(Q^2)$, is known as the generalized form of the forward spin polarizability. Its expression corresponds to the $O(\nu^3)$ term in the dispersion relation of $g_{TT}$:

$$\gamma_{TT}(Q^2) = \frac{1}{2\pi^2} \int_{\nu_{th}}^{\infty} \frac{K(\nu, Q^2)\sigma_{TT}(\nu, Q^2)}{\nu^4} d\nu$$

$$= \frac{16\alpha M^2}{Q^6} \int_{0}^{x_{th}} \left[ g_1(x, Q^2) - \frac{4M^2}{Q^2} x^2 g_2(x, Q^2) \right] x^2 dx.$$  

(3.15)

In the real photon limit ($Q^2 = 0$), $I_{TT}(Q^2)$ and $\gamma_{TT}(Q^2)$ reduce to $-\kappa_N^2/4$ and the forward spin polarizability $\gamma_0$, respectively.

Applying the same procedure to $g_{LT}$, one extracts the sum rule for $I_{LT}(Q^2)$:

$$I_{LT}(Q^2) = \frac{2M^2}{Q^2} \int_{0}^{x_{th}} \left[ g_1(x, Q^2) + g_2(x, Q^2) \right] dx,$$  

(3.16)

and the longitudinal-transverse spin polarizability:

$$\delta_{LT}(Q^2) = \frac{16\alpha M^2}{Q^6} \int_{0}^{x_{th}} \left[ g_1(x, Q^2) + g_2(x, Q^2) \right] x^2 dx.$$  

(3.17)

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In addition, Ji and Osborne [75] have generalized sum rules from the covariant form of the spin-dependent Compton amplitudes \(S_1\) and \(S_2\). These amplitudes can be equivalently written in terms of \(g_{TT}\) and \(g_{LT}\) [73]:

\[
S_1(\nu, Q^2) = \frac{\nu M}{\nu^2 + Q^2} \left[ g_{TT}(\nu, Q^2) + \frac{Q}{\nu} g_{LT}(\nu, Q^2) \right],
\]

\[
S_2(\nu, Q^2) = -\frac{M^2}{\nu^2 + Q^2} \left[ g_{TT}(\nu, Q^2) - \frac{\nu}{Q} g_{LT}(\nu, Q^2) \right].
\]

(3.18)

The leading term from the unsubtracted dispersion relation for \(S_1\) leads to the \(g_1\) sum rule, written as

\[
I_1(Q^2) = \frac{2M^2}{Q^2} \int_0^{x_{\text{th}}} g_1(x, Q^2)dx
= \frac{M^2}{4\pi^2\alpha} \int_{\nu_{\text{th}}}^{\infty} \frac{K(\nu, Q^2)}{\nu^2 + Q^2} \left[ \sigma_{TT}(\nu, Q^2) + \frac{Q}{\nu} \sigma_{LT}(\nu, Q^2) \right] d\nu.
\]

(3.19)

The elastic contribution to the integral is excluded from the sum rule, which amounts to [76]

\[
I_1^\text{el}(Q^2) = \frac{1}{2} \frac{G_M(Q^2)(G_E(Q^2) + \tau G_M(Q^2))}{1 + \tau},
\]

(3.20)

where \(\tau = Q^2/4M^2\), \(G_E\) and \(G_M\) are the Sachs electromagnetic form factors. It is worth noting that this sum rule also recovers \(-\kappa_N^2/4\) at the real photon point. At large \(Q^2\), the elastic contribution is highly suppressed, so \(I_1(Q^2)\) approximately equals \(2M^2\Gamma_1(Q^2)/Q^2\), with \(\Gamma_1(Q^2)\) the \(g_1\) moment defined as

\[
\Gamma(Q^2) = \int_0^1 g_1(x, Q^2)dx.
\]

(3.21)

The second amplitude \(S_2\) is an odd function of \(\nu\). Its unsubtracted dispersion relation reads

\[
\nu \Re S_2(\nu, Q^2) = \frac{2\nu^2}{\pi} \mathcal{P} \int_0^{\infty} \frac{\Im S_2(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu'.
\]

(3.22)
If assuming a Regge behavior for \( S_2 \) that vanishes faster than \( 1/\nu \), the unsubtracted dispersion relation for \( \nu S_2 \), which is even in \( \nu \), can be written as

\[
\text{Re} [\nu S_2(\nu, Q^2)] = \frac{2}{\pi} \mathcal{P} \int_0^\infty \nu' \frac{\text{Im} S_2(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu'. \tag{3.23}
\]

Subtraction between equation (3.22) and equation (3.23) leads to

\[
0 = \int_0^\infty \text{Im} S_2(\nu, Q^2) d\nu = \int_0^1 g_2(x, Q^2) dx. \tag{3.24}
\]

This sum rule is known as the Burkhardt-Cottingham (BC) sum rule [77]. By separating the elastic and inelastic contributions of \( S_2 \), an alternative form of the BC sum rule is given as [73]

\[
I_2(Q^2) = \frac{2M^2}{Q^2} \int_0^{x_{th}} g_2(x, Q^2) dx = \frac{1}{4} G_M(Q^2) (G_M(Q^2) - G_E(Q^2)) \frac{1 + \tau}{1 - \tau}. \tag{3.25}
\]

This sum rule relates the integral of \( g_2 \) over the inelastic region to the Sachs form factors. It is straightforward to obtain the value of \( I_2(0) \), which equals \( \frac{1}{4} \mu_N K_N \).

### 3.3 Operator product expansion

It is shown in the previous section that the dispersive sum rules are intimately connected with the spin-dependent structure functions and their moments. In this section, we continue the discussion of spin-dependent structure functions in Section 2.1 and examine their physical interpretation in the deep inelastic region with the theoretical tool named Operator Product Expansion (OPE). A few model-independent QCD predictions for the moments of the structure functions at high \( Q^2 \), in terms of sum rules, are also presented.

Section 2.2 illustrates the connection between the structure functions and the virtual photoabsorption cross sections. In fact, the hadronic tensor is related to the
forward Compton amplitude for an electromagnetic process, written as [6]

\[ W_{\mu\nu} = \frac{1}{2\pi M} \text{Im} T_{\mu\nu} = \frac{1}{4\pi M} \int e^{iq \cdot x} \langle N| [J_\mu(x), J_\nu(0)] |N\rangle d^4x, \]  

where \( J_\mu \) represents the electromagnetic current operator. The behavior of operator products, such as \( J_\mu(x)J_\nu(0) \), can be studied with OPE [78], which was originally introduced by Wilson [79].

In OPE, the product of two operators at short distances is expressed as a sum over local operators \( O_n \):

\[ \lim_{x \to 0} A(x)B(0) = \sum_n C_n(x)O_n(0), \]  

where \( C_n(x) \) are the coefficient functions that involve powers of four-dimensional spatial vector \( x \) and can be calculated perturbatively in QCD. The convergence of this expansion clearly requires \( x \) to be small, or equivalently, \( Q^2 \) to be large.

For the inclusive lepton scattering, the local operators \( O_n \) are related to the quark field operators and their covariant derivatives. Contributions from the expansion are classified by “twist” defined as \( \tau = d - n \), with \( d \) the energy dimension and \( n \) the spin of the operators. The leading order at twist-2 can be interpreted as the interaction of single active parton, while the higher twist contributions involve more active partons or contracted derivatives and are suppressed by the power of \( 1/Q^2 \).

The \( g_1 \) moment introduced in equation (3.21) can be expressed in terms of the twist expansion via OPE [80]:

\[ \Gamma_1(Q^2) = \sum_{\tau=2,4,\ldots} \frac{\mu_\tau(Q^2)}{(Q^2)^{\tau/2-1}}, \]  

where \( \mu_\tau \) are related to the nucleon matrix elements of quark-gluon operators with twist \( \leq \tau \) and can be calculated in perturbative QCD. The leading twist term of
this expansion is related to the Ellis-Jaffe sum rule [81], and its coefficient \( \mu_2 \) sums over the axial charge for every quark flavor. It is often decomposed into the non-singlet (\( g_A \) for the flavor triplet and \( a_8 \) for the flavor octet) and the singlet (\( \Delta \Sigma \)) contributions, and hence reads [82]

\[
\mu_2(Q^2) = \left( \pm \frac{1}{12} g_A + \frac{1}{36} a_8 \right) + \frac{1}{9} \Delta \Sigma + \mathcal{O}(\alpha_s(Q^2)).
\] (3.29)

where the “+” (“−”) sign is for the proton (neutron) and \( \mathcal{O}(\alpha_s(Q^2)) \) represents the contribution from QCD radiative correction.

The difference of \( g_1 \) moments between the proton and the neutron leaves the flavor non-singlet contribution as

\[
\Gamma^p_1(Q^2) - \Gamma^n_1(Q^2) = \frac{g_A}{6} + O(\alpha_s(Q^2)) + O(1/Q^2).
\] (3.30)

It recovers the Bjorken sum rule [83] in the limit of large \( Q^2 \). The \( O(\alpha_s(Q^2)) \) term with QCD corrections up to the third order is given in Ref. [84], which yields

\[
-\frac{g_A}{6} \sum_{k=1}^{3} c_k \left( \frac{\alpha_s(Q^2)}{\pi} \right)^k,
\] (3.31)

with \( c_1 = 1, c_2 = 3.5833, \) and \( c_3 = 20.2153. \)

The structure function \( g_2 \) can be separated into a twist-2 term (\( g_{2WW} \)) and higher twist contributions (\( \bar{g}_2 \)):

\[
g_2(x, Q^2) = g_{2WW}(x, Q^2) + \bar{g}_2(x, Q^2).
\] (3.32)

Subsequently, \( g_{2WW} \) is determined by the leading twist term of \( g_1 \) via the Wandzura-Wilczek relation [85]:

\[
g_{2WW}(x, Q^2) = -g_1^{LT}(x, Q^2) + \int_x^1 \frac{g_1^{LT}(x, Q^2)}{y} dy,
\] (3.33)
with the assumption of massless quarks. The higher twist contribution of \( g_2 \) possesses the information about the quark-gluon interactions and the quark transverse polarization distribution \([86]\).

### 3.4 Chiral perturbation theory

Perturbative QCD has achieved a great success at high energies where the observables can be calculated with a power expansion on the strong coupling constant \( \alpha_s \). However, this perturbation technique becomes inapplicable in the low energy domain because \( \alpha_s(Q^2) \) is of order one at \( Q^2 < 1 \text{ GeV}^2 \). In fact, the degrees of freedom in QCD can be effectively viewed as composite hadrons instead of quarks and gluons at low energies. Chiral perturbation theory (ChPT) \([87, 88, 89]\) is an effective field theory, which, based on chiral symmetry, constructs the most general effective Lagrangian for the active hadronic degrees of freedom, and describes low energy QCD in a model-independent way.

In this section, we briefly present the framework of ChPT. Comprehensive introductions to this profound topic can be found in Refs. \([90, 91]\).

#### 3.4.1 Chiral symmetry

QCD is a gauge theory for the non-Abelian SU(3) group of the color symmetry. Its Lagrangian for \( N_f \) flavors of quarks is written as

\[
\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu, a} + \sum_{N_f} \bar{q} \left( i \gamma^\mu D_\mu - m_q \right) q, \tag{3.34}
\]

with the gluon field strength tensor

\[
G_{\mu\nu}^a = \partial^\mu A_{\nu}^a - \partial^\nu A_{\mu}^a + g f_{abc} A_{\nu}^b A_{\mu}^c, \tag{3.35}
\]

and the covariant derivative

\[
D_\mu = \partial_\mu - ig \frac{\lambda_a}{2} A_\mu^a, \tag{3.36}
\]

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where \( m_q \) is the quark mass, \( q \) denotes the quark field, \( A_\mu \) represents the gluon vector field and \( g \) is the coupling constant. Here \( \lambda_a \) are the Gell-Mann matrices, and \( T_a = \lambda_a/2 \) are the generators with the structure constants \( f_{abc} \) in the adjoint representation of the SU(3) color symmetry group. Note that the following discussion is restricted for the light quarks, \( i.e., \) the up \((u)\), down \((d)\) and strange \((s)\) quarks, and sometimes only the \( u \) and \( d \) quarks are considered \((N_f = 2)\).

The quark field \( q \) can be separated into its chiral components:

\[
q = (P_L + P_R)q = q_L + q_R, \tag{3.37}
\]

with the left handed \((P_L)\) and right handed \((P_R)\) projection operators defined as

\[
P_L = \frac{1}{2}(1 - \gamma_5), \quad P_R = \frac{1}{2}(1 + \gamma_5). \tag{3.38}
\]

For massless Dirac particles the operator \( P_L \) \((P_R)\) projects to the negative \((positive)\) helicity eigenstate, and hence the chirality is identical to the helicity.

With equation (3.37), the QCD Lagrangian then reads

\[
\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^0 + \mathcal{L}_M, \tag{3.39}
\]

with the mass term

\[
\mathcal{L}_M = - \sum_{N_f} (\bar{q}_R m_q q_L + \bar{q}_L m_q q_R), \tag{3.40}
\]

and the massless term

\[
\mathcal{L}_{QCD}^0 = - \frac{1}{4} G^{a}_{\mu \nu} G^{\mu \nu, a} + \sum_{N_f} (\bar{q}_R i \gamma^\mu D_\mu q_R + \bar{q}_L i \gamma^\mu D_\mu q_L). \tag{3.41}
\]

In the limit that light quark mass vanishes, which is often referred to as the chiral limit, \( \mathcal{L}_{QCD} \) equals \( \mathcal{L}_{QCD}^0 \), and the term involved with quark fields decouples into the left- and right-handed-only components. Consequently, the QCD Lagrangian admits
an SU($N_f$)$_L \times$ SU($N_f$)$_R \times$ U(1)$_V$ symmetry\(^\dagger\) in the chiral limit. The symmetry group of SU($N_f$)$_L \times$ SU($N_f$)$_R$ is the so-called chiral symmetry. Equation (3.40) indicates that the mass term mixes left- and right-handed quark fields, and thus non-zero quark mass explicitly breaks the chiral symmetry.

Considering the linear combinations of the left- and right-handed charge operators from the SU($N_f$)$_L \times$ SU($N_f$)$_R$ group, which are defined as [90]

\[
Q_V^a = Q_R^a + Q_L^a, \quad Q_A^a = Q_R^a - Q_L^a, \quad a = 1, \ldots, N_f^2 - 1.
\] (3.42)

They possess opposite parity and commute with the QCD Hamiltonian in the chiral limit. Subsequently, the existence of degenerate parity doublets is expected in the hadron spectrum, but is not observed experimentally. This fact indicates the QCD ground state is not annihilated by the generator $Q_A^a$ [90]. Moreover, the ground state is necessarily invariant under the vector rotation subgroup of SU($N_f$)$_V$ [92], so that the chiral symmetry group spontaneously breaks down to SU($N_f$)$_V$. According to the Goldstone theorem [93, 94], each axial generator $Q_A^a$ is associated with a massless Goldstone boson. In the case of $N_f = 2$ the pions, as the three lightest pseudoscalar mesons, are identified as these Goldstone bosons\(^\ddagger\), whose finite masses are due to the explicit chiral symmetry breaking.

3.4.2 Effective chiral Lagrangian in mesonic sector

The spontaneous breaking of chiral symmetry results in “unnaturally” light pseudoscalar mesons. This phenomenon is known as the characteristic mass gap in the hadron spectrum, and leads to a reasonable guess that the strong interactions at low energies should be governed by these pseudoscalar mesons, primarily the pions, since they are the softest excitations of the QCD ground state. For low energy QCD

\(^\dagger\)In the classical case such a Lagrangian has an U($N_f$)$_L \times$ U($N_f$)$_R$ symmetry, while on the quantum level the singlet axial-vector current is not conserved.

\(^\ddagger\)With the inclusion of s quark, the Goldstone bosons are the eight lightest pseudoscalar mesons of pions, kaons and $\eta$ meson.
one would construct an effective Lagrangian, which takes the most general form and admits all the symmetries of the original QCD Lagrangian, describing the dynamics of these active hadronic degrees of freedom.

The effective Lagrangian can take the form of

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_{\text{s.b.}},$$

where $\mathcal{L}_0$ is the chiral symmetric term with a spontaneous symmetry breaking to SU($N_f$)$_V$, and $\mathcal{L}_{\text{s.b.}}$ denotes the explicit symmetry breaking term containing the finite quark masses. Considering the small masses of $u$ and $d$ quarks compared to typical hadronic mass scales, such as the masses of the first non-Goldstone meson $m_\rho \approx 770$ MeV and the least massive baryon $m_p \approx 938$ MeV, it is reasonable to treat $\mathcal{L}_{\text{s.b.}}$ as a perturbation to $\mathcal{L}_0$.

In the mesonic sector, the lowest-order of the effective Lagrangian is expressed as

$$L_{\text{eff}}^{(2)} = \frac{F_0^2}{4} \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{F_0^2 B_0}{2} \text{Tr} \left( MU^\dagger + UM^\dagger \right),$$

where $F_0 \approx 93$ MeV is related to the pion-decay constant, $M$ represents the diagonal quark mass matrix, and $U$ is an unitary matrix field which collects the Goldstone bosons and transforms as $U \rightarrow RUL^\dagger$ under the chiral rotations $R \in \text{SU}(N_f)_R$ and $L \in \text{SU}(N_f)_L$. The constant $B_0$ is connected with the non-vanishing chiral quark condensate, at the lowest-order, as $3F_0^2 B_0 = -\langle \bar{q}q \rangle$. Assuming $\langle \bar{q}q \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$, we have the relation of $m_\pi^2 = 2B_0(m_u + m_d)$.

For higher order contributions the effective Lagrangian can be written in terms of the increasing power of derivatives and the finite quark mass. The “standard” chiral counting scenario gives the chiral order counts of $U$ as $O(1)$, $\partial_\mu U$ and external fields as $O(p)$, and $m_q$ as $O(p^2)$ with $p$ the external momentum [95]. Subsequently,
the Lagrangian is organized by the chiral orders as
\[ \mathcal{L}_{\text{eff}} = \sum_{i+2j=d} \mathcal{L}_{i,j}^{(d)}, \quad \mathcal{L}_{i,j}^{(d)} = \mathcal{O}(p^i m_q^j). \] (3.45)

Since the Lorentz indices in the derivatives have to be contracted for a scalar result, only the orders of \( \mathcal{O}(p^{2n}) \) survive, with the leading order at \( \mathcal{O}(p^2) \). It is worth noting that such an expansion only works for a small external momentum, and the convergence is characterized by the parameter \( \Lambda_{CSB} \), which is named chiral symmetry breaking scale and approximately amounts to \( 4\pi F_0 \). Besides, the importance of loop diagrams to the effective Lagrangian is systematically evaluated by Weinberg’s power counting rule [87], which reads
\[ D = 2 + \sum_{n=1}^{\infty} 2N_{2n}(n - 1) + 2N_L, \] (3.46)
with \( N_{2n} \) the number of vertices at chiral order \( 2n \) and \( N_L \) the number of independent loops. This formula relates the diagram structure to the momentum expansion, and thus provides a practical guidance for the order of expansion and the inclusion of loop diagrams at a given dimension \( D \).

3.4.3 Chiral perturbation theory for nucleons

The framework of ChPT can be extended for the interactions of baryons at low energies, in other words, the effective Lagrangian includes the baryon fields. In the case of two flavors (\( u \) and \( d \)), the discussion is narrowed to the nucleons and the pions. For a process with one nucleon in both the initial and final states, the lowest order of \( \pi N \) effective Lagrangian can be written as [96]
\[ \mathcal{L}^{(1)}_{\pi N} = \bar{\Psi} \left( i\gamma^\mu D_\mu - m_{N,0} + \frac{g_{A,0}}{2} \gamma^\mu \gamma_5 u_\mu \right) \Psi, \] (3.47)
with \( u^2 = U, \ u_\mu = iu^\dagger \partial_\mu U u^\dagger \) and \( D_\mu = \partial_\mu + \frac{i}{2}[u^\dagger, \partial_\mu u] \). Here matrix \( \Psi \) contains the complex, four-component Dirac field for each nucleon, \( m_{N,0} \) and \( g_{A,0} \) represent the
nucleon mass and the axial-vector coupling constant in the chiral limit, respectively.

Non-vanishing $m_{N,0}$ value results in a different scenario of the chiral order counting for the baryon fields, which is summarized as [97]

$$\Psi, \bar{\Psi}, D_\mu \Psi = \mathcal{O}(1), \quad (i\gamma^\mu D_\mu - m_{N,0})\Psi = \mathcal{O}(p),$$

$$\gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(1), \quad \gamma_5 = \mathcal{O}(p).$$

(3.48)

Subsequently, the effective Lagrangian in the one-nucleon sector is organized as a low energy expansion:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi}^{(2)} + \mathcal{L}_{\pi}^{(4)} + \cdots + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \cdots.$$  \hspace{1cm} (3.49)

The relation between loop diagrams and the momentum expansion, i.e., the power counting rule, is complicated due to the fact that the nucleon mass is finite in the chiral limit and is comparable to the chiral symmetry breaking scale $\Lambda_{\text{CSB}}$. The one-to-one correspondence between loop diagrams and chiral orders in the mesonic sector does not hold anymore in the baryonic case, and an infinite number of loop diagrams would contribute to a certain chiral order [96]. To overcome this difficulty, the Heavy Baryon ChPT (HBChPT) approach was introduced, which considered the nucleon as extremely heavy. The external momentum is thus separated into a rest mass related term and a small residual, written as $p_\mu = m_N v_\mu + k_\mu$, with a typical choice of $v_\mu = (1, 0, 0, 0)$. As the result of this separation, the effective Lagrangian becomes a two-fold momentum expansion over $1/\Lambda_{\text{CSB}}$ and $1/m_N$, so a consistent power counting scheme is retrieved. More details of this approach can be found in the review article [98].

The low energy expansion in HBChPT sometimes results in a problem with analyticity due to the singularity structure of Green functions. This fact motivated the search for more advanced approaches, which not only have the proper power counting scheme as in HBChPT, but also are analytical for all situations. One of
these approaches is the so-called Infrared Regularization (IR) [99], which separates
the one-loop integral into an infrared singular part and a regular part. The former
obeys the power counting rule, while the latter can be expanded in a power series
and hence be absorbed into the low energy constants of the effective Lagrangian. In
addition, other regularization methods have also been developed, and one can find
more details regarding the recent theoretical progress of the baryonic ChPT in the
review articles [95, 100].

3.5 Phenomenological models

Although ChPT is able to provide low energy QCD predictions, the breaking of
the momentum expansion over the chiral breaking scale limited its application at a
moderate $Q^2$, where the perturbative QCD also experiences a breaking of the $1/Q^2$
expansion. Many aspects of QCD in the non-perturbative and transition regimes,
such as the rich resonant states in the inclusive lepton scattering spectrum, are
still insufficiently understood. Therefore, models or parameterizations based on the
existing experimental data are important to give phenomenological predictions in
this kinematic region.

3.5.1 MAID models

The unitary isobar model MAID [59] is based on phenomenological fits for world data
of pion photo- and electroproduction on the nucleon. The model is valid from the pion
production threshold to $W = 2$ GeV with $Q^2$ less than 3.9 GeV$^2$. The MAID model
consists of contributions from several resonances and a non-resonant background. In
the latest version (MAID2007) [101], the resonance contributions include thirteen 4-
star rated resonances§, and each of them is described by the Breit-Wigner form. The
background parameterization involves mixed pseudovector and pseudoscalar $\pi NN$

§The 4-star rating is defined as “Existence is certain, and properties are at least fairly well
explored.” [19]. For the resonances list please refer to [101].
couplings and $t$-channel vector-meson contributions, and it follows the $K$-matrix approximation introduced in Ref. [102] to maintain the unitarity. For the photo- and electroproduction of other pseudoscalar mesons on the nucleon, one may also use the KAON-MAID [103, 104] and ETA-MAID [62] models, which were developed for kaons and $\eta$-meson, respectively.

