

ASYMPTOTIC BEHAVIOR OF THE BROWNIAN FROG MODEL

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ABSTRACT. We introduce an extension of the frog model to Euclidean space and prove properties for the spread of active particles. The new geometry introduces a phase transition that does not occur for the frog model on the lattice. Fix $r > 0$ and place a particle at each point x of a unit intensity Poisson point process $\mathcal{P} \subseteq \mathbb{R}^d - \mathbb{B}(0, r)$. Around each point in \mathcal{P} , put a ball of radius r . A particle at the origin performs Brownian motion. When it hits the ball around x for some $x \in \mathcal{P}$, new particles begin independent Brownian motions from the centers of the balls in the cluster containing x . Subsequent visits to the cluster do nothing. This waking process continues indefinitely. For r smaller than the critical threshold of continuum percolation, we show that the set of activated sites in \mathcal{P} behaves like a linearly expanding ball. Moreover, in any fixed ball the set of active particles converges to a unit intensity Poisson point process. Lastly, we prove that the model expands at rate at least $t^{2-\epsilon}$ when $d = 2$ and r equals the critical threshold in continuum percolation.

1. INTRODUCTION

The *Brownian frog model* starts with a single active particle at the origin and sleeping particles at the points of a unit intensity Poisson point process $\mathcal{P} \subset \mathbb{R}^d - \mathbb{B}(0, r)$ with $r > 0$ fixed. Let the cluster associated with $x \in \mathcal{P}$ be the largest component of $\cup_{y \in \mathcal{P}} B(y, r)$ containing $B(x, r)$. Active particles perform independent Brownian motion, and sleeping particles become active the first time an active particle is within r of its cluster. All particles in the cluster wake simultaneously.

One way to visualize the process is that each particle is a frog on a lily pad of radius r . If a frog wakes up, it strays from its starting location according to a Brownian motion and wakes sleeping frogs upon perturbing their cluster of lily pads. To avoid an explosion of active particles we must avoid the possibility of an infinite component of overlapping lily pads. This amounts to taking $r < r_d$ with r_d the critical threshold in continuum percolation on \mathbb{R}^d (see [MR96]). In this article we prove that the set of activated sites at time t behaves like a ball of radius $\gamma_r t$. Additionally, we show that the limiting density of awake frogs in a fixed region is Poisson distributed, and we establish that $\gamma_r \rightarrow \infty$ as $r \rightarrow r_2$ when $d = 2$.

This is the continuous analogue of a growing system of random walks known as the *frog model*. The process starts with an active particle at the root of a graph and some number of sleeping particles at all the other vertices. In either continuous or discrete time, the active particle at the root begins a random walk. Whenever an active particle visits a vertex with sleeping particles all of those particles become active and begin their own independent random walk.

The frog model was introduced by K. Ravishankar in 1996 on \mathbb{Z}^d with one particle-per-site and simple random walk paths. Initially it was known as the egg model, but was shortly thereafter relabeled the frog model. The name has stuck, and we continue to use it. The “branching,” i.e. activation of sleeping particles, is constrained by the number of points on each site and the paths of all frogs in the process. This makes for a rather nuanced

process that exhibits behavior different from a single or a branching random walk. Likewise, the Brownian frog model has features somewhere between Brownian motion and branching Brownian motion.

Early papers on the frog model work in the setting suggested by Ravishankar. The paper [TW99] shows that the root is visited infinitely often in all dimensions. Next, [AMP02a, RS04] proved that the set of visited sites has a limiting shape. The latter works in continuous time, so the proofs and theorem statements are slightly different. The past two decades have further investigated the process by modifying the graph, the number of frogs per site, and the random paths followed by frogs. For example, [AMPR01, Pop01] consider the change in transience/recurrence and the limiting shape with a random configuration of frogs. The articles [GS09, DP14] consider the frog model on the lattice where particles perform a biased random walk. Another modification to the random walk path is where frogs take a geometric number of steps then perish. Various phase transitions are explored in [AMP02b, LMP05]. The question of transience and recurrence on trees is explored in [HJJ16b, HJJ16a, JJ16, Ros17]. Recent work has looked at the waking behavior on finite trees ([Her16]), and the passage time to a distinguished set of vertices in \mathbb{Z}^d ([Kub16]). The article [HJJ17] establishes linear expansion of the set of activated frogs on regular trees, and also pins down different regimes for rapid and slow cover time for the frog model on finite trees.

A Brownian frog model on \mathbb{R} is studied in [Ros16]. Active particles have a fixed leftward drift, and are placed according to a Poisson point process with intensity $f(x)$. This paper establishes sharp conditions on f that determine whether the model is transient or not. Note that on the line particles will collide so it is natural to set $r = 0$. In higher dimensions it is necessary to set $r > 0$.

Symmetry ensures that if the Brownian frog model has a limiting shape it will be a ball, which is not the case in the one-per-site discrete frog model on \mathbb{Z}^d for large enough d (see [RS04, Theorem 1.1]). Moreover, [AMP02a, Theorem 1.2] proves that if enough frogs are placed at each site of \mathbb{Z}^d then the limiting shape contains a flat edge. Here we confirm the conjecture from the introduction of [RS04] that there is a limiting shape for the Brownian frog model. We also set the stage for studying the continuous frog model. The hallmark theorems for the frog model ought to have analogues in the Brownian frog model. However, the different geometry introduces several new lines of inquiry. It is natural to consider diffusions other than Brownian motion. For instance, an interesting question would be to consider frogs that move according to an Ornstein-Uhlenbeck process $dB_t - \alpha|X_t|dt$, translated to the location where they wake up. One can then ask whether the shape theorem holds for this more general continuous frog model.

The geometry of the overlapping lily pads is another interesting feature of the Brownian frog model. As r nears the threshold in continuum percolation there will be large clusters of overlapping lily pads. These can cause big jumps that take no time. Like bond percolation, continuum percolation is best understood when $d = 2$. It is known that clusters are almost surely finite in this dimension [MR96]. Nonetheless, they have infinite expected size. We use this and the shape theorem to show the frog model expands superlinearly at criticality (rate $t^{2-\epsilon}$). This has never been observed before. How the frog model expands at criticality for $d > 2$ remains open. It may be a very hard question. Like in bond percolation on \mathbb{Z}^d , it is a major open problem to show that clusters are almost surely finite at criticality in higher dimensions. A more tractable further question is to show that the speed of expansion $\lim_{r \rightarrow r_d^-} \gamma_r = \infty$ for all $d > 2$. We know this is true when $d = 2$ (see Corollary 4).

Formal description and theorem statements. Fix $r > 0$. Let $\mathcal{P} \subseteq \mathbb{R}^d - \mathbb{B}(0, r)$ be a unit intensity Poisson point process. We let $\{(B_t^x)_{t \geq 0} : x \in \mathcal{P} \cup \{0\}\}$ be a family of independent Brownian motions, with $B_0^x = x$. For $x, z \in \mathcal{P} \cup \{0\}$ define $t(x, z) = \inf\{t : \|B_t^x - z\| \leq r\}$ to be the first time the path started at x is within r of z . Following [AMP02a], we define the passage time to z for the process started at x as

$$(1) \quad T(x, z) = \inf \left\{ \sum_{\{x_1, \dots, x_k\} \subseteq \mathcal{P}} t(x_i, x_{i+1}) : x_1 = x \text{ and } x_k = z \text{ for some } k \right\}.$$

We define $Z_z^x(t)$ to be the location at time t of the particle initially at z in the process started from x . This can be written in terms of $T(x, z)$:

$$(2) \quad Z_z^x(t) = \begin{cases} z, & T(x, z) \geq t \\ B_{t-T(x, z)}^z, & T(x, z) < t \end{cases}.$$

Now let $\Gamma(x_0, \mathcal{P}) \subseteq \mathbb{R}^d$ be the *continuum percolation cluster of \mathcal{P} containing x_0* . This is obtained recursively by setting $\Gamma(x_0, \mathcal{P}) = \bigcup_{i=1}^{\infty} \Gamma_i(x_0, \mathcal{P})$ where $\Gamma_0(x_0, \mathcal{P}) = \{x \in \mathcal{P} : \|x - x_0\| \leq r\}$, and for $i \geq 1$

$$\Gamma_i(x_0, \mathcal{P}) = \{x \in \mathcal{P} : \|x - \Gamma_{i-1}(x_0, \mathcal{P})\| \leq r\}.$$

Define

$$\xi_t^x = \{z \in \mathcal{P} \cup \{0\} : T(x, z) \leq t \text{ for some } y \in \Gamma(z, \mathcal{P})\}$$

to be the set of activated sites from $\mathcal{P} \cup \{0\}$ provided the initially active particle is at x . For convenience we set $\xi_t^0 = \xi_t$. Define $A_t = \{Z_z^x(t) : z \in \xi_t\}$ to be the locations of active frogs at time t .

Theorem 1. *For all $d \geq 1$, given $0 < r < r_d$ there exists $0 < \gamma_r < \infty$ such that for all $\epsilon > 0$*

$$\mathbb{B}(0, \gamma_r(1 - \epsilon)t) \cap \mathcal{P} \subseteq \xi_t \subseteq \mathbb{B}(0, \gamma_r(1 + \epsilon)t)$$

almost surely for sufficiently large t .

