Aerodynamic Optimization of Helicopter Rotors using a Harmonic Balance Lifting Surface Technique

by

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Thomas Witelski

Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mechanical Engineering and Materials Science in the Graduate School of Duke University 2018
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Abstract

This thesis concerns the optimization of the aerodynamic performance of conventional helicopter rotors, given a set of design variables to control the rotor’s pitching angle, twist and chord distributions. Two models are presented for use. The lifting line model is a vortex lattice model that uses assumptions on the size and shape of the blade to simplify the model, but is unable to account for unsteady and small aspect ratio effects. The lifting surface model removes these assumptions and allows for a wider variety of accurate solutions, at the cost of overall computational complexity. The lifting surface model is chosen for analysis, and then condensed using static condensation and harmonic balance. The final system is discretized and pertinent values of power, force, and moment calculated using Kelvin’s theorem and the unsteady Bernoulli equation. This system is then optimized in one of two ways: using a direct linear solve if possible, or the open source package IPOPT where necessary. The results of single-point and multi-point optimization demonstrate for low speed forward flight, the lifting line model is sufficient for modeling purposes. As the speed of the rotor increases, more unsteady effects become prominent in the system, and therefore the lifting surface model becomes more necessary. When conducting a chord optimization on the rotor, hysteresis effects and local minima are calculated for the non-linear optimization. The global minima within the set of captured local minima can be found through simple data visualization, and the global minima is confirmed to have similar behavior to the results of lifting line; a large spike in in-
duced power at a critical advance ratio, with a sharp decline in induced power as the rotor flies faster. Within the realm of practical forward flight speeds of a conventional rotor, smooth, continuous results are demonstrated.
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List of Abbreviations and Symbols

When reading the following symbols, it is general rule of thumb that boldfaced quantities are vector or matrix quantities, while normal fonts are indicative of scalars or other operators.

For certain symbols with multiple meanings, it should be apparent how it is being used in context.

Symbols

\[ A \] Influence Coefficient Matrix
\[ A \] Dimensional Area
\[ B \] Force Matrix / Wake-Airfoil Relationship Matrix
\[ B \] Number of Blades
\[ b \] Perscribed Normal Fluid Velocity
\[ C \] Coefficient of Element
\[ c \] Blade Chord
\[ D \] Moment Matrix
\[ E \] Fourier transform
\[ E \] Kinetic Energy
\[ F \] Force Vector
\[ K \] Induced Power Matrix
\[ k \] Reduced Frequency
\( \ell \)  Generic Length

\( M \)  Moment Vector

\( N \)  Number of Element

\( n \)  Normal Vector

\( m, n \)  Integer Multiples

\( P \)  Dimensional Power

\( R \)  Other Constraint Function

\( R \)  Blade Radius

\( r / r \)  Generic location

\( t \)  Time

\( U \)  Local First order Velocity

\( w / w \)  Local Second order Velocity (Induced Velocity)

\( \alpha \)  Angle of Attack

\( \Gamma \)  Circulation Element

\( \Theta \)  Rotor design variables vector

\( \theta \)  Rotor pitch / Rotor twist

\( \lambda \)  Lagrange Multiplier

\( \mu \)  Forward Advance Ratio

\( \Pi \)  Functional

\( \rho \)  Density of the Fluid

\( \psi \)  Azimuthal angle

\( \Omega \)  Rotor angular velocity

Subscripts, Superscripts, and Operators

\( (\bullet)_{1,2,...} \)  Condition 1,2,...

\( (\bullet)_{a} \)  Airfoil values
\((\bullet)\)_c \quad \text{Chord-wise Stations}

\((\bullet)\)\text{cutout} \quad \text{Cutout of blade}

\((\bullet)\)\text{coeff} \quad \text{Coefficients}

\((\bullet)\)_{i,n} \quad \text{Index, quantity at cell i,n}

\((\bullet)\)_L \quad \text{Lift / Lagrange Lift Constraint}

\((\bullet)\)_M \quad \text{Lagrange Moment Constraint}

\((\bullet)\)_{Mx} \quad \text{x-Moment or Rolling Moment}

\((\bullet)\)_{My} \quad \text{y-Moment or Pitching Moment}

\((\bullet)\)_N \quad \text{Last element of a quantity / Near Field quantity}

\((\bullet)\)_{Pi} \quad \text{Induced Power}

\((\bullet)\)\text{R} \quad \text{Required Value / Lagrange Other Constraint / Radial Stations}

\((\bullet)\)\text{shaft} \quad \text{Shaft value}

\((\bullet)\)\text{T} \quad \text{Thrust}

\((\bullet)\)\text{veh} \quad \text{Total Vehicle value}

\((\bullet)\)\text{w} \quad \text{Wake values}

\((\bullet)\)\Gamma_a \quad \text{Derivative with respect to } \Gamma_a

\((\bullet)\)\Gamma_w \quad \text{Derivative with respect to } \Gamma_w

\((\bullet)\)\Theta \quad \text{Derivative with respect to } \Theta

\((\bullet)\)\psi \quad \text{Azimuthal Stations}

\((\bullet)\)\text{a} \quad \text{Modeled Time domain data}

\((\hat{\bullet})\) \quad \text{Spectral domain data / Modified representation of quantity / Unit vector}

\((\bullet)\)^n \quad \text{Time step n}

\Delta(\bullet) \quad \text{Change in a quantity}

\nabla(\bullet) \quad \text{Gradient Operator}

\nabla^2(\bullet) \quad \text{Laplacian operator}
\[ \frac{\partial}{\partial t}(\bullet) \] Time Derivative

\[(\bullet)^{-1} \] Inverse Operation

\[(\bullet)^T \] Transpose Operation

\[||\bullet||\] Magnitude

\[(\bullet) \cdot (\bullet)\] Dot Product Operator

\[(\bullet) \times (\bullet)\] Cross Product Operator

Abbreviations

AR Aspect Ratio \((c/2R)\)

FF Far Field

HHC Higher Harmonic Control

IPOPT Internal Point OPTimizer

LL Lifting Line

LS Lifting Surface

MP Multi-Point

NF Near Field

QP Quadratic Programming

SP Single Point

rev per-rev, used in controls
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Helicopters are unique aircraft that are able to perform a multitude of applications that fixed-wing aircraft cannot. Their defining feature is one or more main rotors with the axis of rotation oriented vertically with respect to the vehicle, making it possible to hover and to land and takeoff vertically – an important capability in certain military and civilian applications. The use of a single main rotor, however, produces a laterally asymmetric wake, which in turn results in high induced power requirements at higher forward flight speeds [3]. Modeling the physics of high speed forward flight is critical to the design and analysis of efficient high speed rotorcraft.

1.1 Previous Work

For propellers in axial flight and rotors in forward flight, an important issue is the minimum power required to generate a required thrust or lift, and the corresponding circulation distribution and blade design that achieves this optimal solution. Classical solutions for the problem of minimum induced loss propellers have been known since the 1900s. Betz [7, 8] found the optimum circulation distribution for a propeller modeled using momentum theory, including swirl in the wake of the rotor.
In momentum theory, the propeller is modeled as an actuator disk, so the effect of blade count is not modeled. Prandtl and Betz [9] used momentum theory with an approximate model of the effect of finite blade counts – and thus discrete wake sheets in the wake of the rotor – to find the optimum circulation distribution on propellers with a finite number of blades. Goldstein [10] cast the optimization problem as a solution to a two-dimensional partial differential equation of the flow in the wake of a propeller that satisfies the Betz [11] optimality condition for propellers. This PDE was solved using a series solution technique to obtain the exact optimal circulation distribution for lightly loaded propellers. Betz [12] further found the optimal circulation distribution for a rotor in hover using momentum theory with swirl.

While optimal solutions for propellers and rotor in hover have been known since the early 1900s, only recently have optimal solutions been known for helicopters in forward flight. For helicopter rotors generating lifting in forward flight, Hall, Yang, and Hall [13] found the optimal circulation for lifting rotors in forward flight. Their analysis used a finite element discretization of the potential equations to solve for the flow between the wake sheets trailing behind the rotor. For optimality, it was determined that “the vorticity distribution in the wake is the same as that produced by the flow past an impermeable membrane...which translates at speed $v$ and shears at angular rate $\omega$...These conditions are similar to Betz’s rigid wake condition for propellers.” This approach is similar to Goldstein’s solution for the potential flow between the helical wake sheets of a propeller. Using this analysis, Hall, Yang, and Hall found the optimal circulation distribution for a three-bladed rotor in forward flight at moderate advance ratios.

These calculations of the optimal circulation distribution for a helicopter in forward flight provides the lower bound on the power required for flight. In general, the optimum circulation distribution will vary along the span of the blade, and will also vary with azimuthal position. It may not be possible to realize the optimum
circulation distribution with a real rotor, since they have a fixed chord and twist distribution along the span of the blades, limited control of the blade pitch, and typically just collective and cyclic controls (0/rev and 1/rev control). Giovanetti and Hall [14] extended the “rubber rotor” optimization to include an additional constraint, i.e., that circulation must be realizable using design variables and control inputs. The aerodynamics were modeled using a vortex-lattice model of the wake, a lifting-line model of the rotor blades, and a drag polar representation of the airfoil section to model viscous effects. The rotor design that minimized the induced power was found using this approach. For conventional rotors, it was found that realizable rotors require significantly more power than rubber rotors. For co-axial counter rotating rotors, the rubber rotor and realizable rotor powers were found to be more comparable, suggesting better efficiency in co-axial rotors like the rotor of the X2 experimental aircraft in Figure 1.3.

Recent approaches like those taken by Giovanetti, Hall, and Peters [3, 14, 15] require the use of numerical models of the rotor and its wake. Unlike the necessarily simplified models that could only undergo specific motions used by Betz and Goldstein, these numerical models allow for more flexibility in the motion of the vehicle,
and allow for more constraints (e.g, aeroelastic, acoustic, and material) to be added to the optimization. CAMRAD [16] is a so-called “comprehensive” numerical modeling program developed by NASA. This include free-wake modeling, large angular motions, and aeroelastic coupling. If less fidelity is required, a lifting line model can be used in its place. The lifting line model is less computationally expensive, and in most forward flight, axial flight, and hover cases has been demonstrated to be sufficient [13, 17, 18, 19].
It is also important to consider the optimization of multiple objectives of the helicopter of the rotor simultaneously. All helicopters undergo mission profiles with different elements: climb, cruise at various speeds, hover, and descent. Johnson [20], Moodie and Yeo [21], and Massaro and Benini [22] have conducted multi-objective work on conventional and compound rotor configurations by investigating a single design point either using a vortex lattice model or a comprehensive code like CAMRAD [20, 22]. They varied the design slightly to check sensitivity to the change in design. Another approach, taken by Rand and Khromov [2, 23], involves a Monte Carlo-like selection of design variables, then selecting the most optimum design points after analysis of all designs is complete. For example, Figure 1.4 depicts the Monte-Carlo simulation of the power of a rotor in hover and cruise; each set of colored points represents the simulation for a given forward flight velocity. After conducting a large amount of simulations for different sets of parameters where the power is calculated for both conditions, points are chosen where changing to another point would necessarily cause an increase in at least one of the powers; this set of points is known as the Pareto frontier, and is representative of the most optimum points of the simulation. These codes were limited to a small number of parameters, since sensitivity testing by finite difference is computationally expensive, and a Monte-Carlo simulation involves the use of several hundred simulations for each parameter.

Studies conducted by Tremolet and Basset [24] and assisted by work done by ONERA [25] have expanded this range of parameters and created the software tool “Concepts of Rotorcraft Enhanced Assessment Through Integrated Optimization Network” or CREATION. CREATION is able to consider a significant number of multi-disciplinary inputs, including various forward flight conditions, material selection, and acoustical effects. Giovanetti [14] solved the multi-point optimization problem by expanding on the work done by Hall & Hall [3] to generate the multi-point optimization of conventional and coaxial rotors considering low and high speed
1.2 Problem Statement

The present work has three main objectives: to improve the fidelity of the current Duke rotor aerodynamics models, to decrease the computational effort required to compute optimal aerodynamic rotor designs, and to improve the techniques used to optimize the system. Towards those ends, we have replaced the lifting line model with an improved lifting surface model of the blades, thus allowing us to capture non-quasi-steady airfoil effects. We also make other general improvements to the model to improve reverse flow regions experienced in high speed forward flight. This increase in fidelity comes with higher computational cost due to the increase in the number of degrees of freedom associated with the model. To reduce the computation cost, we use two strategies. The first is static condensation, in which we condense
out most of the lifting surface degrees of freedom, making using of symmetries of the problem. We also have adapted the use of Harmonic Balance, widely used in the CFD community, to our vortex lattice model. The result is a higher fidelity model of the blade aerodynamics, but with computational cost considerably lower than the previous time-domain lifting line technique.

To improve the optimization techniques of the system, an open source computer package IPOPT is used in conjunction with OpenMP to improve computational speed and accuracy of the optimized parameter space.

1.3 Overview

Chapter 2 describes all mathematical models used in this thesis: the lifting line procedure originally used at Duke, the new modified lifting surface approach, and the model reduction techniques used to make the model more efficient. This chapter also details comparisons between the two models before incorporating optimization.

Single point optimization is introduced in Chapter 3. A variational statement is developed to optimize the power of the system given certain linear constraints. Optimized results of the lifting line and lifting surface models are also compared.

Chapter 4 adds non-linear constraints to single point optimization through incorporation of non-linear chord distribution optimization. IPOPT is integrated into the system to include these design parameters, and comparisons are conducted between optimizations with and without chord optimization.

Multi-point optimization is presented in Chapter 5, where a model using weighted averages is outlined for the optimization procedure.

Chapter 6 will suggest future directions for research.
To solve the problem of rotor optimization, two main issues must be addressed. First, a mathematical model of the rotor must be created that is a good approximation of the physical system. Second, a numerical scheme must be developed that optimizes the design variables that parameterize the mathematical model appropriately. Authors have previously developed a lifting line technique [3, 13, 18] to find the optimal design of rotors and the corresponding performance. However, lifting line theory has a number of limitations. For example, the method is only valid for low blade reduced frequencies. In fact, there is a weak singularity in the model for unsteady flow problems that can produce spurious results if not carefully handled.

To address the limitations of lifting line approach, we present an improved approach that uses a lifting surface technique. Using lifting surface, unsteady, non-quasi-steady effects can be modeled correctly. The forces, moments, and resulting power acting on the blade can be computed directly by integrating unsteady loads on the blades. We will use this method, together with an optimization algorithm, to find the optimum chord distribution on the blades. The added computational complexity of lifting surface will be addressed using two model reduction techniques
inspired by previous research, i.e., Guyan reduction and harmonic balance.

2.1 Terminology

Consider a helicopter in horizontal flight at constant speed. For our analysis, the hub of the rotor will be placed at the center of a Cartesian coordinate system. The x-axis points aft horizontally in the direction of the tail of the aircraft, the y-axis points horizontally out the right side of the aircraft, and the z-axis is in the vertical direction pointing upwards. The rotation of the rotor is taken to be positive counter clockwise when viewed from above, as is common for helicopters manufactured in the United States. The axis of rotation may be tilted fore or aft through an angle of $\alpha_{\text{shaft}}$. The angular position of a rotor blade is measured by the azimuthal angle $\psi$, with $\psi = 0^\circ$ corresponding to the blade in the aft most position. From $0^\circ$ to $180^\circ$ the blade is advancing in the flight direction, while from $180^\circ$ to $360^\circ$ is blade is retreating. Note that the system is periodic with period $2\pi/\Omega B$, where $\Omega$ is the angular velocity of the rotor and $B$ is the number of blades.

The rotor has radius $R$, root cutout $R_{\text{cut}}$, and the blades each have chord distribution $c(r)$, and twist distribution $\theta(r)$.

2.2 Lifting Line Model

For the original lifting line model, the blades of the rotor are assumed to be two-dimensional lines of bound vorticity. As the blades travel through the air in a prescribed manner, the lines representing the blades will leave a wake behind them containing shed and trailing vorticity. For lightly loaded rotors, we can ignore wake roll-up, in which case the wake lies on the trace, the surface swept out by the trailing edge of the blades. The resulting wake is modeled by quadrilateral vortex rings or panels (see Figures 2.1 and 2.2). Each vortex ring has unknown strength $\Gamma_{w,i}$ to be determined.
The quadrilateral vortex rings in the wake will induce a wash (flow velocity) on both the wake and the lifting line. We assume that these induced effects are small enough that wake roll-up can be ignored, and thus, the wake can be modeled as rigid. This lightly loaded assumption holds so long as the motion of the rotor is much larger than the induced velocity \((w \ll U)\). One can show that this assumption is valid when

\[
C_L \ll 2\mu^2
\]  

(2.1)

where \(C_L\) represents the global lift coefficient of the rotor, defined as

\[
C_L = \frac{L}{\rho \pi R^2 (\Omega R)^2}
\]  

(2.2)
and $\mu$ the forward advance ratio of the blade. Thus, the model is valid for moderate to high advance ratios; hover and low speed forward flight cannot be modeled.

