Stiffness and Frequency of Slender Structures: An Experimental Study Utilizing 3D Printing

by

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Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mechanical Engineering and Materials Science in the Graduate School of Duke University 2018
Abstract

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Abstract

This study analyzes the effect of geometric changes to the stiffness and frequency of slender structures. Geometric changes were made by altering the width and length of the structure as well as adding structural components. 3D printing was utilized to create the slender structures which were tested experimentally. Stiffness was determined by finding the slope of the linear region of the structure’s force vs deflection plot. The frequency of the structure was obtained by putting a time series of the structure’s oscillations through a Fast Fourier transform which provides a peak signifying the structures in plane frequency. Additionally, several structures were combined to create a springs in parallel system. Results of analysis show that for a structure with constant material properties that increasing/decreasing the length will lead to an decrease/increase in stiffness and frequency while altering the width of the structure will increase stiffness, but have no effect on frequency. It is also shown that additional structural components added to a simple structure increases its stiffness and frequency. Analysis of the springs in parallel system will give a non-linear force vs deflection plot which is made up of linear regions. The slope of the curve changes when the deflection is equal to the spacing between structures. These results are useful for designing structures to fulfill their requirements in the overall system.
This thesis is dedicated to my family and all of the teachers who have helped along the way.
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Symbols

The symbols below are used throughout this paper.

- $k$: Stiffness
- $F$: Force
- $\delta$: Deflection
- $\Delta$: Spacing between structures
- $C$: Constant for structure geometry
- $E$: Elastic modulus
- $I$: Moment of area
- $L$: Length of structure
- $x(t)$: Time series of oscillations
- $t$: Time
- $f$: Frequency
- $\bar{A}$: Non-dimensional amplitude
- $m$: Mass
- $\rho$: Density
- $A$: Area
- $y_i$: Vertical translation
- $x_i$: Horizontal translation
$\theta_i$ Rotation
$k$ Element stiffness matrix
$K$ Global stiffness matrix
$\lambda$ Eigenvalues
$m$ Element mass matrix
$M$ Global mass matrix
$T$ Transformation matrix
$D$ Diameter
$X$ Cross beam depth
$Y$ Height of column
$Z$ Length of cross beam
$\bar{k}$ Non-dimensional slope

Abbreviations

The abbreviations below are used throughout this paper.

3D Three dimensional
FEA Finite Element Analysis
FFT Fast Fourier transform
CAD Computer-aided design
mm Millimeter
g grams
N Newtons
EXP Experimental results
THEO Theoretical results
ABS Acrylonitrile Butadiene Styrene
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1

Introduction

1.1 Background and Motivation

Simple geometric changes to a structure can lead to noticeable changes in its mechanical properties. It is important to know how these simple changes affect the structure’s properties so that during design the structure can be modeled so that it fulfills the requirements needed of it in the system. It is often best to look at simple, small scale structures to examine the effect of geometric changes on its mechanical properties. Using simple structures allows for greater control during experimentation. While large scale structural components can be manufactured fairly accurately, it is often difficult to create a structure used for lab work that has essentially the same dimensions as its Computer-aided design (CAD) model and does not have any major imperfections such as cracking or large seams. These problems have been eliminated by 3D printing. 3D printing is a powerful tool that is used in a wide variety of fields, it is able to create a solid object from a single file using an additive process by layering plastic to form the complete object [1]. Additive manufacturing has made it possible to perform analysis of simple structures because it allows for a
3D object to be created that is nearly a flawless representation of its CAD model.

1.2 Goals and Approach

The goal of this paper will be to look at how geometric changes to simple structures affect their stiffness and frequency. Analysis was conducted using experimental, FEA simulation and theoretical methods. Stiffness was tested experimentally by loading the structure and measuring the resulting deflection. This analysis allowed for the creation of a force vs deflection plot which provided the stiffness of the structure. Frequency of the analyzed structures was found by collecting a time series of the structure’s oscillations and putting that time series through a Fast Fourier transform (FFT). All of the frequencies of interest are in-plane.

In order to create the structures that were tested experimentally, a CAD model was created in SolidWorks that allowed for the structure to be 3D printed and for FEA simulations to be run. 3D printing was chosen to create the structures because it would allow for a more accurate representation of the CAD model than if the structure was made by hand. The two structures that were analyzed are plane frames and rings. Once analysis of the individual structures was completed, multiple plane frames and multiple rings were combined to create a system that has reinforceable stiffness. In this system the size of the structure as well as the spacing between the structures act as control parameters.

1.3 Outline

To begin the paper, a review of the theory of plane frames and ring structure’s mechanical properties is presented. This review looks at what factors contribute to stiffness and frequency and also introduces the theoretical methods used to analyze single and combined structures. Next, an outline of all the different structures’ geometries and the experimental testing that they underwent is given. The following
chapters give the results of the static and dynamic analysis that was conducted on the single plane frame and ring structures. Experimental, FEA and theoretical results are included. In the next two chapters, the experimental and theoretical results for the parallel frames and nested rings are given. The paper concludes by discussing and interpreting the results found in the proceeding chapters and talking about the role that 3D printing played in experimental testing. Additionally, suggestions are offered about where the nested ring structures could be used in real world systems.
2.1 Stiffness of Structures

The stiffness of a structure describes its ability to resist deformation under applied loading [2]. Graphically, stiffness can be described as the slope of a force vs displacement curve as shown in Figure 2.1. As long as the slope of the force vs deflection curve is linear, called the elastic region, the stiffness of the structure will be constant and the structure will be able to regain its original geometry after undergoing elastic deformations [2]. However once the slope becomes non-linear the structure’s force vs deflection behavior will change. It will undergo non-linear elastic and/or plastic deformations. And in the case of plastic deformations, the structure will not regain its original geometry [3].

\[ F = k \delta \]

\( F \) \quad \text{Force}
\( \delta \) \quad \text{Displacement}
\( k \) \quad \text{Stiffness}

\textbf{Figure 2.1:} Force vs displacement curve showing the linear portion of the curve is proportional to stiffness.
For certain simple structures undergoing flexure, it is possible to approximate the stiffness using the scaling relation \[ (2.1) \]

\[
k \propto C \frac{EI}{L^3}
\]

where \( E \) is the elastic modulus, \( I \) is the moment of area, \( L \) is the length or height of the structure and \( C \) is a constant that is dependent upon geometry, loading and boundary conditions. Since the values for \( C \) and \( E \) are constant, it can be seen that the stiffness of a structure is dependent upon its geometry. For instance increasing the height of the structure will decrease its stiffness, while increasing the cross section dimensions, thus increasing \( I \), will increase the stiffness.

### 2.2 Vibration of Structures

Vibrations describe how a system oscillates about an equilibrium point while being influenced by restoring forces such as elasticity or gravity \[ [5] \]. The number of oscillations per time unit is called the frequency of the system \[ [5] \]. Frequency is a key component in modal analysis which allows for the response of a structure to be analyzed while it is being subjected to excitations \[ [5] \]. Each structure undergoes several

![Figure 2.2](image-url)

**Figure 2.2**: Example of a time series that is put through a FFT to create a plot showing the dimensionless amplitude, \( \bar{A} \), vs frequency, \( f \). The frequency for the system is labeled \( f \) on Figure 2.2(b).
different vibration modes while it is being oscillated. Since the first mode has the lowest frequency, it is often the most important frequency value during design. In order to determine the frequency of a vibrating structure, a time series, shown in Figure 2.2(a), providing the structure’s oscillations per time unit needs to be collected. This time series can then be put through a FFT, Figure 2.2(b), to get the frequency of the structure. A FFT converts data in the time domain into data in the frequency domain [5].

It is often convenient to look at the scaling of frequency to see what factors of the structure affect it. In general, the frequency of the system can be thought of as the relationship between mass, $m$, and stiffness, $k$, of the system [5].

$$f \propto \sqrt{\frac{k}{m}} \quad (2.2a)$$

substituting in $k = EI/L^3$ and $m = \rho AL$ gives the frequency of a structure in flexure

$$f \propto \frac{1}{L^2} \sqrt{\frac{EI}{\rho A}} \quad (2.2b)$$

where $L$ is the structure height, $E$ is the elastic modulus, $I$ is the moment of area, $\rho$ is the density of the structure’s material and $A$ is the cross sectional area. Equation 2.2(b) shows that the major influence on a structure’s frequency is its length. Additionally Equation 2.2(b) also shows that if a width of a structure is doubled, its frequency is not altered because the changes in cross section size end up canceling each other out.

2.3 Plane Frame Geometry

The two types of plane frames analyzed in this paper are portal frames and sway frames, which are a subset of portal frames.
2.3.1 Portal Frame

A portal frame, Figure 2.3(a), is a structure that consists of two vertical columns and a horizontal cross beam. The cross section dimensions of the beam are not significantly different than the cross section dimensions of the columns. One end of each column is fixed where the frame terminates into the support. So if the portal frame is subjected to any external forces, the top corners will experience horizontal and vertical translation as well as rotation while the fixed ends of the columns do not undergo any deformations.

2.3.2 Sway Frame

A sway frame, Figure 2.3(b), has a similar geometry to a portal frame except that the cross beam has a significantly larger cross sectional area than the columns. This allows for the cross beam to be treated as rigid. This rigid member essentially eliminates any vertical translation or rotation, meaning that the beam only deflects in the horizontal direction.

![Portal frame and Sway frame](image)

**Figure 2.3**: Renderings of the single story portal and sway frames analyzed. Deflections from bending are shown.
2.4 Theoretical Analysis Methods for Plane Frames

For some of the frames analyzed in this paper it was possible to calculate the stiffness and frequency using theoretical methods. The sections below outline the methods used.

2.4.1 Stiffness Matrix Method

In order to solve for the stiffness of the portal frame shown in Figure 2.3(a), the stiffness matrix method was used. The stiffness matrix allows for the displacement of certain nodes to be found by solving the system of equations [4, 6]

\[
\begin{bmatrix}
F
0
0
0
0
= \frac{EI}{L^3}
\end{bmatrix}
\begin{bmatrix}
\frac{AL^2}{I} + 12 & 0 & 6L & -\frac{AL^2}{I} & 0 & 0 \\
0 & \frac{AL^2}{I} + 12 & 6L & 0 & -12 & 6L \\
6L & 6L & 8L^2 & 0 & -6L & 2L^2 \\
-\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} + 12 & 0 & 6L \\
0 & 0 & -12 & -6L & 0 & \frac{AL^2}{I} + 12 -6L \\
0 & 0 & 6L & 2L^2 & 6L & -6L & 8L^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
x_2 \\
y_2 \\
x_2 \\
\theta_2
\end{bmatrix}
\]

(2.3)

Once the displacements have been found, they can be used along with the applied force to determine the stiffness of the structure. Appendix A provides more detail about the formulation of the global stiffness matrix and how to solve for deflections.

2.4.2 Eigenvalue Analysis

The theoretical frequency of a portal frame can be found by solving the generalized eigenvalue problem [5]

\[\|[K] - \lambda[M]\| = 0\]  

(2.4a)
where \([K]\) is the stiffness matrix, \(\lambda\) are the systems eigenvalues and \([M]\) is the mass matrix

\[
[M] = \frac{\rho A L}{420} \begin{bmatrix}
296 & 0 & 22L & 70 & 0 & 0 \\
0 & 296 & 22L & 0 & 54 & -13L \\
22L & 22L & 8L^2 & 0 & 13L & -3L^2 \\
70 & 0 & 0 & 296 & 0 & 22L \\
0 & 54 & 13L & 0 & 296 & -22L \\
0 & -13L & -3L^2 & 22L & -22L & 8L^2 \\
\end{bmatrix}
\] (2.4b)

Appendix A gives an overview on how to formulate the mass matrix and also an example of how the theoretical frequency of a portal frame was found.

### 2.4.3 Sway Frame Stiffness

Because of the geometry of the sway frame, the structure only has a single degree of freedom in the horizontal direction, this greatly simplifies the process of determining the theoretical stiffness. Assuming the cross sections of both columns are the same, the stiffness is given as \([7]\)

\[
k = 24 \frac{EI}{L^3}
\] (2.5)

### 2.4.4 Sway Frame Frequency

Like stiffness, the theoretical frequency of sway frames can be found from a single equation \([7]\)

\[
f_1 = \frac{1}{2\pi} \left[ \frac{12\Sigma(EI)_C}{L^3(M + 0.37\Sigma M_C)} \right]^{1/2}
\] (2.6)

Equation 2.6 allows for the frequency of frames with two plus columns to be calculated \([7]\).
2.5 Stiffness of Rings

2.5.1 Hardening vs Softening System

For the experimental analysis of ring stiffness, an attempt was made to keep applied loading and resulting deflections small to ensure that the force vs displacement relationship was linear. However, if larger loads were applied and the linear region was exited the direction of the loading (compressive vs tension) caused the ring to act as a hardening or softening spring [8]. If tensile loading is applied, Figure 2.4(b), to a ring the deflections will occur in the positive y direction and the ring will act as a hardening system. While if a compressive force, Figure 2.4(a), is applied deflection will occur in the negative y direction and the ring will act as a softening system. This will result in the ring having a higher stiffness values in tension than it does in compression.

Figure 2.4: Rendering of ring undergoing stiffness testing. Figure 2.4(a) shows when the ring is subjected to a compressive force, the ring acts as a softening system while Figure 2.4(b) shows the ring subjected to a tensile force causing it to act as a hardening spring.

Figure 2.5 shows a force vs deflection curve for a ring subjected to a positive (tension) and negative (compression) force. The plot shows that when positive forces are applied and the linear region has been exited, there is a disproportionate increase in slope of the line. And the opposite is true when negative forces
are applied; once the curve is no longer linear, the slope of the line progressively decreases.

![Figure 2.5: Force vs displacement curve for a ring subjected to positive (tension) and negative (compression) loading. Positive loading creates a hardening system while negative loading creates a softening system.](image)

**2.5.2 Theoretical Analysis**

A theoretical stiffness value for the stiffness of rings can be found from [9]

\[
 k = \left( \frac{32\pi}{\pi^2 - 8} \right) \frac{EI}{D^3} \quad (2.7)
\]

where \( E \) is the elastic modulus, \( I \) is the moment of area, \( D \) is the diameter and the term in the front of the equation factors in the circular geometry of the ring and loading. More information of the derivation of Equation 2.7 can be found in [9]. It should be noted that this equation has the same general form as Equation 2.1 except the value for \( C \).
2.6 Springs in Parallel

For a system with three springs in parallel, the equivalent stiffness is given by [7]

\[ k_{eq} = k_1 + k_2 + k_3 \]  

(2.8)

where \( k_1 \) is the first spring’s constant, \( k_2 \) is the second spring’s constant and \( k_3 \) is the third spring’s constant. A theoretical system for springs in parallel and its force vs displacement plot can be seen in Figure 2.6. Figure 2.6(b) shows a non-linear curve that consists of three linear regions. The slope of the curve changes twice, at \( \Delta_1 \) and \( \Delta_2 \). Once each new spring is contacted the equivalent stiffness is the sum of all the springs that are in contact with each other up until that point. This theoretical analysis will be used in the context of frames and rings.

![Figure 2.6: A theoretical system to represent a system with springs in parallel and its corresponding force vs deflection curve.](image)
3.1 CAD and Equipment

3.1.1 SolidWorks

All of the structures analyzed in this paper were modeled in SolidWorks. In addition the FEA simulation package in SolidWorks was used to validate the results that were obtained from experimentation. Appendix B gives an overview of the meshing procedure and the static and dynamic FEA simulations that were used.