We will give an overview of this proof momentarily. First, we state our other two theorems. The first is the continuous analogue of [RS04, Theorem 1.3].

Theorem 2. *For any $R < \infty$ as $t \rightarrow \infty$ the locations of awake frogs $A_t \cap \mathbb{B}(0, R)$ converges to a unit intensity Poisson point process.*

This should not be surprising since due to the shape theorem there are a large number of awake frogs each of which has a small probability of being in $\mathbb{B}(0, R)$

In the shape theorem it is natural to work with the growth rate; however in other situations it is easier to study the time constant, $\tilde{\gamma}_r = 1/\gamma_r$. At the critical value the mean cluster size is infinite. Based on this one would expect that time constant $\tilde{\gamma}_r \rightarrow 0$ as $r \rightarrow r_d$. A simple argument, valid in any dimension, shows that it is enough to show $\tilde{\gamma}_r = 0$ at criticality. To prove this in $d = 2$ we will show that the time to wake up large clusters of frogs is relatively small.

Theorem 3. *Suppose $d = 2$ and let R_t be the x -coordinate of the right-most awake frog at time t . If $r = r_2$ then for all $\epsilon > 0$ $R_t/t^{2-\epsilon} \rightarrow \infty$ a.s. as $t \rightarrow \infty$.*

Our argument relies on a result about sponge crossings of a fixed rectangle from [Roy90] that utilizes the duality between closed crossings and vacant cut sets, which only holds in $d = 2$. The idea of the proof is to first use a continuity argument that reduces Theorem 3 to showing the time constant is 0 at criticality. We use the sponge crossing result from [Roy90] along with a BK-inequality to show that a given ball is likely to be part of a cluster that displaces far to the right. These rightward jumps take no time, and Theorem 1 ensures that the frog model does not take too long to discover these large jumps. We remark that we are unsure whether t^2 is the actual asymptotic speed of R_t . A further question is to determine the exponent α such that R_t/t^α has a bounded nonzero limit. A quick corollary that follows from left-continuity is that the speed of expansion diverges as r approaches r_2 .

Corollary 4. *When $d = 2$ it holds that $\lim_{r \rightarrow r_2^-} \gamma_r = \infty$.*

Outline of Theorem 1. As in [AMP02a, RS04] this result relies on the sub-additive ergodic theorem. Although the outline is similar to these previous works, the techniques we use are different. The proof of Theorem 1 relies on three propositions. We list them along with a sketch of each proof. The most important is the existence of a limiting speed in each direction.

Proposition 5. *Fix a unit vector v and let $L_v = \{tv : t \geq 0\}$ be the ray through v emanating from $x_0 = 0$. Let x_n be the point in \mathcal{P} nearest to nv for $n \geq 0$. We have*

$$\lim_{n \rightarrow \infty} \frac{T(x_0, x_n)}{n} = \tilde{\gamma}_r \in [0, \infty).$$

Note that the time-constant $\tilde{\gamma}_r$ does not depend on v .

This is a standard application of Liggett's subadditive ergodic theorem. The main difficulty is showing that $\mathbf{E}T(0, x_1)$ is finite, but this follows from Lemma 16 which is the first step of proving Proposition 7.

In [AMP02a] the frog model can grow at most linearly, and in the continuous time setting of [RS04] it is a minor nuisance, but not a serious obstacle to show the process stays contained in a linearly expanding ball. A concern for the Brownian frog model is that overlapping clusters of frogs all wake simultaneously. So, we might have large jumps that take no time. This introduces some extra work to prove containment in a linearly expanding ball.

Proposition 6. *For $0 < r < r_d$ there exists $R_1 < \infty$ and $c_1 > 0$ such that*

$$\mathbf{P}[\xi_t \subseteq \mathbb{B}(0, R_1 t)] \geq 1 - e^{-c_1 t}$$

for all $t \geq 0$.

We dominate the frog model by a branching process that resets the explored regions at each branching event. We rely on large deviation estimates for the cluster size in continuum percolation and the displacement of Brownian motion. It takes some care to create a branching process that grows slowly, and couples to the Brownian frog model.

A standard ingredient in proving a shape theorem is that the process grows at least linearly.

Proposition 7. *For $0 < r < r_d$, all $\epsilon > 0$, and all $m \geq 1$ there exists an almost surely finite t_m such that*

$$\mathbf{P}[\mathbb{B}(0, (\gamma_r - \epsilon)t) \subseteq \bar{\xi}_t] \geq 1 - t^{-m}$$

for all $t \geq t_m$.

Our proof is somewhat simpler than what was done in [AMP02a] and [RS04]. We write wvhp as short for “with very high probability,” which means that the probability of failure goes to 0 faster than n^{-m} for any m . We break $[-n, n]^d$ into small cubes of size $n^{1-\beta}$ and show that by time $n^{2+\epsilon}$ wvhp a frog has been awoken in each small cube. Given $a > 0$ we can pick D independent of n so that by iterating this construction D times, we have an awake frog in each of a collection of cubes with side n^a . These frogs are enough to finish waking the rest of the frogs in $[n, n]^d$ by time $(D+1)n^{2+\epsilon}$. We then use this lemma and the approximate independence of distant frogs to show that this is enough to hit every point in the ball expanding at rate $\gamma_r - \epsilon$.

Organization of paper. Section 2 establishes the existence of a time-constant $\tilde{\gamma}_r$ in Proposition 5 and uses this along with Proposition 6 and 7 to deduce Theorem 1. Section 3 contains the proof of Proposition 6, that the Brownian frog model is contained in a linearly expanding ball. In Section 4 we prove that the Brownian frog model contains a linearly expanding ball in Proposition 7. Section 5 proves that the locations of active frogs converge to a Poisson process and Section 6 shows Theorem 3, that the process expands superlinearly at criticality when $d = 2$.

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2. EXISTENCE OF A LIMITING SHAPE

We deduce a limiting speed by showing that the passage times between particles on a fixed ray from the origin satisfy the hypotheses of Liggett’s subadditive theorem. Deducing a limiting shape for the Brownian frog model is slightly more convenient than the discrete frog model, since it is already defined on \mathbb{R}^d . This removes the need to interpolate from \mathbb{Z}^d to \mathbb{Q}^d to \mathbb{R}^d . Additionally, radial symmetry ensures that the limiting speed in each direction is the same, and so the final shape must be a ball. We begin by stating Liggett’s subadditive ergodic theorem, then showing the passage times along a ray satisfy the hypotheses.

Theorem 8 (Liggett’s subadditive ergodic theorem). *Suppose that $\{Y(m, n)\}$ is a collection of positive random variables indexed by integers satisfying $0 \leq m < n$ such that*

- (i) $Y(0, n) \leq Y(0, m) + Y(m, n)$ for all $0 \leq m < n$ (subadditivity);
- (ii) The joint distribution of $\{Y(m+1, m+k+1), k \geq 1\}$ is the same as that of $\{Y(m, m+k), k \geq 1\}$ for each $m \geq 0$;
- (iii) For each $k \geq 1$ the sequence of random variables $\{Y(nk, (n+1)k), n \geq 1\}$ is a stationary ergodic process;
- (iv) $\mathbf{E}Y(0, 1) < \infty$.

Then

$$\lim_{n \rightarrow \infty} \frac{Y(0, n)}{n} \rightarrow \gamma \quad a.s.,$$

where $\gamma = \inf_{n \geq 0} \frac{\mathbf{E}Y(0, n)}{n}$.

Proof of Proposition 5. It suffices to prove that the collection $\{T(x_m, x_n)\}_{0 \leq m < n}$ satisfies (i)-(iv) of Theorem 8. For (i), we claim that

$$T(x_m, x_n) \leq T(x_m, x_\ell) + T(x_\ell, x_n) \text{ for all } 0 \leq m \leq \ell < n.$$

If site x_n is reached before site x_ℓ (so that $T(x_m, x_n) \leq T(x_m, x_\ell)$), there is nothing to prove. If that does not happen and $T(x_m, x_n) > T(x_m, x_\ell)$ then the process which began with only the frog at site x_ℓ awake can be coupled to the process originating from site x_m ,

which might have other particles awake at time $T(x_m, x_\ell)$. Thus the remaining time to reach site x_n for the original process can be at most $T(x_\ell, x_n)$, proving (i).

The second point, (ii), follows from radial symmetry and translation invariance of the initial configuration.

To get (iii), for simplicity assume that L_v is the nonnegative x -axis in two dimensions. Let Π_m be the Poisson points in the region $(m - 1/2, m + 1/2) \times \mathbb{R}$. This is an i.i.d. sequence and hence ergodic. Moreover, $Y(nk, (n + 1)k)$ is a shift invariant function of the Π_m and hence is ergodic by [Dur10, Theorem 7.1.3].