2.3 Lifting Line Calculation of Power, Forces, and Moments

The lifting line model is used in conjunction with a Kelvin impulse technique to calculate all forces and moments acting on the rotor, while a kinetic energy argument is used to calculate the induced power. It is assumed that the flow is inviscid, incompressible, and irrotational (except for the wake itself).

The lift and thrust of the system is related to the amount of Kelvin impulse that is deposited into the wake over one period of rotor motion. Consider the Trefftz volume depicted as $\mathcal{V}$ in Figure 2.2. It can be shown [26] that the increase in Kelvin impulse generated in one turn of the wake can be found by computing the Kelvin impulse via an area integral over the surface of the wake, i.e.,

$$\xi = -\rho \int_{\mathcal{W}} \phi n \, dA$$

(2.3)

where $\rho$ represents the fluid density, $\phi$ the potential function, and $n$ the unit normal. The jump in potential across the wake is just $\Gamma$, the circulation. Because the normal vector $n$ on either side of the wake point in opposite directions, Equation (2.3) can be rewritten as

$$\xi = -\rho \int_{\mathcal{W}} \Gamma n \, dA$$

(2.4)

where now the integral is taken over just one side of the wake. Dividing this expression by the period of rotation $T$ gives the average rate of change of Kelvin impulse. Thus, the time average force acting on the rotor is

$$F = \frac{\xi}{T} = \frac{\rho}{T} \int_{\mathcal{W}} \Gamma n \, dA.$$  

(2.5)
Similarly one can find the time averaged moment on the rotor by computing the rate of change of Kelvin angular impulse. Hence,

\[ M = \frac{\rho}{T} \oint_{\Omega} \Gamma \mathbf{r} \times \mathbf{n} \, dA \quad (2.6) \]

where \( \mathbf{r} \) is the moment arm extending from the center of the rotor to where the wake was shed.

To obtain the induced power, an energy approach is taken. The energy deposited into the wake per period is

\[ E = \iiint_{\Omega} \frac{1}{2} \rho |\mathbf{w}|^2 \, d\Omega . \quad (2.7) \]

where the integral is taken over the Trefftz volume, as shown in Figure 2.2. This is a potential flow, so

\[ \mathbf{w} = \nabla \phi , \quad (2.8) \]

with \( \phi \) satisfying

\[ \nabla^2 \phi = 0 . \quad (2.9) \]

Substitution of Equation (2.8) into Equation (2.7), and application of Green’s theorem allows us to convert Equation (2.7) to a surface integral. Dividing by \( T \) gives the rate of energy dissipated into the wake, i.e.,

\[ P_i = \frac{E}{T} = \frac{\rho}{2T} \iiint_{\Omega} |\nabla \phi|^2 \, d\Omega = -\frac{\rho}{2T} \oint_{\Omega} \Gamma \mathbf{w} \cdot \mathbf{n} \, dA . \quad (2.10) \]

Note that the wash is linear in \( \Gamma \), so the induced power is quadratic in the circulation.

While all of the above equations are stated in integral form, it is possible to convert them to a discrete form. The continuous mathematical system that defines the wake is discretized into quadrilateral vortex lattice panels. A Riemann sum is then conducted in lieu of an integral using these wake lattice panels; values such as
the unit normal \( \mathbf{n} \), panel area \( \Delta A \), lever arm \( \mathbf{r} \), and circulation \( \Gamma \) are calculated for each panel, and then summed over the entire wake, i.e.,

\[
F = \sum_{i=1}^{M} \tilde{b}_i \Gamma_i = B \Gamma_w \tag{2.11a}
\]

\[
M = \sum_{i=1}^{M} \tilde{d}_i \Gamma_i = D \Gamma_w \tag{2.11b}
\]

\[
P_i = \frac{\rho}{2T} \sum_i \Gamma_i w_i \Delta A_i = \frac{1}{2} \Gamma_w \mathbf{T} K \Gamma_w \tag{2.11c}
\]

with

\[
\tilde{b}_i = \frac{\rho}{T} \mathbf{n}_i \Delta A_i \tag{2.12a}
\]

\[
\tilde{d}_i = \frac{\rho}{T} \mathbf{r}_i \times \mathbf{n}_i \Delta A_i \tag{2.12b}
\]

\[
K_{ij} = -\frac{\rho}{T} \tilde{w}_{ij} \cdot \mathbf{n}_i \Delta A_i \tag{2.12c}
\]

where \( \tilde{w}_{ij} \) represents the induced velocity in the far field at panel \( i \) due to the influence of a panel \( j \) in the Trefftz volume, as well as all the corresponding panels in the other turns of the wake. Similar to Equation (2.10), the expression for power is quadratic in the circulation of the system.

2.3.1 Inclusion of Viscous Profile Power

The power calculated using Equation (2.10) only includes losses due to induced effects. Viscous effects should be considered to create a more realistic mathematical model of our physical system. A full Navier Stokes analysis would be prohibitively expensive, and would also be difficult to optimize. Instead, we assume that the aerodynamic surfaces have large aspect ratios (i.e. \( c/R \ll 1 \) for a rectangular planform blade), and that the reduced frequency \( k \) is low. Under these assumptions, it is reasonable to assume that the viscous forces can be accurately modeled using a
quasi-steady model. Thus, we use an experimentally obtained steady drag polar to estimate the drag forces on the blades.

As described in Hall & Hall [3], data from the NACA 23012 airfoil at a Reynolds number \( Re = 8.8 \times 10^6 \) was selected for the previous analysis. This data can be seen in Figure 2.3. Alongside the experimental data is a parabolic curve fit, where

\[
    c_d \approx c_{d0} + c_{d2} \left( c_l - c_{l0} \right)^2
\]  

(2.13)

For this particular airfoil, \( c_{d0} = 0.005519 \), \( c_{d2} = 0.00472 \), and \( c_{l0} = 0.1229 \). Figure 2.3 shows that the quadratic interpolation is only reliable over a finite range of lift coefficients. At high angles of attack, the airfoil will stall. Thus, the lift coefficient is limited to a maximum and minimum value.

![Experimental drag polar for the NACA 23012 airfoil](image)

**Figure 2.3**: Experimental drag polar for the NACA 23012 airfoil [4]. A parabolic curve fit, as well as minimum and maximum realistic coefficients of lift, are included for comparison [3].

To integrate the viscous effects into the current model for power, the relationship
between lift coefficient and circulation

\[ c_l = \frac{l}{0.5\rho u_\infty^2 c} = \frac{2\Gamma}{u_\infty c} \]  

is used. Viscous power can now be calculated as the work done by the drag force per period of the wake, divided by the period, so that

\[ P_v = \frac{1}{T} \int_{W} \frac{1}{2} \rho u_\infty^2 c c dA . \]  

Substitution of Equations (2.13) and (2.14) into Equation (2.15) yields

\[ P_v = \frac{\rho}{2T} \int_{W} \left\{ \left( \frac{4c d_2}{c} \right) (\Gamma - \Gamma_0)^2 + (u_\infty^2 c c d_0) \right\} . \]  

The total power of the system can now be represented in matrix form, using the previous result, as the sum of the viscous profile power and induced power, i.e.

\[ P = \frac{1}{2} \Gamma_w^T (K_{inv} + K_{visc}) \Gamma_w - \Gamma_w^T Q + P_{v0} \]  

with

\[ K_{inv,ij} = -\frac{\rho}{T} [\vec{w}]_{ij} \cdot n \Delta A_i , \]  

\[ K_{visc,ij} = \delta_{ij} \frac{\rho}{T} \left[ \frac{4c d_2}{c} \right]_i \Delta A_i , \]  

\[ Q_i = 2\frac{\rho}{T} u_\infty u_{\infty,i} [c_0 c d_2] \Delta A_i , \]  

\[ P_{v0} = \frac{\rho}{2T} \sum_{i=1}^{M} u_\infty^2 u_{\infty,i} [c (c d_0 + c_d c_0)] \Delta A_i . \]  

### 2.4 Lifting Line Direct Analysis Technique

Given design parameters for the system, one can solve directly for the circulation \( \Gamma \) using a realizability condition in the form of a no-penetration boundary condition,
that is, the induced wash on the lifting line of the blades arising from the circulation
distribution must be consistent with the geometry that produces it.

For this condition, at all stations there is a prescribed normal wash $b_i$ that will
be a function of the twist of the blade as well as the motion of the rotor. There will
also be an induced wash caused by all vortex panels in the wake. For a prescribed
wake, the wash is a linear function of the circulation, i.e.,

$$ w = A \Gamma_w, $$  \hspace{1cm} (2.19)

where the entries of $A_{ij}$ are determined by the Biot-Savart Law using the filaments
of panel $j$ and a receiving point on panel $i$. Thus, the no-through-flow condition
becomes

$$ R = A \Gamma_w - b = 0 $$  \hspace{1cm} (2.20)

Note Equation (2.20) is a vector equation describing the no-through-flow condition
at every radial and azimuthal position of a blade. Equation (2.20) can now be solved
for $\Gamma_w$ for a prescribed rotor design and motion. In a subsequent chapter, we discuss
the optimization of the rotor design to minimize power.

2.5 Lifting Line Technique Limitations

There are two key issues with the lifting line technique. First, the assembly of the
matrices to solve the system requires a significant amount of CPU time. The wash
induced by every wake panel must be found at every time on every lifting line element.
Thus, the CPU time required to assemble the influence coefficient matrix goes like
the square of the number of panels in the wake (per turn).

Second, and more important physically, by creating a lifting line model, a pseudo
quasi-steady assumption has been made on the blade. That is, implicitly the airfoil
aerodynamics are modeled with the reduced frequency $k \ll 1$. But, for a rectangular
blade, we have

\[ k = \frac{\omega c}{2U} = \frac{n\Omega c}{2\Omega R} = n\frac{c}{2R} = \frac{n}{AR}, \quad (2.21) \]

where \( \Omega \) is the angular velocity of the rotor, \( n \) the harmonic number of the reduced frequency, \( c \) the chord of the blade, and \( R \) the radius of the blade. This illustrates \( k \sim n/AR \). In words, the blade reduced frequency goes like the harmonic of interest divided by the blade aspect ratio. For typical helicopters, aspect ratios are on the order of 10 [27], and we might reasonably expect harmonics up to three or four to be important. This implies reduced frequencies on the order of 0.4, which is certainly not quasi-steady.

2.6 Lifting Surface Model

To alleviate the low reduced frequency limitation of the lifting line model, we next present a lifting surface model. Similar to the lifting line model, the lifting surface model will have a prescribed wake comprised of shed vorticity and trailing vorticity (denoted again with \( \Gamma_w \)) that is related to the motion of the blade. The key difference – illustrated in Figure 2.4 – is that the blade surfaces are modeled using vortex rings rather than a single vortex filament. These rings, whose strengths are denoted by \( \Gamma_{a,i} \), tessellate the surface of the rotor blades. These panels rotate and translate as the rotor turns and translates through the air. Small scale motion, such as the blade pitching, are modeled using boundary conditions rather than grid deformation.

To connect the blade model to the wake model, it is assumed that the airfoil will create shed and trailing vorticity behind the airfoil, which will also be represented by quadrilateral vortex rings.

The use of a lifting surface model allows us to model non-quasi-steady aerodynamics. Note that we still assume the rotor is lightly loaded with a rigid wake, so the approach is only valid in forward flight (no hover).
2.6.1 Reverse Flow Region Improvements

A phenomenon not included in the wake structure of Figures 2.1 and 2.4 is that of reverse flow. As the blade travels through the retreating side of its motion \((180^\circ \leq \psi \leq 360^\circ)\) there is the possibility for certain radial stations to experience a mean flow over a wing section from the trailing edge to the leading edge. This occurs when

\[
V \cos(\psi) > \Omega r .
\]  

(2.22)

Methods of modeling this phenomenon have been discussed in the literature [15, 28, 29], and the model presented here has adopted a similar approach of physically modifying the wake in these regions. When reverse flow occurs, we trail the wake from the leading edge rather than the trailing edge. For a lifting line model this is not a concern since the rotor is modeled as a set of lines. When using a lifting surface model, alterations must be made to the wake to compensate for this shift. This switch to the leading edge is visualized in Figures 2.5 and 2.6. As the advance ratio increases, this region grows in size until the entire retreating side of the rotor is in reverse flow.
Figure 2.5: Comparison of the wake with and without reverse flow alterations on the retreating side of motion. The changes to the reverse flow wake have been circled for clarity.

2.7 Lifting Surface Calculation of Power, Forces, and Moments

With the blade now mathematically modeled using vortex rings, there is a decision to make on how to compute the forces, moments, and induced power generated by the rotor. It is now possible to compute these quantities directly using a "near field" approach, calculating the lift and moment by a direct integration of surface pressures instead of using far field wake integration.
Using the unsteady incompressible form of Bernoulli’s equation [30], the instantaneous lift on the airfoil is given by

\[
F(t) = \rho \iint_a \left( \mathbf{V}(r) \times \bm{\gamma}_a + \frac{\partial \Gamma}{\partial t} \right) \mathbf{n} \, dA
\]

where \( \mathbf{n} \) represents the blade unit normal, and \( \mathbf{V}(r) \) is the velocity of the fluid with respect to the bound vortex, given by \( \mathbf{V} = \mathbf{U} + \mathbf{w} \). \( \mathbf{U} \) is the first order velocity created by the geometric motion of the system, \( \mathbf{w} \) is the second order induced velocities generated by the induced interactions of airfoil and wake vorticity. The first order force is much larger than the second order force. The second order force will therefore represent a loss and be book-kept as induced power.

There are two terms in Equation (2.23). The first is the familiar “Kutta-Joukowski” lift found in steady aerodynamics problems. The second, involving the time rate of change of circulation, only appears in unsteady problems and arises from the unsteady form of the Bernoulli equation.

The moment can be found by introducing the moment arm \( r \) into the integrand in Equation (2.23), i.e.,

\[
M(t) = \rho \iint_a \left( r \times (\mathbf{U}(r) \times \bm{\gamma}_a) + r \times \frac{\partial \Gamma}{\partial t} \bm{n} \right) \, dA .
\]
To calculate the instantaneous induced power of the system, the work done by second-order induced forces must be calculated. Using Equation (2.23) again, but neglecting first order velocities in favor of second order velocities,

\[ F(t) = \rho \int \left( (w(\Gamma_a, \Gamma_w, r)) \times \gamma_a + \frac{\partial \Gamma}{\partial t} n \right) dA \]  

(2.25)

where \( w \) is the induced velocity induced by the entire vortex system,

\[ w(r_a) = \int_a v_a(r) dA_a + \int_w v_w(r) dA_w \]  

(2.26)

The work done by Equation (2.25) is the dot product of the first order velocity of the system with the induced force, i.e.,

\[ P_i(t) = \rho \int_a U(r) \cdot [w(\Gamma_a, \Gamma_w, r) \times \gamma_a] dA , \]  

(2.27)

which can be averaged over a given time period to obtain an average power if necessary.

Equations (2.23), (2.24) and (2.27) can be discretized in a similar fashion to lifting line. The quadrilateral vortex lattice panels representing the wake and airfoil are used to represent the system in lieu of a continuous system needed for an integral. The values necessary in the integrand for force, moment, and power such as normal vector \( n \), area \( \Delta A \), circulation \( \Gamma \), and velocity \( U \) are calculated for each panel, then combined using a Riemann sum, i.e.,

\[ F = \sum_{i=1}^{N_aN_w} \frac{\Delta t}{T} \left[ b_{1i} \Gamma_{ai} + b_{2i} \hat{\Gamma}_{ai} \right] = (B_{N1} + B_{N2} E) \Gamma_a = B_N \Gamma_a \]  

(2.28a)

\[ M = \sum_{i=1}^{N_aN_w} \frac{\Delta t}{T} \left[ d_{1i} \Gamma_{ai} + d_{2i} \hat{\Gamma}_{ai} \right] = (D_{N1} + D_{N2} E) \Gamma_a = D_N \Gamma_a \]  

(2.28b)
\[
P_i = \sum_{i=1}^{N_a} \sum_{j=1}^{N_w+1} \frac{\Delta t}{T} K_{Nij} \Gamma ai \Gamma j = \Gamma a^T K_N \Gamma \tag{2.28c}
\]

where \( N_a \) is the number of airfoil panels at a given time-step, \( N_w \) is the number of wake panels at a given time-step, \( \Delta t \) is the size of the time step, and \( E \) is a finite difference derivative operator representing the time derivative operator necessary to calculate the unsteady lift. Also,

\[
b_{1i} = \rho U_i \times \sum_{k=1}^{4} d_i \Delta A_i \tag{2.29a}
\]

\[
b_{2i} = \rho n_i \Delta A_i \tag{2.29b}
\]

\[
d_{1i} = \rho r \times \left[ U_i \times \sum_{k=1}^{4} d_i \right] \Delta A_i \tag{2.29c}
\]

\[
d_{2i} = \rho r \times n_i \Delta A_i \tag{2.29d}
\]

\[
K_{Nij} = \rho U_i \cdot \left( \mathbf{v}_{ij} \times \sum_{k=1}^{4} d_i \right) \Delta A_i . \tag{2.29e}
\]

\( \mathbf{v}_{ij} \) is the induced velocity of panel \( j \) on airfoil panel \( i \) and includes the wash induced by both the wake and the rotor blades. \( d \) is the vectoral length of one edge of a vortex ring (vortex filament) as seen in Figure 2.7. In the induced power, the matrix \( K_N \) is non-square; induced effects are calculated from all panels (airfoil and wake), but the effects are only calculated on the airfoil itself. If a square matrix is necessary, \( \Gamma \) can be substituted into the induced power expression, which would introduce zeros into \( K_N \) to make it square. Viscous effects can be included approximately using the quasi-steady drag polar technique from lifting line theory.

