Partial identification and testable restrictions in multi-unit auctions

David McAdams *

Duke Fuqua School of Business and Economics Department, 1 Towerview Road, Durham, NC 27708, United States

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ABSTRACT

Bidders' values in discriminatory and uniform-price auctions are not necessarily point-identified under the assumptions of equilibrium bidding and independent private values, but meaningful policy analysis can proceed from bounds on bidder values. This paper provides upper and lower bounds on the set of values that can rationalize a given distribution of bids, under the additional (and standard) assumption of non-increasing marginal values. Novel testable implications of the best response hypothesis are also provided, again under the assumption of non-increasing marginal values.

1. Introduction

In a multi-unit auction, multiple identical objects are bought, sold, or traded. Applications are wide-ranging and well-studied, e.g. sales auctions of Treasury bonds, quota rights, and IPOs; procurement auctions of electricity; and double auction trading in stock markets and emission permit markets.

Unfortunately, the theory of multi-unit auctions still lags far behind the theory of single-object auctions, with implications for empirical research. The theory of single-object auctions shows that the distribution of bidder values is point-identified from the distribution of bids given independent private values (IPV). Consequently, an econometrician who can confidently estimate the distribution of bids in a given auction can confidently infer the distribution of bidder values in that auction, under the joint hypothesis of equilibrium bidding and IPV. On the other hand, in multi-unit auctions, such as discriminatory and uniform-price auctions, theory provides very little guidance on how to interpret data generated in equilibrium given IPV.

This paper seeks to fill this hole in the literature. First, I show that the distribution of bidder values is *not* necessarily point-identified from the distribution of bids under the assumptions of equilibrium bidding, and asymmetric independent private values. However, if one is willing to impose the additional assumption that bidders have non-increasing marginal values (NIMV), I provide upper and lower bounds on the distribution of bidder values that can be inferred from the distribution of bids in both discriminatory and uniform-price auctions. NIMV is a very natural assumption in some settings. For instance, Chapman et al. (2006) (CMP) argue that banks who bid in the Bank of Canada's short-term cash reserve auctions exhibit NIMV, since any cash acquired through the auction will be put to its highest-value use, and since there are no fixed costs associated with deploying cash acquired through the auction.

The second main contribution of the paper is to provide testable implications of the hypothesis that bidder i's strategy is a best response to the distribution of others' bids, under the assumptions that bidders have IPV, and bidder i has NIMV. (Each bidder's strategy can be tested separately, regardless of whether other players' strategies are best responses.) These testable implications leverage the fact that more than one unit is sold and have no analogue in single-object auction theory.

The introduction continues with a discussion of some related literature. The bulk of the paper then focuses on the discriminatory auction: Section 2 presents a model of the discriminatory auction; Section 3 develops partial identification results; and Section 4 provides a novel set of testable restrictions of the best response hypothesis. Section 5 then extends these results to the uniform-price auction. Section 6 provides an example in which this paper's identification results are used to conduct a hypothetical policy counter-factual in the uniform-price auction. (For a policy counter-factual analysis in the discriminatory auction using this paper's results, see Hortacsu and McAdams (2008).) Section 7 concludes. Proofs of more technical results are in the Appendix.

Related literature. This paper builds upon the extensive literature on non-parametric identification in single-object auctions, especially Guerre et al. (2000) (GPV). (See Athey and Haile (in press) for a comprehensive survey.) GPV pioneered the non-parametric approach of using the first-order conditions of optimal bidding to identify the distribution of bidder values from the distribution of equilibrium bids. In this paper, I exploit such first-order conditions to bound bidders' values, and to generate testable restrictions. However, the problem of identification is more difficult in multi-unit auctions than in single-object auctions, since a "bid" in an auction of S identical units is an S-dimensional demand schedule, while a "value" is an S-dimensional marginal value schedule.

* Tel.: +1 617 253 1306.
E-mail address: david.mcadams@duke.edu.