### 3.5.2 SAID Model

The SAID model [58] is based on the partial-wave analysis of single-pion photo-production data. In this model, the spectrum of $N^*$ and $\Delta^*$ resonances and their characteristics are analyzed by a two-step process: fits for the world data determine the contributions from multipoles, which are then parameterized by the combination of a common background and the corresponding resonant states. The model is applicable within the kinematic range of $Q^2 < 5$ GeV$^2$ and $W_{th} < W < 2$ GeV, where $W_{th}$ represents the pion production threshold. Resonance contributions to the GDH sum rule or the sum rule of nucleon polarizability can be evaluated by the fitted multipole amplitudes in this model.

### 3.6 Existing neutron and $^3$He data at low $Q^2$

Intensive experimental and theoretical efforts have been dedicated to investigate the moments of the nucleon spin dependent structure functions in the form of sum rules over a wide range of $Q^2$. In the non-perturbative regime, theoretical predictions of the moments are available from the ChPT framework, while phenomenological models can describe the $Q^2$ evolution of the moments from the perturbative regime to the non-perturbative regime. Experimental data at low $Q^2$ can thus probe the theories and offer insights in the QCD confinement region.

In this section, we briefly summarize the results of the experiment E94-010 at JLab [105, 106, 107, 33]. The comparison of the dispersive sum rules between the E94-
010 data and the ChPT predictions is one of the main motivations for the experiment E97-110, which extended the tests to a lower $Q^2$ region of 0.02-0.2 GeV$^2$. For a thorough overview on the experimental results of the structure function moments in the low and intermediate $Q^2$ region, one can refer to Refs. [108, 109]. In addition, the recent and relevant theoretical and experimental developments are discussed in the last subsection.

3.6.1 Neutron results of E94-010

E94-010 had measured the cross sections for the inclusive scattering of polarized electrons from a polarized $^3$He target with the beam energy ranging from 0.86 to 5.06 GeV. Its data covered the quasi-elastic region, the nucleon resonance region and partially the DIS region. Generalized GDH integrals from the threshold to $W = 2$ GeV were extracted for both the neutron and $^3$He with $Q^2$ of 0.1-0.9 GeV$^2$.

The E94-010 neutron result of the generalized GDH sum rule was published in 2002 [105], which is shown in Figure 3.3. The DIS contribution to the unmeasured part of the integral is evaluated by the Regge parameterization from Thomas and Bianchi [110], and the nuclear effects for extracting the neutron results from a $^3$He target is taken into account by the calculation from Ciofi degli Atti and Scopetta [111]. Note that the generalized GDH sum rule $I(Q^2)$ in E94-010 is formulated by a direct replacement of the helicity dependent cross sections for real photons with that for virtual photons [65]. $I(Q^2)$ can be expressed in terms of $I_{TT}(Q^2)$ as

$$I(Q^2) = \frac{8\pi^2\alpha}{M^2} I_{TT}(Q^2),$$  \hspace{1cm} (3.50)

and it recovers $-I^{GDH}$ at the real photon point.

The comparison involves two baryon ChPT calculations. Ji et al. [112] calculated the generalized GDH sum rule at order $\mathcal{O}(p^4)$ with the HBChPT approach, while Bernard et al. [113] utilized a Lorentz-invariant formulation of Baryon ChPT with
Figure 3.3: Experimental data of the generalized GDH sum rule for the neutron from E94-010 [105]. The error bars represent statistical uncertainties of the data, and the grey band shows the systematic uncertainties.

the Infrared Regularization (IRBChPT). In the latter approach, the $\Delta$-resonance contribution is evaluated by the relativistic Born graphs, and the contributions from vector mesons (VM) are included in the effective low energy constants [114]. The estimation of the resonance contribution relies on several experimental parameters, which are not well-known and lead to a sizable systematic uncertainty displayed as the yellow band in the plot.

As shown in the figure, the E94-010 data smoothly evolve with an increasing $Q^2$ and approach the HERMES data [115] at large $Q^2$. The behavior of the experimental data is well described by the phenomenology MAID model at $Q^2 > 0.3$ GeV$^2$. At low $Q^2$, the data deviate from the MAID model, but the lowest $Q^2$ point overlaps the ChPT prediction with estimated resonance contributions. This encouraging fact indicates a strong predictive power of ChPT in the low energy region, and it needs to be further tested with more experimental data at lower $Q^2$. 

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Figure 3.4: E94-010 data for the generalized neutron spin polarizabilities of $\gamma_{TT}(Q^2)$ (top) and $\delta_{LT}(Q^2)$ (bottom) [106].

In 2004, the E94-010 collaboration published the experimental results of the neutron spin polarizabilities. Figure 3.4 shows the E94-010 data and the predictions from HBChPT by Kao et al. [116] and IRBChPT by Bernard et al. [113] in the low $Q^2$ region. The IRBChPT calculation with resonance contributions from $\Delta(1230)$ and vector mesons [114] is depicted as the cyan band. For the generalized forward spin polarizability $\gamma_{TT}(Q^2)$, shown in the top panel, a good agreement is found between the data point at the lowest $Q^2$ and the IRBChPT calculation with the effects from resonant states. The bottom plot is for the transverse-longitudinal polarizability $\delta_{LT}(Q^2)$. It is surprising that the lowest $Q^2$ point, unlike the other cases, is significantly inconsistent with both of the ChPT calculations. Therefore, experimental data at lower $Q^2$ are required to further explore this discrepancy.
Figure 3.5: Experimental data of the generalized GDH sum rule for $^3$He from E94-010 [33].

3.6.2 $^3$He results of E94-010

The generalized GDH sum rule for $^3$He from E94-010 was published in 2008 [33]. As shown in Figure 3.5, the experimental data were compared with the MAID model and the quasielastic contribution was estimated by a PWIA calculation [32, 34]. Since $I(Q^2)$ is supposed to recover $-I^{GDH}$ at the real photon point, a drastic turnover is expected at $Q^2 < 0.2$ GeV$^2$. The theoretical calculation also predicts this behavior, though both the PWIA approach and the MAID model are not supposed to be applied for a quantitative result in this kinematic range. Therefore, we are concerned with the $Q^2$ evolution of the $I(Q^2 < 0.2)$ for $^3$He, which can be further investigated by the experimental data at lower $Q^2$. 

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3.6.3 Recent Developments

In the last decade, significant theoretical and experimental developments have been accomplished for the study of the nucleon spin dependent structure functions and their moments in the non-perturbative regime. On the experimental side, three JLab experiments were performed to investigate the moments of the spin dependent structure functions at $Q^2 \ll 1 \text{ GeV}^2$:

- The experiment E08-027 [117] determined the proton $g_2$ and its moments from a polarized proton target;

- The experiment EG4 [118] utilized a polarized proton target and a polarized deuteron target, and extracted the $g_1$ moments for both of the proton and the neutron;

- The experiment E97-110 measured the $g_1$ and $g_2$ moments for the neutron and $^3\text{He}$ from a polarized $^3\text{He}$ target. Its results are discussed in the remainder of this dissertation.

On the theoretical side, the $\delta_{LT}$ discrepancy has motivated the improvement on ChPT calculations. On one hand, Bernard et al. [119] have explicitly incorporated the spin-$\frac{3}{2}$ degree of freedom in a covariant formulation of the BChPT based on the “small scale expansion” [120]. The calculation estimated the low energy constants from the electromagnetic width of the $\Delta$-resonance and is thus free of parameters. Improved descriptions for both the forward spin polarizability and the longitudinal-transverse spin polarizability are found in the calculation up to the third order. However, a full one-loop calculation (fourth order) is necessary to resolve the discrepancy in the $\delta_{LT}$. On the other hand, Lensky et al. [121] provide parameter-free results of the leading-order (LO) and the next-to-leading-order (NLO) BChPT calculations. The $\Delta$ degree of freedom is explicitly included and the contribution is
calculated via the technique called “δ-expansion” [122], in which the corresponding scale is the mass difference $\Delta$ between the $\Delta(1232)$ resonance and the nucleons. Both of the LO ($O(p^3)$) and the NLO ($O(p^4/\Delta)$) calculations exhibit improved agreements in the spin polarizabilities. Figure 3.6 shows the recent ChPT results in comparison with the E94-010 data on the neutron spin polarizabilities. These improved ChPT calculations await the experimental test at lower $Q^2$.

In summary, the predictive power of the baryon ChPT framework needs further experimental verification in terms of the dispersive sum rules at low $Q^2$. Besides, the $Q^2$ evolution of the generalized GDH sum rule for $^3$He is also of great interest because of the expecting turnover to recover the GDH integral at the real photon
point. The low $Q^2$ data from the experiment E97-110 are crucial to investigate these topics. In the following chapters, we report this experiment in detail.
Experiment E97-110 is a precise measurement based on inclusive scattering of longitudinally polarized electrons off a high-pressure $^3$He target polarized in both longitudinal and transverse directions. The experiment aims to determine the spin-dependent structure functions of $g_1(x, Q^2)$ and $g_2(x, Q^2)$ and their lower moments for the neutron and $^3$He, and to test the theoretical predictions from ChPT as well as phenomenology models in the low energy regime.

E97-110 was performed in experimental Hall A at the TJNAF located in Newport News, Virginia. The experimental runs was separated into two groups: the first period in April-May 2003, and the second period in July-August 2003. Another graduate student in the collaboration, N. Ton, has been analyzing the experimental data taken in the first period, which covered the lowest $Q^2$ point at 0.02 GeV$^2$. The detailed analysis procedure and the $^3$He results for the second period data, with $Q^2$ from 0.03 to 0.23 GeV$^2$, are addressed in this dissertation.

During the second period, longitudinally polarized electron beams with energies from 1.1 to 4.4 GeV were utilized. The scattered electrons at an angle of 6.10° or 9.03° were detected by the right-arm Hall A High Resolution Spectrometer (HRS).
For the detection at these forward scattering angles, a septum magnet [123] was placed upstream of the spectrometer and bent the scattered electrons to its entrance. The experimental data of the differential cross sections covered the quasi-elastic and the resonance production regions. The full kinematic coverage of the second period is shown in Figure 4.1.

In this chapter, the experimental apparatuses of E97-110 are discussed in detail. Firstly, we introduce the polarized $^3$He target cell utilized in the experiment. The polarized electron beam at JLab and the beamline in Hall A are discussed in the subsequent sections. Lastly, an introduction to the spectrometer, the detector package,
and the optics calibration is presented.

4.1 Polarized $^3$He target

E97-110 has utilized the JLab polarized $^3$He target system [124] as the effective polarized neutron target. The target system consists of a target cell, a polarizing optic system, two orthogonal sets of Helmholtz coils, and apparatuses for two independent polarimetry methods: the Nuclear Magnetic Resonance (NMR) [125] and the Electron Paramagnetic Resonance (EPR) [126].

![Figure 4.2: The schematic of the JLab $^3$He target system. Transverse Helmholtz coils are hidden in the figure.](image)

As shown in Figure 4.2, the target cell, filled with $^3$He and nitrogen ($N_2$) gas, is placed at the center of the Helmholtz coils. During the operation, its top chamber
is heated by an oven to 170°, and the vaporized rubidium (Rb) in the chamber is continuously illuminated by the circularly polarized laser light for polarizing the $^3$He gas. The polarization angle can be aligned in the 25 Gauss magnetic field produced by the Helmholtz coils, and the polarization can be measured by the two polarimetry methods. The bottom chamber of the target cell is placed along the beam line, allowing incident electrons to pass through and interact with polarized $^3$He nuclei.

The JLab $^3$He target system has been upgraded with the rubidium-potassium hybrid vapor and the narrow band laser [127]. As the result, the typical target polarization increases from 35-40% to 60-65%. In this section, we briefly review the target system at the time when E97-110 was performed, i.e., before the upgrade.

### 4.1.1 Target cells

High-pressure $^3$He gaseous cells with an operation pressure at 10-12 atm were used in E97-110. The cells were designed to minimize the radiation length for the incident and scattered electrons, resulting in 130-140 µm thick glass windows and 700 µm thick glass walls. As shown in Figure 4.3, the cells consist of a spherical pumping chamber and a cylindrical target chamber, which are connected with a transfer tube. The pumping chamber was coated with a small amount of Rb metal, and the $^3$He gas was polarized here through a process known as the Spin Exchange Optical Pumping (SEOP). The polarized $^3$He gas then diffused to the target chamber and interacted with the incident electron beam passing through the chamber.

The scattered electrons at angles of 6° and 9° would be eventually detected. The traversing length of the electrons in the glass cell wall is proportional to $1/\sin \theta$, with $\theta$ the scattering angle, and thus the cell contributes significantly to the total radiation length at small scattering angles. For minimizing this effect, an “ice cone” target cell with an enlarged downstream target chamber was utilized. As shown in Figure 4.4, the scattered electrons at 6° from the chamber center would only traverse
Figure 4.3: Side view of the standard $^3$He target cell in Hall A.

Table 4.1: The characteristics of the target cells used in the second period of E97-110. These cells were studied at the University of Virginia [128].

<table>
<thead>
<tr>
<th>Target Cell</th>
<th>Type</th>
<th>Length</th>
<th>Ave. Density</th>
<th>Relaxation Life Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>Standard</td>
<td>40 cm</td>
<td>8.90 ± 0.13 amg</td>
<td>56 ± 11 hours</td>
</tr>
<tr>
<td>Priapus</td>
<td>Standard</td>
<td>40 cm</td>
<td>8.72 ± 0.13 amg</td>
<td>54 ± 11 hours</td>
</tr>
</tbody>
</table>

the end target window of 200-250 $\mu$m thick. This cell reduces the radiation length from the glass walls by a factor of 30. However, the “ice cone” cell was only used in the first period. The characteristics of the two standard cells used in the second period are listed in table 4.1.

Figure 4.4: Top view of the “ice cone” target cell.
4.1.2 Spin exchange optical pumping

The $^3$He gas in the target cell is polarized via the SEOP process, which can be decomposed into two steps: firstly the Rb vapor is “optically pumped” by circularly polarized photons; then $^3$He nuclei are polarized by “spin-exchange” with the alkali atoms. Since $^3$He nucleus has a negative magnetic moment, the direction of the $^3$He spin polarization is opposite to the magnetic holding field produced by the Helmholtz coils.

The process of optical pumping is shown in Figure 4.5. Circularly polarized photons at wavelength of 794.8 nm can excite the outermost electron in Rb from its ground state of $5S_{1/2}$ to the state of $5P_{1/2}$. Both states are split into $m_J = \pm 1/2$ in an external magnetic field due to the Zeeman effect. The excitation is governed by the selection rules, requiring $\Delta m_J = -1$ and $+1$ for the absorption of left- and right-handed circularly polarized photons, respectively. The JLab polarized $^3$He target system utilizes the right-handed circularly polarized light for inducing the transition from $m_J = -1/2$ at $5S_{1/2}$ to $m_J = +1/2$ at $5P_{1/2}$. The decay of the excited electron is associated with emission of unpolarized photons, which can depolarize other Rb atoms. Therefore, a “non-radiative quenching” technique is applied by adding a small amount of N$_2$ molecules into the cell, de-exciting the Rb via collisions and absorb the energy into N$_2$ vibrational and rotational degrees of freedom. As the light continuously pumps the electrons at the $m_J = -1/2$ sub-level of $5S_{1/2}$, the $m_J = +1/2$ state gradually becomes dominant, so the Rb is polarized.

As illustrated in Figure 4.6, optically pumped Rb atoms then polarize $^3$He nuclei in the pumping chamber via spin exchange [129], in which the hyperfine interactions of the binary collisions dominate[130]. Assuming the SEOP happens in an isolated chamber, the number of $^3$He nuclei (N) with positive (“$\uparrow$”) and negative (“$\downarrow$”) spin
Figure 4.5: Optical pumping for Rb with the right-handed circular polarized light.

alignment is described the dynamic function:

\[
\frac{dN_1}{dt} = -\frac{dN_\perp}{dt} = \frac{\gamma_{SE}}{N^A_\perp + N^A_\parallel} \left( N^A_\parallel N^A_\perp - N^A_\perp N^A_\parallel \right) - \frac{1}{2} (N^A_\parallel - N^A_\perp) \Gamma, \tag{4.1}
\]

in which \(N^A\) is the number of Rb atoms, \(\Gamma\) denotes the \(^3\)He nuclear spin relaxation rate, and \(\gamma_{SE}\) represents the spin exchange rate from the Rb valence electron to the \(^3\)He nuclei. With the \(^3\)He polarization defined as

\[
P = \frac{N_\perp - N_\parallel}{N_\perp + N_\parallel}, \tag{4.2}
\]

it is straightforward to obtain the time derivative of the \(^3\)He polarization:

\[
\frac{dP}{dt} = -(\gamma_{SE} + \Gamma) P + \gamma_{SE} P_A, \tag{4.3}
\]

in which \(P_A\) is the average Rb polarization and is typically > 90% in the JLab target system. The solution to this equation with the boundary condition of \(P = 0\) at \(t = 0\)
leads to the time evolution of the polarization:

\[ P(t) = P_A \frac{\gamma_{SE}}{\gamma_{SE} + \Gamma} \left( 1 - e^{-(\gamma_{SE} + \Gamma)t} \right). \] (4.4)

Thus the maximum polarization is given by

\[ P_{\text{max}} = P_A \frac{\gamma_{SE}}{\gamma_{SE} + \Gamma}. \] (4.5)

This simple deduction introduces the two important factor for characterizing the polarization in target cells: the spin exchange rate and the spin relaxation rate. The former can be expressed in terms of a coefficient \( k_{SE} \) and the number density of Rb \([\text{Rb}]\) as

\[ \gamma_{SE} = k_{SE}[\text{Rb}]. \] (4.6)

The measured value of \( k_{SE} \) was given as \( 6.8 \pm 0.2 \times 10^{-20} \text{ cm}^3/\text{s} \) [131], and \([\text{Rb}]\) can be calculated from a commonly used fit for the Rb vapor pressure as \( p = 10^{4.312 - 4040/T} \) [132], with \( T \) the temperature in Kelvin. Consequently, \( \gamma_{SE} \) is approximately \( (16 \text{ h})^{-1} \) at the operation temperature \( (170^\circ) \). The spin exchange rate can be improved by mixing a certain amount of potassium (K) vapor into the cell [133].

The most important factors contributing to the \(^3\text{He} \) spin relaxation are the \(^3\text{He} \) dipole-dipole interaction [134], the magnetic field gradient-induced relaxation [135], and the relaxation due to the paramagnetic impurities and microscopic fissures on the cell walls. These factors were studied and estimated for the target cells used in E97-110 [136]. In addition, the electron beam induced relaxation was also studied, in which the \(^3\text{He} \) atoms were ionized and then depolarized via hyperfine interactions [128]. The relaxation lifetime, defined as \( \tau = \Gamma^{-1} \), has already been listed in table 4.1.

The target cells used in the experiment actually do not satisfy the isolated chamber assumption. Diffusions of \(^3\text{He} \) gas between the pumping chamber and the target
chamber are involved in the cells, so the target polarization also depends on the diffusion factor [137]. In practice, the polarization is determined by target polarimetry, which is presented in the following subsections.

4.1.3 Target polarimetry

The $^3$He polarization can be measured by the NMR and the EPR methods in the target system. The principle of NMR can be easily understood in a classical picture. Considering a $^3$He nucleus with a magnetic moment $\vec{M}$ in a magnetic field $\vec{B} = B\hat{z}$, the equation of motion for its magnetic moment is written as

$$\frac{d\vec{M}}{dt} = \gamma B \vec{M} \times \hat{z},$$

(4.7)

in which $\gamma = 2\pi \times 3.243$ kHz-Gauss$^{-1}$ [132] is the gyromagnetic ratio for $^3$He. This equation describes the precession of the magnetic moment $\vec{M}$ in the holding field $\vec{B}_0$ at the Larmor frequency $\omega = \gamma B$. In an rotating frame at the angular frequency $\omega_0\hat{z}$, the magnetic field can be replaced by the effective field

$$\vec{B}_{eff} = \frac{\omega - \omega_0}{\gamma} \hat{z}.$$  

(4.8)

**Figure 4.6:** Spin exchange between the Rb valence electron and the $^3$He nucleus.
In practice, the rotation is provided by applying a radio-frequency (RF) field, oscillating as $\vec{B}_1 = B_1(\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t))$ and perpendicular to the holding field. The equation of motion with this RF field then becomes

$$\left(\frac{d\vec{M}}{dt}\right)_{rot} = \vec{M} \times \left(\frac{\omega - \omega_0}{\gamma}\hat{z} + \omega_1\hat{y}'\right),$$

with $\omega_1 = \gamma B_1$ and $\hat{y}'$ the direction of $\vec{B}_1$. Therefore, the $^3$He spin in the rotating frame is precessing about an effective field written as

$$\vec{B}_{eff} = \left(\frac{\omega - \omega_0}{\gamma}\hat{z} + \omega_1\hat{y}'\right).$$

The effective field is pointing to an angle $\theta$ with respect to $\hat{z}$, which can be expressed as

$$\sin \theta = \frac{\omega_1}{\sqrt{(\omega - \omega_0)^2 + \omega_1^2}}.$$

Figure 4.7 illustrates this classical view of this spin alignment. For a NMR measurement, the holding field is swept from $B_{min}$ to $B_{max}$. Assuming $-(\gamma B_{min} - \omega_0) \gg \omega_1$ and $\gamma B_{max} - \omega_0 \gg \omega_1$, the effective field is swept from approximately $-\hat{z}$ to $+\hat{z}$. If the holding field changed slow enough, the $^3$He spin direction can follow the direction of the effective field. It is called the “adiabatic” condition, and can be written as

$$\frac{1}{B_1} \left| \frac{dB}{dt} \right| \ll \gamma B_1.$$

When the holding field reaches the resonance point, $B = \omega_0/\gamma$, the $\hat{z}$-component of the effective field vanishes, and the spin is aligned with $\hat{y}'$. Sweeping around the resonance point involves a spin flip in the $\hat{z}$ direction and induces a signal in the pick-up coils attached to the target chamber. The amplitude of this signal is related to the polarization of $^3$He and the effective field angle as

$$S \propto P \cdot \mu \cdot [^3\text{He}] \cdot \sin \theta,$$
where $P$, $\mu$ and $^3\text{He}$ are the polarization, magnetic moment and number density of $^3\text{He}$, respectively.

Since the $^3\text{He}$ nuclei continuously depolarize via relaxation, the measurement also requires the holding field sweeping to be fast enough so the relaxation is negligible during the measurement. This “fast” condition can be quantified as

$$\frac{1}{B_1} \left| \frac{dB}{dt} \right| \gg \frac{1}{T_1},$$

(4.14)

with $T_1$ the $^3\text{He}$ relaxation rate in the rotating frame. Therefore, the NMR technique, which is required to be adiabatic as well as fast, is named as Adiabatic Fast Passage (AFP) NMR.

The NMR polarimetry requires a calibration before the usage. This is commonly performed with a similar cell but filled with water. The thermal polarization of water

---

**Figure 4.7**: Adiabatic fast passage in a classical view.
is described by the Boltzmann distribution:

$$P_w = \tanh \frac{\mu_p B}{kT}, \quad (4.15)$$

where $\mu_p$ is the magnetic moment of proton in the unit of nuclear magneton, and $k$, $B$ and $T$ represent the Boltzmann constant, the magnetic field, and the water temperature in Kelvin, respectively. The thermal polarization of water is very low ($\approx 7 \times 10^{-9}$ at 295 K). As a result, the calibration process typically takes multiple measurements and obtains the average value of all the results.