Finally, to see (iv), for $d \geq 2$ we apply Lemma 16. This guarantees the the frog process contains a ball of radius $\Omega(\sqrt{t})$ with very high probability. Thus, the expected time to reach x_1 must be finite. For $d = 1$, let τ be the first return to the origin of a Brownian motion. Since $\mathbf{P}[\tau > t] \leq Ct^{-1/2}$ and in a random time with exponential tail we will have at least three awakened particles moving independently, we can conclude that $\mathbf{ET}(0, x_1) < \infty$. \square

Now we explain how the existence of γ_r gives a limiting shape.

Proof of Theorem 1. The existence of $0 < \gamma_r \leq \infty$ follows from Proposition 5. It is finite by Proposition 6, and positive by its definition: $\tilde{\gamma}_r = \inf_{n \geq 1} \mathbf{ET}(0, x_n)/n < \infty$. When $d = 1$ this immediately yields the shape theorem. For $d \geq 2$ a standard shape theorem argument (confer [BG81, DG82, AMP02a, RS04]) where we cover $\mathbb{B}(0, \gamma_r)$ with finitely many discs of radius $\epsilon' R_1 > 0$ and scale by t while applying Proposition 7 gives that $\bar{\xi}_t \subseteq \mathbb{B}(0, (\gamma_r + \epsilon)t)$ for all large t . The fact that $\mathbb{B}(0, (\gamma_r - \epsilon)t) \subseteq \bar{\xi}_t$ is proven in Proposition 7. \square

3. CONTAINMENT IN A BALL

We will show that the set of activated sites does not grow super-linearly. The idea is to dominate it by a branching process that places particles according to the geometry of a cluster in (subcritical) continuum percolation. We begin with an informal description.

The initial particle at 0 starts inside an empty ball of radius $r < r_d$ in the environment \mathcal{P} . It performs Brownian motion until it comes within r of a point $x_0 \in \mathcal{P}$ at time τ . This particle has explored the Brownian sausage around B_t for times in $[0, \tau)$ and found no points from \mathcal{P} in it. To preserve a coupling with the frog model we need to, whenever possible, use the points in \mathcal{P} . Accordingly, we refill this explored region with a fresh Poisson point process, and then sample the continuum cluster at x_0 . If we skipped this replenishment of points, then future clusters we sample would have some dependence on τ and not be identically distributed. After replenishing, the continuum cluster at x_0 will contain K additional centers of balls: x_1, \dots, x_K . The process then branches into $K + 2$ offspring. The '+2' comes from the initial particle at 0 which is placed at $x_{-1} = B_\tau$, and the particle at x_0 .

For each x_i we want to reproduce the environment seen by the particle initially at 0. To do this we need to refill the parts of the continuum cluster outside of $\mathbb{B}(x_i, r)$ with a fresh Poisson point process. We use the same points for all of the x_i , but fill in slightly different regions with them. Now, we let the particle at each x_i explore the modified \mathcal{P} environment until it starts a new branching event which occurs in the same manner. The particles from the x_i are blind to the discoveries of other particles. So it is possible that branching events occur multiple times at the same points. This introduces considerable dependence, but we will see that there is still independence with the branching times and offspring down a single lineage. This is enough for us to deduce that it is a.s. contained in a linearly expanding ball.

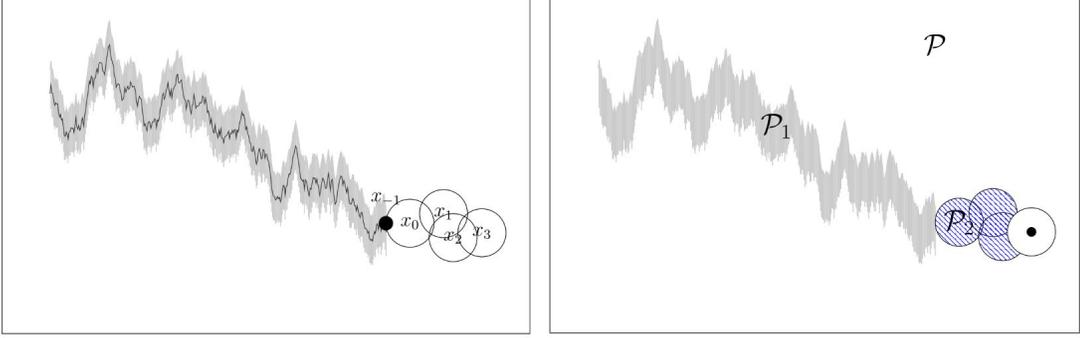


FIGURE 1. The particle at the origin moves until it reaches a cluster. This causes a branching event that places children at x_0, \dots, x_K . In this example $K = 3$. The left panel shows the locations of the offspring. The right panel shows the environment that x_3 sees at birth. The region explored by the parent particle, x_{-1} , is filled by an independent Poisson point process \mathcal{P}_1 , and the offspring cluster minus $B(x_3, r)$, shown with shaded lines, is filled with another independent Poisson process \mathcal{P}_2 . The completely unexplored regions of the space still contain the original point process \mathcal{P} .

3.1. Formal description and properties. We need a way to make branching identically distributed while remembering the points of \mathcal{P} . To do this we will refill explored areas with points from $(\mathcal{P}_i)_{i \geq 1}$, a collection of independent unit Poisson point processes on \mathbb{R}^d .

Recall the definition of $\Gamma(x_0, \mathcal{P}) \subseteq \mathbb{R}^d$ the *continuum percolation cluster of \mathcal{P} containing x_0* . It is obtained recursively by setting $\Gamma(x_0, \mathcal{P}) = \bigcup_{i=1}^{\infty} \Gamma_i(x_0, \mathcal{P})$ where $\Gamma_0(x_0, \mathcal{P}) = \{x \in \mathcal{P} : \|x - x_0\| \leq r\}$, and for $i \geq 1$

$$\Gamma_i(x_0, \mathcal{P}) = \{x \in \mathcal{P} : \|x - \Gamma_{i-1}(x_0, \mathcal{P})\| \leq r\}.$$

We let $\Gamma = \Gamma(0, \tilde{\mathcal{P}})$ be the cluster at the origin for $\tilde{\mathcal{P}}$ an independent unit intensity Poisson point process. We define Γ in terms of $\tilde{\mathcal{P}}$ because \mathcal{P} has a ball at the origin removed.

Let $\tau \in [0, \infty]$ be the time for a Brownian motion started at the origin to be within r of a point in \mathcal{P} . Formally,

$$\tau = \inf\{t : \|B_t - \mathcal{P}\| = r\}.$$

Recall that $\{(B_t^x)_{t \geq 0} : x \in \mathcal{P} \cup \{0\}\}$ is the family of Brownian motion paths followed by frogs in the Brownian frog model. Let $W_t^x = \cup_{0 \leq s \leq t} \mathbb{B}(B_s^x, r)$ be the Brownian sausage of radius r generated by $(B_s^x)_{0 \leq s \leq t}$.

Now we describe the manner in which particles branch. At time τ the particle started from the origin has revealed that $\mathcal{P} \cap W_\tau = \emptyset$ and that $\|B_\tau - x_0\| = r$ for some $x_0 \in \mathcal{P}$. Replace the points in $\mathcal{P} - W_\tau$ with $\mathcal{P}_1 \cap W_\tau$. That is, form the set

$$\mathcal{P}' = (\mathcal{P} - W_\tau) \cup (\mathcal{P}_1 \cap W_\tau).$$

By Lemma 20 (see the appendix) the process \mathcal{P}' is a unit intensity Poisson point process. Let $\Gamma(x_0, \mathcal{P}')$ be the new continuum cluster at x_0 . We place offspring at $x_{-1} := B_\tau$ and at the centers of the balls in this cluster. More precisely, at the points

$$\Gamma(x_0, \mathcal{P}') \cap \mathcal{P}' = \{x_{-1}, x_0, x_1, \dots, x_K\}$$

where $K + 1$ is the number of balls in the cluster. See Figure 1.

To ensure the environment looks the same in distribution for each new particle, we replace $\mathcal{P}' \cap \Gamma(x_0, \mathcal{P}')$ with $\mathcal{P}_2 \cap \Gamma(x_0, \mathcal{P}')$. For the particle at x_i we also delete any points in $\mathbb{B}(x_i, r)$ from the process. This makes it so each new particle has the same environment to explore as the initial particle at the origin. Formally, each particle in $\{x_{-1}, x_0, x_1, \dots, x_K\}$ will move according to a Brownian motion in the environment

$$\mathcal{P}_i'' = \left[(\mathcal{P}' - \Gamma(x_0, \mathcal{P}')) \cup (\mathcal{P}_2 \cap \Gamma(x_0, \mathcal{P}')) \right] - \mathbb{B}(0, x_i), \quad i = -1, 0, 1, \dots, K.$$

Notice that invariance again ensures that \mathcal{P}_i'' is a unit intensity point process on $\mathbb{R}^d - \mathbb{B}(0, x_i)$.

Now, each particle will explore \mathcal{P}_i'' . The particles do not see the areas explored by their siblings. When the particle started at x_i comes within r of a point in \mathcal{P}_i'' it will, independent of its siblings, replace the Brownian sausage it carved out with an unused \mathcal{P}_j . We place $2+K$ offspring as before in a Γ distributed way, then refill the cluster with \mathcal{P}_{i+1} . All particles proceed to branch in this manner indefinitely.