Furthermore, there may be many different marginal value schedules that can rationalize a given bid as a best response. This creates a challenge for identification since, even absent estimation error, an econometrician cannot infer the true distribution of bidder values from the assumption of equilibrium play. My response to this challenge is to focus on the more modest goal of bounding bidder values. This paper therefore relates to the growing literature on partially identified (or "set-identified") models of games. See e.g. Manski (1995) and, in the context of auctions, Haile and Tamer (2003).

The methodology developed here has been applied recently in Chapman et al. (2007) and Hortacsu and McAdams (2008). Also closely related are Hortacsu (2001) and Kastl (2005), who take a non-parametric approach to bound bidder values in discriminatory and uniform-price auctions, respectively.[1] However, neither Hortacsu nor Kastl exploit all of the restrictions imposed on values by the first-order necessary conditions of optimal bidding. Consequently, their bounds are looser than those developed here, weakening any policy conclusions derived from them.

This paper also provides a novel set of testable restrictions of the best response hypothesis in multi-unit auctions. In related work, Kastl (2005) explores the fact that some observed bids in uniform-price auctions cannot be rationalized as best responses. He proposes a model in which bidders incur what I shall call a "complexity cost" when submitting (step-function) bids having more steps, and finds that bidding behavior in his application appears consistent with small, non-zero complexity costs.[2] This paper is complementary to Kastl (2005), since it provides a way to test his model of equilibrium bidding with complexity costs, as well as the standard model of equilibrium bidding without such costs (see Section 4).

2. Model: Discriminatory auction

S identical indivisible objects ("units") are sold to N risk-neutral bidders.[3]

Information. Each bidder i's marginal value schedule (or "value") takes the form $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,S}) \in \mathbf{R}^S$. ($\tilde{\mathbf{v}}_i$ denotes bidder i's value as a random variable, whereas notation $\mathbf{v}_i$ refers to a typical realization of his value. Similar notation is also used for bids, etc.)

Assumption 1. Bidders have independent private values (IPV), where the distribution of $(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_n)$ is common knowledge among the bidders, but unknown to the econometrician.

Auction rules. *Permissible bids.* Each bidder submits a bid $\mathbf{b}_i = (b_{i,1}, \ldots, b_{i,S}) \in \mathbf{R}^S_+$, such that $b_{i,q} \geq b_{i,q'}$ for all $q < q'$. Let $\mathcal{B}$ be the set of all such permissible bids. As shorthand, let $\mathbf{b}_I$ denote a profile of bids made by bidders in the set $I \subset \{1, \ldots, N\}$, and let b^x_I be the xth highest unit-bid made by any bidder in I, with shorthand $b^x_{\{1,\ldots,N\}} \equiv b^x$ for the xth highest unit-bid overall. *Allocation rule.* Quantity is allocated by market-clearing, i.e. the highest S unit-bids win. Let $\tilde{s}_{i,q} \equiv \tilde{b}^{S-q+1}_{-i}$ denote bidder i's residual inverse supply, the competing unit-bid that bidder i views as random and must beat to win a qth unit of the good. Assumption 2 rules out the possibility of ties, so that $b_{i,q}$ is one of the highest S unit-bids iff $b_{i,q} > s_{i,q}$.[4] *Payment rule.* In the discriminatory auction, bidder i pays the sum of his unit-bids on what he wins.

Definition 1 (*Winning Probability*). $w_{i,q}(x) = \Pr(x > \tilde{s}_{i,q})$ is the probability that bidder i wins at least quantity q when bidding price x for a qth unit.

Assumption 2. For each bidder i and each $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$ in the support of his bidding strategy, his winning probability $w_{i,q}(\cdot)$ is continuously differentiable at $b_{i,q}$ for all q.