For the JLab $^3$He target system, the RF field is approximately 0.09 Gauss and oscillates with the frequency of 91 kHz, which leads to the resonance point at 28.06 Gauss. The holding field takes about 5.8 seconds to sweep from 25 to 32 Gauss. The systematic uncertainty of the NMR measurement is at the level of 2%, with the dominant contribution from the calculation of the magnetic flux through the pick-up coils [127].

The EPR polarimetry relies on the Zeeman splitting of the Rb’s ground state in a magnetic field, which can be characterized by the electron paramagnetic resonance frequency, written as

$$\nu_{EPR} = k_z B, \quad (4.16)$$

with the constant $k_z = 0.466$ MHz·G$^{-1}$. Flipping the polarized $^3$He changes the magnetic field by a small amount due to the alignment of the $^3$He magnetic moments, and thence changes the EPR frequency. The $^3$He polarization can be extracted from the observed change of the EPR frequency via

$$P = \frac{\Delta \nu_{EPR}}{k_{EPR}}, \quad (4.17)$$

with $k_{EPR}$ the calibration constant, which is related to the magnetic field, the number density and the magnetic moment of $^3$He, and a few constants. Since the calibration
constant depends on the external magnetic field, the spin flip in EPR is performed by adiabatically and fast sweeping the small RF field, while the holding field remains unchanged.

The main systematic uncertainty of the EPR measurement comes from the estimation of a temperature related factor in the calibration constant, which amounts to approximately 1.5%. Besides, since EPR measurement is performed in the pumping chamber of the target cell, so one has to determine the degradation of polarization during the transport. The analysis regarding the E97-110 target cells can be found in Ref. [128].

During the E97-110 runs, both NMR and EPR measurements had been carried out every six hours, and a linear interpolation between two target polarimetry measurements were used to describe the experimental runs within this time period. The target polarization was found to be 35-45%. However, a large discrepancy (7.5%) was observed between the results from the two methods, which might be due to the field gradient produced by the septum magnet. This discrepancy was carefully studied in Ref. [127], and weighted averages of the EPR and NMR values were applied. The finalized systematic uncertainty of the target polarization amounted from 3% to 5.2% [138].

4.2 The electron accelerator at JLab

The polarized electron beam used in E97-110 was delivered by the Continuous Electron Beam Accelerator Facility (CEBAF), whose schematic is depicted in Figure 4.8. The electron accelerator was originally designed with a maximum beam energy of 4 GeV, but the excellent performance of the RF cavities raised the energy limit to nearly 6 GeV. Recently the CEBAF was upgraded with 5 new high-voltage cryomodules per linac, higher refrigeration capability, enhanced magnets and power supplies, and a tenth arc for the beamline connecting to the new experimental Hall D [139].
The goal of this upgrade is to deliver a 5.5-passes beam with the maximum energy at 12 GeV, so it is referred to as the “12 GeV upgrade”. Since E97-110 was conducted prior to the upgrade, the following discussion is for the 6 GeV CEBAF.

4.2.1 Polarized electron beam

The polarized electron source at CEBAF is a photoemission electron gun with a thin, strained gallium arsenide (GaAs) photocathode [140]. During the beam production, the cathode is illuminated by circularly polarized photons of energy slightly higher than the band gap. Electrons are thus excited from the valence band to the conduction band in the photocathode. Since the photocathode has a negative electron affinity on the surface due to the activation by adding a layer of cesium and nitrogen trifluoride, the electrons reached the conduction band are free to escape from the...
crystal. They are then accelerated by an electric potential of -100 kV and leave the source [141].

Figure 4.9: Excitation from the absorption of left-handed circularly polarized photons in a GaAs semiconductor. The relative transition ratios are printed in the open circles.

The spin polarization of the produced electrons is achieved due to the band structure of the photocathode. The GaAs semiconductor consists of a $S_{1/2}$ conduction band and a $P$-type valence band that splits into $P_{3/2}$ and $P_{1/2}$ states. The excitations between these states subject to the selection rules introduced in Section 4.1, and their transition rates are characterized by the Clebsch-Gordon coefficients. As shown in Figure 4.9, left-handed (right-handed) circularly polarized photons with a proper energy can only excite the electrons from $P_{3/2}$ to $S_{1/2}$, with a 3-to-1 preference to the state of $m_J = +1/2$ ($-1/2$), and lead to a theoretical polarization of $P = (3 - 1) / (3 + 1) = 50\%$. A higher polarization can be achieved with a strained GaAs photocathode, in which the degeneracy of the valence band is removed by the asymmetric crystalline structure [142]. As the result of such an advanced photocathode material, the beam
polarization at JLab typically reaches 75-85%.

As discussed above, the production of polarized electrons necessitates circularly polarized photons. At CEBAF, the circular optical polarization is produced by an electro-optic device containing a birefringent crystal, called Pockels cell. The polarization direction of the photons out of the cell is controlled by the electric field applying on it. This device and an upstream Insertable Half-wave Plate (IHWP) enable a fast flip of the beam helicity, which is crucial for the helicity-dependent measurement of asymmetries or cross section differences. Figure 4.10 illustrates the polarized electron source.

Being generated from the source, the polarized electrons enter the injector subsequently. Before any further acceleration, the polarization angles of the electrons are manipulated by a Wien filter and a solenoid spin rotator [143], so that the delivered beam is longitudinally polarized in the experimental halls. The injector then
accelerates the electrons to 67 MeV and inject them into the north linac.

In the 6 GeV era, each of the two superconducting linacs consists of 20 cryomodules with an average accelerating gradient of 7 MeV/m [144]. Injected electrons are firstly accelerated in the north linac, and then enter into the south linac through a recirculation arc. The maximum energy gain for the accelerated electrons approaches 1.2 GeV per pass of the two linacs. After each pass, the electrons can reenter the linacs through the arc and be accelerated up to 5 passes, or they can be extracted and delivered to the experimental halls.

4.2.2 Beam helicity control

The beam helicity control in E97-110 followed the “G0 scheme” [145], in which the helicity sequence was pseudo-randomly generated in quartet states of “+−−+” and its complement “−++−”. Such a sequence organization is advantageous compared to the pair states (“+−” and its complement “−+”) because it nulls the asymmetry caused by the linear drifts of the beam intensity.

![Diagram of helicity quartet and control signals]

**Figure 4.11**: The helicity quartet and its control signals.

As shown in Figure 4.11, the quartet trigger signal (QRT) defines the beginning of a helicity states quartet. The data-taking for each helicity state is controlled by the
the macro-pulse trigger (MPS), which consists of a “settle” period of approximately 500 µsec and a “stable” period of 1/30 sec. The former indicates the helicity change is under preparation, while the latter signals the validity of the helicity states. The stable period is chosen to be exactly twice as the power line cycle at 60 Hz, and allows a cancellation of the corresponding noise in data-taking. Typically the helicity information is sent to the experimental hall with a delay of eight MPS cycles, but during the second period of E97-110, the report of the helicity states was in-time.

4.3 Beamline in Hall A

Precise information about the beam is crucial to an accelerator based experiment. The basic instrumentation along the Hall A beamline [146] helps the users monitor the status of the polarized electron beam and extract information about the beam, including the position, energy, current, and polarization. Figure 4.12 shows the schematic of the Hall A beamline, and the relevant apparatuses are discussed below.
4.3.1 Beam position monitor

Along the Hall A beamline, two Beam Position Monitors (BPMs) are equipped 7.524 and 1.286 meters upstream of the target, respectively. The BPMs are calibrated by an absolute position measurement with three wire scanners, which are aligned vertically and $\pm 45^\circ$ on a harp frame as shown in Figure 4.13. During the measurement, the harp passes through the beam, and creates a beam profile by recoding its position and the collected charge on the wire (or the intensity of particle showers detected by the surrounding photomultiplier tubes). The beam position is then determined from the fits for the beam profile.

Once calibrated, the BPMs can perform a non-invasive measurement to monitor the beam position and direction at the target location. Each BPM contains four coaxial antenna along the beamline. The induced signal collected in each antenna is inversely proportional to the distance from the beam, and the difference-over-sum of all the signals yields the beam position. The nominal resolution of the BPMs is 20 $\mu$m at the beam current of 10 $\mu$A.
4.3.2  Beam energy measurement

The absolute beam energy are determined by two methods in Hall A[147]: the $ep$ measurement and the arc energy measurement.

In the elastic electron-proton scattering, the kinematic relation between the energy of the incident electron ($E$) and the angles of the scattered electron ($\theta_e$) and the recoil proton ($\theta_p$) is written as

$$E = M \frac{\cos \theta_e + \sin \theta_e / \tan \theta_p - 1}{1 - \cos \theta_p} + O \left( \frac{m^2}{E^2} \right),$$  \hspace{1cm} (4.18)

with $M$ the proton mass and $m$ the electron mass. Along the Hall A beamline, a stand-alone device can simultaneously measure the scattered electron angle and the recoil proton angle from the elastic $ep$ process and hence determine the beam energy. The device, as shown in Figure 4.14, consists of a thin film polyethylene ($\text{CH}_2$) target and two sets of detectors located symmetrically with respect to the beamline. The nominal uncertainty from this method is $\sim 2 \times 10^{-4}$ GeV.

The arc energy measurement has a comparable precision of $2 \times 10^{-4}$ GeV. This method utilizes the relation between an electron’s momentum ($p$) and its deflection angle ($\theta$) in a known magnetic field ($\vec{B}$), which reads:

$$p = \frac{k}{\theta} \int \vec{B} \cdot d\vec{l},$$  \hspace{1cm} (4.19)

with $k = 0.299792 \text{ GeV} \cdot \text{rad} \cdot \text{T}^{-1} \cdot \text{m}^{-1}$. Two simultaneous measurements are required to determine the beam energy in the arc section: one measures the magnetic field integral with eight dipoles and the 9th dipole as a reference, the other measures the bending angle of the arc with wire scanners. The corresponding apparatuses are illustrated in Figure 4.15.

The two basic methods provide high precision information about the beam energy, but they are both invasive and cannot be performed during the experimental runs.
In practice, the beam energy is continuously monitored online by the Tiefenbach method ("Tiefenbach energy"), which relies on the same principle as the arc energy measurement. Instead of a direct measurement in the arc section, this method extracts the angle from the monitored beam position with the magnetic transfer functions. It is thus non-invasive but at a cost of the precision, which approximately amounts to $5 \times 10^{-4}$ GeV.

In E97-110, the beam energy was determined by both the $ep$ measurement and the Tiefenbach measurement. The $ep$ measurement was carried out only for a single run with the nominal beam energy of 2135 MeV. It yielded $E = 2135.67 \pm 0.20\text{stat} \pm 0.46\text{sys}$ MeV. For the other runs, the beam energy was determined from the Tiefenbach measurement. The average beam energies for the second period is summarized in table 4.2.
4.3.3 Beam current monitor

The number of incident electrons (beam charge) is typically determined from continuous beam current measurements. In Hall A, the beam current can be measured by a Unser monitor and two resonant RF cavities inside a temperature stabilized box [146], which is placed 25 meters upstream from the target position. These de-

Table 4.2: Average beam energies for the second period of E97-110.

<table>
<thead>
<tr>
<th>Target Cell</th>
<th>Angle</th>
<th>Beam Energy (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>6.10°</td>
<td>2134.2</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>2134.9</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>2844.8</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>4208.8</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>1147.3</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>2233.9</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>3318.8</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>3775.4</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>4404.2</td>
</tr>
</tbody>
</table>
vices and the corresponding electronics for the data acquisition are called the Beam Current Monitor (BCM) system.

The Unser monitor [148] is a non-invasive, parametric DC current transformer. Its calibration is carried out with a known current passing through a wire inside the beampipe. Since the Unser monitor suffers from significant signal drifts with an operation time longer than several minutes, it is not appropriate for continuous monitoring the beam current. Instead, the device is used for a reference measurement. In addition, the absolute reference is also be provided by a cavity monitor and a Faraday cup at the injector section during the calibration runs.

The RF cavity monitor is a cylindrical waveguide with a high quality factor (∗3000). Its resonant frequency is tuned to coincide with the beam frequency (1497 MHz), so the output signal’s amplitude is proportional to the beam current. The output signal from each cavity monitor is split into two paths: one for the sampled current and the other for the integrated current. The sampling branch of the output signal is fed to a high-precision digital AC voltmeter. This device digitizes the root-mean-square (RMS) of the input signal within one second, and thus represents the average beam current for the corresponding second. The sampled data are recorded every 1-2 second(s) in the data acquisition (DAQ) system. For the integrated beam current, the output signal is processed by a RMS-to-DC converter and a voltage-to-frequency converter. The processed signal is then sent to scalers, whose output accumulates during each experimental run and records the total beam charge. The RMS-to-DC converter has a linear response only for beam currents from approximately 5 to 200 µA, so the signals from low beam currents are strengthened by a set of amplifiers with the gain factor of 3 or 10.

Consequently, the total beam charge ($Q_a$) is obtained from the readout from the

∗The quality factor is a dimensionless factor characterizes the resonance behavior. Higher value indicates narrower bandwidth relative to the resonant frequency and less relative energy loss.
BCM scalers \((N_a)\) within the corresponding time period \((t)\):

\[
Q_a = \frac{N_a - f_a t}{k_a},
\]

in which \(a = 1, 3, 10\) denote the gain factors, \(k_a\) represent the calibration constants and \(f_a\) are the BCM offsets. For E97-110, the beam currents were from 0.5 to 10 \(\mu\)A. In the beam charge analysis, signals at the amplification of \(a = 3\) and \(a = 10\) were used for the beam currents above and below 5 \(\mu\)A, respectively, while signals at \(a = 1\) were only considered as references. The calibration constants and offsets were determined by the aforementioned absolute references during the calibration runs in August 2003. However, a significant drop was found in some of the BCM offsets during the experiment, so the results from calibration runs could not be directly applied in the data analysis. Instead, linear fits were used to describe the drop of the offsets. More details about the beam charge analysis can be found in the technical note by V. Sulkosky [149].

4.3.4 Beam charge asymmetry feedback

During the polarized beam production, imperfections in the Pockels cell generate a small but non-zero linear component in the circular optical polarization and result in an elliptical polarization. Since the polarization ellipse is asymmetric to the flip of the beam helicity, it introduces a helicity correlated asymmetry, namely the Polarization Induced Transport Asymmetry (PITA) [150].

The PITA leads to different amounts of charge in each helicity state, which is unwanted in a helicity-dependent measurement. The charge asymmetry can be minimized by an asymmetry feedback system [151], which consists of the Hall A Proton Parity Experiment (HAPPEX) DAQ system and a rotatable half-wave plate (RHW-P). On one hand, the HAPPEX system continuously monitors the charge asymmetry, and applies an asymmetric offset voltage accordingly on the Pockels cell to cancel the
residual linear polarization. On the other hand, the RHWP, located downstream of the Pockels cell, rotates the direction of the linear component to reduce the charge asymmetry. The charge asymmetry feedback system updated every two minutes in E97-110.

4.3.5 Beam polarimetry

The knowledge of the beam polarization is required in an asymmetry measurement. In Hall A, a Møller polarimeter and a Compton polarimeter can independently perform the measurement of the beam polarization. Since the main sources of the systematic uncertainties are different in these two polarimeters, their results are partly complementary and can be used for a cross calibration.

**The Møller polarimeter**

The schematic of the Møller polarimeter is shown in Figure 4.16. The method is based on the doubly polarized Møller scattering as $e^- + e^- \rightarrow e^- + e^-$. The cross section of this process can be expressed in terms of the polarizations of the incident electron ($P$) and the target electron ($P_t$) as [152]

$$\sigma = \sigma_0 \left( 1 + \sum_{i,j} A_{ij} P^i P_t^j \right), \quad (4.21)$$

in which $i,j = X,Y,Z$ are the axes, $\sigma_0$ represents the unpolarized Møller cross section, and $A_{ij}$ is the analyzing power of the asymmetry term with the polarization projections $P^i$ and $P_t^j$. Assuming the electron beam is passing through the $Z$-axis and the scattering plane coincides with the horizontal plane ($ZX$-plane), the analyzing power can be expressed as

$$A_{ZZ} = - \frac{\sin^2 \theta_{CM} (7 + \cos^2 \theta_{CM})}{(3 + \cos^2 \theta_{CM})^2}, \quad (4.22)$$

$$A_{XX} = - A_{YY} = - \frac{\sin^4 \theta_{CM}}{(3 + \cos^2 \theta_{CM})^2},$$
with $\theta_{CM}$ the scattering angle in the center of mass frame. Note that the analyzing power reaches its maximum value at $\theta_{CM} = 90^\circ$, and amounts to $|A_{ZZ}| = 7/9$ and $|A_{XX}| = |A_{YY}| = 1/9$. In practice, the polarimeter measures helicity-dependent asymmetry instead of the absolute cross sections, and the longitudinal beam polarization is extracted from the $A_{ZZ}$ term with a known target polarization.

A magnetized ferromagnetic foil provides the polarized atomic electron target for the Møller polarimeter. Its polarization is determined by the foil magnetization measurements. The foil target can be tilted in the horizontal plane, resulting in an adjustable polarization direction. For the Møller polarimetry, the beam polarization is determined by an “average of averages” procedure. Firstly the asymmetry measurement at a given target angle is performed twice with the target polarized in the opposite directions, and the resulted average reduces false asymmetries. Then the
measurements are repeated for a target angle mirrored with respect to the beamline, and the average of the asymmetries from the two target angles is taken. Typically the target angles are chosen as approximately ±20° in Hall A, and averaging them cancels the transverse contribution and minimizes the systematic uncertainties from determining the target angles.

Scattered electrons from the Möller process are selected by a magnetic spectrometer in coincidence, which consists of three quadrupole magnets and a dipole magnet. With a proper setting of the fields according to the beam energy, the scattered electrons with 75° < θ_CM < 105° and −5° < φ_CM < 5°, where φ_CM is the azimuthal angle, are selected by the spectrometer. The double-arm electrons are then detected in coincidence by lead-glass calorimeter modules.

The Möller polarimeter typically performs the invasive measurement at a low beam current (0.5 µA). By taking one hour of the beam time, the statistical uncertainty from the measurement is typically 0.2%, and the systematic uncertainty is dominated by the target polarization measurement, which amounts to 3% relative to the polarization.

**The Compton Polarimeter**

The Compton polarimetry measures the asymmetry from the Compton scattering between polarized electrons and circularly polarized photons. The beam polarization can be determined as

\[
P = \frac{A_{exp}}{P_\gamma A_{th}}, \tag{4.23}
\]

with \(A_{exp}\) the measured asymmetry, \(P_\gamma\) the photon polarization, and \(A_{th}\) the Compton analyzing power.

As shown in Figure 4.17, the Compton polarimeter in Hall A contains a magnetic chicane, a photon source and detectors for the scattered photons and electrons. The
chicane consists of four dipoles and deflects the electron beam vertically to intersect with the photon beam. The circular polarization of the photon beam is higher than 99%, and its helicity state can be controlled by a rotatable quarter-wave plate. After the Compton scattering, the backscattered photons are detected by an electromagnet calorimeter, while the scattered electrons are detected by a Silicon Strip Detector (SSD).

The Compton polarization measurement is non-invasive, and thus can be performed during the experimental runs. The typical statistical uncertainty is about 1% in a measurement that lasts one hour. The systematic uncertainty is also at the level of 1%, with a dominant contribution from the low analyzing power in the Compton scattering.

**Beam polarization in E97-110**

For E97-110, the averaged beam polarization from the Møller polarimeter and the Compton polarimeter were 74.7% and 74.9%, respectively [128]. Since Compton
scattering had a statistical uncertainty of 3-5% due to the low beam currents, the beam polarization used in the data analysis was mainly extracted from the Møller polarimeter.

A noticeable beam bleedthrough from Hall C had occurred during the first period, and resulted in a large correction to the beam polarization due to the opposite signs of the polarizations in the two experimental halls. The bleedthrough of polarization was measured and controlled to a few percent level during the second period. These measurements also allowed for correcting this effect to within 1% [128], so the systematic uncertainty of the polarization in the second period was still dominated by the Møller polarimetry, resulted in 3.5%.

4.3.6 Raster

The delivered electron beam in Hall A is typically focused to 100-200 µm at the target position. The heat created by the beam is dangerous for a high-pressure gaseous target with thin glass windows. In order to distribute the heat load, an apparatus called raster can be utilized. The Hall A raster, located 23 meters upstream of the target, consists of a vertical and a horizontal dipole magnets. It can move the beam position by applying the dipole magnetic field.

For E97-110, a triangular raster was used, which uniformly distributed the beam inside a square area with the dipole magnetic field varying in a triangular waveform over time. During the second period, the beam was distributed in a $3 \times 3$ or $4 \times 4$ mm$^2$ area.

4.4 The spectrometers and the septum magnet

The Hall A HRS is a pair of magnetic spectrometers with a nominal resolution of $\delta p/p \approx 10^{-4}$ for the electron momentum. The left-arm (HRS-L) and the right-arm (HRS-R) spectrometers are almost identical, and each of them consists of three
quadrupole magnets (Q) and one field gradient dipole magnet (D), which provides a vertical bending angle of $45^\circ$. The superconducting magnets are organized in a sequence of QQDQ as shown in Figure 4.18. The first quadrupole is convergent in the vertical plane, while the other two quadrupoles provide transverse focussing.

The HRS selects charged particles in a certain kinematic range, and sends them to the detectors. For the detection of electrons, the central momentum is determined by the dipole magnetic field ($B_0$) as

$$p = \sum_i \Gamma_i B_0^i,$$  \hspace{1cm} (4.24)

in which $\Gamma_i$ is the spectrometer constant at the order $i$. The constants can be determined from the elastic scattering of electrons off a carbon foil target [153], with the dominant contribution from the first order ($> 99.9\%$). Table 4.3 summarizes the main characteristics for the HRS.

The minimum accessible angle of the HRS ($12.5^\circ$) is limited due to the physical

\[82\]
Table 4.3: Main characteristics of the HRS [146].

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending angle</td>
<td>$45^\circ$</td>
</tr>
<tr>
<td>Optical length</td>
<td>23.4 m</td>
</tr>
<tr>
<td>Momentum Range</td>
<td>0.3-4.0 GeV</td>
</tr>
<tr>
<td>Momentum acceptance</td>
<td>$\pm 4.5%$</td>
</tr>
<tr>
<td>Momentum resolution</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Dispersion at the focus</td>
<td>12.4 m</td>
</tr>
<tr>
<td>Radial linear magnification</td>
<td>-.25</td>
</tr>
<tr>
<td>HRS-L angular range</td>
<td>12.5$^\circ$-150$^\circ$</td>
</tr>
<tr>
<td>HRS-R angular range</td>
<td>12.5$^\circ$-130$^\circ$</td>
</tr>
<tr>
<td>Horizontal angular acceptance</td>
<td>$\pm 30$ mrad</td>
</tr>
<tr>
<td>Horizontal angular resolution</td>
<td>0.5 mrad</td>
</tr>
<tr>
<td>Vertical angular acceptance</td>
<td>$\pm 60$ mrad</td>
</tr>
<tr>
<td>Vertical angular resolution</td>
<td>1.0 mrad</td>
</tr>
<tr>
<td>Solid angle at $\delta p/p = 0$, $y_0 = 0$</td>
<td>6 msr</td>
</tr>
<tr>
<td>Transverse length acceptance</td>
<td>$\pm 5$ cm</td>
</tr>
<tr>
<td>Transverse position resolution</td>
<td>1 mm</td>
</tr>
</tbody>
</table>

Size of its first quadrupole magnet. However, smaller detection angles may be required by the experiments exploring a low $Q^2$ region. Therefore, a superconducting septum magnet was designed to bend the charged particles with momenta up to 4 GeV and at any angles from $6^\circ$ to $12.5^\circ$ [123]. As depicted in Figure 4.19, the magnet can be installed in front of the HRS and deflect the particles to the spectrometer’s entrance, while the target has to be displaced by 80 cm upstream. Deployment of this equipment reduces the solid angle acceptance to 4.7 msr and slightly degrades the momentum resolution to nominally $< 2 \times 10^{-4}$.

