A Geometrical Exposition of Input-Output Analysis

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A number of important theorems have been developed regarding the qualitative properties that input-output and related economic models must possess in order to be viable. That a number of apparently different notions of viability (for example, all industries can make profits, balanced growth is possible, any set of final demands can be produced) turn out to be equivalent, is one of the more important results. These theorems, normally expressed in the vernacular of the mathematician, have enjoyed only limited recognition, however, despite their proven usefulness and the ever increasing interest in such models. The strategy of this paper will be to exploit the purely geometrical nature of one condition for viability in order to expose the equivalence of seven other important conditions. It is hoped that this relatively simple geometrical exposition will not only tie the main mathematical results together but will also bring these results within the reach of a wider audience.

The input-output model is specified by the (semipositive) matrix of input coefficients1

\[
A = [a_{ij}] = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\]

where \(a_{ij}\) denotes the amount of the \(i\)th good required as an input to produce each unit of the \(j\)th good. The \(j\)th column of \(A\) corresponds, therefore, to the input requirements of the \(j\)th industry (per unit of output). The gross output levels are given by the vector

\[
x = (x_1, x_2, \ldots, x_n)'
\]

where \(x_i\) denotes the gross output of the \(i\)th good. Notice, in this context, that the vector \(Ax\) represents the input requirements associated with gross output levels equal to \(x\) and, thus, the vector

\[
x - Ax = (I - A)x
\]

represents the net output levels corresponding to \(x\).

A prime concern in input-output analysis is the question of whether or not a particular set of final demands can be satisfied. In other words, in what circumstances is the following "weak solvability" condition satisfied?2

1) Weak Solvability: \((I - A)x \geq c\) has a solution \(x \geq 0\) for some set of final demands \(c > 0\).

It is also of interest to determine the conditions under which any set of final demands could be satisfied.

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1 The matrix \(A\) is semipositive if every row and column is a semipositive vector. A semipositive vector is a vector composed of nonnegative elements, at least one of which is strictly positive. Since semipositive matrices form the basis of the von Neumann model and other linear economic models as well as the input-output model, the methods of this paper can be easily extended to more general circumstances.

2 We employ the conventional vector notation that \(a = b\) iff \(a_i = b_i\) for all \(i\), \(a \geq b\) iff \(a_i \geq b_i\) for all \(i\), \(a \geq b\) iff \(a \geq b\) and \(a \neq b\), and \(a > b\) iff \(a > b\) for all \(i\).
2) Strong Solvability: \((I - A)x = c\) has a solution \(x \geq 0\) for any \(c \geq 0\).

A related issue involves the question of whether or not there exist prices at which all industries could operate profitably.

3) Profitability: \(p[I - A] > 0\) has a solution \(p > 0\).

A final concern involves the dynamic specification of an input-output model that outputs of one period become the inputs of the next period, i.e.,

\[Ax_t \leq x_{t-1}\]

denote the input constraints for production in period \(t\). The issue in this context is to determine whether or not growth is possible.

4) Growth: \(Ax_t \leq x_{t-1}, x_t > x_{t-1}\) has a solution \(x_t, x_{t-1} \geq 0\).

The fact that these conditions take the form of restrictions upon the matrix \(A\) is obvious. The fact that these conditions are equivalent to one another and to any of the following additional conditions is less obvious.\(^3\)

5) Hawkins-Simon: The leading principal minors of \([I - A]\) are all positive, i.e.,

\[\det (1 - a_{11}) > 0,\]

\[\det \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix} > 0, \ldots,\]

\[\det [I - A] > 0\]

6) Dominant Diagonal: The matrix \([I - A]\) has a dominant positive diagonal, i.e., there exist numbers (weights) \(d_j > 0\) such that

\[d_j(1 - a_{jj}) > \sum_{i \neq j} d_i a_{ij}, \quad j = 1, \ldots, n\]

7) The dominant root of \(A\) is less than one.\(^4\)

Condition 7) gives rise to interest in the dominant root of semipositive matrices.