Let ζ_t be the birth locations of all particles active at time t in the branching process defined above. We collect a few facts regarding ζ_t . The most important is that it dominates the Brownian frog model.

Lemma 9. *There exists a coupling such that $\xi_t \subseteq \zeta_t$ for all $t \geq 0$.*

Proof. At all steps we use the original point process \mathcal{P} . Additionally we couple the Brownian paths in the canonical way; the frogs in both processes use the same Brownian motions, and extra frogs added into the dominating process use independent Brownian motions. When we refill explored regions of \mathcal{P} with independent points, we are adding additional Brownian motions to the process. Since the process becomes monotonically larger when more particles are introduced, this ensures that $\xi_t \subseteq \zeta_t$ for all t . \square

Let Z_n be the set of birthplaces for particles in the n th generation of the process. For a point $z_n \in Z_n$ let z_{n-1}, \dots, z_1, z_0 be its ancestry back to the original active particle at the origin. So, z_{n-i} is the starting location of the i th ancestor of z_n . Set τ_i to be the time between the birth of z_i and z_{i+1} .

Lemma 10. *$\tau_i \stackrel{d}{=} \tau$ and τ_0, \dots, τ_n are independent.*

Proof. When a particle begins a Brownian motion from z_i it does so in a unit Poisson point process, \mathcal{P}_i'' on $\mathbb{R}^d - \mathbb{B}(z_i, r)$. Thus, the time to be within r of a point in $\tilde{\mathcal{P}}$ is distributed like τ . The τ_i are independent since we have no information about the unexplored areas of \mathcal{P}_i'' , and replace the areas covered by the Brownian sausage around the path started from z_i with an independent Poisson point process. \square

For $1 \leq i \leq n$ let $L(z_{i-1}) \in \mathbb{R}^d$ be the center of the first ball reached by the particle started at z_{i-1} during a branching event. Let X_i be the number of new particles placed around $Z(z_{i-1})$ when branching occurs.

Lemma 11. *It holds that $X_i \stackrel{d}{=} 2 + K_i$ with $K_i \stackrel{d}{=} K$ as in the definition of the branching process. The particles counted by X_i are contained in a neighborhood of $L(z_i)$ distributed like a continuum percolation cluster $\Gamma_i + L(z_i)$, plus a point on the boundary of the ball around $L(z_i)$.*

Proof. The reasoning is similar to the proof of Lemma 10. \square

3.2. Large deviations in the branching process. We break the proof for the upper bound into three steps. In Lemma 12, we estimate the expected number of frogs in generation n . In Lemma 13, we give an upper bound on the maximal generation present in the population at time t . Then we combine the second result with large deviation estimates to bound the maximal distance an active particle has traveled up to time t .

Lemma 12. *Recall that Z_n is the number of particles in the n th generation. Let $\mu = \mathbf{E}X_1$ with X_1 as in Lemma 11. So long as $r < r_d$ it holds that $\mu < \infty$ and $\mathbf{E}Z_n = \mu^n$.*

Proof. By [Pen03, Lemma 10.2] for $r < r_d$ we have $\mu < \infty$. Now, Lemma 11 ensures that particles produce offspring in an identically distributed way. It follows that $\mathbf{E}[Z_{n+1} | Z_n] = \mu Z_n$, hence $\mathbf{E}Z_n = \mu^n$. Note that this computation does not require that the different families are independent. \square

Define the generation number of a particle in ζ_t to be the number of ancestors the particle has. Let N_t be the largest generation number in ζ_t .

Lemma 13. *There exist $\epsilon_0, c > 0$ so the $\mathbf{P}[N_t > t/\epsilon_0] \leq e^{-ct}$ for sufficiently large t .*

Proof. Consider one lineage and let τ_i be the time between the birth of two particles in the same lineage. By Lemma 10, the τ_i are i.i.d. Moreover, we have $\mathbf{P}[\tau_1 = 0] = 0$ since each particle starts in an ball of radius r that contains no other points in the process.

Let $S_n = \tau_1 + \dots + \tau_n$. If $S_n < \epsilon n$, then there must be at least $n/2$ of these τ_i satisfying $\tau_i \leq 2\epsilon$. Notice that Markov's inequality applied to Lemma 12 guarantees $\mathbf{P}[Z_n \geq (3\mu)^n] \leq 3^{-n}$. For an arbitrary $\epsilon > 0$ we condition on this event to obtain

$$(3) \quad \mathbf{P}[N_{\epsilon n} \geq n] \leq 3^{-n} + \mathbf{P}[N_{\epsilon n} \geq n | Z_n < (3\mu)^n].$$

We can apply a union bound to the second summand to obtain

$$\begin{aligned} \mathbf{P}[N_{\epsilon n} \geq n | Z_n < (3\mu)^n] &\leq (3\mu)^n \mathbf{P}[S_n \leq \epsilon n] \\ &\leq (3\mu)^n 2^n \mathbf{P}[\tau_1 \leq 2\epsilon]^{n/2}, \end{aligned}$$

where 2^n is an upper bound on the number of subsets of $\{1, \dots, n\}$ of size at least $n/2$. As $\mathbf{P}[\tau_1 = 0] = 0$ we can choose ϵ_0 such that

$$6\mu \mathbf{P}[\tau_1 \leq 2\epsilon_0]^{1/2} < 1/3.$$

It follows from (3) that $\mathbf{P}[N_{\epsilon_0 n} \geq n] \leq 3^{-n} + 3^{-n} \leq 2^{-n}$ for large enough n . Let $t = \epsilon_0 n$ so that

$$\mathbf{P}[N_t > t/\epsilon_0] = \mathbf{P}[N_{\epsilon_0 n} \geq n] \leq 2^{-n} = 2^{-t/\epsilon_0} \leq e^{-ct}$$

for some $c > 0$. \square

Next, we need a result of Mathew Penrose that the right tail of the number of balls in a continuum cluster decays exponentially.

Lemma 14. *Let X be the number of offspring created at each step in the branching process as in Lemma 11. It holds that $\mathbf{P}[X > k] \leq e^{-ck}$ for some $c > 0$ and large enough k .*

Proof. Lemma 11 ensures that the number of offspring is dominated by two plus the number of particles in a continuum cluster. The exponential bound then follows from the exponential bound on the cluster size in [Pen03, Lemma 10.2]. \square

Now we are in a position to prove that the frog model is contained in a linearly expanding ball.

Proof of Proposition 6. Let $m_t = \max_{x \in \xi_t} \|x\|$ be the furthest activated site from the origin in the frog model, and $M_t = \max_{z \in \zeta_t} \|z\|$ be the furthest birth-site of a particle in the branching process. The coupling in Lemma 9 ensures that $m_t \preceq M_t$. So, it suffices to find a constant $R_1 > 0$ such that $M_t \leq R_1 t$ almost surely for all sufficiently large t .

Let $z \in \zeta_t$ be an arbitrary birth place of an active particle. We need to introduce a few random quantities attached to z that will give us an upper bound on $\|z\|$. First we focus on the Brownian path leading to z .

Suppose that the particle at z was born at time $\tau \leq t$ and let $(B_s)_{s \geq 0}$ be the Brownian path followed by the ancestors of z , and then by the particle at z after time τ . Define $B_t^* = \sup_{s \leq t} \|B_s\|$. Set N_t to be the maximal generation of particles in ζ_t .

Now we turn our attention to the displacement of children during the branching times of ancestors of the particle started from z . Let $J \stackrel{d}{=} 2rX$ with X , the number of particles created at a branching time, distributed as in Lemma 14. Take J_1, J_2, \dots to be i.i.d. copies of J and $(B_s)_{s \geq 0}$ to be an independent Brownian motion.

We claim that

$$(4) \quad \|z\| \preceq B_t^* + \sum_{i=1}^{N_t} J_i.$$

Note that there is dependence between B_t^* and N_t . However, the J_i are independent of both B_t^* and N_t .

To justify (4) we note that the contribution from the Brownian path leading to z is at most B_t^* , and the number of jumps that have occurred is at most N_t . Any given offspring will displace by at most the size of a continuum cluster. This is bounded in distribution by $J = 2rX$, the number of particles in the cluster times the diameter of each disk. The independence assertions regarding the J_i and B_t^* and N_t follow from Lemma 11.

