Analysis and Modeling of Landscape Topography: Statistical Description and Evolution Under Natural and Disturbed Conditions.

by

Sara Bonetti

Department of Civil and Environmental Engineering
Duke University

Date: ______________________

Approved:

___________________________
Amilcare Porporato, Supervisor

___________________________
Gabriel Katul

___________________________
Andrew Bragg

___________________________
Zbigniew Kabala

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Civil and Environmental Engineering in the Graduate School of Duke University

2018
ABSTRACT

Analysis and Modeling of Landscape Topography: Statistical Description and Evolution Under Natural and Disturbed Conditions.

by

Sara Bonetti

Department of Civil and Environmental Engineering
Duke University

Date: __________________________
Approved:

_________________________
Amilcare Porporato, Supervisor

_________________________
Gabriel Katul

_________________________
Andrew Bragg

_________________________
Zbigniew Kabala

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Civil and Environmental Engineering in the Graduate School of Duke University 2018
Abstract

The topographical properties of a landscape and their time evolution are key features of the Earth’s surface, regulating ecosystem functioning in terms of soil properties as well as water and energy budgets, and creating visually diverse and striking patterns across various spatial scales. Furthermore, the natural evolution of a topography under the influence of geologic erosion can be greatly altered by anthropogenic disturbances (e.g., through agriculture, mining, deforestation), with the potential of accelerating soil erosion, causing land degradation and soil fertility losses. Hence, understanding the geomorphological processes driving the evolution of landscapes under natural and disturbed conditions is key not only to define the main factors and feedbacks shaping the Earth’s topography, but also to foresee the consequences of intensive land use and implement optimal strategies of land management and recovery.

This dissertation addresses some key aspects of landscape evolution and stability, with a focus on the statistical description and modeling of hillslope morphologies under natural and disturbed conditions, the theoretical definition of drainage area at regular and non-regular points of the watershed, and the formation of spatially organized ridge and valley patterns.

We start from the analysis of topographic slopes under natural and accelerated soil erosion. Using large topographic datasets from mountain ranges worldwide, we show that the approximate age of a landscape is fingerprinted in the tails of its slope
distributions. We then explore the role of the different processes driving this dynamic smoothing over geologic time scales by means of numerical experiments, showing that the relaxation process is mainly dominated by diffusion. The effect of agricultural-driven soil erosion on hillslope morphology is then investigated, highlighting how the natural aging process can be altered by intensive land use which, at smaller scales, produces key differences in the slope distribution tails. Furthermore, theoretical solutions are derived for the hillslope profile and the associated soil creep and runoff erosion fluxes, and used to link the observed differences in the morphological features of disturbed and undisturbed areas to a disruption of the natural balance between soil creep and runoff erosion mechanisms.

We then move the analysis to the drainage area, an important nonlocal morphometric variable used in a large number of geomorphological and ecohydrological applications. A nonlinear differential equation whose validity is limited to regular points of the watershed is obtained from a continuity equation, and the theory is then extended to critical and singular points by means of both Gauss’ theorem and dynamical systems concepts. Such a link between the drainage area and a continuity equation sets the basis for the subsequent analysis of organized ridge and valley patterns and channel forming instability. The formation of ridge/valley patterns is analyzed by means of numerical experiments in detachment limited conditions, with the identification of various regimes as a function of diffusive soil creep, runoff erosion, and tectonic uplift as well as the specific geomorphic transport law assumed. Lastly, a linear stability analysis of the coupled water and landscape evolution dynamics is outlined to investigate the critical conditions triggering channel formation and the emergence of characteristic valley spacings in relation to the main geomorphological processes involved.
To my family
Contents

Abstract iv
List of Tables xi
List of Figures xii
Acknowledgements xvi

1 Introduction 1

1.1 Motivation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
1.2 Outline . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

2 Dynamic smoothing of mountains 10

2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
2.2 Study sites and data availability . . . . . . . . . . . . . . . . . . . . 16
2.3 Statistical analysis of gradients and slope distributions . . . . . . 19
  2.3.1 Choice of the power-law region in slope distributions . . . . . 21
  2.3.2 Effect of elevation accuracy . . . . . . . . . . . . . . . . . . . . 21
  2.3.3 Binning analysis . . . . . . . . . . . . . . . . . . . . . . . . . . 24
2.4 Time evolution of slope statistics . . . . . . . . . . . . . . . . . . . 24
  2.4.1 Simulation setup . . . . . . . . . . . . . . . . . . . . . . . . . . 29
  2.4.2 Results from landscape evolution modeling . . . . . . . . . . . 31
2.5 Discussion and conclusions . . . . . . . . . . . . . . . . . . . . . . . 33
7 Conclusions

A Supplementary information and additional results for Chapter 2 144
  A.1 Edwards-Wilkinson (EW) and Kardar-Parisi-Zhang (KPZ) equations. 144
  A.2 Additional tables and figures . . . . . . . . . . . . . . . . . . . . . 145

B Landscape aging process:
cross-CZO analysis and effect of DEM grid size 151
  B.1 Study sites: Critical Zone Observatories . . . . . . . . . . . . . . . 152
  B.2 Gradients and slope distributions . . . . . . . . . . . . . . . . . . . . 152
  B.3 Effect of DEM resolution on slope distribution tails . . . . . . . . . 155

C TL and DL regimes:
sensitivity analysis and additional results 158
  C.1 TL and DL regimes: effect of slope and water flow exponents . . . . 158
  C.2 Application to the Calhoun CZO: additional results . . . . . . . . . 159

D Detection of critical points 164

Bibliography 166

Biography 185
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Slope statistics for the twelve case studies</td>
<td>17</td>
</tr>
<tr>
<td>2.2</td>
<td>Parameter values for the landscape evolution model used in the simulated scenarios SC1-SC8</td>
<td>30</td>
</tr>
<tr>
<td>5.1</td>
<td>Parameter values for the first set of simulations ((m = 0.5 \text{ and } n = 1))</td>
<td>116</td>
</tr>
<tr>
<td>5.2</td>
<td>Parameter values for the simulations sets with varying exponents (m) and (n)</td>
<td>117</td>
</tr>
<tr>
<td>6.1</td>
<td>Parameter values used in the linear stability analysis</td>
<td>137</td>
</tr>
<tr>
<td>A.1</td>
<td>Coordinate limits and statistics of the elevation field for the 36 DEMs analyzed</td>
<td>146</td>
</tr>
<tr>
<td>A.2</td>
<td>Gradient and slope statistics for the 36 DEMs analyzed</td>
<td>147</td>
</tr>
<tr>
<td>A.3</td>
<td>Results from the binning analysis</td>
<td>148</td>
</tr>
<tr>
<td>B.1</td>
<td>Locations and statistics of the CZOs analyzed</td>
<td>152</td>
</tr>
<tr>
<td>B.2</td>
<td>Effect of grid resolution on slope statistics at the Calhoun CZO</td>
<td>156</td>
</tr>
</tbody>
</table>
## List of Figures

1.1 Conceptual framework and dissertation outline ................................. 4
1.2 Spatial and temporal scales involved ............................................ 5
2.1 Location of the study sites .......................................................... 11
2.2 Digital Elevation Models of the study sites ..................................... 13
2.3 Probability distributions of gradients for the 36 DEMs analyzed .......... 14
2.4 Probability distributions of slopes for the 36 DEMs analyzed .............. 15
2.5 Slope distributions and landscape age ............................................ 16
2.6 Choice of the power-law region in slope distributions ...................... 22
2.7 Choice of the power-law region and effect of elevation accuracy on the $q$-age relationship ................................................................. 23
2.8 Andean, Alpine, and Himalayan watersheds used in the numerical experiments ................................................................. 24
2.9 Slope distributions for the Alpine watershed for scenarios SC1-SC7 ... 25
2.10 Slope distributions for the Andean watershed for scenarios SC1-SC7 ... 26
2.11 Slope distributions for the Himalayan watershed for scenarios SC1-SC7 27
2.12 Slope values over the domain located in the Andes for scenarios SC1, SC3, SC5, and SC7 ................................................................. 28
2.13 Evolution of the power-law exponent $q$ for scenarios SC1-SC8 ........... 31
2.14 Effect of additive and multiplicative noise ...................................... 34
3.1 The Calhoun Critical Zone Observatory: location and landscape degradation ................................................................. 38
3.2 Shaded relief image and cross sections of disturbed and undisturbed portions of the Calhoun CZO ......................... 40
3.3 Gradient and slope distributions for the disturbed and undisturbed areas .................................................. 42
3.4 Freezing and relaxation times ................................................. 43
3.5 Analytical steady state solutions for the transport limited and detachment limited regimes ........................................... 47
3.6 Hillslope profiles in the disturbed and undisturbed portions ............... 49
3.7 Analytical fit of hillslopes in the undisturbed and disturbed sections . 51
3.8 Partitioning of the undisturbed and disturbed sections based on local surface curvature and dominant erosional processes ...................... 54
3.9 Time evolution of the disturbed hillslope D3 and erosion rates associated with creep and runoff erosion ........................... 57
4.1 Graphical representation and definitions ........................................... 63
4.2 Drainage area at local minima ................................................. 69
4.3 Regular and singular points: conceptual representation .................... 73
4.4 One-dimensional case .......................................................... 75
4.5 Convergent and divergent cones and paraboloids ........................ 77
4.6 Sinusoidal surface ............................................................... 81
4.7 Sinusoidal surface: plan curvature and specific catchment area ........ 83
4.8 Superposition of Gaussian surfaces .......................................... 84
4.9 Surface with a fold ............................................................. 86
4.10 Location of the Calhoun CZO and 3D representation of the smoothed topographic surface analyzed ............................... 88
4.11 Application to a real topographic surface: dynamical systems representation and elevation profiles .......................... 90
4.12 Application to a real topographic surface: computation of drainage area 92
4.13 Specific drainage area and plan curvature along unstable manifolds . 93
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.5</td>
<td>Effect of DEM resolution on slope distributions</td>
<td>155</td>
</tr>
<tr>
<td>B.6</td>
<td>Effect of DEM size on slope distributions</td>
<td>157</td>
</tr>
<tr>
<td>C.1</td>
<td>Effect of variations in slope and water flux exponents: results for the transport-limited regime</td>
<td>159</td>
</tr>
<tr>
<td>C.2</td>
<td>Effect of variations in slope and water flux exponents: results for the detachment limited regime</td>
<td>159</td>
</tr>
<tr>
<td>C.3</td>
<td>Results for the undisturbed hillslopes</td>
<td>160</td>
</tr>
<tr>
<td>C.4</td>
<td>Results for the disturbed hillslopes</td>
<td>161</td>
</tr>
<tr>
<td>C.5</td>
<td>Time evolution of hillslope D1 and erosion rates associated with creep and runoff erosion</td>
<td>162</td>
</tr>
<tr>
<td>C.6</td>
<td>Time evolution of hillslope D2 and erosion rates associated with creep and runoff erosion</td>
<td>162</td>
</tr>
<tr>
<td>C.7</td>
<td>Time evolution of hillslope D4 and erosion rates associated with creep and runoff erosion</td>
<td>163</td>
</tr>
<tr>
<td>C.8</td>
<td>Time evolution of hillslope D5 and erosion rates associated with creep and runoff erosion</td>
<td>163</td>
</tr>
<tr>
<td>D.1</td>
<td>Numerical detection of critical points</td>
<td>165</td>
</tr>
</tbody>
</table>
Acknowledgements

First and foremost, I am deeply grateful to my parents and sister, for the constant love and support regardless of the distance. This dissertation would not have been possible without the mentorship of my advisor, Amilcare Porporato, whose guidance, creativity, knowledge, and criticism have inspired and encouraged me and my research. I would also like to thank my committee members Gaby Katul, Andrew Bragg, and Zbigniew Kabala. Special thanks to Gaby Katul, whose advice, friendship, and support have been simply invaluable. I would also like to thank Daniel Richter, Carlo Camporeale, and Ana Barros for the valuable discussions and suggestions. I am grateful to all the lab mates and friends who accompanied me during these years. A special thank to Gabriele Manoli, words cannot express how thankful I am to him.

I acknowledge fundings from the USDA Agricultural Research Service cooperative agreement 58-383 6408-3-027, and US National Science Foundation (NSF) grants CBET-1033467, EAR-1331846, EAR-384 1316258, EAR-1338694, and the Duke WISNet Grant DGE-1068871.
1

Introduction

“We live in landscapes; we manage landscapes.

We often describe the environment around us in terms of landscapes.

Yet landscapes have long been a scientific blind spot.”

I. R. Noble, 1999

1.1 Motivation

Landscape topography and its time evolution play a crucial role in a variety of eco-hydrological and geomorphological processes, regulating ecosystem functioning and services in general (Figure 1.1). For example, the spatial distribution of energy (e.g., radiation) and water (e.g., soil moisture) variables, which are the main drivers of vegetation and nutrients distribution (Larcher, 2003), is highly dependent on the topographic features of a landscape (e.g., elevation, slope, aspect, curvature, drainage area) (Florinsky and Kuryakova, 1996). The amount of solar radiation intercepted by a surface is affected by local slope and aspect (Rodríguez-Iturbe and Porporato, 2007; Essery and Marks, 2007), while surface and subsurface water redistribution are strongly influenced by microtopographic attributes such as landscape connectivity,
curvature, and slope, thus regulating the local soil moisture available to plants (Anderson and Burt, 1978; Burt and Butcher, 1985; Western et al., 1999; Rinderer et al., 2016; Dymond et al., 2017). Furthermore, topographic heterogeneities and the resulting differences in temperature, radiation, soil moisture availability, and nutrients distribution add complexity to the landscape promoting biodiversity (Marston, 2010) and are responsible for both aboveground (Franklin, 1998; Florinsky and Kuryakova, 1996; Bennie et al., 2008) and belowground (Florinsky et al., 2004; Swallow et al., 2009) species distribution.

While on short time scales the topography of a region can be regarded as a static entity, on longer time scales topographic properties are dynamic quantities evolving through time in relation to various erosional and depositional processes, such as soil creep, runoff erosion, and landslides. The combined action of these geomorphological processes shape the landscape resulting in peculiar hillslope profiles and spatially organized patterns of ridges and valleys (see, e.g., Figure 1.1). In addition, the natural landscape aging process can be greatly altered by human activities (e.g., agriculture, deforestation, mining), with huge impacts on erosional processes and the resulting topography (Hooke, 2000; Tarolli, 2016; Tarolli and Sofia, 2016), and possibly threatening soil productivity and ecosystem services (Foley et al., 2005; Millennium Ecosystem Assessment, 2005; Power, 2010). For example, the rate of soil conversion to intensive agricultural uses has increased in the past century thanks to new technologies introduced by the so-called Green Revolution (Tilman et al., 2002; Richter and Markewitz, 2001). Such an intense use of soils not infrequently exceeds soil resilience to recover to its previous condition (Amundson et al., 2015), so that the rate of erosion in many agricultural areas greatly exceeds soil formation (Montgomery, 2007; Richter, 2007; Ellis et al., 2013; Amundson et al., 2015). This type of disturbance is likely to alter the relaxation process of a hillslope resulting in peculiar changes in the geomorphological properties with respect to natural environ-
ments (Figure 1.2), and has repercussions on various ecosystem services altering soil physical and chemical properties and reducing agricultural potential (Brink et al., 1977; Montgomery, 2007; Power, 2010). While less intensively used soils may retain their properties over long time scales (e.g., soil thickness, moisture, carbon and nutrient content), intensively managed soils are much less able to sustain their original qualities, with the possibility of overshooting stability thresholds with rapid shifts to unsustainable conditions (Rockström et al., 2009; Porporato et al., 2015).

An accurate description of the local (e.g., elevation, slope, aspect) and non-local (e.g., drainage area) topographic features that shape the physical environment, both from a theoretical and a statistical perspective, can be used to incorporate topographic features into regional ecohydrological, morphological, and climatological models to define the different processes that shaped present-day topographies. Moreover, the development of appropriate tools for the statistical description and detection of accelerated soil erosion as well as simple theoretical frameworks accounting for the main ecohydrological mechanisms involved is key to quantify such processes, understanding the underlying dynamics and feedbacks, and ultimately implement strategies for more sustainable land use practices. This dissertation is a contribution to the analysis and modeling of landscape topography and evolution under both natural and disturbed conditions. The analysis is conducted from a statistical perspective in order to identify signatures of the evolution process, as well as by means of theoretical and numerical models to disentangle the main geomorphological processes involved.

1.2 Outline

Figure 1.1 provides an overview of the main topics addressed in this dissertation, ranging from the analysis of topographic features, to their time evolution in relation to various erosional mechanisms, to the effect of disturbances on hillslope morphol-
Figure 1.1: Ecosystem services (e.g., water quality and runoff, vegetation distribution, carbon and nutrients dynamics, soil fertility) as well as hillslope morphologies and spatial topographic patterns are strongly dependent on various local and non-local topographic features and erosional/depositional processes shaping the landscape. Intensive anthropogenic land use (e.g., agriculture) as well as natural disturbances (e.g., climatic) greatly alter the geomorphology of large regions, altering the natural erosional/depositional processes, with repercussions on ecosystem services and hillslope morphologies. Pictures show (a) a shaded relief map of evenly spaced valleys near Bradley, California (modified from Perron and Fagherazzi (2012)), (b) the natural landscapes of the gentle rounded topography of the Blue Ridge Mountains in the subtropical climate of North Carolina and (c) a canyon carved by the Colorado river near Page, Arizona, while pictures (d) and (e) show the massive soil erosion caused by intensive cotton cultivation at the Calhoun Critical Zone Observatory in South Carolina. A reference to the main topics addressed in each thesis chapter is provided in the figure.

ogy and the associated erosional fluxes. Such processes act over a wide range of spatial and temporal scales (Figure 1.2). For example, the formation of the drainage network is a strongly non-local process taking place at regional (≈ 100-1000 m) scale, and considered to be a relatively fast phase occurring during the early stages of the landscape evolution process (Sinclair and Ball, 1996; Banavar et al., 2001). Conversely, the relaxation phase (Fernandes and Dietrich, 1997; Banavar et al., 2001),
Figure 1.2: Under natural conditions, after the main mountain forming event, the landscape evolution is typically characterized by distinctive phases involving various spatial and temporal scales. At the beginning, during the so-called freezing time the main drainage network is formed (Sinclair and Ball, 1996; Willgoose et al., 1991; Banavar et al., 2001). This is a relatively fast time scale after which the main geometry of the drainage network remains almost invariant. Subsequently, during the relaxation time the landscape morphology is smoothed by various geomorphological processes and hillslopes evolve towards an equilibrium form, while the drainage network remains generally invariant. Disturbances are likely to have an impact at the hillslope scale (i.e., small spatial scales) and to alter the relaxation process and the resulting hillslope morphology.

during which hillslopes tend towards an equilibrium morphology while maintaining the planar geometry of the drainage network practically invariant, is considered to be a much longer process mainly acting at the hillslope scale (∼ 10-100 m). Anthropogenic disturbances are likely to have an impact at smaller scales (∼ 1-100 m), with the potential to alter the relaxation process and the resulting topography. In this thesis, the relaxation process under both natural conditions and agricultural-driven
accelerated soil erosion is first analyzed. Subsequently, a theoretical definition of drainage area at regular and non-regular points is derived. Lastly, the formation of organized ridges and valleys patterns as well as channel-forming instabilities are analyzed in relation to the main geomorphological processes acting over the landscape, by means of a numerical and theoretical analysis of the governing equations. Specifically, the dissertation is organized as follows.

Chapter 2 focuses on the natural relaxation process of mountainous landscapes, with the definition of a quantitative relation between topography and geologic history. Slope statistics of different mountain ranges worldwide are analyzed, showing that landscape age is fingerprinted in their distribution tails. The statistical analysis reveals a universal relaxation process, through an algebraic decay progressively replaced by an exponential one, with exponents described by a global monotonic function. The dominant components of this dynamic smoothing are then investigated using a landscape evolution model, showing that the time evolution of slope statistics results from a delicate balance between diffusive soil creep, noise, and advective river incision, with the relaxation phase mainly dominated by diffusion. Results presented in this chapter may suggest ways to formulate reduced order topographic evolution models for ecohydrological, geomorphological, and climatological applications, and to explore similarities in surface evolution in different contexts, a phase during which the statistical mechanical properties of the dynamic smoothing are analyzed.

In Chapter 3 the effect of accelerated erosion on landscape morphology is investigated. As already discussed in the previous section, intensive agricultural land use can have detrimental effects on landscape properties, modifying the geomorphological features of a region and greatly accelerating soil erosion, with consequent fertility loss and reduced agricultural potential. To quantify the effects of such erosional processes on hillslope morphology and gain insight into the underlying dynamics, a twofold approach is adopted. First, a statistical analysis of topographical features
is conducted, with a focus on slope and gradient distributions. The accelerated soil erosion is shown to be fingerprinted in the distribution tails, which provide a clear statistical signature of this human-induced land modification. Theoretical solutions are then derived for the hillslope morphology and the associated creep and runoff erosion fluxes, allowing us to distinguish between the main erosional mechanisms operating in disturbed and undisturbed areas. The relaxation time required for the disturbed hillslopes to reach a quasi-equilibrium condition is also investigated. We focus our application on the landscape at the Calhoun Critical Zone Observatory in the US Southern Piedmont, where disastrous soil erosion followed intensive cotton cultivation, resulting in a highly-degraded ecosystem. The observed differences in hillslope morphologies in disturbed and undisturbed areas are shown to be a direct consequence of the disruption of the natural balance between soil creep and runoff erosion.

In Chapter 4 a theoretical description of drainage area at regular and non regular points of the watershed is provided. The drainage area is an important, nonlocal property of the landscape, which controls surface and subsurface hydrologic fluxes. Its role in numerous ecohydrological and geomorphological applications has given rise to several numerical methods for its computation, but its theoretical analysis has lagged behind. Only recently, an analytical definition for the specific catchment area has been proposed (Gallant and Hutchinson, 2011), with the derivation of a differential equation whose validity is limited to regular points of the watershed. In Chapter 4 it is shown that such a differential equation can be derived from a continuity equation (Chen et al., 2014) and the theory is then extended to critical and singular points both applying Gauss’ theorem and by means of a dynamical systems approach to define basins of attraction of local surface minima. Simple analytical examples as well as applications to more complex topographic surfaces are examined. The theoretical description of topographic features and properties, such
as the drainage area, channel lines, and watershed divides can be broadly adopted to develop and test the numerical algorithms currently used in digital terrain analysis for the computation of the drainage area, as well as for the theoretical analysis of landscape evolution and stability.

Chapter 5 focuses on the analysis of organized patterns of ridges and valleys in detachment limited conditions. Here, numerical simulations are used to assess the role of diffusive soil creep, runoff erosion, and tectonic uplift in the definition of drainage networks and river branching with the identifications of various regimes. The effect of considering different geomorphic transport laws on the observed patterns and the corresponding hypsometric curves is also analyzed. To do this, the runoff-erosion component of the landscape evolution equation is re-casted in terms of specific catchment area, which is shown to be directly linked to the water flow under specific assumptions. This provides a clear mathematical framework which can be further employed for a theoretical analysis of landscape evolution and stability.

Lastly, in Chapter 6 the linear stability analysis of the coupled water continuity and landscape evolution equations is outlined in detachment-limited conditions to understand the processes leading to channel-forming instabilities and subsequent emergence of evenly spaced valleys. The problem has been recently addressed by means of numerical simulations (Perron et al., 2008, 2009) showing the emergence of characteristic valley spacings scaling with the relative proportion of diffusive soil creep and runoff erosion, but a linear stability analysis of the underlying equations is still missing. In this chapter, the landscape evolution equation is coupled to a water continuity equation, which can be mapped into a drainage area equation (see also Chapter 4), thus providing a direct link between the classical theoretical frameworks and the approximations used in numerical simulations (where the drainage area is generally used as a proxy for water flow). The stability analysis is outlined by means of a spectral Galerkin technique with numerical integration. Preliminary results show
that the conditions for the channel-forming instability can be identified as a function of the geomorphological processes involved, with the selection of characteristic valley spacings in relation to the relative proportions of soil creep and runoff erosion.
Dynamic smoothing of mountains

“And how many years can a mountain exist before it is washed to the sea?”

B. Dylan - Blowin’ in the wind (1963)


2.1 Introduction

The qualitative difference between the steep slopes and high, pointed peaks of young mountains, such as the Alps or the Andes, and the rounded landscapes of the old Appalachians or the Urals can be easily noticed even by a casual look at a topographic map. The statistical properties of a landscape (i.e., elevation, gradients, slope, and curvature) mirror its different phases of growth and decay (Hurst et al., 2013). Citing Strahler’s pioneering work (Strahler, 1956): “slope-frequency distributions will differ with differences of stage in the geomorphic cycle of erosion of a region. Studies may
reveal the systematic progression of values of slope-population parameters associated with the developmental cycle, thus making possible further quantification of landform stages”. Indeed, the asymmetry of topographic gradient distributions has been linked to the orogenetic properties of the landscape and the processes driving their evolution (Strahler, 1956; Speight, 1971; Montgomery, 2001; Wolinsky and Pratson, 2005; Vico and Porporato, 2009), depending on whether active rock uplift or depositional processes dominate (Montgomery, 2001). While the relationship between the hypsometric curve and the stage of geomorphic development of the basin has been previously analyzed (Strahler, 1952, 1964; Schumm, 1956; Willgoose, 1994; Willgoose and Hancock, 1998; Demoulin, 2011), a quantitative relation between landscape age and its geomorphological properties (in particular the topographic slope) is still missing. The importance of understanding processes controlling slope distributions at geological time scales goes beyond a pure geomorphological interest, given their role in several hydroclimatic processes, including incident solar radiation and water availability, which ultimately define vegetation patterns and biodiversity in a given landscape. A compact representation of landscape topographic features would be useful to infer its ecological implications over geological time scales.
The initial phase of intense orogenic activity is typically followed by a relaxation phase, during which landscape morphology is shaped by a strong interaction between diffusive soil creep and advective river incision (Tucker and Bras, 1998; Rodríguez-Iturbe and Rinaldo, 2001; Simpson and Schlunegger, 2003; Chen et al., 2014), as well as other erosional mechanisms such as glacial erosion (Harbor, 1992; MacGregor et al., 2000; Egholm et al., 2009) and landslides (Montgomery and Dietrich, 1994; Densmore et al., 1997; Egholm et al., 2013). Landscape evolution models (LEMs) (Willgoose et al., 1991; Tucker and Slingerland, 1994; Braun and Sambridge, 1997; Coulthard, 2001; Perron et al., 2008; Hancock et al., 2010; Egholm et al., 2013) have been used to simulate such processes and explore different aspects of channel network development and its geomorphic and hydrologic statistics (Willgoose et al., 1991; Hancock et al., 2002), landscape self-organization (Rigon et al., 1994; Rinaldo et al., 1995; Rodríguez-Iturbe and Rinaldo, 2001; Perron et al., 2008, 2009), the coupling between surface processes and tectonics (Willett et al., 2001), and the feedbacks between landsliding and fluvial erosion (Egholm et al., 2013). In particular, it has been shown that only a narrow range of diffusivity and fluvial incision parameters is able to produce realistic mountain topographies (Willett et al., 2001; Simpson and Schlunegger, 2003), and that only a limited range of the ratio between characteristic diffusion and advection timescales can produce reasonable valley spacings in soil-mantled landscapes (Perron et al., 2008, 2009). However, less attention has been paid to the relative importance of these two transport mechanisms in defining the evolution of landscape morphology statistics through time. The complex statistical dynamics of the Earth’s topography remain a largely unexplored field, with surface evolution modeling largely focusing on microscale phenomena (Barabási and Stanley, 1995; Krug, 1997; Dodds and Rothman, 2000) and network patterns (Maritan et al., 1996; Rodríguez-Iturbe and Rinaldo, 2001) but only sparse applications to landscape evolution processes (see, e.g., Passalacqua et al. (2006); Pelletier (2007)).
In this chapter a metric that quantitatively relates slope statistics and landscape age is defined and the relative roles of diffusive soil creep, advective river incision, and random component on the evolution of such statistics during the relaxation phase are quantified.
Figure 2.3: Probability distributions of gradients (gray dots) for (APPAL) Appalachians, (OZA) Ozarks, (FRA) Central Massif, (FLINT) Flint Hills, (URAL) Urals, (HAMER) Hamersley Range, (ROCKY) Rockies, (HIMAL) Himalaya, (APENN) Apennines, (ETH) Ethiopian Highlands, (ALP) Alps, and (ANDE) Andes. Gaussian distributions (red lines) are also shown for comparison.
Figure 2.4: Probability distributions of slopes (gray dots) for (APPAL) Appalachians, (OZA) Ozarks, (FRA) Central Massif, (FLINT) Flint Hills, (URAL) Urals, (HAMER) Hamersley Range, (ROCKY) Rockies, (HIMAL) Himalaya, (APENN) Apennines, (ETH) Ethiopian Highlands, (ALP) Alps, and (ANDE) Andes. The power-laws (black line) is computed by fitting the linear tail of the pdf between $10^{-3.2}$ and $10^{-0.95}$ (the equation obtained from the linear fitting is also reported). Rayleigh distribution computed according to Vico and Porporato (2009) (Eq. (2.1)) is also shown for comparison (red lines).
Figure 2.5: (a) Probability distributions of slopes for the Appalachians (APPAL) and the Bolivian Andes (ANDE). (b) Logarithmic plot of the tails of the slope distributions showing the power-law scaling. The mean power law exponent $q$ is the mean between the $q$ values obtained for the three DEMs used for each location. $q$ is computed by fitting the power-law between $p(S) = 10^{-3.2}$ and $p(S) = 10^{-0.95}$ (gray dashed lines) and is higher for the younger mountain range (i.e., Andes), reflecting the fatter tail of the slope distributions. (c) Power law exponent $q$ computed for the 12 study cases as a function of approximate landscape age: diamonds represent the mean value $q$, vertical lines the maximum and minimum values for the DEMs considered, horizontal lines the error bars for the landscape age.

2.2 Study sites and data availability

For the statistical analysis performed in Section 2.3 twelve case studies representing mountain ranges with different ages of uplift are considered (see Figure 2.1, Table 2.1). Four case studies are located in the continental United States: the Appalachians in North Carolina, New York, and Canada, the Flint Hills in Kansas, the Rocky...
Table 2.1: Mean, standard deviation, skewness, and kurtosis of slopes together with the power-law exponent $q$ for the 12 case studies. The overline indicates that these are mean values over the 3 DEMs considered per location. References for the approximate age of the landscape: [a] Hibbard (2000); Wagner et al. (2012); [b] Whitaker and Engelder (2012); Cox (2009); [c] Faure (1995); Ledru et al. (2001); [d] Berzin et al. (1996); Puchkov (1997); [e] Frye (1955); [f] Twidale (1994); [g] English and Johnston (2004); [h] Klootwijk et al. (1992); Yin and Harrison (2000); [i] Burchfiel and Royden (1991); Corrado et al. (1997); [j] Davidson and Rex (1980); Gani et al. (2007); [k] Frisch et al. (2000); [l] Hoke and Garzione (2008).

<table>
<thead>
<tr>
<th>Mountain Range (Abbr.)</th>
<th>$\mu_S$ [m m$^{-1}$]</th>
<th>$\sigma_S$ [m m$^{-1}$]</th>
<th>$sk_S$ [-]</th>
<th>$k_S$ [-]</th>
<th>$q$ [-]</th>
<th>Age [Mya]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appalachian Mts. (APPAL)</td>
<td>0.265</td>
<td>0.154</td>
<td>0.906</td>
<td>4.309</td>
<td>-10.224</td>
<td>360±50</td>
</tr>
<tr>
<td>Ozark Mts. (OZA)</td>
<td>0.187</td>
<td>0.121</td>
<td>0.977</td>
<td>4.151</td>
<td>-10.344</td>
<td>310±10</td>
</tr>
<tr>
<td>Central Massif (FRA)</td>
<td>0.193</td>
<td>0.144</td>
<td>1.297</td>
<td>5.125</td>
<td>-10.167</td>
<td>300±20</td>
</tr>
<tr>
<td>Urals (URAL)</td>
<td>0.120</td>
<td>0.083</td>
<td>1.971</td>
<td>9.363</td>
<td>-8.403</td>
<td>280±20</td>
</tr>
<tr>
<td>Flint Hills (FLINT)</td>
<td>0.058</td>
<td>0.039</td>
<td>1.541</td>
<td>7.767</td>
<td>-8.097</td>
<td>275±20</td>
</tr>
<tr>
<td>Hamersley Range (HAMER)</td>
<td>0.125</td>
<td>0.138</td>
<td>2.088</td>
<td>9.030</td>
<td>-7.232</td>
<td>135±35</td>
</tr>
<tr>
<td>Rocky Mts. (ROCKY)</td>
<td>0.370</td>
<td>0.245</td>
<td>1.672</td>
<td>10.550</td>
<td>-5.565</td>
<td>67.5±12.5</td>
</tr>
<tr>
<td>Himalaya (HIMAL)</td>
<td>0.588</td>
<td>0.289</td>
<td>1.277</td>
<td>8.189</td>
<td>-6.830</td>
<td>60±10</td>
</tr>
<tr>
<td>Apennines (APENN)</td>
<td>0.302</td>
<td>0.196</td>
<td>1.216</td>
<td>7.318</td>
<td>-6.796</td>
<td>40±20</td>
</tr>
<tr>
<td>Ethiopian Highlands (ETH)</td>
<td>0.341</td>
<td>0.221</td>
<td>1.509</td>
<td>9.527</td>
<td>-5.826</td>
<td>28±5</td>
</tr>
<tr>
<td>Alps (ALP)</td>
<td>0.527</td>
<td>0.320</td>
<td>2.067</td>
<td>19.557</td>
<td>-5.281</td>
<td>17±5</td>
</tr>
<tr>
<td>Andes (ANDE)</td>
<td>0.403</td>
<td>0.228</td>
<td>1.264</td>
<td>8.089</td>
<td>-5.995</td>
<td>7.5±1.5</td>
</tr>
</tbody>
</table>

Mountains in Colorado, and Ozark Mountains in Arkansas. Other five study sites characterized by more recent orogenetic processes are considered, comprising the Himalayas in India and Nepal, the Bolivian Andes, the Italian Apennines, the Alps between France, Italy, Switzerland, and Austria, and the Ethiopian Highlands. Finally, three sites with older orogeny were considered, namely the Urals in Russia, the Australian Hamersley Range, and the French Central Massif. The approximate landscape age was assigned considering the period when the main geological mountain-forming events took place. The Appalachians are the product of a series of orogenic events (Hibbard, 2000; Wagner et al., 2012; Hopper et al., 2017), spanning a period that goes from the Ordovician/Silurian to the Pennsylvanian, and are thus assigned an approximate age of 360±50 Mya. The Ozarks were uplifted as a result of the Ouachita orogeny during the Pennsylvanian period (Whitaker and Engelder,
so that its age is assumed equal to 310±10 Mya, while the French Massif Central formed during the Variscan orogeny (Faure, 1995; Ledru et al., 2001) in the Late Paleozoic (Middle Carboniferous to Early Permian), thus placing its age at approximately 300±20 Mya. The Flint Hills were formed during the Permian period (Frye, 1955), thus having an age of approximately 275±20, while the Urals are a Late Paleozoic orogenic belt (Berzin et al., 1996; Puchkov, 1997), that can be dated to approximately 280±20 Mya. The Hamersley Range has an approximate age of 135±35 Mya (Twidale, 1994; Twidale and Campbell, 2005), while the Rocky Mountains were formed during the Laramide orogeny in the Late Cretaceous to Paleocene (English and Johnston, 2004) thus having an age of approximately 67.5±12.5 Mya. The Himalayas were formed by the Indo-Asian collision and can be dated to 60±10 Mya (Yin and Harrison, 2000), the Apennines developed during several tectonic phases and can be dated to approximately 40±20 Mya (Burchfiel and Royden, 1991; Corrado et al., 1997), and the Ethiopian Highlands can be dated to approximately 28±5 Mya (Davidson and Rex, 1980; Gani et al., 2007). The Alps arose as a result of the Alpine orogeny mainly in the Oligocene/Miocene epochs, thus can be dated to approximately 17±5 Mya (Ratschbacher et al., 1991; Frisch et al., 2000), while the Bolivian Andes have an approximate age of 7.5±1.5 Mya (Hoke and Garzione, 2008).