The most important theorems in this regard may be stated as follows:\(^5\)

If \(A\) is a semipositive (indecomposable)\(^6\) matrix and if \(\lambda^*\) and \(x^*\) denote the dominant root and the associated characteristic vector, then:\(^7\)

(a) \(\lambda^* \geq 0\) (\(\lambda^* > 0\))

(b) \(x^* \geq 0\) (\(x^* > 0\))

(c) \(\frac{\partial \lambda^*}{\partial a_{ij}} \geq 0\) \(\left(\frac{\partial \lambda^*}{\partial a_{ij}} > 0\right)\)

\[i, j = 1, \ldots, n\]

(d) \([I - A]^{-1} \geq 0\) \(\text{iff } \lambda > \lambda^*\)

\([I - A]^{-1} > 0\) \(\text{iff } \lambda > \lambda^*\)

(e) \(x^*\) is not necessarily the only semipositive characteristic vector. (\(x^*\) is the only semipositive characteristic vector.)

(f) \([I - A]\) has all leading principal minors positive \(\text{iff } \lambda > \lambda^*\).

The purpose of this note is to explain geometrically conditions 1) through 7) and theorems (a) through (f). This explanation will be illustrated in the context of a two-good model \((n = 2)\) where the matrix \(A\) is

\[(\text{possibly complex})\]

(possibly complex) are referred to as a characteristic root and vector, respectively, of the matrix \(A\) iff \(Ax = \lambda x\).

The dominant root is the root of largest absolute value (the square root of the sum of the squares of the real and imaginary parts of the complex root). In the case of a semipositive matrix this root happens to be real and nonnegative and the associated characteristic vector is real.

\(^3\) These results are referred to as the Perron-Frobenius Theorems. See Edwin Burmeister and A. Rodney Dobell, pp. 437–40 for a compact statement of these and related results.

\(^4\) A matrix is decomposable (indecomposable) if there exists (does not exist) a partition of the set of indices \(\{1, 2, 3, \ldots, n\}\) into nonempty sets \(I\) and \(J\) such that \(a_{ij} = 0\) \(i \in I, j \in J\). If such a partition exists it means that the industries in \(J\) can operate independently of the industries in \(I\). If no such partition exists then every industry requires either directly or indirectly the output of every other industry.

\(^5\) Parentheses enclose those results which depend upon \(A\) being indecomposable as well as semipositive.
indecomposable. The arguments can be immediately generalized to three goods \((n=3)\) and offer considerable insights into the general case, however. The modifications that are necessary when \(A\) is decomposable require no significant changes in approach and will be obvious to the reader.

The following result will be indispensable in this endeavor: If \(a, b \neq 0\) then \(a \cdot b = |a| |b| \cos \theta\), where \(\theta\) is the included angle between the vectors \(a\) and \(b\). Thus:

\[
a \cdot b > 0 \text{ iff } a \text{ and } b \text{ form an acute angle } (0 \leq \theta < 90^\circ)
\]

\[
a \cdot b = 0 \text{ iff } a \text{ and } b \text{ form a right angle } (\theta = 90^\circ)
\]

\[
a \cdot b < 0 \text{ iff } a \text{ and } b \text{ form an obtuse angle } (90^\circ < \theta \leq 180^\circ)
\]

The system that will be illustrated has the input coefficients matrix

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

We will assume initially \(A > 0\) and that \(a_{11}, a_{22} < 1\) (\(a_{12} = 0\) or \(a_{21} = 0\) would correspond to \(A\) being decomposable and \(a_{ii} < 1\) means that the input requirement of an industry for its own output is less than the industry’s total output).

Let us begin by considering the columns of the matrix

\[
[I - A] = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}
\]

The first column has the sign pattern \((+ -)\) and thus belongs to the fourth quadrant while the second column has the sign pattern \((- +)\) and thus belongs to the second quadrant. Beyond this observation there are two interesting possibilities: 1) the cone generated by the two column vectors contains all of the positive (first) quadrant, or 2) the cone generated by the two column vectors contains none of the positive quadrant. The first possibility is illustrated in Figure 1 and the second in Figure 2 (the cone is depicted by the shaded area in both cases). In both illustrations, the first column \((I-A)_{11}\) is labeled \(B_1\) and the second column \((I-A)_{22}\) is labeled \(B_2\).