Since $z \in \zeta_t$ is arbitrary it follows from (4) that

$$(5) \quad M_t \preceq B_t^* + \sum_{i=1}^{N_t} J_i.$$

We turn our attention to providing exponential bounds on the two right-hand terms. The maximum of Brownian motion can be bounded by Schilder's theorem which ensures that for any $c > 0$ there exists $C > 0$ such that for all sufficiently large t we have,

$$(6) \quad \mathbf{P}[B_t^* > Ct] \leq e^{-ct}.$$

We can obtain a similar exponential bound on $\sum_{i=1}^{N_t} J_i$. Recall that Lemma 13 gives ϵ_0 and c' so that $\mathbf{P}[N_t > t/\epsilon_0] \leq e^{-c't}$ for large t . Let $T = \lceil t/\epsilon_0 \rceil$ and $C' > 0$. Using the exponential bound on the right tail of X in Lemma 14 we have

$$\mathbf{P}\left[\sum_{i=1}^{N_t} J_i > C't, N_t \leq T\right] \leq \mathbf{P}\left[\sum_{i=1}^T J_i > C't\right].$$

The sum $J_1 + \dots + J_T$ is a finite sum of i.i.d. random variables with exponential tails. A standard concentration inequality that arises from the moment generating function of J_1 is that for any $c > 0$ there exists $C' > 0$ so that

$$(7) \quad \mathbf{P}[J_1 + \dots + J_T > C'T] \leq e^{-cT}.$$

It follows from the relation at (5) and the bounds in (6) and (7) that for sufficiently large $R_1 > 0$ we have constants $c, c' > 0$ so that

$$\begin{aligned} \mathbf{P}[M_t \geq R_1 t] &\leq \mathbf{P}[B_t^* > (R_1/2)t] + \mathbf{P}\left[\sum_{i=1}^{N_t} J_i > (R_1/2)t\right] \\ &\leq e^{-ct} + e^{-c't}. \end{aligned}$$

This ensures that for all sufficiently large t we have $M_t \leq R_1 t$ completing the proof. \square

4. PASSAGE TIME AND CONTAINING A BALL

First, we will show that the frog model contains a growing small cube. To do this, we will divide the $[-n, n]^d$ cube into smaller cubes using a square mesh. Then we stagger the frog's wake-up times in such a way that there is an active frog at each square in the mesh. By shrinking the size of the mesh, we are able to show that the active frogs spread out and that they will visit all frogs in $[-n, n]^d$ by time $(D+1)n^{2+\epsilon}$ for a constant D . We say that events A_n happen *with extremely high probability (wehp)* if $\mathbf{P}[A_n] \geq 1 - \exp(-n^\gamma)$ for some γ . Alternatively, we say that A_n happens *with very high probability (vwhp)* if $\mathbf{P}[A_n] \geq 1 - n^{-m}$ for any $m < \infty$ and sufficiently large n .

We begin by stating some basic facts of Brownian motion.

Lemma 15. *Let B_t be a one dimensional Brownian motion. If $\ell \geq 1$ is an integer*

$$(8) \quad \mathbf{P}\left[\max_{0 \leq t \leq \ell k^2} |B_t| \leq k\right] \leq (0.7)^\ell$$

$$(9) \quad \mathbf{P}\left[\max_{0 \leq t \leq k^2} B_t \geq \ell k\right] \leq 2e^{-\ell^2/2}$$

Proof. For the first we note that

$$\sup_{x \in [-k, k]} \mathbf{P}_x[B_{k^2} \in [-k, k]] \leq 0.7.$$

Here \mathbf{P}_x is the probability when the particle starts at x . For the second, the reflection principle and a standard normal tail bound gives

$$\mathbf{P}\left[\max_{0 \leq t \leq k^2} B_t \geq \ell k\right] \leq 2\mathbf{P}[B_{k^2} \geq \ell k] \leq 2e^{-\ell^2/2}$$

which proves the desired result. \square

Now we show that given $O(n^{2+\epsilon})$ time to evolve, the Brownian frog model is very likely to wake all of the frogs in a cube of width n . We use the notation $f(n) = \Omega(g(n))$ to mean that there exists C such that for large enough n it holds $f(n) \geq Cg(n)$.

Lemma 16. *Let $T_1 = n^{2+\epsilon}$. There is a constant D so that with whp all frogs in $[-n, n]^d$ are awake at time $(D+1)T_1$.*

Proof. Suppose $d \geq 3$. In this case all events will be wehp. Our proof will be iterative, with D stages. To prepare for this, we will label our Poisson points with independent random variables uniform on $\{1, \dots, D\}$. By Poisson thinning, the frogs labeled i are a Poisson process with intensity $1/D$, call the process \mathcal{P}_i .

Let $\epsilon = 1/(3+d)$ and $\beta = 1/d$. Choose D so that $(1-\beta)^D$ is smaller than $\frac{2+\epsilon}{2d}$. Consider $Q_n = [-n, n]^d$ with an active frog at the origin and sleeping 1-frogs at \mathcal{P}_1 . Run the process for $t_1 := n^{2-\epsilon}$. It follows from [vdBBdH01] that wehp the Brownian sausage with radius r

created by the active frog from the origin has volume $V \geq n^{2-2\epsilon}$ at time $n^{2-\epsilon}$. In addition, wehp the sausage is a subset of $[-n, n]^d$ because the frog is unlikely to have walked outside the box in time $n^{2-\epsilon}$. Since the number of points of \mathcal{P}_1 in a set of volume V is Poisson with mean V/D a standard large deviations estimate for Poisson random variables implies that wehp the number of awakened frogs is at least $n^{2-3\epsilon}$. Call these the active 1-frogs.

Form a minimal covering of Q_n by finitely many disjoint small cubes, $Q_{n_1}^1, Q_{n_1}^2, \dots$, of side length $n_1 := n^{1-\beta}$ and call these the generation 1 (G1) cubes. Call the center of a G1 cube the cube centered at the same point with side length $n_1/2$. Continue the process until $T_1 = n^{2+\epsilon}$. We want to bound the probability that an active 1-frog is in the center of a given G1 cube at time T_1 . A standard hitting estimate for Brownian motion lets us write this probability as

$$\int_{Q_{n_1}^k} \frac{1}{(2\pi s)^{d/2}} e^{-(x-y)^2/2s} dy$$

where s is the time since the 1-frog was activated and x is the initial location of that 1-frog.

Because all active 1-frogs were activated before time t_1 , $s \in [T_1 - t_1, T_1]$. Since the diameter of Q_n is $2\sqrt{d}n$, and all of the active 1-frogs are within Q_n wehp, we have $|x - y| \leq 2\sqrt{d}n$. So,

$$\begin{aligned} \int_{Q_{n_1}^k} \frac{1}{(2\pi s)^{d/2}} e^{-(x-y)^2/2s} dy &\geq (2\pi T_1)^{-d/2} |Q_{n_1}^k| e^{-2dn^2/(T_1-t_1)} \\ &= (2\pi n^{2+\epsilon})^{-d/2} n^{(1-\beta)d} 2^{-d} e^{-2dn^2/(n^{2+\epsilon}-n^{2-\epsilon})} \\ &= \Omega\left(\frac{n^{(1-\beta)d}}{n^{(2-\epsilon)d/2}}\right). \end{aligned}$$

Thus for n large, the probability that there are no 1-frogs in a given generation 1 cube at time t_1 is no more than

$$\begin{aligned} \left(1 - C \frac{n^{(1-\beta)d}}{n^{(2-\epsilon)d/2}}\right)^{n^{2-3\epsilon}} &\approx \exp\left(-Cn^{(2-3\epsilon)+(1-\beta)d-(2-\epsilon)d/2}\right) \\ (10) \qquad \qquad \qquad &= \exp\left(-Cn^{1-3\epsilon-\epsilon d/2}\right) = \exp(-Cn^\delta) \end{aligned}$$

with $\delta > 0$ since $\epsilon = 1/(3+d)$. Since the number of G1 cubes is less than or equal to $(2n^\beta)^d$, it follows that wehp there is a 1-frog in every G1 cube at time T_1 .

At T_1 , put down sleeping 2-frogs at the points in \mathcal{P}_2 . Recall that $n_1 = n^{1-\beta}$, let $n_2 = n_1^{1-\beta}$, $t_2 = T_1 + n_1^{2-\epsilon}$, and $T_2 = T_1 + n_1^{2+\epsilon}$. We know that wehp, there is an active 1-frog in a given G1 cube. Focusing on a specific G1 cube, we let the active 1-frog run and wake up the sleeping 2-frogs until t_2 .

Similar to the first step, between time T_1 and time t_2 , the active 1-frog in this G1 cube will produce wehp a Brownian sausage of volume at least $n_1^{2-2\epsilon}$, waking up, wehp, at least $n_1^{2-3\epsilon}$ of the 2-frogs. Call these newly awakened frogs the active 2-frogs. Again, we allow these active 2-frogs to run from until time T_2 . Defining generation 2 cubes as disjoint cubes of side length $n_1^{1-\beta}$ inside the G1 cube, we can do a very similar calculation to find the probability that at time T_2 there is an active 2-frog in the center of each G2 cube:

$$\int_{Q_{n_1}^k} \frac{1}{(2\pi s)^{d/2}} e^{(x-y)^2/2s_2} dy$$

where $s_2 \in [T_2 - t_2, T_2 - T_1]$ is the time since the 2-frog was activated and x is the initial location of that 2-frog. Using the same restrictions on the maximum distance between x and y . Note a concern might be that the initial 1-frog in each G1 cube starts near the boundary and strays out of the cube in time less than t_2 . We can address this by requiring the more strict event that each G1 frog be inside a cube centered in $Q_{n^{1-\beta}}^k$ but with sidelength $1/2$ that of $Q_{n^{1-\beta}}^k$. This only changes the volume by a constant factor, and thus does not change the conclusion at (10). We get that this new probability must be at least

$$(2\pi(T_2 - T_1))^{-d/2} n_1^{(1-\beta)d} 2^{-d} e^{Cn_1^2/(n_1^{2+\epsilon} - n_1^{2-\epsilon})}.$$

The above quantity is asymptotically larger than

$$\begin{aligned} 2\pi(T_2 - T_1)^{-d/2} n_1^{(1-\beta)d} &\approx \exp(-n_1^{(1-\beta)d+2-3\epsilon-(2+\epsilon)d/2}) \\ &= \exp(-n_1^{\delta}) \\ &= \exp(-n^{\delta_2}) \end{aligned}$$

for some $\delta_2 > 0$. Thus, for large n we have

$$\mathbf{P}[\text{there is an active 2-frog in each } G_2 \text{ cube at time } T_2] = \Omega(\exp(-n^{\delta_2})).$$

So the probability that there is not an active 2-frog in a given G2 cube is once again exponentially small. Because there are $(2n_1^\beta)^d$ G2 cubes, then wehp there is an active 2-frog in the center of every G2 cube.