DEMs were obtained from the NASA Shuttle Radar Topography Mission (SRTM), with a resolution of 1 arc sec, and elevation values having an accuracy to within ±0.5 m. For each location we analyzed three digital elevation models (DEMs) located in various areas of each mountain range, for a total of 36 DEMs, and each DEM covers an area of approximately 3000 km². Furthermore, areas characterized by the presence of extensive glaciers and flat areas were avoided. Figure 2.2 shows the 36 DEMs used in the study. Location (i.e., latitude and longitude) and statistics of the elevation field (i.e., maximum, minimum, mean and standard deviation) are provided in
2.3 Statistical analysis of gradients and slope distributions

Each point $x, y$ is characterized by its elevation $h(x, y)$, and gradients, $\partial_x h$ and $\partial_y h$, from which the local slope is obtained as $S = \sqrt{(\partial_x h)^2 + (\partial_y h)^2}$. Observed gradient distributions are displayed in Figure 2.3, where the Gaussian distribution is also shown for comparison. The parameters of the Gaussian distribution (i.e., mean and variance) have been estimated directly from the fields of partial derivatives and are listed in Table A.2 for the 36 DEMs considered.

Observed slope distributions are displayed in Figure 2.4, where the Rayleigh distribution is also shown for comparison, defined as (Vico and Porporato, 2009)

$$p_R(S) = \frac{S}{\sigma_{\partial h}^2} e^{-\frac{S^2}{2\sigma_{\partial h}^2}},$$

(2.1)

$s$ being the slope and $\sigma_{\partial h}$ is the standard deviation of gradients (Table A.2). Slope statistics (i.e., mean, standard deviation, skewness, kurtosis) for the 36 DEMs are listed in Table A.2, while mean values (averaged over the three DEMs considered per location) are given in Table 2.1. All the slope distributions show a unimodal behavior (Figure 2.4), coherently with previous observations for various landscapes (Speight, 1971; Montgomery, 2001; Wolinsky and Pratson, 2005; Vico and Porporato, 2009). Mean, variance and skewness of the slope distributions appear not to be correlated with landscape age, while their kurtosis tends to be higher for younger landscapes (see Table A.2), suggesting different behaviors of the tails with landscape age. This notable result is corroborated by the analysis of the distribution tails, which are systematically heavier for younger mountains. An example is displayed in Figure 2.5a showing the slope probability density functions (pdfs) for the youngest and oldest mountains analyzed, namely the Appalachians and the Andes. A similar
result is evident also from the gradients, which tend to follow Gaussian distributions for older mountains, while their tails tend to become heavier for younger mountains (see Figure 2.3). For isotropic and uncorrelated Gaussian gradients the pdf of $S$ has a Rayleigh distribution (Vico and Porporato, 2009), which is approached by the observed distributions of the oldest mountains (see Figure 2.4). Conversely, the tails of the slope distributions tend to follow a power law, $p(S) \sim S^q$, as evident from the logarithmic plot in Figure 2.5b, with higher values of the power $q$ (i.e., fatter tails) for areas with more recent orogenetic processes. The value of $q$ is here obtained by performing a linear regression of the tails of the slope distributions in the logarithmic plot for each DEM in the dataset (see Table A.2 and Figure 2.5b). The value of the power-law exponent $q$ resulted to be robust to the binning procedure in the slope pdf analysis, to the error in the DEM elevation accuracy, and to the definition of the limits of the power-law region in the slope distribution (see sensitivity analysis in the following sections).

The mean power law exponent $\bar{q}$ at each location was computed as the mean of the values obtained for the three DEMs analyzed per location and is higher for younger mountains (see, e.g., the -5.99 and -10.22 values in Figure 2.4b for the Andes and the Appalachians, respectively). The results for the twelve case studies are presented in Table 2.1 and Figure 2.5c, showing a strong correlation between the tails of the pdfs and the landscape age, and thus linking the developmental stage of the landscape to the parameters of the slope distributions. Such results appeared to be robust to changes in the regions selected within the different mountain ranges. The linear decay of the exponent with age is rather well defined ($R^2 = 0.88$). Other fitting curves, such as a quadratic function and a sigmoid curve, were also fitted but they did not provide any substantial improvement in the $R^2$ value. This linear approximation of the $q$-Age relationship suggests that the exponent of the slope tails decays at a rate of 0.013 Mya$^{-1}$ (see Figure 2.5c) approaching a Rayleigh distribution.
The observed relation between slope distribution tails and landscape age for different mountain ranges around the world provides a remarkable, quantitative linkage between landscape topography (mainly slope) and its stage of geomorphic development. The faster decay of the tails and the approach towards Gaussian distributions for gradients (and thus Rayleigh distributions for slopes) with age is evocative of the relaxation behavior to equilibrium of classical surface evolution models – in particular the Kardar-Parisi-Zhang (KPZ) and Edwards-Wilkinson (EW) models (Barabási and Stanley, 1995; Krug, 1997), a brief review of which is provided in Section A.1.

2.3.1 Choice of the power-law region in slope distributions

To evaluate the effect of choosing different limits for the definition of the power-law region in the slope distribution tails (namely, $10^{-3.2}$ and $10^{-0.95}$), we evaluated the exponent $q$ assuming a larger and a smaller region of the log-log plot. In particular, the limits where shifted of $\pm 10^{0.4}$ (see Figure 2.6, black and red dashed lines). Results obtained for the three different cases are compared in Figure 2.7a and show that, as long as the fitting of the exponent $q$ is limited to the region of the distribution tail that follows a power law behavior, the value of $q$ is rather insensitive to the specific limits chosen resulting in a robust $q$-Age relationship (see equations in Figure 2.7a).

2.3.2 Effect of elevation accuracy

To evaluate the uncertainty in each DEM estimate of the power-law exponent $q$, we analyzed the effect of adding a random noise to the original DEM. In particular, a uniformly distributed random noise bounded between $\pm 0.5$ m (i.e., the accuracy of the DEMs used) has been added to each DEM. Results are displayed in Figure 2.7b and show that the linear decrease in the power-law estimate is robust to the uncertainty in the DEM elevation values.

21
Figure 2.6: Probability distributions of slopes (gray dots) for (APPAL) Appalachians, (OZA) Ozarks, (FRA) Central Massif, (FLINT) Flint Hills, (URAL) Urals, (HAMER) Hamersley Range, (ROCKY) Rockies, (HIMAL) Himalaya, (APENN) Apennines, (ETH) Ethiopian Highlands, (ALP) Alps, and (ANDE) Andes. The power-law (black, green, and red solid lines) is computed by fitting the linear tail of the pdf between $10^{-3.6}$ and $10^{-0.55}$ (black dashed lines), $10^{-3.2}$ and $10^{-0.95}$ (green dashed lines), and $10^{-2.8}$ and $10^{-1.35}$ (red dashed lines). The $q$ values obtained from the linear fitting are also reported.
Figure 2.7: Power law exponent $q$ computed for the 12 study cases as a function of approximate landscape age: symbols represent the mean value $\bar{q}$, vertical lines the maximum and minimum values for the three DEMs considered per location, horizontal lines the error bars for the landscape age. (a) Choice of power-law region: results are shown for different limits for the linear fitting, namely $10^{-3.6}$ and $10^{-0.55}$ (black), $10^{-3.2}$ and $10^{-0.95}$ (green), and $10^{-2.8}$ and $10^{-1.35}$ (red). (b) Effect of elevation accuracy: blue is used for the original DEM data, while red is for the DEM modified by adding a uniform noise bounded between $\pm 0.5$ m.
2.3.3 Binning analysis

All the results presented in the main text were obtained by computing the slope pdfs with a number of bins \( n_{\text{bins}} \) equal to 100. To evaluate the effect of different bin sizes on the definition of the power-law exponent \( q \), we computed the optimal number of bins according to three different rules (Rice, Doane, and Scott) and computed the value of \( q \) obtained by fitting the power law (between \( 10^{-3.2} \) and \( 10^{-0.95} \)) to the pdfs obtained with the three binning methods. Results are displayed in Table A.3 and show that assuming \( n_{\text{bins}}=100 \) provides a good estimate of the exponent \( q \) (values in Table A.2), since its value is within ± one standard deviation from the mean of the \( q \) values obtained from the three binning formulas.

2.4 Time evolution of slope statistics

Aiming at exploring the possible reasons underlying the apparent universal behavior observed, we undertook a preliminary modeling analysis of the relative importance of
Figure 2.9: Slope distributions for the Alpine watershed for scenarios SC1-SC7 at 17 Mya (approximate landscape age, initial condition), 200 Mya, and 400 Mya. The power-law exponent \( q \) was computed by fitting the power-law between \( 10^{-3.2} \) and \( 10^{-0.95} \) (gray dashed lines).

the effects of diffusion, advection, and random noise on the relaxation process. The model accounts for leading order processes only and is kept intentionally simple to provide a first general description of the main processes driving the slope evolution. Accordingly, the following equation for the time evolution of the surface elevation \( h \) was used:

\[
\frac{\partial h}{\partial t} = D \nabla^2 h - (K + \xi_1) A^m |\nabla h|^n + \xi_2
\]  

(2.2)

where \( t \) is time, \( D \ [L^2 \ T^{-1}] \) is a surface diffusivity, \( K \ [L^{1-2m} \ T^{-1}] \) is a constant depending on channel geometry, rock erodibility, and precipitation rate, \( m \) and \( n \) are model parameters, \( A(x, y, t) \ [L^2] \) is the drainage area (defined as the total catchment area above a point (Tarboton, 1997), resulting in a nonlocal term given by the integral of all areas draining to the \( x, y \) point), \( |\nabla h| \) is the topographic slope, \( \xi_1 \ [L^{1-2m} \ T^{-1}] \) and \( \xi_2 \ [L \ T^{-1}] \) are a multiplicative and an additive noise term, respectively. The first
Figure 2.10: Slope distributions for the Andean watershed for scenarios SC1-SC7 at 7.5 Mya (approximate landscape age, initial condition), 200 Mya, and 400 Mya. The power-law exponent $q$ was computed by fitting the power-law between $10^{-3.2}$ and $10^{-0.95}$ (gray dashed lines).

The term on the right hand side describes the smoothing of the topographic surface due to soil creep (Rodríguez-Iturbe and Rinaldo, 2001; Perron et al., 2008; Chen et al., 2014) and can reproduce erosion ($\nabla^2 h < 0$) or deposition ($\nabla^2 h > 0$). Here a linear diffusion model is used for simplicity and for a direct comparison with the classical models of surface evolution (KPZ and EW). Nonlinear diffusion models are often employed for areas characterized by steep terrain to account for slope failure (e.g., Roering et al. (1999)), but it should be noticed that their use is restricted to areas with slopes lower than a critical value and that they only affect the rate of diffusive erosion, but not the final hillslope morphology. Thus, the effect of possible slope failure is instead surrogated here by means of probabilistic (noise) terms (discussed below). The term $KA^m|\nabla h|^n$ is used to model fluvial erosion (Perron et al., 2008; Chen et al., 2014) and acts as a nonlinear advection term that tends to amplify perturbations...
Figure 2.11: Slope distributions for the Himalayan watershed for scenarios SC1-SC7 at 60 Mya (approximate landscape age, initial condition), 200 Mya, and 400 Mya. The power-law exponent $q$ was computed by fitting the power-law between $10^{-3.2}$ and $10^{-0.95}$ (gray dashed lines).

in the topography and propagate them in the direction of the topographic gradient $\nabla h$. This term can only cause erosion and is both strongly nonlinear (when $n$ is different from one) and strongly nonlocal because of its dependence on the drainage area $A$. The additive noise $\xi_2$ is a local state-independent term, which represents randomness in the landscape evolution due to landslides, heterogeneities in rock and soil properties, rainfall, etc. To evaluate the effect of accounting for more complex models of landslide dynamics, the case of a multiplicative noise $\xi_1$ is also analyzed. This allows us to account for the space-time dependence of landslide processes on the topographic slope and soil moisture content (Montgomery and Dietrich, 1994; Densmore et al., 1997), while retaining their stochastic component. Note that the interaction of this noise with the drainage area $A$ (which is here introduced as a surrogate of soil moisture) makes the multiplicative noise a highly non-local term.
Figure 2.12: Slope values over the the domain located in the Andes for the initial condition (a) and for scenarios SC1 (b, c, d), SC3 (e, f, g), SC5 (h, i, j), and SC7 (k, l, m) at 150, 300, and 400 Mya.

Thus, turning on and off these two random terms allows us to span the extreme cases of complex landscape models of local and additive versus nonlocal and state dependent dynamics, within which more complex random landslide models have been developed (e.g., Densmore et al. (1997); Egholm et al. (2013)).

Comparing Equation (2.2) and the classic KPZ equation (see Section A.1), it is evident that the main differences are in the definition of the advective term $KA^n|\nabla h|^n$. First, the value of the exponent $n$ in the KPZ equation is equal to 2 and derives from retaining the leading-order nonlinearity in the definition of the growth normal to the surface, while here $n$ is a model parameter usually obtained through calibration (Hancock et al., 2002) and typically of order one. Secondly, the drainage area $A$ in
Equation (2.2) makes the advective term strongly nonlocal, while in the KPZ equation the dynamics of \( h \) is assumed to be local thus ruling out any coupling with bulk properties (Krug, 1997). When the advective term is neglected, both Equation (2.2) and the KPZ equation reduce to the EW equation in which the surface growth is only determined by diffusion and noise. Neither the KPZ nor the EW model includes multiplicative noise terms.

Note that the simplified modeling framework adopted here only considers directly diffusive soil creep, advective river incision, and a random component chosen to mimic processes with large uncertainty such as landslide dynamics. Glacial erosion, which creates U-shaped valleys in alpine environments (Harbor, 1992; MacGregor et al., 2000; Herman and Braun, 2008; Egholm et al., 2009), is not explicitly modeled here. However, glacial erosion primarily acts during the early stages of mountain formation while, at mid to low latitudes, it is less likely to occur in the relaxation phase, which is the main focus of the work here. Thus, the possible effects of past glaciations are, at least in part, already encoded in the initial condition of the topography, which was taken from real landscapes where the effect of glaciations was present. Moreover, some mountain ranges considered here (e.g., Southern Appalachians) are believed to have never been glaciated.

2.4.1 Simulation setup

Numerical simulations were performed for three drainage basins in the Andes, Alps, and Himalayas (see Figure 2.8). Equation (2.2) is solved numerically on a regular grid with \( dx = dy = 1 \) arcsec (the arcsec of longitude was adjusted in order to account for its decrease with increasing latitude) using forward differencing in time and centered difference approximations for the spatial derivatives. Inflow/outflow boundary conditions were imposed over the domain. In particular, at the pour point an outflow boundary condition was assumed, while on the topographic divide
Table 2.2: Parameters values for the landscape evolution model used in the simulated scenarios SC1-SC8. Values of \( m = 0.3 \) and \( n = 0.7 \) are kept constant for all cases. Units are \( D \ [\text{m}^2/\text{yr}] \), \( K \ [\text{m}^{1-2m}/\text{yr}] \), \( \xi_2 \ [\text{m}/\text{Mya}] \), \( \xi_1 \ [\text{m}^{1-2m}/\text{Mya}] \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>SC1</th>
<th>SC2</th>
<th>SC3</th>
<th>SC4</th>
<th>SC5</th>
<th>SC6</th>
<th>SC7</th>
<th>SC8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>( 4.5 \times 10^{-6} )</td>
<td>( 4.5 \times 10^{-6} )</td>
<td>( 4.5 \times 10^{-6} )</td>
<td>( 1 \times 10^{-5} )</td>
<td>( 1 \times 10^{-5} )</td>
<td>( 4.5 \times 10^{-6} )</td>
<td>( 8 \times 10^{-6} )</td>
<td>( 8 \times 10^{-6} )</td>
</tr>
<tr>
<td>( K )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 1 \times 10^{-8} )</td>
<td>( 1 \times 10^{-8} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 0 )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
<tr>
<td>( \xi_2 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \xi_1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( [-0.5,0.5] )</td>
<td>( [-0.7,0.7] )</td>
</tr>
<tr>
<td>Himalaya</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 3 \times 10^{-5} )</td>
<td>( 1 \times 10^{-4} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
</tr>
<tr>
<td>( K )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 1 \times 10^{-8} )</td>
<td>( 1 \times 10^{-8} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 0 )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
<tr>
<td>( \xi_2 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( [-50,50] )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \xi_1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( [-0.5,0.5] )</td>
</tr>
<tr>
<td>Alps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 3 \times 10^{-5} )</td>
<td>( 6 \times 10^{-5} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
<td>( 5 \times 10^{-6} )</td>
</tr>
<tr>
<td>( K )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 1.5 \times 10^{-8} )</td>
<td>( 2 \times 10^{-8} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 0 )</td>
<td>( 5 \times 10^{-9} )</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
<tr>
<td>( \xi_2 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( [-50,50] )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \xi_1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( [-0.5,0.5] )</td>
</tr>
</tbody>
</table>

A diffusive process was imposed. The time step (equal to \( 10^4 \text{ yr} \)) was assumed in order to ensure the convergence of both advective and diffusive terms in Equation (2.2). The drainage area \( A \) is computed at each grid point with the D8 algorithm through the Matlab TopoToolbox.

Eight modeling scenarios (SC1-SC8) were analyzed (see Table 2.2 for model parameters). In SC1 the parameters \( K \) and \( D \) were calibrated to have the slope statistics follow the observed pattern. In scenarios SC2-SC3 and SC4-SC5 the advective and diffusive terms were progressively increased, respectively, in order to analyze their relative role on the evolution of slope distributions. Scenario SC6 corresponds to \( K = 0 \), thus reproducing a purely diffusive process. In SC7 only the additive noise term is switched on, while in scenario SC8 the multiplicative noise \( \xi_1 \) is employed in place of the additive one. Parameters \( D, K, m, \) and \( n \) were assumed uniform in space and time and the values adopted here are intended to be representative of averaged quantities whose relative amounts are simply used to quantify the proportional impact of advective and diffusive terms on slope statistics. While \( m \) and \( n \) were kept constants (and equal to literature values (Perron et al., 2008; Chen et al., 2014)), only \( D \) and \( K \) were modified to quantify the relative effects of the two terms. The
Figure 2.13: Evolution of the power-law exponent $q$ over time for scenarios SC1-SC8 for watersheds in the Andes (a), Himalaya (b), and Alps (c) and comparison with data (gray diamonds).

The additive noise term $\xi_2$ is modeled as a uniform noise bounded between $\pm 5 \cdot 10^{-5}$ m/yr, while the multiplicative one ($\xi_1$) was computed as a uniform noise bounded between $\pm 7 \cdot 10^{-7}$, $\pm 5 \cdot 10^{-7}$, and $\pm 5 \cdot 10^{-7}$ m$^{1-2m}$/yr for the basin in the Andes, Himalayas, and Alps, respectively. The value of $\xi_1$ was chosen to obtain a multiplicative term of the same order of magnitude as the additive noise $\xi_2$ used in SC7.

2.4.2 Results from landscape evolution modeling

Slope distributions obtained at different stages for scenarios SC1-SC7 are displayed in Figures 2.9, 2.10, and 2.11, for the watersheds in the Alps, Andes, and Himalayas, respectively. Maps of slope values over the Andean watershed at different stages for scenarios SC1, SC3, SC5, and SC7 are displayed in Figure 2.12 (the qualitative behavior is analogous for the Alpine and Himalayan watersheds, see Figures A.1 and A.2), while results from all scenarios are summarized in Figure 2.13.

When the advective term dominates (SC2-SC3), the strong localized fluvial incision creates concentrated regions of very steep slopes (similar to canyons, see for
example results from SC3 for the Andean watershed in Figure 2.12) resulting in a strong increase in the power-law exponent $q$ due to very heavy-tailed distributions (panels b and c in Figures 2.9, 2.10, 2.11). Furthermore, completely neglecting the advective term (SC6) produces an evolution of slope statistics similar to scenario SC1. A strong increase of the diffusion term tends to smooth the surface unrealistically, leading to a featureless topography where mode and variance of the distribution progressively tend to zero (panels d and e in Figures 2.9, 2.10, 2.11). The introduction of a random noise allows to maintain the shape of the distribution over time even with increasing values of the diffusion term $D$ (panel g in Figures 2.9, 2.10, 2.11). Thus, as the landscape ages, the resulting statistics appear to follow the equilibrium behavior of both KPZ and EW models, with the evolution process mainly driven by diffusion. It is also likely that the noise term is state dependent and thus becomes less important as the landscape evolves towards the ultimate flat configuration. Finally, results from scenario SC8 are shown in Figure 2.14 and compared with results from SC1 and SC7, suggesting that a state-dependent random component allows for a more gradual decrease of the power-law exponent $q$ during the first part of the relaxation phase, although qualitatively the results are not different from the case of the additive noise. This point requires further investigation, including the comparison with more complex landslide modeling frameworks (see, e.g., Densmore et al. (1998); Egholm et al. (2013)).

In terms of the processes analyzed, our results show that in the early phase of smoothing the time evolution of slopes in mountainous landscapes is the outcome of a balance between diffusive soil creep, fluvial incision, and noise, while the relaxation phase is mainly dominated by diffusion. In particular, to obtain a reasonable evolution of slope statistics through time, the relative values of diffusive and advective parameters must be carefully tuned, with the diffusion coefficient $D$ being several order of magnitude higher than the river incision coefficient $K$. Thus, while the
The statistical analysis we performed shows that landscape age is clearly encoded in the topographic properties of the landscape. Gradient and slope distributions start with fat tails (i.e., algebraic decay) and, at older orogenetic stages, tend to become Gaussian and Rayleigh distributed, respectively. The behavior of such statistics is encoded in the exponent of the power law which decreases with age in a fairly regular fashion, providing a novel quantitative link between landscape age and its statistical properties. The time evolution of slope distributions can be reproduced numerically by carefully tuning diffusive, advective, and noise terms, resulting in a
relaxation phase mainly dominated by diffusive soil creep, compared to noise and advective fluvial incision. When contrasted with the narrow range of parameter values in which realistic landscape evolutions are obtained, the robustness of the found relationship between age and tail-slope decay suggests that in nature such balance between erosional and depositional processes is likely achieved through a self-regulating mechanism.
The effect of accelerated soil erosion on hillslope morphology

“What remains now is, compared with those days, like the skeleton of a body wasted by disease: the soil, or at any rate as much of it as is rich and soft, has rolled away, and only the spare body of the land remains.”

*Plato - Critias (111b-c), translated by Waterfield (2008)*

### 3.1 Introduction

Anthropogenic land use has been widely recognized to alter the geomorphology of large regions (Syvitski and Kettner, 2011; Dotterweich, 2013; Tarolli and Sofia, 2016) with tremendous environmental impacts in terms of biodiversity loss, land and water degradation, and ecosystem functioning (Foley et al., 2005; Millennium Ecosystem Assessment, 2005; Power, 2010). The rate of erosion in many agricultural areas highly exceeds soil formation (Hillel, 1998; Hooke, 2000; Montgomery, 2007; Richter, 2007; Ellis et al., 2013; Amundson et al., 2015), averaging 1 to 2 orders of magnitude
above soil production and geological erosion under native vegetation (Montgomery, 2007). The accelerated erosion due to intensive agricultural land use is typically related to the disruption of the natural balance between the various erosional mechanisms operating along a hillslope (Bennett and Lowdermilk, 1938; Sharpe, 1938; Carson and Kirkby, 1972; Montgomery, 2007). In fact, surface runoff and river discharge generally increase when natural vegetation is cleared for agricultural purposes (Hooke, 2000; Foley et al., 2005; Dotterweich, 2013), thus changing the mechanisms driving soil erosion from slow, bio-hydrologically driven creep (Kirkby, 1967), to soil removal by overland and shallow subsurface flow, a much more rapid process (Amundson et al., 2015). Soil erosion depletes soil properties (i.e., soil thickness, moisture, carbon and nutrient content) leading these intensively managed soils to possibility overshooting stability thresholds with rapid shifts to unsustainable conditions (Rockström et al., 2009; Porporato et al., 2015). Examples of large-scale erosional processes related to human activity have been observed worldwide. Some of the most remarkable cases include the massive land degradation caused by cotton plantations in the US Southern Piedmont (Trimble (1985); Richter et al. (2014), see Figure 3.1), the accelerated soil erosion in the Chinese Loess Plateau due to a prolonged intensive human activity (Fu, 1989; Shi and Shao, 2000; Zhao et al., 2013), as well as the massive soil erosion observed in many Mediterranean regions (Martínez-Casasnovas and Sánchez-Bosch, 2000; Zalidis et al., 2002; García-Ruiz, 2010) or the massive Brazilian land degradation due to unsustainable crop and livestock activities (Fearnside, 2001; Merten and Minella, 2013).

High-resolution topography has been recently used to quantify anthropogenic signatures on landscape morphology (Tarolli and Sofia, 2016). Most work focused on the analysis of the topographic footprint of various man-made landforms, such as terraces, roads, canals, and irrigation networks, as well as the effect of mining and urbanization (Brown et al. (2017) and references therein). Time series high resolu-
tion topographic images have also been employed for the quantification of soil surface changes and erosion rates from agricultural areas (e.g., Eltner et al. (2015); Pineux et al. (2017)). The statistical distributions of slopes in a variety of landscapes have been linked to the landscape orogenetic properties and to the main erosional and depositional mechanisms driving their evolution (see, e.g., Speight (1971); Montgomery (2001); Wolinsky and Pratson (2005); Vico and Porporato (2009); Bonetti and Porporato (2017)).

With respect to landscape evolution modeling, early works on the study of hillslope development focused on the theoretical analysis of the transient dynamics, equilibrium conditions, and stability of various geomorphic transport laws (Culling, 1960; Hirano, 1968; Kirkby, 1971; Smith and Bretherton, 1972; Armstrong, 1987; Loewenherz, 1991) and subsequently led to the development of various large-scale numerical models of landscape evolution used to tackle a wide array of geomorphological problems (e.g., Willgoose et al. (1991); Tucker and Bras (1998); Perron et al. (2008); Hancock et al. (2010)). While these large-scale numerical models have proven to be useful tools for understanding the linkages between the various erosional processes and their topographic properties at coarser scales, it is often difficult to extract information about the basic mechanisms responsible for differences among landscapes from their cumbersome numerical outputs (Dietrich et al., 2003).

To analyze the effect of accelerated soil erosion on hillslope morphology, a statistical analysis of the landscape topographic features across various spatial scales is first conducted here to identify signatures of human impact on landscape morphology. Subsequently, theoretical quasi-equilibrium solutions for the hillslope profile and the associated erosional fluxes are derived at the hillslope scale and used to identify the dominant erosional processes responsible for the observed differences between morphological features in natural and disturbed conditions. The transient dynamics are also investigated to gain insight into the characteristic timescales of the relaxation
process. The simple theoretical framework provided here can be extended to further investigate the role of vegetation cover in retaining runoff erosion, and to identify critical conditions of land erosion in relation to different degrees of vegetation cover. An application to the Calhoun Critical Zone Observatory (CZO), a highly degraded landscape in the US Southern Piedmont, shows that the observed differences in hillslope morphologies between disturbed and undisturbed areas are encoded in gradient and slope statistics and are linked to an imbalance between soil creep and runoff erosion processes.

Figure 3.1: The Calhoun Critical Zone Observatory: (a) location and (b,c) landscape degradation (1950 ca., photos from Nelson (2016))

3.2 Study area: the Calhoun Critical Zone Observatory

The Calhoun CZO in South Carolina (Figure 3.1) is an area where decades of intensive agricultural land use resulted in a distinctive topography with gullies, interfluves, hillslopes and significantly eroded areas (Richter et al., 2014; Brecheisen and Richter, 2014; Bastola et al., 2017; Coughlan et al., 2017). According to the leading hydrologist Marvin Hoover, “nowhere in the country have hydrological processes in the soil been altered by past land use to a greater extent than in the South Carolina Piedmont” (Hoover, 1950). In this area, the land degradation process accelerated from the early-19th to early-20th centuries due to unsustainable land-use practices.
for cotton cultivation, which drastically altered soils and the geomorphology of the landscape (Ireland et al., 1939; Hoover, 1950; Trimble, 1985; Richter and Markewitz, 2001). An estimated 17-cm of soil was eroded between 1750 and 1950 from the entire Piedmont from Virginia to Alabama (Trimble, 1985; Richter et al., 2014), resulting in “a miserable panorama of unpainted shacks, rain-gullied fields, straggling fences, and rattle-trap Fords, dirt, poverty, disease, drudgery, and monotony that stretches for a thousand miles across the cotton belt” (Johnson et al., 1935).

The severe soil erosion greatly increased surface and subsurface runoff, causing enormous sedimentation and legacy sediment problems, jeopardizing soil-water storage, and accentuating plant-water stress and plant disease (Richter et al. (2014) and references therein). The massive land degradation and consequent soil fertility loss triggered land abandoned in the early 1900s. During the New Deal era (1933-41), the US Forest Service purchased many properties for the Sumter National Forest, including areas that would become the Calhoun CZO. This state-initiated intervention started a trajectory of regeneration with the overhaul of terraces, extensive tree planting, and gully rehabilitation (Coughlan et al., 2017). The territory is now dominated by a pine forest (both planted and naturally regenerated), has an average rainfall of approximately 1180 mm/yr, and temperatures typically range between -2.4 °C and 33.1 °C (average minimum/maximum for the coldest/warmest months, respectively) (Coughlan et al., 2017).

3.3 Effect of aging and disturbances on landscape statistics

A topographic landscape is characterized by its elevation \( z(x, y, t) \). Its first derivatives, \( \partial_x z \) and \( \partial_y z \), describe the change in elevation in the \( x \) and \( y \) directions, respectively, and are linked to the local slope, defined as the magnitude of the gradient and obtained as \( S = \sqrt{(\partial_x z)^2 + (\partial_y z)^2} \). The analysis of gradient and slope distributions at the Calhoun CZO is conducted here at both larger scales (i.e., regional, \( \approx 10-100 \)
Figure 3.2: (a) Shaded relief image of a subregion (approximately 0.96 km$^2$) of the Calhoun CZO where both disturbed and undisturbed areas are present (see also the 3D representation in Figure 3.3a): on the right the undisturbed area (hardwood forest) is characterized by a smooth topography with gentle slope (see cross section AA’ in panel b), on the left the disturbed area is characterized by the presence of man-made work and deeply incised gullies resulting in steeper slopes (see cross section BB’ in panel c).

m), where the stage of geomorphic development can be directly linked to its slope distributions (Section 3.3.1), and smaller scales ($\approx$ 1-10 m, typical of human disturbances), where significantly eroded and disturbed areas will be shown to result in peculiar changes in slope and gradient distributions (Section 3.3.2).

3.3.1 Landscape aging

The stage of geomorphic development of a landscape is particularly evident in the distribution tails of gradient and slopes, which tend to be heavier for younger landscapes (see Chapter 2). Such a process is characterized by gradient and slope probability density functions (pdfs) with power-law tails progressively approaching Gaussian and Rayleigh distributions with age, respectively,

$$p(S) \sim S^\beta,$$  \hspace{1cm} (3.1)
with higher $\beta$ (i.e., fatter tails) for younger mountains and a linear decay of the exponent with age ($d\beta/dt = 0.013\ \text{Mya}^{-1}$). Results for the Calhoun CZO as well as for other five CZOs located in the continental US and characterized by different stages of geomorphic development are shown in Appendix B, where the rate of decay $d\beta/dt$ is used to infer approximate ages of the various CZOs. Reasonable age estimates are thus obtained, especially for the old landscapes of Calhoun and Shale Hills in the Appalachians, and for the younger topographies of the West and Mid-West. In particular, for the Calhoun CZO the exponent $\beta$ is equal to -9.85, coherently with the old landscape of the Piedmont region and within the error range typical of the Appalachian mountains (see Appendix B ). The value of the power law exponent $\beta$ is robust to changes in the DEM resolution (see analysis in Appendix B ), thus confirming that, at large scales, the statistical properties of a landscape are directly linked to its stage of geomorphic development.