During Experiment E97-110, only one septum magnet was available and paired with the HRS-R. Therefore, the HRS-R was the main detector of this experiment, while the HRS-L was only utilized to monitor the false asymmetries and the beam luminosity based on the elastic electron scattering off a carbon foil target.
4.5 The right-arm detector package

Various characteristics of the charged particles passing through the spectrometer are determined by the HRS detector stack. As illustrated in Figure 4.20, the right-arm detector package is composed of a pair of vertical drift chambers (VDC), two plastic scintillator planes, a gas Cherenkov detector, and a two-layer calorimeter. The DAQ electronics are triggered by the timing information from the two scintillators, and hence record the position and direction of the detected particles, provided by the VDCs, and the particle identification (PID) from the gas Cherenkov detector in conjunction with the two-layer calorimeter.

The HRS-L differs slightly from the HRS-R in the particle identification detectors. The main difference between them lies in the arrangement of the second-layer modules of the calorimeter. Besides, a focal-plane polarimeter can be installed in the HRS-L, and provides measurement for the polarization of the detected protons. In this section, only the right-arm detectors are presented because the E97-110 data were only taken from the HRS-R.
4.5.1 Vertical drift chambers

Two VDCs in the HRS provide high resolution tracking information for charged particles. The lower VDC is placed approximately in coincidence with the spectrometer’s focal plane, while the upper one is placed 335 mm higher and displaced horizontally according to the nominal $45^\circ$ particle trajectory. Each VDC consists of two horizontal wire planes organized in a standard UV configuration. In this configuration, the two sets of sense wires, denoted as U-plane and V-plane, are orthogonal to each other and inclined $45^\circ$ from the dispersive and non-dispersive directions, as shown in Figure 4.21. Each wire plane contains 368 active sense wires, resulting in a space of 4.24 mm between two adjacent wires.

The VDC methodology utilizes the Townsend avalanche [155]. For the HRS VDC, the wire chamber is filled with a standard argon-ethane (62:38) gas mixture. An electric field is applied to the chamber by holding its gold-plated Mylar walls at a
high negative voltage, nominally $-4.0$ kV. The charged particles traversing through
the chamber ionize the gaseous atoms and create ion-electron pairs. The ionized
electrons are then drawn to the nearest wires, which are held at ground potential
and act as anode. The radial electric field surrounding the wire accelerates these
drifting particles, and thence causes a local ionization cascade, yielding a detectable
signal on this wire. Consequently, the position of the charged particle is extracted
from the precisely measured timings of these signals in the wire plane.

According to the design for the HRS [154], electrons at the nominal trajectory
angle ($45^\circ$) will fire four to six wires and generate a cluster with the same number
of hits in each plane. In the extreme case where the trajectory is at an angle of $52^\circ$,
three sense wires will be activated. The timing of each hit is processed by time-
Figure 4.22: Particle trajectory reconstructed from the VDC data. Linear fit for the position on a wire plane (left) and the focal plane angle reconstruction from all wire planes (right). U1, V1 denote the wire planes in the lower VDC, and U2, V2 denote for the upper VDC.

to-digital converters (TDC), and is used to calculate the vertical drifting distance to the corresponding sense wire. In the data analysis, a linear fit for the drifting distances vs. the wire positions leads to the cluster position in the wire plane, and the trajectory angle in the focal plane is reconstructed by the position information from all planes. Figure 4.22 illustrates the tracking reconstruction from the VDC data. The nominal resolutions of the position and the angle in the focal plane are approximately 100 µm and 0.5 mrad, respectively.

4.5.2 Plastic scintillators

The timing information of the traversing particles is provided by the two plastic scintillator planes equipped in the HRS. The two planes are separated by 2 meters, and each of them consists of six thin scintillator paddles, which are overlapped on the edge with the neighboring paddles to avoid any gap in the plane. The thickness of the paddle is minimized (5 mm) to reduce the hadron detection efficiency.
Scintillator materials can be excited by ionizing particles, and emits the absorbed energy in the form of photons. For the detection of the emitted photons, each end of the paddle is attached by a photomultiplier tube (PMT). The paddle is recognized as “fired” only if both of the attached PMTs had signals. The electrical signal from the paddle will be converted to the timing information of the detected particle, which has a nominal resolution of 0.3 ns.

The signals from the scintillator planes are utilized to trigger the DAQ system. In addition, the timing information can determine the time-of-flight between the two planes for the detected particles.

4.5.3 The gas Cherenkov detector

The gas Cherenkov detector [156], mounted between the two scintillator planes, distinguishes the electrons or positrons (selected by the spectrometer) from other charged particles, primarily the pions or the protons. This detector possesses a large volume tank filled with CO$_2$ gas as the radiator. As shown in Figure 4.23, Ten spherical-rectangular mirrors housed in the tank encompass the Cherenkov radiations from the detected particles with trajectory angles of 45° ± 20°, and each focuses the reflected light upon a specialized PMT [157]. The detection of Cherenkov light will thus be converted to electrical signals.

In the radiator, the Cherenkov radiation occurs when a charged particle passing through it at a speed faster than the phase velocity of the light. In order to produce the Cherenkov radiation, the charged particles must satisfy $\beta \geq 1/n$, where $n$ is the radiator’s index of fraction. This requirement directly leads to a threshold for the particle’s momentum, which reads

$$p_{th} = \frac{m_0}{\sqrt{n^2 - 1}},$$

(4.25)

with $m_0$ the particle’s rest mass. The gas Cherenkov detector operates at standard
conditions for temperature and pressure (STP), so the index of fraction in the CO₂
gas is \( n = 1.00041 \). It results in a sufficiently low threshold for electrons (17.84 MeV),
while the thresholds for pions (4.87 GeV) and protons (32.76 GeV) are beyond the
selection of the HRS. Therefore, the signals from the Cherenkov detector identify the
electrons with an efficiency of 99% [146]. In E97-110, the detection efficiency and
the electron selection efficiency of the Cherenkov detector were better than 99.85% (99.67%) and 99.26% (99.66%) for the experimental data at 6° (9°) [158].

4.5.4 The two-layer calorimeter

Complementary PID information can be offered by a homogenous calorimeter made
of lead-glass. The detector is longitudinally segmented into two layers: the “preshower” (first) layer and the “shower” (second) layer. Figure 4.24 shows the layout of the
lead-glass blocks in the HRS-R. In the preshower layer, forty-eight \( 10 \times 10 \times 35 \) cm³
blocks are oriented transversely with respect to the nominal trajectory, while eighty longitudinally-oriented blocks with a size of $15 \times 15 \times 35$ cm$^3$ compose of the second layer. Each of the blocks is viewed by a PMT, converting the Cherenkov radiations from the particle showers to electrical signals. The detector provides a destructive measurement, since the blocks are long enough for an electron to deposit all of its energy into the lead-glass absorber.

In high-energy or nuclear physics, a homogeneous calorimeter is designed to detect the electromagnetic or hadronic cascades created by a particle in a scintillating or non-scintillating Cherenkov medium. The energy or position of the detected particle is thence reconstructed from the shower signals, and $e-\pi$ discrimination can also be achieved by measuring the longitudinal profile of the particle’s energy deposition. The electromagnetic and nuclear interactions of a particle passing through a medium are characterized by the radiation length $X_0$ and the nuclear interaction length $\Lambda_I$, respectively. As shown in Figure 4.25, the radiation length is commonly an order of magnitude shorter than the nuclear interaction length in a material. For the two-
layer calorimeter in the HRS-R, the traversing electron deposits part of its energy in the preshower layer and all of its rest energy in the shower layer, but a pion only deposits a small amount of the energy in both layers. The rejection of pions can thus be achieved by a 2-dimensional cut on the energy deposits in different layers. The corresponding PID parameter can be defined as

$$R_{sh} = \frac{E_{tot}}{p} \times \frac{\log (E_{presh})}{\log (E_{sh})},$$

(4.26)

where $p$ is the particle’s momentum, $E_{tot}$ is the total energy deposit in the calorimeter, and $E_{presh}$ and $E_{sh}$ are the energies in the preshower and shower layers, respectively.

Combining the PID information from the gas Cherenkov detector and the two-layer calorimeter, the HRS-R can achieve a pion suppression factor of $2 \times 10^5$ at $p > 2$ GeV [146]. The performance of the calorimeter in E97-110 was studied by Lu et al. [159], resulting in a detection efficiency higher than 99.76% and the electron cut efficiency $> 99\%$. 

Figure 4.25: Radiation length and nuclear interaction length for chemical elements, reproduced from Ref. [19].

![Figure 4.25: Radiation length and nuclear interaction length for chemical elements, reproduced from Ref. [19].](image-url)
4.5.5 Data acquisition

In the HRS, the triggers for the DAQ system can be generated from the signals of the scintillator planes and the gas Cherenkov detector. For E97-110, the master trigger was formed by the two scintillator planes, denoted as S1 and S2. This trigger required the detection signals from both S1 and S2, as well as a reasonable trajectory of the detected particle as compared to the nominal one (45°). The latter criterion was fulfilled if the indices of the fired paddles in the two scintillator planes differed from each other by 0 or 1, i.e., the $i$-th paddle fired in S1 necessitated a fired S2 paddle at $i$ or $i \pm 1$. In addition to the master trigger, the system utilized a secondary trigger to measure the trigger efficiency (scintillator efficiency). The criteria of this trigger were: only one scintillator plane fired and the Cherenkov detector had a signal. The secondary trigger was thus exclusive to the master one, and indicated a good event that was not detected by one of the scintillators, resulting in the triggering inefficiency. During the experiment, the trigger efficiencies were higher than 99.4% for most of the production runs.

The data acquisition was accomplished by separating the detector signals into two paths. On one hand, the signals were processed by the electronic logic units to generate the trigger, which was then sent to the trigger supervisor (TS). Once accepted, the TS distributed the trigger to all connected electronics and synchronized them to prevent time jitters. Subsequently, the triggered electronics would open a time window that lasts long enough for receiving the detector signals. On the other hand, the detector signals sent to the DAQ electronics were delayed by a certain amount of time (typically hundreds of nanoseconds), so their arrival time could coincide within the data-acquiring window.

Since the DAQ system could not handle additional events when it was processing and recording the current event, the TS would reject any incoming triggers when
the system was “busy”. This rejection led to an incomplete events recording, which was characterized by the DAQ livetime ($LT$). During the experiment, the triggers were counted by several scalers in the DAQ system. Therefore, the livetime can be determined from the total number of triggers ($T_i$) and the accepted number of triggers ($T_{acc,i}$):

$$LT_{\pm}^i = \frac{N_{ps,i}T_{\pm,acc}^i}{T_i},$$

where $i = 1, 2$ denote the trigger type, “±” represent the helicity states, and $N_{ps}$ is the trigger-prescale. The trigger-prescale technique is commonly applied when the event rates are extremely high, in which case only one out of $N_{ps}$ triggers\(^\dagger\) is recognized by the system. The livetime correction is an important factor for the experimental yields, which is presented in the next chapter.

4.6 Spectrometer optics

The position and direction of the detected particles are measured by the detector package in the spectrometer’s focal plane. These focal plane coordinates can be transported back to the target plane coordinates, which are needed by the extraction of the experimental observables at the interaction vertices. This transport process is accomplished with an optics matrix of the spectrometer. In this section, we introduce the definition of the spectrometer optics, and present the procedure to optimize the optics matrix elements for E97-110.

4.6.1 Coordinate systems

Several Cartesian coordinate systems are implemented for the optics study. The common convention for the axes is the right-hand rule, but the other convention can also be used. A particle’s trajectory is described by a set of coordinates, including

\(^\dagger\)Due to the binary nature of the computers, typically $N_{ps} = 2^{ps}$ with $ps$ the prescaling factor.
the in-plane angle $\phi = dy/dz$ and the out-of-plane angle $\theta = dx/dz$, as well as the corresponding displacements of $x$ and $y$. Note that the angular coordinates are indeed the tangents of the angles. Figure 4.26 illustrates the coordinates in a Cartesian system.

The origin of the Hall A laboratory coordinate system (HLCS) is defined as the center of the experimental hall. The $z$-axis is parallel to the beamline and points downstream. The $y$- and $x$-axis are orthogonal to the $z$-axis, with $\hat{y}$ pointing upward and $\hat{x}$ pointing toward the left arm of the HRS. Figure 4.27 shows the three axes defined in the HLCS.

Each arm of the HRS has its own target coordinate system (TCS), defined with a sieve slit placed vertically between the target and the spectrometer. The $z$-axis perpendicularly passes through the midpoint of the central sieve slit hole, pointing away from the target. The $x$-axis points downward. The angle between the two
z-axes in the TCS and the HLCS is the central selection angle of the spectrometer, denoted by $\Theta_0$. In the TCS, the intersection point between the $z$-axis and the sieve slit is positioned at $(0, 0, L)$, with $L$ the distance between the sieve slit and the hall center. Therefore, the TCS and HLCS origins coincide with each other if the sieve slit was perfectly centered. In reality, the displacement of the TCS origin from the hall center should be surveyed before experimental runs. Figure 4.28 depicts the trajectory of an scattered electron in the TCS system.

Detected particle’s spatial and angular coordinates are given by the VDCs. The U and V wire planes can independently determine the coordinates in the detector coordinate system (DCS). As illustrated in Figure 4.22, the tangent of the trajectory angle is obtained from the positions in the wire planes. The relation is written as

$$
\tan \eta_U = \frac{p_{U2} - p_{U1}}{d},
$$

$$
\tan \eta_V = \frac{p_{V2} - p_{V1}}{d},
$$

with $\eta$ the trajectory angle, $p$ the position in the corresponding wire plane, and $d$
Figure 4.28: The beam and scattered electron in the coordinate systems. Different colors indicate different coordinate systems. The subscripts of \(tg\) and \(hl\) designate the axes in the TCS and the HLCS, respectively.

The distance between the two U-planes (or equivalently the two V-planes). The DCS origin is defined as the intersection between the U1 center wire (wire-184) and the projection of the V1 center wire on the U1 plane. The \(z\)-axis is vertically up and perpendicular to the wire planes, and the \(x\)-axis is parallel to the long asymmetry axis of the U1 plane and points toward the beam dump. In an ideal alignment of
the wire planes, the coordinates can be expressed as

\[ x_{\text{det}} = \frac{1}{\sqrt{2}} (p_{V1} + (p_{V1} - d' \tan \eta_V)) , \]

\[ \theta_{\text{det}} = \frac{1}{\sqrt{2}} (\tan \eta_U + \tan \eta_V) , \]

\[ y_{\text{det}} = \frac{1}{\sqrt{2}} (-p_{U1} + (p_{V1} - d' \tan \eta_V)) , \]

\[ \phi_{\text{det}} = \frac{1}{\sqrt{2}} (-\tan \eta_U + \tan \eta_V) , \]

with \( d' \) the distance between U1 and V1. Imperfect alignments in the VDCs result in additional corrections to these coordinates.

The detected coordinates are transformed to the focal plane coordinate system (FCS) before the reconstruction of the particle’s interaction vertex. Such a transform minimizes the dispersive angle \( \theta \) and leads to a faster convergence for optimizing the optics matrix elements. The FCS is defined as rotating the \( z \)-axis of the DCS over an angle of \( \rho \), so the axis is along the local central trajectory. The rotation angle is dependent on the relative deviation of the momentum

\[ \delta = \frac{p}{p_0} - 1 , \]

with \( p \) and \( p_0 \) the particle’s momentum and the central momentum setting of the spectrometer, respectively. The rotation thus directly connects \( \delta \) with the focal plane coordinate \( x_{fp} \). Figure 4.29 shows how the FCS is constructed.

A special rotation at \( \rho_0 = -45^\circ \) is often used to help the transport, in which case the \( z \)-axis in the DCS is rotated to the nominal trajectory, and the coordinates are
Figure 4.29: Rotations for the focal plane coordinate system.

expressed as

\[ x_{\text{tra}} = x_{\text{det}} \cos \rho_0 (1 + \theta_{\text{tra}} \tan \rho_0), \]

\[ \theta_{\text{tra}} = \frac{\theta_{\text{det}} + \tan \rho_0}{1 - \theta_{\text{det}} \tan \rho_0}, \tag{4.31} \]

\[ y_{\text{tra}} = y_{\text{det}} + x_{\text{det}} \phi_{\text{tra}} \sin \rho_0, \]

\[ \phi_{\text{tra}} = \frac{\phi_{\text{det}}}{\cos \rho_0 (1 - \theta_{\text{det}} \tan \rho_0)}. \]

The detected vertex in the focal plane is thence written in the FCS as [160]

\[ x_{fp} = x_{\text{tra}}, \]

\[ \theta_{fp} = \frac{\theta_{\text{det}} + \tan \rho}{1 - \theta_{\text{det}} \tan \rho}, \]

\[ y_{fp} = y_{\text{tra}} - \sum_i C_{i000}^y x_{fp}^i, \tag{4.32} \]

\[ \phi_{fp} = \frac{\phi_{\text{det}} - \sum_i C_{i000}^p x_{fp}^i}{\cos \rho - \theta_{\text{det}} \sin \rho}, \]

\[ \tan \rho = \sum_i C_{i000}^t x_{fp}^i, \]
with the coefficients $C_{y000}^y$, $C_{z000}^p$ and $C_{t000}^t$ determined in the calibration. The coordinate corrections due to the VDC misalignment can be absorbed into these coefficients.

### 4.6.2 Target vertex reconstruction

The spatial and angular coordinates are measured in the DCS, and are then transformed to the FCS. The resulting focal plane coordinates are used to reconstruct the relative momentum deviation and the TCS coordinates, written as

$$\begin{pmatrix} \delta & x & \theta & y & \phi \end{pmatrix}^T_{tg}. \tag{4.33}$$

For reducing the parameter space, the $x_{tg}$ can be effectively zeroed by a precise alignment on the vertical direction ($y_{beam}$ is within 250 µm and the TCS origin is close enough to the hall center) [160].

The transport from the focal plane to the target plane is expressed as

$$\begin{pmatrix} \delta \\ \theta \\ y \\ \phi \end{pmatrix}_{tg} = \begin{pmatrix} \langle \delta|x\rangle & \langle \delta|\theta\rangle & 0 & 0 \\ \langle \theta|x\rangle & \langle \theta|\theta\rangle & 0 & 0 \\ 0 & 0 & \langle y|y\rangle & \langle y|\phi\rangle \\ 0 & 0 & \langle \phi|y\rangle & \langle \phi|\phi\rangle \end{pmatrix} \begin{pmatrix} x \\ \theta \\ y \\ \phi \end{pmatrix}_{fp}, \tag{4.34}$$

with the $4 \times 4$ optics matrix in the first order. The zero elements are due to the midplane symmetry, in which the coordinates in the dispersive plane ($x$, $\theta$) are always orthogonal to those in the transverse plane ($y$, $\phi$). However, for E97-110 the symmetry did not hold due to the equipped septum magnet, and thus the full optics matrix had to be used:

$$\begin{pmatrix} \delta \\ \theta \\ y \\ \phi \end{pmatrix}_{tg} = \begin{pmatrix} \langle \delta|x\rangle & \langle \delta|\theta\rangle & \langle \delta|y\rangle & \langle \delta|\phi\rangle \\ \langle \theta|x\rangle & \langle \theta|\theta\rangle & \langle \theta|y\rangle & \langle \theta|\phi\rangle \\ \langle y|x\rangle & \langle y|\theta\rangle & \langle y|y\rangle & \langle y|\phi\rangle \\ \langle \phi|x\rangle & \langle \phi|\theta\rangle & \langle \phi|y\rangle & \langle \phi|\phi\rangle \end{pmatrix} \begin{pmatrix} x \\ \theta \\ y \\ \phi \end{pmatrix}_{fp}. \tag{4.35}$$
The target coordinates and the focal plane coordinates are thus linked by a set of tensors. For example, the relative momentum deviation \( \delta \) can be written as

\[
\delta = \sum_{j,k,l} D_{jkl}^{\theta, y_f, \phi_{fp}} \theta_{fp} \phi_{fp},
\]

in which the coefficient tensor \( D_{jkl}^{\theta, y_f, \phi_{fp}} \) is a polynomial expansion of \( x_{fp} \):

\[
D_{jkl} = \sum_{i=1}^{m} C_{ijkl}^{D} x_{fp}^i,
\]

with \( C_{ijkl}^{D} \) the matrix elements for \( \delta \). Analogous relations exist for all of the target variables.

Once the TCS coordinates are obtained, the energy of the scattered electron can be directly calculated from \( \delta \), and the scattering angle is extracted from the angular coordinates as

\[
\theta = \arccos \left( \frac{\cos \Theta_0 - \phi_{tg} \sin \Theta_0}{\sqrt{1 + \theta_{tg}^2 + \phi_{tg}^2}} \right).
\]

In addition, the \( y_{tg} \) intimately relates to the reaction point \( z_{react} \), which can be obtained as

\[
z_{react} = -(y_{tg} + D_y) \frac{\cos(\arctan \phi_{tg})}{\sin(\Theta_0 + \arctan \phi_{tg})} + x_{beam} \cot(\Theta_0 + \arctan \phi_{tg}).
\]

Note that the \( z_{react} \) is effectively compressed with the factor of approximately \( \sin \Theta_0 \). For E97-110, the target cell was only about 4 cm long in the \( y_{tg} \) at the scattering angle of 6°. This highly compressed coordinate introduced a difficulty in cutting out the background events from the glass windows, and thus several target collimators were implemented, which are discussed in the next section.

4.6.3 Optics calibration

The optics matrix elements are practically determined in the calibration runs with sieve-slit collimators and foil targets. During the calibrations, the beam position
and energy are provided by the beamline measurements, and the foil target position \((z_{\text{react}})\) is surveyed. The sieve slit has an array of holes, allowing scattered electrons to pass through. The position of each hole is precisely measured by a survey, which is then used to determine the variables for the corresponding target vertex, written as

\[
x_{tg} = x_{\text{sieve}} - L\theta_{tg}, \\
\theta_{tg} = \frac{x_{\text{sieve}} + D_x + y_{\text{beam}}}{L - z_{\text{react}} \cos \Theta_0 - x_{\text{beam}} \sin \Theta_0}, \\
y_{tg} = y_{\text{sieve}} - L\phi_{tg}, \\
\phi_{tg} = \frac{y_{\text{sieve}} + D_y - x_{\text{beam}} \cos \Theta_0 + z_{\text{react}} \sin \Theta_0}{L - z_{\text{react}} \cos \Theta_0 - x_{\text{beam}} \sin \Theta_0},
\]

(4.40)

where \(x_{\text{sieve}}\) and \(y_{\text{sieve}}\) are the hole positions in the TCS, and \(L, D_x\) and \(D_y\) (see Figure 4.28) are the displacements of the sieve slit from the hall center along the \(z\)-, \(x\)- and \(y\)-axis, respectively. The central angle \(\Theta_0\) and the displacements of the sieve slit are directly measured by a survey. For the calibration of \(\delta\), elastic scattering of electrons off a foil target is utilized. The scattered electron energy can be directly determined from equation (2.6), and the energy loss due to the radiation effect is corrected theoretically with the knowledge of the radiation length.