In Figure 1 the cone \(B_2OB_1\) contains all of the positive quadrant. The only other possibilities are that \(B_1\) lies in the extended line \(OB_1\) and thus that the cone \(B_2OB_1\) is simply this extended line (the case in which \([I-A]\) is singular) or that the cone \(B_2OB_1\) contains the negative (third) quadrant (illustrated in Figure 2). There are simply no other possibilities for the two vectors: they either “bend” toward the positive quadrant or they do not bend at all or they bend toward the negative quadrant. If there were three goods, the analogous (three dimensional) cone would either contain the positive orthant (see Figure 3), be a plane or a line, or contain the negative orthant. Similarly, with \(n\) goods the cone either contains the positive orthant, is a linear subspace, or contains the negative orthant. Notice that in all cases the cone either contains all of the positive orthant or none of the positive orthant.

It can now be shown that conditions 1) through 7) are equivalent by arguing that each is equivalent to:

8) The cone generated by the columns of \((I-A)\) contains the positive orthant, i.e.,

9 The cone generated by a set of vectors \(B_1 \ldots B_m \in \mathbb{R}^n\) is defined to be the set \(\{y \in \mathbb{R}^n | y = \sum_{i=1}^{m} a_i B_i, a_i \geq 0, i = 1, \ldots, m\}\).

10 The positive orthant is the set \(\{y \in \mathbb{R}^n | y > 0\}\). The negative orthant is the set \(\{y \in \mathbb{R}^n | y < 0\}\).

11 A linear subspace is a set \(S\) that has the property that if \(x_1, x_2 \in S\) then \(a_1 x_1 + a_2 x_2 \in S\) for all \(a_1, a_2 \in \mathbb{R}\) (all real numbers \(a_1\) and \(a_2\)).
\[ \{ y \in \mathbb{R}^n \mid y = (I - A)x, \ x \geq 0 \} \supset \{ y \in \mathbb{R}^n \mid y > 0 \} \]

Condition 1) states that the cone generated by \([I - A]\) contains at least one element from the positive orthant. Since this cone either contains all of the positive orthant or none of it, condition 1) is equivalent to condition 8). Condition 2) states that the cone generated by \((I - A)\) contains all of the nonnegative orthant (the positive orthant together with the positive axes) and is thus obviously equivalent to condition 8).

Condition 3) states that there exists a positive vector \(p\) that forms a strictly positive scalar product with every column of \([I - A]\). This simply means that \(p\) forms an acute angle with every column of \([I - A]\). To say that every column of \([I - A]\) forms an acute angle with a strictly positive vector is clearly equivalent, moreover, to the statement that the columns of \([I - A]\) bend toward the positive orthant or the statement that the cone generated by the columns of \([I - A]\) contains the positive orthant. Thus condition 3) is equivalent to condition 8). Notice in this regard that the interior of the cone \(EOF\) in Figure 1 represents the set of price solutions.
vectors at which both industries could operate profitably. (The analogous cone $EOF$ in Figure 2, on the other hand, contains only negative prices.)

Condition 4) states that there exist $x_t, x_{t-1}$ such that $x_t > x_{t-1}$ and $Ax_t \leq x_{t-1}$. If we define

$$c = x_t - x_{t-1} > 0$$

we may simplify this condition as follows:

$$x_{t-1} - Ax_t \geq 0$$
$$x_{t-1} - A(x_{t-1} + c) \geq 0$$
$$x_{t-1} - Ax_{t-1} + Ac > 0$$
$$(I - A)x_{t-1} > 0$$

Condition 4) states, therefore, that the cone generated by the columns of $[I - A]$ contains an element from the positive orthant. Thus condition 4) is equivalent to condition 1) which has already been shown to be equivalent to condition 8).

Condition 5) is the most difficult to interpret. For the case of two goods ($n=2$) it requires that

$$\text{det} (1 - a_{11}) > 0,$$
$$\text{det} \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix} = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} > 0$$

Notice that it is necessary for the second-
order determinant to be positive that 
\((1 - a_{22})\) be positive also. A moment’s
consideration reveals that if either 
\((1 - a_{11})\) or 
\((1 - a_{22})\) were negative the corresponding
column of \((I - A)\) would lie in the negative
(third) quadrant and it would be impossible
for the cone generated by the columns of
\((I - A)\) to contain the positive quadrant
irrespective of the position of the other
column. This reasoning is also correct for
the case of three or more goods.