When calculating the forces, moments, and induced power for the lifting surface model, the near field (blade integration) and far field (wake integration from Lifting Line) techniques should both give correct results. In fact, one can show using Kelvin
Figure 2.7: Visualization of the creation of a vortex ring generated on the airfoil. As the ring is traced, the vectors representing the sides of the panel are labeled with the vector $d_i$, which will be used in on-blade calculations.

Impulse and kinetic energy techniques that near field and far field approaches give the same results for forces and power. In previous lifting line computations [3, 14, 17, 31], the blades were assumed to be high aspect ratio, with quasi-steady lift acting along the lifting line. In this high aspect ratio case, a far-field integration alone can be performed to compute the moment as well. However, for finite chord blades, a correct computation of the moment requires some integration over the blade. When conducting the analysis using the lifting surface method, we calculate the moment using a near field approach, and denote the matrix form of the moment integrator with $D_N$, similar to Equation (2.28b).

2.8 Lifting Surface Analysis

Like lifting line, lifting surface can be used to find the aerodynamic behavior of a prescribed rotor design. At each time step, $t_n$, of the lifting surface model, we impose a no through flow condition of the form

\[
A_{a,a}^n \Gamma_a^n + A_{a,\theta}^n \theta^n + \sum_k A_{a,w}^k \Gamma_w^n = b^n, \quad n = 1, 2, \ldots, N \quad (2.30a)
\]

\[
\Gamma_w + A_{w,a}^n \Gamma_a^n, \quad n = 1, 2, \ldots, N. \quad (2.30b)
\]

where $\Theta$ is a set of design parameters that set the linear controls of the system (i.e. blade controls and twist). $A_{a,a,ij}$ is an influence coefficient that gives the normal wash on airfoil panel $i$ due to vortex panel $j$. Similarly $A_{a,w,ij}$ gives the influence
of wake panel \( j \) on airfoil panel \( i \); these influence coefficients are calculated with the Biot-Savart Law as in Equation (2.20). \( A_{a,\theta} \) is the wash influence of all linear controls included in the model (i.e. blade controls and twist), and \( A_{w,a} \) dictates the relationship between wake panels and airfoil panels.

The left side of Equation (2.30a) takes on a form similar to Equation (2.20). Equation (2.30b) gives the relationship between the rate of change of circulation on the blades and the trailing and shed vorticity in the wake, found using Kelvin’s theorem.

While it appears that each time step is individually solvable at this point, Equation (2.30b) couples all the time levels together so one must solve simultaneously for all time steps. Doing so yields a matrix equation of the form

\[
R = \begin{bmatrix}
\tilde{A}_{a,a} & \tilde{A}_{a,w} & A_{a,\theta} \\
\tilde{A}_{w,a} & I & 0
\end{bmatrix} \begin{bmatrix}
\Gamma_a \\
\Gamma_w \\
\theta
\end{bmatrix} - \begin{bmatrix}
b \\
0
\end{bmatrix} = 0.
\] (2.31)

Here, \( \tilde{A}_{a,a} \) is a large block diagonal matrix, with \( N \) identical blocks made up of the individual \( A_{a,a} \) blocks; since \( \tilde{A}_{a,a} \) is more commonly used to solve the system, from here on this matrix will be denoted \( A_{a,a} \). The matrix \( \tilde{A}_{a,w} \), which represents the influence of the wake circulation on the wash on the wing, is fully populated by concatenating the individual \( A_{a,w} \) blocks in rows; since \( \tilde{A}_{a,w} \) is more commonly used to solve the system, from here on this matrix will be denoted \( A_{a,w} \). Finally, \( \tilde{A}_{w,a} \) is sparse, and populated with diagonal entries of the individual \( A_{w,a} \); since \( \tilde{A}_{w,a} \) is more commonly used to solve the system, from here on this matrix will be denoted \( A_{w,a} \).

2.9 Lifting Surface Technique Limitations

While the lifting surface model is a higher fidelity model than lifting line, the model is computationally more complex and requires more CPU time. For example, for
a four-bladed rotor with 32 elements in the radial direction, 8 elements along
the blade chord, and 32 time steps per quarter revolution, there is a total number
of 36,864 unknowns; the corresponding lifting line system would only have 4,608
unknowns. The lifting surface system is much larger than the lifting line system,
resulting in larger computation times.

This increase in computation time is caused by the inclusion of multiple chord-
wise panels, increasing the number of degrees of freedom dramatically. Since the
model requires the blade to be modeled at all time steps, this increase in complexity
due to chord-wise panels is compounded by the number of time steps in the model.
Using two techniques inspired by previous research, it is possible to reduce the com-
plexity of the final solution procedure while still maintaining the fidelity of the lifting
surface model.

2.9.1 Static Condensation

To address the issue of number of chord-wise panels, it is possible to reduce the
number of degrees of freedom necessary in the computation using condensation.
Instead of physically removing elements from the model, multiple elements will be
condensed out of the solution procedure, reducing the number of degrees of freedom
to solve for. Originally this idea was proposed by Guyan as a method to reduce the
size of mass and stiffness matrices in structural analysis, and since then has been
used in many different applications [32, 33, 34, 35] as Guyan reduction, or more
commonly static condensation. Our approach here is conceptually similar to Peter’s
airloads theory [15]. In Peter’s model, the entire airfoil can be described by just two
parameters that are weighted averages of the induced wash on the airfoil. Similarly it
is possible to invoke condensation in the lifting surface model to significantly reduce
the number of degrees of freedom.

To condense the system of equations given by Equation (2.31), the expressions
for some of the elements in $\Gamma_a$ will be written with respect to the other elements. To do this, the first equation of Equation (2.31) will be used to solve for $\Gamma_a$, the $\Gamma_a$ is then substituted into the second equation. Doing so yields

$$[I - A_{w,a}A_{a,a}^{-1}A_{a,w}] \Gamma_w = -A_{w,a}A_{a,a}^{-1} \{b - A_{a,a}\theta\} . \quad (2.32)$$

Recall that $A_{a,a}^{-1}$ is block diagonal, with $N$ identical blocks. So $A_{a,a}^{-1}$ (or its factors) is very inexpensive to compute, resulting in a substantially smaller matrix problem to solve. For our previous example, the number of degrees of freedom is reduced from 36,864 to just 4,096. This reduction produces no loss in accuracy, but a substantial reduction in computational cost.

### 2.9.2 Harmonic Balance

As the rotor moves through the air, it sheds vorticity into the wake at each time step. This leads to a direct relationship between the time rate of change of bound vorticity at that time step and the wake vorticities shed at that instant. Since the wake is convected as the rotor moves through the air, the shed vorticity in the wake will have an induced effect on the next time step’s shed vorticity; this couples all time steps together as seen in Equation (2.31), and leads to very large computational times as the resulting matrix equations are quite large, even after using static condensation as in Equation (2.32).

To further reduce the computational size of the problem, we use a form of harmonic balance [36]. Instead of modeling one full turn of the rotor in the time-domain, we represent the circulation on the rotor as a Fourier series. In complex form, the circulation at discrete times $k$ is given by

$$\Gamma_w^k = \sum_{n=-\infty}^{n=+\infty} \hat{\Gamma}_n e^{i\Omega nk} \quad (2.33)$$

where $\hat{\Gamma}_n$ is the Fourier coefficient associated with the frequency $\Omega n$. Equation (2.33)
is a discrete inverse Fourier transform. There is also a discrete Fourier transform that can take the time data and create the Fourier coefficients. Writing both of these transforms in matrix form, we have

\[ \Gamma_w = E\hat{\Gamma} \quad (2.34a) \]

\[ \hat{\Gamma} = E^{-1}\Gamma_w \quad (2.34b) \]

where \( EE^{-1} = I \). While Equation (2.33) has summation bounds that are infinite, a good approximation is possible using a small number of coefficients in the analysis. Depending on the system, it is possible that a very small number of coefficients need to be used.

With harmonic balance, Equation (2.32) can be reduced even further in size. By replacing \( \Gamma_w \) with the relationship found in Equation (2.34), one obtains

\[ \left[ I - A_{w,a}A^{-1}_{a,a}A_{a,w} \right] E\hat{\Gamma} = -A_{w,a}A^{-1}_{a,a}\{b - A_{a,\theta}\} \quad (2.35) \]

where \( \hat{\Gamma} \) represents the Fourier coefficients of the system. Equation (2.35) is now over-constrained; to alleviate the problem, a least squares approach is taken by projecting Equation (2.35) onto \( E \), giving the final set of equations

\[ E^T \left[ I - A_{w,a}A^{-1}_{a,a}A_{a,w} \right] E\hat{\Gamma} = -E^T A_{w,a}A^{-1}_{a,a}\{b - A_{a,\theta}\} , \quad (2.36) \]

or more succinctly as

\[ \hat{A}_{w,w}\hat{\Gamma} = \hat{b} - A_{w,b}\theta . \quad (2.37) \]

Note that Equation (2.37) is real, despite the intermediate complex operations.

2.10 Lifting Surface & Lifting Line Results

To compare the results of the lifting line and lifting surface techniques, a numerical test was conducted for a given rotor in various forward flight conditions. The geometry tested is a conventional rotor with four blades rotating counter-clockwise. The
Table 2.1: Fixed parameters for blade being tested in this loading line, lifting surface comparison.

<table>
<thead>
<tr>
<th>Element</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>4</td>
</tr>
<tr>
<td>$R$</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha_{\text{shaft}}$</td>
<td>$-5.0^\circ$</td>
</tr>
<tr>
<td>$R_{\text{cutout}}$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>5.0$^\circ$</td>
</tr>
<tr>
<td>$N_R$</td>
<td>20</td>
</tr>
<tr>
<td>$N_c$</td>
<td>4</td>
</tr>
<tr>
<td>$N_\psi$</td>
<td>18 ($\times$ 4)</td>
</tr>
<tr>
<td>$N_{\text{wake}}$</td>
<td>40</td>
</tr>
</tbody>
</table>

Blades have a non-dimensional radius of one. The shaft angle is five degrees, tilted into the forward flight direction. The blade is untwisted, but has $5^\circ$ of collective pitch and $0^\circ$ of cyclic pitch; thus, the rotor is untrimmed. To avoid the complexity of reverse flow regions for this analysis, the rotor has a larger than normal cutout of 50% of the radius to maintain forward flow on all regions of the blade up to an advance ratio of 0.5. The grid has 20 panels in the radial direction, 18 panels per blade in the azimuthal direction, and 4 panels in the chord-wise direction (on the blade). Ten turns of the wake (forty quarter turns) are used to model the wake. These grid parameters, along with a summary of the physical parameters, are included in Table 2.1.

Three elements were varied in this test. The first is the aspect ratio, defined as

$$AR = \frac{c^2}{BA} = \frac{c}{B(R - R_{\text{cutout}})}.$$  \hspace{1cm} (2.38)

An increasing aspect ratio also leads to an overall decrease in the reduced frequency of an oscillatory motion, meaning less unsteady effects experienced by the blade.

The second element is the advance ratio, the non-dimensional parameter comparing the forward flight speed to the rotation rate of the rotor. The advance ratio
is given by

$$\mu = \frac{V}{\Omega R}.$$  \hspace{1cm} (2.39)

To prevent reverse flow for our problem, the advance ratio should be less than \(R_{\text{cutout}}/R = 0.5\). Further, recall the lightly loaded assumption of Equation (2.1). Typical coefficients of lift, as defined by Equation (2.2), for rotors are in the range of 0.01 [29, 37]. Thus the lightly loaded assumption requires

$$C_L \ll 2\mu^2$$

$$\rightarrow \mu \gg 0.07.$$  

With these two bounds, the overall restrictions on the advance ratio are

$$0.07 \ll \mu < 0.5.$$  \hspace{1cm} (2.40)

Finally, the number of Fourier coefficients used in the analysis is \(N_c\). Using more coefficients will capture higher harmonic content, but also increase computational cost. For calculation purposes, \(N_c\) represents the number of harmonics necessary. Each non-zero harmonic has a magnitude and phase, so there are \(2N_c + 1\) physical coefficients.

\subsection*{2.10.1 High Aspect Ratio Case}

In this section, we consider the very high aspect ratio of 100. For this aspect ratio the reduced frequency is extremely low, on the order of \(10^{-3}\). Thus, the flow is essentially quasi-steady. We analyzed this rotor using the present lifting surface analysis, and also the previous lifting line analysis of Giovanetti & Hall [14]. Shown in Figure 2.8 are the computed local circulation values and coefficients of lift; both values are non-dimensionalized according to

$$c_\ell = \frac{\ell}{\rho \pi R^2 (\Omega R)^2}.$$  \hspace{1cm} (2.41a)
\[ \Gamma = \frac{\Gamma_{\rho R^2}}{L} \] (2.41b)

Note the sectional lift and circulation computed using the two approaches agree very well with one another as we would expect in the limit of high aspect ratio and low reduced frequency.
Figure 2.8: Computed non-dimensional circulation $\bar{\Gamma}$ and sectional lift coefficient $c_\ell$ for an untrimmed rotor with aspect ratio 100 in forward flight of $\mu = 0.45$ computed using lifting line [14] and lifting surface with harmonic balance and static condensation theories. Ten harmonics were used in the reduced order model.
Figure 2.9 shows the computed global lift, calculated using one, two, three, and ten harmonics in the analysis. Remarkably, Figures 2.9a and 2.9b demonstrate that the rotor lift is accurately approximated by using just one coefficient. With just three coefficients, the computational lift agrees with converged results within 2%. This is an important result because the rotor aerodynamics can be accurately computed using a small number of harmonics with the harmonic balance approach, resulting in substantial computational savings.

Also shown in Figures 2.9 and 2.10 are the lift and thrust computed using both near field and far field approaches. The results of the two approaches are in excellent agreement, suggesting that both approaches give the correct time-averaged force. From Figure 2.9c it is apparent that once 10 harmonic coefficients are used in the lifting surface model, the near and far field results are effectively the same.

Figure 2.9c shows the computed coefficient of lift using ten harmonics for the near field and far field calculation of lift. The near field and far field results are nearly the same, which is once again expected because of the high aspect ratio of the blade. Similarly, Figure 2.10 shows the computed thrust. Again, the near field and far field results obtained from lifting surface agree with each other to within 5%.

The computed coefficient of induced power is shown in Figure 2.11 as a function of advance ratio, and again in Figure 2.12 normalized by the coefficient of lift squared. The computed power converges rapidly with increasing number of harmonics; with three harmonics all induced power results for the near and far field calculation are within 2% of each other. When comparing the near and far field in Figures 2.11 and 2.12, the near field and far field calculations are in substantial agreement. Looking more closely at Figure 2.11 there is a larger amount of difference between the ten coefficient results for induced power in the near and far field when compared with the ten coefficient results for lift in Figure 2.9, but this is primarily due to the relationship that induced power has with circulation. For lift, \( C_L \sim \Gamma \), while \( C_{Pi} \sim \Gamma^2 \).
(this can easily be seen in the relationships of Equations (2.11) and (2.28)). Because there is more dependence on the circulation, what is a small error in $\Gamma$ will result in the square of that error in the induced power. Nevertheless, even with a larger amount of error three coefficients ends up being close to 2% error for all advance ratios, and convergence is seen around 10 coefficients. With Figure 2.12 it is possible to compare with the Glauert limit [12]. The expression,

$$\frac{C_{P_i}}{C_L^2} = \frac{1}{2\mu}$$

serves as a lower bound. Put simply, this is the smallest value of normalized power loss any vehicle can produce (with a flat wing or rotor). Figure 2.12 shows that the computed power is greater than this bound.