As mentioned above, target variables are known from the optics calibration, so the optics matrix elements can be determined by minimizing an estimator that describes the abberation between the reconstructed variables and the survey results. A common estimator used in this procedure is the \(\chi^2\), defined as

\[
\chi^2_W = \sum_s \left( \frac{\sum_{ijkl} C_{ijkl}^W x_i^l f_p^i \theta_{f_p^i}^j y_{f_p^i}^k \phi_{f_p^i}^l - W_0}{\sigma_{W}^s} \right)^2,
\]

(4.41)

where \(s\) represents the calibration events, \(W\) denotes the target variables, \(W_0\) is the reference from survey, and \(\sigma_{W}^s\) is the corresponding uncertainty. The program
“OPTIMIZE++” [160], based on the function minimizing algorithm MIGRAD [161], was developed for the optimization of the optics matrix elements.

4.6.4 calibration in E97-110

During the optics calibration for E97-110, the acceptance for the target variables of $z_{\text{react}}$ and $\delta$ was studied with a set of 10 mils\(^{\dagger}\) carbon foils. The calibration data for the relative momentum were acquired by a five step scan of the $^{12}$C elastic peak, centered at $\delta = +3\%, +2\%, 0\%, -2\%$ and $-4\%$. The target cell length, $z_{\text{react}} = \pm 20$ cm, were covered by multiple carbon foils located at the corresponding positions. For $6^\circ$ calibration, three carbon foils were located at $-20$, $0$, and $+10$ cm. The absence of the downstream foil at $+20$ cm was due to the limit of the spectrometer acceptance. Optics data from five carbon foils, placed at $\pm 20$, $\pm 10$ and $0$ cm, were taken for the $9^\circ$ calibration, but the data of the foils at $\pm 20$ cm did not participate in the parameter optimization because of the significant statistical uncertainties.

The spatial and angular coordinates were calibrated by two sieve slit collimators with different thicknesses. The thin sieve slit was used to collect the calibration data for the spectrometer momentum setting at the elastic peak. However, the HRS-R momentum setting for the elastic peak at $3.319$ GeV and $9^\circ$ was not achievable, in which case a thicker sieve slit was used for the quasielastic region to suppress punch-through background at lower momentum. Figure 4.30 depicts the sieve slit used in the experiment. On each of the sieve slit, totally 49 holes are arranged in a $7 \times 7$ grid pattern, and define the target variables as in equation (4.40). These holes are $1.4$ mm in diameter, except two outliers with the diameter of $2.7$ mm, which determine the image orientation in the spectrometer’s focal plane. The horizontal spacing for the first four columns (closer to the beamline) is $4.78$ mm, and it is enlarged to $6.12$ mm for the last three columns, while the seven rows are uniformly spaced $13.31$ mm.

\(^{\dagger}\)1 thousandth of 1 inch
Figure 4.30: The sieve slit and hole indices used in E97-110. The plot is rotated by 90°, and the spacings are in mm.

Table 4.4: The displacements of the sieve slits for E97-110.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sieve Slit</th>
<th>L (mm)</th>
<th>$D_x$ (mm)</th>
<th>$D_y$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6°</td>
<td>Thin</td>
<td>799.8</td>
<td>2.0</td>
<td>0.1</td>
</tr>
<tr>
<td>9°</td>
<td>Thin</td>
<td>798.9</td>
<td>1.8</td>
<td>0.3</td>
</tr>
<tr>
<td>9°</td>
<td>Thick</td>
<td>798.6</td>
<td>2.0</td>
<td>-0.4</td>
</tr>
</tbody>
</table>

Precise information about the absolute positions of the BPMs, target and spectrometer displacements from their nominal locations, and the position of the sieve slit central hole had been determined by surveys. The detailed reports on these items can be found in Ref. [162]. The beam positions ($x_{beam}$ and $y_{beam}$) were retrieved from the BPMs calibrated by the harp scan and BPM survey. Table 4.4 and table 4.5 listed the survey results for the sieve slits and the target cells, respectively.

The optimization procedure had a few changes in E97-110 due to the complications introduced by the septum magnet. Firstly, since the $y_{tg}$ acceptance was only constrained by three carbon foils, the fourth or higher order dependence on
Table 4.5: The HLCS displacements of the target cells from the hall center. Results for the foil targets are linearly interpolated between the polarized $^3$He values and the reference cell values.

<table>
<thead>
<tr>
<th>Cell Type</th>
<th>Target</th>
<th>$\Delta x$ (mm)</th>
<th>$\Delta z$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>Polarized $^3$He</td>
<td>0.3</td>
<td>-1.0</td>
</tr>
<tr>
<td></td>
<td>Reference Cell</td>
<td>-0.3</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>Single $^{12}$C Foil</td>
<td>0.1</td>
<td>-0.7</td>
</tr>
<tr>
<td></td>
<td>Multiple $^{12}$C Foils</td>
<td>0.0</td>
<td>-0.6</td>
</tr>
<tr>
<td>Priapus</td>
<td>Polarized $^3$He</td>
<td>0.2</td>
<td>-1.3</td>
</tr>
<tr>
<td></td>
<td>Reference Cell</td>
<td>0.1</td>
<td>-0.4</td>
</tr>
<tr>
<td></td>
<td>Single $^{12}$C Foil</td>
<td>0.2</td>
<td>-1.1</td>
</tr>
<tr>
<td></td>
<td>Multiple $^{12}$C Foils</td>
<td>0.2</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

the corresponding angular coordinate in the focal plane ($\phi_{fp}$) was eliminated for the matrix elements to prevent overfitting issue. Secondly, the 49 sieve slit holes were individually optimized in the E97-110 procedure to expand the data constraints, while the standard procedure often optimized them column by column. However, this expansion led to a difficulty of convergence for jointly optimizing the two angular coordinates, which were then independently optimized. Moreover, the spatial coordinate $x_{tg}$ was also involved in the target variable reconstructions for a better reconstruction of the out-of-plane angle $\theta_{tg}$ [163]. The expression for $x_{tg}$ reads

$$x_{tg} = -\theta_{sieve} \frac{z_{react} \cos \Theta_0}{\cos(\arctan \phi_{sieve})} - y_{beam},$$

(4.42)

in which $\theta_{sieve}$ and $\phi_{sieve}$ are the angular coordinates of an individual sieve slit hole with a given foil position. Lastly, instead of a typical tight momentum cut on the elastic peak, loose momentum cuts of 24-50 MeV in range were applied for the E97-110 calibration data. A tight momentum cut could have rejected more punch-through events from the sieve slit collimators. However, the scattered electrons from the upstream foils at a small angle passed through the NMR coils, and lost more energy due to the increase of the total radiation length. Therefore, the momentum cuts had to be widened to save the elastic events from the upstream carbon foils. The
Table 4.6: The momentum resolutions from the momentum scans during the E97-110 calibration.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Energy (GeV)</th>
<th>δ Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>6°</td>
<td>1.096</td>
<td>8.8 × 10^{-4}</td>
</tr>
<tr>
<td></td>
<td>2.134</td>
<td>4.4 × 10^{-4}</td>
</tr>
<tr>
<td></td>
<td>2.844</td>
<td>4.2 × 10^{-4}</td>
</tr>
<tr>
<td>9°</td>
<td>1.147</td>
<td>8.5 × 10^{-4}</td>
</tr>
<tr>
<td></td>
<td>2.235</td>
<td>5.2 × 10^{-4}</td>
</tr>
</tbody>
</table>

additional energy loss due to the NMR coils also degraded the momentum resolutions of the spectrometer, which were listed in table 4.6.

4.7 Collimators

Target cell windows have been realized as a common background source in the scattering measurements with gas or liquid targets. The effect can be significant due to the large density ratio between the cell material and the gas target. For experiments involved with the HRS, the window background can be removed by software cuts on the reconstructed target variables, primarily $y_{tg}$. However, as mentioned in the previous section, the highly compressed $y_{tg}$ at small scattering angles worsened the resolution of the target position $z_{react}$, and undermined the efficiency of the software cuts. In addition, the bad resolution enhanced the contamination from the cell windows, and led to overwhelmed background events as shown in Figure 4.31, which could hardly be rejected by the software cuts.

In order to minimize the background contributions from the target cell windows, several sets of collimators were implemented in E97-110. A special collimator set, named as “BRI6”, was utilized for the “ice cone” cell at 6° in the first period. For the second period, the sets of “BRS6” and “BRS9” were equipped for the standard cells at 6° and 9°, respectively. The layout of these collimator sets is shown in Figure 4.32. Each set consists of a downstream collimator and an upstream collimator,
Figure 4.31: The reconstructed $z_{\text{react}}$ for the nitrogen target with and without collimators, reproduced from Ref. [158].

which block the background events from the corresponding cell windows. The implementation of the collimators effectively reduced the acceptance of the target positions to about 20 cm.

The septum magnet entrance was also shielded by a 3 cm thick collimator made of tungsten. The tungsten collimator encompassed the sieve slit with an aperture of approximately $100 \times 55 \text{ mm}^2$ for all of the sieve slit holes. This collimator blocked the ambient backgrounds, which might come from the incident electron beam.
Figure 4.32: The layout of target collimators.
Data Analysis

The tracking, momentum, and PID information about the scattered electrons from the process of $^3\text{He}(e,e')$ had been recorded in E97-110. As described in the previous chapters, the spin dependent structure functions could be extracted from the cross section differences at the Born level. In order to access the physics of interest, the E97-110 data had been analyzed in three major steps. Firstly, the experimental cross sections and asymmetries were determined from the recorded events. The relevant factors, including the detector acceptance, target and beam polarizations, nitrogen and cell background contributions and DAQ livetime were taken into account in this step. Then the Born cross sections and asymmetries were “unfolded” from the experimental observables by correcting the radiation effects. Lastly, the experimental results at constant beam energy were interpolated to constant $Q^2$, and the generalized GDH integral at certain $Q^2$ was thence obtained.

The preliminary neutron results from the second period of E97-110 were extracted by V. Sulkosky [158] before I joined the collaboration, in which case the resonance region was the focus. I have analyzed the data for extracting the generalized GDH sum rule of $^3\text{He}$. The quasielastic cross sections are precisely determined, and a more
advanced radiative correction have been carried out to reveal the Born cross sections in the whole inelastic spectrum, which is not only essential for the $^3$He results but also necessary for finalizing the neutron results. This chapter presents the details of the analysis procedure for the second period $^3$He results.

5.1 Experimental asymmetries

The parallel and perpendicular asymmetries can be obtained by reversing the longitudinal beam polarization but fixing the target polarization direction, as shown in equation (2.28), and vice versa [164]. The helicity-dependent measurement in Hall A often takes full advantage of the reversible spin direction for both the beam and the target. In principle, constant false asymmetries are eliminated in the data sets composed of opposite polarizations for both the beam and the target. Given the asymmetry sign designated as “+” or “−”, the raw asymmetry can be expressed as

$$A_{\|,\perp}^{\text{raw}} = \left( \frac{Y^+ - Y^-}{Y^+ + Y^-} \right)_{\|,\perp},$$

with the experimental yield defined as

$$Y^\pm = \frac{N^\pm}{LT^\pm \cdot Q^\pm / e},$$

in which $N^\pm$, $LT^\pm$ and $Q^\pm / e$ are the number of events, the DAQ livetime, and the number of incident electrons determined from the measured beam charge, respectively. These quantities are directly extracted from the recorded data. Sign convention for the polarization configurations is discussed in the following subsection. In equation (5.1), the acceptance and kinematic range should be chosen identically for the data from different configurations, so these factors, as well as the detection efficiency, are cancelled out in the result.
The experimental asymmetry is then obtained by taking into account a dilution factor \( f \) and the polarizations of the beam \( (P_b) \) and the target \( (P_t) \), written as

\[
A_{\parallel,\perp}^{\text{exp}} = \pm \frac{A_{\parallel,\perp}^{\text{raw}}}{f P_t P_b}.
\] (5.3)

The polarizations are determined from the polarimetry measurements, and the dilution factor is discussed in Subsection 5.1.2. These quantities effectively remove the unpolarized contributions to the denominator in equation (5.1). Since the sign convention used in the raw asymmetry may be different to the sign defined in (2.28), the positive or negative sign is chosen accordingly for a consistent result.

5.1.1 Asymmetry sign convention

The sign convention \( (S_A) \) for the measured asymmetry can be viewed as a product of the beam polarization sign \( (S_b) \) and the target polarization sign \( (S_t) \):

\[
S_A = S_b \cdot S_t.
\] (5.4)

For E97-110, the beam polarization sign is determined by an absolute reference and the state of the IHWP in the polarized electron source (see Figure 4.10). The reference \( (S_{\text{ref}}) \) is provided by measuring a known asymmetry. Longitudinal and transverse reference signs were specified by elastic scattering of \(^3\text{He}(\vec{e}, \vec{e}')\) and the \( \Delta(1232) \) resonance, respectively. The IHWP can be either inserted into the optics polarization system, defined as the state of “IN”, or retracted from it, defined as “OUT”. The relative beam polarization sign \( (S_b/S_{\text{ref}}) \) is taken from the Møller polarimeter, but this sign needs to be reversed if the IHWP was in different positions during the experimental run and the polarization measurement. For instance, if the Møller polarimetry measurement resulted in a sign of +1 with the IHWP at “IN”, the relative sign should be −1 if the experimental data were taken when the IHWP was “OUT”.

110
Table 5.1: The target polarization sign $S_t$ in E97-110.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Angle</th>
<th>Holding field direction (HLCS)</th>
<th>$S_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal</td>
<td>$0^\circ$</td>
<td>$+\hat{z}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Longitudinal</td>
<td>$180^\circ$</td>
<td>$-\hat{z}$</td>
<td>$+1$</td>
</tr>
<tr>
<td>Transverse</td>
<td>$90^\circ$</td>
<td>$-\hat{x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Transverse</td>
<td>$270^\circ$</td>
<td>$+\hat{x}$</td>
<td>$+1$</td>
</tr>
</tbody>
</table>

The target polarization sign is defined by the holding field direction in the HLCS. For the longitudinal target polarization, the polarization angle is defined as $0^\circ$ and $S_t = -1$ if the holding field pointed towards the downstream beamline. In the case of transverse polarization, the angle is $90^\circ$ and $S_t = -1$ if the holding field pointed towards the HRS-R. The angle increases by $180^\circ$ and the polarization sign reverses if the target holding field was flipped. Table 5.1 summarizes the sign convention of the target polarization in E97-110. Ideally all of the target polarization angles should be utilized, so a $0^\circ$ and $180^\circ$ set and a $90^\circ$ and $270^\circ$ set can be formed. However, a significantly lower polarization at $90^\circ$ was observed in a previous experiment (E94-010) [165], and thus this configuration was avoided in E97-110.

5.1.2 Dilution factor

Ideally the experimental yield is purely contributed from the electron scattering off the $^{3}\text{He}$ target. However, inelastic contaminations from the nitrogen gas and the glass cell walls cannot be simply separated from the interested reaction channels, i.e., the inclusive scattering of $^{3}\text{He}(e,e')$. The raw asymmetries measured in the experiment can be expressed as

$$A_{raw} = \frac{1}{2} \frac{\Delta Y(^{3}\text{He}) + \Delta Y(N_2) + \Delta Y(\text{cell})}{Y(^{3}\text{He}) + Y(N_2) + Y(\text{cell})}, \quad (5.5)$$

with $\Delta Y$ and $Y$ the yield difference and yield average from different targets in the experimental runs, respectively. The unpolarized yields difference $\Delta Y(N_2) = 0$ and $\Delta Y(\text{cell}) = 0$, so these contaminations “dilute” the real asymmetry value. Therefore,
a dilution factor is introduced to correct this effect.

During the experiment, the dilution factors were determined by dedicated measurements with a reference cell filled with nitrogen gas and an empty cell. These dilution data had approximately the same kinematic coverage as compared to the production data, including the two scattering angles at 6° and 9°. Given the experimental yields (Y') from these runs, the dilution factor for the nitrogen gas is written as

\[ f_{N_2} = 1 - \frac{(Y'_{\text{ref}} - Y'_{\text{empty}})c_\rho}{Y'_{\text{pro}} - Y'_{\text{empty}}}, \quad (5.6) \]

with the ratio of nitrogen number densities between the target cell and the reference cell

\[ c_\rho = \frac{[N_2]}{[N_2]_{\text{ref}}}, \quad (5.7) \]

and the glass cell dilution factor reads

\[ f_{\text{cell}} = 1 - \frac{Y'_{\text{empty}}}{Y'_{\text{pro}} - (Y'_{\text{pro}} - Y'_{\text{empty}})(1 - f_{N_2})}. \quad (5.8) \]

The experimental yields need an additional correction, expressed as

\[ Y' = \frac{Y}{\epsilon_{\text{det}}}, \quad (5.9) \]

with \( \epsilon_{\text{det}} \) the efficiency for scattered electrons to be detected. Here only the VDC tracking efficiency is considered, since all the other detectors have an identical efficiency for electrons at the same kinematics. In the VDC tracking reconstruction, one scattered electron is supposed to produce only one track, but multi-tracks can occur if several particles passed through the wire planes simultaneously or some wires were noisy. Although the good track can be selected from multi-tracks, this procedure introduces an inefficiency. The fractions of multi-tracks were different in the production, the reference and the empty runs, primarily due to the different event rates.
among these types of measurements. Therefore, the tracking efficiency was corrected for the experimental yields with a systematic uncertainty better than 1%.

The dilution factors are essentially fractions of yields from different targets:

\[
f_{N_2} = \frac{Y( ^3\text{He})}{Y( ^3\text{He}) + Y(N_2)},
\]

\[
f_{\text{cell}} = \frac{Y( ^3\text{He})}{Y( ^3\text{He}) + Y(\text{cell})}.
\]

Thus the final factor used in equation (5.3) is the combination of them:

\[
f = \frac{Y( ^3\text{He})}{Y( ^3\text{He}) + Y(N_2) + Y(\text{cell})} = \left( \frac{1}{f_{N_2}} + \frac{1}{f_{\text{cell}}} - 1 \right)^{-1}.
\]

From the dilution factor analysis [166], \( f_{N_2} \) amounts between 0.87-0.94 with an uncertainty better than 0.3%, and \( f_{\text{cell}} \) yields 0.47-0.99. The values of these dilution factors become larger as the scattered electron’s energy (\( E' \)) increases. In the most important kinematic regions, \( i.e., \Delta\text{-resonance for neutron and quasielastic for } ^3\text{He}, f_{N_2} > 0.92 \) and \( f_{\text{cell}} > 0.95 \) for both 6° and 9° data.

5.2 Experimental cross sections

The unpolarized, raw cross sections are determined from the experimental yields and the spectrometer acceptances, expressed as

\[
\sigma_0^{raw} = \frac{Y_0}{\epsilon_{\text{det}}N_T\Delta\Omega\Delta E'},
\]

where \( \Delta\Omega \) is the solid angle acceptance, \( \Delta E' \) is the momentum acceptance, and \( N_T = \rho\Delta Z \) is the target thickness determined from the target density \( \rho \) and the target length acceptance \( \Delta Z \). Here \( Y_0 \) represents the unpolarized experimental yields after background subtraction. The definition of the detection efficiency \( \epsilon_{\text{det}} \) remains unchanged, but for absolute cross section calculation all of the detector efficiencies need to be taken into account.
In order to determine the differential cross section for the $^3\text{He}$ target, the raw cross section needs to be corrected as

$$\frac{d\sigma_0^{\text{exp}}}{d\Omega dE'} = C_{bc} (\sigma_0^{\text{raw}} - C_p \sigma_N^{\text{raw}}),$$

(5.13)

in which $C_{bc}$ represents the bin centering correction for finite acceptances, and $\sigma_N^{\text{raw}}$ is the raw cross section for the nitrogen gas. For simplicity, we abbreviate the differential cross sections to $\sigma$ for the following discussions, so $\sigma_0^{\text{exp}} = \frac{d\sigma_0^{\text{exp}}}{d\Omega dE'}$.

The cross section differences can thus be obtained from the asymmetries and the unpolarized cross sections:

$$\Delta\sigma_0^{\text{exp}} = 2 \cdot A_{\parallel,\perp}^{\text{exp}} \cdot \sigma_0^{\text{exp}}.$$

(5.14)

The extraction of experimental cross sections involves several important corrections including the background subtraction, spectrometer acceptance effects and the bin-centering correction. In the following subsections, these corrections are discussed.

### 5.2.1 Background subtraction

As aforementioned in the asymmetry analysis, the experimental yields from the target cell are not only from the inclusive $^3\text{He}(e,e')$ channels. In that case, the acceptance effects were cancelled and the contaminations from nitrogen gas and glass cell were corrected by a dilution factor. For absolute cross sections the acceptance effects have to be determined, but the contaminations from different processes complicate this study. Thus we firstly performed the empty target subtraction to remove the glass cell backgrounds, and then determined the raw cross sections for the $^3\text{He}$ data and the nitrogen reference data.

### 5.2.2 Acceptance corrections

The acceptance factors of $\Delta\Omega$, $\Delta E'$ and $\Delta Z$ in equation (5.12) are determined from the spectrometer acceptance for the reconstructed target variables $\theta_{tg}$, $\phi_{tg}$, $y_{tg}$ and $\delta$. 114
Since the particle trajectories were complicated due to the spectrometer’s magnetic field, the acceptance was practically studied with a dedicated Monte Carlo simulation named SAMC [167].

The SAMC program incorporates a set of transport functions to describe the particle trajectories from the target plane to the focal plane. Geometrical information about the collimator, the septum magnet and the spectrometer is also included for checking if a particle could pass to the focal plane. Radiation effects including the energy loss via ionization or Bremsstrahlung process can be turned on for a more realistic study, and details of the formula can be found in Ref. [167]. In the simulation, a large amount of events are generated to cover an acceptance range wider than the interested range, then the acceptance effects can be expressed as

\[
\frac{1}{\Delta\Omega\Delta E'\Delta Z} = \frac{N_{\text{gen}}^{\text{MC}}}{N_{\text{acc}}^{\text{MC}}},
\]

in which \(N_{\text{gen}}^{\text{MC}}\) and \(N_{\text{acc}}^{\text{MC}}\) are the numbers of events generated initially and survived after the transport within the acceptance cuts \(\Delta\Omega\Delta E'\Delta Z\), respectively.

The acceptance effects determined from equation (5.15) can be viewed as a normalization factor obtained from the Monte Carlo simulation. The corresponding systematic uncertainty was studied by comparing the geometrical shape of the acceptance between the experimental data and the simulation results, as shown in Figure 5.1. Besides, the stability of cross section results over different acceptance cuts was also studied, and the variation was included in this uncertainty. The finalized systematic uncertainty of the normalization factors ranged from 1% to 2% for all the data sets.

In addition to the general normalization, another two important acceptance corrections were applied for E97-110 data. They were named as “\(\delta\)-discontinuity” correction and “\(\theta_{tg}\)-deficit” correction, respectively.
The $\delta$-discontinuity was observed in the experimental data that the two adjacent momentum settings had different results at the same kinematic point. The averaged difference yielded 2-3%, but it drastically increased as the $\delta$ was further away from its central point, and maximized at 7-12% for $9^\circ$ and 15-18% for $6^\circ$ at the edge of the $\delta$ acceptance, where $\delta = -3.6\%$. This discontinuity was caused by imperfect simulation for the edges of the $\delta$ acceptance, which was considered as a higher order acceptance effect. The discontinuity correction was performed with polynomial fits for the $\delta$ acceptance, and its systematic uncertainty was studied by varying the fitting orders and checking the difference between the corrected cross sections and the averaged cross sections in the overlapping region of the adjacent momentum settings. The final results for systematic uncertainty of the $\delta$ discontinuity correction amounted to 0.8-2.5%. Figure 5.2 shows the systematic study for this higher order correction.

The $\theta_{tg}$-deficit was a higher order effect from the background subtraction, which was performed before acceptance studies. Figure 5.3 shows a prominent peak at the $\theta_{tg}$ variable that can be removed by the background subtraction. However,
Figure 5.2: The discontinuity correction for 2845 MeV data at 6°, provided by [138]. The cross sections with δ discontinuity corrections are compared with the averaged cross sections of overlapped bins from the elastic and the quasi-elastic momentum settings.