Sufficient conditions for higher order
determinants to be positive are more
troublesome. Routine calculation reveals,
however, that the determinant of \((I - A)\)
for the case of two goods is (except for
sign) the area of the parallelogram
determined by the two columns of
\((I - A)\).\(^{12}\)
That is, in Figure 1 the determinant of
\((I - A)\) is equal to the area of parallelo-
gram \(B_0 O B_2 D\) and in Figure 2 the determi-
nant of \((I - A)\) is equal to \(-\) the area
\(B_1 O B_2 D\). The sign in each case depends
upon whether the area is obtained by
"sweeping" counterclockwise from \(B_1\)
to \(B_2\) (positive sign) or by sweeping
clockwise from \(B_1\) to \(B_2\) (negative sign).\(^{13}\)
Thus requiring that both determinants be posi-
tive is equivalent to requiring that the
cone generated by the (two) columns of
\((I - A)\) contain the positive quadrant.

The case of three goods is only slightly
more difficult. Here the determinant of
\((I - A)\) can be interpreted (except for
sign) as the volume of the parallelepiped
formed by the (three) columns of \((I - A)\).\(^{14}\)
(This parallelepiped is illustrated in Fig-
ure 3.) The second-order leading principal
minor is moreover (except for sign) the
area of the parallelogram obtained by pro-
jecting \((I - A) \cdot 0(I - A) \cdot D\) onto the plane
formed by the first and second coordinate
axes.

This second-order leading principal mi-
nor is positive if the direction of movement
from \((I - A)_{11}\) to \((I - A)_{22}\) is counterclock-
wise in the plane formed by the first and
second coordinate axes.\(^{15}\) If the first-
and second-order leading principal minors are
positive the sign of the determinant of
\((I - A)\) depends upon whether the direc-
tion of movement from \((I - A)_{11}\) to
\((I - A)_{22}\) is counterclockwise (positive
sign) or clockwise (negative sign) in the
plane formed by the second and third
coordinate axes.

The three goods case with all leading
principal minors positive is illustrated in
Figure 3. That the first, second, and third-
order leading principal minors are positive
is illustrated, respectively, by the following
observations: \((1 - a_{11})\) is positive, the
direction of sweep is counterclockwise from
\((I - A)_{11}\) to \((I - A)_{22}\), and the directions
of sweep from \((I - A)_{11}\) to \((I - A)_{22}\) and
from \((I - A)_{22}\) to \((I - A)_{33}\) are both
clockwise.\(^{16}\) These conditions are
equally clear to the statement that the
cone generated by \((I - A)\) contains the
positive orthant.\(^{17}\)

\(^{12}\) If \((I - A)\) were singular, the parallelogram would
degenerate (collapse) into a line segment which has zero
area.

\(^{13}\) The order \(B_1\) to \(B_2\) is crucial. Recall that switching
columns changes the sign of the determinant.

\(^{14}\) If \((I - A)\) were singular the parallelepiped would
degenerate (collapse) into a parallelogram which has
zero volume. See Michael Spivak, p. 83, for a general
discussion of the determinant as an "oriented" volume.

\(^{15}\) There is a simple "rule of thumb" for determining
the direction of counterclockwise sweep. Point the
thumb of your right hand along the positive direction of
one axis and the fingers curl in the direction of coun-
terclockwise sweep (positive orientation) for movements
in the perpendicular plane.

\(^{16}\) The sign of the determinant of \((I - A)\) is deter-
mined in general by interchanging columns of \((I - A)\)
until the resulting matrix has a normal orientation (until
every column lies in the direction of counterclockwise
sweep from the preceding column). If the number of re-
quired "switches" is odd, then the sign of the original
determinant is negative; otherwise, the sign is positive.
Note that in Figure 2 \((I - A)\) has a normal orientation
to begin with and thus requires zero switches.

\(^{17}\) It should be noted that if the leading principal
minors are all positive, then all principal minors must
be positive. In Figure 2, for example, the remaining
Similar reasoning (with hyper-parallel-epipeds) would establish the same result for an arbitrary number of goods and we may conclude that condition 5) is equivalent to condition 8).

Condition 6) states that there exists a vector

$$d = (d_1, \ldots, d_n) > 0$$

such that $$d(I - A) > 0$$.