3.1.2 Equipment

Once modeling was completed, the structures were created using a Fortus 250mc 3D printer. All structures were printed using the “solid interior” setting which created an object with the least amount of air pockets in its interior. The printer used Acrylonitrile Butadiene Styrene (ABS) plastic which has an Elastic modulus of $E \approx 2$ GPa. In reality the value of $E$ is usually given in a range of 1.9-2.1 GPa, this could lead to discrepencies between the experimental and FEA/theory results depending on the actual value of $E$ used to print the structures. ABS is a common plastic to use during 3D printing because it is mechanically strong, resistant to high temperatures
and also cost effective [1]. Its one major downside is that it is susceptible to warping
[1]. Using 3D printing was beneficial because it created an object that had almost
identical dimensions to the SolidWorks model. Plastic was layered down with a slice
height of 0.0070 which was the highest accuracy setting the printer offered [10].

Three main pieces of equipment were used during experimental testing. Deflec-
tions and time series of oscillations were measured using a Micro-Epsilon optoNCDT
laser which has a measuring rate of 750 Hz [11]. An acA630-750uc Basler ace digital
camera was used to capture the deflections of the rings. This camera has a resolution
of 640 px x 480 px and a frame rate of 751 fps [12]. Loading was applied to the rings
from an Omega DFG35 Force Gauge which had a maximum capacity of 50.00 N and
a resolution of 0.05 N [13].

3.2 Plane Frame Geometry

The following sections outline the geometry of the plane frames analyzed and also
list the testing that was performed on them. In addition, renderings of some of the
frames are given. The mass that is listed is the weight of the columns and cross
beams, the mass of the support is excluded.

3.2.1 Single Story Plane Frames

Table 3.1: Approximate dimensions, mass and analysis performed on the single story
plane frames.

<table>
<thead>
<tr>
<th>Frame</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>Z (mm)</th>
<th>b (mm)</th>
<th>mass (g)</th>
<th>Static</th>
<th>Dynamic</th>
</tr>
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<tr>
<td>A</td>
<td>2</td>
<td>50</td>
<td>50</td>
<td>20</td>
<td>6.84</td>
<td>✓</td>
<td>-</td>
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<td>B</td>
<td>2</td>
<td>100</td>
<td>50</td>
<td>20</td>
<td>11.12</td>
<td>✓</td>
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<td>150</td>
<td>50</td>
<td>20</td>
<td>22.88</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>D</td>
<td>8</td>
<td>50</td>
<td>50</td>
<td>20</td>
<td>11.31</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>E</td>
<td>8</td>
<td>100</td>
<td>50</td>
<td>20</td>
<td>15.47</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>F</td>
<td>8</td>
<td>150</td>
<td>50</td>
<td>20</td>
<td>19.67</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>G</td>
<td>2</td>
<td>100</td>
<td>50</td>
<td>10</td>
<td>6.18</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>L</td>
<td>2</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>11.75</td>
<td>-</td>
<td>✓</td>
</tr>
<tr>
<td>M</td>
<td>8</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>21.30</td>
<td>-</td>
<td>✓</td>
</tr>
<tr>
<td>O</td>
<td>2</td>
<td>100</td>
<td>100</td>
<td>10</td>
<td>6.45</td>
<td>-</td>
<td>✓</td>
</tr>
</tbody>
</table>
Table 3.1 lists the dimensions, mass and testing undertaken for all of the single story frames with two columns. Example geometries of the single story portal and sway frames analyzed are shown in Figure 3.1.

![Figure 3.1: Renderings of the single story portal and sway frames analyzed.](image)

Figure 3.1: Renderings of the single story portal and sway frames analyzed. Y corresponds to the column height, Z corresponds to the cross beam length, and X corresponds to the thickness of the cross beam. Frames have a 2 mm column thickness and 2 mm fillets.

### 3.2.2 Single Story Plane Frames - Additional Columns

![Figure 3.2: Renderings of the single story sway frames with additional columns analyzed.](image)

Figure 3.2: Renderings of the single story sway frames with additional columns analyzed. Y corresponds to the column height, Z corresponds to the cross beam length, and X corresponds to the thickness of the cross beam. Frames have a 2 mm column thickness and 2 mm fillets.

Example geometries of the single story sway frames analyzed are shown in Figure 3.2. Table 3.2 lists the dimensions, mass and testing undergone for all of the single
story frames with three plus columns. All frames with three plus columns were sway frames.

Table 3.2: Approximate dimensions, mass and analysis performed on single story plane frames with three plus columns.

<table>
<thead>
<tr>
<th>Frame</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>Z (mm)</th>
<th>b (mm)</th>
<th>columns</th>
<th>mass (g)</th>
<th>Static</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>8</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>3</td>
<td>25.82</td>
<td>-</td>
<td>✓</td>
</tr>
<tr>
<td>Q</td>
<td>8</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>6</td>
<td>37.33</td>
<td>-</td>
<td>✓</td>
</tr>
</tbody>
</table>

3.2.3 Two Story Plane Frames

![Two story frame renders](image)

**Figure 3.3**: Renderings of the two story portal and sway frames analyzed. Y corresponds to the column height, Z corresponds to the cross beam length, and X corresponds to the thickness of the cross beam. Frames have a 2 mm column thickness and 2 mm fillets. Figure 3.3(b) shows cross bracing in the top panel of the frame.

Example geometries of the two story frames analyzed are shown in Figure 3.3. Table 3.3 lists the dimensions, mass and testing undergone for all of the two story frames analyzed.
Table 3.3: Approximate dimensions, mass and analysis performed on two story plane frames.

<table>
<thead>
<tr>
<th>Frame</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>Z (mm)</th>
<th>b (mm)</th>
<th>cross bracing</th>
<th>mass (g)</th>
<th>Static</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>2</td>
<td>150</td>
<td>50</td>
<td>20</td>
<td>-</td>
<td>17.90</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>I</td>
<td>8</td>
<td>150</td>
<td>50</td>
<td>20</td>
<td>-</td>
<td>25.86</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>J</td>
<td>8</td>
<td>150</td>
<td>50</td>
<td>20</td>
<td>Bottom</td>
<td>29.06</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>K</td>
<td>8</td>
<td>150</td>
<td>50</td>
<td>20</td>
<td>Top</td>
<td>29.16</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>N</td>
<td>2</td>
<td>200</td>
<td>100</td>
<td>20</td>
<td>-</td>
<td>26.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.3 Ring Geometry

Table 3.4: Numbering, approximate dimensions, mass and analysis performed on rings.

<table>
<thead>
<tr>
<th>Ring</th>
<th>D (mm)</th>
<th>b (mm)</th>
<th>mass (g)</th>
<th>Static</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>240(_{10})</td>
<td>240</td>
<td>10</td>
<td>15.36</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>230(_{10})</td>
<td>230</td>
<td>10</td>
<td>14.77</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>220(_{10})</td>
<td>220</td>
<td>10</td>
<td>14.09</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>210(_{10})</td>
<td>210</td>
<td>10</td>
<td>13.44</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>200(_{10})</td>
<td>200</td>
<td>10</td>
<td>13.04</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>190(_{10})</td>
<td>190</td>
<td>10</td>
<td>12.22</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>180(_{10})</td>
<td>180</td>
<td>10</td>
<td>11.54</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>170(_{10})</td>
<td>170</td>
<td>10</td>
<td>10.89</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>160(_{10})</td>
<td>160</td>
<td>10</td>
<td>10.22</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>150(_{10})</td>
<td>150</td>
<td>10</td>
<td>9.63</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>140(_{10})</td>
<td>140</td>
<td>10</td>
<td>8.95</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>130(_{10})</td>
<td>130</td>
<td>10</td>
<td>8.33</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>120(_{10})</td>
<td>120</td>
<td>10</td>
<td>7.68</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>110(_{10})</td>
<td>110</td>
<td>10</td>
<td>7.04</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>100(_{10})</td>
<td>100</td>
<td>10</td>
<td>6.48</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>240(_{20})</td>
<td>240</td>
<td>20</td>
<td>30.80</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>230(_{20})</td>
<td>230</td>
<td>20</td>
<td>29.62</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>220(_{20})</td>
<td>220</td>
<td>20</td>
<td>28.25</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>210(_{20})</td>
<td>210</td>
<td>20</td>
<td>26.97</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>200(_{20})</td>
<td>200</td>
<td>20</td>
<td>25.71</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>190(_{20})</td>
<td>190</td>
<td>20</td>
<td>24.48</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>180(_{20})</td>
<td>180</td>
<td>20</td>
<td>23.13</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>170(_{20})</td>
<td>170</td>
<td>20</td>
<td>21.82</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>160(_{20})</td>
<td>160</td>
<td>20</td>
<td>20.49</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>150(_{20})</td>
<td>150</td>
<td>20</td>
<td>19.27</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>140(_{20})</td>
<td>140</td>
<td>20</td>
<td>17.94</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>130(_{20})</td>
<td>130</td>
<td>20</td>
<td>16.65</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>120(_{20})</td>
<td>120</td>
<td>20</td>
<td>15.37</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>110(_{20})</td>
<td>110</td>
<td>20</td>
<td>14.08</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>100(_{20})</td>
<td>100</td>
<td>20</td>
<td>12.84</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
Two sets of rings were analyzed, one set had a width of 10 mm the other set had a 20 mm width. The thickness of all the rings was 2 mm. Diameters of the rings were incremented by 10 mm and ranged from a maximum of 240 mm to a minimum of 100 mm. The diameter is the distance from center point to center point on the ring’s thickness. Table 3.4 lists the numbering system, approximate dimensions, mass and analysis performed on all the rings analyzed. Figure 3.4 provides a rendering of the analyzed rings.

3.4 Parallel Frames

The parallel frame system consisted of three identical frames with equal spacing ($\Delta_1 = \Delta_2$). Each frame was 10 mm wide and had an 8 mm thick cross beam, meaning the system underwent sway motion. Two parameters were varied in the systems, the column length $Y$ and the spacing between frames $\Delta$. Spacing between frames was varied by extruding the edges of the cross beam to decrease the gap. The rounded edges on the frames shown in Figure 3.5 were included to allow for slipping between connection points as loading increased. Table 3.5 shows the parameters in the three groupings analyzed.
Table 3.5: Groups analyzed for the frames in parallel system. $Y$ is column length and $\Delta$ is spacing.

<table>
<thead>
<tr>
<th>Group</th>
<th>$Y$ (mm)</th>
<th>$\Delta$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>7.10</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>14.20</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>7.10</td>
</tr>
</tbody>
</table>

3.5 Nested Rings

Each nested ring group consisted of three rings that all had the same width. The same rings that were used for individual testing were also used in nested testing. In the majority of the tests, the diameters of the rings were consecutive, giving a constant separation between rings of 6 mm. Four tests were run using every other ring diameter which gave a spacing of 16 mm. Parameters that were varied within the system were the ring diameter $D$, spacing between rings $\Delta$, and method of loading.

![Diagram of nested rings](image)

**Figure 3.6**: Rendering of nested system. Parameters varied in system were ring diameter $D$, spacing $\Delta_1$, $\Delta_2$, and the method of loading (compressive or tension).

The following table displays the nested systems that were analyzed. Included are the diameters of the three rings in the system, width of the rings, spacing between rings and the loading that was applied. Compressive loads were applied using a point load and flat bar.
Table 3.6: Nested ring group with diameters in nest, width of rings, spacing between rings and loading applied.

<table>
<thead>
<tr>
<th>Nest</th>
<th>Diameters in nest (mm)</th>
<th>b (mm)</th>
<th>∆ (mm)</th>
<th>Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>120 110</td>
<td>100</td>
<td>10</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>2</td>
<td>150 140</td>
<td>130</td>
<td>10</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>3</td>
<td>180 170</td>
<td>160</td>
<td>10</td>
<td>6 flat bar, point load, tension</td>
</tr>
<tr>
<td>4</td>
<td>210 200</td>
<td>190</td>
<td>10</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>5</td>
<td>240 230</td>
<td>220</td>
<td>10</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>6</td>
<td>170 150</td>
<td>130</td>
<td>10</td>
<td>16 flat bar, point load</td>
</tr>
<tr>
<td>7</td>
<td>120 110</td>
<td>100</td>
<td>20</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>8</td>
<td>150 140</td>
<td>130</td>
<td>20</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>9</td>
<td>180 170</td>
<td>160</td>
<td>20</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>10</td>
<td>210 200</td>
<td>190</td>
<td>20</td>
<td>6 flat bar, point load</td>
</tr>
<tr>
<td>11</td>
<td>240 230</td>
<td>220</td>
<td>10</td>
<td>6 flat bar, point load, tension</td>
</tr>
<tr>
<td>12</td>
<td>240 220</td>
<td>200</td>
<td>20</td>
<td>16 flat bar, point load</td>
</tr>
</tbody>
</table>

20
4.1 Experimental Procedure

Stiffness of the plane frames was found by applying load to the frame and then measuring the resulting deflections with a laser. Positive deflections were taken to be along the positive $x$ axis. Boundary conditions for the frames were fixed at one end and free at the other. Figure 4.1 shows the experimental set up for a single story sway frame.

\[ \text{Figure 4.1: Rendering of experimental set up to test stiffness of plane frames. Force } F \text{ was applied at the corner of the frame which resulted in the frame deflecting distance } \delta. \]
Loading was applied using a force control technique by hanging slotted weights at one of the corners of the frame. The initial distance from the laser to the frame served as a datum. As loading was incremented the resulting displacements were measured, the difference was then taken between these displacements and the datum to see what the deflection was from the frames unloaded position. Displacements were collected for the loading and unloading of the slotted weights. The measured deflection was then plotted against its corresponding force to create a force vs deflection plot. This data was then fitted with a trend line which gave the experimental stiffness of the frame.

4.2 Results

The results for the plane frame stiffness will be broken up into single and two story frames.

4.2.1 Single Story Plane Frames

Table 4.1: Stiffness values for single story frames from various analysis methods. Stiffness values were found using Equation 2.3 while Sway values were found using Equation 2.5. Dimensions for the frames can be found in Table 3.1.

<table>
<thead>
<tr>
<th>Frame</th>
<th>EXP</th>
<th>FEA</th>
<th>Sway</th>
<th>Stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.12</td>
<td>4.11</td>
<td>-</td>
<td>3.58</td>
</tr>
<tr>
<td>B</td>
<td>0.63</td>
<td>0.58</td>
<td>-</td>
<td>0.52</td>
</tr>
<tr>
<td>C</td>
<td>0.21</td>
<td>0.18</td>
<td>-</td>
<td>0.16</td>
</tr>
<tr>
<td>D</td>
<td>6.76</td>
<td>5.80</td>
<td>5.12</td>
<td>5.07</td>
</tr>
<tr>
<td>E</td>
<td>0.73</td>
<td>0.69</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>F</td>
<td>0.23</td>
<td>0.20</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>G</td>
<td>0.31</td>
<td>0.28</td>
<td>-</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 4.1 shows the stiffness values obtained from experimental, FEA simulation and theoretical analysis. Frame B will serve as a baseline to compare the effect that changes to frame geometry have on stiffness. Figure 4.2 demonstrates the effect of
width on stiffness by comparing the slopes of Frame B and Frame G. Additionally, Figure 4.3 shows the deflected shape of Frame B.