This section has demonstrated that the present lifting surface method exhibits modal convergence and that the method agrees well with lifting line in the limit of high aspect ratio. This suggests that the harmonic balance lifting surface method, while computationally very efficient, has the accuracy of a time domain approach.

2.10.2 Low Aspect Ratio Case

Next, we consider a rotor with an aspect ratio of 10. Due to the direct relationship between aspect ratio and the reduced frequency, this will increase all reduced frequencies by a multiple of 10, creating reduced frequencies of the order of $10^{-1}$ to $10^{-2}$. Unsteady effects become important around this reduced frequency range, and it is exactly those effects that the lifting surface technique was designed to model.

We analyzed these rotors using the previous lifting line technique, as well as the present lifting surface technique, with both near and far field analyses. Shown in Figure 2.13 are the computed circulation and coefficient of lift in the rotor wake at a relatively high advance ratio $\mu = 0.45$. On the advancing size, the peak non-dimensional circulation is 15% larger in the lifting line rotor of Figure 2.13b as
compared with the same maximum non-dimensional circulation on the lifting surface rotor of Figure 2.13c. Further, while the contours have a similar basic shape, the lifting line rotor has certain irregular details not present in the non-dimensional circulation of the lifting surface rotor. These differences suggest that when analyzing lower aspect rotors, differences on the order of 10\% can be expected between the lifting line and lifting surface approaches. These differences are primarily due to the nature of instantaneous lift – and therefore $\Gamma$ – generation in lifting line. This instantaneous change is what can cause the irregular edges of non-dimensional circulation in Figure 2.13b. In a physical system this kind of instantaneous change is not possible; the lifting surface model smooths these edges in the solution since lifting surface circulation cannot instantaneously change. Beyond instantaneous change, when comparing the two maximum contours of circulation there is a $2^\circ - 4^\circ$ phase shift difference in $\psi$, primarily due to the aforementioned unsteady effects. Because of these differences at the local level, differences on the same order are expected in global quantities such as lift and power.
Figure 2.13: Computed non-dimensional circulation $\bar{\Gamma}$ and sectional lift coefficient $C_\ell$ for an untrimmed rotor with aspect ratio 10 in forward flight of $\mu = 0.45$ computed using lifting line [14] and lifting surface with harmonic balance and static condensation theories. Ten harmonics were used in the reduced order model.
The computed global lift and thrust are visualized in Figures 2.14 and 2.15 as a function of advance ratio for lifting line and lifting surface with a far field and near field approach. Looking at lift in Figure 2.14 as a function of advance ratio for these analyses, using only one coefficient does not result in accurate answers when compared with lifting line. This seems to suggest that higher harmonics are necessary to model the physical system at lower aspect ratios.

Figure 2.14 also demonstrates a large difference between the results at 3 harmonics and 10 harmonics. When looking individually at the near field or far field results, around \( \mu = 0.35 \) there is a steep divergence between the 10 coefficient model and the 3 coefficient model. This suggests that more coefficients are necessary for the model at lower aspect ratios or higher advance ratios. Once 5 coefficients were used in the model of the system, this steep divergence when compared to the 10 coefficient model disappeared, and similar to the lift of the high aspect ratio rotor in Figure 2.9 there was almost no difference between the near field and far field results when comparing the 10 or 5 coefficient models. All of this suggests that for lower aspect ratio rotors it is necessary to use more coefficients; approximately 5 coefficients is sufficient to model a rotor with an aspect ratio on the order of 10.

Finally, the induced power was computed and is presented in Figures 2.16 and 2.17, as a coefficient and normalized by the lift coefficient squared. Similar to the conclusions drawn from Figure 2.14, approximately 5 coefficients is sufficient to model the power of the system. When comparing the 5 coefficient results of induced power to the lifting line results, there is a 15% difference in value. This difference is in alignment with the differences on the local level in Figure 2.13, but is in conflict with the negligible difference between lifting line and lifting surface in normalized induced power seen in Figure 2.17. This negligible difference can be explained by comparing the differences in lift in Figure 2.14 and power in Figure 2.16; the induced power of lifting surface is approximately 15% lower than the induced power of lifting
line, while the lift of lifting surface is approximately 5%-10% lower than the lift of lifting line. Proportionally, the numerator and denominator of the normalized power loss have changes that cancel each other out, resulting in approximately the same normalized power as lifting line.

2.11 Conclusions

Using the lifting surface technique, together with Guyan reduction and harmonic balance, results in reduced computational complexity with a possible loss in accuracy on the order of 10%. At high aspect ratios, lifting line and lifting surface are in good agreement as expected. As the aspect ratio decreases, some notable differences in the solutions appear. These differences are due to the increase in the reduced frequency with blade chord. Lifting surface can model unsteady airfoil effects, while lifting line cannot. This difference in approach is what causes the results for power, lift, and thrust to be approximately 10% when using lifting surface versus lifting line. With regard to harmonic balance, in almost all high aspect ratio cases, and a majority of low aspect ratio cases, $N_{\text{coeff}} = 3$ is sufficient to model the rotor, with errors less than 2% of the converged forces and induced power. In some cases, more harmonics are required, but good convergence is achieved in all cases for $N_{\text{coeff}} \leq 10$. 
Figure 2.9: $C_L$ for AR=100 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.10: $C_T$ for AR=100 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.11: $C_{Pi}$ for AR=100 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.12: Non-dimensional power loss for AR=100 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.14: $C_L$ for AR=10 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.15: $C_T$ for AR=10 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{\text{coeff}}$, used in the reduced order model.
Figure 2.16: $C_{Pi}$ for AR=10 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{coeff}$, used in the reduced order model.
Figure 2.17: Non-dimensional power loss for AR=10 blade in untrimmed flight; lifting line theory [14] and lifting surface with various numbers of harmonics, $N_{\text{coeff}}$, used in the reduced order model.
In this chapter, we present an optimization procedure to minimize the total power loss – using the models for force and power described in Chapter 2 – of a given rotor system subject to constraints on lift, trim and realizability. The Kuhn-Tucker conditions are presented, and then used to develop a variational statement that can be used with the rotor system. A variational statement is formed for the lifting line and lifting surface model with equality constraints to find the optimum linear design parameters (blade twist and pitch) that minimize induced power. Because the power is quadratic and the constraints are linear, the problem is a so-called quadratic programming problem. The constrained optimization is thus found by solving a set of linear equations resulting in a unique optimization. Results of the optimization using these techniques are presented for both lifting line and lifting surface models. We demonstrate harmonic convergence of the harmonic balance method.
3.1 Description of Optimization

We seek to find the minimum total power of a conventional rotor, $P$, for a given forward flight velocity determined by the advance ratio, $\mu$. First, we will consider a system where the only parameters in the design space are circulation, $\Gamma$, and will only impose two constraints: a required lift and trim equality constraints. This result is known as the “Rubber Rotor” solution, since no constraint is place on the actual geometry of the blade. Next, we will add the blade twist and pitch into the system as design parameters, and add another constraint on the realizability of the rotor by enforcing a geometric, no penetration boundary condition on the blade. We will assume that the blade twist and pitch are small, resulting in a linearized no penetration boundary condition.

In Chapter 4, we will introduce chord distribution as a design parameter. This will create multiple nonlinearities throughout the system, as well as other inequality constraints, which will be discussed later.

3.2 Theoretical Optimization Procedure

The optimizations needed for this analysis all take the form

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0 \\
& \quad h_j(x) = 0
\end{align*}$$

(3.1)

where $x$ is the set of design variables for the given problem, $g_i$ is the $i$th inequality constraint imposed on the system, and $h_j$ is the $j$th equality constraint imposed on the system. To solve this system, the Karush-Kuhn-Tucker (KKT) conditions are applied [38]. This includes the stationary requirements, given by

$$-\nabla f = \sum_{i=1}^{m} \mu_i \nabla g_i + \sum_{j=1}^{\ell} \lambda_j h_j.$$ 

(3.2)
Here, μ & λ are inequality and equality constants, respectively. The primal feasibility requirements are given by
\[ g_i(x^*) \leq 0 \quad \text{for} \quad i = 1, \ldots, m \] (3.3)
\[ h_j(x^*) = 0 \quad \text{for} \quad j = 1, \ldots, \ell \] (3.4)

Finally, we require
\[ \mu_i \geq 0 \quad \text{for} \quad i = 1, \ldots, m \] (3.5)

This insures that the inequality constraint is only active if the optimum solution would otherwise lie in the infeasible region.

Using the KKT conditions, it is possible to rewrite the stationary and primal feasibility requirements into one concise variational statement. By writing the functional
\[ \Pi = f(x) + \lambda^T h(x) + \mu^T g(x) \] (3.6)
the stationary conditions can be written as \( \frac{\partial \Pi}{\partial x} = 0 \), and the primal requirements can be written as \( \frac{\partial \Pi}{\partial \lambda} = 0 \) and \( \frac{\partial \Pi}{\partial \mu} = 0 \). This can be expressed as
\[
\begin{bmatrix}
\frac{\partial \Pi}{\partial x} \\
\frac{\partial \Pi}{\partial \lambda} \\
\frac{\partial \Pi}{\partial \mu}
\end{bmatrix} = 0
\] (3.7)

This representation of the KKT conditions can also be connected to the fundamental lemma of the calculus of variations. By taking the variation of the functional \( \Pi \) one obtains
\[ \delta \Pi = \delta x \left( \frac{\partial \Pi}{\partial x} \right) + \delta \lambda \left( \frac{\partial \Pi}{\partial \lambda} \right) + \delta \mu \left( \frac{\partial \Pi}{\partial \mu} \right) \] (3.8)

Applying the information from Equation (3.7), one can infer that the variation of \( \Pi \) is zero, since all elements in parenthesis are zero. The fact that \( \delta \Pi = 0 \) is zero is therefore independent of the small variation of the design variables \( x \) or the multipliers \( \lambda \) and \( \mu \).
The dual feasibility requirement is not included in this representation. For equality constraint problems, this is not an issue since \( \mu = 0 \) is only related to inequality constraints. When inequality constraints must be included, the usage of slack variables avoids the inclusion of the dual feasibility requirements. In essence, the inequality is transformed into

\[
g(x) + s = 0
\]  

(3.9)

where \( s \) represents the slack variable for the given constraint and \( s_i \geq 0 \ \forall \ i \). As long as all values of \( s \geq 0 \) then it is guaranteed that \( g(x) \leq 0 \). When \( s_i > 0 \) the constraint is inactive since it is not pushing the equality; when \( s_i = 0 \) the constraint is said to be active, since this is the smallest possible value \( s_i \) can have before \( x \) becomes infeasible. Slack variables become their own unknown in the system, increasing the size of the design space.

### 3.3 Lifting Line Optimization Process

To optimize the lifting line model, the system will first be translated into the functional

\[
\Pi = P + \lambda_F^T \cdot (F - F_R) + \lambda_M^T \cdot (M - M_R)
\]

(3.10)

where the variable to be optimized will be the total power, \( P = P_t + P_v \), subject to constraints on the global forces and moments, \( F_R \) and \( M_R \). Substitution of Equations (2.11) and (2.17) into Equation (3.10) gives

\[
\Pi = \frac{1}{2} \Gamma_w^T K \Gamma_w - \Gamma_w^T Q + P_{v0} + \lambda_F^T \cdot (B \Gamma_w - F_R) + \lambda_M^T \cdot (D \Gamma_w - M_R)
\]

(3.11)

where \( K = (K_{inv} + K_{visc}) \). Taking the variation of Equation (3.11) and setting the result to zero (\( \delta \Pi = 0 \)) for arbitrary values of \( \delta \Gamma_w \) and \( \delta \lambda \) gives the desired necessary
conditions for optimality, i.e.

\[
\begin{bmatrix}
\frac{1}{2}(K + K^T) & B^T & D^T \\
B & 0 & 0 \\
D & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Gamma_w \\
\lambda_F \\
\lambda_M
\end{bmatrix}
= \begin{bmatrix}
Q \\
F_R \\
M_R
\end{bmatrix}.
\]  

(3.12)

Note that because of the linear nature of the lift and moment constraints, as well as the quadratic nature of the induced power constraint, the end result of the variational approach is a linear system of equations. Finding the optimum solution requires just one linear matrix solve to find the optimum circulation distribution \(\Gamma_w\).

Note this approach finds the optimal circulation in the wake, \(\Gamma_w\), and the corresponding Lagrange multipliers \(\lambda\), without regard to the controls or twist distributions required to achieve this optimum. This result is therefore known as the “Rubber Rotor” result, since no constraint is placed on the actual geometry of the blade. The optimum circulation in this case may not be realizable.

### 3.3.1 Optimization Including Geometry

To include the influence of the blade geometry, a modification to the original variational statement is made to include an extra set of constraints \(R\),

\[
\Pi = \frac{1}{2}\Gamma_w^T K \Gamma_w - \Gamma_w^T Q + P_{v0} + \lambda_F^T \cdot (B \Gamma_w - F_R) + \lambda_M^T \cdot (D \Gamma_w - M_R) + \lambda_R^T \cdot (R(\Gamma_w, \Theta))
\]  

(3.13)

where \(\Theta\) represents the design constraints of the blade such as twist distribution, chord distribution, and cyclic controls. \(R\) is a realizability constraint that describes the relationship between the blade geometry and the resulting circulation distribution. With the addition of a constraint that may be nonlinear, the resulting
optimality conditions may be nonlinear. In general, they will take the form

\[
\begin{bmatrix}
\frac{1}{2}(K + K^T) & B^T & D^T & R^T_{\Gamma_w} \\
B & 0 & 0 & 0 \\
D & 0 & 0 & 0 \\
0 & 0 & 0 & R^T_{\Theta}
\end{bmatrix}
\begin{bmatrix}
\Gamma_w \\
\lambda_F \\
\lambda_M \\
\lambda_R
\end{bmatrix} =
\begin{bmatrix}
0 \\
F_R \\
M_R \\
0
\end{bmatrix}
\tag{3.14a}
\]

\[
R(\Gamma_w, \Theta) = 0 .
\tag{3.14b}
\]

In previous work [3, 14, 18, 31] \( R \) was a realizability constraint based on lifting line theory, that is Equation (3.14b) describes the solution \( \Gamma_w \) resulting from a set of rotor design variables \( \Theta \), as found using a lifting line model. Here \( \Theta \) can include design variables such as the blade chord distribution, twist distribution, collective and cyclic controls, etc. In general, the resulting set of equations are nonlinear. The realizability constraint imposed is the same as the direct analysis technique for lifting line outlined in Chapter 2; in this case, \( \theta \) is treated as an unknown and the equation \( R \) is substituted into Equation (3.14).

Note that the values of \( A \) and \( b \) found in \( R \) will depend nonlinearly on the geometry of the system, \( \Theta \). For example, the prescribed value of the flow will depend not on the twist angle \( \theta \), but on \( \sin \theta \). For the purposes of this analysis, we assume small angle approximations, so these nonlinearities become weak and the variational statement remains linear. This assumption makes the realizability constraint \( R \) linear in the design space. Physically, this may result in poor answers near the hub of the rotor, since larger twist angles are experienced in this region.

3.4 Lifting Surface Optimization Process

Similarly to the lifting line problem, we define the optimal rotor using a variational statement. As described in Chapter 2, we use the far field approach to calculate the
forces and power in the variational statement, but the near field moment calculation will be used. Modification must now be made to Equation (3.13) computing the moments using the vortex strengths on the airfoil surface, $\Gamma_a$, rather than the far field circulation, $\Gamma_w$, i.e.,

$$\Pi = \frac{1}{2} \Gamma^T_w K \Gamma_w - \Gamma^T_w Q + P_{v0} + \lambda^T_F \cdot (B \Gamma_w - F_R) + \lambda^T_M \cdot (D_N \Gamma_a - M_R)$$

$$+ \lambda^T_R \cdot (R(\Gamma_w, \Gamma_a, \Theta)).$$

(3.15)

Adding the additional degrees of freedom associated with the blade surface elements increases the number of unknowns, and hence the size of the matrix equation to be solved.

Next, taking the variation of Equation (3.15) and setting the result to zero gives the (nonlinear) equations describing the necessary conditions for optimality, i.e.,

$$\begin{bmatrix}
\frac{1}{2} (K + K^T) & 0 & B^T & 0 & R^T_{\Gamma_w} \\
0 & 0 & 0 & D^T_N & R^T_{\Gamma_a} \\
B & 0 & 0 & 0 & 0 \\
0 & D_N & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & R^T_\Theta
\end{bmatrix} \begin{bmatrix}
\Gamma_w \\
\Gamma_a \\
\lambda_F \\
\lambda_M \\
\lambda_R
\end{bmatrix} = \begin{bmatrix}
Q \\
0 \\
F_R \\
M_R \\
0
\end{bmatrix},$$

(3.16a)

$$R(\Gamma_w, \Gamma_a, \Theta) = 0.$$  (3.16b)

It is apparent from Equation (3.16) that the function $R$ plays an increasingly larger role in the matrix solution space as we add fidelity to the model, making the calculation more complex.