This condition is obviously equivalent to condition 3) which has already been discussed.

Condition 7) requires that the largest real root of $$A$$ be less than one. To see this, first consider the case of two goods illustrated in Figure 1. The dominant root is
the largest real number $\lambda$ such that

$$ \det \begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix} = 0 $$

Notice that for $\lambda = 1$ the determinant is equal to the area $B_1OB_2D$ (in Figure 1) which is certainly not zero. It cannot be the case, therefore, that $\lambda = 1$ is the dominant root.

Suppose now that we increase the "trial" value of $\lambda$ to some value greater than one. This has the effect of bending the columns of $[\lambda I - A]$ further toward the positive quadrant and is illustrated in Figure 1 by the shifts from $OB_1$ to $OB_1'$ and from $OB_2$ to $OB_2'$. It is apparent that further increasing the trial value of $\lambda$ will never make the determinant of $[\lambda I - A]$ equal to zero. The only case in which the determinant is zero occurs when both columns of $[\lambda I - A]$ lie in the same line (when the parallelogram collapses into a line segment). Increasing $\lambda$ above 1 in Figure 1 simply bends these columns the wrong way.

Since any value of $\lambda$ above one cannot be the dominant root, consider the effect of reducing $\lambda$ below one. This has the effect of bending the columns of $[\lambda I - A]$ away from the positive quadrant and toward the negative quadrant and is illustrated by the shifts from $OB_1$ to $OB_1'$ and from $OB_2$ to $OB_2'$. In fact, when $\lambda$ is zero both columns of $[\lambda I - A]$ belong to the negative quadrant. It must be the case, therefore, that at some value of $\lambda$ between one and zero the columns of $[\lambda I - A]$ cease to bend toward the positive quadrant and begin to bend toward the negative quadrant. At this point the columns of $[\lambda I - A]$ lie in the same line—$B_1'OB_2'$ in Figure 1. This value of $\lambda$ is, of course, the dominant root and is denoted by $\lambda^*$. That is, $\lambda^*$ appears at the point where the dominant diagonal vanishes.

18 $\lambda$ is a solution of $Ax = \lambda x$ iff $\det[\lambda I - A] = 0$. See Hadley.

Similar reasoning applied to Figure 2 leads to the conclusion that the dominant root must be greater than one when the cone contains the negative quadrant. The only other possibility, that $[I - A]$ is singular, corresponds to a dominant root equal to one. It is clear, therefore, that in the case of two goods the dominant root of $A$ being less than one is equivalent to the cone generated by the columns of $[I - A]$ containing the positive quadrant.

For the case of three goods the reasoning is quite analogous. Here the problem is to choose $\lambda$ so that the columns of $[\lambda I - A]$ lie in a plane. If the cone generated by the columns of $(I - A)$ contains the positive orthant, we must choose $\lambda$ less than one in order to bend these columns away from the positive orthant and toward the negative orthant (see Figure 3). If the columns of $[I - A]$ generate a plane, then $[I - A]$ is singular and the dominant root is one. Lastly, if the columns of $[I - A]$ generate a cone which contains the negative orthant, then we must choose $\lambda$ greater than one in order to bend these columns away from the negative orthant and toward the positive orthant. It is clear that this reasoning holds for an arbitrary number of goods and we may conclude that condition 7) is equivalent to condition 8).

An indirect implication of the above analysis is that the dominant root is nonnegative in all cases—Theorem (a). This can be seen as follows. For any negative $\lambda$ the columns of $[\lambda I - A]$ belong to the nonpositive orthant. On the other hand, by making $\lambda$ sufficiently positive $[\lambda I - A]$ can be made to generate a cone which contains the positive orthant (each column of $\lambda I - A$ approaches the respective positive coordinate axis as $\lambda$ becomes larger and larger). There must exist a real characteristic root, therefore, between the negative and positive values of $\lambda$ (which

19 The nonpositive orthant is the set $\{y \in \mathbb{R}^n \mid y \leq 0\}$. 
corresponds to the point at which the columns of \( [\lambda I - A] \) bend neither toward
the negative nor the positive orthants. Since this is true for any negative \( \lambda \) this
root must be nonnegative. That is, there
must exist \( \lambda^* \) greater than or equal to zero
which satisfies these conditions.\(^{20}\)