![Figure 4.2: Force vs deflection plot of Frame B vs Frame G to compare the effect of width on stiffness.](image1)

![Figure 4.3: Deflected Frame B.](image2)

The plot shows that doubling the width of the frame results in the stiffness for the frame also being doubled. This confirms the scaling relationship in Equation 2.1. If the width, \( b \), is doubled then \( I \) will increase by a factor of 2 since \( I = (1/12)bd^3 \), meaning that there is a linear relationship between the width of a plane frame and its stiffness.
Figure 4.4: Force vs deflection plots to compare the effect of column length and cross beam depth on stiffness. (a) Shows portal frames with column lengths Y=50 mm, 100 mm, 150 mm. (b) Shows sway frames with column lengths Y=50 mm, 100 mm, 150 mm.
Plots in Figure 4.4 were used to examine the effect of column length and cross beam depth on stiffness. Figure 4.4(a) plots data for portal frames with column lengths of $Y=50$ mm, 100 mm, 150 mm while Figure 4.4(b) plots sway frames with column lengths of $Y=50$ mm, 100 mm, 150 mm. As expected, these plots show that an increase in column length makes the frame less stiff. Figure 4.4 also shows that if the column length is fixed, a sway frame will be stiffer than its portal frame counterpart. This is due to the fact that the sway frames have a cross beam with a larger $I$ value which will increase their overall stiffness inhibiting rotation at the corners [4]. However the plots show that this additional stiffness from a thicker cross beam decreases with an increasing column length. Frame A and Frame D have column lengths of 50 mm and there is a noticeable difference in their stiffness values. However Frame C and Frame F have column lengths of 150 mm and there is only a slight difference in their stiffness. This shows that length is the dominant factor in stiffness.

Table 4.2 provides a comparison of the effect of geometry changes compared to Frame B as well as the difference in the results obtained from experimental, FEA and theory analysis.

Table 4.2: Comparison of geometry changes on single story frame stiffness and difference in analysis methods [4].

<table>
<thead>
<tr>
<th>Frame</th>
<th>$(k/k_b)_{EXP}$</th>
<th>$(k/k_b)_{FEA}$</th>
<th>$(k/k_b)_{THEO}$</th>
<th>$k_{EXP}/k_{FEA}$</th>
<th>$k_{THEO}/k_{FEA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6.55</td>
<td>7.11</td>
<td>6.89</td>
<td>1.00</td>
<td>1.15</td>
</tr>
<tr>
<td>B</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>C</td>
<td>0.34</td>
<td>0.31</td>
<td>0.31</td>
<td>1.20</td>
<td>1.31</td>
</tr>
<tr>
<td>D</td>
<td>10.73</td>
<td>10.05</td>
<td>9.76</td>
<td>1.17</td>
<td>1.33</td>
</tr>
<tr>
<td>E</td>
<td>1.16</td>
<td>1.19</td>
<td>1.23</td>
<td>1.06</td>
<td>1.15</td>
</tr>
<tr>
<td>F</td>
<td>0.36</td>
<td>0.35</td>
<td>0.36</td>
<td>1.15</td>
<td>1.22</td>
</tr>
<tr>
<td>G</td>
<td>0.48</td>
<td>0.48</td>
<td>0.50</td>
<td>1.10</td>
<td>1.18</td>
</tr>
</tbody>
</table>

In comparing the sway frames, it can be seen that Frame F is $0.3 \ (1.5^{-3})$ times
the stiffness of Frame E and the stiffness of Frame D is $8 \times (0.5^{-3})$ times the stiffness of Frame E [4]. These results hold true with scaling in Equation 2.1 which shows that the stiffness of a frame goes with $L^{-3}$. A comparison of the FEA and theoretical results show that the FEA matches up better with the experimental results than theory. One explanation for this is because the 3D printed model was the file that was used in SolidWorks FEA simulation so the same structure was being analyzed while the theoretical analysis was not based on any model, just a theoretical frame.

The final piece of analysis that was done was a non-dimensionalization of the stiffness data for the single story sway frame. Figure 4.5 shows the non-dimensionalized force ($FY^2/EI$) vs the non-dimensionalized deflection ($\delta/Y$).

![Figure 4.5: Non-dimensionalized force vs deflection plot of single story sway frames. $\bar{k}$ is the non-dimensionalized slope.](image)

It can be seen that the slope of the line through the non-dimensionalized data is
$k = 28.78$. This value is slightly higher than the coefficient in Equation 2.5 which states that the slope should be 24, showing that there is a good match between experimental and theory analysis.

4.2.2 Two Story Plane Frames

A summary of the stiffness values for the two story plane frames obtained from experimental, FEA and theory analysis are shown in Table 4.3. Figure 4.6 shows the deflected shapes for the two story frames while Figure 4.7 shows the force vs displacement data collected.

Table 4.3: Summary and comparison of stiffness data for two story frames [4]. Stiffness values were found using Equation 2.3 while Sway values were found using Equation 2.5. Dimensions for the frames can be found in Table 3.3.

<table>
<thead>
<tr>
<th>Frame</th>
<th>EXP</th>
<th>FEA</th>
<th>Sway</th>
<th>Stiffness</th>
<th>$k_{EXP}/k_{FEA}$</th>
<th>$k_{EXP}/k_{THEO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>0.54</td>
<td>0.52</td>
<td>-</td>
<td>-</td>
<td>1.05</td>
<td>-</td>
</tr>
<tr>
<td>$I$</td>
<td>0.96</td>
<td>0.96</td>
<td>-</td>
<td>1.14</td>
<td>1.00</td>
<td>0.84</td>
</tr>
<tr>
<td>$J$</td>
<td>1.88</td>
<td>1.86</td>
<td>1.52</td>
<td>-</td>
<td>1.01</td>
<td>1.24</td>
</tr>
<tr>
<td>$K$</td>
<td>1.79</td>
<td>1.91</td>
<td>-</td>
<td>-</td>
<td>0.94</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 4.6: Deflected shapes for two story plane frames. (a) Shows the portal frame undergoing rotation at the connection of the cross beam and columns. (b) Shows that the top panel deflects as one because it is rigid.
Figure 4.7: Force vs deflection plots to compare the effect of cross beam depth and cross bracing on stiffness. (a) Shows two story frames with length $Y=150$ mm and cross beam depth of $X = 2$ and 8 mm. (b) Shows two story sway frame with cross bracing in the top and bottom panel. Cross bracing is 2 mm thick.
Stiffness values found experimentally match up well with the FEA results. A comparison of Frame J and Frame K show that placing cross bracing in either the top or bottom panel of the two story frame resulted in frames with similar stiffness values. This can also be seen by looking at the slopes of the lines in Figure 4.7(b). Additionally Figure 4.7(a) reiterates the result that was found for the single story frame, cross beam depth does increase the stiffness of the frame.
5.1 Experimental Procedure

Frequency values for the plane frames were obtained by oscillating the frames and collecting a time series of their oscillations with a laser. Figure 5.1 shows the oscillation trace of a single story sway frame. This laser was able to capture a location at an increment of 0.00133 seconds. This time series was then put through a FFT to

Figure 5.1: Rendering of experimental set up to test frequency of plane frames. All oscillations occurred along the $x$ axis. Oscillation was considered to be positive if it was along the positive $x$ axis.
obtain the frequency value(s) for the frames. The FFT algorithm was implemented in MATLAB and was adapted from [14]. Results of the FFT were a plot that gave the dimensionless amplitude, $\bar{A}$, of the frame oscillations vs the frequency, $f$, of the frame in Hertz. For the single story frames, the only vibration mode of interest was when the cross beam translated along the $x$ axis. In some of the testing for the two story frames, two modes of vibration were captured through the FFT. Mode 1 was taken to be when both the cross beams were oscillating in phase while Mode 2 was when the cross beams were oscillating out of phase. Boundary conditions were the same as they were during static testing, fixed at the bottom and free at the top.

5.2 Results

Results will be given for single story frames, single story frames with additional columns and two story frames.

5.2.1 Single Story Plane Frames

Figure 5.2 shows the time series and FFT for Frame B.

![Figure 5.2: Results of frequency analysis of Frame B. (a) Time series showing exponential decay. (b) In plane frequency for Frame B of $f = 54.28$ Hz.](image)
It can be seen from looking at the time series in Figure 5.2(a) that the frame oscillates in a smooth, exponentially decaying manner due to the elasticity of the ABS plastic. In this study damping is small, so it may be ignored. Oscillations will continue until the frame returns to its at rest position. Table 5.1 shows the experimental, FEA and theoretical frequency values that were found for the single story plane frames while Table 5.2 shows a comparison of geometric variations and experimental, FEA and theory results. Frame B will serve as the baseline geometry to gauge what effect geometric changes have on frequency.

Table 5.1: Frequency values for single story frames from various analysis methods. Eigenvalue values were found using Equation 2.4a while Sway values were found using Equation 2.6. Dimensions for the frames can be found in Table 3.1.

<table>
<thead>
<tr>
<th>Frame</th>
<th>EXP</th>
<th>FEA</th>
<th>Sway</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>54.28</td>
<td>50.85</td>
<td>-</td>
<td>52.11</td>
</tr>
<tr>
<td>C</td>
<td>26.93</td>
<td>24.96</td>
<td>-</td>
<td>25.53</td>
</tr>
<tr>
<td>E</td>
<td>43.60</td>
<td>37.69</td>
<td>38.08</td>
<td>37.72</td>
</tr>
<tr>
<td>F</td>
<td>22.28</td>
<td>19.13</td>
<td>19.45</td>
<td>19.28</td>
</tr>
<tr>
<td>G</td>
<td>52.61</td>
<td>50.34</td>
<td>-</td>
<td>52.11</td>
</tr>
<tr>
<td>L</td>
<td>41.33</td>
<td>40.25</td>
<td>-</td>
<td>41.03</td>
</tr>
<tr>
<td>M</td>
<td>32.02</td>
<td>29.06</td>
<td>28.95</td>
<td>28.64</td>
</tr>
<tr>
<td>O</td>
<td>39.64</td>
<td>39.94</td>
<td>-</td>
<td>41.03</td>
</tr>
</tbody>
</table>

Table 5.2: Comparison of geometric changes to frequency and experimental, FEA and theoretical results.

<table>
<thead>
<tr>
<th>Frame</th>
<th>( \frac{f}{f_b} ) EXP</th>
<th>( \frac{f}{f_b} ) FEA</th>
<th>( \frac{f}{f_b} ) THEO</th>
<th>( \frac{f_{\text{EXP}}}{f_{\text{FEA}}} )</th>
<th>( \frac{f_{\text{THEO}}}{f_{\text{FEA}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.07</td>
<td>1.04</td>
</tr>
<tr>
<td>C</td>
<td>0.50</td>
<td>0.49</td>
<td>0.49</td>
<td>1.08</td>
<td>1.05</td>
</tr>
<tr>
<td>E</td>
<td>0.80</td>
<td>0.74</td>
<td>0.72</td>
<td>1.16</td>
<td>1.16</td>
</tr>
<tr>
<td>F</td>
<td>0.41</td>
<td>0.38</td>
<td>0.37</td>
<td>1.16</td>
<td>1.16</td>
</tr>
<tr>
<td>G</td>
<td>0.97</td>
<td>0.99</td>
<td>1.00</td>
<td>1.05</td>
<td>1.01</td>
</tr>
<tr>
<td>L</td>
<td>0.76</td>
<td>0.79</td>
<td>0.79</td>
<td>1.03</td>
<td>1.01</td>
</tr>
<tr>
<td>M</td>
<td>0.59</td>
<td>0.57</td>
<td>0.55</td>
<td>1.10</td>
<td>1.12</td>
</tr>
<tr>
<td>O</td>
<td>0.73</td>
<td>0.79</td>
<td>0.79</td>
<td>0.99</td>
<td>0.97</td>
</tr>
</tbody>
</table>

An attempt was made to measure the frequency of the frames with column length of 50 mm, however the frequency was too high and the time series could not be
properly captured by the laser. Instead Frames L, M and O were used to look at the frequency of a frame whose cross beam length is the same length as its columns, meaning that the beam and columns both had a length of 100 mm. A comparison of the frequencies of Frame B vs Frame G and Frame L vs Frame O shows that increasing the width of a structure does not increase its frequency. The scaling relation in Equation 2.2b suggests this result, the increase in width is canceled out because of the $I$ term in the numerator and the $A$ term in the denominator. Table 5.2 also shows that the FEA and theory results match up well with the experimental results. Another result is that sway frames have a lower frequency than a portal frame with the same dimensions. This is because the mass of the cross beam is larger which causes the frame to oscillate slower.

Figure 5.3 shows the mode shapes from the SolidWorks FEA simulation for the single story portal and sway frame. A comparison of the two figures shows that the two frames have similar mode shapes. Like the deflected shape found from stiffness testing, the only difference between the two is that the portal frame undergoes rotation at the corners while the sway frame beam is rigid.

![Mode shapes for single story plane frames. (a) Portal frame mode shape. (b) Sway frame mode shape.](image)
5.2.2 Single Story Frames - Additional Columns

Figure 5.4 shows the time series and FFT for Frame Q. The time series shows that the frame has very small amplitudes of oscillations due to the additional columns and initial conditions.

![Figure 5.4: Results of frequency analysis of Frame Q. (a) Time series showing exponential decay. (b) In plane frequency for Frame Q of \( f = 50.42 \) Hz.](image)

An exponential decay is still present, however it is not as obvious because the amplitude of oscillations are so small. Table 5.3 shows the experimental, FEA and theoretical frequency values that were found for the single story plane frames and compares analysis methods.

Table 5.3: Frequency values for single story frames with additional columns from various analysis methods. Sway values were found using Equation 2.6. Dimensions for the frames can be found in Table 3.1.

<table>
<thead>
<tr>
<th>Frame</th>
<th>Frequency (Hz)</th>
<th>Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EXP</td>
<td>FEA</td>
</tr>
<tr>
<td>( P )</td>
<td>39.24</td>
<td>34.44</td>
</tr>
<tr>
<td>( Q )</td>
<td>50.42</td>
<td>44.34</td>
</tr>
</tbody>
</table>

Additional columns increase the frequency of the frame because the added columns contribute more to an increase in stiffness than they increase the mass [4]. It can also be seen from the comparison of the frequencies for Frame \( M \) (Table 5.1) and Frame \( P \) that the added column increases frequency of the frame by 20% [4].
Figure 5.5 shows the mode shape for Frame Q resembles the mode shape of the sway frame shown in Figure 5.3(b) except it has additional columns.

5.2.3 Two Story Plane Frames

Figure 5.6 shows the time series and FFT for Frame N. The time series in Figure 5.6(a) show that the oscillation peaks do not decay in an exponential manner like they did for the single story plane frames. It appears as though two consecutive oscillations have the same amplitude, then there is a decay. This is due to multiple modes of vibration participating during oscillations. Figure 5.6(b) shows that the FFT captures the second mode, which has a higher value than the first mode and
smaller amplitude. Table 5.4 shows the experimental, FEA and theoretical results for the two story plane frames while Table 5.5 compares the experimental results with FEA and theory.

Table 5.4: Frequency values for two story frames from various analysis methods. See Table 3.3 for frame dimensions. Eigenvalue results were found using Equation 2.4a.