Payoffs. Bidder i's interim expected payoff in the discriminatory auction takes the form $\Pi_i(\mathbf{b}_i, \mathbf{v}_i; \tilde{\mathbf{s}}_i) = \sum_{q=1}^{S} \Pi_{i,q}(b_{i,q}, v_{i,q}; \tilde{s}_{i,q})$, where each "unit-payoff" function

$$\Pi_{i,q}(b_{i,q}, v_{i,q}; \tilde{s}_{i,q}) = w_{i,q}(b_{i,q})(v_{i,q} - b_{i,q}) \tag{1}$$

depends on others' strategies only through the distribution of bidder i's residual inverse supply for quantity q. Assumption 2 implies that the following derivatives are well-defined and continuous in $b_{i,q}$:

$$\frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,q})}{\mathrm{d}b_{i,q}} = w'_{i,q}(b_{i,q})(v_{i,q} - b_{i,q}) - w_{i,q}(b_{i,q}) \tag{2}$$

$$\frac{\mathrm{d}^2\Pi_{i,q}(b_{i,q}, v_{i,q})}{\mathrm{d}b_{i,q}\mathrm{d}v_{i,q}} = w'_{i,q}(b_{i,q}) \geq 0. \tag{3}$$

Assumption 3. $w_{i,q}(b_{i,q}) > 0$ and $w'_{i,q}(b_{i,q}) > 0$ for all i, $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$, and q.

Assumption 3 combines two simplifying assumptions. First, bidder i has some chance of winning all S units, i.e. $w_{i,q}(b_{i,q}) > 0$ for all q.[5] Second, bidder i faces no "gaps" in the distribution of competing bids for any unit, i.e. $w'_{i,q}(b_{i,q}) > 0$ for all q. This assumption entails some significant loss of generality, as gaps can and do occur in practice. See the longer working-paper version of this paper, McAdams (2007), for an extension of the present analysis to settings with gaps. For present purposes, I will briefly discuss the possibility of gaps, and present an example showing why gaps can arise when a bidder plays a best response.

Gaps. Formally, say that bidder i faces a "gap" at price p for quantity q if $w'_{i,q}(p) = 0$ and $w_{i,q}(p) > 0$. Bidding in a gap is never a best response in the first-price auction (i.e. the discriminatory auction when $S = 1$) since a bidder can, on the margin, lower his payments without lowering his probability of winning by bidding slightly less. However, in discriminatory auctions of two or more units, a bidder's best response may well be to bid in a gap for some quantities. (Bidding in gaps arises even more naturally in uniform-price auctions, since some winning bids never set the price.)

[1] Kang and Puller (2008) follows Hortacsu (2001), but imposes an additional restriction that each bidder has the same marginal value for all quantities for which he bids the same price. For a parametric structural analysis, see Fevrier et al. (2001).

[2] Especially in uniform-price auctions, one would expect any best response to utilize the maximal number of steps permitted by the auction rules. However, this is rarely the case. In the uniform-price Czech Treasury auctions studied in Kastl (2005), bidders could make bids having up to ten steps, but no one ever used more than nine. Similarly, in the uniform-price electricity procurement auctions studied by Hortacsu and Puller (2008), "only one firm ever used the maximum number of steps, and that only occurred once for that firm". (Private communication with Steven Puller.)

[3] As in McAdams (2006), the analysis can be adapted to procurement auctions and double auctions by allowing for non-zero bidder endowments.

[4] The analysis can be easily adapted to settings in which ties occur with positive probability, as when there is a finite grid of prices.

[5] This assumption is innocuous. If bidder i wins up to quantity $\overline{Q}$ units with positive probability, but never wins $\overline{Q} + 1$ units, then the identifying and bounding inequalities developed here apply to all marginal values on quantities $1, \ldots, \overline{Q}$. For quantities greater than $\overline{Q}$, it is relatively straightforward to derive an upper bound on marginal values, based on the observation that bidder i must not have preferred to raise his bid enough to win such quantities with positive probability.