Notice that the columns of \( [\lambda^* I - A] \)
generate a degenerate or flat cone (a linear
subspace). (This cone is illustrated by
the infinite line through \( B^*_1OB^*_2 \) in Figure 1 or
Figure 2.) The subset of this cone that
would be obtained by considering only
those nonnegative weights for the columns
of \( [\lambda^* I - A] \) that sum to one, i.e.,
\[
\begin{align*}
\{ y &\mid y = \sum_{i=1}^{n} \alpha_i (\lambda^* I - A)x_i \}, \\
\alpha_i &\geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1 \}
\end{align*}
\]
is a convex polyhedral set whose vertices
are formed by the columns of \( [\lambda^* I - A] \)
and which contains the origin. (This subset
is illustrated by the line segment
\( B^*_1OB^*_2 \) in Figure 1 or Figure 2 and would
be a triangle passing through the origin in
Figure 3.) That set of weights \( x^*_i \) which
generates the origin (causes the sum of the
weighted columns to be the origin) is a
semipositive vector which satisfies
\[
\sum_{i=1}^{n} x^*_i (\lambda^* I - A)x_i = 0
\]
and is thus the semipositive characteristic
vector associated with the dominant root
—Theorem (b).\(^{21}\)

Theorem (c) follows from the fact that
increasing any element of \( A \) either has no
effect upon the (degenerate) cone gener-
ated by the columns of \( [\lambda^* I - A] \) (possible
only if \( A \) is decomposable) or it causes
these columns to bend toward the negative
orthant. For example, increasing either
\( a_{11} \) or \( a_{21} \) in Figure 1 or Figure 2 rotates
the first column of \( (\lambda^* I - A) \) toward the
negative quadrant. To bend these columns
back toward the positive orthant (to ob-
tain the dominant root of the new matrix)
requires, therefore, increasing \( \lambda \) above the
value of the original dominant root.

Theorems (d) and (f) both involve the
fact that increasing \( \lambda \) above \( \lambda^* \) causes
the columns of \( [\lambda I - A] \) to bend toward
the positive orthant. This means that \( \lambda \) being
greater than \( \lambda^* \) is equivalent to the cone
generated by the columns of \( [\lambda I - A] \)
containing the positive orthant. The discus-
sion of condition 5) revealed, however,
that the latter condition is necessary and
sufficient for all leading principal minors
of \( [\lambda I - A] \) to be positive. Thus \( \lambda \) being
greater than \( \lambda^* \) is equivalent to all leading
principal minors of \( [\lambda I - A] \) being positive
—Theorem (f).

Theorem (d) says, in essence, that the
inverse of \( [\lambda I - A] \) is semipositive if and
only if the cone generated by the columns
of \( [\lambda I - A] \) contains the positive orthant.
To see this, let us construct the inverse of
such a matrix. Notice that in Figure 1 the
columns of \( [I - A] \) generate a cone which
contains the positive orthant. We may
determine the rows of \( [I - A]^{-1} \) by the
following method. The fact that
\[
[I - A]^{-1}[I - A] = I
\]
Means that the first row of \( [I - A]^{-1} \) must
form an acute angle with the first column
of \( [I - A] \) (the product of these vectors
must be one) and a right angle with the
second column of \( [I - A] \) (the product of
these vectors must be zero). The first row
of \( [I - A]^{-1} \) must lie, therefore, somewhere
along the extended line \( OF \) in Figure 1.

\(^{20}\) The fact that \( \lambda^* \) is strictly positive when \( A \) is in-
decomposable is suggested by Figures 1, 2, and 3. For
\( \lambda^* \) actually to be zero it is necessary that \( A \) be decom-
posable and singular.