Equation (2.31) can be used in place of $R$ in Equation (3.16) to include geometry as a constraint in the optimization. Like Equation (2.20) for lifting line, Equation (2.31) makes a small angle assumption for twist and pitch. This allows Equation (3.16) to become linear.
3.5 Numerical Optimization Procedure

After substituting Equations (2.20) and (2.31) into Equations (3.14) and (3.16), respectively, the variational statements created are quadratic in the design space (i.e. $\Gamma$, $\Theta$, and $\lambda$). Problems of this type can be solved numerically in one iteration using Quadratic Programming (QP).

QP is a process used to solve a quadratic optimization problem with linear constraints, i.e.

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2}x^T Ax + c^T x \\
\text{subject to} & \quad Bx \leq b
\end{align*}
\]

where $x$ represents the design space to be optimized. Linear optimization statements like this come from the lifting line and lifting surface approach, as long as no nonlinear variables such as chord are represented. Further, all of these optimization statements have equality constraints, so it is possible simplify Equation (3.17) further to

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2}x^T Ax + c^T x \\
\text{subject to} & \quad Bx = b
\end{align*}
\]

By creating the functional $\Pi$ of the problem statement in Equation (3.18) and taking the variation, one obtains

\[
\delta \Pi = \delta x^T \left[ \frac{1}{2} (A + A^T) x + c + B^T \lambda \right] + \delta \lambda^T [B x - b].
\]

As described previously, $\delta \Pi = 0$ must be independent of other variations, such as $\delta x^T$, equaling zero. This creates the system of equations

\[
\begin{bmatrix}
A & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
-c \\
b
\end{bmatrix}
\]
which can be solved directly as long as the matrix is not singular. This direct solve approach is accurate due to the nature of the QP problem statement; there is no issue regarding local or global optimization since a quadratic function only has one minimum. If the constraints are also linear, the product of the constraints with the Lagrange multipliers in $\Pi$ creates another quadratic term in the functional. For example, if a linear constraint involving the design variables $D\Theta$ is added to Equation (3.18) the resulting variational statement takes the form

$$\begin{bmatrix}
\frac{A+A^T}{2} & B^T & D^T \\
B & 0 & 0 \\
D & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\Theta \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
-c \\
0 \\
b
\end{bmatrix}, \quad (3.21)$$

which is similar to the variational statement seen in Equation (3.12).

### 3.6 Rotor Test Cases

The blade used in these test cases is the same as the rotor described in Chapter 2, with two differences. First, more turns of the wake are included in the model (from 40 to 50) to ensure the induced effects of the wake are accurately modeled. Second, to generate a more realistic blade, the cutout has been decreased from 50% to 10%; as before, the non-dimensional radius will still be measured from the hub of the rotor, i.e., $R = R/c$. This rotor will experience reverse flow for $\mu > 0.1$ on portions of the retreating blade.

The rotor is trimmed in pitch and roll, and has a required lift coefficient of $C_{l,\text{req}} = 0.00925$. The conventional rectangular blade has a chord of 0.091, creating a blade with aspect ratio 11 and solidity (and thrust weighted solidity) of $\sigma = \sigma_{\text{TW}} = 0.1157$. This gives $C_l/\sigma = 0.08$. These rotor parameters, as well as a summary of grid and physical parameters, are included in Table 3.1.

The parameters of advance ratio and number of Fourier coefficients will also vary in these analyses. Since the design of the rotor is not set a priori the twist and
controls of the rotor will also change. Use of cyclic controls trims the rotor in pitch, roll, and lift, while higher harmonic controls allow the wake distribution to become more efficient. These designs are included as coefficients in the series

$$\theta_{\text{controls}}(\psi) = \sum_{n=0}^{N_{\text{rev}}} [A_n \cos(n\psi)] + \sum_{n=1}^{N_{\text{rev}}} [B_n \sin(n\psi)]$$ (3.22)

where $N_{\text{rev}}$ represents the number of higher harmonic controls included. The 1/rev control includes the collective term $A_0$, and cyclic terms $A_1$ and $B_1$. As the blade moves azimuthally, these coefficients change the root control of the blade creating more or less pitch angle. The radial twist distribution is constructed using Legendre polynomials along the length of the blade using

$$\theta_{\text{twist}}(r_i) = \sum_{n=1}^{N_{\text{twist}}} \Theta_{\text{twist}} L_n(\hat{r}_i)$$ (3.23)

where $N_{\text{twist}}$ represents the number of Legendre polynomials used in the model, $L_n$ is the nth normalized Legendre polynomial, and $\hat{r}_i$ the normalized radial position of a given panel,

$$\hat{r}_i = \frac{r_i - \frac{1}{2}(R + R_{\text{cutout}})}{\frac{1}{2}(R - R_{\text{cutout}})}.$$ (3.24)
This scales the radial distribution to be between -1 and 1, which is the required domain to use the normalized Legendre polynomials in a linear independent series expansion. The zeroth Legendre polynomial is also absent from this representation since it is already included in the collective term $A_0$.

### 3.6.1 Harmonic Balance Convergence

To establish the appropriate number of harmonics to use in the harmonic balance technique, the blade described was analyzed for various values of $N_{coeff}$. The results of this study are seen for a variety of advance ratios in Figure 3.1, and in more detail for an advance ratio of $\mu = 0.40$ in Figure 3.2.

Figure 3.1 shows the optimized normalized induced power as a function of the advance ratio for various values of per-rev control ($N_{coeff}$). Physically, the induced power increases towards a critical advance ratio around $\mu = 0.70$, and then decreases roughly with $1/\mu$. As described by Ormiston [39], this behavior is caused by the growing reverse flow region on the retreating side of the blade. As more of the retreating blade is experiencing negative lift, the pitch and twist must generate enough positive lift on the outer portion of the blade to counter the positive lift on the advancing blade to maintain trim; this generates large pitch and twist values, which in turn generate larger induced powers.

Once three harmonics are used, all results are within 10% error when compared with the highest level of harmonics. Once five harmonics are used, the results are nearly indistinguishable from 10 harmonics; this supports the conclusion that only 5 harmonics are required in the model reduction to obtain precise results. Another trend seen in Figure 3.1 is the variance of the plots as the per-rev controls and $\mu$ change. For the rubber rotor and 1/rev optimal, there is little difference between the 3 harmonics and 5 harmonics curves at low $\mu$. As $\mu$ increases, this grows to a point where there is at least a 10% difference in normalized induced power. This
demonstrates that more higher harmonic content is present in the solution as the rotor flies faster or uses more control inputs.

This last conclusion can be investigated further in Figure 3.2, which plots the non-dimensional circulation, defined as

$$\hat{\Gamma} = \frac{\rho \Omega R^2 \Gamma}{L},$$  \hspace{1cm} (3.25)

and the local non-dimensional lift, defined as

$$C_\ell = \frac{\ell}{\rho \pi R^2 (\Omega R)^2},$$  \hspace{1cm} (3.26)

in the rotor frame of reference for 3, 5, and 10 harmonics in the reduced order model. An advance ratio of $\mu = 0.4$ and 3/rev controls were chosen to reflect a realistic forward flight velocity, while still capturing a significant difference in the models. A large positive circulation is developed in all models on the retreating side around $r/R = 0.6$; as explained previously, this circulation is necessary to balance the negative lift due to reverse flow, as well as the positive lift on the advancing side of the blade (which can also be seen in the local lift coefficients of Figure 3.2).

When only 3 harmonics are used, there is an asymmetry to the circulation and lift on the retreating side of the blade that is altered when using 5 or 10 harmonics. Further, a prominent spike of lift on the advancing side of the rotor is present in the 5 and 10 harmonic models that is not as refined in the 3 harmonic model. Given there are at least three peaks in the models of Figures 3.2e and 3.2f, then using three harmonics in the model is the bare minimum to begin capturing these features. Looking closer at Figure 3.1, once 5 harmonics are used any set of controls or $\mu$ values are well within 1% of the 10 harmonics result, suggesting that the features seen in the 5 and 10 harmonic results of Figure 3.2 are the largest amount of harmonic content the model will need to accommodate. For all of these reasons, all future results will use 5 harmonics, i.e. $N_{\text{coeff}} = 5$. 

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**Figure 3.1**: Normalized induced power as a function of advance ratio; various amounts of controls and $N_{\text{coeff}}$.

(a) $N_c = 3, \bar{\Gamma}$  (b) $N_c = 5, \bar{\Gamma}$  (c) $N_c = 10, \bar{\Gamma}$

(d) $N_c = 3, C_l$  (e) $N_c = 5, C_l$  (f) $N_c = 10, C_l$

**Figure 3.2**: Comparisons of Non-dimensional Circulation ($\bar{\Gamma}$), and sectional Coefficient of Lift ($C_l$) for $\mu = 0.40$ and 3/rev controls. Moving from left to right, various harmonics used in the reduced order model $N_{\text{coeff}}$. 

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3.6.2 Single-Point Optimization

The rectangular rotor is now optimized at a single forward flight speed, with the given constraints of lift, trim, and no-penetration on the blade. As stated previously, all computations are run with 5 harmonics in the reduced order model to ensure accurately modeling the lift, moment, and power of the rotor.

The comparison of induced power at the various advance ratios can be seen in Figure 3.3. Each color represents a different set of per-rev controls, while the dotted line and solid line represent no-twist and twist, respectively. As more controls and twist are used in the design, Figure 3.3 demonstrates induced power improvements. This is expected since adding more flexibility to the design allows the rotor to mimic the “Rubber Rotor”. Increasing from 2/rev to 3/rev controls at an advance ratio of 0.65 results in a 30% decrease in normalized power; by adding twist this decreases the power another 5%. According to theory, the induced power at the critical advance ratio for 1/rev control is infinite; 1/rev control can not increase the pitch on the retreating blade to combat the growing reverse flow region without decreasing the pitch on the advancing side.

When the rotor is at lower flight speeds, the resulting optimized normalized power is only about 5% more than the limiting rubber rotor result even with just 1/rev controls. At higher advance ratios, the benefits of using higher harmonic controls and a fixed optimum twist becomes dramatic. However, as one approaches 3/rev controls, diminishing returns result. Around the critical advance ratio where the most improvement is experienced, the difference between using 1/rev and 2/rev controls is close to 95%, and from 2/rev to 3/rev the difference is around 40%. This trend of marginal decreases in improvement suggests that using 4/rev, 5/rev, or even 10/rev results will lead to less improvement of the normalized induced power. Beyond the mathematical model, it is also difficult and expensive to implement a conventional
rotor with large amounts of higher harmonics controls for pitch, especially when there are other static parameters that are easier to vary like twist and chord distribution. This leads to a conclusion that a rotor practically doesn’t need more than 3/rev controls to experience the most efficient improvements in induced power.

The blade pitch angle as a function of azimuth and the optimum twist distribution as a function of normalized radius for certain forward flight conditions are visualized in Figures 3.4 and 3.5. Before the critical advance ratio at about $\mu_{\text{crit}} = 0.70$, the controls of Figure 3.4 demonstrate roughly 1/rev behavior, with a small control on the advancing side of the motion, and a larger angle on the retreating side to compensate for less inflow velocity; this ensures lift is roughly equal on the two sides of the rotor, thus keeping it in trim. As the advance ratio crosses this critical value, the controls become extremely negative on the retreating side; the definition of positive angle of attack has switched at this point so the blade must rotate the opposite direction in order to continue generating positive lift. A similar abrupt change is present in the twist of Figure 3.5, albeit to a lesser degree. Before the critical advance ratio, the blade is twisted negatively due to the reverse flow at the root, and becomes positive outboard of the reverse flow region, decreasing to near zero at the tip. As the advance ratio increases towards the critical advance ratio, the peak of the twist adjusts to match the reverse flow region on the retreating side of the blade. Beyond the critical advance ratio, the twist at the root becomes positive and monotonically decreasing.

When combining the twist and controls of Figures 3.4 and 3.5, one can obtain the total pitching angle of the blade. For $\mu < \mu_{\text{crit}}$, the total is well within the small angle assumption requirements outlined in Chapter 2. For $\mu < \mu_{\text{crit}}$, certain locations on the retreating side of the blade have an angle much larger than the small angle assumption allows, suggesting the need to model large angles of attack effects. However, considering the conventional rotor will never realistically fly beyond $\mu = 0.50$, this is more of a theoretical exercise than a practical one.
**Figure 3.3:** Normalized induced power as a function of advance ratio; multiple per-rev controls and twist illustrated.

**Figure 3.4:** Control evolution of 3/rev controls with twist at various advance ratios.
Lifting Line Comparisons

Similar results to those above have been found by Hall [18], Hall & Giovanetti [18], and Giovanetti [31] by using a similar optimization technique with a lifting line model. This section compares the results of this approach with the results of Giovannetti [31] which used a lifting line model similar to the one described in Chapter 2.

First, the normalized powers are compared over multiple advance ratios in Figure 3.6. Especially for lower advance ratios, there is near perfect agreement between the current lifting surface model and the lifting line model of Giovanetti. Differences begin to arise at the higher, post-critical advance ratios. These differences are caused by two variations in the models: the lifting surface model physically models the blade and can more accurately capture wake-blade interactions near this point, and the lifting line model of Giovanetti employs a slightly different tactic to viscous forces, opting for lookup tables over the quadratic drag polar implemented in this code. Since viscous forces tend to increase with the square of the velocity, viscous effects rise in prominence in the section where Giovannetti’s results differ from the current model, suggesting the difference in viscous terms could have a substantial effect. The controls and twist are compared between lifting surface and lifting line in Figures 3.7 and 3.8 for an advance ratio of 0.4. Unsurprisingly, the results demonstrate significant similarities between the predicted optimal blade pitch for the lifting line and lifting surface theories.

3.6.3 Model Reduction Timing Comparisons

To better understand the improvements of static condensation and harmonic balance from a timing perspective, a single point optimization was conducted at a moderate advance ratio of $\mu = 0.40$, and timed for three different cases: once without any model reduction techniques, once with static condensation, once with harmonic balance, and once with static condensation and harmonic balance. These timing results are
Table 3.2: Timing analysis conducted for $\mu = 0.40$ and various amounts of model order reduction.

<table>
<thead>
<tr>
<th>Reduction Used</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>5.2</td>
</tr>
<tr>
<td>Static Condensation</td>
<td>4.9</td>
</tr>
<tr>
<td>Harmonic Balance</td>
<td>4.0</td>
</tr>
<tr>
<td>Static Condensation &amp; Harmonic Balance</td>
<td>3.6</td>
</tr>
</tbody>
</table>

tabulated in Table 3.2, where each row represents the average of five separate timings for the given reduction. By using both static condensation and harmonic balance, a 30% reduction in time is experienced. Most of this savings comes from harmonic balance, which has reduced the size of the system from the order of thousands to the order of hundreds. While there is significant startup time in the form of matrix multiplication, this only has a complexity that is $O(n^2)$, while the solve itself is typically $O(n^3)$.

3.7 Conclusions

Using the model reduction techniques of Guyan reduction and harmonic balance lead to significant improvements in computational speed by reducing the size of the system. At least for the purposes of designing minimum power rotors, most of the relevant physics can be modeled using just a few harmonics, often as few as three. Results obtained with the small angle assumption to find the blade pitch and twist design parameters are similar to lifting line in power and design, with differences on the order of 10%, a value that is in agreement with the non-optimized results of Chapter 2.
**Figure 3.5:** Twist evolution of 3/rev controls with twist at various advance ratios.

**Figure 3.6:** Normalized induced power as a function of advance ratio; comparison with Lifting Line results.
Figure 3.7: Control comparison of 1/rev and 3/rev controls with twist at \( \mu = 0.4 \); comparison with Lifting Line results.

Figure 3.8: Twist comparison of 1/rev and 3/rev controls with twist at \( \mu = 0.4 \); comparison with Lifting Line results.
In Chapter 3, we optimized the twist distribution of rotor blades, along with the collective, cyclic, and higher harmonic blade pitch control, to find minimum loss designs for rotors in forward flight. Such QP problems have cost functions that are quadratic in the unknown design variables and circulation, with linear constraints.

In this chapter, we perform similar single point optimizations, but now add the chord distribution along the span of the blade as a design variable. The result is a (nearly) quadratic cost function with nonlinear constraints. We solve the resulting nonlinear programming problem using the open source optimization code IPOPT, which uses gradient based search algorithms to find local minima.