3.3.2 Effect of disturbances

The effect of disturbances becomes apparent at smaller scales ($\approx 1$-10 m). Figure 3.2a shows a smaller portion of the Calhoun CZO ($\approx 0.96\ \text{km}^2$), where both an undisturbed and a disturbed portion are present. The former is characterized by the presence of hardwood forest and has a smoother topography with gentler slopes, while the latter has deep gullies with steeper slopes, as evident from the two cross sections compared in Figure 3.2 (see also the 3D representation in Figure 3.3a). These different topographical features reflect themselves in the gradient and slope distributions, whose tails tend to be fatter for the disturbed portion due to the observed steeper slopes (Figure 3.3). This results in gradient pdfs being well approximated by Gaussian and Laplace distributions for the undisturbed and disturbed areas, respectively (Figure 3.3b,c). In terms of slope distribution tails, the undisturbed area has a very low $\beta$ value ($\beta = -9.27$, typical of this old Piedmont landscape, as discussed
3.4 Effect of disturbances on hillslope evolution

The evolution of a landscape is typically characterized by two distinctive phases (Figure 3.4). Initially, the drainage network is formed over a relatively fast time scale, the so-called freezing time (Sinclair and Ball, 1996; Banavar et al., 2001), after which the planar geometry of the network remains practically fixed. Over a much longer time scale, called relaxation time (Fernandes and Dietrich, 1997; Banavar et al., 2001), a second phase takes place, during which erosion and creep keep acting on the landscape while the drainage network is preserved. During this second phase, hillslopes evolve towards a quasi-equilibrium form as long as boundary conditions, geomorphic processes, and uplift rates are not changed (Hack, 1960; Kirkby, 1971; Hack, 1975; Fernandes and Dietrich, 1997; Willett and Brandon, 2002). While a two-dimensional framework is clearly necessary to model the landscape evolution during
the network-forming phase, a one-dimensional analysis is sufficient to capture the main processes and feedback mechanisms involved at the hillslope scale during the relaxation phase (Fernandes and Dietrich, 1997). Disturbances linked to erosion and land cover change are not expected to alter the main large-scale drainage network, but are likely to have an impact on the resulting hillslope profile at the end of the relaxation phase.

Focusing our attention to the relaxation phase and one-dimensional analysis, the mathematical approach introduced here is based on the analysis of the coupled dynamics of surface water and landscape evolution, for which conservation equations are written at the hillslope scale (Kirkby, 1971; Smith and Bretherton, 1972; Armstrong, 1987; Fowler, 2011) and are meant to capture the approach to quasi-equilibrium hillslope profiles resulting from erosion and creep. We consider a symmetric hillslope inclined towards a fixed boundary at \( x = x_L \), with divide at \( x = 0 \), and \( \frac{\partial h}{\partial x} < 0 \). On this hillslope, the time evolution of the water height \( h \) [L] is expressed according
to the continuity equation (Smith and Bretherton, 1972; Smith, 2010; Fowler, 2011; Chen et al., 2014)

\[
\frac{\partial h}{\partial t} = R - \frac{\partial q}{\partial x},
\]  

(3.2)

where \( t \) [T] is time, \( q \) [L\(^2\) T\(^{-1}\)] the water flux, and \( R \) [L T\(^{-1}\)] the rainfall rate effectively contributing to runoff production (i.e., the total rainfall reduced by losses due to evapotranspiration, infiltration and leakage). It is assumed here that water flows in the direction opposite to the topographic gradient. Further assuming steady state conditions, constant unitary rainfall rate, and water flowing at constant unitary speed, the two-dimensional form of Eq. (3.2) reduces to an equation for the specific catchment area (see Chapter 4).

The time evolution of the surface topography is given by the conservation equation

\[
\frac{\partial z}{\partial t} = U - \frac{\partial f}{\partial x} = U - \frac{\partial}{\partial x} (f_d + f_c)
\]  

(3.3)

where \( z \) [L] is the surface elevation, \( U \) [L T\(^{-1}\)] the tectonic uplift rate, and \( f \) [L\(^2\) T\(^{-1}\)] the sediment flux. The total sediment flux is the sum of fluxes associated with runoff erosion/channelized flow (\( f_c \)) and mass movement of sediment/creep/colluvial processes (\( f_d \)). We neglect here any mass movement associated with landslide dynamics, thus assuming that the landscape is characterized by moderate to small gradients, where slope failure is mostly insignificant.

Soil creep is generally hypothesized to be driven by processes such as bioturbation, frost heaving, and wetting/drying, and it is the dominant process in soil-mantled landscapes with moderate topographic gradients (Carson and Kirkby, 1972; Perron et al., 2008). Field studies (e.g., Schumm (1967); McKeen et al. (1993); Small et al. (1999); Dietrich et al. (2003)) provided empirical evidence for a linear increase of \( f_d \) with the landscape gradient, so that the time-averaged downslope soil creep flux
is generally assumed to be proportional to the topographic gradient (Culling, 1960, 1963),

\[ f_d = -D \frac{\partial z}{\partial x}, \quad (3.4) \]

where the diffusion coefficient \( D \) [L^2 T^{-1}] is assumed to be constant in space and time. This is a reasonable assumption on short hillslopes (e.g., 20-100 m long), without significant variations in bedrock geology, soil properties, and vegetation cover (Fernandes and Dietrich, 1997). The diffusion term obtained by modeling the soil creep according to Eq. (3.4) tends to smoothen the surface by dampening out perturbations in the topography and is argued to be the predominant erosional mechanisms on hillslopes (Fernandes and Dietrich, 1997; Perron et al., 2008; Chen et al., 2014).

The runoff erosion/river incision component \( f_c \) is a function of the water flow and topographic slope, resulting in a term that tends to amplify perturbations in the topography and is responsible for channel and valley formation. The runoff erosion term can have different formulations, namely detachment-limited (DL) and transport-limited (TL). In the former regime, the sediment transport rate is limited by the rate of detachment and sediment entrainment (i.e., all eroded material is transported outside the considered domain with no sediment redeposition), while in the latter regime the sediment rate is limited by the transport capacity (Dietrich et al., 2003). TL hillslopes are traditionally defined as those that are soil-covered (weathering is efficient at producing debris, but transport processes are inefficient at removing it from the slopes), while DL hillslopes are those that are not soil-covered (Pelletier, 2012). From a modeling perspective, in TL models erosion and deposition by flowing water is related to the divergence of sediment flux (see, e.g., Smith and Bretherton (1972); Willgoose et al. (1991); Tarboton et al. (1992)), with \( f_c \) expressed
as (Pelletier, 2012)

\[ f_c = K_T q^{m_T} \left| \frac{\partial z}{\partial x} \right|^{n_T}, \quad (3.5) \]

so that the landscape evolution equation becomes

\[ \frac{\partial z}{\partial t} = U + D \frac{\partial^2 z}{\partial x^2} - K_T \frac{\partial}{\partial x} \left( q^{m_T} \left| \frac{\partial z}{\partial x} \right|^{n_T} \right), \quad (3.6) \]

where the coefficient \( K_T \) \([L^{2-2m_T} T^{m_T-1}]\) is assumed constant in space and time (again, a reasonable assumption over short hillslopes with homogeneous properties).

Conversely, in DL models the rate of fluvial/slope-wash erosion is directly related to the shear stress, power, or velocity of the flow (Howard, 1994; Perron et al., 2008), so that in this case

\[ \frac{\partial f_c}{\partial x} = K_D q^{m_D} \left| \frac{\partial z}{\partial x} \right|^{n_D}, \quad (3.7) \]

with the time evolution of the topographic surface \( z \) reading

\[ \frac{\partial z}{\partial t} = U + D \frac{\partial^2 z}{\partial x^2} - K_D q^{m_D} \left| \frac{\partial z}{\partial x} \right|^{n_D}, \quad (3.8) \]

where \( K_D \) \([L^{1-2m_D} T^{m_T-1}]\) is also assumed constant in space and time.

3.4.1 Steady state solutions

Steady state solutions representing the quasi-equilibrium profiles at the end of the relaxation phase are derived assuming zero sediment and water fluxes at the divide (i.e., \( \frac{\partial z}{\partial x} = q = 0 \) at \( x = 0 \)), and a fixed surface level \( z = z_L \) at \( x = x_L \). Such boundary conditions are consistent with those assumed in previous studies (e.g., Smith and Bretherton (1972); Loewenherz (1991); Fowler (2011)). In particular, the fixed location of the channel (i.e., \( x_L \)) derives from assuming that during the relaxation phase the planar geometry of the drainage network remains practically invariant (Figure 3.4).
Figure 3.5: Analytical steady state solutions for the transport limited (TL, solid lines) and detachment limited (DL, dashed lines) cases: elevation profile $z_0$, slope $S_0$, curvature $\partial^2 z_0/\partial x^2$, soil creep $f_d$ and runoff erosion $f_c$ sediment fluxes. Results are computed assuming $R = K_a = U = 1$, and for different values of the diffusion coefficient $D$. In the TL case plots are shown for $m_T = 2$ (i.e., logarithmic solution, Eq. (3.16)).

Imposing zero sediment and water fluxes at the ridge, the steady state solution of Eq. (3.2) and (3.3) provides water and sediment fluxes linearly increasing with the distance from the divide,

$$q_0 = Rx \quad (3.9)$$

$$f_0 = Ux \quad (3.10)$$

where subscript 0 denotes steady state, and both rainfall and uplift rate are assumed to be constant in space and time.
From an analysis of the generic steady state solution (Eq. (3.9)-(3.10)), a condition for the concavity/convexity of hillslopes can be derived (Smith and Bretherton, 1972), which will be employed in the following section to analytically derive the inflection point in TL conditions. Differentiating both sides of $Ux = f_0(S_0, q_0)$ with respect to $x$ gives

$$U = \frac{\partial f_0}{\partial S_0} \frac{\partial S_0}{\partial x} + \frac{\partial f_0}{\partial q_0} \frac{\partial q_0}{\partial x}. \quad (3.11)$$

Inserting Eq. (3.9) and (3.10), the previous equation can be rewritten as

$$x \frac{\partial f_0}{\partial S_0} \frac{\partial S_0}{\partial x} = f_0 - q_0 \frac{\partial f_0}{\partial q_0}. \quad (3.12)$$

The concavity/convexity of the topographic surface is given by the sign of $\frac{\partial S_0}{\partial x}$ and, observing that $\frac{\partial f}{\partial S} > 0$ (i.e., the sediment transport is assumed to be an increasing function of the topographic slope, coherently with both TL and DL formulations), it reduces to an analysis of the sign of $f_0 - q_0 \frac{\partial f_0}{\partial q_0}$: when $f_0/q_0 > \frac{\partial f_0}{\partial q_0}$ the hillslope is concave down, while it is concave up when $f_0/q_0 < \frac{\partial f_0}{\partial q_0}$.

**Transport limited (TL) conditions**

In TL conditions the sediment flux is given by $f = -D \frac{\partial z}{\partial x} + K_a \left[ \frac{\partial z}{\partial x} \right]^{n_T} q^{m_T}$. To allow mathematical tractability, we assume $n_T = 1$ (numerical results obtained for different values of $n_T$ are provided in Appendix C and discussed below). For the geometry assumed here, $|\partial z/\partial x| = -\partial z/\partial x$, so that the steady state sediment flux becomes

$$Ux = -D \frac{\partial z_0}{\partial x} - K_a \frac{\partial z_0}{\partial x} (Rx)^{m_T}, \quad (3.13)$$

from which the local slope is obtained as

$$S_0(x) = \frac{\partial z_0}{\partial x} = \frac{Ux}{KR^{m_T}x^{m_T} + D}. \quad (3.14)$$
Integrating to find $z_0(x)$ and imposing the fixed level boundary condition at $x = x_L$ provides the steady state profile

$$
z_0(x) = z_L + \frac{U}{2D} \left( x_L^2 \mathcal{H}_1 \left( 1, \frac{2}{m_T}; 1 + \frac{2}{m_T}; \frac{K_a R^{m_T} x_L^{m_T}}{D} \right) - \right)

$$

$$
x^2 \mathcal{H}_1 \left( 1, \frac{2}{m_T}; 1 + \frac{2}{m_T}; \frac{K_a R^{m_T} x_L^{m_T}}{D} \right) \right),

(3.15)

where $\mathcal{H}_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function (Abramowitz and Stegun, 1964).

Note that when $m_T = 2$ the logarithmic solution provided by Loewenherz (1991) is recovered (Abramowitz and Stegun, 1964), namely

$$
z_0(x) = z_L - \frac{U}{K_T R^{2} x_L^2} \left( \ln \left( 1 + \frac{K_T R^2 x_L^2}{D} \right) - \ln \left( 1 + \frac{K_a R^{2} x_L^2}{D} \right) \right).

(3.16)

The local surface curvature is equal to

$$
\frac{\partial^2 z_0(x)}{\partial x^2} = \frac{-DU + K_a (m_T - 1) U R^{m_T} x^{m_T}}{(D + K_a R^{m_T} x^{m_T})^2}

(3.17)
from which the inflection point \( x_i = \left( D/(K_T(m_T - 1)R^{m_T}) \right)^{1/m_T} \) can be obtained. Analogously, the inflection point can be derived from the concavity/convexity relation (Eq. (6.10)) imposing \( f_0 = q_0 \partial f_0/\partial q_0 \). Note that to have \( x_i \) real the exponent \( m_T \) must be \( > 1 \), so that when \( m_T < 1 \) the profile is always concave down (i.e., no inflection point).

The fluxes associated to both soil creep and runoff erosion can be computed as

\[
\begin{align*}
  f_d(x) &= -D \frac{\partial z_0}{\partial x} = \frac{DU_x}{D + K_T R^{m_T} x^{m_T}} \quad (3.18) \\
  f_c(x) &= -K a q_0^{m_T} \frac{\partial z_0}{\partial x} = U x - f_d = \frac{K_T U x(R x)^{m_T}}{D + K_T (R x)^{m_T}}. \quad (3.19)
\end{align*}
\]

An example of the behavior of the solution is shown in Figure 3.5 for different values of the diffusion coefficient \( D \). When the diffusion term dominates over the advective one, the entire profile is convex, while reducing the soil creep component allows us to obtain a hillslope morphology convex at the divide and transitioning to concave as the valley head is approached. In this case, the upper convexity is related to soil creep erosional processes, while the basal concavity results from the runoff erosion component (Carson and Kirkby, 1972), as evident from the comparison of \( f_d \) and \( f_c \) fluxes in Figure 3.5d,e. As noted by Smith and Bretherton (1972), both transport mechanisms (soil creep and river incision) are necessary to create steady state profiles that have both a convex and a concave region. The diffusive flux increases/decreases with \( x \) in the convex/concave portion of the hillslope, while the runoff erosion term increases with the distance from the divide.

Numerical steady state solutions for different values of the exponents \( n_T \) and \( m_T \) are provided in Appendix C. Decreasing either exponent reduces the runoff erosion component, thus providing hillslope profiles mainly concave down. Solutions are particularly sensitive to variations in the water flow exponent \( m_T \): doubling its value, while maintaining constant all other parameters, can substantially change
the hillslope morphology from entirely concave down to a profile with both convex and concave portions. Conversely, variations in the slope exponent $n_T$ result in modifications of the mean landscape elevation, without changing much the overall shape (in terms of concavity/convexity) of the profile.

Figure 3.7: (a-d) Analytical fit of hillslopes in the undisturbed (U1, U3) and disturbed (D1, D3) sections (see Figure 3.6 for location): blue symbols are digitized values, blue line is the fitted DL analytical profile given by Eq. (3.20), obtained assuming $U = 2 \cdot 10^{-5}$ m/year and $R = 1100$ mm/year. Fitting parameters $D$ and $K_D$ have units of m$^2$ yr$^{-1}$ and m$^{-1}$, respectively. The secondary axis shows the surface curvature for the fitted hillslope (Eq. (3.22)) and the yellow shading highlights the concave-up portion of the hillslope (i.e., positive curvature). (e-h) Slope values computed from the digitized section (symbols) are compared to the analytical solution, Eq. (3.21). (i-l) Theoretical soil creep (dashed line) and runoff erosion (solid line) sediment fluxes (Eq. (3.23) and (3.24)). Green and red shading represent $f_d$ and $f_c$ dominated portions, respectively. Results for the other hillslopes are provided in Appendix C.
**Detachment limited (DL) conditions**

Analytical results for the DL regime are derived assuming $n_d = m_d = 1$. Numerical steady state solutions for different values of the exponents are provided in Appendix C and also discussed below. Imposing the boundary conditions at $x = 0$ and $x = x_L$ and making use of $|\partial z/\partial x| = -\partial z/\partial x$, the steady state solution to Eq. (3.8) is given by

$$z_0(x) = z_L + \frac{U}{2D} \left( -x^2_0 H_2 \left( 1, 1; \frac{3}{2}, 2; -\frac{K_D R x^2}{2D} \right) + \right. $$

$$+ \left. x^2_0 H_2 \left( 1, 1; \frac{3}{2}, 2; -\frac{K_D R x^2}{2D} \right) \right), \quad (3.20)$$

where $H_2(\cdot, \cdot; \cdot, \cdot)$ is the generalized hypergeometric function (Prudnikov et al., 1986). From Eq. (3.20) the slope and curvature can be derived,

$$S_0(x) = -\frac{\partial z_0}{\partial x} = \frac{\sqrt{2UD} \left( \sqrt{\frac{K_D R x}{2D}} \right)}{\sqrt{DK_D R}} \quad (3.21)$$

$$\frac{\partial^2 z_0(x)}{\partial x^2} = \frac{U \left( -\sqrt{D} + \sqrt{2K_D R x} D \left( \sqrt{\frac{K_D R x}{2D}} \right) \right)}{D^{3/2}}, \quad (3.22)$$

where $D(\cdot)$ is the Dawson’s integral (Abramowitz and Stegun, 1964). In this case the inflection point cannot be derived analytically, neither from the curvature nor from the concavity/convexity relation.

For the DL case, the soil creep and runoff erosion fluxes can be obtained as

$$f_d(x) = -D \frac{\partial z_0}{\partial x} = \frac{\sqrt{2UD} \left( \sqrt{\frac{K_D R x}{2D}} \right)}{\sqrt{K_D R}} \quad (3.23)$$

$$f_c(x) = \int_0^x K_D q_0 \frac{\partial z_0}{\partial x} \, dx = Ux - f_d = Ux - \frac{\sqrt{2UD} \left( \sqrt{\frac{K_D R x}{2D}} \right)}{\sqrt{K_D R}}. \quad (3.24)$$

An example of the behavior of the solution is shown in Figure 3.5 for different values of the diffusion coefficient $D$. As we would expect from previous numerical
studies on TL and DL models (see, e.g., Pelletier (2012)), hillslopes in DL and TL models display similar morphologies (see comparison in Figure 3.5a). Again, higher values of the diffusion coefficient $D$ provide more convex topographies, transitioning to convex/concave profiles as $D$ is reduced and the runoff erosion flux becomes predominant on the lower part of the hillslope. The similarity between the TL and DL steady state morphologies derives from imposing a fixed level at $z = z_L$, so that the deposition zone at the base of the hillslope is basically negligible for the TL model. This results in the runoff terms being practically equivalent in the two formulations, despite being represented by a divergence and a sink term in TL and DL regimes, respectively. This will allow us to focus our attention on the relative roles of soil creep and runoff erosion (see application to Calhoun CZO in Section 3.5), instead of the specific formulation of the runoff erosion term.

Numerical steady state profiles obtained for different values of the exponents $m_D$ and $n_D$ are shown in Appendix C. In this case, the solution is more sensitive to the exponent of the slope (i.e., $n_D$): changes in $n_D$ while keeping constant all the other parameters can shift the morphology from convex to one with both concave and convex portions, while variations in the water flux exponent $m_D$ only modify the elevation of the surface, but not the overall shape.

3.5 Hillslope analysis of the Calhoun CZO landscape

Given that TL and DL conditions provide similar morphologies and behaviors in relation to the basically negligible deposition in TL regime, the following analysis is based on the results from the DL model (but analogous considerations can be done using the solution obtained for the TL model). We consider two cross-sections (Figure 3.6a) in the disturbed and undisturbed areas, respectively. From a visual inspection of the hillslopes in the two cases, it is evident that the morphologies are different between undisturbed and disturbed portions, the first being mainly convex,
the second having a convex portion near the divide and transitioning to concave downhill. Five hillslopes per each section are isolated (D1-5 and U1-5 in Figure 3.6b,c) and the DL elevation profile (Eq. (3.20)) is fitted to each hillslope. The fitting parameters are the soil creep and runoff erosion coefficients ($D$ and $K_D$), while we assumed $U = 2 \cdot 10^{-5}$ m/year (Pavich, 1989; Markewich et al., 1990) and $R = 1100$ mm/year for all the hillslopes. This allows us to distinguish between the creep and runoff erosion contributions to the definition of the observed hillslope morphologies. Note that the value of $R$ used here is approximately equal to the mean annual rainfall at the Calhoun CZO, while corrections for evapotranspiration and infiltration losses were neglected. This results in $K_D$ and $D$ being effective parameters: accounting for losses will likely alter their magnitudes but not the relative amounts of runoff erosion and soil creep fluxes (the main focus of the work here).

The results of the analysis are shown in Figure 3.7 (see also Appendix C), provid-

![Figure 3.8](image-url)
ing a good agreement between the analytical steady state and the observed hillslope morphology, in terms of both elevation profile (panels a-d) and local slope values (panels e-h). As already observed, hillslopes in the undisturbed portion are mostly convex, while profiles in the disturbed portion are convex near the divide and transition to concave with increasing distance towards the valley head (see yellow shading in panels a-d, showing the concave-up portion, where the local curvature computed by means of the analytical solution becomes positive). The theoretical solution also provides a good estimate of slope values (Figure 3.7e-h), which tend to display higher maximum values in the disturbed portion, coherently with the results from the statistical analysis (Section 3.3.2). Such different morphologies are a result of the different dominating erosional processes along disturbed and undisturbed hillslopes (see panels i-l in Figure 3.7). In fact, while the convex undisturbed sections are dominated by soil creep fluxes, the sediment flux associated with runoff erosion prevails where hillslopes become concave up (i.e., in the lower portion of the disturbed profiles). This allows us to partition the hillslopes not only based on their topographical properties, but also according to the dominant erosional mechanism, as exemplified in Figure 3.8 where $f_c$- and $f_d$-dominated portions are distinguished. Theoretical results confirm that, in the disturbed areas, the accelerated soil erosion produces landforms characterized by higher slopes and more concave-up portions as a result of higher runoff erosion fluxes (compared to soil creep) along the hillslopes, as theorized by Gilbert (1909).

It is also interesting to analyze the time evolution of disturbed hillslopes to gain insight into the relaxation time required for their equilibrium adjustment (i.e., the time necessary to re-establish a quasi-equilibrium state under natural erosion). Eq. (3.8) is numerically solved starting from an initial condition equal to the fitted disturbed hillslopes (D1-5). We assume $U = 2 \cdot 10^{-5}$ m/year, $R = 1100$ mm/year, $n_D = m_D = 1$, while the coefficients $D$ and $K_D$ are set equal to the average values of
the parameters obtained through the fitting procedure for the undisturbed sections (i.e., $D = 0.0035 \text{ m}^2 \text{ yr}^{-1}$ and $K_D = 7.19 \cdot 10^{-7} \text{ m}^{-1}$). This is done to mimic the effect of hillslope restoration to the original vegetated state, in order to analyze the time required for the hillslope to reach morphological features typical of undisturbed old landscapes. Results are shown in Figure 3.9 for hillslope D3 (see Appendix C for results pertaining to the other four hillslopes). Numerical simulations show that two time scales can be distinguished. At the beginning the diffusive soil creep reduces the higher frequency components of the surface by quickly filling the concave-up portion of the profile resulting in a concave down morphology (for all the hillslopes this happens within less than 3000 years), while the time required for the hillslope to actually reach steady state is much longer (between $\approx 20000$ and $100000$ years depending on the initial condition and dimension of the hillslope considered). These values are in agreement with those obtained by Fernandes and Dietrich (1997), where the relaxation times from numerical experiments were found to range between 70000 years and 1 Ma for hillslopes 25 and 100 m long, respectively (note that results were there obtained assuming a constant downcutting rate as boundary condition).

3.6 Discussion and conclusions

The statistical analysis of gradient and slope distributions showed that the natural aging process related to mechanisms of geological erosion is well encoded in the distribution tails at larger scales (see also Chapter 2). Such a natural process can be altered by agricultural-driven accelerated erosion which, at smaller scales, produces key differences in the distribution tails. The observed differences in the morphological features of disturbed and undisturbed areas were shown to be related to a disruption of the natural balance between soil creep and runoff erosion mechanisms. To quantify such an imbalance, we derived theoretical steady state solutions for the hillslope morphology and the related sediment fluxes. These analytical solutions allowed us
Figure 3.9: (a) Time evolution of the disturbed hillslope D3 and (b-c) erosion rates associated with creep \( \dot{r}_d \) and runoff erosion \( \dot{r}_c \). Results are obtained assuming 
\[ D = 0.0035 \text{ m}^2 \text{ yr}^{-1} \] and 
\[ K_D = 7.19 \cdot 10^{-7} \text{ m}^{-1} \] (i.e., equal to the average values obtained from fitting the undisturbed sections, Figure 3.7), 
\[ U = 2 \cdot 10^{-5} \text{ m/yr}, \] and 
\[ R = 1100 \text{ mm/year}. \]

To reconstruct hillslope morphologies as well as to define the main erosional processes acting on different portions of the hillslope, clearly quantifying the relative increase in runoff erosion (compared to soil creep) needed to obtain the convex/concave hillslope profiles observed in the disturbed areas of the Calhoun CZO. The analysis of the relaxation process highlighted the emergence of two time scales, the first associated with the rapid removal of the higher frequency components of the surface topography and consequent development of a concave-down hillslope, the second related to the time required to reach a steady state condition where the uplift term is balanced by runoff erosion and soil creep fluxes. The relaxation time resulted to vary with the initial topography as well as hillslope length, coherently with previous studies.
The statistical description of landscape topographic features can be used to up-scale various plot-level processes providing coarse-scale descriptions of the local topography useful for a variety of ecohydrological, geomorphological, and climatic applications (Vico and Porporato, 2009; Riveros-Iregui and McGlynn, 2009; Bartlett et al., 2013; Wang et al., 2015). Furthermore, the integration of this type of analysis with time series high-resolution imaging can be used to directly relate erosion rates computed from satellite images (Eltner et al., 2015; Pineux et al., 2017) with the variation in the statistical properties of the landscape.

The shape of the hillslope as well as its relaxation time can be further altered by different climatic and land cover conditions (Ahnert, 1987; Fernandes and Dietrich, 1997). In particular, variations in the frequency and amount of precipitation are likely to change water fluxes as well as soil erodibility, while vegetation plays a key role in retaining runoff erosion, favoring infiltration and evapotranspiration losses. The mathematical framework used here directly accounts for a water continuity equation, which can be extended to include more detailed ecohydrological processes (e.g., through appropriate modeling of evapotranspiration and infiltration losses), thus allowing the study of equilibrium topographies, transient dynamics, and channel forming instability under various levels of vegetation cover and climatic forcings. Furthermore, disturbances not only alter the geomorphological properties of the landscape, but have a number of legacy effects such as an impact on biogeochemical cycles (Quinton et al., 2010; Viaud et al., 2010). Future analyses will consider the coupling of these dynamics.
Theoretical definition of drainage area at regular and non-regular points

“Water does not move from place to place unless it is drawn by a lower position. Lowness therefore serves as a magnet for water.”

L. Da Vinci (1452-1519) - Translated by McCurdy (1906)


4.1 Introduction

More than a century ago, J. C. Maxwell stated the importance of “an exact knowledge of the first elements of physical geography” and observed the prevalence of “loose notions on the subject” (Maxwell, 1870). Since then, the geomorphological and eco-hydrological literature has been replete with references to topographic entities such as drainage divides, valley lines, channel heads and drainage area, but most of these basic elements of physical geography still lack a sound mathematical description, such
that Maxwell’s statement remains relevant even today. Among these morphometric variables, the drainage (or catchment) area $A$ - defined as the horizontally projected integral of all areas draining to a point (Dingman, 2015) - is an extremely important nonlocal variable playing a role in several geomorphological and ecohydrological processes related to surface and subsurface water redistribution. For example, it is often used as an approximation of channel discharge in landscape evolution models (Perron et al., 2008; Rodríguez-Iturbe and Rinaldo, 2001; Chen et al., 2014; Bonetti and Porporato, 2017) and is related to soil moisture in combination with other topographic variables (Beven and Kirkby, 1979; Barling et al., 1994; Iverson et al., 1997; Summerell et al., 2004; Murphy et al., 2009). Furthermore, water-content indices based on drainage area have been adopted in the definition of landslide susceptibility (Montgomery and Dietrich, 1994; Borga et al., 1998), for the determination of vegetation patterns, and for biodiversity mapping (e.g., Moore et al. (1993); Moody and Meentemeyer (2001); Svenning et al. (2004); Zinko et al. (2005); Shoutis et al. (2010); Kuglerová et al. (2014)).

Despite its importance and the numerous algorithms developed for its numerical computation from regular grid Digital Elevation Models (DEMs) (O’Callaghan and Mark, 1984; Jenson and Domingue, 1988; Fairfield and Leymarie, 1991; Freeman, 1991; Quinn et al., 1991; Costa-Cabral and Burges, 1994; Tarboton, 1997; Orlandini et al., 2003; Seibert and McGlynn, 2007; Orlandini and Moretti, 2009), a first theoretical description of the drainage area has been provided only recently, with the derivation of a differential equation for the evaluation of the specific catchment area $a$ (Gallant and Hutchinson, 2011). The differential equation proposed therein has been used as a benchmark to test some numerical algorithms (namely, D8 (O’Callaghan and Mark, 1984; Jenson and Domingue, 1988), DEMON (Costa-Cabral and Burges, 1994), and D∞ (Tarboton, 1997)), showing that all the tested methods tend to overestimate $a$ in divergent terrain (ridges and hilltops), with the D∞ method being
overall the most accurate. More recently, a has been obtained by mapping it into the steady-state water level that would be obtained assuming a uniform (in space and time) and unitary rainfall rate transported downslope at a constant speed in the direction opposite to the gradient (Chen et al., 2014).

The theoretical expression proposed by Gallant and Hutchinson (2011) can be employed only at regular points of the watershed, thus not providing any insight in those areas characterized by either zero slope or singularities of the topographic field where different slope lines coalesce (i.e., flat areas, local maxima and minima, and ridge and valley lines when treated as singularities of the topographic surface). The first methods to deal with these non-regular points of the topographic surface were theorized in the nineteenth century, with the development of both local and nonlocal methods for the delineation of drainage lines (de Saint-Venant, 1852; Cayley, 1859; Maxwell, 1870; Boussinesq, 1871; Breton de Champ, 1877). These early works mainly focused on methods for the characterization of ridge and valley lines, while not directly dealing with the computation of the drainage area. Local ridge and valley definitions are based on differential geometry principles (de Saint-Venant, 1852; Boussinesq, 1871; Breton de Champ, 1877; Haralick, 1983; Gauch and Pizer, 1993; Eberly et al., 1994; Kweon and Kanade, 1994; López et al., 2000; Minár et al., 2013) but provide loci of ridge and valley lines that are interrupted in many points due to the presence of critical points and minor flat horizontal areas (Florinsky, 2012), thus not providing any insight in the treatment of non-regular points of the topographic surface. Nonlocal methods, on the other hand, are based on the pioneering work by Cayley (1859), who explicitly focused on special points of the surface, such as local elevation maxima, minima, and saddle points. He observed that, in general, there are only two special slope lines passing by a saddle, and they are a ridge and a valley line. From a dynamical systems perspective, these two special slope lines are the stable and unstable manifolds of the saddle point (Strogatz, 2014; Gilmore, 1993; Argyris et al.,
The work by Cayley (1859) was continued by Maxwell (1870), who stated that through each point of the surface passes a slope line going from a maximum to a minimum of the surface and defined basins and hills as districts whose slope lines come from the same minimum or run to the same maximum, respectively. This procedure allows the definition of ridges as “slope districts boundaries” (Nackman, 1984; Griffin et al., 1992; Rosin et al., 1992; Steger, 1999). Techniques based on the delineation of slope district boundaries have recently gained attention within the computer vision community for the analysis of image intensity functions (see, e.g., Griffin et al. (1992); Rosin et al. (1992); Rosin (1995)). However, they can be broadly applied to digital terrain analysis (Koenderink and van Doorn, 1993; Dawes and Short, 1994; Koenderink and van Doorn, 1994; Steger, 1999; Serrat et al., 2000; Florinsky, 2012) for the detection of ridge and valley lines (representing watershed divides and channels, respectively) and, as will be investigated in this work, for developing and testing models and algorithms for the numerical calculation of the drainage area at non-regular points. In addition, skeleton construction techniques have been shown to allow a fully automated recognition in contour-based DEMs of the complex topographic structures that can be found in real landscapes (Moretti and Orlandini, 2008).

The goal of this chapter is to provide a theoretical framework to define the drainage area at both regular and non-regular points of the topographic surface. The differential equation proposed by Gallant and Hutchinson (2011) is here derived in a more intuitive way from the water continuity equation defined by Chen et al. (2014). The theory is then extended to non-regular points of the surface by means of a dynamical systems approach that builds on the work of Cayley (1859) and Maxwell (1870). We hope that a theoretical definition of drainage area at regular and non-regular points of a given landscape may help to test and improve currently used numerical algorithms, as well as be used for the analysis of landscape evolution and
The chapter is organized as follows. The mathematical formulation for the definition of the drainage area at both regular and non-regular points is presented in Sections 4.2 and 4.3, respectively. In Section 4.4 some applications to special cases are provided, from simple analytical examples to more complex surfaces including critical and singular points. The main focus here is on continuous surfaces, while an extension to discrete surfaces (e.g., DEMs) is briefly discussed in Section 4.5 with an application to a real topography.

**Figure 4.1:** Graphical representation and definitions: contour lines ($W$, blue), slope lines ($S$, black) and projected slope lines ($L$, red) for a two dimensional topographic surface $z = f(x, y)$. A 2D surface is here defined as a single-valued function $z$ having a 2D domain ($x, y$ being the horizontal Cartesian coordinates). The drainage area $A$ pertaining to a contour segment of length $w$ at a certain arc-distance $l$ from the hilltop is the area between the two projected slope lines (green shaded area).
4.2 Drainage area at regular points

4.2.1 Terminology and basic definitions

Consider a topographic surface defined by a single-valued function \( z = f(x, y) \), \( z \) being the elevation and \( x, y \) the horizontal Cartesian coordinates (with \( x = xi + yj \)). While both continuous and discrete representations are used to describe a surface topography (either of which, however, is an approximation of real land surface complexities), here we focus on a continuous description. On this topographic surface two families of lines can be defined, namely contour and slope lines (Cayley, 1859; Florinsky, 2012).