<table>
<thead>
<tr>
<th>Frame</th>
<th>Mode 1 EXP</th>
<th>Mode 1 FEA</th>
<th>Mode 1 Eigenvalue</th>
<th>Mode 2 EXP</th>
<th>Mode 2 FEA</th>
<th>Mode 2 Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>41.82</td>
<td>40.61</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>I</td>
<td>43.68</td>
<td>38.09</td>
<td>42.41</td>
<td>121.39</td>
<td>101.48</td>
<td>111.04</td>
</tr>
<tr>
<td>J</td>
<td>70.77</td>
<td>63.70</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>K</td>
<td>44.04</td>
<td>40.30</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>N (top)</td>
<td>20.87</td>
<td>19.11</td>
<td>-</td>
<td>68.51</td>
<td>62.07</td>
<td>-</td>
</tr>
<tr>
<td>N (bottom)</td>
<td>20.55</td>
<td>19.11</td>
<td>-</td>
<td>68.31</td>
<td>62.07</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.5: Comparison of frequency values for two story frames from various analysis methods.

<table>
<thead>
<tr>
<th>Frame</th>
<th>Mode 1 (f_{\text{EXP}}/f_{\text{FEA}})</th>
<th>Mode 1 (f_{\text{EXP}}/f_{\text{THEO}})</th>
<th>Mode 2 (f_{\text{EXP}}/f_{\text{FEA}})</th>
<th>Mode 2 (f_{\text{EXP}}/f_{\text{THEO}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1.03</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>I</td>
<td>1.15</td>
<td>1.03</td>
<td>1.20</td>
<td>1.09</td>
</tr>
<tr>
<td>J</td>
<td>1.11</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>K</td>
<td>1.09</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>N (top)</td>
<td>1.09</td>
<td>-</td>
<td>1.10</td>
<td>-</td>
</tr>
<tr>
<td>N (bottom)</td>
<td>1.08</td>
<td>-</td>
<td>1.10</td>
<td>-</td>
</tr>
</tbody>
</table>

Frame N was tested twice to see if there would be any difference in frequency from causing oscillations to start from the middle cross beam versus starting from the top cross beam. The results show however that this made no difference. Comparing the effect of cross bracing on frequency shows that placing cross bracing in the bottom panel of a two story frame (Frame J) makes the frequency much higher than if the cross bracing is in the top panel (Frame K). This is because the panel that has cross bracing is going to have a higher effective mass which will influence frequency. So if cross bracing is in the top panel, oscillations will be longer lowering the frequency.
Table 5.5 shows that there is good correlation between the experimental, FEA and theory results. Figure 5.7 displays the first and second mode shapes for a two story portal frame.

![Mode shapes for Frame N.](image)

(a) Mode shape 1

(b) Mode shape 2

**Figure 5.7:** Mode shapes for *Frame N*. (a) Mode shape 1 has both cross beams going in the same direction along the $x$ axis during oscillations. (b) Mode shape 2 has the cross beams going in opposite directions along the $x$ axis during oscillation.
6.1 Experimental Procedure

Experimental testing of the ring’s stiffness was done following the same basic procedure as was used to determine the plane frame stiffness, load the ring and measure the resulting deflections. Figure 6.1 shows the rendering of the experimental set up. Compressive forces caused deflections down the $y$ axis while the tensile load caused deflections in the opposite direction.

![Diagram](https://via.placeholder.com/150)

**Figure 6.1:** Rendering of experimental set up to determine stiffness of rings. (a) Compressive point load was applied causing negative deflections. (b) Tensile point load was applied causing positive deflections.
caused the ring to displace up the y axis and were taken to be positive deflections. Loading was applied by using a force gauge which served as a displacement control method.

Deflections of the ring were found by taking a picture of the deflected ring after loading increment was applied and then was put through an image processing program in MATLAB to determine how much vertical displacement it underwent from its initial position. This program also plotted the applied load vs the resulting deflection and fit a trendline through the data to give the stiffness of the ring. Appendix D gives an example of the MATLAB program for Ring 24010.

Boundary conditions during testing had the ring fixed along a thin line over its width and free everywhere else. Before testing began it was believed that the boundary conditions at the fixed location of the ring would change as more compressive load was applied. The thought was that as more loading was applied a larger portion of the bottom of the ring would contact the table so it would switch from essentially a point to a flat surface. To try to duplicate this “flattening out,” testing was also conducted with a flat bar compressive load in addition to the point load. However for the levels of deflection required during testing, the ring did not undergo enough deflections on its bottom surface to flatten out and the boundary condition stayed fixed along one position, as a result the stiffness value for the ring in compression from a flat bar and point load were identical. The results presented in this chapter are for the point load in compression and tension. Results of the flat bar testing can be seen in Appendix C.

6.2 Results

Tables 6.1 and 6.2 list the stiffness values for the rings from experimental, FEA and theoretical analysis. Figure 6.2 displays examples of the dimensional and non-dimensional force vs deflection plots created to determine the ring stiffness.
Table 6.1: Stiffness (N/mm) of rings tested from experimental, FEA and theoretical analysis. Experimental and FEA analysis loads were applied using a point load. Dimensions for the rings can be seen in Table 3.4. Theoretical results were found using Equation 2.7.

<table>
<thead>
<tr>
<th>Ring</th>
<th>EXP</th>
<th>FEA</th>
<th>THEO</th>
<th>( k_{\text{EXP}} / k_{\text{FEA}} )</th>
<th>( k_{\text{EXP}} / k_{\text{THEO}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>240(_{10})</td>
<td>0.06</td>
<td>0.05</td>
<td>0.05</td>
<td>1.19</td>
<td>1.23</td>
</tr>
<tr>
<td>230(_{10})</td>
<td>0.07</td>
<td>0.06</td>
<td>0.06</td>
<td>1.13</td>
<td>1.14</td>
</tr>
<tr>
<td>220(_{10})</td>
<td>0.08</td>
<td>0.07</td>
<td>0.07</td>
<td>0.16</td>
<td>1.18</td>
</tr>
<tr>
<td>210(_{10})</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
<td>1.04</td>
<td>1.03</td>
</tr>
<tr>
<td>200(_{10})</td>
<td>0.10</td>
<td>0.09</td>
<td>0.09</td>
<td>1.10</td>
<td>1.13</td>
</tr>
<tr>
<td>190(_{10})</td>
<td>0.11</td>
<td>0.11</td>
<td>0.10</td>
<td>1.09</td>
<td>1.10</td>
</tr>
<tr>
<td>180(_{10})</td>
<td>0.14</td>
<td>0.12</td>
<td>0.12</td>
<td>1.14</td>
<td>1.16</td>
</tr>
<tr>
<td>170(_{10})</td>
<td>0.18</td>
<td>0.15</td>
<td>0.15</td>
<td>1.17</td>
<td>1.22</td>
</tr>
<tr>
<td>160(_{10})</td>
<td>0.20</td>
<td>0.18</td>
<td>0.18</td>
<td>1.14</td>
<td>1.16</td>
</tr>
<tr>
<td>150(_{10})</td>
<td>0.24</td>
<td>0.22</td>
<td>0.21</td>
<td>1.12</td>
<td>1.15</td>
</tr>
<tr>
<td>140(_{10})</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.97</td>
<td>0.99</td>
</tr>
<tr>
<td>130(_{10})</td>
<td>0.32</td>
<td>0.33</td>
<td>0.33</td>
<td>0.97</td>
<td>0.98</td>
</tr>
<tr>
<td>120(_{10})</td>
<td>0.38</td>
<td>0.42</td>
<td>0.41</td>
<td>0.91</td>
<td>0.92</td>
</tr>
<tr>
<td>110(_{10})</td>
<td>0.50</td>
<td>0.55</td>
<td>0.54</td>
<td>0.92</td>
<td>0.94</td>
</tr>
<tr>
<td>100(_{10})</td>
<td>0.91</td>
<td>0.73</td>
<td>0.72</td>
<td>1.25</td>
<td>1.27</td>
</tr>
<tr>
<td>240(_{20})</td>
<td>0.12</td>
<td>0.11</td>
<td>0.10</td>
<td>1.14</td>
<td>1.16</td>
</tr>
<tr>
<td>230(_{20})</td>
<td>0.14</td>
<td>0.12</td>
<td>0.12</td>
<td>1.17</td>
<td>1.19</td>
</tr>
<tr>
<td>220(_{20})</td>
<td>0.17</td>
<td>0.14</td>
<td>0.13</td>
<td>1.22</td>
<td>1.25</td>
</tr>
<tr>
<td>210(_{20})</td>
<td>0.21</td>
<td>0.16</td>
<td>0.15</td>
<td>1.32</td>
<td>1.33</td>
</tr>
<tr>
<td>200(_{20})</td>
<td>0.20</td>
<td>0.18</td>
<td>0.18</td>
<td>1.12</td>
<td>1.14</td>
</tr>
<tr>
<td>190(_{20})</td>
<td>0.25</td>
<td>0.22</td>
<td>0.21</td>
<td>1.14</td>
<td>1.17</td>
</tr>
<tr>
<td>180(_{20})</td>
<td>0.29</td>
<td>0.25</td>
<td>0.25</td>
<td>1.17</td>
<td>1.18</td>
</tr>
<tr>
<td>170(_{20})</td>
<td>0.31</td>
<td>0.29</td>
<td>0.29</td>
<td>1.06</td>
<td>1.07</td>
</tr>
<tr>
<td>160(_{20})</td>
<td>0.41</td>
<td>0.35</td>
<td>0.35</td>
<td>1.16</td>
<td>1.17</td>
</tr>
<tr>
<td>150(_{20})</td>
<td>0.49</td>
<td>0.43</td>
<td>0.42</td>
<td>1.15</td>
<td>1.16</td>
</tr>
<tr>
<td>140(_{20})</td>
<td>0.60</td>
<td>0.55</td>
<td>0.52</td>
<td>1.10</td>
<td>1.15</td>
</tr>
<tr>
<td>130(_{20})</td>
<td>0.77</td>
<td>0.69</td>
<td>0.65</td>
<td>1.11</td>
<td>1.18</td>
</tr>
<tr>
<td>120(_{20})</td>
<td>0.98</td>
<td>0.86</td>
<td>0.83</td>
<td>1.15</td>
<td>1.19</td>
</tr>
<tr>
<td>110(_{20})</td>
<td>1.11</td>
<td>1.11</td>
<td>1.08</td>
<td>0.99</td>
<td>1.03</td>
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<tr>
<td>100(_{20})</td>
<td>1.83</td>
<td>1.45</td>
<td>1.43</td>
<td>1.26</td>
<td>1.28</td>
</tr>
</tbody>
</table>
Table 6.2: Stiffness (N/mm) of rings tested from experimental, FEA and theoretical analysis. Experimental and FEA analysis loads were applied using a tensile load. Dimensions for the rings can be seen in Table 3.4. Theoretical results were found using Equation 2.7.

<table>
<thead>
<tr>
<th>Ring</th>
<th>EXP</th>
<th>FEA</th>
<th>THEO</th>
<th>$k_{\text{EXP}}/k_{\text{FEA}}$</th>
<th>$k_{\text{EXP}}/k_{\text{THEO}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>240</td>
<td>0.08</td>
<td>0.05</td>
<td>0.05</td>
<td>1.55</td>
<td>1.59</td>
</tr>
<tr>
<td>230</td>
<td>0.09</td>
<td>0.06</td>
<td>0.06</td>
<td>1.58</td>
<td>1.60</td>
</tr>
<tr>
<td>220</td>
<td>0.12</td>
<td>0.07</td>
<td>0.07</td>
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<td>1.73</td>
</tr>
<tr>
<td>210</td>
<td>0.13</td>
<td>0.08</td>
<td>0.08</td>
<td>1.68</td>
<td>1.66</td>
</tr>
<tr>
<td>200</td>
<td>0.14</td>
<td>0.09</td>
<td>0.09</td>
<td>1.56</td>
<td>1.61</td>
</tr>
<tr>
<td>190</td>
<td>0.17</td>
<td>0.11</td>
<td>0.10</td>
<td>1.64</td>
<td>1.65</td>
</tr>
<tr>
<td>180</td>
<td>0.17</td>
<td>0.12</td>
<td>0.12</td>
<td>1.33</td>
<td>1.35</td>
</tr>
<tr>
<td>170</td>
<td>0.23</td>
<td>0.15</td>
<td>0.15</td>
<td>1.53</td>
<td>1.58</td>
</tr>
<tr>
<td>160</td>
<td>0.35</td>
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<td>0.18</td>
<td>1.95</td>
<td>1.98</td>
</tr>
<tr>
<td>150</td>
<td>0.37</td>
<td>0.22</td>
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<td>1.76</td>
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<td>0.26</td>
<td>1.41</td>
<td>1.43</td>
</tr>
<tr>
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<td>0.33</td>
<td>0.33</td>
<td>0.96</td>
<td>0.97</td>
</tr>
<tr>
<td>120</td>
<td>0.51</td>
<td>0.42</td>
<td>0.41</td>
<td>1.22</td>
<td>1.24</td>
</tr>
<tr>
<td>110</td>
<td>0.80</td>
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<td>1.48</td>
<td>1.49</td>
</tr>
<tr>
<td>100</td>
<td>1.19</td>
<td>0.73</td>
<td>0.72</td>
<td>1.63</td>
<td>1.66</td>
</tr>
<tr>
<td>240</td>
<td>0.17</td>
<td>0.11</td>
<td>0.10</td>
<td>1.60</td>
<td>1.62</td>
</tr>
<tr>
<td>230</td>
<td>0.18</td>
<td>0.12</td>
<td>0.12</td>
<td>1.54</td>
<td>1.57</td>
</tr>
<tr>
<td>220</td>
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<td>0.14</td>
<td>0.13</td>
<td>1.72</td>
<td>1.76</td>
</tr>
<tr>
<td>210</td>
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<td>0.16</td>
<td>0.15</td>
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<td>1.66</td>
</tr>
<tr>
<td>200</td>
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<td>0.18</td>
<td>0.18</td>
<td>1.65</td>
<td>1.68</td>
</tr>
<tr>
<td>190</td>
<td>0.36</td>
<td>0.22</td>
<td>0.21</td>
<td>1.67</td>
<td>1.72</td>
</tr>
<tr>
<td>180</td>
<td>0.49</td>
<td>0.25</td>
<td>0.25</td>
<td>1.98</td>
<td>2.00</td>
</tr>
<tr>
<td>170</td>
<td>0.54</td>
<td>0.29</td>
<td>0.29</td>
<td>1.83</td>
<td>1.85</td>
</tr>
<tr>
<td>160</td>
<td>0.66</td>
<td>0.35</td>
<td>0.35</td>
<td>1.88</td>
<td>1.90</td>
</tr>
<tr>
<td>150</td>
<td>0.68</td>
<td>0.43</td>
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<td>1.57</td>
<td>1.59</td>
</tr>
<tr>
<td>140</td>
<td>0.96</td>
<td>0.55</td>
<td>0.52</td>
<td>1.75</td>
<td>1.83</td>
</tr>
<tr>
<td>130</td>
<td>1.34</td>
<td>0.69</td>
<td>0.65</td>
<td>1.97</td>
<td>2.05</td>
</tr>
<tr>
<td>120</td>
<td>1.67</td>
<td>0.86</td>
<td>0.83</td>
<td>1.95</td>
<td>2.02</td>
</tr>
<tr>
<td>110</td>
<td>1.82</td>
<td>1.11</td>
<td>1.08</td>
<td>1.63</td>
<td>1.69</td>
</tr>
<tr>
<td>100</td>
<td>2.32</td>
<td>1.45</td>
<td>1.43</td>
<td>1.60</td>
<td>1.62</td>
</tr>
</tbody>
</table>
Figure 6.2: Force vs deflection curves for 10 mm wide rings under point load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of $\bar{k} = 56.63$. 