It is important to note here that the thinning of the frogs at the beginning was necessary to ensure that we did not reuse frogs in multiple places/steps at the same time and to provide the independent environments for the 1-frogs within the G1 cubes.

We iterate this process with $n_k = n_{k-1}^{1-\beta}$, $t_k = T_{k-1} + n_{k-1}^{2-\epsilon}$ and $T_k = T_{k-1} + n_{k-1}^{2+\epsilon}$ for k up to D . We get the same results at each cube size iteration. The end result is to say that wehp we have an active D -frog in the center of every generation D cube of length $n^{(1-\beta)^D}$ wehp. Because we introduced frogs in stages rather than placing them all in the process from the beginning, the frogs in this modified process wake up at the same time or later than they would in the true frog model. Therefore, if this process contains a cube of all awake frogs, we can be certain that the original process contains the same cube of all awake frogs at that time as well.

Now we have an active frog within $n^{(1-\beta)^D}$ of every point in $[-n, n]$. For notational simplicity, define $b := d(1-\beta)^D$. All that is left is to bound the probability that a frog within n^b of a point y will hit $\mathbb{B}(y, r)$ by T_1 . To do this, divide $[-n, n]^d$ into tiny cubes of side r/\sqrt{d} , so that if y is in a tiny cube, $\mathbb{B}(y, r)$ contains the entire tiny cube. Inside each tiny cube put a tiny ball of radius $q = r/2\sqrt{d}$. Then we look at the probability that a Brownian motion starting at x with $|x| = n^b$ hits $\mathbb{B}(0, r)$ before exiting a larger ball $\mathbb{B}(0, n^{2b})$. Because of our initial choice of D , (9) says that wehp the exit time of the ball $\mathbb{B}(0, n^{2b})$ is greater than T_1 . Letting $S_q = \inf\{t > 0: B_t = |q|, |B_0| = n^b\}$, then from a standard exit time probability, we get the lower bound

$$\mathbf{P}[S_q < T_1] \geq \frac{1}{2} \frac{n^{(2-d)b} - n^{(2-d)2b}}{q^{2-d} - n^{(2-d)2b}}$$

where the $\frac{1}{2}$ is added as a convenient and conservative estimate that the Brownian motion did not leave $\mathbb{B}(0, n^{2b})$.

So given a fixed tiny cube, the probability no awake frog within distance n^b of it hits the tiny ball within time T_1 is no larger than

$$\left(1 - \frac{1}{2} \frac{n^{(2-d)b} - n^{(2-d)2b}}{q^{2-d} - n^{(2-d)2b}}\right)^{n^{(d-1)b}} \approx \exp\left(-Cn^{(2-d)b+(d-1)b}\right).$$

where the number of frogs is given by finding the number of cubes of side length $n^{(b-\beta)^D}$ within n^b of a point. Since the exponent for n is positive, this implies that if we divide $[-n, n]^d$ into cubes of side r/\sqrt{d} then wehp each tiny cube is visited by an awake frog by time $T_1 + T_D$, and hence all the frogs in $[-n, n]^d$ are awake by time $(D+1)T_1$.

In $d=2$, from [vdBBdH01] we see that the probability that the volume of the sausage is being smaller than $bt/\log t$ is only $\exp(-\gamma(b)\log n)$, where $\gamma(b) \rightarrow \infty$ as $b \rightarrow 0$. So, instead of the sausage volume being at least $n^{2-2\epsilon}$ wehp, this only holds wvhp. Since the number of squares used in the construction is only polynomial in n , this wvhp result is enough to repeat the previous proof. \square

This result is enough to meet the conditions of Theorem 8 and to get the existence of a speed, but the statement of Lemma 16 is not strong enough to say that the process containing a linearly expanding ball. However, we will now build on this result to get the containment of a linearly expanding ball as desired.

Proof of Proposition 7. Let $L = \{te_1 : t \geq 0\}$ with e_1 the standard unit vector. For $n \geq 0$ let x_n be closest point in \mathcal{P} to ne_1 . Let $s = d/2 + \alpha/2$ with $\alpha > 0$ small and let

$$R_n = [-n^s, n + n^s] \times [-n^s, n^s]^{(d-1)}.$$

By (8) wvhp

- (i) no frog from outside R_n wakes up a frog at x_0, x_1, \dots, x_n before time n . (to see this, note that at some time before n it would have to be on the boundary of R_n), and
- (ii) no frog that starts in R_n moves by n^r before time n .

Let $q = n^{1/2+\alpha}$. When the good event in (ii) occurs, the passage times $T(x_i, x_j)$ and $T(x_k, x_\ell)$ are independent for $k - j \geq q$ because the frogs used in finding the first passage time will not be able to travel far enough to influence the second passage time. This can be made precise by coupling to independent random variables.

Let $m = \lfloor n^\alpha \rfloor$. Let $Y_i = T(x_{(i-1)m}, x_{im})$ for $i \leq n/m$. If m is large then $EY_i \leq (\gamma_r + \epsilon)m$ by subadditivity. Define the random variable

$$\bar{Y}_i = Y_i \mathbf{1}\{Y_i \leq n^{\alpha(2+\epsilon)}\},$$

and note that by Lemma 16 wvhp $\bar{Y}_i = Y_i$ for all $1 \leq i \leq n/m$. Let $Z_i = \bar{Y}_i - E\bar{Y}_i$ and observe that $E\bar{Y}_i \leq EY_i \leq (\gamma_r + \epsilon)m$, and $|Z_i| \leq n^{\alpha(2+\epsilon)}$. In the definition of Z_i , we are subtracting off the mean of Y_i , but because it is bounded by something of order $n^{2\alpha}$, the dominant part of Z_i is still of order $n^{\alpha(2+\epsilon)}$. For $k \leq n^{1/2}$ consider the sum

$$S_k = \sum_{j=0}^{\lfloor n^{1/2}/m \rfloor} Z_{jn^{1/2}+k}.$$

In words, S_k represents how far the passage time deviates from the mean for a selection of points along the ray L . In S_k , we only add up passage times between points that are at

least $n^{1/2+\alpha}$ apart; so, on the good event (ii) for each fixed k the variables being summed are independent, mean 0, and bounded by $n^{\alpha(2+\epsilon)}$. So, if p is a positive integer

$$\mathbf{E}|S_k|^{2p} = O\left((n^{1/2}/m)^p n^{2p\alpha(2+\epsilon)}\right).$$

To see this, note that the p th power of the sum has terms of the form $Z_{i_1}^{p_1} \dots Z_{i_k}^{p_k}$ with $p_1 + \dots + p_k = 2p$. If any $p_i = 1$, then the expected value of the product vanishes. Thus, the largest contribution to the polynomial comes when all $p_i = 2$. This is because $\binom{n^{1/2}/m}{p} \approx (n^{1/2}/m)^p$ is the asymptotically largest binomial coefficient. Using our bound $Z_i \leq n^{\alpha(2+\epsilon)}$, the dominant terms (all $p_i = 2$) sum to be no larger than

$$(n^{1/2}/m)^p n^{2p\alpha(2+\epsilon)},$$

as claimed. The $O(\cdot)$ accounts for the smaller order terms. Now, it follows from Markov's inequality that

$$\mathbf{P}[|S_k| > \epsilon n^{1/2}] \leq \epsilon^{-2p} n^{-\alpha p} n^{p[2\alpha(2+\epsilon)-1/2]} = \epsilon^{-2p} n^{-\beta p}$$

where $\beta = \alpha + 1/2 - 2\alpha(2 + \epsilon)$. If p is large enough then

$$\mathbf{P}\left[\sum_{k=1}^{n^{1/2}} |S_k| > \epsilon n\right] \leq n^{-\beta p/2}.$$

Tracing back through the definitions

$$\mathbf{P}[T(x_0, x_n) \leq n(\gamma_r + 2\epsilon)] \geq 1 - n^{-\beta p/2}.$$

If we let $N = \ell n$, then

$$\mathbf{P}[T(x_0, x_{kn}) \leq kn(\gamma_r + 2\epsilon) \text{ for all } 1 \leq k \leq \ell] \geq 1 - \ell n^{-\beta p/2}.$$

The bound is the worst when $k = 1$ and it is used ℓ times. Given a starting point we can wvhp cover a ball of radius $(\epsilon N)^{1/(2+\epsilon)}$ wvhp in time ϵN . We can rewrite $2n = (\epsilon \ell n)^{1/(2+\epsilon)}$ when $\ell = 2^{2+\epsilon} n^{1+\epsilon}/\epsilon$. If p is large enough and we add enough lines then all points in the ball of radius N are within $(\epsilon N)^{1/(2+\epsilon)}$ of the points on our lines so with high probability the ball is covered at time $N(\gamma_r + 3\epsilon)$ and the proof is complete. Note that to choose the lines put the ball in a cube and then pick the lines so that they are sufficiently close on the boundary of the cube. They will be even closer inside the ball. \square

5. LOCAL CONVERGENCE

We will now prove a theorem about the distribution of active frogs at time t for large t . For simplicity, we prove the result for $d = 1$; the same argument with slightly harder calculus proves the result in higher dimensions.