4.1 Optimization Statement

The optimization problem here is the same as in Chapter 3, except the design variables $\Theta$ now include $c(r)$, the chord distribution along the span of the rotor blades. Unlike blade twist and pitch, which appears linearly in the realizability constraints
if a small angle approximation is used, blade chord is highly nonlinear. For example, the matrix $A_{aa}$ in Equation (2.31) that describes the induced wash on the blade due to the circulation on the wake scales roughly like $1/c$ [30], making the trim and realizability constraints nonlinear. All matrices in Equations (3.14a) and (3.16) must now become functions of the chord parameters (i.e. $K(\theta_c)$, $B(\theta_c)$, etc). The highly nonlinear nature of these functions makes them difficult to express analytically; it is therefore necessary to use finite differences to calculate the derivatives required for optimization.

4.1.1 Practical Constraints on the Chord of a Blade

When optimizing the chord distribution of a blade, the practicality of the result must also be considered. With faster forward flight velocities, the large dynamic pressure on the advancing side of the blade makes large chord distributions unnecessary to maintain a prescribed lift. These small chord distributions are not practical for a rotor for two reasons: first, the internal stress on a blade scales with $1/c$, so a small chord will result in large internal stresses that can cause permanent damage to a rotor. Second, at lower forward flight speeds the small chord rotor is unable to maintain the given lift, making the rotor impractical in scenarios like hover.

Two methods are used to make the constructed rotor more realistic. First, a chord inequality constraint is placed on the chord design space so

$$c(r_i) \geq c_{\text{min}} \quad (4.1)$$

where $c_{\text{min}}$ is chosen to be a small, but acceptable chord value. Second, a constraint is placed on thrust weighted solidity of the rotor, $\sigma_{\text{TW}}$, so

$$\sigma_{\text{TW}} = \frac{3}{R^3} \int_{R_{CO}}^{R} r^2 \sigma(r) \, dr \geq \sigma_{\text{TW, min}} \quad (4.2)$$

As described in Johnson, the thrust weighted solidity is a measure of how effective
a rotor is at hovering [29]. Constraining the system to a minimum thrust weighted solidity ensures the rotor designed for cruise here has sufficient solidity in hover.

Equation (4.2) weights the chord distribution more heavily at the tip of the blade; since the circulation tends towards zero at the tip, using Equation (4.2) without modification can cause unusual blade shapes at the tip. The expression for \( \sigma_{TW} \) is therefore modified slightly to go to zero at the blade tip, i.e.

\[
\sigma_{TWM} = \frac{3}{R^3} \int_{R_CO}^{R} \sigma(r) \cdot \{1 - \exp\left[-\left(1 - \frac{r}{R}\right) / \varepsilon\right]\} \, dr 
\]

(4.3)

where \( \varepsilon \) is a penetration depth that indicates roughly where the roll off begins (typically 0.10). The addition of the exponential roll off at the tip prevents artificially large tip chords.

These additional inequality constraints are included in the optimization statement through \( R \) with slack variables, as explained in Chapter 3. The slack variables introduce another layer of complexity to Equation (3.16) since the slack variables must be positive in the solution. For a simple system, this is not hard to impose analytically, but given the nonlinearity of the rotor system described earlier, the optimum of the rotor system when including chord distribution as a design variable will have to be solved with an iterative procedure.

### 4.2 Numerical Optimization Procedure

Chapter 3 introduced the QP optimization procedure to solve linear, single point optimization problems. The rotor optimization statements of Chapter 3 are quadratic in the design and multiplier variables, allowing the optimum to be calculated from one linear matrix equation (i.e. Equation (3.21)). Due to the nonlinearities introduced by the inclusion of chord distribution design variables, as well as inequality constraints, the resulting optimization problem is no longer QP. Alternatives to this method must
be considered.

A popular iterative approach is to use what are known as gradient free, heuristic methods. These techniques rely on probability, evolution, and sometimes randomness to find constrained optimums. Methods include the following: random search techniques, genetic algorithms, and simulated annealing [40]. Open source packages also exist that use heuristic processes such as NOMAD [41, 42], which uses a mesh refinement technique to isolate the optimum point in a given system. These methods do not scale well with the parameter space, however, and certain algorithms are only designed to solve very specific problems.

Instead, an iterative approach based on gradient methods is considered. The simplest of these approaches is a Newton iteration, where we seek to

$$\text{minimize } \phi(x, \Theta)$$

subject to

$$R(x, \Theta) = 0.$$ (4.4)

Creating the functional and taking the variation of Equation (4.4) creates

$$\begin{bmatrix} \Pi_x \\ \Pi_\Theta \\ R \end{bmatrix} = 0$$ (4.5)

which reverts back to the QP approach if the objective function $\phi$ is quadratic and the constraints $R$ are linear in the design space. Newton iteration linearizes Equation (4.5) to solve the system in a set of iterations. This creates the iterative matrix equation

$$\begin{bmatrix} \Pi_{xx} & \Pi_{x\Theta} & R_x^T \\ \Pi_{\Theta x} & \Pi_{\Theta\Theta} & R_{\Theta}^T \\ R_x & R_{\Theta} & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \Theta \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \Pi_{x0} \\ \Pi_{\Theta0} \\ R_0 \end{bmatrix}$$ (4.6)

where $\Delta x, \Delta \Theta$, and $\Delta \lambda$ represent the changes in physical variables, design space variables, and Lagrange multipliers at a given iteration. Equation (4.6) will be
iterated until the right-hand side becomes the zero vector, at which point the system is converged.

Newton iteration encounters three problems concerning practicality: (1) the creation of the matrix on the left hand side of Equation (4.6) and its solution is computationally expensive; (2) the step-size can overstep the solution, which can sometimes lead to a non-converged solution; and (3) the existence of local minima can cause convergence to a non-global minima.

4.2.1 Description of the Open Source optimizer IPOPT

IPOPT [43], short for Interal Point OPTimizer, is an open source Quasi-Newton approach to solving optimization problems with the form of Equation (4.4). It uses a boundary term, known as $\mu$, to restate all constraints (inequality, equality, and boundary constraints) as a set of equality constraints, then optimizes the system using gradient based search algorithms.

To conduct an optimization with IPOPT, the user must provide the program with the objective function, $\phi$, the constraints, $R$, the gradient of $\phi$, the Jacobian of $R$, and any bounds on physical or design variables $x$ and $\Theta$. IPOPT then conducts the optimization and outputs the optimum solution for the system. $\phi$, $R$, and $\nabla \phi$ can easily be provided analytically by the optimization statements outlined in Chapter 2. The Jacobian of $R$ is calculated analytically where possible, and using a finite difference scheme for the chord design variables $\Theta_c$. As the geometry of the rotor changes with changes in $\Theta_c$, IPOPT modifies the geometry of the wake in order to correctly calculate the coefficient matrices necessary in the constraints of the optimization.

Figure 4.1 lists the important steps of an IPOPT iteration, but some elements warrant further discussion.

The left hand side of Equation (4.6) that IPOPT constructs is created by the
START
\[
\min_{x \in \mathbb{R}^n} f(x) \\
\quad c(x) = 0 \\
\quad x \geq 0
\]

Rewrite system using barrier term
\[
\min_{x \in \mathbb{R}^n} \phi_\mu(x) = f(x) - \mu \sum_{i=1}^{m} \ln(x_i) \\
\quad c(x) = 0
\]

Set barrier parameter $\mu$ to moderate value, then solve system
\[
\nabla f(x) + \nabla c(x) y - z = 0 \\
\quad c(x) = 0 \\
\quad XZ\epsilon - \mu c = 0 \\
\quad x, z \geq 0
\]

Determine Direction of Iteration
Wronskian Required: Use L-BGSV over multiple steps
Inertia of Matrix not guaranteed: Properly set $\delta$

\[
\begin{bmatrix}
W_k + X_k^{-1}Z_k + \delta I & \nabla c(x_k) \\
-\nabla c(x_k)^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x_k \\
\Delta \delta_k
\end{bmatrix}
= -\begin{bmatrix}
\nabla \phi_\mu(x_k) + \nabla c(x_k)y_k \\
\nabla c(x_k)
\end{bmatrix}
\]

Choose Step Size to improve solution
Set direction from previous step, iterate over $\alpha$
Set improvement criteria: Use IPOPT Filter Method
- Has objective function $\phi$ improved?
- Has the norm of constraints $c$ improved?

\[x_k + \alpha_k \Delta x_k = x_{k+1}\]

Is the barrier modified equation sufficiently solved?

YES

Is $\mu$ sufficiently small?

FINAAL SYSTEM OUTPUT

Figure 4.1: Flow Diagram of Decisions made by IPOPT
Figure 4.2: Depiction of Newton Iteration Overshoot; the solution oscillates back and forth instead of making progress towards the minimum in the center [45]. This is primarily caused by poor step size control.

Jacobian of the constraints $\mathbf{R}$ and the Hessian of the overall functional $\Pi$. It is trivial to compute zeroth derivatives by simply evaluating. Likewise, it is easy to calculate the Jacobian of a given vector by using a finite difference scheme with computational complexity $O(n)$. Calculating the Hessian, however, is extremely time consuming to generate accurate results. Each entry in the gradient of Equation (4.5) must have its derivative evaluated three times, placing the evaluation on the order of $O(n^2)$ if the exact functional relationship for the Hessian is known. The exact relationship in this constraint is extremely complicated to compute so it is rarely used, meaning the computational cost can increase even farther depending on how the Hessian and gradient of the function are calculated. IPOPT avoids this computational problem by generating an approximation to the Hessian using the Broyden, Fletcher, Goldfarb, Shanno (BFGS) method [44]. This procedure uses information already calculated in the gradient and functional at previous iterations to create an approximation for the Hessian at the given iteration.

When conducting a Newton iteration, each iteration must provide a step direction and step size. The quality of the step direction is determined by the quality of the provided Hessian, but the step size must also be considered. If the step size at each iteration is not controlled, oscillatory behavior as seen in Figure 4.2 can occur.
IPOPT controls the step size by picking intelligent step sizes to ensure convergence to the center through the Armijo step size rule [46]. This ensures each iteration has a functional that makes “sufficient progress” towards the minimum, and assists in damping the oscillations of Figure 4.2.

4.2.2 IPOPT & QP Comparisons

Using IPOPT, we compute the single point optimal solution for the same rotor considered in Chapter 3; for this comparison, we only consider the “rubber rotor” and linear design variables of twist and pitch in order to make the problem QP. Shown in Figure 4.3 is the induced power as a function of advance ratio for the IPOPT analysis as well as the linear, QP approach. The analysis was conducted for the “rubber rotor”, as well as twist and pitch design variables. These results show that the QP approach and IPOPT package find the same (unique) optimum at all forward flight velocities. This is intuitive since these problems are within the solution feasibility of each of the methods. Solving with the QP approach takes one iteration, while IPOPT requires at least 5 iterations that take on the order of seconds. When solving more complicated rotor systems with nonlinear design variables (i.e. $\Theta_c$), QP is unable to generate a solution, while IPOPT finds a solution within 100 iterations; these iterations take significantly longer than the previous iterations due to the finite difference computation of the Jacobian of $R$.

While IPOPT makes significant improvements over a standard Newton iteration, the method is still susceptible to local minima. Figure 4.4 illustrates the optimization of a given function with multiple local minima using IPOPT and other open source packages. Instead of converging to the absolute minimum, IPOPT converges to a nearby local minimum closest to its starting point. At the same time, other packages like NOMAD are able to converge to the known absolute minimum in the center. Even with the improvements of IPOPT, assuring convergence to an absolute
Figure 4.3: Comparison of Direct Linear Solve using the QP approach, and the commercial package IPOPT. Both optimizations were conducted on the rubber rotor and linear controls and twist systems seen in Chapter 2.

minimum is something gradient based methods can not guarantee.

4.3 Rotor Test Cases

We now apply the methods described above to a series of example problems. The rotor used in these test cases is the same as the rotor described in Chapter 3 with the same lift and trim constraints. Additionally, the chord distribution will be constrained to have a local chord no smaller than 0.1% of the radius, and a thrust weighted solidity no smaller than $\sigma_{TW,\text{min}} = 0.1157$ to maintain feasibility of the rotor design in hover. These updated grid constraints, as well as a summary of grid and physical parameters, are given in Table 4.1. We varied the parameters of advance ratio, number of Fourier coefficients, and twist and control parameters in these analyses. The nonlinear chord design parameters, $\Theta_c$, will also vary. Similar to the twist design, the chord will be represented as

$$c(r_i) = \sum_{n=1}^{N_c} \Theta_{cn} L_n(\hat{r}_i)$$

(4.7)
Figure 4.4: Visualization of IPOPT minimum versus global minimum calculated theoretically and computationally using other methods [47]. The large red circle in the middle is the absolute minimum, indicating that IPOPT has found a local minimum of the system.

where $N_c$ represents the number of Legendre polynomials used in the model, $L_n$ is the $n$th normalized Legendre polynomial, and $\hat{r}_i$ the normalized radial position of a given panel, as defined in Chapter 3.

4.3.1 Local Results: $\mu=0.25$ & $\mu=0.50$

To begin the analysis, we will analyze the circulation, lift, pitch, and chord distribution of the rotor in two forward flight conditions: a relatively low forward flight of $\mu = 0.25$ and a faster forward flight velocity of $\mu = 0.50$. Figure 4.5 depicts the non-dimensional circulation results plotted in the vehicle frame of reference. Each
Table 4.1: Fixed parameters for blade being tested in all future sections.

<table>
<thead>
<tr>
<th>Element</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>4</td>
</tr>
<tr>
<td>$R$</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha_{\text{shaft}}$</td>
<td>$-5.0^\circ$</td>
</tr>
<tr>
<td>$R_{\text{cutout}}$</td>
<td>0.10</td>
</tr>
<tr>
<td>$C_{l,\text{req}}$</td>
<td>0.00925</td>
</tr>
<tr>
<td>$c_{\text{min}}$</td>
<td>0.001</td>
</tr>
<tr>
<td>$\sigma_{\text{TW, min}}$</td>
<td>0.1157</td>
</tr>
<tr>
<td>$N_R$</td>
<td>20</td>
</tr>
<tr>
<td>$N_c$</td>
<td>8</td>
</tr>
<tr>
<td>$N_\psi$</td>
<td>18 ($\times 4$)</td>
</tr>
<tr>
<td>$N_{\text{wake}}$</td>
<td>50</td>
</tr>
</tbody>
</table>

The rotor was optimized for twist and chord distribution, and the different columns depict 1/rev and 3/rev controls. The rubber rotor solution is included for comparison in the right most column. Unlike the 1/rev results of Chapter 3, when including chord distribution the 1/rev non-dimensional circulation has all the important features depicted in the rubber rotor solution’s contours of non-dimensional circulation. For example, when flying at $\mu = 0.5$ the rubber rotor experiences a large circulation on the retreating side of the rotor; most of this is captured with 1/rev control, then refined with 3/rev controls.

Figure 4.5 also depicts the modeling of unsteady effects in the model. When 3/rev controls are used with $\mu = 0.50$ the retreating side of circulation has a 5 degree phase lag when compared with the retreating side of the rubber rotor results, which were collected using a method based on the lifting line approach and therefore are unable to account for unsteady effects\(^1\). Considering the reduced frequency of the system is approximately 0.3, this phase lag is expected.

Looking at the blade design variables directly, Figures 4.6 and 4.8 show the

\(^1\) Said another way, the lifting line method assumes the aspect ratio of the blade approaches infinity, causing the overall reduced frequency to go to zero.
chord distributions of the rotor for the low and high speed forward flight velocities. Figure 4.6 is optimized for conventional (1/rev) controls, while Figure 4.8 is optimized for 3/rev controls. Regardless of the number of controls, the slow speed forward flight case has a large chord at the hub, remains relatively constant until 40% of the non-dimensional radius, then tapers off to less than 5% of the original chord value found at the hub. As the rotor begins to fly faster, the chord distribution changes to have a relatively small chord at the root, increasing to a maximum at approximately 65% of the non-dimensional radius, then tapering off towards the tip of the rotor. When comparing the conventional and higher harmonic control chord distributions more closely, it is apparent that there is more similarity between the $\mu = 0.50$ case than the $\mu = 0.25$ case; the 1/rev $\mu = 0.25$ case has a dip in the chord distribution around $r/R = 0.25$, while the 3/rev $\mu = 0.25$ case seems to have a slight growth in this location.