(a) Dimensional plot

(b) Non-Dimensional plot
Tables 6.1 and 6.2 show that there is a significant difference between the stiffness found from applying a compressive force versus the stiffness found from applying a tension force, some of the stiffness values for tension are double compressive stiffness values. There are two potential reasons for this result. The first is the fact that the ring can act as a hardening or softening system outlined in Section 2.5.1. When the ring is in tension it flattens. After the initial force vs deflection plots were created, they were reviewed and truncated if the points became non-linear at the end. Even after linearization the slope values for tension were still higher. Another potential reason for this discrepancy was the part that was used to apply the tensile loads to the ring. This part, the blue object shown in Figure 6.4(a), was a cantilever to hook on to the ring and was made of ABS plastic. And while the part was noticeably stiffer than the rings, it still acted as a spring and would deflect as the ring was pulled up. This problem did not arise when a point load was applied because the part applying the load was not subjected to a pulling force. Additionally, the data shows that doubling the width of the ring doubles its stiffness, which was shown in Equation 2.1.

A comparison of the theory and FEA results to the experimental results show that the point load values match up closely while the tension values do not match. This is again because of hardening vs softening behavior of the ring. The FEA simulation in SolidWorks as well as the theoretical equation for ring stiffness given in Equation 2.7 are all linear theory. They do not factor in the non-linearities that arise when deflections grow too large.

Figure 6.2 gives a visual of the force vs deflection curves that were used to find the stiffness values for the 10 mm wide rings under a point load. The axis label in Figure 6.2(a) are negative because a downward force was being applied causing the deflections to be in the negative direction. Figure 6.2(a) shows the trend that the slope of the force vs deflection curve gets steeper with a decrease in diameter.
Figure 6.3: Stiffness vs diameter curves for 20 mm wide rings. (a) Stiffness values obtained from point load testing vs diameter. (b) Stiffness values obtained from tension testing vs diameter.
as expected. Figure 6.2(b) shows the non-dimensionalized force \( (FD^2/EI) \) vs the non-dimensionalized deflection \( (\delta/Y) \). The purpose of this plot is to compare the experimental results with theory. According to Equation 2.7 the slope of a non-dimensionalized force vs deflection plot for a ring should be \( \bar{k} \approx 53.77 \). It can be seen from the slope of the non-dimensionalized plot that the experimental data gave \( \bar{k} = 56.63 \). This shows that theoretical values and experimental data match up well for the 10 mm wide rings under point load.

To demonstrate stiffness’s dependence on length, Figure 6.3 was created to show how the stiffness of the individual rings change with diameter. Each marker is the stiffness for an individual ring with a specific diameter. As was previously mentioned the point load testing values match up much better with the theory and FEA values than the tensile values do. However both curves show that the difference between experimental and FEA become larger as the diameter of the ring decreases. This is most likely due to the fact that larger loads had to be applied to the rings with smaller diameters to get them to noticeably deflect. However these larger loads lead to nonlinearities in the data due to disproportionately large deflections. Figure 6.3 can also be used to match the experimental data with the scaling shown in Equation 2.1. For a fixed \( E, I \) and \( C \) for a ring with diameter \( D \), the scaling becomes

\[
 k \propto \frac{1}{D^3} 
\]  
(6.1)

Plotting Equation 6.1 would give an inverse cubic which is the general form of the curves shown in Figure 6.3.

Deflected shapes obtained from stiffness testing for Ring 160_{10} under tensile loading is shown below.
Figure 6.4: Deflected shape of Ring $160_{10}$ under tensile load. (a) Experimental testing procedure for tension load. (b) Deflected shape of ring for all the loads it was subjected to from MATLAB image processing.
7

Frequency of Rings

7.1 Experimental Procedure

Analysis of ring frequency was focused on finding the lowest two in-plane vibration modes of the ring with a fixed point at the bottom. Mode 1 oscillation was taken to be when the ring oscillated in a “side-to-side” motion along the $x$ axis. While Mode 2 oscillations were taken to be in a “up-down” motion along the $y$ axis. Figure 7.1

**Figure 7.1**: Mode shapes of ring oscillations. (a) Mode shape 1 was a “side-to-side” motion along the $x$ axis with positive oscillation along the positive $x$ axis. (b) Mode shape 2 was a “up-down” motion along the $y$ axis with positive oscillation to be taken along the positive $y$ axis.
shows a rendering of how the rings oscillated during experimentation. The same procedure outlined in Section 5.1 was followed during experimental analysis: collect the time series of the ring’s oscillations, put the time series through a FFT algorithm to get a plot of the amplitude versus the frequency. Experimental results were only compared with FEA simulation, this is because an adequate analytical equation could not be found.

Each ring was tested twice to examine the effect of two different types of boundary conditions. Results presented in the following sections are for the boundary conditions shown in Figure 7.1 which had the ring freely standing but was fixed to the surface by a thin line along its width. Additional testing was done with similar boundary condition except that the ring was fixed at the top and free everywhere else, so it was suspended. Two different boundary conditions were tested to see if they influenced the frequency, but the results were almost identical between the two boundary conditions.

7.2 Results

![Figure 7.2: Results of frequency analysis for Ring 17010 Mode 2. (a) Time series showing decaying oscillation. (b) Mode 2 frequency for Ring 17010 of f2 = 30.81 Hz.](image)
Figure 7.2 shows the results of experimental Mode 2 frequency analysis for Ring 170. Like the frames, the time series show that the ring oscillations eventually decay due to damping. The time series in Figure 7.2(a) has the same form as the

Table 7.1: Mode 1 and Mode 2 frequency values from experimental and FEA analysis and comparison of analysis methods. Ring dimensions can be seen in Table 3.4.

<table>
<thead>
<tr>
<th>Ring</th>
<th>EXP Frequency (Hz)</th>
<th>FEA Frequency(Hz)</th>
<th>(f_{\text{EXP}}/f_{\text{FEA}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>240 10</td>
<td>4.40</td>
<td>14.68</td>
<td>5.80</td>
</tr>
<tr>
<td>230 10</td>
<td>4.40</td>
<td>16.14</td>
<td>5.56</td>
</tr>
<tr>
<td>220 10</td>
<td>5.87</td>
<td>17.61</td>
<td>6.07</td>
</tr>
<tr>
<td>210 10</td>
<td>5.87</td>
<td>19.08</td>
<td>7.82</td>
</tr>
<tr>
<td>200 10</td>
<td>7.33</td>
<td>22.73</td>
<td>7.58</td>
</tr>
<tr>
<td>190 10</td>
<td>7.34</td>
<td>23.47</td>
<td>8.13</td>
</tr>
<tr>
<td>180 10</td>
<td>8.81</td>
<td>26.43</td>
<td>9.11</td>
</tr>
<tr>
<td>170 10</td>
<td>10.28</td>
<td>29.35</td>
<td>10.18</td>
</tr>
<tr>
<td>160 10</td>
<td>11.74</td>
<td>33.77</td>
<td>11.50</td>
</tr>
<tr>
<td>150 10</td>
<td>13.21</td>
<td>41.08</td>
<td>13.20</td>
</tr>
<tr>
<td>140 10</td>
<td>14.68</td>
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</tr>
<tr>
<td>130 10</td>
<td>17.61</td>
<td>52.85</td>
<td>17.42</td>
</tr>
<tr>
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<td>22.01</td>
<td>63.11</td>
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</tr>
<tr>
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<td>26.43</td>
<td>74.87</td>
<td>24.35</td>
</tr>
<tr>
<td>100 10</td>
<td>32.29</td>
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<tr>
<td>100 20</td>
<td>33.77</td>
<td>91.32</td>
<td>29.48</td>
</tr>
</tbody>
</table>

49
time series for the two story frames, the amplitudes do not decay in a smooth exponential manner. This is because both vibration modes are present even though the ring was oscillated to excite the Mode 2 frequency.

Table 7.1 displays the frequency values from experimental and FEA analysis as well as a comparison of the two for the rings. There are two trends shown in Table 7.1 that match what was found for the frequency of the plane frames in Chapter 5. The first is that decreasing the height of the structure (the diameter of the rings) increases the frequency. While the second is that increasing the width of the ring does not alter its frequency, which is also shown in the scaling relation in Equation 2.2b.

On the whole the results seem to line up with what was expected and also follow the results found in the analysis of plane frame frequencies and the experimental and FEA results match up well. There does appear to be some irregularities in the Mode 1 experimental results for the rings with the larger diameters. For instance

![Figure 7.3: Mode shape obtained from frequency FEA analysis for Ring 16010. (a) Mode shape 1 was a “side-to-side” motion along the x axis. (b) Mode 2 shape was “up-down” motion along the y axis.]
Figure 7.4: Frequency vs diameter curves to show dependence of frequency on diameter for 20 mm wide rings. (a) Mode 1 frequency vs diameter for 20 mm wide rings. (b) Mode 2 frequency vs diameter for 20 mm wide rings.
Ring 240₁₀ and Ring 230₁₀ have the same Mode 1 frequency for two decimal places. This does not make sense because the rings do not have the same diameter and therefore should have a different frequency. The most likely reason for this discrepancy is the relationship between FFT length and sampling rate [15].

Additional results from frequency analysis can be seen in Figure 7.3 and Figure 7.4. Figure 7.3 shows the mode shapes obtained from SolidWorks FEA simulation for Ring 160₁₀. These mode shapes match the ones shown in the experimental setup in Figure 7.1. The curves shown in Figure 7.4 were created to equate the scaling relationship in Equation 2.2b to the experimental and FEA results visually. For a fixed $E$, $I$, $\rho$ and $A$, Equation 2.2b can be used to relate the ring’s frequency with its diameter $D$.

$$f \propto \frac{1}{D^2} \quad (7.1)$$

Plotting Equation 7.1 would give an inverse cubic curve, which is the form of the curves in Figure 7.4.
8.1 Experimental Procedure

Stiffness of the parallel frame system was tested by applying loading to the first frame in the system and then measuring the resulting deflection of the frame with increased loading. Figure 8.1 shows a rendering of the experimental set-up.

![Figure 8.1: Rendering of experimental set up to test stiffness of parallel frame system. Positive deflections were taken to be along the x axis.](image-url)
Loading was applied using the force gauge as a displacement control method. As Section 3.4 mentioned, the height of the columns and spacing between frames were the two control parameters within the system. A force deflection plot was created by following the same procedure outlined in Section 4.1. The resulting curve resembled Figure 2.6(b). In addition to the experimental data, theory curves for the system were found using the equation for equivalent spring stiffness shown in Equation 2.8. The values that were used for $k_1$, $k_2$ and $k_3$ were found using the formula for the stiffness of sway frames shown in Equation 2.5. These values were then plotted and fit with a trend line to compare to the experimental results.

8.2 Results

The deflected shape for the Group 1 parallel frame system is shown in Figure 8.2. As expected, each frame deflects less the farther away from the load it is within the system. This shows an increase in stiffness within the total system as more frames are contacted. Figure 8.2 also shows that the contact point between the frames is not quite in the middle of each cross beam. This is because as more loading is applied the columns get stretched in the horizontal direction causing the cross beam to move slightly downwards.

![Figure 8.2: Deflected shape of Group 1 parallel frame system from experimental testing.](image_url)
Figure 8.3: Force vs deflection plot for parallel frame Group 1 system. Column length $Y = 100$ mm, spacing between frames $\Delta_1 = \Delta_2 = 7.1$ mm.

Figure 8.4: Force vs deflection plot for parallel frame Group 2 system. Column length $Y = 100$ mm, spacing between frames $\Delta_1 = \Delta_2 = 14.2$ mm.
Figures 8.3 - 8.5 show the force vs deflection curves that were created for the three parallel frame groups. As predicted by Figure 2.6(b), the plots show non-linear curves made up of three linear segments. The slope of the curves change at the deflection value that corresponds to the spacing between the frames, $\delta = \Delta$. A comparison of the experimental data and the theory values show that the same general trends in the curve, with a more noticeable difference occurring when the second and third frames are contacted.

Table 8.1 summarizes the stiffness values from the groups of parallel frames. As the plots show the stiffness of the overall system increased as more frames were contacted. A comparison of the Group 1 and Group 2 results show some discrepancies, even though frames had the same column lengths. The differences occurred when the second and third frames were contacted. Since the frame spacing was larger in Group 2, larger forces had to be applied to get the frames to contact each other. This likely led to non-linear effects due to large displacements. In theory the slope
Table 8.1: Stiffness (N/mm) values for the three parallel frame systems tested. See Table 3.5 for system control parameters.

<table>
<thead>
<tr>
<th>Group</th>
<th>Frame 1</th>
<th></th>
<th>Frame 1 + 2</th>
<th></th>
<th>Frame 1 + 2 + 3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EXP</td>
<td>THEO</td>
<td>EXP</td>
<td>THEO</td>
<td>EXP</td>
<td>THEO</td>
</tr>
<tr>
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<td>0.64</td>
<td>1.08</td>
<td>0.96</td>
</tr>
<tr>
<td>2</td>
<td>0.39</td>
<td>0.32</td>
<td>0.83</td>
<td>0.64</td>
<td>1.26</td>
<td>0.96</td>
</tr>
<tr>
<td>3</td>
<td>0.11</td>
<td>0.09</td>
<td>0.20</td>
<td>0.19</td>
<td>0.35</td>
<td>0.28</td>
</tr>
</tbody>
</table>

values should be the exact same, the only difference would be the deflection location of where the slope changes. A comparison of Group 1 vs Group 2 shows the same result that was found in Section 4.2.1, the system with the column length of $Y = 150$ mm is less stiff by $0.3 (1.5^{-3})$. 
9.1 Experimental Procedure

Experimental testing to determine the stiffness of the nested ring system was found by applying load to the system and then measuring the resulting deflection. Figure 9.1 shows a rendering of the experimental set up for the nested ring systems. The same coordinate system and loading procedure that was outlined in Section 6.1.

\[ \text{Figure 9.1: Rendering of experimental set up to determine stiffness of nested rings.} \]

(a) Compressive point load was applied causing negative deflections. (b) Tensile point load was applied causing positive deflections.
was used for the testing of the nested ring systems. Displacements were measured using a laser while deflections were found following the process outlined in Section 4.1. Displacements were taken for the outermost ring if a compressive force was applied and were taken for the innermost ring if a tensile force was applied.

Like the stiffness of the individual rings, the nested systems were tested under point and flat bar compressive loads. However the results were virtually the same so this chapter will only focus on the point load and tensile load data. Additional plots can be seen in Appendix E. Boundary conditions were similar to individual ring testing except the nested system had three rings so a larger clamp had to be used to ensure that the rings all stayed in the same plane and that they remained fixed to the table. Force vs deflection plots were created using the same process as given in Section 8.1. The theoretical values that were used for $k_1$, $k_2$ and $k_3$ can be found in Table 6.1 and Table 6.2.