Proof of Theorem 2. Let $d = 1$. By Theorem 1, for large t all of the frogs are awakened near the edge of a growing ball. It follows from tail estimates for Brownian motion that the probability some frog awakened at a distance more than $t^{2/3}$ from the origin will end up in $[a, b]$ at time t tends to 0. The expected number of frogs that end up in $[a, b]$ at time t is

$$(11) \quad \mathbf{E}[|A_t \cap [a, b]|] = \int_a^b \sum_{x \in \mathcal{P} \cap [-t^{2/3}, t^{2/3}]} p_{t-s(x)}(x, y) dy$$

where \mathcal{P} is the Poisson point process of frog positions, $s(x)$ is the time the frog at x wakes up, and $p_{t-s(x)}(x, y)$ is the hitting probability. Since Theorem 1 ensures that the process expands linearly, there is a $C < \infty$ so that as $t \rightarrow \infty$ we have

$$\mathbf{P} \left[\sup_{|x| \leq t^{2/3}} s(x) \leq Ct^{2/3} \right] \rightarrow 1.$$

Let $t_1 = t - Ct^{2/3}$. Let $W(y)$ denote the sum in (11) for fixed y . Observe that

$$(12) \quad W(y) \leq \frac{1}{\sqrt{2\pi t_1}} \sum_{x \in \mathcal{P} \cap [-t^{2/3}, t^{2/3}]} e^{-(y-x)^2/2t}.$$

And, for a lower bound we use

$$(13) \quad W(y) \geq \frac{1}{\sqrt{2\pi t}} \sum_{x \in \mathcal{P} \cap [-t^{2/3}, t^{2/3}]} e^{-(y-x)^2/2t_1}.$$

Note that the constants in front of each of these bounds have the same asymptotic behavior. Concentrating on (12), we change variables to $z = x/\sqrt{t}$ and let $t_1 \sim t$ for large values of t and get

$$(2\pi t)^{-1/2} \sum_{z \in t^{-1/2} \mathcal{P} \cap [-t^{1/6}, t^{1/6}]} \exp(-(yt^{-1/2} - z)^2/2).$$

We want to show that as $t \rightarrow \infty$ this converges almost surely to the integral of the normal density which is 1. To do this we replace the integrand $f_t(y) = \exp(-(yt^{-1/2} - z)^2/2)$ by

$$g_{t,\delta} = \inf_{y \in [k\delta, (k+1)\delta]} f_t(y) \quad \text{on } [k\delta, (k+1)\delta].$$

Let $g_\delta(y) = \lim_{t \rightarrow \infty} g_{t,\delta}(y)$. Using the strong law of large numbers for the Poisson process (see [Dur99, Theorem 3.1])

$$(2\pi t)^{-1/2} \sum_{z \in t^{-1/2} \mathcal{P} \cap [-t^{1/6}, t^{1/6}]} g_{t,\delta}(z) \rightarrow \int_{-\infty}^{\infty} g_\delta(y) dy \quad \text{a.s.}$$

Letting $\delta = 2^{-n} \rightarrow 0$,

$$\int_{-\infty}^{\infty} g_\delta(y) dy \rightarrow \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy = 1$$

almost surely. So we have

$$\liminf_{t \rightarrow \infty} (2\pi t)^{-1/2} \sum_{z \in t^{-1/2} \mathcal{P} \cap [-t^{1/6}, t^{1/6}]} \exp(-(yt^{-1/2} - z)^2/2) \leq 1$$

almost surely. The result for the limsup is similar. We now have that $W(y) \rightarrow 1$ which means that

$$\mathbf{E}[|A_t \cap [a, b]|] = \int_a^b W(y) dy \rightarrow \int_a^b 1 dy = b - a,$$

as desired.

At this point we have shown expected number of frogs in $[a, b]$ at time t , given the initial configuration, converges to $b - a$ almost surely. Since each frog has a small probability of ending up in the interval, a standard Poisson approximation result implies, see e.g., [Dur10,

Theorem 3.6.6], that for almost every initial configuration the number of frogs in $[a, b]$ converges in distribution to a $\text{Poisson}(b - a)$ random variable.

In formula (10) write $N_t(a, b)$ for the number of awake frogs in $[a, b]$. To prove that the number of frogs in disjoint intervals are independent, consider two disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$. If we condition on the number of frogs that end up in the first interval then this removes a small number of frogs from the overall population, and slightly increases the probability the others (which are independent of each other) end up in the second interval. That is

$$\mathbf{E}[N_t(a_2, b_2) \mid N_t(a_1, b_1) = j] = \int_a^b \sum_{x \in \mathcal{P} \cap [-t^{2/3}, t^{2/3}] \setminus K} p_{t-s(x)}(x, y) dy$$

where the sum no longer includes K , the random set of initial locations of the j frogs that are in $[a_1, b_1]$ at time t . Repeating the previous proof we conclude that conditional on $N_t(a_1, b_1) = j$, $N_t(a_2, b_2)$ converges to a Poisson with mean $b_2 - a_2$. The proof extends easily to n intervals. \square

6. SPEED AT CRITICALITY WHEN $d = 2$

We start by showing the rightmost active site moves at rate at least $t^{2-\epsilon}$. After this, we make an observation about the limiting behavior of γ_r for all dimensions $d \geq 2$. Combining this observation with superlinear expansion in $d = 2$ we obtain Corollary 4.

6.1. Proof of superlinear expansion. We restrict our discussion to the case $d = 2$. A *sponge crossing* of an $\ell \times m$ rectangle is a sequence of points x_0, \dots, x_k in the Poisson process so that x_0 is within distance r of the left side, x_k is within distance r of the right side, and for $1 \leq j \leq k$ we have $d(x_{j-1}, x_j) \leq r$. Let $p(n, 3n)$ be the probability that there is a crossing of $[0, n] \times [0, 3n]$. When $d = 2$ this probability is bounded away from zero.

Lemma 17. *When $r = r_2$, then there is a $\delta > 0$ so that $p(n, 3n) \geq \delta > 0$ for all n .*

Proof. [Roy90, Lemma 3.2] shows that if $p(n, 3n)$ vanishes then the size of the cluster containing the origin has an exponential tail. This cannot be the case at $r = r_2$ since the expected size of the cluster at the origin is infinite (see [MR96]). \square

We can use this to estimate the probability of a crossing from a given ball.

Lemma 18. *The probability q_{n-r} that a ball centered at $\mathbf{x} = (x_1, x_2)$ is the first point of a sponge crossing of $[x_1, x_1 + n - r] \times \mathbb{R}$ is at least cn^{-1} for some $c > 0$.*

Proof. There are constants $0 < C, \tilde{C} < \infty$ so that the probability there are more than Cn balls that touch the left side of the rectangle is smaller than $\exp(-\tilde{C}n)$ for large n . Since the probability of crossing $[x_1, x_1 + n - r] \times \mathbb{R}$ is greater than $p(n - r, 3(n - r))$, One of the balls that touches the left side has to be the left-most ball in the crossing so

$$p(n, 3n) \leq Cnq_{n-r}.$$

Here we are going a distance of $n - r$ because a ball with center in the width r rectangle has a harder time being the start of crossing the strip of width n than does a ball with its center in the left wall does of crossing a strip of width $(n - r)$. \square

Now we are ready to prove that the time constant $\tilde{\gamma}_{r_2}$ is zero.

Proof of Theorem 3. Set $m = R_t$ and suppose that the awake frog is at $\mathbf{x} = (m, x_2)$. Pick $b < 1 < a < 2b$. We will call a “success” the event that the set $F_t = \mathbb{B}(\mathbf{x}, m^b) \cap \{\mathbf{y}: y_1 \geq x_1\}$ contains a point $\mathbf{y} \in \mathcal{P}$ which is the left-most point in a chain of balls whose end reaches the line $x = m + m^a$. The probability of such a chain occurring at a $\mathbf{y} \in \mathcal{P} \cap F_t$ is bounded below by the probability of crossing the box $[y_1, y_1 + m^a] \times \mathbb{R}$. By Lemma 18 this is at least cm^{-a} . The volume in which we are searching is $|F_t| = \Omega(m^{2b})$, so the expected number of successes, S , is given by $\mathbf{E}S \geq Cm^{2b-a}$.