Example 1. Two bidders participate in a discriminatory auction of two units. Bidder 2's bid takes the form $\tilde{\mathbf{b}}_2 = (2x, x)$, where $x \sim U[0, 1]$. Thus, bidder 1's residual supply has the property that $\tilde{s}_{1,1} \sim U[0, 1]$ while $\tilde{s}_{1,2} \sim U[0, 2]$. To win a first unit for certain, bidder 1 only needs to bid one on that unit. To have more than a 50% chance of winning a second unit, however, bidder 1 must bid more than one on *both* units. Consequently, when bidder 1's marginal value for the second unit is high enough, his best response will be to bid more than one on the first unit. For example, when bidder 1 has value $\mathbf{v}_1 = (5, 5)$, his (unique) best response is to bid (1.5, 1.5), in which case his first unit-bid is in a gap. □

3. Partial identification in discriminatory auctions

How much can be inferred about the unobserved distribution of bidder values $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_N)$ from a given distribution of bids $\tilde{\mathbf{b}} = (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_N)$ in the discriminatory auction, given that those bids arise in equilibrium? To focus on this identification question, this section abstracts from issues associated with estimating the distribution of $\tilde{\mathbf{b}}$.

Definition 2. The distribution of values $\tilde{\mathbf{v}}$ *rationalizes* a given distribution of bids $\tilde{\mathbf{b}}$ if, when values are distributed as $\tilde{\mathbf{v}}$, there exists a Bayesian Nash equilibrium profile of (possibly mixed) strategies $\sigma = (\sigma_1, \ldots, \sigma_N)$ such that $\sigma(\tilde{\mathbf{v}})$ is distributed as $\tilde{\mathbf{b}}$.

For each bidder i and permissible bid $\mathbf{b}_i \in \mathcal{B}$, define

$$V_i^{BR}(\mathbf{b}_i) = \left\{\mathbf{v}_i : \Pi_i(\mathbf{b}_i, \mathbf{v}_i) \geq \Pi_i(\hat{\mathbf{b}}_i, \mathbf{v}_i) \text{ for all } \hat{\mathbf{b}}_i \in \mathcal{B}\right\}. \quad (4)$$

$V_i^{BR}(\mathbf{b}_i)$ is the set of values given which bid $\mathbf{b}_i$ is a best response. A given distribution of bids $\tilde{\mathbf{b}}$ can be rationalized by some distribution of values iff $V_i^{BR}(\mathbf{b}_i) \neq \emptyset$ for all i and all $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$. Indeed, when this is the case, let $\psi = (\psi_1, \ldots, \psi_N)$ be any function mapping bids into the set of values given which those bids are a best response, i.e. $\psi_i(\mathbf{b}_i) \in V_i^{BR}(\mathbf{b}_i)$. Then $\tilde{\mathbf{v}} = (\psi_1(\tilde{\mathbf{b}}_1), \ldots, \psi_N(\tilde{\mathbf{b}}_N))$ rationalizes the observed distribution of bids $\tilde{\mathbf{b}}$.[6] Finally, if there is more than one distribution of values that rationalizes a given distribution of bids, any convex combination of those distributions also rationalizes it. (Fact 1 follows immediately from the definition of the sets $V_i^{BR}(\mathbf{b}_i)$.)

Fact 1. The distribution of values $\tilde{\mathbf{v}}$ rationalizes the distribution of bids $\tilde{\mathbf{b}}$ iff $\tilde{\mathbf{v}} \in Co(\{\psi(\tilde{\mathbf{b}}) : \psi_i(\mathbf{b}) \in V_i^{BR}(\mathbf{b}_i)$ for all i and $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)\})$, where $Co(X)$ denotes the convex hull of X.