\(^{21}\) In Figure 1, notice that \( B^*_1 \) and
\( \langle |B^*_1| / |B^*_2| \rangle B^*_2 \) have the same length. Thus
\( B^*_1 + \langle |B^*_1| / |B^*_2| \rangle B^*_2 = 0 \)
and \( x^* = (1, |B^*_1| / |B^*_2|) > 0 \).
Similar reasoning reveals that the second row of $[I-A]^{-1}$ must lie along $OE$. Notice that this inverse matrix must be positive (or, perhaps, semipositive if $A$ were decomposable).\footnote{22 If $a_{21}$ were zero, for example, the first column of $[I-A]$ would lie along the positive first axis and the second row of $[I-A]^{-1}$ would lie along the positive second axis (and the lower left-hand corner of the inverse is zero). The partition of $\{1, 2\}$ is given in this case by $I=\{2\}, J=\{1\}$.}

For an arbitrary number of goods one obtains a row of $[\lambda I-A]^{-1}$ by finding a vector that forms a zero scalar product (a right angle) with all columns of $[\lambda I-A]$ save one and a positive scalar product (an acute angle) with that one. Thus every row of the inverse must form a nonnegative scalar product (a nonobtuse angle) with every vector in the cone spanned by the columns of $[\lambda I-A]$. Since the cone spanned by the columns of $[\lambda I-A]$ contains the semipositive orthant if and only if $\lambda$ is greater than $\lambda^*$, it follows that every row of the inverse must form a nonnegative scalar product (a nonobtuse angle) with every semipositive vector—that every row of the inverse is semipositive—if and only if $\lambda$ is greater than $\lambda^*$. Thus for any number of goods, $[\lambda I-A]^{-1}$ being semipositive is equivalent to $\lambda$ being greater than $\lambda^*$—Theorem (d).

To gain insight into the issue of decomposability consider the case in which the columns of $[I-A]$ span the positive orthant. The fact that

$$x = [I-A]^{-1}c$$

where $x$ is the vector of gross output levels required to satisfy the vector of final demands $c$, means that

$$\frac{\partial x_i}{\partial c_{ij}} = [I-A]^{-1}_{ij} \quad i, j = 1, \ldots, n$$

may be interpreted as the increase in the gross output of the $i$th industry required to satisfy an increment in the final demand for the $j$th good. Now, if $A$ is decomposable then there exists a partition of $\{1, 2, 3, \ldots, n\}$ into nonempty sets $I$ and $J$ such that\footnote{23 Recall that $I$ and $J$ partition $\{1, 2, \ldots, n\}$ iff $I \cap J = \emptyset$ and $I \cup J = \{1, 2, \ldots, n\}$.} $a_{ij} = 0$, $i \in I, j \in J$. This means that the columns of $[I-A]$ corresponding to the indices in $I$ have zeros in the rows corresponding to the indices in $I$ or simply that the $J$ columns of $[I-A]$ lie in a linear subspace formed by the $J$ positive coordinate axes. The $I$ rows of $[I-A]^{-1}$ must be perpendicular (orthogonal) to each of the $J$ columns of $[I-A]$, moreover. This means that the $I$ rows of $[I-A]^{-1}$ must belong to the linear subspace formed by the $I$ coordinate axes or that the $I$ rows of $[I-A]^{-1}$ must have zeros in the columns corresponding to the indices in $J$. Thus

$$[I-A]^{-1}_{ij} = 0 \quad i \in I, j \in J$$

and no change is required in the gross outputs of any of the $I$ industries to increase final output in any of the $J$ industries.

An example should clarify this. Consider the case of three goods where $[I-A]$ has the sign pattern:

$$\begin{bmatrix} + & - & - \\ - & + & - \\ 0 & 0 & + \end{bmatrix}$$

Clearly $A$ is decomposable and $I=\{3\}, J=\{1, 2\}$ form the required partition. Notice that the first and second columns (the $J$ columns) of $[I-A]$ lie in the plane (the linear subspace) formed by the first and second coordinate axes. For the third row of $[I-A]^{-1}$ to be perpendicular to both the first and second columns of $[I-A]$ it must be perpendicular to the plane determined by the first and second coordinate axes. The only possibility is that the third row (the $I$ rows) of $[I-A]^{-1}$ must lie along the third coordinate axis
(the linear subspace formed by the \( I \) coordinate axes). This third row must, therefore, have zeros in the first and second places, i.e.,
\[
[I - A]_{31}^1 = [I - A]_{32}^1 = 0
\]