A contour line \( \mathcal{W} \) is a set of points obtained from the intersection of the topographic surface with a horizontal plane, thus being a plane curve with equation \( z(x, y) = \text{const} \) (blue lines in Figure 4.1). A slope (or gradient) line (\( \mathcal{S} \) in Figure 4.1) is a curve on the topographic surface for which, at every point, the direction of the tangent vector coincides with the direction of the tangential component of the gravitational force (Florinsky, 2012). The projections of slope lines on a horizontal plane (\( \mathcal{L} \) in Figure 4.1) are the lines normally considered in terrain analysis (Gallant and Hutchinson, 2011; Maxwell, 1870; Moretti and Orlandini, 2008). Any contour line is orthogonal to both slope lines and slope line projections on a horizontal plane (Florinsky, 2012). Slope lines are topographic attributes indicating flow lines in overland flows that are purely driven by gravity, as often assumed (Costa-Cabral and Burges, 1994; Tarboton, 1997). If overland flow is also driven by diffusional and inertial effects, however, slope and flow lines do not coincide and a clear distinction between the two concepts is needed (Gallant and Hutchinson, 2011; Orlandini et al., 2014).

The points of the topographic surface where the local slope \( |\nabla z| = \sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2} \) is equal to zero are special points (also called critical points),
and they represent either peaks (local maxima), sinks (local minima), or saddles (mountain passes). Minima, saddles, and maxima are also referred to as Morse critical points of index 0, 1, and 2, respectively (Gilmore, 1993). At critical points neither contour nor slope lines can be defined. Other important points of the watershed in relation to the definition of channels are those points where the local slope is not defined. These are singular points of the surface and are characterized, for example, by the coalescence of slope lines forming a channel.

The unit tangent vector to a slope line in every point determines the vector field

$$v(x, y) = -\frac{\nabla z}{|\nabla z|},$$

which is defined at all points where the topographic surface is differentiable and has nonzero slope. The direction of $v$ gives the direction of maximal decrease in elevation among all possible directions. In fact, it can be shown that the direction of maximal decrease of elevation (i.e., direction of steepest descent) is a slope line direction (i.e., $-\nabla z$) for each nonspecial point of the topographic surface (Florinsky, 2012).

The drainage area $A$ is a nonlocal variable defined as the horizontal projection of the land surface enclosed between two slope lines that originate at a hilltop (either a common hilltop as exemplified in Figure 4.1, or distinct ones as shown in Figure 4.2 and discussed in the following sections) and are bounded at the lower end by a contour segment of length $w$ (Gallant and Hutchinson, 2011). The specific catchment area $a$ is defined (Gallant and Hutchinson, 2011) as the ratio between the drainage area $A$ and the length of the contour segment $w$ in the limit of $w \to 0$,

$$a = \lim_{w \to 0} \frac{A}{w}. $$

The specific drainage area $a$ (also referred to as specific catchment area) has units of length and diverges at all those points of the topographic surface where the contour
lines coalesce (namely, critical and singular points of the surface). However, at these non-regular points, while $a$ can not be determined, it is still possible to define the drainage area $A$ as the integral of all areas draining to the point of interest. This will be discussed in Section 4.3 where theoretical definitions of $A$ at non-regular points are provided.

### 4.2.2 Governing equation for the specific catchment area

An intuitive way to derive an equation for $a$ at regular points is now outlined, in part following Chen et al. (2014). Considering the hypothetical collection of water due to a uniform and steady rainfall $r$ over a topographic surface, the continuity equation for this water flow reads

$$\frac{\partial h}{\partial t} + \nabla \cdot (hu) = r,$$

where $t$ is time, $h$ is the water depth and $u$ is the flow velocity. Note that the use of the water depth $h$ is here introduced as a conceptual analogy to guide intuition, but the actual flow of water is not being modeled.

At steady state conditions ($\frac{\partial h}{\partial t} = 0$) and assuming that the flow is fed by a constant unitary rainfall rate falling vertically on the topographic surface, the continuity equation reduces to $\nabla \cdot (hu) = 1$. If water is further assumed to flow at constant unitary speed in the opposite direction of the landscape gradient, the velocity $u$ is defined by Equation (4.1) (i.e., $u = v$), which can be shown to be the expression of the tangent vector to a flow line (Florinsky, 2012). In these conditions it is rather intuitive that the water height is simply the specific catchment area, $h = a$, and the continuity equation becomes an equation for the specific drainage area,

$$\nabla \cdot (av) = 1.$$  \hspace{1cm} (4.4)

This equation corresponds to Equation (2) in Chen et al. (2014) (although in that
paper the expression mistakenly uses $A$ instead of $a$; an equation equivalent to (4.4) was stated without derivation by Hutchinson et al. (2013)).

It is interesting to transform the above equation into a local system of coordinates, defined by the slope lines and contour lines, allowing to make contact with the work of Gallant and Hutchinson (2011). Accordingly, we begin by writing Equation (4.4) as

\[(\nabla a) \cdot \mathbf{v} + a(\nabla \cdot \mathbf{v}) = 1, \]  

(4.5)

where the term $\nabla \cdot \mathbf{v}$ is the divergence of the unit normal vector to a contour line. As such it can be shown to represent the curvature of a contour line, which is called plan (or contour) curvature $k_c$ (this can be shown by simply computing the divergence of the vector $\mathbf{v}$ defined by Equation (4.1)(Shary, 1995; Florinsky, 2012)). The curvature of the contour line is a quantitative measure of flow convergence and divergence over the topographic surface. Upon inserting $\nabla \cdot \mathbf{v} = k_c$, Equation (4.5) can be written as

\[(\nabla a) \cdot \mathbf{v} + ak_c = 1. \]  

(4.6)

The term $(\nabla a) \cdot \mathbf{v}$ describes the variation of $a$ along the projected slope line $l$ (whose direction is given by the vector $\mathbf{v}$), so that $(\nabla a) \cdot \mathbf{v} = \partial a / \partial l$. Thus, Equation (4.6) reduces to

\[\frac{\partial a}{\partial l} = 1 - k_c a, \]  

(4.7)

which is the same expression derived by Gallant and Hutchinson (2011). Equation (4.7) allows us to compute the specific catchment area $a$ at any regular point of the topographic surface by construction of slope lines and integration of Equation (4.7) along the flow line from the hilltop (i.e., at $l = 0$) to the point of interest, imposing $a = 0$ at $l = 0$.

Equation (4.7) is a first order inhomogeneous differential equation, which has a
solution of the form (Bender and Orszag, 1999)

\[ a(l) = \frac{C_1}{I(l)} + \frac{1}{I(l)} \int^l I(l')dl' \]  

(4.8)

where \( I(l) = \exp \int^l k_c(l')dl' \) is an integrating factor and \( C_1 \) is an integration constant, whose value is obtained imposing \( a = 0 \) at \( l = 0 \) (i.e., at the hilltop).

In Cartesian coordinates, the plan (or contour) curvature \( k_c \) is given by (Florinsky, 2012)

\[ k_c = -\frac{\partial_{xx}z(\partial_yz)^2 + \partial_{yy}z(\partial_xz)^2 - 2\partial_{xy}z\partial_xz\partial_yz}{|\nabla z|^3} \]  

(4.9)

where \( \partial_{xx}z = \partial^2z/\partial x^2 \), \( \partial_{yy}z = \partial^2z/\partial y^2 \), \( \partial_{xy}z = \partial^2z/\partial x\partial y \). The plan curvature is negative for valleys, where flow converges, and positive for ridges, where flow diverges. Note that the plan curvature tends to infinity near the top of a hill or bottom of a pit (i.e., at critical points), so that at these points Equation (4.7) is not defined. The plan curvature must be distinguished from the profile (or vertical) curvature, which is the curvature of the surface as we move in the gradient direction (i.e., along a slope line) (Shary, 1995; Florinsky, 2012). The profile curvature of the surface does not play any role in the definition of the specific catchment area \( a \) (see Equation (4.7)). However, when the “real” drainage area (i.e., the actual area of the 3D surface, not projected on the horizontal) is needed, the profile curvature would actually matter as it would be necessary to know how the surface curves in the three dimensional space (as encoded in the second fundamental form (do Carmo, 1976; Frankel, 2012)).

4.3 Drainage area at non-regular points

Equation (4.7) can not be used to define the specific drainage area \( a \) at critical and singular points, where the plan curvature is not defined resulting in \( a \) being
Figure 4.2: Elevation contour plot (white = high, dark gray = low) of a gradient system obtained by translating and scaling Gaussian distributions; black dashed lines are contour lines. Its dynamical systems representation is also shown: symbols represent surface maxima, minima and saddles. Red and blue lines are the stable and unstable manifolds, respectively, and were computed from integration of the vector field $\pm \nabla z$ (+ sign for the stable manifold, $-$ sign for the unstable manifold), starting from small perturbations around each saddle point in the principal directions. Yellow lines in panel (a) are slope lines. In panel (a) Gauss’ theorem is applied to the surface $A$ enclosed between the ridge line (i.e., the stable manifold) and the contour line $W$ (black solid line), with $n$ being the unit normal vector (positive outward) to the surface boundaries: the area of the surface $A$ is the drainage area pertaining to $W$. In panel (b) the green area represents the drainage area $A$ for the minimum enclosed by the red ridge lines (i.e., the basin of attraction of the minimum).

a singular function. However, as already noticed, at non-regular points it is still possible to define the drainage area $A$ as the integral of all areas draining to the point of interest. Theoretical definitions of $A$ at both critical and singular points are now introduced, based on the application of Gauss’ theorem, as well as building on the work of Cayley (1859) and Maxwell (1870) for the delineation of ridge and valley lines in the watershed.
4.3.1 Critical points

At critical points two methods can be outlined to define the drainage area $A$. In the first case, Gauss' theorem is employed to define $A$ as the flux of $a$ across a closed contour line. Alternatively, ridge and valley lines are constructed (Cayley, 1859; Maxwell, 1870) and used to define $A$ as the basin of attraction of each surface minimum.

Following the first method, a definition of $A$ pertaining to any closed curve $C$ can be obtained applying Gauss' theorem to the surface $A_C$ enclosed by $C$. Introducing a vector field $a = a v$, the application of Gauss' theorem results in

$$\iint_{A_C} (\nabla \cdot a) dA_C = \oint_C (a \cdot n) dC,$$

$n$ being the outward-pointing unit normal to the curve $C$ (see Figure 4.2a). Since $\nabla \cdot a = 1$ (Eq. (4.5)), the drainage area $A_C$ pertinent to a closed curve $C$ is given by the integral of all $a$ along $C$,

$$A_C = \oint_C (a \cdot n) dC. \tag{4.11}$$

In particular, Eq. (4.11) can be applied to the specific case of a closed contour line $W$ enclosing a minimum located at $x_{min}$ (see Figure 4.2a), resulting in $A_W = \int_W a(W') dW'$ (note that here $a \cdot n = a$, due to the orthogonality between contour and projected slope lines, along which $a$ is defined). The limit of this integral where the area enclosed by $W$ shrinks to the point $x_{min}$ provides the drainage area $A$ at the minimum, assuming that the surface is simply connected in the neighborhood of the minimum. This definition provides a formal way to compute $A$ at critical points through an integral of the specific drainage areas $a$ associated with each slope line that runs into the minimum considered. The computation of $a$ requires the knowledge
of the location of surface maxima and minima and the integration of Equation (4.7) along the projected slope lines from each maximum to the minimum of interest.

The specific drainage area \( a \) can be integrated along \( w \) to compute the drainage area \( A \) pertaining to any contour line bounded between \( w_1 \) and \( w_2 \) (equivalently to Eq. (2) in Gallant and Hutchinson (2011)), resulting in

\[
A(w_1, w_2) = \int_{w_1}^{w_2} a(w)dw. \tag{4.12}
\]

An alternative and more intuitive definition of \( A \) at surface minima can be delineated by partitioning the surface in basins of attraction of each minimum. In fact, the vector field described by the opposite of the elevation gradient (i.e., \(-\nabla z\)) defines a gradient system and, as such, can be analyzed from a dynamical systems perspective. To this purpose, one should first note that, for a gradient system, critical points have real eigenvalues (no spirals or centers) and closed orbits are ruled out (Strogatz, 2014), so that critical points can only be saddles, stable or unstable fixed points of the surface. On a topographic surface, where the water flow is defined by the opposite of the gradient field (i.e., \(-\nabla z\)), stable fixed points represent local elevation minima, unstable fixed points are local maxima, and saddle points represent, for example, mountain passes. Once the critical points are identified, their nature is determined by the Jacobian matrix of the vector field \(-\nabla z\) (Strogatz, 2014).

To compute \( A \) at surface minima, it is then possible to partition the phase space given by the position vector \( \mathbf{x} \) and the vector field \(-\nabla z\) in different regions, representing the basins of attraction of each minimum. Assuming to evaluate the solution of Eq. (4.3) with \( \mathbf{u} = \mathbf{v} \), the basin of attraction of a surface minimum would be the set of points attracted by the minimum for \( t \to \infty \) (Strogatz, 2014). Given a stable fixed point (i.e., a local elevation minimum), it is possible to define its basin of attraction, whose projection on the horizontal surface corresponds to the drainage area \( A \) pertaining to that minimum of the surface. The boundary of the basin of
attraction can be found in terms of separatrices, which are the stable manifolds of the saddle points and represent ridge lines of the topographic surface (Steger, 1999; Gilmore, 1993; Strogatz, 2014). Thus, the entire topography can be partitioned by connecting ridge lines (found as stable manifolds of the saddle points) and each of these regions will encompass a minimum of the topographic surface: the drainage area pertaining to each minimum is the horizontally projected area of the basin of attraction of each minimum.

An example of this second definition based on partitioning the topography in basins of attraction of local minima is shown in Figure 4.2b, where a contour plot of a gradient system is depicted together with its fixed points. Local elevation maxima and minima are connected to saddle points through the stable and unstable manifolds, respectively. The basin of attraction of each minimum, projected on the horizontal surface, gives the total drainage area $A$ pertaining to that minimum (green shaded area in Figure 4.2b).

4.3.2 Singular points

We now focus on surfaces with singularities (e.g., folds), which are loci of points of the topography where the gradient is not defined. While only one slope line passes by each regular point of the surface (Figure 4.3a and 4.3c), at singular points slope lines from different hillslopes merge together forming a cusp (i.e., channel, see Figure 4.3b and 4.3d): this results in $a$ being a singular function, compared to $A$ (see Figure 4.3). Furthermore, singular points are not necessarily minima of the topographic surface so that neither the dynamical systems concept of the basin of attraction of surface minima nor Gauss’ theorem is applicable.

The behavior of $A$ and $a$ around a singularity is depicted in Figure 4.3d and compared to the case of regular points (Figure 4.3c). Here, the drainage area $A$ associated with a segment of contour line $w$ is intentionally assumed to be the same
Figure 4.3: Conceptual representation of (a) regular points of the surface (for each point of the surface passes only one slope line) and (b) singular points where slope lines converge without the presence of any local minimum or saddle, but due to singularities of the topographic surface (cusp). The horizontal projection of the surface is also shown (c-d) and the values of $a$ and $A$ along a contour line of length $w_2 - w_1$ are plotted: note that while $a$ is a smooth function for case (a-c), it becomes singular at the fold in case (b-d). In the case of the cusp $A$ is a step function, as a finite area is instantaneously added at the singularity (green shaded area in panel (d)).

for the case with and without the singularity to highlight the different behavior in the two cases. For regular points $a$ is a smooth function along the segment of contour line, resulting in a gradual increase of $A$, computed as the integral of $a$ along $w$ (Eq. (4.12), Figure 4.3c). On the other hand, because of the singularity $A$ results in a step function, due to the finite area that is instantaneously added (i.e., the green shaded area in Figure 4.3d). Thus, the specific drainage area $a$ at the singularity is proportional to a Dirac delta function (Figure 4.3d). This is evident from Eq. (4.2) where, in the limit of $w \to 0$, $A$ remains finite, so that the ratio $A/w$ goes to infinity.
4.4 Special cases

A number of explanatory cases are presented in this section to discuss specific features of the definition of the drainage area at regular, critical and singular points of topographic surfaces. A simple 1D case is illustrated to show how the specific area is defined as projected on the horizontal plane and highlight the behavior of the drainage area at critical points. The case of the planar slope and the convergent/divergent cone already presented in (Gallant and Hutchinson, 2011) are then reviewed. These elementary examples, for which analytical solutions are available, provide useful insights into the role of the plan curvature in the computation of $a$. In addition, results for a paraboloid underline the importance of the plan curvature as a mechanism of flow convergence/divergence in the calculation of the drainage area, while demonstrating that profile curvature does not play any role. The case of a 2D sinusoidal surface, characterized by both convergent and divergent areas, is then analyzed, along with the case of the superposition of Gaussian functions. These surfaces allow us to analyze the behaviour of more complex topographies, characterized by the presence of ridges, valleys, and multiple critical points. We conclude this section with the case of a folded surface with singularities.

4.4.1 One-dimensional case

A 1D case is first analyzed as a simple illustration of the fact that the drainage area is defined as projected on the horizontal plane and to highlight the singular character of the specific drainage area at local minima. In 1D, $A$ coincides with the specific drainage area $a$. In any one-dimensional case the drainage area is simply the distance (projected on the horizontal axis) of each point $x$ from the closest peak, given that there is no change in gradient sign going from the point $x$ to the peak. In fact, in this case the topographic surface is only a function of $x$ and there is no plan curvature
to be defined. Thus, Equation (4.7) simply reduces to $da = dl$, so that the specific drainage area $a$ is equal to the length of the projected slope line $l$ starting from a maximum and going to a minimum. Note that the slope line is defined projected on the horizontal surface, so that $a$ is the projected arclength going from each maximum to the minimum.

As an example, consider the elevation field given by the superposition of two sinusoidal waves (Figure 4.4a). The corresponding specific drainage area $a$ is shown in Figure 4.4b, where it is evident that $a$ is a singular function at local minima of the surface where flows from different hillslopes converge and are summed.
4.4.2 Planar surface

A first simple, yet explanatory, example is given by the planar surface \( z = x + y \). In this case, the plan curvature is equal to zero, thus the specific catchment area at a point of the surface is equal to the length of the streamline from the ridge to the point (imposing \( a = 0 \) at \( l = 0 \) as boundary condition),

\[
da = dl \rightarrow a = l.
\] (4.13)

Thus, when the surface has zero plan curvature the specific drainage area \( a \) at a point coincides with the length of the projected slope line \( l \) going from the point to the hilltop. For the planar surface, the catchment area \( A \) draining to a segment of contour line with length \( w \) is simply equal to a rectangle of area \( aw \).

4.4.3 Cone

The second example considers the divergent conic surface \( z = -\sqrt{x^2 + y^2} \) (Figure 4.5a). The vector field (4.1) is

\[
\mathbf{v}(x, y) = \frac{xi + yj}{\sqrt{x^2 + y^2}},
\] (4.14)

where \( i \) and \( j \) are unit vectors in the coordinate directions. The vector field (4.14) provides the following equation for the streamlines

\[
\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{y}{x} \rightarrow y = Cx,
\] (4.15)

where the integration constant \( C \) can be found by imposing the passage through a generic point \( (x_0, y_0) \), so that the equation for the streamlines becomes

\[
y = \frac{y_0}{x_0} x.
\] (4.16)
Figure 4.5: Convergent and divergent cones (a,b) and paraboloids (c,d). Blue lines are contour lines, red lines are slope lines. The green shaded area $A$ is the drainage area pertaining to the contour segment of length $w$. The specific drainage area $a$ increases linearly from the center outward in the case of a divergent cone/paraboloid (panel (e), Eq. (4.20)), while for the convergent cone/paraboloid $a$ grows hyperbolically as $r \to 0$ (panel (f), Eq. (4.24)).
The arclength $l$ from the hilltop $(0, 0)$ of the projected streamline is then equal to the radius $r$

$$l = \int_0^{x_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^{x_0} \sqrt{1 + \left(\frac{y_0}{x_0}\right)^2} \, dx = \sqrt{x_0^2 + y_0^2} = r. \quad (4.17)$$

The plan curvature (Equation (4.9)) for the divergent conic surface is positive and equal to $k_c = 1/r = 1/l$, so that the specific catchment area $a$ reads

$$\frac{da}{dl} = 1 - k_c a = 1 - \frac{a}{l} \quad (4.18)$$

which, upon integration, gives

$$a = \frac{l}{2} + \frac{C}{l}, \quad (4.19)$$

as derived by Gallant and Hutchinson (2011). To find the integration constant $C$ we define $a$ as the ratio between the area of the sector and its arclength

$$a = \frac{\theta r^2/2}{\theta r} = \frac{r}{2} = \frac{r}{2} + \frac{C}{r} \rightarrow C = 0 \rightarrow a(r) = r/2. \quad (4.20)$$

Analogously, the integration constant could be found imposing $a = 0$ at $l = 0$.

In the case of a convergent conic surface, $z = \sqrt{x^2 + y^2}$ (Figure 4.5b), the vector field

$$\mathbf{v}(x, y) = \frac{-xi + yj}{\sqrt{x^2 + y^2}} \quad (4.21)$$

provides the following equation for the streamlines

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{y}{x} \rightarrow y = Cx \rightarrow y = \frac{y_0}{x_0} x. \quad (4.22)$$

The plan curvature for the convergent conic surface is negative and equal to $k_c = -1/r = -1/(R-l)$, where $l$ is again defined from uphill to downhill, so that $r = R-l$, $R$ being the distance of the uphill point to the center of the cone. The specific
catchment area $a$ computed according to Equation (4.6) is given by $da/dl = 1 - k_c a = 1 + a/(R - l)$, which, upon integration, results in the same expression obtained by Gallant and Hutchinson (2011)

$$a = \frac{-l^2/2 + lR}{R - l} + \frac{C}{R - l}. \quad (4.23)$$

Imposing that $a = 0$ at $l = 0$, $C = 0$, and the expression for $a$ in the case of a convergent cone is finally

$$a = \frac{R^2 - r^2}{2r}, \quad (4.24)$$

which is obviously what we would obtain simply dividing the area of the sector by its arclength. Thus, while in the case of a divergent cone $a$ increases linearly from the center outward (Eq. (4.20), Figure 4.5e), for the convergent cone the $r/2$ term is modified by a hyperbolic growth term as the center of the cone is approached, resulting in a highly nonlinear increase of $a$ towards the surface minimum (Eq. (4.24), Figure 4.5f).

At the critical point $(0, 0)$ (surface minimum) the specific drainage area $a$ is not defined, as $k_c \rightarrow -\infty$. The singular behavior of $a$ at the minimum is also evident from the definition of $a$ provided by Eq. (4.2), where the denominator $w$ goes to zero (as the contour line at the minimum would reduce to a point). However, it is possible to compute the drainage area $A$ at $(0, 0)$ applying Gauss’ theorem (Eq. (4.11)). Considering a contour line $W$ at a distance $r$ from the minimum (here contour lines are simply circles of radius $r$), the specific drainage area at each point along the contour line is equal to a constant value (Eq. (4.24)) and the length of the contour line is $2\pi r$, resulting in

$$A = \lim_{r \rightarrow 0} \frac{R^2 - r^2}{2r} - 2\pi r = \pi R^2, \quad (4.25)$$

the projected area of the entire cone of radius $R$, as expected.
4.4.4 Paraboloid

It is instructive to compare the previous example with the case of divergent/convergent paraboloid, to show that only the convergence/divergence of slope lines (described by \( k_c \)) plays a role in the definition of \( a \), while the curvature of the slope lines (called profile curvature) does not affect the value of \( a \) (as we would expect from Eq. (4.7), which is defined in terms projected slope lines).

For the divergent paraboloid described by the equation \( z = -x^2 - y^2 \) (Figure 4.5c), the vector field is the same as for the divergent cone (Eq. (4.14)) and again the projected slope line length \( l \) is equal to \( r \). Furthermore, the plan curvature results in \( k_c = 1/r \), so that the final equation for the specific catchment area \( a \) is the same as the one for the divergent cone, Eq. (4.20). Analogously, the case of the convergent paraboloid \( z = x^2 + y^2 \) (Figure 4.5d) results in the same equation of the convergent cone, Eq. (4.24). The equivalent behaviour of the cone and the paraboloid is also evident from inspection of Figure 4.5 again showing that, since the drainage area is defined projected on the horizontal surface, the only surface curvature that plays a role in its definition is the plan curvature. The profile (or vertical) curvature of the surface (i.e., the curvature of the surface as we move in the gradient direction along the slope line (Shary, 1995; Florinsky, 2012)) does not play any role in the definition of the specific catchment area \( a \). It becomes important only in those applications where the “actual” drainage area (i.e., not projected on the horizontal) needs to be defined.

4.4.5 Sinusoidal surface

The 2D sinusoidal surface (Figure 4.6),

\[
z(x, y) = \sin(x - \pi/2) + \sin(y - \pi/2)
\]  

(4.26)
Figure 4.6: (a) 3D representation of the sinusoidal surface given by Equation (4.26): symbols represent critical points ($\triangle = \text{maxima}$, $\bigtriangledown = \text{minima}$, $\bigcirc = \text{saddles}$), while red and blue lines are the stable and unstable manifolds, respectively. (b) Contour plot of the elevation field given by Equation (4.26): black dashed lines are contour lines, while the green shaded area is the total area draining to the local minimum located in $(0, 0)$. (c) Vector field given by Equation (4.27) and slope lines computed according to Equation (4.29) for different values of the integration constant $C$. 
is characterized by both convergent and divergent areas, as well as multiple critical points (surface minima, maxima, and saddles). The vector field $\mathbf{v}$ is

$$\mathbf{v}(x, y) = -\frac{(\sin x)\mathbf{i} + (\sin y)\mathbf{j}}{\sqrt{\sin^2 x + \sin^2 y}}. \quad (4.27)$$

The streamlines can be derived as

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{\sin y}{\sin x}, \quad (4.28)$$

and integrated to give

$$y = 2 \arccot \left( e^{-C} \cot \left( \frac{x}{2} \right) \right). \quad (4.29)$$

By imposing the passage through a generic point $(x_0, y_0)$, $C(x_0, y_0) = -\ln \left( \cot \left( \frac{x_0}{2} \right) \tan \left( \frac{x_0}{2} \right) \right)$. Combining this expression for $C$ with Equation (4.29) provides an equation for the streamline passing for each point, $y = 2 \arccot \left( \cot \left( \frac{x_0}{2} \right) \tan \left( \frac{x_0}{2} \right) \cot \left( \frac{y_0}{2} \right) \right)$. The behaviour of the streamlines and the values of the integration constant for a subset of the topographic surface (namely, between $0 < x < \pi$ and $0 < y < \pi$) are shown in Figure 4.6c. The specific catchment area $a$ computed by integration of Equation (4.7) along the slope line going from the hilltop in $(\pi, \pi)$ to each point of the subset between $0 < x < \pi$ and $0 < y < \pi$ is shown in Figure 4.7d.

The specific catchment area can be analytically derived along the central slope line passing by $(\pi/2, \pi/2)$, and having equation $y = x$. On this slope line,

$$l = \sqrt{(\pi - x)^2 + (\pi - y)^2} = \sqrt{2(\pi - x)^2} = \sqrt{2}(\pi - x), \quad (4.30)$$

while the plan curvature is

$$k_c(x) = -\frac{\cot x}{\sqrt{2}} \rightarrow k_c(l) = -\frac{\cot(\pi - l/\sqrt{2})}{\sqrt{2}}. \quad (4.31)$$
Using (4.31) in (4.7),

\[
\frac{da}{dl} = 1 + \frac{a \cot(\pi - l/\sqrt{2})}{\sqrt{2}}
\]  

(4.32)

which, upon integration, gives

\[
a(l) = -\sqrt{2} \cot \left( \frac{l}{\sqrt{2}} \right) + C \csc \left( \frac{l}{\sqrt{2}} \right).
\]

(4.33)

To find the integration constant \(C\) we impose that \(a \to 0\) as \(l \to 0\), which provides \(C = \sqrt{2}\). Thus, the specific drainage area along the central streamline reduces to

\[
a(l) = \sqrt{2} \tan \left( \frac{l}{2\sqrt{2}} \right).
\]

(4.34)

The behavior of \(k_c\) and \(a\) for the central slope line is shown in Figure 4.7b and 4.7c, showing a slow increase of \(a\) in the first divergent part \((l < \sqrt{2}\pi/2, k_c > 0)\), while it rapidly grows as slope lines converge \((k_c < 0)\) to the local minimum.

The critical points of the surface described by Equation (4.26) are given by \(x_c = n\pi, y_c = n\pi\), with \(n \in \mathbb{Z}\) (i.e., imposing \(-\nabla z = 0\)), and their nature is defined by the Jacobian matrix of the vector field \(-\nabla z\) (see symbols in Figure 4.6). Stable and unstable manifolds for each saddle point can be further computed (red and blue lines in Figure 4.6) and provide ridge and valley lines, respectively. The basin of attraction
Figure 4.8: 3D surface plots (a-c) and vector field (d-f) for the superposition of Gaussian surfaces (Eq. (4.35)). For the three surfaces parameters were set equal to: (a,d) $A = 1, B = C = -1$; (b,e) $A = 1, B = -1, C = 0$; (c,f) $A = 1, B = 0, C = -1$. In panels (d-f) the colorbar represents the surface elevation (blue = low, yellow = high), while symbols are critical points of the surface ($\Delta =$ maxima $\nabla =$ minima, $\bigcirc =$ saddles). Red lines in panels (d,f) represent the stable manifolds (separatrices) connecting maximum and saddle and delineating the area draining to the minimum close to $x = y = -1$.

For each minimum is then defined by connecting maxima and saddles through the stable manifolds of the saddle points. The projection on the horizontal plane of the basin of attraction is the drainage area pertinent to each minimum (as described in Section 4.3, see green area in Figure 4.6b).
4.4.6 Superposition of Gaussian surfaces

A more complex surface where analytical solutions are not available is provided by a superposition of three Gaussian functions,

\[ z = Ae^{-\frac{(x^2+y^2)}{2}} + Be^{-\frac{(x+b)^2 + (y+c)^2}{2d^2}} + Ce^{-\frac{(x+c)^2 + (y+c)^2}{2}}, \]

where \( b = 1, c = 3, \) and \( d = 0.4. \) In particular, the three cases depicted in Figure 4.8 are analyzed: one surface obtained by superimposing three Gaussian functions \((A = 1, B = C = -1, \) Figure 4.8a), and two surfaces given by the sum of only two Gaussian functions \((C = 0 \) and \( B = 0, \) respectively, Figure 4.8b-c). The vector field obtained by means of Equation (4.1) is plotted in Figure 4.8 for each case.

For these Gaussian surfaces the location and nature (i.e., minimum, maximum, or saddle) of critical points can not be determined analytically and are found according to the procedure described in Appendix D, based on fitting a second order polynomial at every surface point. Once saddle points are identified, it is possible to plot the stable manifolds connecting saddles to the surface maxima, thus partitioning the surface in basins of attraction for each minimum. In the first case analyzed (Figures 4.8a,d) the separatrices divide the entire domain in two basins of attraction, defining the area draining to the two minima. For the second case (Figure 4.8b,e) the only critical points are a minimum and a maximum (no saddles are identified), so that the entire domain is the drainage area for the minimum. In the third case (Figure 4.8c,f), only the minimum has a well defined basin of attraction, while the remaining domain flattens in the limit \( x, y \to \infty. \)

4.4.7 Fold

This last example deals with singular points of the topography, where slope lines coalesce. A simple idealization of a topographic surface with singularities is given by
the following function (depicted in Figure 4.9),

$$z(x, y) = \begin{cases} 
\sqrt{x^2 + x^2 + y} & \text{if } y \leq 0 \\
\sqrt{x^2 + y^2 + x^2 + y^2 + y} & \text{if } y > 0,
\end{cases}$$

(4.36)
the gradient of which is

\[
\nabla z(x, y) = \begin{cases} 
  \left( 2x + \frac{x}{\sqrt{x^2+y^2}} \right) i + j & \text{if } y \leq 0 \\
  \left( 2x + \frac{x}{\sqrt{x^2+y^2}} \right) i + \left( 1 + 2y + \frac{y}{\sqrt{x^2+y^2}} \right) j & \text{if } y > 0.
\end{cases}
\] (4.37)

The surface defined by Eq. (4.36) is singular at \( x = 0 \) for \( y \leq 0 \). In fact, here the gradient (Eq. (4.37)) is not defined and it changes sign passing through \( x = 0 \), where the surface forms a cusp with a behavior analogous to the one exemplified in Figure 4.3b. At these points the specific catchment area \( a \) is a singular function \((k_c \rightarrow -\infty \) in Eq. (4.7)), while \( A \) can still be determined as discussed in Section 4.3.2 (see also Figure 4.3). An example is provided in Figure 4.9 where \( a \) and \( A \) are computed along a contour line \( w \) comprised between the points \( P_1 \) and \( P_2 \) (blue line in Figure 4.9b). It is here assumed that the domain is bounded between \( x = \pm 2 \) and \( y = 2 \) (i.e., ridges location). Along \( w \) the surface is singular in \( P \). The specific drainage area \( a \) is computed by integration of Eq. (4.7) along the projected slope lines going from the ridge to each point of the contour segment (Figure 4.9c), while \( A \) is evaluated as the integral of \( a \) along \( w \) (Eq. (4.12)). At the singularity, the drainage area \( A \) is a step function (similarly to the conceptualization of Figure 4.3d), as the whole area above the point \( P \) is instantaneously added (green shaded area in Figure 4.9b). The specific drainage area increases from \( P_1 \) to \( P \) as a result of both the increased length of the projected slope line and the more convergent terrain (more negative \( k_c \) values, as shown in Figure 4.9c). At \( P \), \( a \) results proportional to a Dirac delta function. From \( P \) to \( P_2 \), \( a \) mirrors the behaviour between \( P_1 \) and \( P \) as a result of geometric symmetry around \( x = 0 \).

### 4.5 Application to a real topographic surface

The analysis of real topographies requires moving from a continuous to a discrete representation of the surface, imposed by the finite resolution of a DEM. Topographic
data are discrete approximations of terrain surfaces (typical elevation models are regular grid DEMs, triangulated irregular networks, or contour-based DEMs) and the computation of the drainage area for these discrete surfaces requires the development of appropriate numerical models. It is beyond the scope here to even attempt a discussion of a rigorous translation of the mathematical concepts above to numerical algorithms, although some peculiar differences between discrete and continuous surfaces are discussed while presenting the results of a test case. The latter considers a portion (approximately 700 m by 700 m) of the Calhoun Critical Zone Observatory (CZO) located in South Carolina (Figure 4.10a), where a 1 m resolution DEM is available (National Center for Airborne Laser Mapping (NCALM), 2016).