Moving onto twist, Figures 4.7 and 4.9 show the blade pitch distribution as a function of azimuthal angle for the high and low speed forward flight velocities. Figure 4.7 is optimized for 1/rev controls, while Figure 4.9 is optimized for 3/rev controls. For both 1/rev and 3/rev controls, a small pitch angle is expected on the retreating side, growing to a maximum on the retreating side of the blade. The $\mu = 0.50$ case experiences a higher maximum on the retreating side, once again due to the behavior of the system as it approaches the critical advance ratio. When comparing the 1/rev and 3/rev pitch controls at $\mu = 0.25$ there is very little difference. This is expected according to the results of Chapter 3; at low advance ratios the benefits of higher harmonic controls are not apparent, and so 1/rev pitch result will look very similar to 3/rev pitch results.
Figure 4.5: Comparisons of Nondimensional Circulation ($\tilde{\Gamma}$) at $\mu = 0.25$ (above) and $\mu = 0.50$ (below) for twist, chord, and various per-rev controls optimization. The rotor is flying upward and rotating counter-clockwise. The seemingly different cutout sizes are due to the different sized blades creating a different vehicle reference frame wake profile. The rubber rotor is included in the right most column for comparison.

Figure 4.6: Optimal chord design as a function of non-dimensional radius of the rotor for conventional (1/rev) pitch controls. An advance radio of $\mu = 0.25$ is shown in dotted lines, while a higher advance ratio of $\mu = 0.50$ is depicted in a solid line. The rotor shaft is depicted, along with the rotor cutout, for reference.
4.3.2 Advance Ratio Sweep; Discovery of Local Minima

Moving to a global level, Figures 4.10 to 4.12 show the normalized induced power as a function of advance ratio for chord distribution, pitch and twist distribution in the design space. It is apparent from these plots that there is a jaggedness to the nonlinear design optimization. This jaggedness is due to the way these results were gathered, as well as how IPOPT finds the optimum of a given problem.

Each plot demonstrates different local minima of the system, which were collected using various initial conditions to prime the system. Each set of local minima have been designated as branches, and their representative chord distributions are plotted alongside their label. The optimization for a rectangular planform blade is also included for comparison.

Instead of one continuous curve, multiple branches are found, each with their own unique chord design. Some chord designs make sense in context. The low speed variants of the blade all tend to have large root chords that decrease as $\mu$ increases towards 1. On the other hand, some branches have nonsensical chord designs. For example, branch 4 always has a chord distribution that oscillates from small to large to small over the course of the blade. As expected, since the problem is highly nonlinear Newton iteration can generate local minima instead of the global minima.

These issues of local minima can be better visualized through Figures 4.13 to 4.15, which depict the normalized induced power hysteresis effect of the branch diagram. For these plots, the results were obtained in two different directions: $\mu = 0.0 \rightarrow 1.2$, and $\mu = 1.2 \rightarrow 0.0$. The arrows depict the motion of each of these solutions, and dotted lines represent jumps from one solution to the next. At low $\mu$ values, there is no difference between the low-to-high and high-to-low approaches, suggesting one global minimum. Moving past these values of $\mu$ generates several jumps to different solutions, suggesting multiple local minima of the solution, which then converges
back to one solution at high $\mu$ values.

It is not possible to know with certainty that the smallest local minima depicted in Figures 4.10 to 4.12 are the global minima using this procedure. However, it is possible to compare the branch diagrams of Figures 4.10 to 4.12 to results of a similar optimization when starting each advance ratio at a baseline, rectangular rotor. Results of this comparison are shown in Figures 4.16 to 4.18, where the scattered data depicts the “baseline rotor” results. In almost every instance of the rectangular rotor initialization approach, the baseline rotor converged to the smallest local minimum, suggesting with more certainty that these local minima are also global minima. While there are certain places the baseline rotor did not converge to the smallest branch at a given $\mu$, these are few and far between.

It is also apparent from these baseline rotor figures that there are certain points around the critical advance ratio where one branch becomes the dominant minimum of the problem. This tends to generate “kinks” in the curve where a discontinuity in the derivative is experienced. While kinks of this nature did not occur in results without chord optimization, optimization in Chapter 3 was quadratic, guaranteeing one analytical optimum point, suggesting that not including chord optimization should generate continuous results.

4.3.3 Summary of Results, Physical Significance

The summary of the optimization of the rotor system’s normalized induced power with twist distribution, chord distribution, and controls in the parameter space is shown in Figure 4.19 as a function of the increasing forward flight velocity. Figure 4.19 depicts the optimization with 1/rev, 2/rev, and 3/rev controls, while also optimizing the twist and chord distribution. To compare with the results of Chapter 3, the optimized normalized induced power when using a rectangular chord distribution are depicted using dotted lines, as well as the ”rubber rotor“ optimum which
serves as an extreme lower bound. It is clear that by including chord distribution optimization, there is approximately a 40% improvement in induced power at moderate advance ratios. This improvement is most apparent near a critical advance ratio of $\mu = 0.70$, and then tapers off to 10% or less as the advance ratio increase to 1.2.

4.4 Summary & Conclusions

The inclusion of the non-linear effect of chord optimization changed the state of the system to include multiple minima at various $\mu$. Unlike the results of Chapter 3, using chord optimization led to the discovery of multiple branches of local minima, each with their own distinct design.

While these non-linear results including branches are interesting theoretically, practically speaking all local minima found vary smoothly for any realistic value of $\mu$ lower than 0.5.
**Figure 4.7:** Control comparison at $\mu = 0.25$ and $\mu = 0.50$ for 1/rev pitch controls.

**Figure 4.8:** Optimal chord design as a function of non-dimensional radius of the rotor for 3/rev pitch controls. An advance ratio of $\mu = 0.25$ is shown in dotted lines, while a higher advance ratio of $\mu = 0.50$ is depicted in a solid line. The rotor shaft is depicted, along with the rotor cutout, for reference.
Figure 4.9: Control comparison at $\mu = 0.25$ and $\mu = 0.50$ for 3/rev pitch controls.

Figure 4.10: Normalized induced power as a function of advance ratio; branch diagram for 1/rev controls, twist, and chord optimization. The optimized chords are also included for each branch.
Figure 4.11: Normalized induced power as a function of advance ratio; branch diagram for 2/rev controls, twist, and chord optimization. The optimized chords are also included for each branch.

Figure 4.12: Normalized induced power as a function of advance ratio; branch diagram for 3/rev controls, twist, and chord optimization. The optimized chords are also included for each branch.
Figure 4.13: Normalized induced power as a function of advance ratio; branch diagram for 1/rev controls, twist, and chord optimization. Hysteresis effects depicted as the initial condition of the next iteration is chosen to be the previous iteration’s final result.

Figure 4.14: Normalized induced power as a function of advance ratio; branch diagram for 2/rev controls, twist, and chord optimization. Hysteresis effects depicted as the initial condition of the next iteration is chosen to be the previous iteration’s final result.
Figure 4.15: Normalized induced power as a function of advance ratio; branch diagram for 3/rev controls, twist, and chord optimization. Hysteresis effects depicted as the initial condition of the next iteration is chosen to be the previous iteration’s final result.

Figure 4.16: Normalized induced power as a function of advance ratio; branch diagram for 1/rev controls, twist, and chord optimization. Results when starting all iterations with a baseline, rectangular rotor are included for comparison.
Figure 4.17: Normalized induced power as a function of advance ratio; branch diagram for 2/rev controls, twist, and chord optimization. Results when starting all iterations with a baseline, rectangular rotor are included for comparison.

Figure 4.18: Normalized induced power as a function of advance ratio; branch diagram for 3/rev controls, twist, and chord optimization. Results when starting all iterations with a baseline, rectangular rotor are included for comparison.
Figure 4.19: Twist, chord distribution, and 1/rev, 2/rev, and 3/rev control optimization of normalized induced power at various forward flight speeds. Twist and control optimization results from Chapter 3 are included for comparison, as well as the "rubber rotor" solution.
Chapter 2 described the optimization procedure to determine the optimal blade controls, twist, and chord distribution for a helicopter rotor in forward flight. This optimization procedure was applied at one advance ratio at a time. In the design of a practical rotor, however, multiple operating conditions must be considered. For example, one might consider optimizing for two different forward flight speeds simultaneously, or even between a cruise and climb condition. In this chapter, we present a method to find the optimal design of a rotor whose measure of performance is the weighted sum of the power required at two different operating conditions in cruise. Unlike the sensitivity and Monte-Carlo methods described in Chapter 1, we use a method inspired by weighted averages, as presented in Gass and Saaty [48], and Zadeh [49], where the problem is posed as a constrained optimization of the weighted sum of the two flight conditions.

5.1 Optimization Approach

The goal of this multipoint optimization is to generate the Pareto frontier for two flight conditions. The Pareto frontier is the set of designs where it is impossible to
improve the performance of one flight condition without making the performance of the other flight condition worse [50]. This is visualized in Figure 1.4, where power required during hover is plotted on one axis, and power required in cruise is plotted on the other. To create this frontier, an optimization using a modified Lagrangian power is developed to evaluate the performance of the rotor model with respect to two flight conditions.

To perform the single constrained optimization, the problem was cast as a variational statement by adjoining the objective function (the total power) with the product of the constraints and their multipliers, that is,

\[
\Pi = \frac{1}{2} \Gamma_w^T K \Gamma_w - \Gamma_w^T Q + P_{v0} + \lambda_F^T \cdot (B \Gamma_w - F_R) + \lambda_M^T \cdot (D_N \Gamma_a - M_R) + \lambda_R^T \cdot (R(\Gamma_w, \Gamma_a, \Theta))
\]  

(3.15)

where \( M_R \) and \( F_R \) are prescribed forces and moments. Taking the variation of the above expression, then solving the variational statement results in the optimal solution for the unknown circulation and design parameters.

To perform a multipoint optimization, a similar procedure is conducted where two flight conditions are considered simultaneously, for example, at two different advance ratios. The circulations at each flight condition, \( \Gamma_1 \) and \( \Gamma_2 \), as well as their own set of controls, denoted \( \Theta_{\text{controls}1} \) and \( \Theta_{\text{controls}2} \), are in general different at the two flight conditions. However, the built in blade geometry (twist and chord) is the same for the two flight conditions. This creates an unknown circulation for the multipoint optimization of the form

\[
\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix},
\]  

(5.1)
and a design parameter vector of the form

$$
\Theta = \begin{bmatrix}
\Theta_{\text{twist}} \\
\Theta_{\text{chord}} \\
\Theta_{\text{controls}_1} \\
\Theta_{\text{controls}_2}
\end{bmatrix}.
$$

(5.2)

Since the two flight conditions are at different velocities, the wakes generated are different, so separate vortex lattice models must be used for each. This means that the matrices that make up the objective function as well as the constraints – such as $K$, $B$, $D_N$, and all the $A$ influence matrices – will differ. We will use the matrix subscripts 1 or 2 to denote the two different flight conditions.

For the objective function, we will use a weighted sum of the power losses for the two flight conditions. For flight condition 1, we use a weight of $\alpha$, and for flight condition 2 we use a weight of $(1 - \alpha)$ to ensure that the sum of the weights is always 1. By varying the weighting parameter $\alpha$, we change the relative importance placed on optimizing each flight condition.

As in the single point optimization, this multipoint optimization can be stated in a variational form; the resulting Lagrangian is given by

$$
\Pi = \alpha (P_{\text{ind}1} + P_v) + (1 - \alpha) (P_{\text{ind}2} + P_v) \\
+ \lambda^T_{F_1} \cdot (B_1 \Gamma_{w1} - F_{R1}) + \lambda^T_{F_2} \cdot (B_2 \Gamma_{w2} - F_{R2}) \\
+ \lambda^T_{M_1} \cdot (D_{N1} \Gamma_{a1} - M_{R1}) + \lambda^T_{M_2} \cdot (D_{N2} \Gamma_{a2} - M_{R2}) \\
+ \lambda^T_{R_1} \cdot (R_1(\Gamma_{w1}, \Gamma_{a1}, \Theta)) + \lambda^T_{R_2} \cdot (R_2(\Gamma_{w2}, \Gamma_{a2}, \Theta)) ,
$$

(5.3)

where

$$
B_1 \Gamma_{w1} = F_{R1} ,
$$

(5.4a)

$$
B_2 \Gamma_{w2} = F_{R2} ,
$$

(5.4b)

$$
D_{N1} \Gamma_{a1} = M_{R1} ,
$$

(5.4c)
\[ D_{N_2} \Gamma_{a_2} = M_{R_2}, \quad (5.4d) \]
\[ R_1(\Gamma_{w_1}, \Gamma_{a_1}, \Theta) = 0, \quad (5.4e) \]
\[ R_2(\Gamma_{w_2}, \Gamma_{a_2}, \Theta) = 0. \quad (5.4f) \]

Note that \( R \) has the possibility of containing nonlinear or inequality constraints, and all matrices are a function of \( \Theta_{\text{chord}} \). The system will be solved for various weights \( \alpha \) to generate the optimized induced power at flight conditions 1 & 2. Each set of powers will then be plotted with each flight condition on a separate axis, akin to Figure 1.4, to graphically construct the Pareto frontier.

5.2 Rotor Test Cases

We now consider the multi-point optimization of a conventional rotor. The rotor is now optimized at two flight objectives using the above approach. The geometry and constraints used in this test are the same as the test rotor in Chapter 4 and are outlined in Table 4.1. The rotor is optimized at two different flight objectives: a forward flight velocity of \( \mu = 0.25 \) and a forward flight velocity of \( \mu = 0.50 \). All other constraints, such as lift, trim, and realizability, hold for both conditions. The values for the weight, \( \alpha \), will be varied between 0.0 to 1.0 to capture as much of the Pareto frontier as possible.

5.2.1 Linear Optimization

Figure 5.1 shows the computed Pareto frontier of the rotor when using only the design parameters twist and pitch control. The normalized induced power of the optimized design parameters for the low speed case is plotted on the x-axis, while the normalized induced power of the high speed case is plotted on the y-axis, and each set of blade pitch controls is shown with a different color. As higher controls are used, the frontier begins to move towards the origin; this is not surprising since adding
more degrees of freedom to the system should lead to a more optimum solution, nearer the origin. There is more movement in the frontiers on the high speed $y$-axis. Since there is overall more power consumption in the high speed case, it is logical that most improvement in the design space should go towards the high speed case to keep the weighted average low. On an absolute scale, however, there is little variance in power as $\alpha$ is varied. Within the entire frontier there is never more than a 5% change in normalized power between any two points. This suggests that the one design variable the two conditions have in common – the twist – does not have a significant impact on the value of the weighted average used in the optimization.

5.2.2 Nonlinear Optimization

Now we consider the inclusion of chord optimization in the design of the rotor. Much like the single-point nonlinear optimization of Chapter 4, the Pareto frontiers constructed for nonlinear design parameters demonstrate local minima. Their branches, along with a small representation of the branch’s chord designs, is plotted in Figures 5.2 to 5.4. These plots show the Pareto frontier for the low speed and high speed forward flight conditions for levels of higher harmonic control. Each branch has its own unique chord design. The branches that favor the low-speed case tend to have larger root chords, while the branches that favor the high-speed case tend to have smaller root chords with a larger chord value near $r/R = 0.65$. The low-speed branches have similar designs to the low speed single point optimization results of Chapter 4, while the high-speed branches have similar designs to the high speed single point optimization results of Chapter 4. These similar results tend to occur at values of $\alpha$ close to 0.0 and 1.0, since this will weight the optimization entirely to one flight speed. When $alpha$ is closer to 0.5, both conditions must be considered simultaneously, and a transition region develops between the design dominated by the high speed and low speed flight cases. This transition region is known as a knee, and
is locally an ideal location to choose a multipoint design. In this region, the chord design transitions to a more favorable high-speed chord distribution. This change allows for high-speed induced powers closer to the optimum according to Chapter 4, while sacrificing the performance of the low-speed induced power.

When transitioning from the low-speed forward flight case to the high-speed forward flight case, Figures 5.2 to 5.4 shows that the first primary change to the chord distribution comes in the form of a chord reduction at the hub of the rotor. This change is prioritized by the optimization because a large chord near the root tends to create large values of power in the high-speed forward flight case. As explained in Chapter 4, this is due to the development of the reverse flow region on the retreating side of the blade at forward flight velocities close to $\mu_{\text{crit}}$.

Figures 5.2 to 5.4 also show gaps between the different branches of the Pareto frontier. This issue of captuability is discussed by numerous authors, including Censor [51] and Messac [5]. While these authors dispute the regions of capturability, both agree that regions of negative concavity in the Pareto frontier are uncapturable using a weighted sum method. To illustrate this, Figure 5.5 demonstrates a possible completed Pareto frontier. Certain points on the frontier have been highlighted, in particular the points $c_1$ and $c_2$ separate a region of negative concavity. According to Messac & Censor, this region is not capturable using the methods presented.