Theorem (e) requires some consideration of the other characteristic roots and vectors of \( A \).\(^2\) For the case of two goods illustrated in Figure 1 or Figure 2, the other root corresponds to the point at which \((\lambda I - A)_{11}\) and \((\lambda I - A)_{12}\) "cross" in the negative quadrant. This crossing must occur since \((\lambda I - A)_{11}\) rotates clockwise through the negative quadrant toward the negative first axis as \(\lambda\) becomes smaller and smaller (more and more negative) and \((\lambda I - A)_{12}\) rotates counterclockwise through the negative quadrant toward the negative second axis as \(\lambda\) becomes smaller and smaller. This crossing is illustrated by the vectors \(B_{1}^{**}\) and \(B_{2}^{**}\) in Figures 1 and 2. Notice that the characteristic vector corresponding to this root (the weights which make the columns \(B_{1}^{**}\) and \(B_{2}^{**}\) sum to the origin) cannot be semipositive since the origin does not belong to the line segment \(B_{1}^{**}B_{2}^{**}\).

The case of three goods is quite similar. As \(\lambda\) becomes smaller and smaller, \((\lambda I - A)_{11}\) (in Figure 3) rotates through the negative orthant toward the negative first axis, \((\lambda I - A)_{12}\) rotates through the negative orthant toward the negative second axis, and \((\lambda I - A)_{13}\) rotates through the negative orthant toward the negative third axis. Either there are two crossings of the type that one vector crosses the plane determined by the other two (the other two roots are real and distinct), or one vector coincides with the plane determined by the other two but does not actually cross it (two repeated real roots), or there are no crossings at all (two conjugate complex roots).\(^2\) Any other real characteristic roots and vectors must correspond, therefore, to real crossings in the negative orthant. This means that the convex polyhedral set whose vertices are determined by the columns of \([\lambda I - A]\) (when evaluated at any other real root) cannot contain the origin. Any other real characteristic vector (the weights which make these columns sum to the origin) cannot, therefore, be semipositive. This reasoning holds for an arbitrary number of goods and we may conclude that \(x^{*}\) is the only semipositive characteristic vector—Theorem (e).

It should be noted in passing that \(\lambda^{*}\) and \(x^{*}\) have an economic interpretation. The search for balanced growth
\[
x_{t} = \sigma x_{t-1}
\]
with full employment
\[
A x_{t} = x_{t-1}
\]
leads to the realization that \(\sigma = 1/\lambda^{*}\) is the maximum balanced growth factor and \(x^{*}\) represents the balanced growth proportions. This means that \(1/\lambda^{*} - 1\) can be regarded as the maximum (percentage) growth rate.

Consider, alternatively, the problem of determining the profit rate of the \(j\)th industry when prices are given by the (row) vector \(p\). This profit rate is obtained by finding the discount rate, \(r\), which makes the present value of receipts equal to the present value of production costs, i.e., the solution \(r\) to
\[
\frac{p_{j}}{1 + r} - p A_{j} = 0
\]
One naturally wonders if there exists an equilibrium vector of prices such that the profit rate is the same in all industries.

\(^2\) The latter possibility is due to the fact that the plane formed by any two vectors rotates as it sweeps through the negative orthant. It is therefore possible for this plane to "roll over" the other vector without ever actually crossing it.
That is, is there a set of prices $p$ and a common profit rate $r$ that satisfies

$$\frac{p}{1+r} - pA = 0$$

or, more simply,

$$p\left[\frac{1}{1+r}I - A\right] = 0$$

Clearly, $\frac{1}{1+r}=\lambda^*$ or $r=\frac{1}{\lambda^*}-1$ is the required common profit rate and $p=p^*$ ($p^*$ is the characteristic row vector associated with the dominant root $\lambda^*$) is the required equilibrium price vector.

The vector $p^*$ can be obtained geometrically by noticing that $p^*$ must form a right angle with every column of $[\lambda^*I-A]$. For the case of two goods illustrated in Figure 1 or Figure 2, $p^*$ must lie along line $OG$. Notice that $p^*$ is positive in this case and that in general the same propositions can be established for this dominant row vector as were established for the dominant column vector $x^*$.

Finally notice that if $\lambda^*$ is less than one, the maximum growth and profit rate, $1/\lambda^*-1$, is positive while if $\lambda^*$ is greater than one, the maximum growth and profit rate is negative (see Figures 1 and 2).

REFERENCES