The imperfect subtraction created a depression in the θtg < 0 spectrum as shown in Figure 5.4. According to equation (4.38), the scattering angle is a symmetric function on the out-of-plane angular coordinate θtg, i.e., θ(θtg) = θ(−θtg). Thus the negative and positive θtg spectra should be symmetric. For correcting the negative θtg deficit, the elastic data from carbon foils at θtg > 0 were firstly analyzed and compared with the simulation, showing a good agreement. Then the negative θtg acceptance was corrected according to the simulation results. The size of this correction ranged from 0.8% to 4.3% for the whole data sets, and the associated systematic uncertainty was determined by taking the difference of cross sections between the positive and
negative halves of $\theta_{tg}$, which resulted in around 1%.

5.2.3 Bin-centering correction

The experimental results extracted from equation (5.12) and (5.13) are indeed the averaged cross sections over a finite acceptance bin of $\Omega \Delta E'$, denoted as $\bar{\sigma}_{\text{bin}}$. According to the Mean Value Theorem (MVT), this average value equals to the differential cross section at a certain point within the acceptance range. However, this point does not necessarily coincide with the acceptance bin center or the yields-weighted mean center. For a cross section spectrum at the constant beam energy and a fixed scattering angle, the bin-centering correction was studied to determine the correction factor $f_i^{bc}$ for a given point $(\theta_c, \nu_i)$ inside the $i$th acceptance bin, so the differential cross section can be recovered as

$$\sigma(\theta_c, \nu_i, E) = f_i^{bc} \bar{\sigma}_i.$$  \hspace{1cm} (5.16)
The bin-centering correction for E97-110 was studied with the unpolarized data. Firstly, various binnings for the scattering angle $\theta$ and the energy transfer $\nu$ were tried to test the $\sigma_0$ variation. As a result, the variation was found to be predominantly from the angular acceptance. For a careful study of the angular bin-centering, the unpolarized cross sections were calculated over the whole solid angle acceptance with 5 mrad $\theta$-bins, whose sizes were minimized for negligible finite acceptance effects as compared to the statistical uncertainties. Quadratic and linear fits were then utilized to describe the angular dependence of the $\sigma_0$, and the process was repeated for various binnings for $\nu$. Lastly, the averaged cross sections were centered to a given angle $\theta_c$ according to these fits, as illustrated in Figure 5.5.

Due to the large statistical uncertainties of the asymmetry data, this bin-centering correction could not be performed for the measured asymmetries. Therefore, the extracted unpolarized cross sections were centered to the average angles of the asymme-
Figure 5.5: The bin centering study with 5 mrad $\theta$-binning and 0.5 MeV $\nu$-binning for 2135 MeV data at 6°, provided by [138]. The left plot shows the data at $\nu = 1.25$ MeV, and the right one is for $\nu = 5.25$ MeV.

tries, amounted to 6.10° and 9.03°. The bin-centering factors at these central angles yielded $f_{bc}^{3\text{He}} = 95.4$-98.8% for $^3\text{He}$ data and $f_{bc}^{\text{N}} = 94.7$-98.8% for nitrogen data. The systematic uncertainty of this procedure was less than 0.4%, primarily contributed from the fit dependency.

5.2.4 Acceptance cuts and systematic uncertainties

For the final results of the experimental cross sections, the acceptance cuts were chosen as

- $|\delta| < 3.6°$;
- $|\theta_{tg}| < 20$ mrad;
- $|\phi_{tg}| < 6$ mrad;
- $|y_{tg}| < 0.8$ cm.

The above mentioned systematic uncertainties had been combined in quadrature to obtain the total systematic uncertainty for the acceptance, yielded 1.8-3.0%.
Table 5.2: Systematic uncertainties for experimental cross sections, provided by [138].

<table>
<thead>
<tr>
<th>Source</th>
<th>Systematic Uncertainty (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target density</td>
<td>1.6</td>
</tr>
<tr>
<td>VDC tracking efficiency</td>
<td>1.0</td>
</tr>
<tr>
<td>Beam charge</td>
<td>1.0</td>
</tr>
<tr>
<td>Detector efficiencies</td>
<td>1.4-1.9</td>
</tr>
<tr>
<td>$\nu$-dependent stability</td>
<td>0.0-1.25</td>
</tr>
<tr>
<td>Spectrometer acceptance</td>
<td>2.0-3.0</td>
</tr>
<tr>
<td>Beam polarization</td>
<td>3.5</td>
</tr>
<tr>
<td>Target polarization</td>
<td>3.0-5.2</td>
</tr>
</tbody>
</table>

The acceptance cuts for experimental asymmetries were loosen for a better statistical uncertainty, resulted in

- $|\delta| < 4.5\%$;
- $|\theta_{tg}| < 50$ mrad;
- $|\phi_{tg}| < 30$ mrad;
- $|z_{react}| < 30$ cm.

The first order of acceptance effects were cancelled in the asymmetry extraction, and thus the systematic uncertainty was assumed negligible as compared to the statistical uncertainty.

The finalized systematic uncertainties for the experimental cross sections are listed in Table 5.2.

5.3 Radiative correction

As discussed in Chapter 2, the structure functions can be obtained from the inclusive cross sections or cross section differences at the Born level. However, in the real world the observed experimental cross sections do not equal to the Born cross sections. Generally three major effects deviate the experimental cross sections from the Born values:
• Loop diagram contributions. In addition to the diagram illustrated in Figure 2.1, higher order Feynman diagrams also contribute to the scattering process. Typically only the next-to-leading-order diagrams ($\mathcal{O}(\alpha^4)$) are considered, including the self-energy, the vacuum polarization, and the vertex correction diagrams. Figure 5.6 illustrates the one-loop diagrams. These higher order contributions need to be removed from the observed cross sections to recover the Born term.

• Ionization energy loss of electrons passing through the materials before and after the interaction vertex. This energy loss effectively changes the kinematics at the vertex, denoted as

\[
E_i = E - \Delta E_{i \text{ioni}},
\]
\[
E_f = E' + \Delta E_{f \text{ioni}},
\]  

(5.17)

where $E$ is the beam energy, $E'$ is the detected energy of the scattered electron, and $E_i$ and $E_f$ are the initial and final energies of the interaction after the corresponding correction for the ionization loss $\Delta E_{i \text{ioni}}$ and $\Delta E_{f \text{ioni}}$, respectively. The energy loss due to ionization is expected to be small as compared to the beam energies at the level of GeV.

• Photon radiation via Bremsstrahlung process. The interaction involves acceleration of electrons and thus produces Bremsstrahlung photons. The effects are further categorized into the internal and the external Bremsstrahlung. The former occurs during the interactions between the electrons and the target, while the latter is caused by passage of electrons through materials such as the glass cell wall. The internal Bremsstrahlung diagrams are depicted in Figure 5.6. Analogous to the ionization effect, the energy loss through Bremsstrahlung radiation changes the initial and final kinematics observed in the experiment.
The above effects are called the “radiation effects”, and the Born cross section is folded in these effects. Before the extraction of the structure functions, the radiative correction is performed to obtain the Born level results for the differential cross sections and the cross section differences. The correction is generally expressed as

\[
\sigma_0^{\text{Born}} = \sigma_0^{\text{exp}} + \Delta\sigma_{RC},
\]

\[
\Delta\sigma_{||}^{\text{Born}} = \Delta\sigma_{||}^{\text{exp}} + \delta(\Delta\sigma_{||})_{RC},
\]

in which \(\Delta\sigma_{RC}\) and \(\delta(\Delta\sigma_{||})_{RC}\) represent the radiative corrections for unpolarized cross sections and polarized cross section differences, respectively.
Figure 5.7: The kinematic regions of the measured data and the corresponding radiative corrections. In the plot of beam energy vs. scattering energy, precise knowledge about the radiative effects requires cross sections in a triangular region for the data set at a constant beam energy.

Due to the energy loss in the radiation effects, the measured $\sigma^{exp}$ at a certain kinematic point $(E, E')$ is affected by the "real" cross sections in a range of $(E_i \leq E, E_f \leq E')$, as depicted in Figure 5.7. Therefore, precise knowledge of the radiative corrections always require models or theoretical calculations for the unmeasured region. Fortunately, the E97-110 data measured in a wide kinematic range offer a data-driven approach to carry out the radiative corrections. The model dependency can be well constrained for all the data sets except the ones with the lowest beam energies. In the following subsections, we present the radiative correction procedure.
for E97-110 in details.

5.3.1 Radiation and collisional thickness

For high-energy electrons (> 50 MeV), the energy loss per unit depth in a material via Bremsstrahlung is written as

$$\frac{dE}{dx} \approx -[N] \int_0^1 \frac{u d\sigma(u)}{du} du \equiv \sigma_{rad}, \quad (5.19)$$

where \([N]\) is the number density of the material atoms, and \(u \equiv \nu'/E\) with \(\nu'\) the radiated photon energy and \(E\) the initial electron energy. This integrated spectrum of photon emissions can be approximately expressed as [168]

$$\sigma_{rad}(Z) = 4\alpha r_e^2 \left[ Z^2 (L_{rad}(Z) - f(\alpha Z)) + Z L'_{rad}(Z) \right], \quad (5.20)$$

with \(r_e \approx 2.81794\) fm the classical electron radius and \(Z\) the charge of the nucleus. The Coulomb correction function \(f(\alpha Z)\) sums over an infinite series, and it has a truncated version [169]:

$$f(a) = a^2 \left[ \frac{1}{1 + a^2} + 0.20206 - 0.0369a^2 + 0.0083a^4 - 0.002a^6 \right], \quad (5.21)$$

with a good accuracy for \(a \equiv \alpha Z < \frac{2}{3}\). The radiation logarithms are given as \(L_{rad}(Z) = \ln(184.15Z^{-1/3})\) and \(L'_{rad}(Z) = \ln(1194Z^{-2/3})\) for \(Z \geq 4\), and the best estimates for \(Z < 4\) in Ref. [168] are used for light nuclei.

The accumulative energy loss is written as

$$E(x) = E_0 e^{-[N]\sigma_{rad}x}, \quad (5.22)$$

and thus it can be characterized by the radiation length defined as

$$X_0 = \frac{\rho}{[N]\sigma_{rad}(Z)} = \frac{A}{N_A \sigma_{rad}(Z)}. \quad (5.23)$$

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Table 5.3: Radiation thickness for the second period of E97-110. \( t_b \) is for the upstream (before) radiation thickness and \( t_a \) is for the downstream.

<table>
<thead>
<tr>
<th>Target Cell</th>
<th>Angle</th>
<th>( t_b )</th>
<th>( t_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>6.10°</td>
<td>2.116 \times 10^{-3}</td>
<td>6.492 \times 10^{-2}</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>2.071 \times 10^{-3}</td>
<td>6.293 \times 10^{-2}</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>2.073 \times 10^{-3}</td>
<td>4.492 \times 10^{-2}</td>
</tr>
</tbody>
</table>

where \( \rho \) represents the mass density, \( A \) is the atomic mass number and \( N_A \) is the the Avogadro constant. For a composite material in which all the atoms are uniformly distributed, the radiation length is obtained as

\[
X_0 = \left( \sum_i w_i \frac{l_j}{X_0^i} \right)^{-1},
\]

(5.24)

with \( w_i \) and \( X_0^i \) the mass fraction and radiation length for \( i \)th element, respectively.

As a prerequisite of the radiative correction, the total radiation length of the materials traversed by incident electrons or scattered electrons have to be determined for the calculation of the external Bremsstrahlung effects. Since the particle trajectories were selected by the spectrometer and hence were almost deterministic (with a small variation inside the acceptance), the radiation length before and after the interactions could be simply calculated from the experimental geometry without a tracking simulation. Figure 5.8 shows the passage of particles through the matters in E97-110. In the calculation of radiative effects, a dimensionless parameter known as radiation thickness is used to describe the total radiation length upstream and downstream of target central point, which is defined as

\[
t = \sum_j \frac{\rho_j l_j}{X_0^j},
\]

(5.25)

in which \( l_j \) is the passage length in the \( j \)th material. Table 5.3 summarizes the radiation thickness calculated in the Ref. [170].
Figure 5.8: Geometry information about the radiation length, reproduced from [170].
The ionization energy loss of massive charged particles passing through a matter can be viewed as a series of energy transfer to atomic electrons via collisions, and thus it is often referred to as collisional loss. The energy distribution after $\xi$ unit of length is given by the Landau distribution \[171\]:

$$f(\xi, \Delta E) = \frac{1}{\xi 2\pi j} \int_{r-j\infty}^{r+j\infty} e^{\ln(u)+\lambda u} du,$$  \hspace{1cm} (5.26)

with an arbitrary real constant $r$ and the Landau variable $\lambda$ defined as

$$\lambda = \frac{\Delta E - \langle \Delta E \rangle}{\xi} - \beta^2 - \ln \left( \frac{\xi}{E_m} \right) - 1 + \gamma_E,$$  \hspace{1cm} (5.27)

where $\xi$ is the collisional thickness in the unit of energy, $\langle \Delta E \rangle$ represents the mean energy loss, $E_m = m_e(\gamma - 1)$ is the maximum energy transfer in a single collision for electrons, and $\gamma_E \approx 0.5772$ is the Euler constant. The mean energy loss is given by the Bethe-Bloch equation, and the most probable energy loss is obtained when $f(x, \Delta E)$ reaches its maximum. It is obvious that the material dependency is included in the collisional thickness, which is defined as

$$\xi = 153.4 \frac{Z}{\beta^2 A} \rho \delta x \text{ keV},$$  \hspace{1cm} (5.28)

with $\delta x$ the material thickness. Table 5.4 lists the related variables for characterizing ionization energy loss, and more details can be found in the technical note \[170\].

Landau distribution has an assumption for the typical energy loss in a single collision, which should be much less than $E_m$ but larger than the binding energy of inner electrons. The latter assumption restricts its application for gaseous materials. In addition, it is known that the Landau distribution fails to describe experimental results for ultra thin materials \[19\]. In this work, two parameters were checked to assure the applicability of Landau formalism for our data: the significance parameter
Table 5.4: Ionization energy loss parameters for the second period of E97-110.

<table>
<thead>
<tr>
<th>Target Cell</th>
<th>Angle</th>
<th>type</th>
<th>ξ (keV)</th>
<th>(∆E)_mp (MeV)</th>
<th>⟨∆E⟩ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>6.10°</td>
<td>Before</td>
<td>7.550</td>
<td>0.137</td>
<td>0.240</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>148.1</td>
<td>2.861</td>
<td>4.289</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>Before</td>
<td>7.407</td>
<td>0.136</td>
<td>0.237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>144.1</td>
<td>2.780</td>
<td>4.184</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>Before</td>
<td>7.417</td>
<td>0.136</td>
<td>0.237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>106.9</td>
<td>2.054</td>
<td>3.144</td>
</tr>
</tbody>
</table>

Table 5.5: Parameters for checking the applicability of Landau theory. κ is given in range since it is dependent on the beam energy.

<table>
<thead>
<tr>
<th>Target Cell</th>
<th>Angle</th>
<th>type</th>
<th>κ</th>
<th>ξ/I₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penelope</td>
<td>6.10°</td>
<td>Before</td>
<td>&lt; 0.01</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>&lt; 0.01</td>
<td>1139</td>
</tr>
<tr>
<td>Priapus</td>
<td>6.10°</td>
<td>Before</td>
<td>&lt; 0.01</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>&lt; 0.01</td>
<td>1108</td>
</tr>
<tr>
<td>Priapus</td>
<td>9.03°</td>
<td>Before</td>
<td>&lt; 0.01</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After</td>
<td>&lt; 0.01</td>
<td>822</td>
</tr>
</tbody>
</table>

κ = ξ/E_m and ξ/I₀ with I₀ ≈ 10·Z eV the mean ionization potential of an atom. We have used the criteria in the GEANT4 simulation toolkit [172], given that κ < 0.01 and ξ/I₀ > 50 for using the Landau distribution. Table 5.5 lists the values for these two parameters. Note that the total collisional thickness is used in this evaluation, but the uncertainty from this approximation is assumed negligible since the ionization energy loss itself is a small effect as compared to the beam energy.

As shown in Figure 5.6, two polystyrene foams were installed to thermally isolate the septum from the target chamber. During the experiment, it was found that ice accumulated on the polystyrene foam due to the low operational temperature for the superconducting septum magnet. The existence of the ice introduces additional materials in the passage of scattered electrons. Its thickness was studied with the elastic data from ¹²C foil target. In this study, arbitrary values of ice thickness were tested in the simulation, and the W – M_T spectrum was compared between the simulation results and the experimental data. As a result, the 5.0 mm thick ice in the
simulation well reproduced the data, and thus this additional material is included in the radiative corrections. The details of the ice study can be found in the technical note [173].

The material thickness is not perfectly known, and thus the uncertainty contributes to the systematic uncertainty for the estimates of the radiation thickness and the collisional thickness. For the materials before the target center, the total systematic uncertainty results in 3.1%, with the dominant contribution from the beryllium (Be) and glass windows. The systematic uncertainty of the thickness after the vertex amounts to 1.7%, and the main source is the glass wall. The estimate for the ice thickness relies on the study with simulations, so it is assigned a conservatively large uncertainty at 30%.

5.3.2 Elastic radiative tail

At Born level, elastic scattering peaked at a certain kinematic point, and thus can be well separated from other kinematic regions. However, the radiative effects of this channel produce a long tail that falls into the inelastic spectrum. For E97-110 we are not interested in the elastic scattering, and thus the first step in the radiative correction is to subtract the elastic radiative tail (ERT) from the observed cross sections.

As shown in Figure 5.9, the elastic tail contribution is significant at low $Q^2$, and thus a dedicated simulation program was developed to generate the elastic radiative tail. This program has incorporated the internal and external radiative effects, the spectrometer acceptances and the punch through effects from the cell collimators. The study of ERT subtraction mostly relied on the unpolarized data, since they have much better statistical uncertainties in the elastic and quasi-elastic regions, and then the ERT for polarized cross section differences were generated based on the results from unpolarized ERT study.
The simulation for ERT

The differential cross section for unpolarized elastic electron-$^3$He scattering follows equation 2.12 with a charge factor of $Z^2$. The $^3$He electromagnetic form factors in Ref. [174] are used in the study. The simulation implements three recipes for the internal radiative effects: the Mo&Tsai (MT) approach [175], the MT approach without the energy peaking approximation from X.Yan (XY) [176], and the covariant formalism from Akushevich&Shumeiko (AS) [177].

In the MT approach, the loop-diagrams contribution directly changes the Born cross sections, and results in no energy loss, thus the effects can be accounted as a correction factor multiplied to the Born cross section. The internal Bremsstrahlung effects are calculated by a modified Bethe-Heitler formula. Although the so-called angle peaking approximation, in which the photons are emitted along the directions of incoming and outgoing electrons, is often applied to simplify the calculation of the “exact” formula for ERT (B.25 in Ref.[175]), here we directly integrate over all possible emission angles and thus the ERT results is without any peaking approxi-
mations.

The XY formula (Appendix A in Ref. [176]) was derived for removing the energy peaking approximation used in the inelastic radiative corrections. However, it inherits the formalism with angle peaking approximation from MT, which is unwanted in the elastic tail generation, as the angle peaking approximation leads to a large error when $E' > \frac{1}{3} E$ for the ERT [175]. Consequently, the XY approach is only utilized to test the effects of peaking approximations in the ERT.

The internal Bremsstrahlung diagrams are calculated from QED in the covariant formalism from AS. The infrared divergence in the real photon emission is handled by the Bardin&Shumeiko approach [178], in which the cross section is divided into an infrared divergent term and an infrared free term. The former one is a factorized term on the Born cross section, and the divergence is cancelled out with the virtual photon corrections (vertex correction and vacuum polarization). For generating ERT via AS approach, the well-developed POLRAD code [179] was used.

The external radiation effects can either be incorporated in the MT approach with equivalent radiator method (ERM), in which the internal Bremsstrahlung is treated equivalently as two identical external radiators sandwiching the interaction vertex. However, the ERM formula is derived with peaking approximations for energy and angle (or only angle in the XY approach). Thus the program generated ERT based on a Monte-Carlo simulation for the external radiation effects. In the simulation, scattered electrons were generated from the differential cross section with the internal radiative corrections, and the vertices were uniformly distributed in $z_{\text{react}} = \pm 20$ cm. In addition to the radiation thickness mentioned in the previous subsection, the electron passage in the cell collimators were also simulated. The cell collimators blocked the particles only if their radiation thickness contributions were reasonably large ($> 10$), and thus the electrons touching the edge of the collimators would “punch-through” and enter the spectrometer acceptance. The acceptance simulation
took the results from the acceptance study with SAMC. Fits of 5th order polynomials were utilized to describe the rising and falling edges of the acceptance, while the flat region were fitted by 2nd order polynomials. For the energy loss in the external materials, Landau theory was utilized for the ionization energy loss, and the energy loss due to external Bremsstrahlung was given as [180]

\[ I_{ext}(E_0, \Delta E, t) = \frac{b t}{\Gamma(1 + b t) \Delta E} \left( \frac{\Delta E}{E} \right)^{b t} \phi \left( \frac{\Delta E}{E_0} \right), \quad (5.29) \]

with

\[ b = \frac{4}{3} \left[ 1 + \frac{1}{9} \ln(1194Z^{-2/3}) + Z \ln(184.15Z^{-1/3}) \right], \quad (5.30) \]

where \( t \) is the radiation thickness, \( \phi(x) = 1 - x + \frac{3}{4}x^2 \) is the screening factor, and \( \Gamma(z) \) is the so-called Gamma function defined as

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \quad (5.31) \]

The direction change of the electrons passing through the materials was approximated with the multiple scattering formula, in which the angle after multiple scatters follows a Gaussian distribution with a root-mean-square as [19]

\[ \delta \theta \approx \frac{13.6 \text{ MeV}}{E} Z \sqrt{\frac{t}{t(1 + 0.038 \ln(t))}}. \quad (5.32) \]

In the final results, the ERT contributions in unpolarized cross sections and polarized cross section differences were subtracted by the AS approach. Cross checks with the results from other approaches determined the systematic uncertainty of the internal radiative correction. Before the ERT subtraction, the simulated tails were smoothed by cubic spline (piecewise polynomial) fits to eliminate statistical fluctuations, and the statistical uncertainties of the simulation results, amounted to < 1%, were integrated into the subtraction uncertainty. Figure 5.10-5.13 show the
ERT subtraction for the four lowest $Q^2$ data sets, and Figure 5.14 shows the higher $Q^2$ data sets.

**Figure 5.10**: Elastic radiative tail subtraction for the data set of 1147 MeV @ 9°. A zoomed-in view for the quasielastic region is shown in the right plot (same for Figure 5.11, 5.12, 5.13).

**Figure 5.11**: Elastic radiative tail subtraction for the data set of 2234 MeV @ 9°.
Systematic uncertainty study

It is obvious that the ERT has a significant contribution in the quasielastic region at low $Q^2$, which is the most important kinematic region in our work. Although the ERT also dominates the region where $E'$ is small, this region only contributes a small amount to the final GDH integral due to the energy weighting, and thus it
is less concerned. The systematic uncertainty of the ERT is carefully studied for reliable results based on this subtraction, which consists of three major components: the uncertainty of radiation thickness, the uncertainty of theoretical calculation for internal effects, and the imperfect model descriptions for \(^3\)He form factors, external radiation effects, and the spectrometer acceptance.

The uncertainty of radiation thickness is studied with varied input in the simulation program. As mentioned in the previous subsection, the uncertainties for the materials upstream and downstream of the target center are 3.1% and 1.7%, respec-
tively, and the ice thickness has an uncertainty of ±1.5 mm. The simulation program was tested with varied input of radiation thickness and ice thickness, and the variation of the output ERT was assigned as the systematic uncertainty, which resulted in 1-5%. Figure 5.15 shows the ERT for 2135 MeV at 6° with 2, 4, 6 mm ice on the polystyrene foams.