As a first step to make the translation from the previous theory to the discrete case, a moving average of size 20 m by 20 m was applied to smooth the topographic surface (Figure 4.10b) in order to limit the number of critical points but still provide an informative test case. The computation of the specific catchment area \( a \) by means of Eq. (4.7) is feasible on the entire domain, excluding critical points of the surface where \( k_c \to \pm \infty \). This requires the construction of slope lines passing through each
DEM grid cell, with computational costs much higher than the commonly used flow routing algorithms (Gallant and Hutchinson, 2011). An example of this calculation is provided for a subregion of the DEM relative to a contour line of length \( w \) (see Figure 4.11a for location of the subregion and Figure 4.12 for results). After computing the slope line going from each point of the contour line to the local surface maximum, Eq. (4.7) is integrated along each projected slope line, from the local maximum to the contour segment \( w \) and imposing \( a = 0 \) at the hilltop. The values of the plan curvature \( k_c \) and the specific drainage area \( a \) along these slope lines are shown in Figures 4.12a and 4.12b. Near the hilltop \( k_c \) is positive (i.e., divergent topography) and the specific drainage area along the slope line slowly increases. Moving downhill, as the plan curvature decreases and becomes negative (i.e., convergent topography), \( a \) rapidly grows (note the higher values of \( a \) in the central slope lines characterized by highly convergent terrain). The behaviour of both \( a \) and \( A \) along the contour line \( w \) is also shown in Figures 4.12c and 4.12d (analogous to the conceptual representation of Figure 4.3). \( A \) is here computed integrating \( a \) along \( w \), according to Eq. (4.12). The central part of the contour line is characterized by a strong increase of \( a \) due to the convergent terrain configuration, resulting also in a steep increase of \( A \) (Figures 4.12c and 4.12d), not too dissimilar from Figure 4.3c. The value of \( A \) computed by means of Equations (4.12) and (4.7) is also in good agreement with the area of the polygon enclosed by \( w \) and the slope lines (Figure 4.12d), confirming the validity of Eq. (4.7) for the computation of the specific drainage area. Gauss’ theorem was here applied to find \( A \) along a contour line of length \( w \), but the same procedure can be applied to compute the drainage area pertaining to any local minimum. In this case a closed contour encompassing the minimum must be considered, as discussed in Section 4.3.1.

Regarding the computation of \( A \) at critical points, the dynamical systems concepts introduced in Section 4.3.1 can also be applied to define the drainage area
at each minimum as its basin of attraction as well as to find ridge and valley lines based on the stable and unstable manifolds of saddle points. Critical points of the surface are shown in Figure 4.11. Their location and nature were identified according to the procedure delineated in Appendix D. From each saddle the stable and unstable manifolds can be constructed and provide a way to define ridge and valley lines (as defined by Cayley (1859) and Maxwell (1870)) on the entire domain: stable manifolds represent ridge lines connecting saddle points and local maxima, while unstable manifolds are valley lines connecting saddle points to local surface minima (see Figure 4.11). It is important to note, however, that while stable and unstable manifolds can be used to identify some ridge and valley lines, they do not account for those ridges and valleys characterized by coalescence of streamlines without any
associated saddle point (Dawes and Short, 1994) (e.g., the example of the fold). Once the stable manifolds are detected, the entire domain can be partitioned in basins of attraction of each surface minimum, thus allowing the reconstruction of the drainage area pertaining to each local surface minimum (Gilmore, 1993).

When moving from a continuous to a discrete surface, the detection of those valley lines related to singularities of the surface translates into finding a suitable criterion to infer the coalescence of streamlines. With reference to the three cross-sections depicted in Figure 4.11b, the increasing curvature of the topographic surface as slope lines merge together suggests that the discrete counterpart of the fold might be identified, at least in this case, based on local surface curvature. Figure 4.13 shows the behavior of $a$ and $k_c$ along four unstable manifolds. When these slope lines merge together (points A and B in Figure 4.13a) a decrease of $k_c$ (indicating convergence of streamlines) is observed, which translates in a sudden increase of $a$ (see insets in Figures 4.13b and 4.13c). Such an increase is even more evident as the minimum is approached, where more slope lines coalesce into higher order channels.

4.6 Discussion and Conclusions

Despite the large number of numerical algorithms developed for its computation and the manifold of its geomorphological and ecohydrological applications, the specific drainage area has been lacking an analytical definition until recently (Gallant and Hutchinson, 2011). Its differential equation for regular points (Gallant and Hutchinson, 2011) was here re-derived in a simpler and more intuitive way from a steady state continuity equation. The definition of the drainage area was then extended to critical and singular points of the topographic surface.

The theoretical tools used here can be easily extended to discrete surfaces. For singular points the discrete nature of the surface requires an ad hoc definition of singularities. This coalescence of streamlines is related to the detection of channel lines.
Figure 4.12: Computation of $a$ and $A$ along a contour line $w$ (see Figure 4.11a for location of $w$). Panels (a) and (b) show the contour curvature $k_c$ (Eq. (4.9)) and the specific drainage area $a$ (obtained by integration of Eq. (4.7)) along the slope lines going from discrete points of $w$ to the local maximum (red triangle). Black solid lines in panels (c) and (d) show the values of $a$ and $A$ along the contour line $w$ (analogously to the conceptual representation of Figure 4.3d). $A$ is computed by integration of $a$ along the contour line $w$ and is compared to the total area of the polygon enclosed by $w$ and the two external projected slope lines, i.e., the area inside the green lines in Figure 4.11 (gray dashed line, computed using Matlab polyarea function).

in digital topographies and is crucial for hydrological and geomorphological applications (Montgomery and Foufoula-Georgiou, 1993). While in the continuous case such behaviour is easily identifiable by the singularity of the surface (e.g., the example of the cusp in Section 4.4.7), for a discrete surface it depends on the definition of empirical criteria. On the one hand, in geomorphology the automatic detection of channel heads and river networks from discrete DEMs (see, e.g., O’Callaghan and Mark (1984); Montgomery and Dietrich (1988); Tarboton et al. (1991); Montgomery and Foufoula-Georgiou (1993); Rodríguez-Iturbe and Rinaldo (2001); Heine et al. (2004)) is typically based on imposing either a constant or a slope-dependent threshold on the drainage area. Within the computer vision literature, on the other
Figure 4.13: Behavior of (b) $a$ and (c) $k_c$ along the unstable manifolds (a) of saddle points 1-4 (circles). The blue triangle represents a surface minimum. Insets show the sudden variation of $a$ and $k_c$ where the four slope lines merge together (point B). Note that the four streamlines from the minimum to point B are very close to each other but do not overlay completely due to the discrete nature of the surface analyzed, resulting in small differences in the progressive distance from the minimum to point B (in panels b and c).

Hand, the channel delineation has been based on high convergence of streamlines (as theorized by Rothe (1915) for valley lines). In this case, those grid cells that are crossed by a minimum number of streamlines are identified as channel lines (for an application see (López and Serrat, 1996)). The sudden variation of $a$ and $k_c$ where slope lines merge together (Figures 4.13b and 4.13c) suggests that the two methods for automatic detection of channels from DEMs are not too dissimilar. In fact, as it is evident from Figure 4.13c, the coalescence of streamlines translates in an increase of $a$, so that the detection of channel lines in terms of either merging streamlines or critical support areas (used as a surrogate for overland flow responsible for erosion and sediment transport) is intimately connected.

We hope this work may be useful towards a better “knowledge of the first elements
of physical geography” (Maxwell, 1870). The theoretical definitions provided here can be used as a benchmark for the evaluation of current numerical methods for the definition of drainage area, as well as for advancing the theoretical analysis of landscape evolution dynamics and stability and channel formation theory.
Organized ridge and valley patterns in detachment-limited conditions

“Stillicidi casus lapidem cavat”

(the fall of drippings hollows a stone)

Titus Lucretius Carus - De rerum natura, translated by Rouse (1924)

5.1 Introduction

The organized spatial distribution of ridges and valleys is one of the most visually striking features of a landscape. The description of channel formation and river branching as well as the analysis of the underlying mechanisms have long fascinated the geomorphological and hydrological communities, leading to the development of a rich body of work on the statistical, theoretical, and numerical analysis of landscape organization. Early works focused on the study of the coupled dynamics of water and sediment transport to identify stability conditions for incipient valley formation (Smith and Bretherton, 1972; Loewenherz, 1991; Izumi and Parker, 1995). Research also delved into the statistical description of river networks, with the empirical derivation of multiple scaling laws (Rodríguez-Iturbe and Rinaldo, 2001; Dodds
and Rothman, 2000), the definition of stream ordering systems for the river basin characterization (e.g., Horton (1945); Strahler (1952); Shreve (1966)), and the development of channel network models based on optimality principles (Rigon et al., 1993; Rodríguez-Iturbe and Rinaldo, 2001). However, while these analyses have proven useful to understand and describe the spatial organization and governing laws of river networks, they do not provide much insight on the geomorphological processes and feedback mechanisms acting over time in the formation of the observed ridge and valley patterns (Perron et al., 2012).

In the past few decades a large number of numerical landscape evolution models capable of generating variably dissected landscapes have been developed and used for different applications (e.g., Willgoose et al. (1991); Tucker and Slingerland (1994); Coulthard (2001); Perron et al. (2008); Hancock et al. (2010); Chen et al. (2014)). However, only recently, landscape evolution models in detachment-limited (DL) regime have been employed for the analysis of first-order valleys spacing (Perron et al., 2008, 2009) and branching river networks (Perron et al., 2012), whose dynamics have been linked to the main erosional mechanisms acting on the topography. This has been done by introducing a dimensionless parameter (denoted as a Péclet number, in analogy with advection/diffusion processes) describing the relative proportions of diffusive soil creep and runoff erosion. These works provided an important step forward in the study of spatially organized patterns of ridges and valleys, but they require some critical appraisal. In particular, the runoff erosion rate is assumed to be a function of the drainage area $A$, which is used as a proxy for surface runoff. While a proportionality between runoff erosion and drainage area is supported by a number of experimental studies (e.g., Leopold and Miller (1956); Flint (1974)), such an approximation has numerical and theoretical consequences. First, $A$ is usually computed by means of flow-routing algorithms, which are known to be grid size dependent (Schoorl et al., 2000; Pelletier, 2012). This resulted in $ad$
hoc approximations of the landscape evolution equation and the introduction of additional empirical relationships (e.g., Perron et al. (2008); Pelletier (2012)), without a sound theoretical justification. Secondly, the use of $A$ without a formal theoretical link to the water flow neglects “the coupling of the developing landscape with the flow rate, a fundamental step in the understanding of landscape self-organization” (Rodríguez-Iturbe and Rinaldo, 2001).

In this work, the runoff erosion term in DL conditions is re-written as a function of the specific drainage area $a$, which will be shown to be directly linked to a continuity equation (see also Chapter 4). This allows us both to obtain grid-independent solutions without the introduction of additional parameters, and to provide a clear mathematical framework which can be used for the theoretical analysis of channel forming instability in DL regime. Furthermore, numerical simulations are used to assess the resulting ridge and valley patterns as a function of diffusive soil creep, runoff erosion, and uplift rate. Various regimes are identified, and the modifications of the observed patterns and the corresponding hypsometric curves (Langbein, 1947; Strahler, 1952) in relation to different geomorphic transport laws are investigated. While the work here is not intended as an exhaustive analysis of landscape evolution and organization in DL conditions, we hope it will provide a solid ground for the theoretical and numerical analysis of landscape dynamics and self-organization.

5.2 Landscape evolution in detachment limited conditions

The time evolution of the surface elevation $z(x, y, t)$ is described by the continuity equation

$$\frac{\partial z}{\partial t} = U - \nabla \cdot f = U - \nabla \cdot (f_d + f_c) \quad (5.1)$$

where $t$ is time, $U$ is an uplift rate, and $f$ is the total sediment flux, given by the sum of fluxes related to runoff erosion/channelized flow ($f_c$) and mass movement
of sediment/creep/colluvial processes \( (f_d) \). The soil creep flux is assumed to be proportional to the topographic gradient (Culling, 1960, 1963), so that \( f_d = -D \nabla z \), \( D \) being a diffusion coefficient. In DL conditions (Howard, 1994; Perron et al., 2008) it is assumed that all eroded material is transported outside the model domain, so that no sediment redeposition occurs, and the divergence of the runoff erosion flux \( f_c \) is directly proportional to the shear stress, which in turn is a function of the local slope and the water flux. Under these conditions, the runoff erosion flux can be approximated as \( f_c = K_a |\nabla z|^n q^m \), where \( K_a \) is a coefficient, and \( m \) and \( n \) are model parameters. With these approximations for the soil creep and runoff erosion components, Eq. (5.1) reads

\[
\frac{\partial z}{\partial t} = D \nabla^2 z - K_a |\nabla z|^n q^m + U. \tag{5.2}
\]

The soil creep flux results in a diffusion term which tends to smoothen the surface, while the runoff erosion component becomes a sink term which excavates the topography proportionally to the local slope and the water flux. Note that in DL conditions continuity of sediments is not guaranteed, as the runoff erosion term excavates the surface with no redeposition of the eroded material (which is assumed to be instantaneously transported outside the domain). This differs from transport-limited formulations (see Chapter 3 for a description of the governing equations) used in early landscape evolution models (e.g., Willgoose et al. (1991); Tarboton et al. (1992)) and works on landscape stability (Smith and Bretherton, 1972; Loewenherz, 1991; Fowler, 2011), where the runoff erosion flux is modeled as the divergence of the shear stress, thus ensuring continuity of sediments.

The surface water can be modeled according to the continuity equation

\[
\frac{\partial h}{\partial t} = R - \nabla \cdot (qn) \tag{5.3}
\]

where \( h \) is the water height, \( n \) the direction of the flow, and \( R \) the rainfall rate
effectively contributing to runoff production. At steady state conditions, assuming a constant and unitary rainfall rate and water flowing at unitary speed in the direction opposite to the landscape gradient (i.e., $\mathbf{n} = -\nabla z/|\nabla z|$), Eq. (5.3) reduces to an equation for the specific catchment area (see also Chapter 4),

$$
-\nabla \cdot \left( a \frac{\nabla z}{|\nabla z|} \right) = 1.
$$

(5.4)

Thus, in these conditions, the magnitude of the water flux $q$ coincides with that of the water height $h$ (as the flow is assumed to have unitary velocity) and the term $q$ in Eq. (5.2) can be approximated with the specific catchment area, $q \approx a$, so that the resulting landscape evolution equation reads

$$
\frac{\partial z}{\partial t} = D\nabla^2 z - K_a a^m |\nabla z|^n + U.
$$

(5.5)

It is important to observe that in most landscape evolution models (e.g., Perron et al. (2008); Tucker and Hancock (2010); Pelletier (2012); Chen et al. (2014)) the water flux in Eq. (5.2) is approximated with the total drainage area $A$ (i.e., the runoff erosion term reads $K_a A^m |\nabla z|^n$), based on a number of empirical relations between the water flux and the total contributing area (see, e.g., Leopold and Miller (1956); Flint (1974); Rodríguez-Iturbe and Rinaldo (2001); Perron et al. (2008)). The value of $A$ is generally evaluated using flow-routing algorithms (e.g., D8, D∞). This results in $A$ being a grid-dependent variable, so that results form the landscape evolution model are affected by the specific grid resolution. To bypass this issue, the drainage area $A$ is often modified to account for the channel width (see, e.g., Perron et al. (2008); Pelletier (2012)), but this results in ad hoc approximations of Eq. (5.2) and the introduction of additional empirical parameters, with little theoretical justification. The use of $a$ as a proxy for the water flux $q$, instead, allows to avoid any grid-dependence of the resulting topography, while being directly linked to the water flux through a continuity equation.
5.2.1 Nondimensional analysis

To quantify the relative proportion of soil creep, runoff erosion, and uplift in shaping the landscape morphology and network branching, a dimensional analysis of Eq. (5.5) is performed (in part following Perron et al. (2008)). This allows us to obtain a dimensionless parameter describing the bulk properties of the system, which can be used to identify different regimes in the formation of patterns of ridges and valleys over the considered domain. To do this we write the following dimensionless quantities

\[ \hat{t} = \frac{t}{D^{1/2}}, \quad \hat{x} = \frac{x}{l}, \quad \hat{y} = \frac{y}{l}, \]
\[ \hat{z} = \frac{z}{U l^{1-1/2}}, \quad \hat{a} = \frac{a}{l}. \]  

(5.6)

where \( l \) is the horizontal length of the square domain considered here (see simulation setup in Section 5.2.2). Upon inserting these quantities in Eq. (5.2), its nondimensional form becomes

\[ \frac{\partial \hat{z}}{\partial \hat{t}} = \hat{\nabla}^2 \hat{z} - \chi \hat{a}^m |\hat{\nabla} \hat{z}|^n + 1 \]  

(5.7)

where

\[ \chi = \frac{K_a l^{m+n}}{D^n U^{1-n}} \]  

(5.8)

is a dimensionless parameter describing the bulk properties of the system in terms of runoff erosion (encoded in \( K_a \) and \( l \)), soil creep (encoded in the diffusion coefficient \( D \)), and uplift \( U \). The quantity \( \chi \) differs from the Péclet number (Pe) introduced by Perron et al. (2008) in the use of \( l \) (i.e., the size of the square domain) instead of the drainage basin length, and in the use of the uplift rate \( U \) instead of the relief maximum height. The use of the domain size as a horizontal length scale allows us to define \( \chi \) in terms of bulk variables of the system, in contrast with the use of the drainage basin length which makes Pe a local variable (i.e., specific to the drainage
basin considered). Furthermore, defining $\chi$ in terms of the uplift rate $U$ (instead of the main relief height) not only enables the definition of $\chi$ as a function of solely input variables (while both basin length and relief height are outputs of the model), but also allows us to fully explore the role of the uplift rate on the formation of patterns of channels and ridges. In fact, based on Eq. (5.8), we can argue that when the slope exponent $n$ is equal to 1, the relative proportion of runoff erosion and soil creep is independent of the uplift rate. However, if $n > 1$ the uplift acts to increase the runoff erosion component, while for $n < 1$ it enhances the diffusion component of the system. This will likely result in different drainage network patterns, as will be investigated in the following sections.

Note that the dimensional analysis introduced here is based on the assumption of a square domain with size $l = l_x = l_y$, being $l_x$ and $l_y$ the domain lengths in the $x$ and $y$ directions, respectively. The case of rectangular domains with $l_x \neq l_y$ will be discussed in Section 5.3.3.

5.2.2 Simulation setup and modeling scenarios

Numerical simulations were performed on a regular square grid of size $l$, using forward differencing in time and centered difference approximations for the spatial derivatives. The total drainage area $A$ was computed at each grid point with the $D\infty$ algorithm using the Matlab TopoToolbox. The specific catchment area $a$ was then approximated as $A/dx$ (Tarboton, 1997), being $dx$ the grid size.

A first set of simulations was run assuming $n = 1$ and $m = 0.5$ (see model parameters in Table 5.1). Different simulation runs were performed to identify the critical values of $\chi$ at which the system transitions to different valley/ridge patterns. Here simulations were run assuming a constant uplift rate $U$ over the entire domain and imposing a constant downcutting rate $W$ at the four boundaries (analogously to Perron et al. (2012)). In this case, a dynamic steady state is reached when each
point in the topography is lowering at the same rate (note that this is equivalent to fixing the boundary domain and increasing the value of the uplift rate). The value of the downcutting rate $W$ was chosen to be everywhere greater that the runoff erosion rate. We run a number of simulations having a domain size of $l = 100$ m and same downcutting rate $W=500$ m/yr and covering all the different valley/ridge patterns (i.e., regimes). Additionally, simulations were run considering different domain sizes (e.g., 50, 75, 152, and 200 m) to show that the final ridge/valley pattern is embedded solely in the value of $\chi$.

Additionally, to assess the sensitivity of the network pattern to the slope and water flux exponents, simulations were performed changing the values of $n$ and $m$. In particular, four sets of simulations were performed (i.e., increasing/decreasing both $m$ and $n$), and assuming fixed domain boundaries (i.e., down-cutting rate equal to zero). Parameters for these simulation sets are provided in Table 5.2. Note that the range of $\chi$ values explored in each modeling set was chosen to avoid numerical instabilities.

Lastly, simulations were run over rectangular domains to evaluate the effect of boundary conditions as well as domain shape and size on the resulting ridge/valley networks. Simulations were run assuming $m = 0.5$, $n = 1$ (same as the first set of simulations), and $\chi = 200$ over domains of fixed length in the $x$ direction (i.e., $l_x = 100$ m), and varying length $l_y$ in the $y$ direction. Specifically, we ran simulations for $\beta = 0.5, 1, 1.5, 2, 2.5,$ and $5$, where $\beta$ is a shape factor defined as the ratio between the two horizontal length scales $l_y$ and $l_x$, namely $\beta = l_y/l_x$.

5.3 Ridge and valley patterns under detachment limited conditions

5.3.1 Results for $m = 0.5$ and $n = 1$

Simulation results for the scenarios with $m = 0.5$ and $n = 1$ are shown in Figure 5.1. For low values of the dimensionless parameter $\chi$ (i.e., $\lesssim 30$) no channels are
Figure 5.1: Simulation results obtained for $m = 0.5$ and $n = 1$ (parameter values are shown in Table 5.1). (a) Highest Strahler order (red) and number of main channels on each domain side (blue) for different values of the dimensionless parameter $\chi$. Based on the number of channels and the Strahler order nine regimes can be identified (I-IX) with distinctively different ridge/valley patterns (b-j): brown corresponds to ridge lines and green to valleys (to better illustrate the ridge and valley structure we show here the difference between the drainage area $A$ and its value computed over the flipped topography $-z$).

Figure 5.2: (a) Normalized hypsometric curves obtained for $\chi = 20, 125, 180, \text{ and } 340$: when no secondary branching is observed (i.e., $\chi \leq 160$) the hypsometric curve is concave up, while after the first secondary branching is formed it transitions to a shape concave up for higher elevations and concave down at low elevations. Panels b-e show 3d plots steady state topography for the four cases, the color represents surface elevation (red = high, blue = low). Results are shown here for $m = 0.5$ and $n = 1$ (same as Figure 5.1), parameter values for these simulations are given in Table 5.1.
formed and the topography evolves to a smooth surface dominated by the diffusive soil creep (Figure 5.1b). As the runoff erosion coefficient is increased the system progressively develops one, three, and five channels on each side of the square domain for $30 \leq \chi \leq 58$, $58 \leq \chi \leq 97$, and $97 \leq \chi \leq 155$, respectively (Figure 5.1c-e). When $\chi$ is increased above $\approx 155$ the central channels develop secondary branches, with the main central channel becoming of Strahler order two (Figure 5.1f). As $\chi$ is further increased seven channels can be observed originating on each side of the domain, and the main central channel further branches (Figure 5.1g-j). Simulations were run here up to $\chi = 374.5$, with the central channel developing several secondary branches and becoming of Strahler order five. Thus, for $m = 0.5$ and $n = 1$ and within the
Figure 5.4: Ridge/valley patterns (brown = ridge, green = valley) for $\chi = 40$ (a-c), 125 (e-g), and 200 (i-k) and $W = 50$ (a, e, i), 500 (b, f, j), and 5000 (c, g, k) m/yr. The corresponding normalized hypsometric curves are shown in panel d, h, and l for $\chi = 40$, 125, and 200, respectively.

range of $\chi$ values explored here, nine regimes can be identified based on the number of channels forming on each side of the square domain and the maximum Strahler order (Figure 5.1 and Table 5.1).

As already discussed in the previous sections, using the specific drainage area $a$ as a proxy for the water flux ensures the modeled ridge/valley patterns to be robust to the grid size, while theoretically justified in terms of a continuity equation. To assess the independence of such drainage patterns on the specific grid resolution chosen in the simulations, additional numerical tests were performed. In particular, simulations were run over the square domain of size $l = 100$ m for $\chi = 40$, 125, and 200 assuming grid sizes of 0.5 and 2 m. Results are compared to those obtained for $dx = 1$ in Figure 5.3, confirming that the resulting patterns and hypsometric curves
Figure 5.5: Simulation results obtained for different values of the slope and runoff exponents (i.e., $n$ and $m$): maximum Strahler order and number of channels on each domain side as a function of $\chi$ are shown in panel a and b, respectively. 2d representations of ridge and valley patterns are shown in Figure 5.7-5.10.

As the topography transitions from a smooth topography to a progressively more dissected one, the shape of the hypsographic curve varies from concave up to one with a concave down portion for low elevations (see Figure 5.2). In particular, channel formation (with no secondary branching) causes the hypsometric curve to progressively lower as a result of the lower altitudes observed in the topography, while maintaining a concave up profile (see case $\chi = 125$ in Figure 5.2). As secondary branches develop, the hypsometric curve transitions to a concave/convex one, with the convex portion...
Figure 5.6: Normalized hypsometric curves obtained for different values of the slope and runoff exponents (i.e., \( n \) and \( m \)). The black line represents the no branching solution (no channels are formed), the dashed gray lines represents cases in which channels are formed but no secondary branching is observed, while dashed and solid red lines show results with secondary branching. 2d representations of the planar geometric patterns of ridges and valleys for these cases are shown in Figure 5.7-5.10.

At lower altitudes becoming more evident as \( \chi \) is increased (see results for \( \chi = 180 \) and \( \chi = 340 \) in Figure 5.2).

As we shall see from Eq (5.8) when \( n = 1 \) the value of \( \chi \) is independent of the uplift rate (or analogously, the downcutting rate imposed at the boundaries). To validate this statement additional simulations were run for \( \chi = 40, 125, \) and \( 200 \) and considering downcutting rates at the boundaries of 50, 500, and 5000 m/yr. Results (Figure 5.4) confirm that the geometry of ridge and valley lines as well as the normalized hypsometric curves are independent of the value of the uplift rate when \( n = 1 \).
Figure 5.7: Ridge (brown) and valley (green) patterns for simulation scenarios with $m = 0.5$ and $n = 1.3$. Simulation parameters are provided in Table 5.2.

Figure 5.8: Ridge (brown) and valley (green) patterns for simulation scenarios with $m = 0.5$ and $n = 0.7$. Simulation parameters are provided in Table 5.2.
5.3.2 Effect of variations in the exponents $m$ and $n$

To investigate the effect of using different geomorphic transport laws for the runoff erosion term (i.e., changing the slope and water flow exponents), simulations were run modifying $n$ and $m$ by $\pm 0.3$ and $\pm 0.2$, respectively. Results are shown in Figures 5.5-5.10. When the value of $n$ is different from unity, the resulting ridge/valley patterns depend on the uplift rate $U$ (see Eq. (5.8)). When $n$ is increased the system displays channelization and secondary branching for higher values of $\chi$ (i.e., points are shifted to the right in Figure 5.5), with a more dissected planar geometry characterized by narrower valleys (Figure 5.7). A decrease in the value of $n$ makes the system develop smoother geometries, with wider valleys and the first secondary branching developing when only three channels per each side of the domain are present (see Figure 5.8). This results in a hypsometric curve with a more pronounced basal convexity as $n$ is increased above unity, as a consequence of the progressively more dissected topography (see Figure 5.6a,c).
Conversely, an increase/decrease in $m$ results in an opposite behavior of the system. In fact, when $m$ is increased (Figure 5.9) the system develops secondary branching when only three channels are present on each side of the domain, with the formation of less numerous but wider valleys and a reduced basal convexity in the hypsometric curve (Figure 5.6b). On the other end, a decrease in the value of the exponent $m$ results in a more dissected landscape, with narrower valleys and the development of up to nine channels per domain side before the first secondary branching occurs (Figure 5.10). This results in a more pronounced transition of the

Figure 5.10: Ridge (brown) and valley (green) patterns for simulation scenarios with $m = 0.3$ and $n = 1$. Simulation parameters are provided in Table 5.2.
Figure 5.11: Valley networks obtained for rectangular domains of varying dimension. The x dimension is kept constant and equal to 100 m, while the y dimension is changed to have shape factors $\beta = l_y/l_x$ of (a) 0.5, (b) 1, (c) 1.5, (d) 2, (e) 2.5, and (f) 5. All results are obtained for $\chi = 200$ (panel b is the same simulation as Figure 5.1g, parameter values for all the simulations are the same and given in Table 5.1). Color code denotes the drainage area, with green corresponding to valleys and brown to ridgelines.

Hypsometric curve to a concave down shape for low altitudes (see Figure 5.10d).

5.3.3 Ridge/valley patterns over rectangular domains

To assess possible boundary effects on the observed networks and secondary branching, simulations were run over rectangular domains of various dimensions. Results are shown in Figure 5.11 and 5.12. The drainage network shown in Figure 5.11b is the result obtained over a square domain ($\beta = 1$) for $\chi = 200$: in this case the system
is in regime VI (see Figure 5.1) with five channels per each side of the domain and maximum Strahler order equal to two (i.e., one secondary branching). Reducing the shape factor $\beta$ does not provide the system with a drainage area sufficient to actually develop large valleys so that only parallel channels are formed (Figure 5.11a). This is related to the fact that $\chi$ (Eq. (5.8)) should be computed using as length scale $l$ the most constraining size of the domain. In fact, for this simulation the actual value of $\chi$ (i.e., computed with $l_y = 50$ m) would be 70.7, meaning that the system is in regime III (i.e., three channels per side, which is actually observed on the shorter side of the domain, Figure 5.11a).

Increasing $\beta$ above unity (Figure 5.11c-f) provides patterns where the valley on the shorter side of the domain (i.e., $l_x = 100$ m) still develops only one secondary branching (coherently with the expected behavior for $\chi = 200$). However, the number of channels on $l_x$ can be higher than five, as the valley have more freedom to adjust
with the rest of the geometry, thus partially breaking the symmetry of the square case. Conversely, on the longer side of the domain, channels tend to develop more secondary branches (i.e., Strahler order > 2). This is a result of the fact that, while on the square domain the two main ridgelines are necessarily formed along the diagonals thus providing a fixed and equal partition of $A$ on the four sides of the domain, the valley developing on longer sides of rectangular domains have more space to widen and the collection of additional drainage area allows further branching.

The normalized hypsometric curves for rectangular domains of shape factor $\beta = 0.5$, 2, and 5 are compared with that of the square domain (i.e., $\beta = 1$) in Figure 5.12. For $\beta = 0.5$ the hypsometric curve is mainly concave up (as we would expect given that there is no secondary branching), with a slight convexity for very high elevations. As $\beta$ is increased above unity, the progressively more dissected landscape (i.e., with multiple secondary branches) results in a more pronounced basal convexity of the normalized hypsometric curve (see, for example, results for $\beta = 5$).

5.4 Discussion and conclusions

The analysis performed here shows that predictable ridge/valley patterns are formed in DL conditions over a square domain and that different regimes can be distinguished in relation to a dimensionless parameter ($\chi$) accounting for the relative proportions of runoff erosion, soil creep and uplift. The observed patterns were shown to depend on the geomorphic transport law considered, with varying levels of channelization and branching as a function of the slope and water flow exponents, $n$ and $m$. Thus, while the overall behavior of the system is always an increase in channelization and subsequent branching as the runoff erosion component predominates over the diffusive soil creep, the spacing between valleys as well as the number of secondary branches is strongly dependent upon the specific geomorphic transport law considered. Furthermore, when the slope exponent $n \neq 1$, the resulting planar geometry
of the drainage network is highly affected by the uplift rate, as encoded in the dependence of $\chi$ on $U$. This is a key aspect in order to properly disentangle the role of different geomorphological processes in shaping the spatial organization of drainage basins, which was not accounted for in the Péclet number proposed by Perron et al. (2008).

The normalized hypsometric curves of the steady state topographies were shown to display different shapes based on the value of $\chi$, thus reflecting their dependence on the level of channelization and branching. For small $\chi$ values the shape is mainly concave up, while, as the landscape becomes more dissected due to an increase of the runoff erosion component with respect to soil creep, the hypsometric curve displays a progressively more pronounced basal convexity which reflects the degree of branching within the domain (Willgoose and Hancock, 1998). In early works on the study of hypsometric curves these differences in shape were mainly related to the stage of geomorphic development of the basin, with the basal convexity considered to be a peculiar feature of young stages of the landscape, and the concave shape being typical of older landscapes (Strahler, 1952, 1964). However, as noted by Willgoose and Hancock (1998) and shown in the work here, the shape of the hypsometric curve is mainly related to the level of channelization and branching observed in the system and thus to the dominant erosional mechanisms acting on the landscape (i.e., interplay between runoff erosion, soil creep, and uplift). Furthermore, shape and size of the basin have been shown to play a role in the delineation of the hypsometric curve (see results for rectangular experiments). This is related to the fact that a smaller domain will have a higher proportion of its area dominated by diffusion, so that its hypsometric curve will be likely to display a less pronounced basal convexity, as observed by Willgoose and Hancock (1998).

Lastly, the theoretical framework provided here directly links the water flow term in the runoff erosion component to the specific catchment area through a water con-
tuity equation. This is a key point to set the basis for a theoretical analysis of incipient channel formation and secondary branching (e.g., following Smith and Bretherton (1972); Fowler (2011)), which requires accounting for the coupled dynamics of topographic surface evolution and water transport.
Table 5.1: Parameter values for the first set of simulations ($m = 0.5$ and $n = 1$). Bold faced rows show parameter values for the nine ridge and valley patterns displayed in Figure 5.1.