Characterizing each set $V_i^{BR}(\mathbf{b}_i)$ can be difficult in practice. For each possible $\mathbf{v}_i \in \mathbf{R}^S$, one must solve a challenging multi-dimensional constrained optimization problem, to determine whether $\mathbf{b}_i$ is a best response given value $\mathbf{v}_i$. To make further progress, consider the *larger* set of values that satisfy all "first-order conditions" associated with bid $\mathbf{b}_i$:

$$V_i^{\text{FOC}}(\mathbf{b}_i) = \left\{\mathbf{v}_i : \limsup_{\alpha \to 0} \frac{\Pi_i(\alpha\hat{\mathbf{b}}_i + (1-\alpha)\mathbf{b}_i, \mathbf{v}_i) - \Pi_i(\mathbf{b}_i, \mathbf{v}_i)}{\alpha|\hat{\mathbf{b}}_i - \mathbf{b}_i|} \leq 0 \quad \text{for all } \hat{\mathbf{b}}_i \in \mathcal{B}\right\}. \quad (5)$$

These first-order conditions impose a set of *necessary conditions* that any distribution of values must satisfy, to rationalize a given distribution of bids.

Fact 2. The distribution of values $\tilde{\mathbf{v}}$ rationalizes the distribution of bids $\tilde{\mathbf{b}}$ only if $\tilde{\mathbf{v}} \in Co\{\psi(\tilde{\mathbf{b}}) : \psi_i(\mathbf{b}) \in V_i^{\text{FOC}}(\mathbf{b}_i)$ for all i and $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)\}$.

3.1. *First-order conditions of optimal bidding*

Fix a bidder i and fix a bid $\mathbf{b}_i$ in the support of his equilibrium distribution of bids $\tilde{\mathbf{b}}_i$. This section characterizes $V_i^{\text{FOC}}(\mathbf{b}_i)$. First, a useful definition.

Definition 3 (*Indifference Levels*). For every $X \subset \{1, \ldots, S\}$, let $v_{i,X}^*(\mathbf{b}_i)$ be the "indifference level of marginal values" such that bidder i is indifferent on the margin between raising or lowering his unit-bids on all quantities in X when his marginal value equals $v_{i,X}^*(\mathbf{b}_i)$ on all quantities in X, implicitly defined by

$$\sum_{q \in X} \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,X}^*(\mathbf{b}_i))}{\mathrm{d}b_{i,q}} = 0. \quad (6)$$

Note from (2) that

$$v_{i,q}^*(\mathbf{b}_i) = b_{i,q} + \frac{w_{i,q}(b_{i,q})}{w_{i,q}'(b_{i,q})} \quad (7)$$

for all individual quantities $q = 1, \ldots, S$.

Lemma 1. *For every bid* $(v_{i,1}^*(\mathbf{b}_i), \ldots, v_{i,S}^*(\mathbf{b}_i)) \in V_i^{\text{FOC}}(\mathbf{b}_i)$.

Proof. When $v_{i,q} = v_{i,q}^*(\mathbf{b}_i)$, bidder i is indifferent on the margin between raising or lowering his qth unit-bid. Consequently, when $v_{i,q} = v_{i,q}^*(\mathbf{b}_i)$ for all q, bidder i is indifferent on the margin to *all* local deviations, so that all of the first-order conditions of optimal bidding are trivially satisfied. □

When just one unit is sold in the first-price auction, $v_{i,1}^*(\mathbf{b}_i)$ is the only marginal value given which bidder i does not strictly prefer on the margin to raise or lower his bid. Thus, the first-price auction with independent private values is point-identified by the first-order conditions of optimal bidding (Guerre et al., 2000). Similarly, in the discriminatory auction, $(v_{i,1}^*(\mathbf{b}_i), \ldots, v_{i,S}^*(\mathbf{b}_i))$ is the only marginal value schedule that can rationalize bidder i's bid schedule $\mathbf{b}_i$ if that bid schedule is strictly decreasing in quantity. On the other hand, if bidder i has bid the same price for several units, many different marginal value schedules can potentially rationalize his bid as a best response.