![ERT for 2135 MeV at 6° with different ice thicknesses](image)

**Figure 5.15:** Elastic radiative tail with different thickness of ice in the simulation. The right plot shows the results normalized to the ERT with 4 mm ice.

The uncertainty from the internal radiation effects lies in different formulae used for the real photon emissions in different approaches, while the factorized part for the virtual photon corrections are calculated at a basic QED level. We tested the difference of the internal radiative corrections from all three approaches. A maximum of 5% difference between MT and AS results was observed in our kinematic region, which was assigned as the corresponding systematic uncertainty. Figure 5.16 shows the test of internal radiative corrections.

The imperfect model description could not be simply studied, and thus we conser-
Figure 5.16: The ERT with internal radiative correction only. The right plot shows the results normalized to the ERT from the MT approach.

We conservatively estimated the uncertainties: a 5% uncertainty was assigned for the $^3$He form factor parameterisation, 3% was assigned for the external energy loss and multiple scattering simulation, and 3% for the punch-through effect and acceptance related uncertainty. All the different components were added in quadrature to determine the total systematic uncertainty. As a result, the systematic uncertainty of the ERT was better than 10%, and its contribution to the final results was dependent on the kinematics and maximized for the 1147 MeV data at 9°.

5.3.3 Radiative correction for inelastic spectrum

Unlike the elastic scattering, the inelastic region is a continuum spectrum, and thus the radiative correction is essentially an integration of all the possible final states that can contribute to the measured differential cross section at a certain kinematic point. Based on the angle peaking approximation for photon emissions and the ERM, the
integration can be expressed as

\[ \sigma_{\text{rad}}(E_s, E_p) = \int \frac{dl}{L} \int_{E_{\text{min}}'}^{E_{\text{max}}'} \frac{dE_p'}{E_p'} dE_p' I(E_s, E_p', l) \sigma(E_s, E_p') I(E_p', E_p, L - l), \]  

(5.33)

in which \( \sigma_{\text{rad}}(E_s, E_p) \) is the observed cross section with incident electron energy \( E_s \) and detected electron energy \( E_p \). The integration boundaries are defined by the elastic kinematics, written as

\[
E_{\text{min}}' = \frac{E_p}{1 - \frac{2E_p}{M^2} \sin^2 \frac{\theta}{2}}, \quad E_{\text{max}}' = \frac{E_s'}{1 + \frac{2E_p}{M^2} \sin^2 \frac{\theta}{2}}.
\]  

(5.34)

The electron paths before the interaction \((l)\) and after the interaction \((L - l)\), define the corresponding external radiation thickness \( t \) and collisional thickness \( \xi \). In the ERM, the internal Bremsstrahlung effects are equivalently viewed as an additional amount of the radiation thickness before and after the interaction, written as [180]

\[ t_r = b^{-1} \frac{\alpha}{\pi} \left( \ln \frac{Q^2}{m^2} - 1 \right). \]  

(5.35)

Thus \( t' = t_{\text{ext}} + t_r \) is the effective radiation length for both internal and external Bremsstrahlung effects. The energy loss can be characterized by the probability function [176]

\[ I(E, E', l) = \frac{1}{\Gamma(1 + b_t l')} \left( \frac{\Delta E}{E} \right)^{bt'_l} \left[ \frac{b_t l'}{\Delta E} \phi \left( \frac{\Delta E}{E} \right) + \frac{\xi}{\Delta E^2} \right], \]  

(5.36)

where \( b, \phi \) and \( \Gamma(z) \) have been introduced in equation (5.29). Note that the ionization energy loss is also included in this formula, and thus it has an additional contribution from \( \xi \) as compared to its original form in (5.29). The Born cross section with corrections that involve no energy loss is expressed as

\[ \bar{\sigma}(E, E') = \frac{1}{\Gamma(1 + bt_s)\Gamma(1 + bt_p)} \sigma_{\text{born}}(E, E')(1 + \delta_{\text{HO}}) \]  

(5.37)
where $\delta_{HO}$ denotes the factorized corrections from higher order diagrams, and $[\Gamma(1 + bt_s)\Gamma(1 + bt_p)]^{-1}$ is a normalization factor from the external Bremsstrahlung effects. The higher order correction includes

$$\delta_{HO} = \delta_{vert} + \delta_{vac} + \delta_{sch} + \delta_0,$$  \hspace{1cm} (5.38)

with vertex correction term

$$\delta_{vert} = \frac{2\alpha}{\pi} \left( \frac{3}{4} \ln \frac{Q^2}{m^2} - 1 \right),$$  \hspace{1cm} (5.39)

vacuum polarization term

$$\delta_{vac} = \frac{2\alpha}{\pi} \left( \frac{1}{3} \ln \frac{Q^2}{m^2} - \frac{5}{9} \right),$$  \hspace{1cm} (5.40)

Schwinger term

$$\delta_{sch} = \frac{\alpha}{\pi} \left[ \frac{\pi^2}{6} - \Phi \left( \cos^2 \theta \frac{2}{\pi} \right) \right],$$  \hspace{1cm} (5.41)

and a correction term for peaking approximations

$$\delta_0 = \frac{\alpha}{\pi} \left[ \Phi \left( \frac{E_s - E_p}{E_s} \right) + \Phi \left( \frac{E_p - E_s}{E_p} \right) \right].$$  \hspace{1cm} (5.42)

Here $\Phi(x)$ is the Spence function, also known as dilogarithm, defined as

$$\Phi(x) = -\int_0^\infty \frac{\ln |1 - y|}{y} dy.$$  \hspace{1cm} (5.43)

The formalism connects the observed cross section with the Born cross section. Thus the objective of the radiative correction is to calculate the integrals and “unfold” the Born cross section from the observed quantities. In the AS approach a covariant formula is used [177], but the unfolding procedure is essentially the same, in which the phase space of radiative photons needs to be integrated so that the effects due to kinematic shifts can be determined.
In practice, the interaction vertex is always assumed as the target center, and thus the integral over the electron path in equation (5.33) is reduced to a fixed point, where

\[ t_s = t_b + t_r, \quad t_p = t_a + t_r, \]

\[ \xi_s = \xi_b, \quad \xi_p = \xi_a, \]

with the pre-calculated radiation thickness and collisional thickness. The assumption should be reasonably good, because the backgrounds from cell windows have been subtracted, and the gas target itself contributes a negligible amount to these quantities. The 2-dimensional integral over \( E'_s \) and \( E'_p \) is often decoupled into two 1-dimensional integrals for \( E'_s \) and \( E'_p \), respectively, via the energy peaking approximation [181]. However, for this work we used the formula from XY [176] to perform the full 2-dimensional integral and avoided the energy peaking approximation.

Since the energy loss probability has a singularity at \( \Delta E = 0 \), the integral is decomposed into

\[ \sigma_{rad} = \sigma_r + \sigma_s + \sigma_p + \sigma_{sp}, \]

with the regular integral

\[ \sigma_r(E_s, E_p) = \int_{E_s_{\text{min}}}^{E_s - \Delta_s} dE'_s \int_{E_p + \Delta_p}^{E_p_{\text{max}}} dE'_p I(E_s, E'_s, l_s) \bar{\sigma}(E'_s, E'_p) I(E'_p, E_p, l_p), \]

the two integrals with one singularity

\[ \sigma_s(E_s, E_p) = I_{sgl}(E_s, \Delta_s, l_s) \int_{E_p + \Delta_p}^{E_p_{\text{max}}} \bar{\sigma}(E_s, E'_p) I(E'_p, E_p, l_p), \]

\[ \sigma_p(E_s, E_p) = I_{sgl}(E_p + \Delta_p, \Delta_p, l_p) \int_{E_p_{\text{min}}}^{E_s - \Delta_s} I(E_s, E'_s, l_s) \bar{\sigma}(E'_s, E_p + \Delta_p), \]

and the integral for the low energy corner

\[ \sigma_{sp}(E_s, E_p) = I_{sgl}(E_s, \Delta_s, l_s) \bar{\sigma}(E_s, E_p + \Delta_p) I_{sgl}(E_p + \Delta_p, \Delta_p, l_s). \]
Here $I_{\text{sgl}}(E, \Delta E, l)$ denotes the factor in the singularity integral:

$$I_{\text{sgl}}(E, \Delta, l) f(E) = \int_{E-\Delta}^{E} dE' I(E, E', l) f(E'),$$

$$= \frac{f(E)}{\Gamma(1+bl_l)} \left( \frac{\Delta}{E} \right)^{bl_l} \left( 1 - \frac{\xi_l}{(1-bl_l)\Delta} \right).$$

(5.49)

The free parameters $\Delta_s$ and $\Delta_p$ need to be small as compared to the energies $E_s$ and $E_p$, but satisfy

$$\Delta_s \gg \frac{\xi_s}{1-bl_s}, \quad \text{and} \quad \Delta_p \gg \frac{\xi_p}{1-bl_p},$$

(5.50)

for a good approximation of the collisional loss. In this work, both of the parameters are chosen as 1.0 MeV.

**Iterative radiative correction**

The experimental cross section subtracted by the ERT yields the $\sigma_{\text{rad}}$ in equation 5.45 for the inelastic spectrum. However, the relation between $\sigma_{\text{rad}}$ and $\sigma_{\text{born}}$ is not a one-to-one mapping. Typically phenomenological models or theoretical calculations for the Born cross section are used to determine the radiative correction, which is then applied to the measured data. Thus this method gives model dependent results for the radiative correction. In the E97-110 analysis, the model dependency was minimized by an iterative method that takes full advantage of the experiment’s wide kinematic coverage.

During the iterative correction, the measured cross sections are initially treated as Born cross section, and build a “pseudo-model”. In this pseudo-model, the Born cross section values are extracted by a quadratic interpolation between the measured data points. For the unmeasured region (see Figure 5.7), the values can be determined from an extrapolation or from other models. Then the radiative correction can be calculated with the pseudo-model. In each iteration, the radiative correction
results are applied to the measured cross section, and then update the pseudo-model with the latest unfolded data. The iteration stops when the results from two consecutive iterations converge within a desired precision. In the data analysis, the results converged at a precision $< 0.1\%$ with less than 50 iterations.

Once the results are converged, the radiative corrections $\Delta \sigma_{RC}$ and $\delta (\Delta \sigma_{\parallel,\perp})_{RC}$ in equation (5.18) can be determined from the difference between the initial and final values of the pseudo-model. The corrections are then applied to the experimental data to unfold the Born cross sections.

**Smoothing and Modeling**

Since the experimental cross sections were used to build the pseudo-model in the radiative correction, we had to smooth the data because nonphysical micro structures due to statistical fluctuations would be enhanced in the iterative process. This smoothing process was carried out with fits for the experimental data.

The functional forms for the fitting were chosen according to the kinematic regions. The quasielastic region was described by a Gaussian distribution and a Landau distribution, and the Breit-Wigner shape fitted the $\Delta (1232)$, $N_1^* (1500)$ and $N_2^* (1700)$ resonances. A common background was fitted by a 2nd order polynomial in addition to the discrete states. The maximum log likelihood with the resonance location and width as Bayesian estimates was utilized as the estimator for the fits, and the goodness of the fit was evaluated by the reduced $\chi^2$. Figure 5.17 and 5.18 show the examples of the fits.

The unpolarized cross section, normalized to the Mott cross section, and the spin dependent structure functions can be obtained by spline fits inside the kinematic coverage of this pseudo-model. However, the radiative correction for the data set at the lowest beam energy, *i.e.*, the 2135 MeV data at 6° and 1147 MeV data at 9°, requires the physics information in an unmeasured region. For providing the
unpolarized cross sections in the unmeasured region, the model from P. Bosted [5] (PB) was utilized. The PB fits were scaled so its quasielastic peak height equaled that of the lowest energy data, and the scaling factor was updated for every iteration. Figure 5.19 shows the comparison between the scaled PB fits and the unfolded data. The quadratic extrapolation for the unmeasured region was also tested to estimate the systematic uncertainty for this model dependency. The two methods yielded $< 3\%$ difference in the radiative corrections for the lowest energy data, and $< 0.5\%$
Figure 5.19: Comparison between P. Bosted fits and the unpolarized data at the lowest beam energy. The model is scaled according to the maximum height of data.

for higher energy data. In the case of polarized cross section differences, no reliable model exists for the whole inelastic spectrum of $^3$He in this kinematic region, and thus the extrapolation was used with a systematic uncertainty assigned as 3%. In addition, different orders of the spline interpolation, including linear, quadratic and cubic, were tested in the pseudo-model, and < 2% differences were observed in the final radiative corrections.

Systematic uncertainty study

Analogous to the systematic study in the ERT subtraction, multiple terms in the inelastic radiative correction were investigated. They are:

- Approach and Model dependencies. The radiative correction results from the three different approaches were compared, and the largest deviation was assigned as the systematic uncertainty. Besides, the uncertainty from the above-mentioned modeling was included in this term.

- Free parameter variation. The free parameters were chosen as 1 MeV for both $\Delta_s$ and $\Delta_p$, we tested the results stability with varying these parameters in the range of $1 \pm 0.5$ MeV, and the difference was treated as the systematic
uncertainty for this term.

- Central angle variation. The cross section results were bin-centered to the central angle, and the kinematic calculation in the inelastic radiative correction was based on this central angle. For the systematic uncertainty regarding the central angle, we changed the central angle by $\pm 0.10^\circ$, and accounted the results variation into the systematic uncertainty.

- Radiation thickness and ice thickness. The thickness uncertainties were incorporated in the radiative calculation, and thus the uncertainty was propagated to the final radiative corrections.

All these terms were added in quadrature for the total systematic uncertainty of the inelastic radiative correction. The dominant terms were found to be the approach and model dependencies and the thickness uncertainties, while the uncertainties from the free parameters and the central angle were negligible. Although the absolute total uncertainty maximizes at the quasielastic peak, the relative uncertainty at that point was $< 5\%$. However, the radiative correction in the dip region, located between the quasielastic peak and the $\Delta(1232)$ resonance, was overwhelmingly affected by the large quasielastic peak at low $Q^2$. The radiative correction yielded a large relative uncertainty in that region, which could exceed 20\% for the lowest energy data. Figure 5.20 shows the decomposed uncertainty terms for the unpolarized data of 2135 MeV at 6°.

5.4 Born cross sections

The Born cross sections were obtained from the experimental cross sections with the radiative corrections as shown in equation (5.18). The radiative corrections included the ERT effects and the inelastic terms, and the systematic uncertainties from the radiative corrections directly propagate to the final results with the addition of the
correction terms. Figure 5.21-5.23 show the results of the Born level cross sections. In these plots, the error bars on the data points represent the statistical uncertainty, and the grey band represents the systematic uncertainty. Note that the polarized cross section differences are rebinned with a larger $\nu$ bin size for a smoother behavior.
Figure 5.21: Unpolarized cross section and polarized cross section differences after radiative corrections, 2135, 2845, and 4209 MeV at 6°.
Figure 5.22: Unpolarized cross section and polarized cross section differences after radiative corrections, 1147, 2234, and 3319 MeV at 9°.
Figure 5.23: Unpolarized cross section and polarized cross section differences after radiative corrections, 3775 and 4404 MeV at 9°.

5.5 Constant $Q^2$ interpolation

With the Born cross sections, we are able to extract the structure functions from the experimental data. Since the experimental data were measured with constant beam energy, and had different $Q^2$ over the whole spectrum, the data needed to be interpolated to constant $Q^2$ for obtaining the moments of the structure functions. The constant $Q^2$ points were determined based on three principles for the interpolation:

- No extrapolation allowed for the unmeasured kinematic region, the constant $Q^2$ data should always be interpolation of experimental data.
- For $^3$He results, the $Q^2$ values are chosen as close as possible to the quasielastic
Table 5.6: The constant $Q^2$ data sets for the second period of E97-110.

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>Reference data set</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>1147 MeV at 9°</td>
</tr>
<tr>
<td>0.050</td>
<td>2135 MeV at 6°</td>
</tr>
<tr>
<td>0.088</td>
<td>2845 MeV at 6°</td>
</tr>
<tr>
<td>0.118</td>
<td>2234 MeV at 9°</td>
</tr>
<tr>
<td>0.230</td>
<td>3319 MeV at 9°</td>
</tr>
</tbody>
</table>

region, since the generalized GDH integral is dominated by this region at low $Q^2$. For neutron results, the dominant contribution is the $\Delta(1232)$ resonance, which is used as the $Q^2$ reference.

- One experimental data set at constant beam energy corresponds to one constant $Q^2$ data set, so the over usage of experimental data is avoided.

The production of constant $Q^2$ spectrum involves a series of two-step spline interpolations on $Q^2$ and another kinematic variable, denoted as $k$, which is discussed later. Firstly, the value of kinematic variable $k$ is chosen according to an experimental data point, designated as $k_c$. Then $k_c$ is interpolated in each data set with a constant beam energy, yielding a data point at $(k_c, Q^2(E_i, \theta_i, k_c))$, with $E_i$ and $\theta_i$ the corresponding beam energy and scattering angle, respectively. Lastly the interpolation of $Q^2$ in this set of $k_c$ points generates the data point at $(k_c, Q^2)$, where $Q^2$ is the designated constant value. Repeating this procedure for all the data points in a reference data set projects its spectrum to a constant $Q^2$ value. As a result, five constant $Q^2$ spectra were interpolated for the five experimental sets at beam energies $\leq 3319$ MeV, ranged from 0.032 GeV$^2$ to 0.230 GeV$^2$ as listed in Table 5.6. The 9° data at 3775 MeV and 4404 MeV, and the 6° data at 4209 Mev were only utilized as the interpolation constraints, since they did not cover the quasielastic region and the full resonance region. Figure 5.24 shows the kinematic plot for the constant $Q^2$ interpolation.
5.5.1 Interpolation variable

At first glance, the kinematic variable is presumably $\nu$ or $W$. However, neither of them is a good interpolation variable for the spin dependent structure functions in the quasielastic region, as shown in Figure 5.25. In this region, the maxima and crossover points of different spectra are not aligned in $W$ or $\nu$, and thus their complicated behaviors over $Q^2$ may result in large errors from spline interpolations.

For a more reliable interpolation in the quasielastic region, the nuclear scaling variable is introduced, written as [182]

$$y(\nu, Q^2) = \frac{M_A + \nu}{W_A^2} \left( \sqrt{\Lambda^2 - M_{A-1}^2 W_A^2} - \sqrt{\nu^2 + Q^2 \Lambda} \right),$$  \hspace{1cm} (5.51)
with the kinematic variable

\[ \Lambda = \frac{1}{2} \left( M_{A-1}^2 - M^2 + W_A^2 \right), \quad (5.52) \]

and the invariant mass

\[ W_A^2 = M_A^2 + 2M_A \nu - Q^2, \quad (5.53) \]

where \( M \) is the nucleon mass, \( M_A \) is the nuclear target mass, and \( M_{A-1} \) represents the mass of the residual nucleus after the nucleon is knocked out. Note the variable \( y \) is finite only when

\[ W_A \geq M + M_{A-1}, \quad (5.54) \]

or equivalently

\[ \nu^* \geq E_s + \frac{E_s^2}{2M_A}, \quad (5.55) \]

with \( \nu^* = \nu - Q^2/2M_A \) the excitation energy from the virtual photon, and \( E_s = M_{A-1} + M - M_A \) the nuclear separation energy. The nuclear scaling behavior was found that the cross section of electron-nucleus scattering can be factorized to a general function \( F(\nu, Q^2) \) and the cross section of electron-nucleon scattering, in which the function \( F \) only depends on a single variable \( y(\nu, Q^2) \) at a relatively high \( Q^2 \). The expression of \( y \) in equation (5.51) was derived by Pace&Salmé [183] in
the framework of PWIA. Although the PWIA approach is expected to have a large error within our kinematic range, as mentioned in Section 2.3, the spin dependent structure functions still exhibit a good alignment with the kinematic variable $y$, and thus this variable is used for the interpolation in the quasielastic region. Figure 5.26 shows the $g_1$ and $g_2$ fits for the experimental data at constant beam energies with the interpolation variables.

For the final $^3$He results, it was observed that the generalized GDH integrand $K\sigma_{TT}/\nu^2$ aligned well with the scaled nuclear $y$-scaling in the quasielastic region,
Figure 5.27: The interpolation variable for generalized sum rule integrand of $I_{TT}$ (left) and $I_{LT}$ (right) in the quasielastic region.

defined as

$$y' = \frac{y - y_{th}}{y_m - y_{th}},$$

in which $y_m = 200$ MeV is chosen to define the range of the $y'$ interpolation, and $y_{th}$ represents the nuclear scaling variable at the two body-breakup threshold. The variable $y'$ is advantageous for the interpolation of $I_{TT}$ integrand because it has a fixed value at $y' = 0$ for the integral threshold. However, other integrands, such as that of $I_{LT}$, do not well behave with $y'$, and thus only $I_{TT}$ was directly interpolated with the variable. Figure 5.27 shows the integrand over $y'$ in the quasielastic region.

5.5.2 Uncertainty propagation

The systematic uncertainty from the interpolation was studied with different degrees of the spline interpolation. Since the constant $Q^2$ point was close to the experimental data set in the quasielastic region, the interpolation uncertainty in this region was negligible, and the difference between the quadratic spline and cubic spline was assigned as the systematic uncertainty in the resonance region.
The interpolated value of the physics quantity can be expressed as

$$ x = \sum_{i=1}^{N+1} C_i x_i, $$

for a spline interpolation of $N$ degrees, where $x_i$ represents the value of the $i$th participated experimental point, and $C_i$ denotes the corresponding interpolation coefficients. The statistical and systematic uncertainties propagate as

$$ \sigma_{stat}^2 = \sum_{i=1}^{N+1} (C_i \sigma_{stat,i})^2, $$

$$ \sigma_{syst} = \sum_{i=1}^{N+1} |C_i \sigma_{syst,i}|, $$

in which the most conservative estimation for the systematic uncertainty propagation is applied.
Results and Conclusion

The physics of interest can be extracted from the Born cross sections. In this chapter, we present the $^3$He results for the second period of E97-110, including the spin dependent structure functions $g_1$ and $g_2$, the virtual photoabsorption cross sections $\sigma_{TT}$ and $\sigma_{LT}$, and the generalized GDH sum rule $I_{TT}(Q^2)$.

6.1 Spin dependent structure functions

The spin dependent structure functions are directly obtained from equations of (2.26) and (2.27) at $\cos \phi = 1$, written as

\[
g_1 = \frac{MQ^2}{4\alpha^2} \frac{\nu E}{(E - \nu)(2E - \nu)} \left[ \Delta \sigma_{\parallel} + \tan \frac{\theta}{2} \Delta \sigma_{\perp} \right],
\]

\[
g_2 = \frac{MQ^2}{4\alpha^2} \frac{\nu^2}{2(E - \nu)(2E - \nu)} \left[ -\Delta \sigma_{\parallel} + \frac{E + (E - \nu) \cos \theta}{(E - \nu) \sin \theta} \Delta \sigma_{\perp} \right].
\]  

6.1.1 Constant beam energy

At small scattering angles, $g_1$ is predominately contributed by $\Delta \sigma_{\parallel}$, while $g_2$ is almost only dependent on $\Delta \sigma_{\perp}$ because of the kinematic factors. Figure 6.1-6.3 show the spin dependent structure functions at constant beam energies.
Figure 6.1: Spin dependent structure functions for incident beam energies of 2135, 2845, and 4209 MeV at 6°.
Figure 6.2: Spin dependent structure functions for incident beam energies of 1147, 2234, and 3319 MeV at 9°.
Since the spectrometer settings of momentum for the perpendicular polarization configuration lacked the setting at the elastic kinematic, $\sigma_\perp$ data had a smaller $\nu$ coverage as compared to the data of $\sigma_\parallel$ at the low $\nu$ end. In order to fully utilize the experimental data, the contributions from $\sigma_\perp$ in the unmeasured region were estimated by extrapolation of the fits, and a systematic uncertainty of 20% was assigned for this estimate.