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$K_a$ [1/(m$^m$ yr)]</th>
<th>$D$ [m$^2$/yr]</th>
<th>$U$ [m/yr]</th>
<th>$W$ [m/yr]</th>
<th>$l$ [m]</th>
<th>$dx$</th>
<th>$N_{Strahler}$</th>
<th>$N_{Channels}$</th>
<th>Regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>1.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-4}$</td>
<td>5.10$^{-5}$</td>
<td>0.05</td>
<td>50</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>I</td>
</tr>
<tr>
<td>20.0</td>
<td>1.0 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>I</td>
</tr>
<tr>
<td>28.3</td>
<td>4.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>50</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>I</td>
</tr>
<tr>
<td>32.5</td>
<td>4.6 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>50</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>II</td>
</tr>
<tr>
<td>37.5</td>
<td>1.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>152</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>II</td>
</tr>
<tr>
<td>40.0</td>
<td>2.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>II</td>
</tr>
<tr>
<td>44.2</td>
<td>6.3 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>0.05</td>
<td>50</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>II</td>
</tr>
<tr>
<td>56.6</td>
<td>1.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>200</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>II</td>
</tr>
<tr>
<td>60.0</td>
<td>3.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>62.5</td>
<td>3.1 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>70.0</td>
<td>3.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>80.0</td>
<td>4.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>90.0</td>
<td>4.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>95.0</td>
<td>4.8 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>III</td>
</tr>
<tr>
<td>100.0</td>
<td>5.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>IV</td>
</tr>
<tr>
<td>110.0</td>
<td>5.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>IV</td>
</tr>
<tr>
<td>125.0</td>
<td>6.3 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>IV</td>
</tr>
<tr>
<td>141.4</td>
<td>2.0 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>0.05</td>
<td>50</td>
<td>0.5</td>
<td>1</td>
<td>5</td>
<td>IV</td>
</tr>
<tr>
<td>150.0</td>
<td>7.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>IV</td>
</tr>
<tr>
<td>160.0</td>
<td>8.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>V</td>
</tr>
<tr>
<td>180.0</td>
<td>9.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>V</td>
</tr>
<tr>
<td>190.0</td>
<td>9.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>VI</td>
</tr>
<tr>
<td>200.0</td>
<td>1.0 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>VI</td>
</tr>
<tr>
<td>220.0</td>
<td>1.1 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>VI</td>
</tr>
<tr>
<td>234.2</td>
<td>6.3 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>0.05</td>
<td>152</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>VII</td>
</tr>
<tr>
<td>240.0</td>
<td>1.2 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>VII</td>
</tr>
<tr>
<td>259.8</td>
<td>2.0 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>5</td>
<td>75</td>
<td>0.5</td>
<td>3</td>
<td>7</td>
<td>VII</td>
</tr>
<tr>
<td>272.8</td>
<td>2.1 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>75</td>
<td>0.5</td>
<td>3</td>
<td>7</td>
<td>VII</td>
</tr>
<tr>
<td>282.8</td>
<td>5.0 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>100</td>
<td>200</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>VIII</td>
</tr>
<tr>
<td>311.1</td>
<td>5.5 $\cdot$ 10$^{-4}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>200</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>VIII</td>
</tr>
<tr>
<td>320.0</td>
<td>1.6 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>100</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>VIII</td>
</tr>
<tr>
<td>340.0</td>
<td>1.7 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>1000</td>
<td>100</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>IX</td>
</tr>
<tr>
<td>374.8</td>
<td>1.0 $\cdot$ 10$^{-3}$</td>
<td>5.10$^{-3}$</td>
<td>5.10$^{-5}$</td>
<td>500</td>
<td>152</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>IX</td>
</tr>
</tbody>
</table>
Table 5.2: Parameter values for the simulations sets with varying exponents $m$ and $n$ (results for these simulations are displayed in Figure 5.1-5.10).

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$K_a$ [1/(m$^m$ yr)]</th>
<th>$D$ [m$^2$/yr]</th>
<th>$U$ [m/yr]</th>
<th>$W$ [m/yr]</th>
<th>$l$ [m]</th>
<th>$dx$</th>
<th>$N_{Strahler}$</th>
<th>$N_{Channels}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$n = 1.3$</td>
<td>$m = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>63.2</td>
<td>1.00$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>94.9</td>
<td>1.50$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>189.7</td>
<td>3.00$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>389.6</td>
<td>6.16$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>594.5</td>
<td>9.40$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>632.5</td>
<td>1.50$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>948.7</td>
<td>1.50$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>1264.9</td>
<td>2.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>1897.4</td>
<td>3.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n = 0.7$</td>
<td>$m = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.3</td>
<td>2.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20.5</td>
<td>4.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>30.8</td>
<td>6.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>41.1</td>
<td>8.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>43.7</td>
<td>8.50$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>46.2</td>
<td>9.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>51.4</td>
<td>1.00$\times$10$^{-2}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>56.5</td>
<td>1.10$\times$10$^{-2}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n = 1$</td>
<td>$m = 0.7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>2.00$\times$10$^{-5}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50.2</td>
<td>1.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>100.5</td>
<td>2.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>125.6</td>
<td>2.50$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>150.7</td>
<td>3.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>175.8</td>
<td>3.50$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>185.9</td>
<td>3.70$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>5</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n = 1$</td>
<td>$m = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.9</td>
<td>2.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>47.8</td>
<td>6.00$\times$10$^{-4}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>87.6</td>
<td>1.10$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>119.4</td>
<td>1.50$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>159.2</td>
<td>2.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>199.1</td>
<td>2.50$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>222.9</td>
<td>2.80$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>254.8</td>
<td>3.20$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>278.7</td>
<td>3.50$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>318.5</td>
<td>4.00$\times$10$^{-3}$</td>
<td>5$\times$10$^{-3}$</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>
Channel formation in detachment-limited conditions: a spectral approach to linear stability analysis

“Mountains may be due to two different causes. Either they are the effects of upheavals of the crust of the earth, such as might occur during a violent earthquake, or they are the effect of water, which, cutting itself a new route, has denuded the valleys, the strata being of different kinds, some soft, some hard. [...] It would require a long period of time for all such changes to be accomplished, during which the mountains themselves might be somewhat diminished in size.

Avicenna, from Toulmin and Goodfield (1965)

6.1 Introduction

Landscapes are characterized by many self-organized patterns with the emergence of characteristic valley spacings (see, e.g., Figure 6.1). The definition of the hydrophysical processes driving the development of such peculiar landforms requires an understanding of the mechanisms of channel initiation (Rodríguez-Iturbe and Rinaldo,
2001), which has been the subject of research for more than 40 years. Early works on channel forming instabilities considered the coupled dynamics of water transport and landscape evolution in a transport-limited (TL) regime, in which the erosion rate is related to the divergence of the sediment flux. Among these, the pioneering work by Smith and Bretherton (1972), denoted hereafter as SB72, first discussed the stability of an erodible surface showing that, if the sediment flux is a function of the water flux and local slope, concave up portions of a hillslope are unstable to lateral perturbations, thus developing channels. However, their analysis does not allow a characteristic wavelength selection as it predicts an unbounded increase of the perturbations at high wave numbers (see also Fowler (2011)). The reason for this failure is attributed to the assumption of a normal flow model, in which water flows down the topographic gradient rather than down the gradient of either the energy or water surface (Smith, 2010). Loewenherz (1991) carried out a formal linear stability analysis of the SB72 model using WKB theory and introduced a modification of the sediment transport law represented by a spatially averaged term which, although artificial and with little physical justification (Smith, 2010; Fowler, 2011), provides a mathematical solution to the instability problem and allows a wavelength selection at intermediate wave numbers. Subsequently, Loewenherz-Lawrence (1994) used a hydrodynamic model showing that the deficiency of the SB72 model lies in the kinematic flow approximation which neglects the difference between the gradients of land and water surfaces, and obtained results that are damped at large wave numbers. Subsequent theoretical investigations derived valley spacings from a linear analysis employing two-dimensional quasi-steady forms of St. Venant equations for shallow water flow and a detachment-limited (DL) formulation for the landscape evolution, where the surface is eroded proportionally to the downstream and lateral components of water velocity (Izumi and Parker, 1995, 2000; Parker and Izumi, 2000). In particular, Izumi and Parker (1995) described channel initiation as an “upstream-driven”
process in which the discharge increases in the downstream direction up to a critical point which triggers erosion. The case of “downstream-driven” erosion is analyzed in Izumi and Parker (2000) and Parker and Izumi (2000), where the growth rate of perturbations is described as a function of the upstream Froude number and the resistance coefficient, but the solutions are derived under very constrained assumptions with limited applicability to real environments (Smith, 2010).

Recently, the problem of formation of evenly spaced valleys has been tackled from a numerical standpoint, with the analysis of a large number of numerical solutions of landscape evolution models in DL conditions (Perron et al., 2008, 2009). In these studies landscape evolution is modeled accounting for diffusive soil creep, tectonic uplift, and DL runoff erosion where the erosion rate is assumed to be proportional to the local topographic slope and the drainage area (used as a proxy for water flow). Numerical simulations show the emergence of characteristic valley spacings scaling with a ratio of the characteristic diffusive and runoff erosion timescales (analogous to a Péclet number). However, while numerical experiments can model the emergence of characteristic valley spacings, a theoretical linear stability analysis of the underlying equations is still missing.

In this Chapter, the linear stability of the coupled dynamics of surface water flow and landscape evolution in DL conditions is outlined. The surface water is modeled in terms of a continuity equation, which can be mapped into an equation for the drainage area thus providing a formal link with commonly used numerical models (e.g., Perron et al. (2008); Tucker and Hancock (2010); Chen et al. (2014)) and allowing a direct comparison between theoretical results and numerical experiments. The landscape evolution equation considers the coupled dynamics of tectonic uplift, diffusive soil creep, and DL runoff erosion. The mathematical model introduced by SB72 is first revisited together with its linear stability analysis (following Fowler (2011)) to introduce terminology, methods, and discuss the underlying assumptions.
and limitations. Subsequently, the stability analysis for the DL regime is outlined with the introduction of a numerical discretization of spectral type able to resolve a generalized eigenvalue problem (see, e.g., Canuto et al. (2006); Camporeale et al. (2012)). Preliminary results show that this type of analysis allows the identification of a most unstable wavelength, with the selection of a characteristic valley spacing based on the relative proportions of soil creep and runoff erosion terms.

6.2 Mathematical framework

The mathematical framework used here for the analysis of channel forming instabilities builds upon the work of Smith and Bretherton (1972) (see also Fowler (2011)), and considers the coupled dynamics of surface water and landscape evolution described by the system of equations

\[
\frac{\partial h}{\partial t} = R - \nabla \cdot q
\]

\[
\frac{\partial z}{\partial t} = U - \nabla \cdot f = U - \nabla \cdot (f_d + f_e)
\]
where $t$ is time, $h$ the surface water height, $R$ the rainfall rate, $q$ the water flux, $U$ the uplift rate, and $f$ the sediment flux. The total sediment flux is the sum of fluxes related to runoff erosion/channelized flow ($f_c$) and mass movement of sediment/creep/colluvial processes ($f_d$). Soil creep is a phenomenon driven by processes such as bioturbation, frost heaving, and wetting/drying (Perron et al., 2008) and it is hypothesized that its time-averaged downslope flux is proportional to the topographic gradient (Culling, 1960, 1963), namely $f_d = -D\nabla z$. For the runoff erosion component, both transport (TL) and detachment (DL) limited models exist (see also Chapter 3). In TL models erosion and deposition by flowing water is related to the divergence of sediment flux which, in turn, is related to the shear stress, power, or velocity of the flow (e.g., Smith and Bretherton (1972); Willgoose et al. (1991); Tarboton et al. (1992)),

$$f_c = K_t h^m |\nabla z|^n t \quad (6.3)$$

Conversely, in DL models the rate of fluvial/slope-wash erosion is directly related to the shear stress, power, or velocity of the flow (Howard, 1994)

$$\nabla \cdot (f_c) = K_d h^n d |\nabla z|^n d. \quad (6.4)$$

Lastly, it is assumed here that water flows in the direction opposite to the topographic gradient, so that $q = q n$, being $q$ the discharge of water per unit width and $n$ the unit vector in the direction opposite to the land surface gradient,

$$n = -\frac{\nabla z}{|\nabla z|}. \quad (6.5)$$

6.3 Transport-limited regime (SB72)

The salient features of the stability analysis proposed by Smith and Bretherton (1972) are now presented (in part following Fowler (2011)) to provide the theoretical
background used for the subsequent analysis in DL regime as well as to discuss the main assumptions and limitations.

6.3.1 Assumptions, boundary conditions, and steady state solution

The approach by SB72 further assumes that

i) no water losses (e.g., infiltration and evaporation) occur,

ii) water is in quasi-equilibrium at the time scale at which surface erosion occurs (i.e., $\partial h/\partial t = 0$),

iii) the sediment flux $f$ moves down the topographic gradient (i.e., vector field is $f = f n$),

iv) $f$ is a function of local slope $S$ and water discharge $q$, so that the transport law is $f = f(q, S)$, with $f(q, S) > 0$, $\partial f/\partial S > 0$, and $\partial f/\partial q > 0$ for all $q, S > 0$. Note that the sediment flux is here kept in its generic form $f(q, S)$, without any specific form of the geomorphic transport laws being assumed. Under these assumptions the water and sediment continuity equations (Eq. (6.1)-(6.2)) reduce to

$$\nabla \cdot (qn) = R$$ \hspace{1cm} (6.6)

$$\frac{\partial z}{\partial t} = -\nabla \cdot (f(q, S)n) + U. \hspace{1cm} (6.7)$$

The assumed geometry consists in a symmetric hillslope inclined towards a fixed boundary at $x = x_0$, with divide at $x = 0$, and $\frac{\partial z}{\partial x} < 0$. Boundary conditions are zero water and sediment fluxes at the divide (i.e., $f = q = 0$ at $x = 0$), and a fixed surface level $z = 0$ at $x = x_0$.

Steady state solutions of the one-dimensional problem are

$$q_0 = Rx \hspace{1cm} (6.8)$$

$$f_0 = Ux \hspace{1cm} (6.9)$$

where subscript 0 denotes steady state, and $R$ and $U$ are assumed to be constants.

The goal of the analysis conducted by SB72 is to find the conditions under which the basic steady state described by Eq. (6.8)-(6.9) is unstable.
Before delving into the details of the stability analysis, a condition for the concavity/convexity of hillslopes must be introduced. This, as will be seen in the following sections, is a crucial point in the stability arguments proposed by SB72. Differentiating both sides of \( Ux = f_0(S_0, q_0) \) and making use of Eq. (6.8) and (6.9) gives

\[
\frac{\partial f_0}{\partial S_0} \frac{\partial S_0}{\partial x} = f_0 - q_0 \frac{\partial f_0}{\partial q_0}\,.
\]

(6.10)

The sign of the curvature of the topographic surface is given by the sign of \( \frac{\partial S_0}{\partial x} \) and, given that \( \frac{\partial f}{\partial S} > 0 \), it reduces to an analysis of the sign of \( f_0 - q_0 \frac{\partial f_0}{\partial q_0} \): when \( f_0/q_0 > \frac{\partial f_0}{\partial q_0} \) the hillslope is concave down, while it is concave up when \( f_0/q_0 < \frac{\partial f_0}{\partial q_0} \).

6.3.2 Stability of the one-dimensional problem

We shall see that for any transport law \( f = f(S, q) \), the constant form profile, denoted as \( z_0 \), is stable to perturbations in the \( x \) direction. We perturb the steady state as

\[
z = z_0 + \tilde{z}
\]

(6.11)

\[
q = q_0 + \tilde{q}
\]

(6.12)

\[
f = f_0 + \tilde{f}.
\]

(6.13)

where tilde denotes perturbations. Boundary conditions for the perturbations are assumed homogeneous, so that \( \tilde{q} = \tilde{f} = 0 \) at \( x = 0 \) and \( \tilde{z} = 0 \) at \( x = x_0 \). Perturbing the one-dimensional water continuity equation provides

\[
\frac{\partial q_0}{\partial x} + \frac{\partial \tilde{q}}{\partial x} = R \rightarrow \frac{\partial \tilde{q}}{\partial x} = 0
\]

(6.14)
which, upon integration and using the boundary condition \( \tilde{q} = 0 \) at \( x = 0 \), gives \( \tilde{q} = 0 \). For the landscape evolution equation (6.7) we first Taylor expand \( f(S,q) \)

\[
\begin{align*}
 f(S,q) &= f(S_0, q_0) + \frac{\partial f_0}{\partial S_0}(S - S_0) + \frac{\partial f_0}{\partial q_0}(q - q_0) \\
 &= f_0 + \frac{\partial f_0}{\partial S_0} \left( -\frac{\partial \tilde{z}}{\partial x} \right) + \frac{\partial f_0}{\partial q_0} \tilde{q} \\
 &= f_0 - \frac{\partial f_0}{\partial S_0} \frac{\partial \tilde{z}}{\partial x},
\end{align*}
\]

(6.15)

where we made use of \( \tilde{q} = 0 \) and \( S = |\nabla z| = S_0 - \frac{\partial \tilde{z}}{\partial x} \) (see Eq. (6.21) below), so that the perturbed sediment equation becomes

\[
\begin{align*}
 \frac{\partial z_0}{\partial t} + \frac{\partial \tilde{z}}{\partial t} &= -\frac{\partial (Ux)}{\partial x} - \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial S_0} \left( -\frac{\partial \tilde{z}}{\partial x} \right) \right) + U \\
 &\rightarrow \frac{\partial \tilde{z}}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial S_0} \frac{\partial \tilde{z}}{\partial x} \right).
\end{align*}
\]

(6.16)

Multiplying both sides by \( \tilde{z} \) and integrating between 0 and \( x_0 \) gives

\[
\frac{\partial}{\partial t} \int_0^{x_0} \frac{\tilde{z}^2}{2} \, dx = \int_0^{x_0} \tilde{z} \frac{\partial \tilde{z}}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial S_0} \right) \, dx + \int_0^{x_0} \tilde{z} \frac{\partial^2 \tilde{z}}{\partial x^2} \frac{\partial f_0}{\partial S_0} \, dx
\]

(6.17)

Integrating by parts and using the boundary conditions we can rewrite the first term on the r.h.s. of the previous equation as

\[
\int_0^{x_0} \tilde{z} \frac{\partial \tilde{z}}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial S_0} \right) \, dx = -\int_0^{x_0} \left( \frac{\partial \tilde{z}}{\partial x} \right)^2 \frac{\partial f_0}{\partial S_0} \, dx - \int_0^{x_0} \tilde{z} \frac{\partial \tilde{z}}{\partial x} \frac{\partial f_0}{\partial S_0} \, dx.
\]

(6.18)

Upon inserting this expression in (6.17) it follows that

\[
\frac{\partial}{\partial t} \int_0^{x_0} \frac{\tilde{z}^2}{2} \, dx = -\int_0^{x_0} \left( \frac{\partial \tilde{z}}{\partial x} \right)^2 \frac{\partial f_0}{\partial S_0} \, dx
\]

(6.19)

where \( \int_0^{x_0} \left( \frac{\partial \tilde{z}}{\partial x} \right)^2 \frac{\partial f_0}{\partial S_0} \, dx > 0 \). If one regards \( \tilde{z}^2/2 \) as the energy of a perturbation, then \( \int_0^{x_0} \tilde{z}^2/2 \) is total energy of all the perturbations, which is thus decreasing over time.
(i.e., the system is stable to perturbations in the $x$ direction). Furthermore, note that $\tilde{q} = 0$ so there is no perturbation in the flux of water at any point.

### 6.3.3 Stability to lateral perturbations

We now move to the study of the stability of the steady state solutions to perturbations in the $y$ direction. Denoting perturbations by an overtilde (see Eq. (6.11)-(6.13)) and being $S_0 = -\frac{\partial z_0}{\partial x}$, we have

$$\nabla z = \left(\frac{\partial z_0}{\partial x} + \frac{\partial \tilde{z}}{\partial x}\right) i + \frac{\partial \tilde{z}}{\partial y} j = \left(-S_0 + \frac{\partial \tilde{z}}{\partial x}\right) i + \frac{\partial \tilde{z}}{\partial y} j \quad (6.20)$$

and

$$|\nabla z| = S_0 - \frac{\partial \tilde{z}}{\partial x} \quad (6.21)$$

where higher order terms were neglected. Thus, the linearized downslope unit vector $n$ (Eq. (6.5)) becomes

$$n = i - \frac{\frac{\partial \tilde{z}}{\partial y}}{S_0} j. \quad (6.22)$$

This linear approximation can be thus employed to write perturbed two-dimensional linearized equations, namely

$$\frac{\partial \tilde{q}}{\partial x} = \frac{q_0}{S_0} \frac{\partial^2 \tilde{z}}{\partial y^2} \quad (6.23)$$

and

$$\frac{\partial \tilde{z}}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial f_0}{\partial S_0} \frac{\partial \tilde{z}}{\partial x}\right) - \tilde{q} \frac{\partial}{\partial x} \left(\frac{\partial f_0}{\partial q_0}\right) + \frac{1}{S_0} \left(f_0 - q_0 \frac{\partial f_0}{\partial q_0}\right) \frac{\partial^2 \tilde{z}}{\partial y^2}. \quad (6.24)$$

We now use normal mode analysis to study the stability of the basic state to perturbations, which are then written in the form

$$\tilde{q} = \phi(x)e^{iky+\sigma t} \quad (6.25)$$

$$\tilde{z} = \psi(x)e^{iky+\sigma t}. \quad (6.26)$$
Inserting these expressions in Eq. (6.23) and (6.24) gives

\[
\frac{\partial \phi}{\partial x} = -\frac{q_0}{S_0} k^2 \psi
\]  \hspace{1cm} (6.27)

\[
\sigma \psi = \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial x} \frac{\partial \psi}{\partial S_0} \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \phi \frac{\partial f_0}{\partial q_0} \right) - k^2 \psi \phi
\]  \hspace{1cm} (6.28)

which can be combined providing the following equation,

\[
\frac{\sigma S_0}{q_0} \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial f_0}{\partial S_0} \frac{\partial \phi}{\partial x}(\frac{S_0}{q_0}) \right) + k^2 \phi \frac{\partial}{\partial x} \left( \phi \frac{\partial f_0}{\partial q_0} \right) + k^2 \left( \frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} \right) \frac{\partial \phi}{\partial x}.
\]  \hspace{1cm} (6.29)

The boundary conditions applied to this system are \( \phi = \frac{\partial \psi}{\partial x} = 0 \) at \( x = 0 \) (zero water flux at the ridge) and \( \psi = 0 \) at \( x = 1 \) (fixed zero level at \( x = x_0 = 1 \)).

To provide an idea of the solution it is assumed that \( q_0, S_0, f_0, \phi_0 \) and \( \frac{\partial f_0}{\partial S_0} \) are positive constants (Fowler, 2011), so that Eq. (6.29) can be rearranged as

\[
\frac{\partial \phi}{\partial x} \left( \frac{S_0}{q_0} k^2 - \left( \frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} \right) \right) - \frac{\partial^3 \phi}{\partial x^3} \frac{\partial f_0}{\partial S_0} k^2 = \Lambda \frac{\partial \phi}{\partial x} - \frac{\partial^3 \phi}{\partial x^3} \frac{\partial f_0}{\partial S_0} k^2 = 0
\]  \hspace{1cm} (6.30)

where \( \Lambda = \frac{S_0}{q_0} k^2 - \left( \frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} \right) \). Solving for \( \phi \) gives three independent solutions, namely \( \phi = \text{const} \) and \( \phi = e^{\pm kx \sqrt{\Lambda/(\partial f_0/\partial S_0)}} \), of which the final solution is a linear combination, \( \phi = c_1 + c_2 e^{kx \sqrt{\Lambda/(\partial f_0/\partial S_0)}} + c_3 e^{-kx \sqrt{\Lambda/(\partial f_0/\partial S_0)}} \). Imposing the boundary condition \( \phi = 0 \) at \( x = 0 \) gives \( c_1 = -c_2 - c_3 \). The conditions \( \frac{\partial \psi}{\partial x} = 0 \) at \( x = 0 \) corresponds to \( \frac{\partial^2 \phi}{\partial x^2} = 0 \) (from Eq. (6.27) and assuming \( S_0 \) and \( q_0 \) are positive constants). Differentiating the solution twice gives \( k^2 \Lambda/(\partial f_0/\partial S_0)c_2 + k^2 \Lambda/(\partial f_0/\partial S_0)c_3 = 0 \rightarrow c_2 = -c_3 \) and, as a consequence, \( c_1 = 0 \). The solution can be rewritten as

\[
\phi = c_2 2 \sinh \left( kx \sqrt{\Lambda/(\partial f_0/\partial S_0)} \right) \hspace{1cm} (6.31)
\]
Lastly, the condition $\psi = 0$ at $x = 1$ corresponds to $\partial \phi / \partial x = 0$, which gives

$$c_2 \sqrt{\frac{-\Lambda}{(\partial f_0 / \partial S_0)^2}} k \cos \left( \sqrt{\frac{-\Lambda}{(\partial f_0 / \partial S_0)^2}} k \right) = 0. \quad (6.32)$$

This last condition requires that $\Lambda < 0$ and $\sqrt{\frac{-\Lambda}{(\partial f_0 / \partial S_0)^2}} k = (m + \frac{1}{2})\pi$ for integer $m$, so that

$$\sigma = \frac{q_0}{S_0} \left( \frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} \right) k^2 - \frac{\partial f_0}{\partial S_0} \left( m + \frac{1}{2} \right)^2 \pi^2. \quad (6.33)$$

Note that if $k = 0$ then $\sigma < 0$, meaning that the system is stable if not perturbed in the $y$ direction, coherently with results from the one-dimensional stability analysis. The system becomes unstable if $\sigma > 0$, i.e.

$$\frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} > \frac{S_0 \delta f_0}{\delta S_0} \frac{(m + \frac{1}{2})^2 \pi^2}{q_0 k^2} \quad (6.34)$$

and the most unstable mode corresponds to $m = 0$, namely

$$\frac{\partial f_0}{\partial q_0} - \frac{f_0}{q_0} > \frac{S_0 \delta f_0}{\delta S_0} \frac{\pi^2}{4q_0 k^2}. \quad (6.35)$$

Note that, as $k \to \infty$, the problem reduces to a study of the concavity/convexity of the hillslope, with concave-up hillslopes being unstable to perturbations. When the instability criterion is satisfied the growth rate $\sigma \sim k^2$, meaning that the process is characterized by negative diffusion resulting in an unbounded increase of $\sigma$ at high wave numbers. As discussed by Fowler (2011) this shortcoming might be related to some of the underlying assumptions. First, the purely kinematic assumption might have removed some stabilizing terms, so that accounting for the difference between water and land surface gradients might dampen the solution at high wave numbers (this motivated some subsequent works, e.g. Loewenherz-Lawrence (1994); Izumi and Parker (1995)). Secondly, the solution derived here assumes that all coefficients
are constant, while an extension of this result to \( x \) dependent coefficients might allow a wave number selection. The case of variable coefficients is addressed in the following section for the analysis of the DL case.

### 6.4 Detachment-limited regime

The analysis of the DL case is performed assuming \( n_d = m_d = 1 \) and quasi-equilibrium conditions for the surface water, so that the system of equations (6.1)-(6.2) reduces to

\[
\nabla \cdot (qn) = R \tag{6.36}
\]

\[
\frac{\partial z}{\partial t} = D\nabla^2 z - K_a q |\nabla z| + U \tag{6.37}
\]

with boundary conditions

\[
q = 0 \quad \text{at} \quad x = 0 \tag{6.38}
\]

\[
z = 0 \quad \text{at} \quad x = x_0 \tag{6.39}
\]

\[
\frac{\partial z}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{6.40}
\]

where the last condition is used to impose zero sediment flux at the ridge. Note that the geometry assumed here is the same as the TL case, i.e. a symmetric hillslope inclined towards a fixed boundary at \( x = x_0 \), with divide at \( x = 0 \), and \( \frac{\partial z}{\partial x} < 0 \).

The one-dimensional steady state solution (see also Chapter 3) is

\[
q_0 = Rx \tag{6.41}
\]

\[
z_0 = \frac{U \left(-x^2_0 \mathcal{H}_2 \left(1, 1; \frac{3}{2}, 2; -\frac{K_a R x^2}{2D}\right) + x^2_0 \mathcal{H}_2 \left(1, 1; \frac{3}{2}, 2; -\frac{K_a R x^2}{2D}\right)\right)}{2D} \tag{6.42}
\]

where \( x_0 \mathcal{H}_2(.,.,.,.) \) is the generalized hypergeometric function (Prudnikov et al., 129).
The local slope can also be derived, namely

\[ S_0(x) = -\frac{\partial z_0}{\partial x} = \frac{\sqrt{2UD\left(\frac{\sqrt{KaR_s}}{\sqrt{2D}}\right)}}{\sqrt{DK_aR}} \] (6.43)

where \( D(.) \) is the Dawson’s integral (Abramowitz and Stegun, 1964).

6.4.1 Stability of the one-dimensional problem

We start from perturbing the steady state as \( q = q_0 + \tilde{q} \) and \( z = z_0 + \tilde{z} \) and adopting homogeneous boundary conditions, namely \( \tilde{q} = \frac{\partial \tilde{z}}{\partial x} = 0 \) at \( x = 0 \) and \( \tilde{z} = 0 \) at \( x = x_0 \).

The perturbed water continuity equation becomes

\[ \frac{\partial}{\partial x}(q_0 + \tilde{q}) = R + \frac{\partial \tilde{q}}{\partial x} = R \rightarrow \frac{\partial \tilde{q}}{\partial x} = 0 \rightarrow \tilde{q} = 0 \] (6.44)

where the last equality is obtained imposing the boundary condition in \( \tilde{q} \). The linearized perturbed sediment equation reads

\[ \frac{\partial \tilde{z}}{\partial t} = D \frac{\partial^2 \tilde{z}}{\partial x^2} + K_a q_0 \frac{\partial \tilde{z}}{\partial x} \] (6.45)

which can be written as energy of the perturbations multiplying both sides by \( \tilde{z} \) and integrating between 0 and \( x_0 \),

\[ \int_0^{x_0} \tilde{z} \frac{\partial \tilde{z}}{\partial t} dx = \frac{\partial}{\partial t} \int_0^{x_0} \frac{\tilde{z}^2}{2} dx = D \int_0^{x_0} \tilde{z} \frac{\partial^2 \tilde{z}}{\partial x^2} dx + K_a \int_0^{x_0} q_0 \tilde{z} \frac{\partial \tilde{z}}{\partial x} dx. \] (6.46)

Integrating by parts and applying the boundary conditions provides the time variation of the energy of the perturbations,

\[ \frac{\partial}{\partial t} \int_0^{x_0} \frac{\tilde{z}^2}{2} dx = -D \int_0^{x_0} \left( \frac{\partial \tilde{z}}{\partial x} \right)^2 dx - \frac{K_a R}{2} \int_0^{x_0} \tilde{z}^2 dx \] (6.47)

which is \( < 0 \), meaning that the energy of the perturbations decreases over time. Hence, the system is stable to perturbations in the \( x \) direction.
6.4.2 Stability to lateral perturbations

To study the stability of the system to perturbations in the $y$ direction, we start from writing linearized perturbed water and landscape evolution equations. Recalling the linear approximations for the slope and vector $n$ (Eq. (6.21) - (6.22)), the perturbed system reads

\[
\frac{\partial \tilde{q}}{\partial x} = \frac{q_0}{S_0} \frac{\partial^2 \tilde{z}}{\partial y^2} \quad (6.48)
\]

\[
\frac{\partial \tilde{z}}{\partial t} = D \frac{\partial^2 \tilde{z}}{\partial x^2} + D \frac{\partial^2 \tilde{z}}{\partial y^2} - K_a S_0 \tilde{q} + K_a \frac{\partial \tilde{z}}{\partial x} q_0. \quad (6.49)
\]

Using normal mode analysis and writing the perturbations as Eq. (6.25) - (6.26) gives

\[
\frac{\partial \phi}{\partial x} = -\frac{q_0 k^2}{S_0} \psi \quad (6.50)
\]

\[
\sigma \psi = D \frac{\partial^2 \psi}{\partial x^2} - k^2 D \psi - K_a S_0 \phi + K_a q_0 \frac{\partial \tilde{z}}{\partial x}. \quad (6.51)
\]

The boundary condition $\tilde{q} = 0$ at $x = 0$ translates into $\phi(0) = 0$, while $\tilde{z} = 0$ at $x = x_0$ gives $\left. \frac{S_0 \phi}{q_0 k^2} \right|_{x=x_0} = 0$, which reduces to imposing $\phi'(x_0) = 0$ as $S_0(x_0)/q_0(x_0)$ is a positive constant. Note that, for ease of notation, primes will be used hereafter to indicate derivatives with respect to $x$. Lastly, $\frac{\tilde{z}}{\partial x} = 0$ at $x = 0$ gives

\[
\left( \frac{S_0 \phi'}{q_0} \right)' \bigg|_{x=0} = \left( \frac{S_0 \phi'}{q_0} \right)' q_0 - \frac{\phi'}{q_0} S_0 \phi' \bigg|_{x=0} = \frac{S_0 \phi'}{q_0} q_0 + \frac{\phi'}{q_0} S_0 \phi' q_0 - \frac{\phi'}{q_0} S_0 \phi' \bigg|_{x=0} = 0 \quad (6.52)
\]
and, using Eq. (6.43) and (6.41), we have

\[
\lim_{x \to 0} \frac{S_0}{q_0} = \frac{U}{DR} \tag{6.53}
\]

\[
\lim_{x \to 0} \frac{S_0q_0 - q_0^2S_0}{q_0^2} = 0 \tag{6.54}
\]

so that the boundary condition reduces to \( \phi''(0) = 0 \).

We now proceed writing the system of equations (6.50)-(6.51) in nondimensional form. Observing that the variables have the following units

\[
\phi = \left[ \frac{L^2}{T} \right], \quad x = [L], \quad \sigma = \left[ \frac{1}{T} \right], \quad k = \left[ \frac{1}{L} \right], \quad K_a = \left[ \frac{1}{L} \right]
\]

\[
R = \left[ \frac{L}{T} \right], \quad q_0 = \left[ \frac{L^2}{T} \right], \quad D = \left[ \frac{L^2}{T} \right], \quad \psi = [L], \quad U = \left[ \frac{L}{T} \right]
\]

and introducing the dimensionless quantities (denoted by an overhat)

\[
\hat{\phi} = \frac{R}{K_a} \hat{\phi}, \quad \hat{\psi} = \frac{Ux_0^2}{D} \hat{\psi}, \quad x = x_0 \hat{x}, \quad k = \frac{k}{x_0}, \quad \sigma = DK_a^2 \hat{\sigma}
\]

(6.56)

provides the following nondimensional system

\[
\hat{\psi} = -\frac{\sqrt{2}}{\beta \sqrt{Pe}} \frac{\hat{\phi}'}{k^2 \hat{x}} D \left( \sqrt{\frac{Pe}{2}} \right)
\]

(6.57)

\[
\beta^2 \hat{\sigma} \hat{\psi} = D \hat{\psi}'' - \frac{\sqrt{2Pe}}{\beta} \frac{1}{\sqrt{\frac{Pe}{2}}} D \left( \sqrt{\frac{Pe}{2}} \hat{x} \right) \hat{\phi} + Pe \hat{x} \hat{\psi}'
\]

(6.58)

where \( Pe = \frac{RK_a x_0^2}{D} \) and \( \beta = K_a x_0 \) are dimensionless parameters. Note that Pe is analogous to the Péclet number used in Perron et al. (2009), the only difference being the value of \( x_0 \) which is here assumed to be the entire hillslope length, while it is equal to the drainage basin length in Perron et al. (2009).