Figures 5.6 to 5.8 shows the hysteresis effect encountered while attempting to collect the branches described earlier. The system was started at a baseline, rectangular rotor, and was modified in two ways: from $\alpha = 0.0 \rightarrow 1.0$, and $\alpha = 1.0 \rightarrow 0.0$. The arrows on the plot indicate the two directions of the $\alpha$ iteration, with the former starting at the bottom right, and the latter starting at the top left. At certain points where major jumps in normalized power occurred, curved dotted arrows are used to indicate where the jump started and ended. Similar to the single-point optimization, there are significant jumps between the branches that are caused by an extremely
small change in $\alpha$. However, unlike the single-point optimization there are not substantial differences between all the branches. Looking at the chord designs of each branch, there is a similarity between the chord design for branch 1 and branch 2, as well as branch 3 and branch 4. This suggests that these curves are actually related to one another, instead of being separate solutions.

To confirm this, the same data in Figures 5.2 to 5.4 is plotted against $\alpha$ in Figure 5.9, where the various colors represent the different branches seen in Figures 5.2 to 5.4, and the line styles represent different levels of per-rev control. Figure 5.9 is similar to the plots detailed in Chapter 4, where it is easier to spot locations where one value of $\alpha$ gives multiple minima. As expected, when comparing branches 1 & 2 or branches 3 & 4 Figure 5.9 illustrates continuity between the two branches, suggesting they are part of the same minimum. Comparing between branches 1 & 2 and branches 3 & 4, however, there are substantial jumps or “kinks” between the two sets of branches. This confirms that the set formed by branches 1 & 2 are unique when compared to the set formed by branches 3 & 4.

Figure 5.9 also helps to confirm the theory regarding capturability. There is an intersection between the two unique sets of curves for each per-rev control; this point is the location where, for a given $\alpha$, there are two unique designs with the same weighted normalized power. According to Messac [5], previous sources have incorrectly assumed it is not possible with the weighted averages method to capture any part of the frontier past this point. This is confirmed by the fact that parts of the curve that have constructed the Pareto frontier in Figures 5.2 to 5.4 are beyond this equality point. Messac also states that once the curvature of the frontier becomes negative, it is not possible to capture the frontier using a weighted sum of cost functions. This is also confirmed by Figures 5.2 to 5.4 since the large gap missing would have to be filled in with a section of curve that has negative curvature. Possible visualizations of this part of the curve can be seen in Figure 5.10, where
the dotted black lines represent the bounding box of the curve, and the red and blue lines give possible suggestions for its orientation.

In summary, the Pareto frontiers for chord distribution, twist distribution, and 1/rev, 2/rev, and 3/rev controls optimization are shown in Figure 5.11, where the y-axis contains the power of the high speed case, and the x-axis contains the power of the low speed case. Each color represents a different set of per-rev controls, and all curves have twist and chord optimization included. The weighting parameter, \( \alpha \), starts at zero in the bottom right hand corner weighting entirely towards the high speed case, and moves towards 1 in the top left weighting entirely towards the low-speed case. For comparison, the linear design curves of Figure 5.1 are illustrated as dashed purple lines. When comparing the design curves with chord optimization to the curves without chord optimization, there are substantial differences. First, using chord optimization has pushed all of the curves towards the origin by a large margin, suggesting a more optimal solution. A larger shift overall has occurred in the y-axis, which is not surprising. According to the single point optimization results of Chapters 3 and 4, there is a larger discrepancy between the ideal, “rubber rotor” result and the high speed forward flight induced power, suggesting there is more room for improvement in the optimized normalized power when adding degrees of freedom to the parameter space.

The design curves with chord optimization vary more widely than the curves without chord optimization. This suggests that chord optimization can have a more substantial effect on the performance of the system. From a practical design standpoint, however, one would not want to operate at the top of these curves; moving from the minimum y-value on the curve to the maximum y-value leads to a doubling of the high speed power, for a benefit of a little over 2% in the low-speed power. Instead, operating at the cusp of these designs nearest the origin where substantial improvements occur in the lower speed case while negligible change occurs in the
Figure 5.1: Pareto frontier of rotor system given two forward flight conditions; optimized using linear design variables of controls and twist, and a constant rectangular blade planform.

Figure 5.2: Pareto frontier of two forward flight conditions; branch diagram for 1/rev controls, twist, and chord distribution optimization. Depictions of the optimized chord distribution are included for each branch.
Figure 5.3: Pareto frontier of two forward flight conditions; branch diagram for 2/rev controls, twist, and chord distribution optimization. Depictions of the optimized chord distribution are included for each branch.

Figure 5.4: Pareto frontier of two forward flight conditions; branch diagram for 3/rev controls, twist, and chord distribution optimization. Depictions of the optimized chord distribution are included for each branch.
Figure 5.5: Depiction of Pareto frontier with concave and convex pieces, as seen in Messac [5].

Figure 5.6: Pareto frontier of two forward flight conditions; branch diagram for 1/rev controls, twist, and chord distribution optimization. Hysteresis effects depicted by iterating over $\alpha$ and using the previous iteration’s final result as the initial condition of the next iteration. Jumps are depicted with dotted lines in locations where large changes occurred in the frontier for small changes in $\alpha$. 
high speed case is more viable.

5.3 Conclusions

When conducting a multipoint optimization using pitch control and twist design parameters, there is little variance between the designs on the Pareto frontier. Effectively, there is one multipoint design for the system, regardless of $\alpha$.

The same effect of local minima and hysteresis seen in Chapter 4 were found when conducting a multipoint optimization with chord optimization. While theoretically interesting, it is important to note that the practical portion of the Pareto frontier has been captured by branch 1. The knee of the Pareto frontier is captured by this branch, and operating on any other location of the Pareto frontier would lead to significant tradeoffs in one of the flight conditions.

Finally, theories regarding a link between concavity and capturability of the Pareto frontier presented by Messac were shown to be in agreement with the data. To capture these regions, alternative multipoint optimization techniques must be considered.
Figure 5.7: Pareto frontier of two forward flight conditions; branch diagram for 2/rev controls, twist, and chord distribution optimization. Hysteresis effects depicted by iterating over $\alpha$ and using the previous iteration’s final result as the initial condition of the next iteration. Jumps are depicted with dotted lines in locations where large changes occurred in the frontier for small changes in $\alpha$. 

$$\mu = 0.25 \text{ Normalized Power, } \frac{C_P}{C_L^2}$$

$$\mu = 0.50 \text{ Normalized Power, } \frac{C_P}{C_L^2}$$
Figure 5.8: Pareto frontier of two forward flight conditions; branch diagram for 3/rev controls, twist, and chord distribution optimization. Hysteresis effects depicted by iterating over $\alpha$ and using the previous iteration’s final result as the initial condition of the next iteration. Jumps are depicted with dotted lines in locations where large changes occurred in the frontier for small changes in $\alpha$.

Figure 5.9: Weighted normalized power versus weighting factor, $\alpha$, for 1/rev, 2/rev, and 3/rev controls, twist distribution, and chord distribution optimization. The different style lines indicate different sets of per-rev controls, while the colors are identical to the branch colors seen in previous figures.
Figure 5.10: Depiction of possible non-convex pieces of the Pareto frontier, as described by Messac [5].

Figure 5.11: Pareto frontiers with design variables of chord distribution, twist distribution, and 1/rev, 2/rev, and 3/rev controls. Linear design optimization results are included for comparison.
6

Conclusion

6.1 Summary & Conclusions

The goal of this thesis is to develop improved theories and analysis tools to solve the problem of power optimization in rotary aircraft. At high advance ratios, it has been shown that conventional rotors experience dramatic power increases. The tools developed here can be used to understand the physics of this problem, and reduce aerodynamic losses through using multipoint design techniques. Chapter 2 describes the models used in these analyses, including a lifting line analysis previously developed at Duke. While the lifting line analysis is simple in technique, the method has some shortcomings, chief among them the inability to effectively model some unsteady aerodynamic effects of the rotating blades. The lifting surface analyses presented gives improved unsteady aerodynamic modeling capability, as well as introducing more accurate means to calculate the aerodynamic moments. This technique builds on the success of lifting line, but at the cost of increased computational complexity due to its increase in the number of vortex rings and collocation points in the model. Two approaches were used to reduce the computational cost
of the listing surface model. Parallel processing was implemented using OpenMP to reduce the computational time required to compute the unsteady solution. Also, static condensation and harmonic balance were introduced to decrease the size of the computational model from tens of thousands of degrees of freedom to just a few thousand or less. Unsteady aerodynamic computations that originally would take hours to complete, now take only minutes. Harmonic balance has the added benefit of giving physical insight into the problem, demonstrating the importance of each harmonic in the rotor system. For example, we found that at most 5 harmonics are necessary to accurately model the time average forces and moments with acceptable accuracy.

The results of a single point optimization with a constant chord were described in Chapter 3. The theoretical optimization procedure, including variational statements and constraints, was described. The result was a problem that was solved using Quadratic Programming (QP). The QP approach is extremely fast – requiring only one matrix solve – but only lends itself to quadratic objective functions with linear equality constraints. The rotor test cases of Chapter 3 demonstrated three key findings. First, dramatic power reductions were demonstrated with the use of increased amounts of per-rev blade pitch control as well as fixed twist distributions on the rotor. These results were compared to lifting line and showed that the lifting line model and lifting surface model are in good agreement at practical $\mu$ values. Where the two models begin to differ – at higher $\mu$ values – the advance ratio is in a regime that not only violates some of the assumptions of lifting line, but is outside the range of $\mu$ values in which conventional helicopters operate.

Non-constant chord distribution designs were studied in Chapter 4, and the procedures and algorithms developed for linear, quadratic optimizations were modified to analyze generate, nonlinear optimization problems. Due to the highly nonlinear nature of the problem, the QP approach can not be used, and instead IPOPT, a
A gradient based open source optimization package, was used. Results of the non-linear chord optimization demonstrated a hysteresis effect with multiple solutions representing different local minima. These local minima were collected by starting at chord distributions similar to the surrounding branches, then optimizations conducted starting from a conventional, rectangular rotor were used to increase the probability that the lowest local minimum at a given $\mu$ was the global minimum.

Multipoint optimization was studied in Chapter 5, where two cruise objectives at different $\mu$ were considered simultaneously in one optimization. These results were collected in what is known as a Pareto frontier, and these frontiers corroborate with the results found in Chapter 4. The same effect of branches and hysteresis was observed in the multipoint optimization, which led to uncapturable points on the Pareto frontier as described by Messac.

An important point is that no gradient-based method can guarantee that a minimum found is a global minimum or that such a minimum will be physically practical. Thus, the end user must use logic and common sense to aid in their determination of the state of a given optimized point. If the chord distribution is extremely impractical, or the twist and controls are not feasible the user should seek to use another starting point to determine if another, more physically practical minima is present, or add constraints to the system to force a more realistic solution.

6.2 Future Work

6.2.1 Blade Dynamics

One of the largest benefits to the lifting surface model is the inclusion of a physically realistic blade planform. In future work, it should be possible to add blade dynamics to the optimization equations. To optimize a rotor with blade dynamics as a constraint, a new term is added to the variational statement so the functional
to be made stationary takes the form

\[
\Pi = \frac{1}{2} \Gamma_W^T K \Gamma_W + \lambda_T^T \cdot (B \Gamma_W - F_R) + \lambda_M^T \cdot (D \Gamma_W - M_R) + \\
\lambda_R^T \cdot (R(\Gamma_W, \Gamma_{airfoil}, \Theta)) + \lambda_D^T \cdot S(\Gamma, \Theta, q, \dot{q}, \ddot{q}) .
\]  

The equation \( S \) represents the blade dynamics model of the system, and \( q \) the state vector describing the blade motion. The vibrating motion of the blades is coupled with the unsteady aerodynamics through the function \( R \), and the aerodynamics are couple to the structural dynamics through the function \( S \). Clearly, the optimized planforms found in Chapters 4 and 5 that have extremely small chords are not practical or physically realistic. This shows that the inclusion of structural dynamic constraints is necessary for realistic designs.

6.2.2 Application to General Geometries

The entirety of this thesis is concerned with the analysis of a conventional, single rotor helicopter. However, even the simplest of helicopters have a main rotor and a smaller tail rotor, while many new designs today have two counter-rotating rotors. Because of this, it is necessary to generalize the code to more general geometries. However, this is a simple extension of the model already created by Hall & Giovanetti [3, 14]. Instead of only one aerodynamic device being represented in \( \Gamma_a \) and \( \Gamma_w \), multiple aerodynamic devices will be put in place. The inclusion of more aerodynamic devices will increase the computational complexity of the model, but the model reduction techniques of Chapter 2 will reduce the complexity significantly. Essentially this work will be a natural extension of the model that has already been created.

6.2.3 Stall Modeling

As described in Chapter 2, the current model uses a quasi-steady quadratic drag polar based on experimental data to calculate a drag coefficient for each spanwise
station. This model works well for steady flight and small angles of attack, but as seen in the results of Chapters 3 to 5, the flow is always unsteady in forward flight, and there are certain flight conditions where the angles of attack are moderate in size. It is therefore necessary to consider alternatives to the quadratic drag polar.

An extension of this procedure was explored by Giovanetti [14], where in lieu of a quadratic fit to the data an experimental drag polar valid for all angles of attack between $-180^\circ$ to $180^\circ$ is used. This would be a natural extension to the current implementation, and would allow for modeling of static stall in the reverse flow region. This is important, since as described by Giovanetti, an optimized rotor should not prevent stall altogether, as it may be necessary to allow stall to occur in certain locations on the blade to obtain the optimum overall circulation distribution.

The drag polar approach will capture stall due to a steady angle of attack, but not dynamic stall. This phenomenon is a viscous effect caused by the shedding of a vortex at the leading edge of an airfoil. As the vortex travels along the suction surface of the airfoil, it increases lift past the stall limit as seen in Figure 6.1. After a certain amount of time, this vortex leaves the airfoil, causing a dramatic decrease in lift. To model this effect, the ONERA stall model can be implemented [25]. The ONERA model modifies the traditional lift statement, instead writing

$$L = \frac{1}{2} \rho AU^2 (C_{z,1} + C_{z,2})$$

(6.2)

with the parameters $C_{z,1}$ and $C_{z,2}$ derived from the differential equations

$$\dot{C}_{z,1} + \lambda C_{z,1} = \lambda C_{z,L} + (\lambda s + \sigma) \dot{\alpha} + s\alpha$$

(6.3a)

$$\ddot{C}_{z,2} + a\dot{C}_{z,2} + rC_{z,2} = - (r\Delta C_z + E\dot{\alpha})$$

(6.3b)

The values of $\lambda, s, \sigma, a, r, E, \Delta C_z$ are all determined experimentally. The value of $C_{z,L}$ is the value of lift expected from a non-stalled model (as seen in Figure 6.2). The
ONERA model adds additional degrees of freedom to the system to allow dynamic stall to occur, which also increases the computational complexity of the system.

![Figure 6.1](image.png)  

**Figure 6.1:** Visualization of dynamic stall; note that the lift is able to surpass the traditional static limit of stall [6]

### 6.2.4 Improved Multi-Objective Search Algorithm

As discussed in Chapter 5, the weighted average objective function used for multi-point optimization has difficulty capturing concave portions of the Pareto frontier. According to Messac, this difficulty is caused by the functional representation of the optimization being a maximum at the point. This maximum is not capturable by a minimization procedure. Different algorithms must be considered to capture these parts of the frontier. There are three primary branches of multi-objective algorithms that will be considered: weight-based approaches, preemptive approaches, and exhaustive approaches.

Weight-based approaches have already been considered in this paper through the
use of weighted averages. However, as described by Messac [5], other approaches can be taken to fit a concave portion of the curve. For example, the approach known as weighted compromise programming (WCP) takes an objective function of the form

$$ J(x) = \sum_{i=1}^{m} w_i \{ \mu_i(x) \}^{c_i} $$

where \( \mu_i \) are the individual objective functions, and \( c_i \) represents a power chosen by the user. There is an added layer of complexity due to the additional parameters \( c_i \). The user does not often know what the correct value for these parameters are \textit{a priori}, and trial and error must be used to find a suitable value for \( c_i \).

Preemptive approaches [52] are approaches that attempt to remove the additional weighting parameters and create a nonlinear function based on the priorities of the user. Stated mathematically, the objective would take the form

$$ J = J_{pf}(\mu(x)) $$

where \( \mu \) represents all the possible objectives being considered. This technique is also sometimes referred to as physical programming.
Finally, there is the possibility of conducting an exhaustive approach. As the name suggests, the user conducts a multitude of design space points, then using a technique like the normalized constraint method [53] find the points of that search that are on the Pareto frontier.

Of these three possibilities, the modified weight-based methods show the most promise. The preemptive approaches are unnecessarily complicated for this approach since there are only two flight conditions being considered, and the exhaustive method is unsustainable due to the dimension of the full parameter space.
Bibliography