Two successes must have disjoint paths, for if they overlapped one would fail to be the left-most point of its chain. Thus, the van den Berg Kesten inequality (see [MR96, Theorem 2.3]) implies that successes are negatively correlated and the variance is smaller than that for independent random variables. Since we are dealing with Bernoulli random variables, the variance is smaller than the mean. Using Chebyshev’s inequality, it follows that

$$\mathbf{P}[S = 0] \leq \mathbf{P}[|S - \mathbf{E}S| \leq ES] \leq \frac{\mathbf{E}S}{(\mathbf{E}S)^2} \leq Cm^{a-2b}.$$

Let $M_1 = m$. After a long path is activated we will have an awake frog with first coordinate $M_2 \geq M_1^a$. By the shape theorem this will take time $O(M_1^b)$ and if we let $0 < \delta < 2b - a$ the failure probability is $O(m^{-\delta})$ for large m . After second step of our construction the first coordinate of the awake frog will be at $M_3 \geq M_2^a \geq (M_1^a)^a = M_1^{a^2}$. The time taken will be $M_2^b = o(M_2^a)$ and the error probability will be $O(M_2^{-\delta}) = O(M_1^{-a\delta})$. A union bound ensures that the probability of some failure in the chain is

$$O\left(\sum_{k=0}^{\infty} M_1^{-a^k \delta}\right).$$

If M_1 is large this probability can be bounded by an arbitrarily small $\epsilon > 0$. So Theorem 1 implies that with high probability the frog model crosses the hyperplane $\{\mathbf{x}: x_1 \geq M_1^a\}$ in a time that is $O(M_1^b)$. This continues for the larger M_i and thus the frog model is expanding at rate $t^{a/b}$. As $0 < b < 1 < a < 2b$ the exponent can be made arbitrarily close to 2. \square

6.2. Proof of Corollary 4. As r approaches the critical percolation value, the number of frogs that wake up in each connected component is getting large. From this, one might expect that the speed of the expansion is going to infinity. We make this intuition precise in dimension 2 with a statement about $\tilde{\gamma}_r$, the time constant from Proposition 5.

Lemma 19. *For all $d \geq 2$ it holds that if $\tilde{\gamma}_{r_d} = 0$, then $\lim_{r \rightarrow r_d} \tilde{\gamma}_r = 0$.*

Proof. $f_k(r) = \mathbf{E}T(x_0, x_k)/k$ is continuous so $g_k(r) = \min_{1 \leq j \leq k} f_j(r)$ is continuous, and hence $\tilde{\gamma}_r = \lim_k g_k(r)$ is upper-semicontinuous (a decreasing limit of continuous functions). Being upper-semicontinuous

$$\limsup_{r \rightarrow r_d} \tilde{\gamma}_r \leq \tilde{\gamma}_{r_d} = 0$$

which proves the desired result. \square

Proof of Corollary 4. This follows from Lemma 19 and Theorem 3. Indeed, let $x_n = (x_n(1), x_n(2))$ be the point from \mathcal{P} nearest to the ne_1 , and set $H(0, n) = \inf\{t: R_t \geq x_n(1)\}$ to be the first time R_t exceeds the first-coordinate of the n th point. For $r < r_c$ Theorem 1 applies and we have a ball as the limiting shape. It follows from convexity of the ball that the hitting time of the half plane converges to the hitting points along the x -axis:

$$\lim_{n \rightarrow \infty} \frac{H(0, n)}{n} = \lim_{n \rightarrow \infty} \frac{T(0, n)}{n} = \tilde{\gamma}_r.$$

By Lemma 19 these limits must also agree at $r = r_c$. Theorem 3 implies that this limit is 0. \square

APPENDIX A. BROWNIAN SURGERY OF A POISSON POINT PROCESS

We would like to show that we obtain a Poisson point process when we run B_t until it hits a point in \mathcal{P} , delete this point, and replace the Brownian sausage carved out with a new Poisson point process. To this end, let $\{G_t\}_{t>0}$ be a family of open sets in \mathbb{R}^d such that

$$G_0 = \{0\}, \quad G_s \subset G_t, \quad \text{for } s < t$$

and having Lebesgue measure

$$|G_t| = t, \quad \forall t > 0$$

and such that $|\partial G_t| = 0$ for all t , where ∂G_t is the boundary of G_t . Let μ be a Poisson random measure on \mathbb{R}^d with uniform unit intensity. Let ν be an independent Poisson random measure on \mathbb{R}^d , also with uniform unit intensity. Define a random time

$$\tau = \inf\{t \geq 0 \mid \mu(\overline{G_t}) \geq 1\},$$

which is the first time that $\overline{G_t}$ contains one of the poisson points in the μ process.

Then τ is exponentially distributed:

$$\mathbf{P}(\tau > s) = \mathbf{P}(\mu(\overline{G_s}) = 0) = e^{-|G_s|} = e^{-s}.$$

Because the boundary has measure zero, with probability one, we have only one point in $\overline{G_\tau}$:

$$\mu(\overline{G_\tau}) = 1$$

and it lies on the boundary ∂G_τ . Then define a new point process by the measure η , as follows:

$$\eta(A) = \mu(A \cap (\overline{G_\tau})^C) + \nu(A \cap G_\tau)$$

Thus, we use points of the ν process inside G_τ , and those of the μ process in $(\overline{G_\tau})^C$. Notice, the point on the boundary is in neither set G_τ nor $(\overline{G_\tau})^C$, so it has been excluded in this construction.

Lemma 20. *η is a Poisson random measure with unit intensity.*

Proof. Conditioning on τ (which is exponentially distributed), we compute

$$(14) \quad \mathbf{P}(\eta(A) = k) = \int_0^\infty \mathbf{P}(\eta(A) = k \mid \tau = s) e^{-s} ds$$

Conditioned on $\tau = s$, $\eta(A)$ is Poisson with rate $|A|$, since it is the sum of *independent* Poisson random variables

$$\mu(A \cap (\overline{G_s})^C) + \nu(A \cap G_s).$$

having rates $|A \cap (\overline{G_s})^C|$ and $|A \cap G_s|$, respectively. Therefore,

$$(15) \quad \mathbf{P}(\eta(A) = k) = e^{-|A|} \frac{|A|^k}{k!} \int_0^\infty e^{-s} ds = e^{-|A|} \frac{|A|^k}{k!}.$$

So, $\eta(A)$ is Poisson with rate $|A|$. We now use characteristic functions to show independence.

Suppose A and B are two disjoint measurable sets in \mathbb{R}^d ; we now show that $\eta(A)$ and $\eta(B)$ are independent. For any $\theta, \delta \in \mathbb{R}$, we have

$$\begin{aligned} \mathbf{E}[e^{i\theta\eta(A)} e^{i\delta\eta(B)}] &= \int_0^\infty \mathbf{E}[e^{i\theta\eta(A)} e^{i\delta\eta(B)} \mid \tau = s] e^{-s} ds \\ (16) \qquad &= \int_0^\infty \mathbf{E}[e^{i\theta(\mu(A \cap (\bar{G}_s)^C) + \nu(A \cap G_s))} e^{i\delta(\mu(B \cap (\bar{G}_s)^C) + \nu(B \cap G_s))} \mid \tau = s] e^{-s} ds \end{aligned}$$

The terms in the expectation are independent since the respective sets are all disjoint:

$$\begin{aligned} &\mathbf{E}[e^{i\theta(\mu(A \cap (\bar{G}_s)^C) + \nu(A \cap G_s))} e^{i\delta(\mu(B \cap (\bar{G}_s)^C) + \nu(B \cap G_s))} \mid \tau = s] \\ &= \mathbf{E}[e^{i\theta\mu(A \cap (\bar{G}_s)^C)}] \mathbf{E}[e^{i\theta\nu(A \cap G_s)}] \mathbf{E}[e^{i\delta\mu(B \cap (\bar{G}_s)^C)}] \mathbf{E}[e^{i\delta\nu(B \cap G_s)}] \\ &= e^{|A \cap (\bar{G}_s)^C|(e^\theta - 1)} e^{|A \cap G_s|(e^\theta - 1)} e^{|B \cap (\bar{G}_s)^C|(e^\delta - 1)} e^{|B \cap G_s|(e^\delta - 1)} \\ (17) \qquad &= e^{|A|(e^\theta - 1)} e^{|B|(e^\delta - 1)} \end{aligned}$$

which is independent of s . The last expression is the product of the characteristic functions of two Poissons. This shows that

$$\mathbf{E}[e^{i\theta\eta(A)} e^{i\delta\eta(B)}] = e^{|A|(e^\theta - 1)} e^{|B|(e^\delta - 1)} = \mathbf{E}[e^{i\theta\eta(A)}] \mathbf{E}[e^{i\delta\eta(B)}]$$

This implies that $\eta(A)$ and $\eta(B)$ are independent. □

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