6.1.2 Constant $Q^2$

As mentioned in the previous chapter, the interpolation of $g_1$ and $g_2$ to constant $Q^2$ involves the nuclear-$y$ variable in the quasielastic region, and the invariant mass $W$. 

**Figure 6.3:** Spin dependent structure functions for incident beam energies of 3775 and 4404 MeV at $9^\circ$. 

![Graph showing structure functions for 3775 MeV and 4404 MeV at 9°](image)
in the resonance region. The resulted $g_1$ and $g_2$ are shown in Figure 6.4 and 6.5.

**Figure 6.4:** Spin dependent structure functions for $Q^2 = 0.032$, 0.050, and 0.088 GeV$^2$. 
Figure 6.5: Spin dependent structure functions for $Q^2 = 0.118$ and 0.230 GeV$^2$.

At low $Q^2$, the structure functions show a symmetric behavior for the ∆ (1232) resonance region, which indicates the transverse-longitudinal cross section $\sigma_{LT} \propto (g_1 + g_2)$ is highly suppressed in this region. For high energies (beam energy $\geq 3775$ MeV) data, $g_1$ is consistently non-zero and tends to increase as the energy transfer increases, while the behavior of $g_2$ cannot be unambiguously determined from the data due to the large statistical uncertainties.

6.2 Virtual photoabsorption cross sections

The spin structure is related to the virtual photoabsorption cross sections of $\sigma_{TT}$ and $\sigma_{LT}$ in terms of the expression in equation (2.37). Figure 6.6-6.8 depict the two
cross sections at constant beam energies. Note that we have used the convention $K_G = \sqrt{\nu^2 + Q^2}$ for extracting these cross sections, and the error bars in the figures represent the quadrature-sum of the statistical and systematic uncertainties.

**Figure 6.6**: Virtual photoabsorption cross sections for incident beam energies of 2135, 2845, and 4209 MeV at 6°. Error bars represent total uncertainties.
Figure 6.7: Virtual photoabsorption cross sections for incident beam energies of 1147, 2234, and 3319 MeV at 9°. Error bars represent total uncertainties.
Figure 6.8: Virtual photoabsorption cross sections for incident beam energies of 3775 and 4404 MeV at 9°. Error bars represent total uncertainties.

At low energies, the transverse-transverse term $\sigma_{TT}$ displays a large positive value around the quasielastic peak, and starts to drop drastically as $\nu$ decreases. This behavior plays an important role in the evolution of $I_{TT}(Q^2)$, which is dominated by the quasielastic contribution due to the weighting of $K_G/\nu^2$ for its integrand. At low $Q^2$, the negative contribution near the threshold exceeds the positive contribution from the quasielastic peak, and thus the expected “turn-over” point, as mentioned in Section 3.6, appears in $I_{TT}(Q^2)$. Such a behavior is discussed with details in the next section. The main feature for $\sigma_{LT}$ is the highly suppressed $\Delta$ (1232) resonance region. Besides, $\sigma_{LT}$ shows a large negative value in the quasielastic region, and the
magnitude increases as the $Q^2$ drops, which is similar to the observation in E94-110 [65]. This behavior is partially due to the increasing kinematic factor of $Q/\nu$ near the two-body breakup threshold:

$$\frac{Q}{\nu_{2BB}} \approx \left( \frac{5.5 \text{ MeV}}{Q} + \frac{Q}{2M_T} \right)^{-1},$$

(6.2)
as it turns over at $Q^2 \approx 0.031 \text{ GeV}^2$, which is lower than the $Q^2$ values of our data points. $\sigma_{LT}$ obviously vanishes at $Q^2 = 0$, but it will be with interest to see whether the behavior is simply due to the kinematic factor, or because of the cancellation of $g_1$ and $g_2$ at some point. The lowest $Q^2$ data from the first period of E97-110 ($Q^2 = 0.02 \text{ GeV}^2$) may provide further insight on this term.

The virtual photoabsorption cross sections are interpolated to constant $Q^2$ in the form of generalized GDH integrand. The results are shown in Figure 6.9 and 6.10. In our kinematic coverage, the contributions beyond quasielastic region are almost negligible, except a noticeable amount from the $\Delta$ (1232) resonance at $Q^2 = 0.23 \text{ GeV}^2$. At a moderate $Q^2$ (0.118 or 0.23 GeV$^2$), the integrand peaks in the quasielastic region, and leads to a positive GDH integral. With the decrease of the $Q^2$ values, the height of the quasielastic peak increases, but the integrand near the threshold decreases and gradually becomes the dominant contribution to the GDH integral. This phenomenon suggests significant contributions from the electrodisintegration process at low $Q^2$, analogous to the photodisintegration in the GDH integrand for the real photons [184].
Figure 6.9: Generalized GDH integrand at $Q^2 = 0.032$, 0.050, and 0.088 GeV$^2$. Error bars represent total uncertainties.
Figure 6.10: Generalized GDH integrand at $Q^2 = 0.118$ and $0.230$ GeV$^2$. Error bars represent total uncertainties.

6.3 Generalized GDH sum rule

The generalized GDH sum rule for $^3$He, as expressed in equation (3.50), is calculated with numerical integration of the extracted GDH integrand. The uncertainties of data were propagated according to equation (5.58), and the integrand fits were utilized to study the uncertainty from the numerical integration methods. The GDH integrals are listed in table 6.1. The integration starts at the two-body breakup threshold, and ends at $W_{max}$, which is calculated with the nucleon mass. The unmeasured contribution from higher $W$ is evaluated by MAID2007 model [101] with the effec-
Table 6.1: The generalized GDH integrals for $^3$He from the second period of E97-110.

<table>
<thead>
<tr>
<th>$Q^2$ (GeV$^2$)</th>
<th>$W_{max}$ (MeV)</th>
<th>$I_{GDH}(Q^2)$ (µb)</th>
<th>$\sigma_{stat}$ (µb)</th>
<th>$\sigma_{syst}$ (µb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032</td>
<td>1470</td>
<td>-17.79</td>
<td>295.75</td>
<td>604.05</td>
</tr>
<tr>
<td>0.050</td>
<td>1770</td>
<td>532.54</td>
<td>170.67</td>
<td>460.76</td>
</tr>
<tr>
<td>0.088</td>
<td>2000</td>
<td>1097.05</td>
<td>76.82</td>
<td>315.72</td>
</tr>
<tr>
<td>0.118</td>
<td>1790</td>
<td>1322.22</td>
<td>69.28</td>
<td>254.10</td>
</tr>
<tr>
<td>0.230</td>
<td>1950</td>
<td>565.71</td>
<td>24.66</td>
<td>69.84</td>
</tr>
</tbody>
</table>

tive polarization approximation ($W < 2$ GeV) and the Regge parameterisation from Thomas&Bianchi [110] ($W > 2$ GeV). The unmeasured contributions at high $W$ are negligible for $Q^2 < 0.1$ GeV$^2$, while their maximum is $< 4\%$ for $Q^2 = 0.23$ GeV$^2$, and thus they are integrated into the systematic uncertainties. The systematic uncertainties at low $Q^2$ are dominated by the ERT subtractions, while the uncertainties from inelastic radiative corrections and the experimental cross sections become more significant at higher $Q^2$.

Figure 6.11 displays the results from E97-110 as compared to the E94-010 results. Consistent results are observed for the overlapping kinematic points between the two experiments. A “turn-over” behavior is found for the generalized GDH integral for $^3$He, which is located at around $Q^2 = 0.1$ GeV$^2$. As we have seen in the section for the generalized GDH integrand, this phenomenon is caused by the significantly negative contribution from the electrodisintegration channel. Although the lowest $Q^2$ point at 0.032 GeV$^2$ is still away from the GDH sum rule at the real photon point, the trend clearly indicates the connection between the sum rule and its generalized form.

6.4 Summary and Conclusion

The main goal of experiment E97-110 is to investigate the $Q^2$-evolution of the generalized GDH sum rule for the neutron and $^3$He in the low $Q^2$ region. In this work, the $^3$He results were extracted for the second period, with the $Q^2$ values ranging from
Figure 6.11: The generalized GDH integrals for $^3$He. The blue point at $Q^2 = 0$ shows the GDH sum rule for real photons. E94-110 data points are provided by [185].

0.032 to 0.23 GeV$^2$. The spin structure of $^3$He is examined with the spin dependent structure functions and the virtual photoabsorption cross sections. In this experiment, the generalized GDH sum rule of $^3$He is firstly measured for $Q^2 < 0.1$ GeV$^2$; and the experimental data reveal the “turn-over” point of $I_{GDH}(Q^2)$ at around 0.1 GeV$^2$. In addition, our data exhibit a sharp decrease of $I_{GDH}(Q^2)$ at lower $Q^2$, and hence suggest the recovery of the GDH sum rule at the real photon point. Investigation in the extremely low $Q^2$ region ($< 0.01$ GeV$^2$) may be required to fully understand the evolution of $I_{GDH}(Q^2)$ and its contributions from the electrodisintegration channels. However, one is expected to encounter experimental difficulties from isolating the quasielastic data from the overwhelming contribution from the radiative tails of the elastic electron-nucleus scattering.
Appendix A

PRad experiment at JLab

In addition to E97-110, I have also participated the “high precision measurement of the proton charge radius” (PRad experiment, E12-11-106*) at JLab during my graduate study. The experiment aims to determine the proton charge radius with a sub-percent precision based on the measurement of elastic electron-proton scattering at very low $Q^2$. My thesis project was originally the PRad experiment, which was expected to be performed before 2015. However, due to the schedule conflict with the 12 GeV upgrade of CEBAF, the PRad experiment was eventually conducted in May-June 2016, so I had switched my thesis project to the data analysis of E97-110. This appendix briefly introduces the PRad project, and summarizes my contributions to the experiment.

*Spokespersons: D. Dutta, H. Gao, A. Gasparian, M. Khandaker.
A.1 Introduction

A.1.1 Radius puzzle

The root-mean-square (rms) proton charge radius is an important quantity to understand the structure of proton. Precise knowledge of the proton charge radius is required by high precision tests of quantum electrodynamics (QED) calculations for the hydrogen Lamb shift. Recently, measurements of muonic hydrogen ($\mu p$) Lamb shift by the CREMA collaboration at PSI reported the most precise value of proton charge radius, which was $r_p = 0.84087 \pm 0.00039$ fm [186]. However, the CREMA result were 5.6$\sigma$ away from the CODATA 2014 value [187], given as $0.8751 \pm 0.0061$ fm. The CODATA value is a compilation of the results from hydrogen Lamb shift measurements and elastic electron-proton ($ep$) scattering experiments. Figure A.1 depicts the CREMA results [188, 186] and the CODATA 2014 value [187], as well as the recent experimental results, including the hydrogen spectroscopy measurement by Beyer et al. [189], the updated global fit with the form factors measured at JLab by Zhan et al. [190], and the $ep$ scattering measurement at Mainz Microtron by Bernauer et al. [191].

This discrepancy, known as the “proton radius puzzle”, is of great interest to atomic, particle, and nuclear physicists. Many efforts have been devoted to resolve the puzzle: theoretical calculations of $\mu p$ energy levels were refined, which were summarized in Refs. [192, 193]; a muon-proton scattering measurement was approved at PSI [194] to study the lepton universality; new hydrogen spectroscopy measurements have been performed [189, 195]. A thorough investigation on the discrepancy also requires a high precision measurement based on the elastic $ep$ scattering, in which the dominant sources of the systematic uncertainty are different from the previous $ep$ experiments. For this purpose, the PRad experiment has been carried out in Hall B at JLab. In the remainder of this appendix, we present the PRad experimental
A.1.2 Extraction of the proton charge radius

The proton charge radius is typically determined by two experimental techniques: the hydrogen spectroscopy and the elastic $ep$ scattering measurement.

Thanks to the development of the optical apparatus and the improved precision of transition measurements in the last decades, the rms charge radius of the proton can be reciprocally determined by the measured hydrogen Lamb shift in conjunction with the state-of-the-art QED calculations. However, the precision of this method is limited by the small contribution from the proton finite size, while the high precision achieved in the $\mu p$ spectroscopy measurement is due to the fact that the finite size effect is significantly enhanced in the bound system of $\mu p$.

On the other hand, the proton charge radius can be extracted from the electric form factor of the proton, which, as introduced in Section 2.1, are typically measured in the elastic $ep$ scattering experiments. Since the Sachs form factors can be interpreted as the Fourier transforms of the spatial distributions $\rho(r)$ of the charge
and magnetization, the $Q^2$ expansion of the electric form factor $G_E^p$ is written as

$$G_E^p(Q^2) = 1 - \frac{Q^2 \langle r^2 \rangle}{6} + \frac{Q^4 \langle r^4 \rangle}{120} - \ldots,$$  \hspace{1cm} (A.1)

where

$$\langle r^{2n} \rangle = \int r^{2n}\rho_E(r)4\pi r^2 dr,$$  \hspace{1cm} (A.2)

and

$$\int \rho_E(r)4\pi r^2 dr = 1.$$  \hspace{1cm} (A.3)

Comparing equation (A.1) with the Taylor expansion of $G_E^p$ at $Q^2 = 0$ (with the boundary condition $G_E^p(Q^2 = 0) = 1$), the rms charge radius is given by:

$$\langle r^2 \rangle = -6 \frac{dG_E^p(Q^2)}{dQ^2} \bigg|_{Q^2=0}.$$  \hspace{1cm} (A.4)

Therefore, the proton charge radius can be directly determined by the slope of the fit for the form factor $G_E^p$ at $Q^2 \to 0$.

In practice, the measured $G_E^p$ data in the low $Q^2$ region are not sufficient to extract the proton charge radius with a linear fit, i.e., neglecting the higher order terms of $\langle r^{2n} \rangle$. Therefore, the extraction of proton charge radius needs to include the data of form factors at not-so-low $Q^2$ and the contribution of the $Q^4 \langle r^4 \rangle$ term cannot be neglected. For instances, Sick analyzed the world data in the range of $Q^2 < 0.62 \text{ GeV}^2$ [196]; Bernauer et al. extracted the proton charge radius from the recent Mainz data with $Q^2$ up to 1 GeV$^2$ [191].

However, the treatment of $\langle r^4 \rangle$ or higher order terms depends on the proton charge distribution, which is not understood at the current stage. For example, $\langle r^4 \rangle$ equals $\frac{5}{3} \langle r^2 \rangle^2$, $\frac{5}{2} \langle r^2 \rangle^2$ and $1 + \frac{10}{3} \langle r^2 \rangle^2$ with the Gaussian, the exponential and the Yukawa distribution, respectively [197]. A model independent extraction of proton charge radius requires the data of form factors in the $Q^2 \to 0$ region where the $\langle r^2 \rangle$
term dominates. Therefore, the PRad experiment focuses on the measurement of cross sections for the $ep$ elastic scattering at very low $Q^2$. A detailed study of the radius extraction in the PRad kinematic range can be found in Ref. [198].

A.2 Experimental Setup

The PRad experiment has measured the cross section of unpolarized elastic $ep$ scattering in an unprecedented low $Q^2$ region of $2 \times 10^{-4}$ to $6 \times 10^{-2}$ GeV$^2$. As shown in Figure A.2, the experiment utilizes a windowless hydrogen gas-flow target, which removes a typical background source of all previous $ep$ experiments – the elastic or quasi-elastic scattering off the nuclei in the target windows. A five-meters-long, two stage vacuum box is placed between the target chamber and the detectors. The external material traversed by the incident beam or the scattered electrons, which is another typical background source, is minimized by connecting the vacuum box to the beampipe at both ends.

The PRad detector package encompasses a Hybrid Calorimeter (HyCal), which is a composite of an inner sector and four outer sectors, and two Gas Electron Multiplier (GEM) chambers. The inner sector of HyCal is a $34 \times 34$ array, with a $2 \times 2$ central hole, of $2.075 \times 2.075 \times 18$ cm$^3$ PbWO$_4$ modules, and each of its outer sectors contains $6 \times 24$ or $24 \times 6$ lead glass modules with the size of $3.815 \times 3.815 \times 45$ cm$^3$. The nominal energy and position resolutions of the PbWO$_4$ sectors are $2.6%/\sqrt{E(\text{GeV})}$ and $2.5 \text{ mm}/\sqrt{E(\text{GeV})}$, respectively, while the resolutions of the lead glass sectors are approximately doubled. Two GEM chambers are mounted on the HyCal box to cover the entire geometric acceptance of HyCal. The GEM detector improves the position resolution to $< 0.1$ mm, and hence significantly reduces the systematic uncertainty in determining $Q^2$.

Scattered electrons from both Møller and $ep$ elastic scattering have been simul-
The schematic of the PRad experiment (not in scale).

Figure A.2: The schematic of the PRad experiment (not in scale).

taneously measured during the experiment. Within the kinematic coverage of this experiment, these two types of events can be easily separated by their energies. The cross sections of $ep$ elastic scattering are thus normalized to those of the well known Møller process, so the systematic uncertainty on determining the luminosity is significantly reduced in this ratio measurement.

The experiment will eventually extract the electric form factor ($G_E$) from the measured cross section data. According to the Rosenbluth formula, the reduced differential cross section is written as

$$
\sigma_R = \left( \frac{d\sigma}{d\Omega} \right)_{\text{exp}} \left( \frac{\sigma_{\text{Mott}}}{\epsilon} - \frac{\tau}{1 + \tau} \right) = \frac{\epsilon}{T} G_E^2 + G_M^2
$$

(A.5)

where $\tau = Q^2/4M_p^2$, and $\epsilon = \left[1 + 2(1 + \tau)\tan^2\theta/2\right]^{-1}$. The reduced cross section is dominated by the contribution from electric form factor $G_E$ at low $Q^2$. In the lowest $Q^2$ region that can only be achievable with the 1.1 GeV beam, the $G_M$ contribution ranges from 0.015% to 0.06%, and hence is negligible. At higher $Q^2$, as the contribution of $G_M$ becomes noticeable, though still at percent level, the $G_M$ contribution can be estimated by existing parameterizations.
A.3 DAQ System

I have led the efforts of building the DAQ system for the PRad experiment. This section briefly summarizes the PRad DAQ system, and more details can be found in the technical notes [199].

The PRad DAQ system consists of three subsystems: the tagger counter system which provides the information about the photon beam utilized during the calibration runs; the DAQ system for the calorimeter detector (HyCal) which also produces the main physics triggers; the DAQ system for the two GEM detectors. The HyCal DAQ system is consisted of 3 Fastbus crates with 10 ADC1881M boards in each, the integrated charge within a 280 ns time window of the electrical signal from PMTs attached to each module is converted to the digital signal by the ADC board and integrated into the data stream. The 1728 modules are divided into 52 timing groups, and the timing information of each group is measured by the CAEN v1190 TDC module. The main trigger is also formed by the HyCal signals. Several linear sum NIM units are utilized to generate the total sum of the HyCal module signals, and this signal is then sent to a discriminator with a certain threshold voltage corresponding to approximately 30% of the incident beam energy to produce the main trigger. The GEM DAQ system is called the Scalable Readout System (SRS). It consists of 64 APV25 chips, and each APV25 possesses 128 ADC channels to convert the electrical signal from the GEM strips to digital signals. In addition, the beam properties measured by the Hall B beamline apparatus and the pressure of the vacuum tank and the target chamber are integrated into the data flow via the Experimental Physics and Industrial Control System (EPICS).

The livetime of the DAQ system can be determined from two independent measurements. On one hand, the numbers of accepted triggers and received triggers are recorded by scalers. On the other hand, the DAQ system sends out a busy signal
when it is not ready to handle an incoming trigger, and the busy time is measured by recording the total counts and the counts gated with this signal from a 5 MHz pulse generator.

A.4 Monte-Carlo Simulation

I developed a Monte-Carlo simulation program for the PRad experiment based on GEANT4 toolkit [172, 200, 201]. In the simulation study prior to the performance of the experiment, the primary background source was found to be the beam halo electrons scattered off the nuclei in the target cell structures. To suppress the background, the diameter of the target cell was enlarged from 8 mm to 2 inches, and empty target runs were added into the run plan for background subtraction. The simulation showed that the statistical uncertainties on the cross sections will be less than 0.5% within the approved beam time, and the systematic uncertainty due to the subtraction could be controlled within 0.4%. These numbers have been already achieved in the preliminary analysis of the real experimental data.

For the radiative corrections, event generators with two independent theoretical recipes for the internal radiative effects were implemented in the simulation program. One is based on the complete QED calculations for the leading order diagrams in [202], which goes beyond the ultra-relativistic approximations (URA), and the other is called ESEPP [203], whose first order Bremsstrahlung effect is calculated without soft-photon approximation or URA. The external radiative effects are taken into account by the GEANT4 built-in models or data for the effect of particles passing through materials [204].

As shown in Figure A.3, a detailed HyCal geometry was implemented in the simulation to study the signal digitization, the cluster reconstruction, and the detector acceptance. In the simulation, the GEM detectors were built as an ideal detector with the geometries of the frames and foils, and the recorded position was smeared
by a normal distribution with $\sigma = 0.1$ mm. The simulation proved that the GEM detectors could significantly refine the systematic uncertainties of determining the polar angle (or $Q^2$) acceptance, and the tracking information between the HyCal and GEM could partially reject the background from the target cell structures. This study motivated the implementation of the GEM detectors in the experimental design, resulting in an improved precision of the measured cross sections.

A.5 Data Analysis

For the PRad data analysis, I have developed the analysis software that reconstructs the digital signals from HyCal modules and GEM strips to the position and energy of each incident particle. The reconstruction methods and its uncertainty studied with Monte-Carlo simulations are summarized in the technical note [205]. In addition to the software development, I have also directly contributed to the data analysis, including the DAQ livetime determination, the zero-suppression for pedestal data, the cosmic and electrical noise rejection, and the radiative correction.

In the current analysis, we have obtained the preliminary Born cross sections from
Figure A.4: The preliminary cross section of PRad 1.1 GeV data, provided by [206].

Figure A.5: The preliminary cross section of PRad 2.2 GeV data, provided by [206].

the experimental data, as shown in Figure A.4 and A.5. The statistical uncertainties are approximately 0.3% and 0.18% for 1.1 GeV and 2.2 GeV data, respectively. While the preliminary systematic uncertainties are conservatively estimated as 0.8-2.0%, which will be improved and finalized in the near future.
A.6 Summary

The large experimental discrepancy on one of the fundamental quantities, the proton charge radius, has puzzled the community since 2010. The PRad experiment is uniquely designed to investigate this puzzle by extracting the proton charge radius from the electric form factor measured in an unprecedentedly low $Q^2$ region. The experiment also minimizes several typical systematic uncertainties in the previous $ep$ scattering measurements. I have made important contributions, in both aspects of hardware and software, to the design, the preparation, and the data analysis of this experiment. We are expecting a direct impact on the “proton radius puzzle” from the final PRad results in the near future.
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Biography

Chao Peng was born on 17 August 1987 in Zhuzhou, Hunan province, China. Peng received a bachelor of engineering degree in engineering physics from Tsinghua University (THU) in July, 2009, and participated in the Low Temperature Plasma Lab in THU as a research associate. In the fall of 2010, Peng attended Duke University and joined the Medium Energy Physics Group in the department of physics and began the graduate studies in experimental nuclear physics under the guidance of Prof. Haiyan Gao.

Selected awards


5. JSA Fellowship for Graduate Student, JLab, 2014.

6. JSA Fellowship for Graduate Student, JLab, 2015.