We introduce the change of variables \( z = 2\hat{x} - 1 \), in order to restrict the domain
to \( z \in [-1, 1] \) (note that \( \frac{\partial^n}{\partial x^n} = 2^n \frac{\partial^n}{\partial z^n} \)) and combine the previous equations to have

\[
\frac{\sqrt{2}}{\beta \sqrt{P_e}} D \left( \frac{\sqrt{P_e}}{2 \sqrt{2}} (z + 1) \right) (z + 1)^2 \hat{\phi}'' + \\
+ \frac{\hat{\phi}''}{\beta} \left( \frac{(z + 1)^2}{\beta} \right) - \frac{2 \sqrt{2}(z + 1) D \left( \frac{\sqrt{P_e}}{2 \sqrt{2}} (z + 1) \right)}{\beta \sqrt{P_e}} - \frac{\sqrt{2}\beta^2(z + 1)^2}{4 \beta \sqrt{P_e}} D \left( \frac{\sqrt{P_e}}{2 \sqrt{2}} (z + 1) \right) + \\
+ \frac{\sqrt{2}\beta(z + 1)^2}{4 \sqrt{P_e}} D \left( \frac{\sqrt{P_e}}{2 \sqrt{2}} (z + 1) \right) \hat{\phi}' = 0
\]

which can be rewritten as

\[
\gamma_1(z) \hat{\phi}(z) + \gamma_2(z) \hat{\phi}'(z) + \gamma_3(z) \hat{\phi}''(z) + \gamma_4(z) \hat{\phi}'''(z) + \gamma_5(z) \hat{\phi}''''(z) = 0
\] (6.60)

where

\[
\begin{align*}
\gamma_1(z) &= a_1 D (a_0(z + 1)) (z + 1)^2 \\
\gamma_2(z) &= a_2(z + 1)^2 + a_3(z + 1) D (a_0(z + 1)) + a_4(z + 1)^3 D (a_0(z + 1)) \\
\gamma_3(z) &= -a_2(z + 1) - a_3 D (a_0(z + 1)) + a_5(z + 1)^2 D (a_0(z + 1)) \\
\gamma_4(z) &= a_6(z + 1)^3 D (a_0(z + 1)) \\
\gamma_5(z) &= a_7(z + 1)^2 D (a_0(z + 1))
\end{align*}
\] (6.61-6.65)

and

\[
\begin{align*}
a_0 &= \frac{\sqrt{P_e}}{2 \sqrt{2}} \\
a_1 &= \frac{\sqrt{2}}{\beta \sqrt{P_e}} \\
a_2 &= \frac{1}{\beta} \\
a_3 &= -\frac{2 \sqrt{2} \beta^2}{\beta \sqrt{P_e}} \\
a_4 &= -\frac{\sqrt{2} \beta^2}{4 \beta} \\
a_5 &= -\frac{\sqrt{2} \beta^2}{4 \beta \sqrt{P_e}} \\
a_6 &= \frac{\sqrt{2} \beta^2 \sqrt{P_e}}{16 \beta} \\
a_7 &= \frac{\sqrt{2} \beta}{4 \sqrt{P_e}}
\end{align*}
\] (6.66)

We now seek numerical solutions to the stability problem by means of a Legendre-Galerkin approach with Gaussian quadrature (Canuto et al., 2006).
The weak formulation is obtained by multiplying both sides of Eq. (6.60) by a generic test function \( v(z) \) and integrating over the interval \((-1,1)\), which gives

\[
\left( \gamma_1(z) \hat{\phi}'''(z), v(z) \right) + \left( \gamma_2(z) \hat{\phi}''(z), v(z) \right) + \left( \gamma_3(z) \hat{\phi}'(z), v(z) \right) + \left( \gamma_4(z) \hat{\phi}(z), v(z) \right) = \sigma \left( \gamma_5(z) \hat{\phi}'(z), v(z) \right)
\]

(6.67)

where \((f(z), g(z)) := \int_{-1}^{1} f(z') g(z') dz'\) denotes the inner product between two functions. The third and second order terms in Eq. (6.67) are then manipulated using integration by parts, providing

\[
\left( \hat{\phi}''' \gamma_1, v_j \right) = \hat{\phi}''(1) \gamma_1(1) v(1) - \hat{\phi}''(-1) \gamma_1(-1) v(-1) - \\
- \hat{\phi}'(1) \left[ \gamma'_1(1) v(1) + \gamma_1(1) v'(1) \right] + \hat{\phi}'(-1) \left[ \gamma'_1(-1) v(-1) + \gamma_1(-1) v'(-1) \right] + \\
+ \left( \hat{\phi}', \gamma''_1 v + \gamma'_1 v' \right) + \left( \hat{\phi}', \gamma'_1 v' + \gamma_1 v''_j \right)
\]

(6.68)

and

\[
\left( \hat{\phi}'' \gamma_2, v(z) \right) = \hat{\phi}'(1) \gamma_2(1) v(1) - \hat{\phi}'(-1) \gamma_2(-1) v(-1) - \\
- \left( \hat{\phi}'(z), \gamma'_2(z) v + \gamma_2(z) v' \right).
\]

(6.69)

For ease of notation the \( z \) dependence was omitted in the previous expressions.

The resulting eigenvalue problem is then discretized by means of a spectral Galerkin technique with numerical quadrature used to approximate the inner products (see Canuto et al. (2006)). The integrals in Eq. (6.67) are computed by a Gauss-type quadrature which, for a given weight function \( w(z) \) approximates a generic integral as

\[
\int_{-1}^{1} f(z) dz \sim \sum_{j=0}^{M} f(z_j) w(z_j)
\]

(6.70)
where the nodes $z_j$ are the roots of the $(M+1)$th orthogonal polynomial $P_{M+1}$ with respect to the weight $w$, and the weights $w_j$ are the solution of the linear system (Canuto et al., 2006)

$$
\sum_{j=0}^{M} (z_j)^k w_j = \int_{-1}^{1} z^k w(z) dz \quad (6.71)
$$

with $0 \leq k \leq N$. In the numerical results presented here we used the algorithm proposed by Swarztrauber (2003) to compute points and weights for the numerical quadrature.

**Trial and test functions**

We seek solutions to Eq. (6.67) of the form

$$
\phi(x) = \sum_{i=1}^{N} u_i(z) \alpha_i + u_0(z) \alpha_0 + u_{-1}(z) \alpha_{-1} = \sum_{i=-1}^{\infty} u_i(z) \alpha_i \quad (6.72)
$$

where $u_i$ are trial functions. To cast the discrete eigenvalue problem in the form $Ax = \sigma Bx$, we introduce suitable bases in terms of Legendre polynomials (Shen, 1994; Canuto et al., 2006; Camporeale et al., 2012). The values of $u_i$ for $i = 1, N$ are obtained here by integrating once each Legendre polynomial, which gives

$$
u_i(z) = \frac{L_{i+2}(z) - L_i(z)}{\sqrt{2(2i + 3)}} \quad (6.73)
$$

$$u'_i(z) = \sqrt{\frac{2i + 3}{2}} L_{i+1}(z) \quad (6.74)
$$

$$u_i(\pm 1) = 0. \quad (6.75)
$$

$$
\text{The trial function } u_0 \text{ is introduced to satisfy boundary conditions while } u_{-1} \text{ is necessary to allow the system to have constant solutions, and are respectively equal}
$$
to

\[ u_0(z) = \frac{-z^2}{4} + \frac{z}{2} + \frac{3}{4} \]  
(6.77)

\[ u_{-1}(z) = \frac{1 + z}{2} \]  
(6.78)

with

\[ u'_0(1) = 0 \]  
(6.79)

\[ u_0(1) = 1 \]  
(6.80)

\[ u_0(-1) = 0 \]  
(6.81)

\[ u_{-1}(-1) = 0. \]  
(6.82)

Imposing (6.75), (6.81), and (6.82) satisfies \( \phi(-1) = 0 \), while Eq. (6.80) ensures that \( \phi(1) \neq 0 \). The boundary conditions \( \phi''(-1) = 0 \) and \( \phi'(1) = 0 \) will be imposed in strong and weak form, respectively.

Test functions are obtained integrating twice each Legendre polynomial,

\[ v_j = \sqrt{j + \frac{3}{2}} \left( \frac{L_{j+3} - L_{j-1}}{(2j + 3)(2j + 5)} - \frac{L_{j+1} - L_{j-1}}{(2j + 1)(2j + 3)} \right), \]  
\( j \in [1, N - 1] \) \( (6.83) \)

\[ v'_j = \frac{L_{j+2} - L_j}{\sqrt{2(2j + 3)}} \]  
(6.84)

\[ v''_j = \sqrt{\frac{2j + 3}{2}} L_{j+1} \]  
(6.85)

\[ v_j(\pm 1) = v'_j(\pm 1) = 0 \]  
(6.86)

We further add a low-degree polynomial to avoid vanishing boundary conditions,

\[ v_{-1}(z) = \frac{z^3}{4} + \frac{z^2}{4} - \frac{z}{4} - \frac{1}{4} \]  
(6.87)

\[ v'_{-1}(z) = \frac{3}{4} z^2 + \frac{z}{2} - \frac{1}{4} \]  
(6.88)
Table 6.1: Parameter values used in the stability analysis. The behavior of the growth rate $\sigma$ as a function of the wavenumber $k$ for different Pe values is shown in Figure 6.2.

<table>
<thead>
<tr>
<th>$R$ [m/yr]</th>
<th>$D$ [m$^2$/yr]</th>
<th>$x_0$ [m]</th>
<th>$K_a$ [1/m]</th>
<th>Pe</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.005</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>0.01</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>0.015</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>16.7</td>
</tr>
<tr>
<td>1</td>
<td>0.02</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>12.5</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>0.03</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>8.3</td>
</tr>
<tr>
<td>1</td>
<td>0.035</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>7.1</td>
</tr>
<tr>
<td>1</td>
<td>0.04</td>
<td>50</td>
<td>$1 \cdot 10^{-4}$</td>
<td>6.3</td>
</tr>
</tbody>
</table>

which satisfies

\begin{align*}
v_{-1}(\pm 1) &= 0 \tag{6.89} \\
v'_{-1}(-1) &= 0 \tag{6.90} \\
v'_{-1}(1) &= 1. \tag{6.91}
\end{align*}

**Preliminary results**

We numerically solved the generalized eigenvalue problem and tracked the behavior of the maximum eigenvalue, representing the growth rate of the most unstable mode, for different values of wavenumber $k$ and dimensionless parameter Pe. Parameter values are provided in Table 6.1 and results are shown in Figure 6.2. As the value of Pe increases (i.e., the runoff erosion component increases or, conversely, the diffusive soil creep decreases) the characteristic wavenumber increases, meaning that the spacing between valley tend to narrower, resulting in a more dissected hillslope, coherently with numerical results from Perron et al. (2008).

6.5 Discussion and conclusions

In this Chapter the numerical stability analysis of the coupled dynamics of surface water and landscape evolution in detachment limited regime was outlined. Prelimi-
Figure 6.2: Growth rate $\sigma$ as a function of the wavenumber $k$ obtained for different values of the dimensionless parameter $Pe$. Results were computed here with number of modes $N = 20$ and number of Gauss points $M = 100$.

Nary results showed the selection of a characteristic wave length, depending on the relative rates of runoff erosion and diffusive soil creep (through the dimensionless parameter $Pe$). The surface water was here described in terms of a continuity equation, in which water is assumed to flow in the direction opposite to the topographic gradient. Previous studies in transport-limited conditions suggested that the lack of wavelength selection in the SB72 model was related to such a normal flow approximation (Loewenherz-Lawrence, 1994; Smith, 2010; Fowler, 2011). Only Fowler (2011) suggested that this might be related to the constant coefficients assumption which was necessary to obtain an analytic solution. The analysis here, which allowed to account for the spatial dependence of the basic state on $x$ (i.e., $q_0(x)$ and $S_0(x)$), supports this hypothesis.

The spectral method proposed here could be further employed for the study of
more complex systems, such as to analyze the effect of vegetation cover on channel formation. This could be done by introducing a loss term in the water continuity equation which accounts for the reduction in surface runoff due to the presence of vegetation.
In this dissertation various aspects of landscape evolution under natural and disturbed conditions were examined, with a focus on the statistical representation of its features, as well as the analytical and numerical modeling of the main processes involved.

We first showed that the landscape aging process (i.e., the relaxation phase) is encoded in its gradient and slope distributions, which in young stages are characterized by fat tails (i.e., algebraic decay) and, at older orogenetic stages, tend to become Gaussian and Rayleigh distributed, respectively. In particular, the tail of the slope distributions is well approximated by a power-law, whose exponent decreases with age in a regular fashion, thus providing a quantitative link between landscape age and its statistical properties. This natural aging process can be altered by agricultural-driven accelerated erosion which, at smaller scales, produces key differences in the distribution tails. Such a systematic analysis of topographic slopes and the development of a general statistical theory providing a compact representation of topographic features in disturbed and undisturbed areas can be useful in a variety of applications, including the analysis of runoff and infiltration mechanisms,
surface energy balances, and biogeochemical processes, as well as to scale-up plot-based knowledge to the landscape level (Vico and Porporato, 2009; Riveros-Iregui and McGlynn, 2009; Bartlett et al., 2013; Wang et al., 2015). The observed differences in slope morphologies and behavior under natural and disturbed conditions were linked to the main geomorphological processes involved. In fact, the time evolution of slope statistics during the relaxation process can be reproduced numerically by carefully tuning diffusive, advective, and noise terms, showing that the relaxation phase is mainly dominated by diffusive soil creep, compared to noise and advective fluvial incision (Chapter 2). Conversely, at smaller spatial and temporal scales, the different morphologies of disturbed hillslopes are linked to a disruption of the natural balance between diffusive soil creep and runoff erosion (Chapter 3). Understanding processes controlling hillslope development in human-impacted landscapes is key to quantify soil erosion under various levels of vegetation cover and land use practices.

Subsequently, a theoretical definition for the drainage area was provided (Chapter 4). The drainage area is an important morphometrical variable employed in a variety of geomorphological and ecohydrological applications, ranging from vegetation distribution mapping to the analysis of landslide dynamics and runoff erosion. The differential equation for regular points proposed by Gallant and Hutchinson (2011) was re-derived from a continuity equation and the theory was then extended to critical and singular points of the topographic surface. Such a theoretical description directly links the drainage area to a water continuity equation, showing that under the assumption of unitary rainfall rate and water flowing at unitary speed in the direction opposite to the topographic gradient, the steady state form of the water continuity equation can be mapped into a specific drainage area equation. This linkage was used in Chapter 5 to re-cast the landscape evolution model in detachment-limited regime in terms of specific drainage area, thus providing a mathematical framework capable of producing grid-independent solutions while being justified from a theoretical
standpoint. The landscape evolution model was then used to study the emergence of organized ridge and valley patterns over a square domain, with the identification of various regimes based on the relative proportions of soil creep, runoff erosion, and tectonic uplift. Such spatial patterns and the behavior of the corresponding hypsometric curves were shown to change with the specific geomorphic transport law assumed as well as with the shape and size of the domain.

Lastly, we moved the analysis to the study of channel initiation and formation of evenly spaced valleys in detachment limited regime (Chapter 6). A stability analysis of the coupled water and landscape evolution dynamics was outlined by means of a spectral Galerkin method with numerical integration. Such a technique allowed us to account for the space dependence of the coefficients of the perturbed equation (which was neglected in previous works in transport limited regime, see Fowler (2011)), thus obtaining preliminary results that allow the identification of a characteristic wavelength. Furthermore, the surface water dynamics were here modeled according to a continuity equation where water flows down the topographic gradient. This is remarkable as it allowed us to obtain a valley spacing selection without introducing more complex representations of the water flow (as done in early works - e.g., Loewenherz-Lawrence (1994); Izumi and Parker (1995)). Such a water continuity equation can be directly mapped into a drainage area equation (see also Chapters 4 and 5), thus allowing a direct comparison with results from commonly used numerical models (e.g., Perron et al. (2008, 2009)).

Future research should focus on the coupling between the geomorphological processes studied here and the main biotic factors involved, such as vegetation effects. In fact, vegetation has the potential to greatly impact the landscape as it affects soil formation processes and weathering (Gabet et al., 2003), surface and subsurface hydrologic fluxes (Rodríguez-Iturbe and Porporato, 2007), as well as slope stability and erosion rates (Dietrich and Perron, 2006). This could be done by directly coupling
the water and surface evolution processes to soil moisture dynamics to account for
the role of vegetation in favoring water losses through infiltration and evapotranspiration thus diminishing runoff and subsequent soil erosion. Furthermore, geomorphic transport laws should be modified to account for the role of vegetation roots in consolidating the soil. Future work should also focus on the impacts of soil erosion on the biogeochemical cycles (Quinton et al., 2010; Viaud et al., 2010), as carbon and nutrient dynamics are strongly dependent on the rate of soil erosion. The development of appropriate modeling tools for the analysis of the coupled geomorphological and ecohydrological processes is key to understand the underlying feedbacks and mechanisms, the ecosystem response to future climatic and anthropogenic changes, and, ultimately, to directly link sustainability concepts to ecosystem functioning and guide the implementation of optimal strategies of land management and recovery.
Appendix A

Supplementary information and additional results for Chapter 2

A.1 Edwards-Wilkinson (EW) and Kardar-Parisi-Zhang (KPZ) equations.

In the physics literature, the growth and erosion of surfaces is usually modeled by means of the KPZ equation, which was originally proposed for the description of the growth of atomic surfaces by sputter deposition (Kardar et al., 1986; Pelletier, 2007). Consider a surface elevation \( h(x,t) \), where \( x \) is position and \( t \) is time. The KPZ equation describes the surface evolution according to the following equation (Kardar et al., 1986; Barabási and Stanley, 1995; Krug, 1997; Dodds and Rothman, 2000; Pelletier, 2007)

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} |\nabla h|^2 + \eta(x,t). \quad (A.1)
\]

The first term on the r.h.s. describes the surface relaxation caused by diffusion. This term tends to smooth the surface, by eroding it where it is concave down \( (\nabla^2 h < 0) \) and aggrading it where is concave-up \( (\nabla^2 h > 0) \). The second term on the r.h.s. is a non-linear term that can either erode \( (\lambda < 0) \) or grow \( (\lambda > 0) \) the surface in
a direction normal to the surface and it erodes/grows the surface where the slope $|\nabla h|$ is largest. In geomorphological applications, since this term is associated with erosion, the coefficient $\lambda$ must be $< 0$. Finally, the third term on the r.h.s. is a random noise, uncorrelated in space and time, with zero mean and finite variance. In geomorphic applications it represents time- and space-dependent inhomogeneities in forcings and material properties (e.g., rainfall rate, soil type, etc.).

When the non-linear term is neglected, the KPZ equation reduces to the EW equation (Edwards and Wilkinson, 1982),

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(x, t)$$  \hspace{1cm} (A.2)

which was the first continuum equation used to describe the growth of interfaces by particle deposition (Barabási and Stanley, 1995).

A.2 Additional tables and figures

Location (latitude and longitude) of the 36 DEMS analyzed in Chapter 2 are given in Table A.1 together with the statistics of the elevation field. Gradients and slope statistics as well as the values of the power-law exponents are provided in Table A.2, while results from the binning analysis are given in Table A.3. Figures A.1 and A.2 show the spatial distribution of slope values obtained at different simulation times from scenarios SC1, SC3, SC5, and SC7 for the Alpine and Himalayan watersheds, respectively.
Table A.1: Coordinates limits and statistics of the elevation field \( h \) (maximum \( h_{\text{max}} \), minimum \( h_{\text{min}} \), mean \( \mu_h \), standard deviation \( \sigma_h \)) for the 36 DEMs analyzed.

<table>
<thead>
<tr>
<th>Mountain Range</th>
<th>Longitude</th>
<th>Latitude</th>
<th>( h_{\text{max}} ) [m]</th>
<th>( h_{\text{min}} ) [m]</th>
<th>( \mu_h ) [m]</th>
<th>( \sigma_h ) [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appalachian Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APPAL1 (NC)</td>
<td>-83.5,-83</td>
<td>35.2,35.7</td>
<td>2029.00</td>
<td>494.00</td>
<td>1047.588</td>
<td>296.102</td>
</tr>
<tr>
<td>APPAL2 (NY)</td>
<td>-74.8,-74.3</td>
<td>41.9,42.4</td>
<td>1275.00</td>
<td>223.00</td>
<td>661.081</td>
<td>148.185</td>
</tr>
<tr>
<td>APPAL3 (CAN)</td>
<td>-66.1,-65.6</td>
<td>48.5,49</td>
<td>1276.00</td>
<td>75.00</td>
<td>550.679</td>
<td>155.141</td>
</tr>
<tr>
<td>Ozark Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OZA1 (AR)</td>
<td>-94.1,-93.6</td>
<td>35.5,35.7</td>
<td>765.00</td>
<td>152.00</td>
<td>492.883</td>
<td>105.046</td>
</tr>
<tr>
<td>OZA2 (AR)</td>
<td>-93.5,-93</td>
<td>35.5,35.7</td>
<td>790.00</td>
<td>130.00</td>
<td>454.96</td>
<td>138.555</td>
</tr>
<tr>
<td>OZA3 (AR)</td>
<td>-93,-92.5</td>
<td>35.4,35.9</td>
<td>688.00</td>
<td>128.00</td>
<td>383.496</td>
<td>118.764</td>
</tr>
<tr>
<td>Central Massif</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FRA1 (FRA)</td>
<td>4.4,5</td>
<td>44.7,45.2</td>
<td>1740.00</td>
<td>343.00</td>
<td>995.885</td>
<td>214.422</td>
</tr>
<tr>
<td>FRA2 (FRA)</td>
<td>3.3,5</td>
<td>44.7,45.2</td>
<td>1494.00</td>
<td>441.00</td>
<td>1004.855</td>
<td>173.047</td>
</tr>
<tr>
<td>FRA3 (FRA)</td>
<td>2.6,3.1</td>
<td>45.2,45.7</td>
<td>1862.00</td>
<td>457.00</td>
<td>1029.190</td>
<td>185.211</td>
</tr>
<tr>
<td>Flint Hills</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FLINT1 (KS)</td>
<td>-96.8,-96.3</td>
<td>37.2,37.7</td>
<td>516.00</td>
<td>272.00</td>
<td>407.364</td>
<td>46.645</td>
</tr>
<tr>
<td>FLINT2 (KS)</td>
<td>-96.8,-96.3</td>
<td>37.8,38.3</td>
<td>547.00</td>
<td>291.00</td>
<td>429.487</td>
<td>31.684</td>
</tr>
<tr>
<td>FLINT3 (KS)</td>
<td>-96.6,-96.1</td>
<td>38.5,39</td>
<td>493.00</td>
<td>318.00</td>
<td>414.345</td>
<td>29.779</td>
</tr>
<tr>
<td>Urals</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>URAL1 (RUS)</td>
<td>57.5,57.5</td>
<td>53.2,53.7</td>
<td>953.00</td>
<td>333.00</td>
<td>587.602</td>
<td>94.409</td>
</tr>
<tr>
<td>URAL2 (RUS)</td>
<td>57.8,58.3</td>
<td>54.1,54.5</td>
<td>1634.00</td>
<td>352.00</td>
<td>774.598</td>
<td>189.995</td>
</tr>
<tr>
<td>URAL3 (RUS)</td>
<td>58.3,58.8</td>
<td>58.8,59.3</td>
<td>1119.00</td>
<td>227.00</td>
<td>505.780</td>
<td>117.342</td>
</tr>
<tr>
<td>Hamersley Range</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HAMER1 (AUS)</td>
<td>116.5,117</td>
<td>-22.4,-21.9</td>
<td>999.00</td>
<td>282.00</td>
<td>492.562</td>
<td>83.034</td>
</tr>
<tr>
<td>HAMER2 (AUS)</td>
<td>118.5,118.5</td>
<td>-22.8,-23.1</td>
<td>1230.00</td>
<td>442.00</td>
<td>766.669</td>
<td>93.459</td>
</tr>
<tr>
<td>HAMER3 (AUS)</td>
<td>118.5,119</td>
<td>-23.1,-22.6</td>
<td>1247.00</td>
<td>579.00</td>
<td>757.051</td>
<td>93.459</td>
</tr>
<tr>
<td>Rocky Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ROCKY1 (CO)</td>
<td>-105.9,-105.4</td>
<td>40.1,40.6</td>
<td>4333.00</td>
<td>2013.00</td>
<td>2989.198</td>
<td>401.777</td>
</tr>
<tr>
<td>ROCKY2 (CO)</td>
<td>-107.8,-107.3</td>
<td>37.4,37.9</td>
<td>4271.00</td>
<td>2027.00</td>
<td>3342.073</td>
<td>399.256</td>
</tr>
<tr>
<td>ROCKY3 (CO)</td>
<td>-107.2,-106.7</td>
<td>38.3,38.9</td>
<td>4332.00</td>
<td>2175.00</td>
<td>3175.633</td>
<td>389.477</td>
</tr>
<tr>
<td>Himalayas</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HIMAL1 (IND)</td>
<td>78.7,79.2</td>
<td>30.2,30.7</td>
<td>4794.00</td>
<td>496.00</td>
<td>2020.381</td>
<td>775.235</td>
</tr>
<tr>
<td>HIMAL2 (IND)</td>
<td>81.3,81.8</td>
<td>29.2,29.7</td>
<td>5485.00</td>
<td>758.00</td>
<td>2576.348</td>
<td>819.664</td>
</tr>
<tr>
<td>HIMAL3 (IND)</td>
<td>79.5,80</td>
<td>29.7,30.2</td>
<td>5402.00</td>
<td>673.00</td>
<td>1911.398</td>
<td>674.768</td>
</tr>
<tr>
<td>Apennines</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APENN1 (ITA)</td>
<td>13.1,13.5</td>
<td>42.3,42.9</td>
<td>2623.00</td>
<td>255.00</td>
<td>1126.304</td>
<td>340.597</td>
</tr>
<tr>
<td>APENN2 (ITA)</td>
<td>13.7,14.2</td>
<td>41.6,42.1</td>
<td>2783.00</td>
<td>285.00</td>
<td>1193.074</td>
<td>475.819</td>
</tr>
<tr>
<td>APENN3 (ITA)</td>
<td>14.2,14.7</td>
<td>41.4,41.9</td>
<td>2040.00</td>
<td>124.00</td>
<td>780.801</td>
<td>283.915</td>
</tr>
<tr>
<td>Ethiopian Highlands</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ETIH1 (ETH)</td>
<td>38.6,39.1</td>
<td>10.7,11.2</td>
<td>4234.00</td>
<td>1348.00</td>
<td>2585.402</td>
<td>540.641</td>
</tr>
<tr>
<td>ETIH2 (ETH)</td>
<td>38.3,38.5</td>
<td>11.5,11.5</td>
<td>4116.00</td>
<td>1555.00</td>
<td>2605.396</td>
<td>423.470</td>
</tr>
<tr>
<td>ETIH3 (ETH)</td>
<td>38.7,39.2</td>
<td>12.1,12.5</td>
<td>4265.00</td>
<td>1474.00</td>
<td>2337.666</td>
<td>433.271</td>
</tr>
<tr>
<td>Alps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ALP1 (ITA-FRA)</td>
<td>6.6,7.1</td>
<td>44.5,45.5</td>
<td>3771.00</td>
<td>817.00</td>
<td>2190.809</td>
<td>441.151</td>
</tr>
<tr>
<td>ALP2 (ITA-AUT)</td>
<td>11.9,12.4</td>
<td>46.4,46.9</td>
<td>3248.00</td>
<td>533.00</td>
<td>1808.427</td>
<td>459.637</td>
</tr>
<tr>
<td>ALP3 (ITA-CHE)</td>
<td>10.3,10.8</td>
<td>46.3,46.8</td>
<td>3872.00</td>
<td>669.00</td>
<td>2239.925</td>
<td>591.545</td>
</tr>
<tr>
<td>Andes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ANDE1 (BOL)</td>
<td>-66.5,-65.5</td>
<td>-19.3,-18.8</td>
<td>4920.00</td>
<td>2656.00</td>
<td>3778.283</td>
<td>393.842</td>
</tr>
<tr>
<td>ANDE2 (BOL)</td>
<td>-65.5,-65</td>
<td>-19.5,-19</td>
<td>4092.00</td>
<td>2110.00</td>
<td>2946.186</td>
<td>350.390</td>
</tr>
<tr>
<td>ANDE3 (BOL)</td>
<td>-66.1,-65.6</td>
<td>-18.7,18.2</td>
<td>4946.00</td>
<td>1797.00</td>
<td>3088.878</td>
<td>604.766</td>
</tr>
</tbody>
</table>
Table A.2: Gradients and slope statistics for the 36 DEMs analyzed: mean and standard deviation of gradients distributions ($\mu_{\hat{h}}, \sigma_{\hat{h}}$), mean, standard deviation, skewness, and kurtosis of slope distributions ($\mu_S, \sigma_S, sk_S, k_S$) and value of the power-law exponent $q$.

<table>
<thead>
<tr>
<th>Mountain Range</th>
<th>$\mu_{\hat{h}}$ [m/m]</th>
<th>$\sigma_{\hat{h}}$ [m/m]</th>
<th>$\mu_S$ [m/m]</th>
<th>$\sigma_S$ [m/m]</th>
<th>$sk_S [-]$</th>
<th>$k_S [-]$</th>
<th>$q [-]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appalachian Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APPAL1 (NC)</td>
<td>0.00306</td>
<td>0.281</td>
<td>0.36</td>
<td>0.169</td>
<td>0.373</td>
<td>3.285</td>
<td>-11.03</td>
</tr>
<tr>
<td>APPAL2 (NY)</td>
<td>-0.00071</td>
<td>0.188</td>
<td>0.229</td>
<td>0.135</td>
<td>0.828</td>
<td>3.778</td>
<td>-10.31</td>
</tr>
<tr>
<td>APPAL3 (CAN)</td>
<td>0.00326</td>
<td>0.184</td>
<td>0.206</td>
<td>0.157</td>
<td>1.517</td>
<td>5.865</td>
<td>-9.33</td>
</tr>
<tr>
<td>Ozark Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OZA1 (AR)</td>
<td>0.00162</td>
<td>0.153</td>
<td>0.183</td>
<td>0.116</td>
<td>0.953</td>
<td>4.109</td>
<td>-10.7</td>
</tr>
<tr>
<td>OZA2 (AR)</td>
<td>0.00038</td>
<td>0.18</td>
<td>0.217</td>
<td>0.131</td>
<td>0.904</td>
<td>3.998</td>
<td>-10.45</td>
</tr>
<tr>
<td>OZA3 (AR)</td>
<td>0.00085</td>
<td>0.14</td>
<td>0.161</td>
<td>0.114</td>
<td>1.075</td>
<td>4.346</td>
<td>-9.55</td>
</tr>
<tr>
<td>Central Massif</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FRA1 (FRA)</td>
<td>-0.00279</td>
<td>0.204</td>
<td>0.239</td>
<td>0.161</td>
<td>0.767</td>
<td>3.008</td>
<td>-11.81</td>
</tr>
<tr>
<td>FRA2 (FRA)</td>
<td>-0.00315</td>
<td>0.149</td>
<td>0.167</td>
<td>0.128</td>
<td>1.505</td>
<td>5.792</td>
<td>-10.18</td>
</tr>
<tr>
<td>FRA3 (FRA)</td>
<td>-0.00297</td>
<td>0.157</td>
<td>0.173</td>
<td>0.14</td>
<td>1.619</td>
<td>6.578</td>
<td>-8.51</td>
</tr>
<tr>
<td>Flint Hills</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FLINT1 (KS)</td>
<td>-0.00034</td>
<td>0.048</td>
<td>0.057</td>
<td>0.038</td>
<td>1.552</td>
<td>7.461</td>
<td>-9.24</td>
</tr>
<tr>
<td>FLINT2 (KS)</td>
<td>-0.00031</td>
<td>0.051</td>
<td>0.061</td>
<td>0.04</td>
<td>1.446</td>
<td>7.95</td>
<td>-8.02</td>
</tr>
<tr>
<td>FLINT3 (KS)</td>
<td>-0.00049</td>
<td>0.048</td>
<td>0.056</td>
<td>0.038</td>
<td>1.624</td>
<td>7.891</td>
<td>-7.03</td>
</tr>
<tr>
<td>Ural</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>URAL1 (RUS)</td>
<td>-0.00101</td>
<td>0.114</td>
<td>0.137</td>
<td>0.085</td>
<td>1.536</td>
<td>7.333</td>
<td>-7.34</td>
</tr>
<tr>
<td>URAL2 (RUS)</td>
<td>0.00452</td>
<td>0.116</td>
<td>0.134</td>
<td>0.095</td>
<td>1.754</td>
<td>7.142</td>
<td>-10.86</td>
</tr>
<tr>
<td>URAL3 (RUS)</td>
<td>0.00238</td>
<td>0.079</td>
<td>0.088</td>
<td>0.068</td>
<td>2.622</td>
<td>13.614</td>
<td>-7.01</td>
</tr>
<tr>
<td>Hamersley Range</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HAMER1 (AUS)</td>
<td>0.00028</td>
<td>0.115</td>
<td>0.117</td>
<td>0.112</td>
<td>2.041</td>
<td>8.7</td>
<td>-7.21</td>
</tr>
<tr>
<td>HAMER2 (AUS)</td>
<td>-0.00054</td>
<td>0.139</td>
<td>0.129</td>
<td>0.149</td>
<td>2.111</td>
<td>9.188</td>
<td>-6.3</td>
</tr>
<tr>
<td>HAMER3 (AUS)</td>
<td>-0.00127</td>
<td>0.117</td>
<td>0.129</td>
<td>0.15</td>
<td>2.112</td>
<td>9.201</td>
<td>-8.18</td>
</tr>
<tr>
<td>Rocky Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ROCKY1 (CO)</td>
<td>-0.00809</td>
<td>0.272</td>
<td>0.316</td>
<td>0.221</td>
<td>2.06</td>
<td>14.228</td>
<td>-5.02</td>
</tr>
<tr>
<td>ROCKY2 (CO)</td>
<td>0.01045</td>
<td>0.345</td>
<td>0.408</td>
<td>0.268</td>
<td>1.632</td>
<td>9.602</td>
<td>-5.58</td>
</tr>
<tr>
<td>ROCKY3 (CO)</td>
<td>0.00186</td>
<td>0.324</td>
<td>0.387</td>
<td>0.245</td>
<td>1.326</td>
<td>7.819</td>
<td>-6.09</td>
</tr>
<tr>
<td>Himalayas</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HIMAL1 (IND)</td>
<td>0.02443</td>
<td>0.45</td>
<td>0.576</td>
<td>0.273</td>
<td>1.117</td>
<td>6.662</td>
<td>-7.01</td>
</tr>
<tr>
<td>HIMAL2 (NPL)</td>
<td>0.01281</td>
<td>0.513</td>
<td>0.656</td>
<td>0.311</td>
<td>1.458</td>
<td>10.891</td>
<td>-6.83</td>
</tr>
<tr>
<td>HIMAL3 (IND)</td>
<td>0.02206</td>
<td>0.425</td>
<td>0.533</td>
<td>0.281</td>
<td>1.255</td>
<td>7.015</td>
<td>-6.65</td>
</tr>
<tr>
<td>Apennines</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APENN1 (ITA)</td>
<td>-0.00091</td>
<td>0.283</td>
<td>0.341</td>
<td>0.209</td>
<td>0.808</td>
<td>4.884</td>
<td>-7.29</td>
</tr>
<tr>
<td>APENN2 (ITA)</td>
<td>0.00193</td>
<td>0.282</td>
<td>0.334</td>
<td>0.22</td>
<td>1.225</td>
<td>8.204</td>
<td>-6.03</td>
</tr>
<tr>
<td>APENN3 (ITA)</td>
<td>-0.00401</td>
<td>0.195</td>
<td>0.231</td>
<td>0.151</td>
<td>1.615</td>
<td>8.866</td>
<td>-7.07</td>
</tr>
<tr>
<td>Ethiopian Highlands</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ETH1 (ETH)</td>
<td>0.01365</td>
<td>0.299</td>
<td>0.351</td>
<td>0.237</td>
<td>1.7</td>
<td>11.456</td>
<td>-5.17</td>
</tr>
<tr>
<td>ETH2 (ETH)</td>
<td>-0.0017</td>
<td>0.246</td>
<td>0.284</td>
<td>0.201</td>
<td>1.716</td>
<td>10.448</td>
<td>-5.81</td>
</tr>
<tr>
<td>ETH3 (ETH)</td>
<td>0.00633</td>
<td>0.317</td>
<td>0.388</td>
<td>0.223</td>
<td>1.112</td>
<td>6.677</td>
<td>-6.5</td>
</tr>
<tr>
<td>Alps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ALP1 (ITA-FRA)</td>
<td>0.00154</td>
<td>0.426</td>
<td>0.524</td>
<td>0.296</td>
<td>2.053</td>
<td>15.014</td>
<td>-5.22</td>
</tr>
<tr>
<td>ALP2 (ITA-AUT)</td>
<td>0.00743</td>
<td>0.455</td>
<td>0.528</td>
<td>0.367</td>
<td>2.291</td>
<td>14.87</td>
<td>-4.93</td>
</tr>
<tr>
<td>ALP3 (ITA-CHE)</td>
<td>0.0037</td>
<td>0.428</td>
<td>0.53</td>
<td>0.293</td>
<td>1.857</td>
<td>28.786</td>
<td>-5.69</td>
</tr>
<tr>
<td>Andes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ANDE1 (BOL)</td>
<td>-0.00636</td>
<td>0.294</td>
<td>0.358</td>
<td>0.211</td>
<td>1.311</td>
<td>8.674</td>
<td>-5.74</td>
</tr>
<tr>
<td>ANDE2 (BOL)</td>
<td>-0.0119</td>
<td>0.312</td>
<td>0.375</td>
<td>0.231</td>
<td>1.42</td>
<td>8.644</td>
<td>-5.78</td>
</tr>
<tr>
<td>ANDE3 (BOL)</td>
<td>-0.01878</td>
<td>0.377</td>
<td>0.477</td>
<td>0.241</td>
<td>1.063</td>
<td>6.949</td>
<td>-6.47</td>
</tr>
</tbody>
</table>
Table A.3: Optimal number of bins ($n_{\text{bins}}$) computed with Rice rule, Doane’s formula, and Scott’s rule for each DEM analyzed in the twelve case studies. The value of $q$ obtained by fitting the power law tail of the distributions for the different values of $n_{\text{bins}}$ is also reported. $\mu_q$ is the mean of these three $q$ values and $\sigma_q$ is their standard deviation.