9.2 Results

Figure 9.2: Images of experimental testing to determine stiffness of Nest 5. (a) Shows compressive loading. (b) Shows tensile loading. Laser marker denotes where measurements were taken. Table 3.6 lists the control parameters for this nest.
Figure 9.3: Force vs deflection plot for Nest 1 system. Ring diameters 120, 110, 100 mm, spacing between frames $\Delta = 6$ mm. Loading applied using point load.

Figure 9.4: Force vs deflection plot for Nest 12 system. Ring diameters 240, 220, 200 mm, spacing between frames $\Delta = 16$ mm. Loading applied using point load.
Figure 9.5: Force vs deflection plot for Nest 3 system. Ring diameters 180, 170, 160 mm, spacing between frames $\Delta = 6$ mm. Loading applied using tensile load.

Figure 9.2 shows the deflected shapes for the Nest 5 system when a point load and tensile load are applied. The laser markings denote where the deflections were measured from. It can be seen from Figure 9.2(a) that the outside ring deflects the most while a compressive load is on the system while Figure 9.2(b) shows that the inside ring deflects the most when the system is in tension. Unlike the parallel frame systems the contact point remains stationary between all three rings due to symmetry.

Figures 9.3 - 9.5 resemble the force vs deflection plots for the parallel frame systems, they are non-linear curves made up of three linear regions. Altering the size of the ring diameters within the system changes the deflection value where the slope of the curve changes. For instance Figure 9.3 has a spacing between rings of 6 mm so the slope of the line changes at $\Delta = 6$ mm while the nest in Figure 9.4 has spacing of 16 mm so the slope of the line changes at $\Delta = 16$ mm. The curve shown in Figure 9.5 displays the same discrepancy that was found for the single ring stiffness.
values in tension and compression. It appears that greater separation happens the more rings that come into contact with each other. This makes sense because the differences will keep building upon each other.

Table 9.1 lists the experimental and theoretical stiffness values of all the ring nests analyzed. Results are for when the system had one, two, or three rings contacted. A comparison of the point and tension load values give the same result as the single ring stiffness analysis, the tension stiffness values are noticeably higher than the compression stiffness values. The reasoning for this was discussed in Section 6.1.

Table 9.1: Stiffness (N/mm) of nested rings from experimental and theoretical analysis. Experimental loading was applied using either a compressive load or a tensile load. System control parameters are shown in Table 3.6. Theoretical values were found using Equation 2.7 and Equation 2.8.

<table>
<thead>
<tr>
<th>Nest</th>
<th>Loading</th>
<th>EXP</th>
<th>THEO</th>
<th>EXP</th>
<th>THEO</th>
<th>EXP</th>
<th>THEO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>point</td>
<td>0.48</td>
<td>0.41</td>
<td>1.02</td>
<td>0.95</td>
<td>1.51</td>
<td>1.67</td>
</tr>
<tr>
<td>2</td>
<td>point</td>
<td>0.25</td>
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</tr>
<tr>
<td>3</td>
<td>point</td>
<td>0.13</td>
<td>0.12</td>
<td>0.30</td>
<td>0.27</td>
<td>0.44</td>
<td>0.44</td>
</tr>
<tr>
<td>3</td>
<td>tension</td>
<td>0.23</td>
<td>0.12</td>
<td>0.53</td>
<td>0.27</td>
<td>0.79</td>
<td>0.44</td>
</tr>
<tr>
<td>4</td>
<td>point</td>
<td>0.09</td>
<td>0.08</td>
<td>0.18</td>
<td>0.17</td>
<td>0.28</td>
<td>0.27</td>
</tr>
<tr>
<td>5</td>
<td>point</td>
<td>0.09</td>
<td>0.05</td>
<td>0.16</td>
<td>0.11</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
<td>point</td>
<td>0.15</td>
<td>0.15</td>
<td>0.34</td>
<td>0.36</td>
<td>0.62</td>
<td>0.68</td>
</tr>
<tr>
<td>7</td>
<td>point</td>
<td>1.06</td>
<td>0.83</td>
<td>2.21</td>
<td>1.91</td>
<td>3.13</td>
<td>3.34</td>
</tr>
<tr>
<td>8</td>
<td>point</td>
<td>0.54</td>
<td>0.42</td>
<td>1.00</td>
<td>0.98</td>
<td>1.59</td>
<td>1.63</td>
</tr>
<tr>
<td>9</td>
<td>point</td>
<td>0.30</td>
<td>0.25</td>
<td>0.63</td>
<td>0.54</td>
<td>0.92</td>
<td>0.89</td>
</tr>
<tr>
<td>10</td>
<td>point</td>
<td>0.15</td>
<td>0.20</td>
<td>0.33</td>
<td>0.41</td>
<td>0.54</td>
<td>0.54</td>
</tr>
<tr>
<td>11</td>
<td>point</td>
<td>0.13</td>
<td>0.10</td>
<td>0.30</td>
<td>0.22</td>
<td>0.39</td>
<td>0.36</td>
</tr>
<tr>
<td>11</td>
<td>tension</td>
<td>0.23</td>
<td>0.10</td>
<td>0.54</td>
<td>0.22</td>
<td>0.75</td>
<td>0.36</td>
</tr>
<tr>
<td>12</td>
<td>point</td>
<td>0.11</td>
<td>0.10</td>
<td>0.25</td>
<td>0.24</td>
<td>0.44</td>
<td>0.42</td>
</tr>
</tbody>
</table>
10.1 Summary

Results of analysis show that for a simple structure with a constant material the structure’s geometry, specifically its length, is the dominant factor in stiffness and frequency. Decreasing the structure’s height will lead to an increase in stiffness and frequency while lengthening the structure will have the opposite effect. A structure’s width also factors into how stiff it is, but it does not alter frequency because the increase in stiffness is canceled out by the increase in mass. Additionally it was found that adding structural components will result in an increased stiffness and frequency. These results show that there are options while designing a structure to fulfill its role in a system.

One of the most surprising results of the analysis was the difference in the stiffness values found when an individual ring or ring nest was subjected to tensile or compressive loads. While [8] talked about how the ring would act differently depending on the direction of the deflection, it was not assumed that the difference between stiffness values would be so large. These results show that non-linear effects arose during experimental testing. This caused the force vs deflection curve
to become non-linear at the end which did not give an accurate trend line through the data used to determine stiffness. If the experiment was to be repeated in the future smaller loads would be applied to ensure deflections remained linear. This of course would be difficult for smaller rings which need to have larger forces applied to get noticeable deflections. Knowing this about a ring is useful because it can be used in a system that needs a component to act as a hardening or softening spring depending on the direction of deflection.

The parallel frames and nested ring systems allowed for the experimental analysis of systems that could be treated as springs in parallel. In these systems, spacing between structures and the length of the individual structures served as control parameters within the system. Altering the spacing between the structures controlled when the structures contacted each other thus increasing the stiffness of the system. Additionally individual structural components within the system can have their geometry designed to lead to an increase in stiffness within the system by the desired amount. It is important to make sure spacing between components does not get too large otherwise non-linear effects will occur which will makes it difficult to determine the correct stiffness of the system. Structural systems that can be treated as springs in parallel are important during design because they create a system that can have its stiffness tailored.

10.2 Comparison of Experimental Methods

During experimental analysis, different methods were used to apply loading and also measure deflections. As was mentioned in the previous chapters, loading was applied using slotted weights to act as a force control method or a load cell to act as a displacement control method. Each technique had its pros and cons. Slotted weights allowed for more accuracy when determining the exact load that was applied. The load cell that was used was only accurate to 0.05 N. Additionally the ABS plastic
has creep which would cause the displayed value on the force gauge to change while a load was applied. This was especially problematic when loading was applied to the rings with the smaller diameters because their higher stiffness gave more resistance to the force gauge. However it was much easier to apply loading using the force gauge than the slotted weights. If the weights were applied, then modifications to the structure had to be added to accommodate them.

As Section 6.2 mentioned, the hook used to apply tension loads to the rings was also a source error during experimentation. This hook, while stiffer than the rings, did deflect from the force of the ring resisting the tensile load. This meant that there was an additional spring in the system that was not planned for. If this testing was to be repeated, the tensile load would be applied with an object made of a stiffer material and would not be a cantilever.

To measure deflections during experimental testing, both a laser and digital camera were used. Looking back at the data that was collected from both techniques, it appears as though the laser data was better. This is most likely due to the fact that it was more complicated to use the digital camera than the laser. During testing, it was important that the camera was positioned correctly and focused on the center of the rings. Additionally the pictures had to be calibrated to convert the units of pixels to millimeters to determine the deflections. Both of these could contribute to a loss in the accuracy of deflection values.

10.3 Comparison of Analysis Methods

For the most part the results obtained from experimental analysis provided assurance that the FEA and theory results were reasonable (with the exception of the stiffness values for rings in tension). The result from the frequency analysis seemed to match up better between experimental and FEA. Obviously there is some difference and there are numerous factors that can affect the experimental results. One of the
largest reasons is most likely due to the accuracy of the 3D printer. While the 3D printer used was extremely accurate, there were differences between the CAD model and the structure that was printed. For example the thickness of the frame columns and ring thickness in SolidWorks were both 2 mm, but their measured thickness value was in the range of 2.1-2.4 mm. This is a small difference, but it does create a difference between the experimental and FEA/theory values. This is going to affect the moment of inertia term while calculating stiffness. A value of \( d = 2.00 \text{ mm} \) is used during FEA/theory analysis while the actual value is \( d \approx 2.02 \text{ mm} \). This difference is noteworthy since the thickness term is cubed in \( I = (1/12)bd^3 \).

10.4 Role of 3D printing

3D printing was an extremely useful tool to utilize to create the structures analyzed in this paper. It allowed for more accurate and time efficient creation of the structures modeled in SolidWorks. Creating the structures by hand whether it was building them by wood or plastic or even machining them would not have been as accurate. It would be impossible to create a structure that has any noticeable seams between the components in the parts. Additionally, the structures that come out of the 3D printer all have the same material properties and interior makeup because of the additive process used to create them. Another benefit of the 3D printer is the time that it saves to create the structures. While it can take many hours to print to the structures, the machine does all the work so little human interaction is necessary. Additionally using the 3D printer allowed for multiple frames and rings to be printed simultaneously. Another benefit of utilizing 3D printing is that it allows for a comparison between experimental results to be made without having to worry about the differences in how the structures were built. This is because the differences between the CAD model and 3D printed object are uniform across all the structures printed.
10.5 Future Work & Applications

The work presented in this paper could be extended in many ways. These could range from additional geometric changes to individual ring structures to adding additional components to the parallel frame and nested ring systems, or frames could be designed so they contacted rings. An interesting idea would be to test the frequency of the parallel frame and nested ring systems. The parallel frame systems would resemble a Newtons cradle while the nested ring system would be a more complicated problem to analyze.

One potential use for the nested rings analyzed would be for energy absorption. The rings in the nest could be designed to create a system that acts as a hardening spring. So if a dynamic load was applied to the rings it would encounter more resistance, slowing down the object, each time an additional ring was contacted.

10.6 Final Thoughts

Completing the work in this study has been beneficial for several reasons. The first is that it helps to give concrete physical examples to the theories that are taught in courses. In addition it also helped to learn how to analyze data to determine differences between experimental and theoretical results. And most importantly it taught the author how to analyze and fix problems within a study that will be beneficial in their career.
Appendix A

Stiffness Matrix and Eigenvalue Analysis

The following sections give an overview of how to formulate the stiffness and mass matrix for the portal frame shown in Figure A.1(a). In addition calculations are presented to demonstrate how the theoretical stiffness and frequency was found for Frame B.

Figure A.1: Rendering of Frame B used to calculate theoretical stiffness and frequency.
A.1 Displacements and Numbering

Figure A.1(b) shows the displacements that Frame B undergoes when it is subjected to an applied load at one of its corners. As Section 2.3.1 stated, the frame will undergo horizontal and vertical translation as well as rotation at each of its fixed points. The four nodes correspond to the four corners of the frame, while the three elements are the two columns and cross beam that make up the frame.

A.2 Transformation Matrix

In order for the global matrices to be assembled, the element stiffness and mass matrices need to be transformed into their global coordinates. This can be done by multiplying the element matrices by the transformation matrix given in Equation A.1. A derivation of this transformation matrix can be found in [16].

\[ [T] = \begin{bmatrix}
    l & m & 0 & 0 & 0 & 0 \\
    -m & l & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & l & m & 0 \\
    0 & 0 & 0 & -m & l & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix} \quad (A.1) \]

where \( l = \cos\theta \) and \( m = \sin\theta \). \( \theta \) is the angle between the structural member and the horizontal axis.

A.3 Matrix Assembly

In the following sections the stiffness and mass matrices will be found for the individual elements. The process followed to create these matrices can be found in [16, 17, 18].
A.3.1 Element 1 - \( \theta = 90^\circ \)

Element Stiffness Matrix:

\[
[k_1] = \begin{bmatrix}
\frac{A_C E_C}{L_C} & 0 & 0 & -\frac{A_C E_C}{L_C} & 0 & 0 \\
0 & \frac{12E_C I_C}{L_C^3} & \frac{6E_C I_C}{L_C^2} & 0 & -\frac{12E_C I_C}{L_C^2} & \frac{6E_C I_C}{L_C} \\
0 & \frac{6E_C I_C}{L_C^2} & \frac{4E_C I_C}{L_C} & 0 & -\frac{6E_C I_C}{L_C} & \frac{2E_C I_C}{L_C} \\
-\frac{A_C E_C}{L_C} & 0 & 0 & \frac{A_C E_C}{L_C} & 0 & 0 \\
0 & \frac{-12E_C I_C}{L_C^3} & \frac{-6E_C I_C}{L_C^2} & 0 & \frac{12E_C I_C}{L_C^2} & \frac{-6E_C I_C}{L_C} \\
0 & \frac{6E_C I_C}{L_C^2} & \frac{2E_C I_C}{L_C} & 0 & -\frac{6E_C I_C}{L_C} & \frac{4E_C I_C}{L_C}
\end{bmatrix} \quad (A.2a)
\]

Global Stiffness Matrix:

\[
[K_1] = [T]^T [k_1] [T] \quad (A.2b)
\]

\[
[K_1] = \begin{bmatrix}
\frac{12E_C I_C}{L_C^3} & 0 & -\frac{6E_C I_C}{L_C^2} & -\frac{12E_C I_C}{L_C^2} & 0 & -\frac{6E_C I_C}{L_C} \\
0 & \frac{A_C E_C}{L_C} & 0 & 0 & -\frac{A_C E_C}{L_C} & 0 \\
-\frac{6E_C I_C}{L_C^2} & 0 & \frac{4E_C I_C}{L_C} & \frac{6E_C I_C}{L_C^2} & 0 & \frac{2E_C I_C}{L_C} \\
-\frac{12E_C I_C}{L_C^2} & 0 & \frac{6E_C I_C}{L_C^2} & \frac{12E_C I_C}{L_C^2} & 0 & \frac{6E_C I_C}{L_C} \\
0 & -\frac{A_C E_C}{L_C} & 0 & 0 & \frac{A_C E_C}{L_C} & 0 \\
-\frac{6E_C I_C}{L_C^2} & 0 & \frac{2E_C I_C}{L_C} & \frac{6E_C I_C}{L_C^2} & 0 & \frac{4E_C I_C}{L_C}
\end{bmatrix} \quad (A.2c)
\]
Element Mass Matrix:

\[
[m_1] = \begin{bmatrix}
\frac{AC_{LC}\rho}{3} & 0 & 0 & \frac{AC_{LC}\rho}{6} & 0 & 0 \\
0 & \frac{13AC_{LC}\rho}{35} & \frac{11AC_{L}\rho}{210} & 0 & \frac{9AC_{LC}\rho}{70} & -\frac{13AC_{L}\rho}{420} \\
0 & \frac{11AC_{L}\rho}{210} & \frac{AC_{L}\rho}{105} & 0 & \frac{13AC_{L}\rho}{420} & -\frac{AC_{L}\rho}{140} \\
\frac{AC_{LC}\rho}{6} & 0 & 0 & \frac{AC_{LC}\rho}{3} & 0 & 0 \\
0 & \frac{9AC_{LC}\rho}{70} & \frac{13AC_{L}\rho}{420} & 0 & \frac{13AC_{LC}\rho}{35} & -\frac{11AC_{L}\rho}{210} \\
0 & \frac{-13AC_{L}\rho}{420} & \frac{AC_{L}\rho}{140} & 0 & \frac{-11AC_{L}\rho}{210} & \frac{AC_{L}\rho}{105}
\end{bmatrix}
\]  
(A.3a)

Global Mass Matrix:

\[
[M_1] = [T]^T[m_1][T]
\]  
(A.3b)

\[
[M_1] = \begin{bmatrix}
\frac{13AC_{LC}\rho}{35} & 0 & -\frac{11AC_{L}\rho}{210} & \frac{9AC_{LC}\rho}{70} & 0 & \frac{13AC_{L}\rho}{420} \\
0 & \frac{AC_{LC}\rho}{6} & 0 & 0 & \frac{AC_{LC}\rho}{3} & 0 \\
-\frac{11AC_{L}\rho}{210} & 0 & \frac{AC_{L}\rho}{105} & -\frac{13AC_{L}\rho}{420} & 0 & \frac{AC_{L}\rho}{140} \\
\frac{9AC_{LC}\rho}{70} & 0 & -\frac{13AC_{L}\rho}{420} & \frac{13AC_{LC}\rho}{35} & 0 & \frac{11AC_{L}\rho}{210} \\
0 & \frac{AC_{LC}\rho}{6} & 0 & 0 & \frac{AC_{LC}\rho}{3} & 0 \\
\frac{13AC_{L}\rho}{420} & 0 & -\frac{AC_{L}\rho}{140} & \frac{11AC_{L}\rho}{210} & 0 & \frac{AC_{L}\rho}{105}
\end{bmatrix}
\]  
(A.3c)
A.3.2 Element 2 - \( \theta = 0^\circ \)

Element Stiffness Matrix:

\[
[k_2] = \begin{bmatrix}
\frac{A_B E_B}{L_B} & 0 & 0 & -\frac{A_B E_B}{L_B} & 0 & 0 \\
0 & \frac{12 E_B I_B}{L_B} & \frac{6 E_B I_B}{L_B} & 0 & -\frac{12 E_B I_B}{L_B} & \frac{6 E_B I_B}{L_B} \\
0 & \frac{6 E_B I_B}{L_B} & \frac{4 E_B I_B}{L_B} & 0 & -\frac{6 E_B I_B}{L_B} & 2\frac{E_B I_B}{L_B} \\
-\frac{A_B E_B}{L_B} & 0 & 0 & \frac{A_B E_B}{L_B} & 0 & 0 \\
0 & -\frac{12 E_B I_B}{L_B} & -\frac{6 E_B I_B}{L_B} & 0 & \frac{12 E_B I_B}{L_B} & -\frac{6 E_B I_B}{L_B} \\
0 & \frac{6 E_B I_B}{L_B} & 2\frac{E_B I_B}{L_B} & 0 & -\frac{6 E_B I_B}{L_B} & \frac{4 E_B I_C}{L_B}
\end{bmatrix}
\]  

(A.4a)

Global Stiffness Matrix:

\[
[K_2] = [T]^T[k_2][T]
\]  

(A.4b)

\[
[K_2] = \begin{bmatrix}
\frac{A_B E_B}{L_B} & 0 & 0 & -\frac{A_B E_B}{L_B} & 0 & 0 \\
0 & \frac{12 E_B I_B}{L_B} & \frac{6 E_B I_B}{L_B} & 0 & -\frac{12 E_B I_B}{L_B} & \frac{6 E_B I_B}{L_B} \\
0 & \frac{6 E_B I_B}{L_B} & \frac{4 E_B I_B}{L_B} & 0 & -\frac{6 E_B I_B}{L_B} & 2\frac{E_B I_B}{L_B} \\
-\frac{A_B E_B}{L_B} & 0 & 0 & \frac{A_B E_B}{L_B} & 0 & 0 \\
0 & -\frac{12 E_B I_B}{L_B} & -\frac{6 E_B I_B}{L_B} & 0 & \frac{12 E_B I_B}{L_B} & -\frac{6 E_B I_B}{L_B} \\
0 & \frac{6 E_B I_B}{L_B} & 2\frac{E_B I_B}{L_B} & 0 & -\frac{6 E_B I_B}{L_B} & \frac{4 E_B I_C}{L_B}
\end{bmatrix}
\]  

(A.4c)
Element Mass Matrix:

\[ [m_2] = \begin{bmatrix}
\frac{A_B L_B \rho}{3} & 0 & 0 & \frac{A_B L_B \rho}{6} & 0 & 0 \\
0 & \frac{13A_B L_B \rho}{35} & \frac{11A_B L_B^2 \rho}{210} & 0 & \frac{9A_B L_B \rho}{70} & -\frac{13A_B L_B^2 \rho}{420} \\
0 & \frac{11A_B L_B^2 \rho}{210} & \frac{A_B L_B^3 \rho}{105} & 0 & \frac{13A_B L_B^2 \rho}{420} & -\frac{A_B L_B^3 \rho}{140} \\
\frac{A_B L_B \rho}{6} & 0 & 0 & \frac{A_B L_B \rho}{3} & 0 & 0 \\
0 & \frac{9A_B L_B \rho}{70} & \frac{13A_B L_B^2 \rho}{420} & 0 & \frac{13A_B L_B \rho}{35} & -\frac{11A_B L_B^2 \rho}{210} \\
0 & \frac{13A_B L_B^2 \rho}{420} & \frac{A_B L_B^3 \rho}{140} & 0 & \frac{11A_B L_B^2 \rho}{210} & \frac{A_B L_B^3 \rho}{105}
\end{bmatrix} \quad (A.5a) \]

Global Mass Matrix:

\[ [M_2] = [T]^T[m_2][T] \quad (A.5b) \]

\[ [M_2] = \begin{bmatrix}
\frac{A_B L_B \rho}{3} & 0 & 0 & \frac{A_B L_B \rho}{6} & 0 & 0 \\
0 & \frac{13A_B L_B \rho}{35} & \frac{11A_B L_B^2 \rho}{210} & 0 & \frac{9A_B L_B \rho}{70} & -\frac{13A_B L_B^2 \rho}{420} \\
0 & \frac{11A_B L_B^2 \rho}{210} & \frac{A_B L_B^3 \rho}{105} & 0 & \frac{13A_B L_B^2 \rho}{420} & -\frac{A_B L_B^3 \rho}{140} \\
\frac{A_B L_B \rho}{6} & 0 & 0 & \frac{A_B L_B \rho}{3} & 0 & 0 \\
0 & \frac{9A_B L_B \rho}{70} & \frac{13A_B L_B^2 \rho}{420} & 0 & \frac{13A_B L_B \rho}{35} & -\frac{11A_B L_B^2 \rho}{210} \\
0 & \frac{13A_B L_B^2 \rho}{420} & \frac{A_B L_B^3 \rho}{140} & 0 & \frac{11A_B L_B^2 \rho}{210} & \frac{A_B L_B^3 \rho}{105}
\end{bmatrix} \quad (A.5c) \]
A.3.3 Element 3 - \( \theta = 270^\circ \)

Element Stiffness Matrix:

\[
[k_3] = \begin{bmatrix}
\frac{A_c E_c}{L_c} & 0 & 0 & -\frac{A_c E_c}{L_c} & 0 & 0 \\
0 & \frac{12E_c I_c}{L_c^3} & \frac{6E_c I_c}{L_c^2} & 0 & -\frac{12E_c I_c}{L_c^3} & \frac{6E_c I_c}{L_c^2} \\
0 & \frac{6E_c I_c}{L_c^3} & \frac{4E_c I_c}{L_c^2} & 0 & -\frac{6E_c I_c}{L_c^3} & \frac{2E_c I_c}{L_c} \\
-\frac{A_c E_c}{L_c} & 0 & 0 & \frac{A_c E_c}{L_c} & 0 & 0 \\
0 & -\frac{12E_c I_c}{L_c^3} & -\frac{6E_c I_c}{L_c^2} & 0 & \frac{12E_c I_c}{L_c^3} & -\frac{6E_c I_c}{L_c^2} \\
0 & \frac{6E_c I_c}{L_c^3} & \frac{2E_c I_c}{L_c^2} & 0 & -\frac{6E_c I_c}{L_c^3} & \frac{4E_c I_c}{L_c} \\
\end{bmatrix} \quad (A.6a)
\]

Global Stiffness Matrix:

\[
[K_3] = [T]^T [k_3] [T] \quad (A.6b)
\]

\[
[K_3] = \begin{bmatrix}
\frac{12E_c I_c}{L_c^3} & 0 & \frac{6E_c I_c}{L_c^2} & -\frac{12E_c I_c}{L_c^3} & 0 & \frac{6E_c I_c}{L_c^2} \\
0 & \frac{A_c E_c}{L_c} & 0 & 0 & -\frac{A_c E_c}{L_c} & 0 \\
\frac{6E_c I_c}{L_c^3} & 0 & \frac{4E_c I_c}{L_c} & -\frac{6E_c I_c}{L_c^3} & 0 & \frac{2E_c I_c}{L_c} \\
-\frac{12E_c I_c}{L_c^3} & 0 & -\frac{6E_c I_c}{L_c^2} & \frac{12E_c I_c}{L_c^3} & 0 & -\frac{6E_c I_c}{L_c^2} \\
0 & -\frac{A_c E_c}{L_c} & 0 & 0 & \frac{A_c E_c}{L_c} & 0 \\
\frac{6E_c I_c}{L_c^3} & 0 & \frac{2E_c I_c}{L_c} & -\frac{6E_c I_c}{L_c^3} & 0 & \frac{4E_c I_c}{L_c} \\
\end{bmatrix} \quad (A.6c)
\]
Element Mass Matrix:

\[
[m_3] = \begin{bmatrix}
\frac{A_C L_C \rho}{3} & 0 & 0 & \frac{A_C L_C \rho}{6} & 0 & 0 \\
0 & \frac{13 A_C L_C \rho}{35} & \frac{11 A_C L_C^2 \rho}{210} & 0 & \frac{9 A_C L_C \rho}{70} & -\frac{13 A_C L_C^2 \rho}{420} \\
0 & \frac{11 A_C L_C^2 \rho}{210} & \frac{A_C L_C^2 \rho}{105} & 0 & \frac{13 A_C L_C^2 \rho}{420} & -\frac{A_C L_C^3 \rho}{140} \\
\frac{A_C L_C \rho}{6} & 0 & 0 & \frac{A_C L_C \rho}{3} & 0 & 0 \\
0 & \frac{9 A_C L_C \rho}{70} & \frac{13 A_C L_C^2 \rho}{420} & 0 & \frac{13 A_C L_C \rho}{35} & -\frac{11 A_C L_C^2 \rho}{210} \\
0 & \frac{13 A_C L_C^2 \rho}{420} & \frac{A_C L_C^3 \rho}{140} & 0 & \frac{11 A_C L_C^2 \rho}{210} & \frac{A_C L_C^3 \rho}{105}
\end{bmatrix}
\]  

(A.7a)

Global Mass Matrix:

\[
[M_3] = [T]^T [m_3] [T]
\]  

(A.7b)

\[
[M_3] = \begin{bmatrix}
\frac{13 A_C L_C \rho}{35} & 0 & \frac{11 A_C L_C^2 \rho}{210} & \frac{9 A_C L_C \rho}{70} & 0 & -\frac{13 A_C L_C^2 \rho}{420} \\
0 & \frac{A_C L_C \rho}{3} & 0 & 0 & \frac{A_C L_C \rho}{6} & 0 \\
\frac{11 A_C L_C^2 \rho}{210} & 0 & \frac{A_C L_C^2 \rho}{105} & \frac{13 A_C L_C^2 \rho}{420} & 0 & -\frac{A_C L_C^3 \rho}{140} \\
\frac{9 A_C L_C \rho}{70} & 0 & \frac{13 A_C L_C^2 \rho}{420} & \frac{13 A_C L_C \rho}{35} & 0 & -\frac{11 A_C L_C^2 \rho}{210} \\
0 & \frac{A_C L_C \rho}{6} & 0 & 0 & \frac{A_C L_C \rho}{3} & 0 \\
-\frac{13 A_C L_C^2 \rho}{420} & 0 & \frac{A_C L_C^3 \rho}{140} & \frac{11 A_C L_C^2 \rho}{210} & 0 & \frac{A_C L_C^3 \rho}{105}
\end{bmatrix}
\]  

(A.7c)
A.4 Global Matrices

The global stiffness matrix, $[K]$, and mass matrix, $[M]$, can be assembled using the procedures in [19]. Figure A.1(b) shows that the frame is fixed at the location where the two columns terminate into the support. This leads to the boundary conditions $y_1 = x_1 = \theta_1 = x_4 = y_4 = \theta_4 = 0$ at those locations. Applying these boundary conditions to the global matrices reduces their size from a 12x12 to a 6x6 [19]. Equation A.8 shows the relationship between the stiffness matrix and the force and displacement vectors.

$$[F] = [K][d] \quad (A.8)$$

A.5 Stiffness Matrix Method

To solve for the stiffness of Frame B, a theoretical force can be applied to the frame allowing for the displacement vector to be solved for. The parameters that were used to solve for the theoretical stiffness of Frame B were $F = 1.00$ N, $E_C = E_B = 2.00(10^9)$ N/m$^2$, $L_B = 0.05$ m, $L_C = 0.10$ m, $I_B = I_C = 1.33(10^{-11})$ m$^4$, $A_B = A_C = 4.00(10^{-05})$ m$^2$. Substituting these values into the force vector and stiffness matrix, the displacement vector is found to be

$$[d] = \begin{bmatrix} 1.93 \\ 0.00 \\ -7.25 \\ 1.93 \\ 0.00 \\ -7.25 \end{bmatrix} \text{m} \quad (A.9)$$

The stiffness of the system can be found by dividing the applied force by the displacement it caused along the x-axis.

$$k = F/x_2 = 0.52 \text{ N/mm} \quad (A.10)$$
A.6 Eigenvalue Analysis

The theoretical frequency of Frame B can be found by solving the eigenvalue problem [5, 4]

$$\|[K] - \lambda[M]\| = 0$$  \hspace{1cm} (A.11)

where $[K]$ is the stiffness matrix, $[M]$ is the mass matrix and $\lambda$ represents the eigenvalues for the system. The lowest eigenvalue will correspond to the first mode of vibration for the frame [5]. MATLAB’s built-in eig(K,M) function was used to solve for the systems eigenvalues which gave a Mode 1 frequency value of

$$\omega_1 = \sqrt{\lambda_1} = 52.11 \text{ Hz}$$  \hspace{1cm} (A.12)
Appendix B

SolidWorks FEA Simulation Procedures

B.1 Meshing

B.1.1 Plane Frames

Two types of meshes were considered during the FEA simulations run in SolidWorks. The first was a standard mesh that consisted of tetrahedral elements [20]. The second option is curvature based mesh which also creates a mesh with tetrahedral elements, but it places more elements in higher curvature areas [20]. The final mesh that was used during simulation on plane frames is shown in Figure B.1.