<table>
<thead>
<tr>
<th>Mountain Range</th>
<th>$n_{\text{bins}}$</th>
<th>$q$</th>
<th>$n_{\text{bins}}$</th>
<th>$q$</th>
<th>$n_{\text{bins}}$</th>
<th>$q$</th>
<th>$\mu_q \pm \sigma_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appalachian Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APPAL1 (NC)</td>
<td>296</td>
<td>-11.03</td>
<td>31</td>
<td>-10.89</td>
<td>729</td>
<td>-10.81</td>
<td>-10.91 ± 0.11</td>
</tr>
<tr>
<td>APPAL2 (NY)</td>
<td>296</td>
<td>-9.92</td>
<td>32</td>
<td>-10.51</td>
<td>498</td>
<td>-9.54</td>
<td>-9.99 ± 0.49</td>
</tr>
<tr>
<td>APPAL3 (CAN)</td>
<td>296</td>
<td>-9.1</td>
<td>33</td>
<td>-9.51</td>
<td>578</td>
<td>-8.86</td>
<td>-9.16 ± 0.33</td>
</tr>
<tr>
<td>Ozark Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OZA1 (AR)</td>
<td>296</td>
<td>-10.69</td>
<td>33</td>
<td>-10.51</td>
<td>417</td>
<td>-10.07</td>
<td>-10.42 ± 0.32</td>
</tr>
<tr>
<td>OZA2 (AR)</td>
<td>296</td>
<td>-10.15</td>
<td>33</td>
<td>-10.62</td>
<td>576</td>
<td>-9.98</td>
<td>-10.25 ± 0.33</td>
</tr>
<tr>
<td>OZA3 (AR)</td>
<td>296</td>
<td>-9.15</td>
<td>33</td>
<td>-9.52</td>
<td>415</td>
<td>-8.28</td>
<td>-8.98 ± 0.64</td>
</tr>
<tr>
<td>Central Massif</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FRA1 (FRA)</td>
<td>296</td>
<td>-11.7</td>
<td>32</td>
<td>-11.97</td>
<td>476</td>
<td>-11.3</td>
<td>-11.66 ± 0.34</td>
</tr>
<tr>
<td>FRA2 (FRA)</td>
<td>296</td>
<td>-9.87</td>
<td>33</td>
<td>-10.17</td>
<td>548</td>
<td>-8.43</td>
<td>-9.49 ± 0.93</td>
</tr>
<tr>
<td>FRA3 (FRA)</td>
<td>296</td>
<td>-8.5</td>
<td>33</td>
<td>-8.68</td>
<td>651</td>
<td>-7.87</td>
<td>-8.35 ± 0.43</td>
</tr>
<tr>
<td>Flint Hills</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FLINT1 (KS)</td>
<td>296</td>
<td>-8.46</td>
<td>33</td>
<td>-9.26</td>
<td>1016</td>
<td>-7.17</td>
<td>-8.30 ± 1.05</td>
</tr>
<tr>
<td>FLINT2 (KS)</td>
<td>296</td>
<td>-7.58</td>
<td>33</td>
<td>-8.23</td>
<td>1186</td>
<td>-5.59</td>
<td>-7.13 ± 1.38</td>
</tr>
<tr>
<td>FLINT3 (KS)</td>
<td>296</td>
<td>-6.57</td>
<td>33</td>
<td>-7.57</td>
<td>683</td>
<td>-6.88</td>
<td>-7.01 ± 0.51</td>
</tr>
<tr>
<td>Urals</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>URA1 (RUS)</td>
<td>296</td>
<td>-6.92</td>
<td>33</td>
<td>-7.48</td>
<td>659</td>
<td>-5.84</td>
<td>-6.75 ± 0.83</td>
</tr>
<tr>
<td>URA2 (RUS)</td>
<td>296</td>
<td>-10.17</td>
<td>33</td>
<td>-11.04</td>
<td>536</td>
<td>-8.7</td>
<td>-9.97 ± 1.18</td>
</tr>
<tr>
<td>URA3 (RUS)</td>
<td>296</td>
<td>-6.28</td>
<td>34</td>
<td>-7.72</td>
<td>813</td>
<td>-5.81</td>
<td>-6.6 ± 1.00</td>
</tr>
<tr>
<td>Hamersley Range</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HAMER1 (AUS)</td>
<td>296</td>
<td>-7.08</td>
<td>34</td>
<td>-7.14</td>
<td>632</td>
<td>-6.37</td>
<td>-6.86 ± 0.43</td>
</tr>
<tr>
<td>HAMER2 (AUS)</td>
<td>296</td>
<td>-6.22</td>
<td>34</td>
<td>-6.25</td>
<td>778</td>
<td>-5.97</td>
<td>-6.15 ± 0.15</td>
</tr>
<tr>
<td>HAMER3 (AUS)</td>
<td>296</td>
<td>-6.18</td>
<td>34</td>
<td>-6.23</td>
<td>781</td>
<td>-5.99</td>
<td>-6.13 ± 0.13</td>
</tr>
<tr>
<td>Rocky Mts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ROCKY1 (CO)</td>
<td>296</td>
<td>-5.01</td>
<td>34</td>
<td>-5.06</td>
<td>1101</td>
<td>-4.93</td>
<td>-5.00 ± 0.07</td>
</tr>
<tr>
<td>ROCKY2 (CO)</td>
<td>296</td>
<td>-5.54</td>
<td>33</td>
<td>-5.62</td>
<td>851</td>
<td>-5.51</td>
<td>-5.56 ± 0.06</td>
</tr>
<tr>
<td>ROCKY3 (CO)</td>
<td>296</td>
<td>-6.09</td>
<td>33</td>
<td>-6.16</td>
<td>1128</td>
<td>-5.96</td>
<td>-6.07 ± 0.10</td>
</tr>
<tr>
<td>Himalayas</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HIMAL1 (IND)</td>
<td>296</td>
<td>-6.99</td>
<td>33</td>
<td>-7.01</td>
<td>758</td>
<td>-6.85</td>
<td>-6.95 ± 0.09</td>
</tr>
<tr>
<td>HIMAL2 (NPL)</td>
<td>296</td>
<td>-6.76</td>
<td>33</td>
<td>-6.9</td>
<td>1248</td>
<td>-6.67</td>
<td>-6.78 ± 0.12</td>
</tr>
<tr>
<td>HIMAL3 (IND)</td>
<td>296</td>
<td>-6.64</td>
<td>33</td>
<td>-6.66</td>
<td>814</td>
<td>-6.52</td>
<td>-6.61 ± 0.08</td>
</tr>
<tr>
<td>Apennines</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>APENN1 (ITA)</td>
<td>296</td>
<td>-7.23</td>
<td>32</td>
<td>-7.3</td>
<td>959</td>
<td>-6.84</td>
<td>-7.12 ± 0.25</td>
</tr>
<tr>
<td>APENN2 (ITA)</td>
<td>296</td>
<td>-6.03</td>
<td>33</td>
<td>-6.11</td>
<td>930</td>
<td>-5.89</td>
<td>-6.01 ± 0.11</td>
</tr>
<tr>
<td>APENN3 (ITA)</td>
<td>296</td>
<td>-6.96</td>
<td>33</td>
<td>-7.18</td>
<td>970</td>
<td>-6.59</td>
<td>-6.91 ± 0.30</td>
</tr>
<tr>
<td>Ethiopian Highlands</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ETH1 (ETH)</td>
<td>296</td>
<td>-5.1</td>
<td>33</td>
<td>-5.19</td>
<td>926</td>
<td>-5.03</td>
<td>-5.11 ± 0.08</td>
</tr>
<tr>
<td>ETH2 (ETH)</td>
<td>296</td>
<td>-5.81</td>
<td>33</td>
<td>-5.85</td>
<td>1083</td>
<td>-5.62</td>
<td>-5.76 ± 0.12</td>
</tr>
<tr>
<td>ETH3 (ETH)</td>
<td>296</td>
<td>-6.49</td>
<td>33</td>
<td>-6.56</td>
<td>904</td>
<td>-6.39</td>
<td>-6.48 ± 0.09</td>
</tr>
<tr>
<td>Alps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ALP1 (ITA-FRA)</td>
<td>296</td>
<td>-5.18</td>
<td>34</td>
<td>-5.24</td>
<td>1112</td>
<td>-5.13</td>
<td>-5.18 ± 0.06</td>
</tr>
<tr>
<td>ALP2 (ITA-AUT)</td>
<td>296</td>
<td>-4.83</td>
<td>34</td>
<td>-4.92</td>
<td>1106</td>
<td>-4.86</td>
<td>-4.87 ± 0.05</td>
</tr>
<tr>
<td>ALP3 (ITA-CHE)</td>
<td>296</td>
<td>-5.64</td>
<td>34</td>
<td>-5.88</td>
<td>2436</td>
<td>-5.55</td>
<td>-5.69 ± 0.17</td>
</tr>
<tr>
<td>Andes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ANDE1 (BOL)</td>
<td>296</td>
<td>-5.72</td>
<td>33</td>
<td>-5.85</td>
<td>919</td>
<td>-5.5</td>
<td>-5.69 ± 0.18</td>
</tr>
<tr>
<td>ANDE2 (BOL)</td>
<td>296</td>
<td>-5.73</td>
<td>33</td>
<td>-5.8</td>
<td>826</td>
<td>-5.69</td>
<td>-5.74 ± 0.06</td>
</tr>
<tr>
<td>ANDE3 (BOL)</td>
<td>296</td>
<td>-6.39</td>
<td>33</td>
<td>-6.47</td>
<td>927</td>
<td>-6.33</td>
<td>-6.39 ± 0.07</td>
</tr>
</tbody>
</table>
Figure A.1: Slope values over the domain located in the Alps for the initial condition (a) and for scenarios SC1 (b, c, d), SC3 (e, f, g), SC5 (h, i, j), and SC7 (k, l, m) at 150, 300, and 400 Mya.
Figure A.2: Slope values over the domain located in the Himalayas for the initial condition (a) and for scenarios SC1 (b, c, d), SC3 (e, f, g), SC5 (h, i, j), and SC7 (k, l, m) at 150, 300, and 400 Mya.
Appendix B

Landscape aging process:
cross-CZO analysis and effect of DEM grid size

In this Appendix, a number of details about the statistical analysis of the landscape aging process discussed in Chapter 3 are presented. First, information about the Critical Zone Observatories (CZO) analyzed and results for the gradient and slope distribution analysis are provided. Subsequently, the effect of different DEM sizes and resolutions on slope statistics is discussed.

\textbf{Figure B.1:} Locations of the CZOs analyzed.
Table B.1: Locations and statistics of the CZOs analyzed. Latitude and longitude for the central position of each DEM are given (each DEM covers an area of 0.5 by 0.5 decimal degrees). Values of elevation mean $\mu_z$ and standard deviation $\sigma_z$, slope mean $\mu_s$, standard deviation $\sigma_S$, skewness $sk_S$, and kurtosis $k_S$, and power-law exponent $\beta$ for each DEM analyzed are also provided.

<table>
<thead>
<tr>
<th>Observatory</th>
<th>Lat.</th>
<th>Long.</th>
<th>$\mu_z$ [m]</th>
<th>$\sigma_z$ [m]</th>
<th>$\mu_S$ [m/m]</th>
<th>$\sigma_S$ [m/m]</th>
<th>$sk_S$ [-]</th>
<th>$k_S$ [-]</th>
<th>$\beta$ [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boulder Creek, CO</td>
<td>39.5</td>
<td>-105.55</td>
<td>2846.01</td>
<td>457.19</td>
<td>0.32</td>
<td>0.21</td>
<td>1.85</td>
<td>12.86</td>
<td>-5.35</td>
</tr>
<tr>
<td>Santa Catalina, AZ</td>
<td>32.45</td>
<td>-110.75</td>
<td>1187.88</td>
<td>379.11</td>
<td>0.19</td>
<td>0.20</td>
<td>1.97</td>
<td>10.39</td>
<td>-6.12</td>
</tr>
<tr>
<td>Southern Sierra, CA</td>
<td>36.75</td>
<td>-118.65</td>
<td>4194</td>
<td>419</td>
<td>0.46</td>
<td>0.31</td>
<td>2.01</td>
<td>12.84</td>
<td>-5.01</td>
</tr>
<tr>
<td>Reynolds Creek, ID</td>
<td>43.05</td>
<td>-116.85</td>
<td>1577.63</td>
<td>257.76</td>
<td>0.19</td>
<td>0.15</td>
<td>1.29</td>
<td>5.46</td>
<td>-8.66</td>
</tr>
<tr>
<td>Calhoun, SC</td>
<td>34.55</td>
<td>-81.75</td>
<td>164.48</td>
<td>26.18</td>
<td>0.08</td>
<td>0.05</td>
<td>1.14</td>
<td>5.24</td>
<td>-9.85</td>
</tr>
<tr>
<td>Shale Hills, PA</td>
<td>40.65</td>
<td>-77.85</td>
<td>364.68</td>
<td>114.39</td>
<td>0.16</td>
<td>0.13</td>
<td>1.34</td>
<td>5.01</td>
<td>-10.66</td>
</tr>
</tbody>
</table>

B.1 Study sites: Critical Zone Observatories

The six CZOs analyzed are located in the Continental US (see Figure B.1). In addition to the Calhoun CZO, other five observatories were analyzed (DEM are shown in Figure B.2): the Shale Hills CZO located within the Valley and Ridge Physiographic Province of the Appalachian Mountains in central Pennsylvania, the Boulder Creek CZO in the southern Rocky Mountains, the Reynolds Creek CZO in Southwest Idaho, the Southern Sierra CZO in California, and the Catalina-Jemez CZO in the Santa Catalina Mountains, Arizona. For each CZO, we used a NASA SRTM Digital Elevation Model (DEM) covering an area of approximately 3000 km$^2$, with a resolution of 1-arcsec and elevation values having an accuracy to within ±0.5 m, coherently with the analysis performed in Chapter 2.

B.2 Gradients and slope distributions

We studied the gradients and slope distributions of the Calhoun CZO, as well as of other five CZOs (listed in Table B.1) located in the continental United States and characterized by different stages of geomorphic development (see previous section).

No clear trend with age is observed in mean, variance, and skewness of the slope distributions, while their kurtosis tends to be higher for the younger landscapes of the West and Mid-West (see Table B.1), coherently with results presented in
Figure B.2: DEMs of the six CZOs analyzed.

Figure B.3: (a-e) Gradients ($\partial_z$) and (f-j) slope ($S$) distributions (gray symbols) of the CZOs (30-m DEMs) are compared to Gaussian and Rayleigh distributions (blue lines), respectively. The logarithmic plot of slope distributions shows the power law scaling of the pdf tail: the power-law exponents $\beta$ are computed by fitting the power law tail (black line) between $p(S) = 10^{-3.2}$ and $p(S) = 10^{-0.95}$ (dashed lines).

Chapter 2. Figure B.3 shows gradients and slope distributions for the additional five CZOs analyzed: results are compared to Gaussian and Rayleigh distributions and the exponents of the power law tails are computed. The tails of gradients and slope distributions (shown in Figure B.4 for the Calhoun CZO and Figure B.3 for the other observatories) tend to approach Gaussian and Rayleigh distributions with age, with
Figure B.4: (a) Gradient ($\partial_z$) and (b) slope ($S$) distributions (gray symbols) for the Calhoun CZO (30-m DEM) are compared to Gaussian and Rayleigh distributions (blue lines), respectively (results for the other five CZOs are shown in the SI). The logarithmic plot of the slope distributions shows the power law scaling of the pdf tail: the power-law exponent $\beta = -9.85$ is computed by fitting the power law tail (black line) between $p(S) = 10^{-3.2}$ and $p(S) = 10^{-0.95}$ (dashed lines), as in Chapter 2 (denoted as BP17 in the figure). (c) The power-law exponents computed for all the CZOs are used to infer approximate landscape ages based on the rate of decay $d\beta/dT$ (black line). Gray diamonds show results from BP17 (dark gray for the Appalachians).

systematically higher values of the power-law exponent $\beta$ for younger landscapes. In particular, for the Calhoun CZO the exponent $\beta$ is equal to -9.85 (Figure B.4b), coherently with the old landscape of the Piedmont region and within the error range typical of the Appalachian mountains (Figure B.4c).

Using the rate of landscape aging provided in Chapter 2, we computed approximate ages for the landscapes of the six CZOs analyzed (Figure B.4c), obtaining reasonable age estimates, especially for the old landscapes of Calhoun and Shale Hills in the Appalachians, and for the younger topographies of the West and Mid-West.

154
Figure B.5: (a) Spatial extent of the 1 and 3 arcsec resolution DEMs. (b) A 1-m resolution LiDAR topography is available on a smaller portion of the domain. (c) Slope distributions for the 1-m (orange), 1-arcsec (gray), and 3-arcsec (orange) DEMs: power-laws were fitted to the tails of the distributions, resulting in very similar $\beta$ values for the three resolutions analyzed. The power-law exponent is computed by fitting the power law tail between $p(S) = 10^{-1.4}$ and $p(S) = 10^{-1.8}$, $p(S) = 10^{-3.2}$ and $p(S) = 10^{-0.95}$, $p(S) = 10^{-2.8}$ and $p(S) = 10^{-0.2}$ for the 1-m, 1-arcsec, and 3-arcsec DEM, respectively (dashed lines).

B.3 Effect of DEM resolution on slope distribution tails

To analyze the effect of DEM resolution on slope statistics, in addition to the 1-arcsec resolution DEM, we use a NASA SRTM 3-arcsec resolution DEM covering the same area. Furthermore, a smaller 1-m resolution LiDAR topography is available for a smaller area of the Calhoun CZO (see Figure B.5a,b).

As expected, mean slope values tend to increase with increasing resolution (Table B.2), creating a smoother landscape (Thompson et al., 2001; Guoan et al., 2003). Analogously, standard deviation, skewness, and kurtosis tend to increase with the DEM resolution in relation to the more moderate slope values in lower resolution DEMs, as evident from the slope distributions shown in Figure B.5c. The higher resolution DEMs display higher values of slopes, allowing to detect small scale features that are smoothed away in the lower resolution DEMs, thus shifting the interval of slope values where the power-law behavior is displayed (Figure B.5c, dashed lines).
Table B.2: Mean $\mu_S$, standard deviation $\sigma_S$, skewness $sk_S$, and kurtosis $k_S$ of slopes together with the power-law exponent $\beta$ computed for the three Calhoun CZO DEMs having different grid resolutions.

<table>
<thead>
<tr>
<th>DEM resolution</th>
<th>$\mu_S$ [m m$^{-1}$]</th>
<th>$\sigma_S$ [m m$^{-1}$]</th>
<th>$sk_S$</th>
<th>$k_S$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 arcsec</td>
<td>0.051</td>
<td>0.029</td>
<td>0.939</td>
<td>4.281</td>
<td>-9.79</td>
</tr>
<tr>
<td>1 arcsec</td>
<td>0.078</td>
<td>0.047</td>
<td>1.141</td>
<td>5.242</td>
<td>-9.85</td>
</tr>
<tr>
<td>1 m</td>
<td>0.139</td>
<td>0.139</td>
<td>3.254</td>
<td>16.481</td>
<td>-10</td>
</tr>
</tbody>
</table>

It is interesting to note, however, that the values of the power-law exponent $\beta$ are comparable among the three cases (-10, -9.85, and -9.79 for the 1-m, 1-arcsec, and 3-arcsec DEMs, respectively), suggesting that the behavior of the tails is a quite robust metric across various ranges of resolutions. Thus, while the moments of the slope distributions show a neat correlation with the DEM resolution, the value of the power-law exponent $\beta$ seems to be a consistent measure of landscape geomorphological stage.

In the DEM resolution analysis, the 1-m resolution DEM covers a smaller area compared to the 1- and 3-arcsec DEMs. To assess the effect of accounting for a smaller portion of the DEM, we computed the slope distribution for the 1-arcsec DEM considering only the points inside the smaller domain (blue pdf in Figure B.6). The pdf resembles the one on the bigger domain, showing that the overall behavior in different areas of the domain is the same and the results of the 1-arcsec DEM can be regarded as an average behavior of the landscape. We also computed the slope pdf (green symbols in Figure B.6) obtained randomly sampling from the 1-m DEM a number of points equal to the points of the 1-arcsec DEM on the smaller domain. Again, the slope pdf resembles the one obtained considering all the points in the 1-m DEM. Note, however, that when a smaller number of points is considered, the full tail of the distribution cannot be fully recovered, as some of the higher (and less frequent) slope values are neglected.
Figure B.6: Effect of DEM size on slope distributions. Slope pdfs computed from 1-arcsec and 1m resolution DEMs (gray and orange, respectively). The blue pdf is obtained computing the slope statistics of the 1 arcsec DEM on the smaller domain of $\approx 80$ km$^2$. Lastly, the slope pdf obtained randomly sampling from the 1-m DEM is shown in green.
Appendix C

TL and DL regimes: sensitivity analysis and additional results

In this Appendix numerical steady state profiles for different values of the exponents $m$ and $n$ in TL and DL regimes (see Chapter 3) are provided. Furthermore, additional results from the application of the mathematical model to the Calhoun CZO are presented.

C.1 TL and DL regimes: effect of slope and water flow exponents

Numerical steady state profiles obtained for different values of the slope and water flow exponents in TL and DL conditions are shown in Figure C.1 and C.2, respectively.
Figure C.1: Effect of changing $n_T$ (top panels) and $m_T$ (bottom panels) on the steady state profiles for the TL case: results are shown for $U = R = K_a = 1$ and different values of the diffusion coefficient $D$.

Figure C.2: Effect of changing $n_D$ (top panels) and $m_D$ (bottom panels) on the steady state profiles for the DL case: results are shown for $U = R = K_a = 1$ and different values of the diffusion coefficient $D$.

C.2 Application to the Calhoun CZO: additional results

Results obtained from fitting the DL analytical steady state solution to the five hillslopes in the undisturbed and disturbed areas are shown in Figure C.3 and C.4, respectively. The time evolution of disturbed hillslopes D1, D2, D4, and D5 and the associated soil creep and runoff erosion rates are shown in Figures C.5-C.8. The
Figure C.3: (a-e) Hillslopes in the undisturbed section: blue symbols are digitized values, blue line is the fitted DL analytical profile, obtained assuming $U = 2 \cdot 10^{-5}$ m/year and $R = 1100$ mm/year. Fitting parameters $D$ and $K_D$ have units of $m^2$ yr$^{-1}$ and yr$^{-1}$, respectively. The secondary axis shows the surface curvature for the fitted hillslope and the yellow shading highlights the concave-up portion of the hillslope (i.e., positive curvature). (f-j) Slope values computed from the digitized section (symbols) are compared to the analytical solution. (k-o) Theoretical soil creep (dashed line) and runoff erosion (solid line) sediment fluxes. Green and red shading represent $f_D$ and $f_c$ dominated portions, respectively.

The landscape evolution equation in DL conditions is numerically solved starting from an initial condition equal to the fitted disturbed hillslopes (D1-5), and assuming $U = 2 \cdot 10^{-5}$ m/yr, $R = 1100$ mm/yr, and $n_D = m_D = 1$. The coefficients $D$ and $K_D$ were set equal to the average values of the parameters obtained through the fitting procedure for the undisturbed sections (i.e., $D = 0.0035$ m$^2$ yr$^{-1}$ and $K_D = 7.19 \cdot 10^{-7}$ yr$^{-1}$).
Figure C.4: (a-e) Hillslopes in the disturbed section: blue symbols are digitized values, blue line is the fitted DL analytical profile, obtained assuming $U = 2 \cdot 10^{-5}$ m/year and $R = 1100$ mm/year. Fitting parameters $D$ and $K_D$ have units of m$^2$ yr$^{-1}$ and yr$^{-1}$, respectively. The secondary axis shows the surface curvature for the fitted hillslope and the yellow shading highlights the concave-up portion of the hillslope (i.e., positive curvature). (f-j) Slope values computed from the digitized section (symbols) are compared to the analytical solution. (k-o) Theoretical soil creep (dashed line) and runoff erosion (solid line) sediment fluxes. Green and red shading represent $f_d$ and $f_c$ dominated portions, respectively.
Figure C.5: Time evolution of (a) hillslope D1 and (b-c) erosion rates associated with creep ($r_d$) and runoff erosion ($r_c$). Results are obtained assuming $D = 0.0035 \text{ m}^2 \text{ yr}^{-1}$ and $K_D = 7.19 \cdot 10^{-7} \text{ yr}^{-1}$ (i.e., average values obtained from fitting the undisturbed sections), $U = 2 \cdot 10^{-5} \text{ m/yr}$, and $R = 1100 \text{ mm/yr}$.

Figure C.6: Time evolution of (a) hillslope D2 and (b-c) erosion rates associated with creep ($r_d$) and runoff erosion ($r_c$). Results are obtained assuming $D = 0.0035 \text{ m}^2 \text{ yr}^{-1}$ and $K_D = 7.19 \cdot 10^{-7} \text{ yr}^{-1}$ (i.e., average values obtained from fitting the undisturbed sections), $U = 2 \cdot 10^{-5} \text{ m/yr}$, and $R = 1100 \text{ mm/yr}$.
Figure C.7: Time evolution of (a) hillslope D4 and (b-c) erosion rates associated with creep ($r_d$) and runoff erosion ($r_c$). Results are obtained assuming $D = 0.0035 \text{ m}^2 \text{ yr}^{-1}$ and $K_D = 7.19 \times 10^{-7} \text{ yr}^{-1}$ (i.e., average values obtained from fitting the undisturbed sections), $U = 2 \times 10^{-5} \text{ m/yr}$, and $R = 1100 \text{ mm/yr}$.

Figure C.8: Time evolution of (a) hillslope D5 and (b,c) erosion rates associated with creep ($r_d$) and runoff erosion ($r_c$). Results are obtained assuming $D = 0.0035 \text{ m}^2 \text{ yr}^{-1}$ and $K_D = 7.19 \times 10^{-7} \text{ yr}^{-1}$ (i.e., average values obtained from fitting the undisturbed sections), $U = 2 \times 10^{-5} \text{ m/yr}$, and $R = 1100 \text{ mm/yr}$. 

163
Appendix D

Detection of critical points

Where the locations of the critical points in Chapter 4 could not be computed analytically, the following numerical procedure is adopted. At each point \((x_g, y_g)\) of the discretized surface a second order polynomial is fitted to the nine points comprising \((x_g, y_g)\) and the eight neighbors (see, for example, Figure D.1b,c). The polynomial surface has equation

\[
Z(x, y) = p_1 + p_2 x + p_3 y + p_4 x^2 + p_5 xy + p_6 y^2. \tag{D.1}
\]

For each surface point the critical point \((x_c, y_c)\) of the fitted surface can be easily found imposing \(-\nabla Z = 0\), where

\[
- \nabla Z = - (p_2 + 2p_4 x + p_5 y)i - (p_3 + 2p_6 y + p_5 x)j. \tag{D.2}
\]

If the critical point lies between \(x_g \pm dx/2\) and \(y_g \pm dy/2\) then the grid point is recognized as a critical point (Figure D.1a). The nature of the critical point is then defined by means of the Jacobian of the vector field (D.2)

\[
J(x_c, y_c) = \begin{bmatrix}
-2p_4 & -p_5 \\
-p_3 & -2p_6
\end{bmatrix}. \tag{D.3}
\]
Figure D.1: Numerical detection of critical points: (a) if the critical point of the fitted polynomial surface lies between $x_g \pm dx/2$ and $y_g \pm dy/2$ (i.e., gray shaded region) then the grid point $(x_g, y_g)$ is recognized as a critical point. Examples of the fitted second order polynomials (red) at a local maximum (b) and saddle (c).

which has trace $\tau = -2(p_4 + p_6)$ and determinant $\Delta = 4p_4p_6 - p_5^2$. When $\Delta < 0$ the critical point is a saddle, when $\Delta > 0$ and $\tau > 0$ it is a surface maximum, and when $\Delta > 0$ and $\tau < 0$ it is a surface minimum.
Bibliography


Boussinesq, J. (1871), “Sur une propriété remarquable des points ou les lignes de plus grande pente d’une surface ont leurs plans osculateurs verticaux, et sur la différence qui existe généralement, à la surface de la terre, entre les lignes de faîte et de thalweg et celles le long desquelles la pente du sol est un minimum,” *CR Paris*, 73, 1368.


182


Biography

Sara Bonetti was born in Villafranca di Verona, Verona (Italy) on March 14th, 1987. Sara earned a Bachelor of Science Degree in Civil Engineering in 2009 and a Master of Science Degree in Civil Engineering in 2011, both at University of Padova (Italy). In 2012 she worked as a civil engineer for the company M3E s.r.l. and as a researcher at University of Padova (Dept. of Mathematics). In November 2012 she moved to Durham, NC (USA) working for one year as a research assistant at Duke University (Nicholas School of the Environment) modeling soil-plant-atmosphere interactions (Manoli et al., 2014; Bonetti et al., 2015). She then returned to Italy, where she attained a professional license to practice as civil engineer and worked as a research assistant (March-May 2014) at University of Padova (Dept. of Civil and Environmental Engineering) on modeling crop yield (Manoli et al., 2015). In August 2014 Sara began her PhD at the Dept. of Civil and Environmental Engineering at Duke University. Initial work focused on the development of a minimalist plant diversity model (Bonetti et al., 2017a). Eventually, Sara started working on landscape statistics and evolution (Bonetti and Porporato, 2017; Bonetti et al., 2018b,a). She also worked on the derivation of Manning equation (Bonetti et al., 2017b) and plant competition (Manoli et al., 2017). In 2018 Sara won the Prof. Utku Award (highest distinction) for best pre-Ph.D. peer-reviewed journal paper (Bonetti and Porporato, 2017). Sara will start a post-doc at ETH Zurich, Switzerland (Dept. of Soil and Terrestrial Environmental Physics) in August 2018.