To determine the proper mesh size to use for FEA analysis on the plane frames, a mesh convergence study was performed on Frame B. The mesh convergence study was run by finding the
resulting displacement of Frame B that corresponded to a load of 2.01 N for different size meshes. Results of the simulation were judged off a “true solution” which was the displacement that Frame B underwent during experimental testing when it was subjected to a load of 2.01 N. It was ultimately determined that a curvature based mesh with an element size of 4.00 mm should be used because it had essentially no percent change from trial to trial. Results of the curvature based mesh convergence study are shown in Table B.1.

Table B.1: Mesh convergence study for the curvature based mesh run on Frame B to determine the proper mesh size to use on the plane frames during FEA simulation.

<table>
<thead>
<tr>
<th>Element Size (mm)</th>
<th>Displacement (mm)</th>
<th>% Change</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.50</td>
<td>3.45</td>
<td>-</td>
<td>8.00%</td>
</tr>
<tr>
<td>6.00</td>
<td>3.46</td>
<td>0.12%</td>
<td>8.00%</td>
</tr>
<tr>
<td>5.50</td>
<td>3.46</td>
<td>0.14%</td>
<td>8.00%</td>
</tr>
<tr>
<td>5.00</td>
<td>3.47</td>
<td>0.29%</td>
<td>8.00%</td>
</tr>
<tr>
<td>4.5</td>
<td>3.48</td>
<td>0.09%</td>
<td>8.00%</td>
</tr>
<tr>
<td>4.00</td>
<td>3.48</td>
<td>0.00%</td>
<td>8.00%</td>
</tr>
</tbody>
</table>

True Solution = 3.21 mm

B.1.2 Rings

All of the rings that were analyzed in SolidWorks used beam elements as a mesh. The number of elements and there location was automatically generated by SolidWorks. An attempt was made to use tetrahedral elements, but the results did not match the experimental data so beam elements were used. Figure B.2 shows an example of the mesh that was used during simulation.

Figure B.2: Beam element mesh on Ring 160\textsubscript{20} used in SolidWorks static and frequency simulation.
B.2 Static Simulation

SolidWork’s static simulation was used to determine the computational stiffness of the structures. The study was run by applying a force to the structure and running the simulation to see what the maximum displacement was. This data was then plotted against the force and fit with a trend line, whose slope was the stiffness of the structure. Issues of large displacements resulting in a non-linear curve were a non-issue because the static simulation uses all linear theory.

B.3 Frequency Simulation

SolidWork’s frequency simulation was used to determine the computational value for all of the necessary vibration modes of the structures. Results of the study give the numerical value for the frequency as well as the mode shapes of the structure.
Appendix C

Ring Stiffness Plots

C.1 10mm Wide Rings

Figure C.1: Force vs deflection curves for 10 mm wide rings under flat bar load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of $\bar{k} = 51.73$. 
Figure C.2: Force vs deflection curves for 10 mm wide rings under tensile load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of $\bar{k} = 83.95$.

C.2 20mm Wide Rings

Figure C.3: Force vs deflection curves for 20 mm wide rings under point load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of $\bar{k} = 61.33$. 
Figure C.4: Force vs deflection curves for 20 mm wide rings under flat bar load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of \( \bar{k} = 57.41 \).

Figure C.5: Force vs deflection curves for 20 mm wide rings under tensile load. (a) Dimensional plot showing the increase in the slope of the trendline as the diameter of the ring decreases. (b) Dimensionless plot of force vs deflection to give a slope of \( \bar{k} = 100.92 \).
C.3 Stiffness Values from Flat Bar Loading

Table C.1: Stiffness (N/mm) of rings tested from experimental and theoretical analysis. Experimental loading was applied using a flat bar. Dimensions for the rings can be seen in Table 3.3.

<table>
<thead>
<tr>
<th>Ring</th>
<th>EXP</th>
<th>THEO</th>
<th>( k_{\text{EXP}} / k_{\text{THEO}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>240(_{10})</td>
<td>0.05</td>
<td>0.05</td>
<td>1.05</td>
</tr>
<tr>
<td>230(_{10})</td>
<td>0.06</td>
<td>0.06</td>
<td>0.95</td>
</tr>
<tr>
<td>220(_{10})</td>
<td>0.07</td>
<td>0.07</td>
<td>1.00</td>
</tr>
<tr>
<td>210(_{10})</td>
<td>0.08</td>
<td>0.08</td>
<td>1.01</td>
</tr>
<tr>
<td>200(_{10})</td>
<td>0.08</td>
<td>0.95</td>
<td>0.92</td>
</tr>
<tr>
<td>190(_{10})</td>
<td>0.10</td>
<td>0.10</td>
<td>0.97</td>
</tr>
<tr>
<td>180(_{10})</td>
<td>0.12</td>
<td>0.12</td>
<td>0.99</td>
</tr>
<tr>
<td>170(_{10})</td>
<td>0.13</td>
<td>0.15</td>
<td>0.92</td>
</tr>
<tr>
<td>160(_{10})</td>
<td>0.17</td>
<td>0.18</td>
<td>0.98</td>
</tr>
<tr>
<td>150(_{10})</td>
<td>0.19</td>
<td>0.21</td>
<td>0.91</td>
</tr>
<tr>
<td>140(_{10})</td>
<td>0.25</td>
<td>0.26</td>
<td>0.97</td>
</tr>
<tr>
<td>130(_{10})</td>
<td>0.30</td>
<td>0.33</td>
<td>0.91</td>
</tr>
<tr>
<td>120(_{10})</td>
<td>0.38</td>
<td>0.41</td>
<td>0.91</td>
</tr>
<tr>
<td>110(_{10})</td>
<td>0.44</td>
<td>0.54</td>
<td>0.82</td>
</tr>
<tr>
<td>100(_{10})</td>
<td>0.72</td>
<td>0.72</td>
<td>1.00</td>
</tr>
<tr>
<td>240(_{20})</td>
<td>0.12</td>
<td>0.10</td>
<td>1.11</td>
</tr>
<tr>
<td>230(_{20})</td>
<td>0.13</td>
<td>0.12</td>
<td>1.07</td>
</tr>
<tr>
<td>220(_{20})</td>
<td>0.15</td>
<td>0.13</td>
<td>1.12</td>
</tr>
<tr>
<td>210(_{20})</td>
<td>0.17</td>
<td>0.15</td>
<td>1.10</td>
</tr>
<tr>
<td>200(_{20})</td>
<td>0.18</td>
<td>0.18</td>
<td>0.99</td>
</tr>
<tr>
<td>190(_{20})</td>
<td>0.23</td>
<td>0.21</td>
<td>1.08</td>
</tr>
<tr>
<td>180(_{20})</td>
<td>0.27</td>
<td>0.25</td>
<td>1.08</td>
</tr>
<tr>
<td>170(_{20})</td>
<td>0.31</td>
<td>0.29</td>
<td>1.06</td>
</tr>
<tr>
<td>160(_{20})</td>
<td>0.41</td>
<td>0.35</td>
<td>1.18</td>
</tr>
<tr>
<td>150(_{20})</td>
<td>0.43</td>
<td>0.42</td>
<td>1.01</td>
</tr>
<tr>
<td>140(_{20})</td>
<td>0.57</td>
<td>0.52</td>
<td>1.10</td>
</tr>
<tr>
<td>130(_{20})</td>
<td>0.72</td>
<td>0.65</td>
<td>1.10</td>
</tr>
<tr>
<td>120(_{20})</td>
<td>0.93</td>
<td>0.83</td>
<td>1.13</td>
</tr>
<tr>
<td>110(_{20})</td>
<td>1.30</td>
<td>1.08</td>
<td>1.21</td>
</tr>
<tr>
<td>100(_{20})</td>
<td>1.50</td>
<td>1.43</td>
<td>1.04</td>
</tr>
</tbody>
</table>
This program plots the force vs displacement for ring subjected to axial and tension loads. It also plots the shape of deflected shape.

%Values used to isolate the cell in the cell array produced from bwboundaries that contained the inside perimeter of the ring
arrayValue_max = 2000; %Upper limit for row value for the cell array
arrayValue_min = 800; %Lower limit for row value for the cell array

%The undeformed 240.10 was used to calibrate the conversion from pixel %deflection to mm deflection. The undeformed inside diameter of Ring B is %199.5mm and 325 pixels
diameter_pixels = 430;
diameter_mm = 238;
pixeltomm_conversion = diameter_mm/diameter_pixels; % Conversion factor

% Loads pictures for Ring i
contents = dir('*.PNG');

% Loop through all picture of Ring i under various applied loads. Calculates
% the vertical and horizontal diameter of Ring under given load
hold on;
for i = 1:numel(contents)

%%% Finding load value corresponding to Ring (i) %%%
% Extracts the string containing the name of picture i from struct contents
picture_name = contents(i,1).name;

% Segments the picture name to only contain values of the applied load
load_str = picture_name(1:end-4);

% Converts the string containing the load value to numeric value
load(i) = str2num(load_str);

% Loading Image (i) in folder
baseFilename = contents(i).name;

% Converting Image to RGB
I_RGB = imread(baseFilename);
% imshow(I_RGB)
% Im = imtool(I_RGB)

% Convert the Image to Gray Scale
I_GRAY = rgb2gray(I_RGB);
% imshow(I_GRAY)
% imshow(I_GRAY)

% Threshold Gray Scale Image to get binary image
ThresholdValue = 0.30; % Chosen by trial and error to get best BW image

I\_BW = \text{im2bw}(I\_GRAY, \text{ThresholdValue});

% \text{imshow}(I\_BW)

% \text{bwboundaries} traces the boundaries of the binary image and produces a cell
% \text{array that contains matrices with the (x,y) coordinates of the different
% \text{perimeters for objects within the picture
} boundaries = \text{bwboundaries}(I\_BW);

% Loop is used to determine the cell in boundaries where the (x,y) data for
% the inside perimeter of the ring is contained
for \ j = 1:length(boundaries)
    x = \text{cellfun}('\text{size}', \text{boundaries}(j),1);
    if x < \text{arrayValue\_max} && x > \text{arrayValue\_min}
        break
    end
end

% Converts cell array to a matrix
ring = \text{cell2mat}(\text{boundaries}(j));

%%%Plotting Shape of Rings%%%
colorvec = \text{hsv}(\text{length}(contents));
\text{set}(\text{gca},'\text{Ydir}','\text{reverse'});
\text{set}(\text{gca},'\text{Xdir}','\text{reverse'});
\text{plot}(\text{ring}(:,2),\text{ring}(:,1),'\text{Color}',\text{colorvec}(i,:));
\text{xlabel}('\text{Horizontal Diameter (pixels')}','\text{interpreter}','\text{latex'},
'\text{fontsize}',14);
\text{ylabel}('\text{Vertical Diameter (pixels')}','\text{interpreter}','\text{latex'},
'\text{fontsize}',14);
LegendInfo\{i\} = [''$F$' = num2str(load(i)) 'N'];
\text{xlim}([0 500]);
\text{ylim}([0 500]);

% Max and min values of the ring in pixels
xmin = \text{min}(\text{ring}(:,2));
%Calculate the horizontal and vertical diameters based on max & min values
% Vertical Pixel Diameter (i) = ymax - ymin;
% Horizontal Pixel Diameter (i) = xmax - xmin;

end
hold off;

%%%Create legend for defromed Rings plot%%% 
I = legend(LegendInfo);
set(I,'interpreter', 'latex');

%%%Convert diameters from pixels to mm%%% 
Vertical_mmDiameter = Vertical_PixelDiameter* pixelto毫米_conversion;
Horizontal_mmDiameter = Horizontal_PixelDiameter* pixelto毫米_conversion;

%%%Calculate the deflection vertical and horizontal deflection of each Ring%%%
% by taking the difference of the undeformed diameter and the deformed diameters
Vertical_Deflection = (Vertical_mmDiameter - Vertical_mmDiameter(1,1))*(-1);
Horizontal_Deflection = (Horizontal_mmDiameter - Horizontal_mmDiameter(1,1))*(1);

%%%Plot the force vs displacement curve%%% 

% Vertical
figure;
hold on;
scatter(Vertical_Deflection, load , 'b*');
Vxy_constants = polyfit(Vertical_Deflection, load, 1);
trendline = Vxy_constants(1)*Vertical_Deflection + Vxy_constants(2);
plot(Vertical_Deflection,trendline,'k');

theString = sprintf('y = %.3f x + %.3f', Vxy_constants(1), Vxy_constants(2));

text(12, 0.5, theString,'interpreter','latex','fontsize',12);

xlabel('Displacement (mm)','interpreter','latex','fontsize',14);

ylabel('Force (N)','interpreter','latex','fontsize',14);

title({'\makebox[4in][c]{Force vs Displacement - 240_{10}}', ...
       '\makebox[4in][c]{Vertical Displacement, Compressive Force }
   '},'Interpreter','latex','fontsize',16);

hold off;

%%%%Print Ring Stiffness to Command Window%%%%

V = sprintf('k = %.3f N/mm', Vxy_constants(1)); %Vertical Stiffness

disp(V);

toc; %End timer
Appendix E

Nested Ring Plots

E.1 10mm Wide Rings

![Graph showing force vs deflection plot for Nest 1 system subjected to flat bar loading. Dimensions of nest given in Table 3.6.]

\( k_{\text{exp}1} = 0.48 \text{ N/mm} \)
\( k_{\text{exp}1+2} = 1.02 \text{ N/mm} \)
\( k_{\text{theo}1+2+3} = 1.67 \text{ N/mm} \)

**Figure E.1:** Force vs deflection plot for Nest 1 system subjected to flat bar loading. Dimensions of nest given in Table 3.6.
Figure E.2: Force vs deflection plots for Nest 2 system. Dimensions of nest given in Table 3.6.

Figure E.3: Force vs deflection plots for Nest 6 system. Dimensions of nest given in Table 3.6.
Figure E.4: Force vs deflection plots for Nest 3 system. Dimensions of nest given in Table 3.6.

Figure E.5: Force vs deflection plots for Nest 4 system. Dimensions of nest given in Table 3.6.
Figure E.6: Force vs deflection plots for Nest 5 system. Dimensions of nest given in Table 3.6.

E.2 20mm Wide Rings

Figure E.7: Force vs deflection plots for Nest 7 system. Dimensions of nest given in Table 3.6.
Figure E.8: Force vs deflection plots for Nest 8 system. Dimensions of nest given in Table 3.6.

Figure E.9: Force vs deflection plots for Nest 9 system. Dimensions of nest given in Table 3.6.
Figure E.10: Force vs deflection plots for Nest 12 system. Dimensions of nest given in Table 3.6.

Figure E.11: Force vs deflection plots for Nest 10 system. Dimensions of nest given in Table 3.6.
Figure E.12: Force vs deflection plots for Nest 11 system. Dimensions of nest given in Table 3.6.
Bibliography