Theorem 1. *Consider any bid* $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$. *Marginal value schedule* $\mathbf{v}_i \in V_i^{\text{FOC}}(\mathbf{b}_i)$ *iff there exists non-negative constants* $c_1, \ldots, c_{S-1} \geq 0$ *such that*

$$v_{i,1} = v_{i,1}^*(\mathbf{b}_i) - c_1 w_{i,2}'(b_{i,2}) \quad (8)$$

$$v_{i,q} = v_{i,q}^*(\mathbf{b}_i) - c_q w_{i,q+1}'(b_{i,q+1}) + c_{q-1} w_{i,q-1}'(b_{i,q-1}) \quad \text{for } q = 2, \ldots, S-1 \quad (9)$$

$$v_{i,S} = v_{i,S}^*(\mathbf{b}_i) + c_{S-1} w_{i,S-1}'(b_{i,S-1}) \quad (10)$$

and such that $c_q = 0$ *for all* q *satisfying* $b_{i,q} > b_{i,q+1}$.

Example 2. Two bidders with independent private values submit equilibrium bids in a discriminatory auction of two units. Bidder 2's bid takes the form $\tilde{\mathbf{b}}_2 = (x, x)$, where $x \sim U[0, 1]$. Thus, bidder 1's residual inverse supply for each quantity has the property that $\tilde{s}_{1,1} = \tilde{s}_{1,2} \sim U[0, 1]$. Since $w_{1,q}(b_{1,q}) = b_{1,q}$ for all $b_{1,q} \in [0, 1]$,

[6] If each ψ_i is one-to-one, then the equilibrium strategies that map this distribution of values into $\tilde{\mathbf{b}}$ are pure strategies taking the form $\sigma_i(\mathbf{v}_i) = \psi_i^{-1}(\mathbf{v}_i)$ for each i. More generally, when ψ_i^{-1} is a correspondence, each bidder i will mix among the elements of $\psi_i^{-1}(\mathbf{v}_i)$ given values $\mathbf{v}_i$.

his expected payoff takes the form $\Pi_1(\mathbf{b}_1, \mathbf{v}_1) = b_{1,1}(v_{1,1} - b_{1,1}) + b_{1,2}(v_{1,2} - b_{1,2})$.

Suppose that bidder 1 is observed submitting the bid $(1/2, 1/2)$. What marginal value schedules could rationalize this bid as a best response? Note from (7) that $v^*_{1,1}(\mathbf{b}_1) = 2b_{1,1}$ and $v^*_{1,2}(\mathbf{b}_1) = 2b_{1,2}$. Thus, by Theorem 1, $V_1^{\mathrm{FOC}}((1/2, 1/2)) = \{\mathbf{v}_1 : v_{1,1} = 1 - c_1$ and $v_{1,2} = 1 + c_1$ for some $c_1 \geq 0\}$. In particular, when bidder 1 submits the bid $(1/2, 1/2)$, there is no way to rule out the possibility that (i) he has constant marginal values ($v_{1,1} = v_{1,2} = 1$), (ii) he has no value for one unit without a second unit ($v_{1,1} = 0$ and $v_{1,2} = 2$), or (iii) he faces a fixed cost of consumption that exceeds the marginal consumption value of the first unit ($v_{1,1} < 0$ and $v_{1,2} = 2 - v_{1,1}$). □

3.2. Tight bounds under non-increasing marginal values

This section provides *tight* upper and lower bounds on marginal values in $V_i^{\mathrm{FOC}}(\mathbf{b}_i)$, if one is willing to assume that bidder i's marginal values are non-increasing in quantity.

Definition 4 (*NIMV*). Bidder i has non-increasing marginal values (NIMV) if $v_{i,q} \geq v_{i,q'}$ for all $q < q'$. Let $V^{\mathrm{NIMV}} \subset \mathbf{R}^S$ be the set of non-increasing marginal value schedules.

Definition 5 (*Steps*). For each bid-level b, let $Q(b) = \{q \in \{1, \ldots, S\} : b_{i,q} = b\}$ be the "step" of quantities that have been bid at that price in bid $\mathbf{b}_i$. (Notation for $\mathbf{b}_i$ is suppressed since the bid is fixed throughout the analysis.) Note that (i) $q \in Q(b_{i,q})$ for all q, (ii) $q < \max Q(b_{i,q})$ iff $b_{i,q} = b_{i,q+1}$, and (iii) $q > \min Q(b_{i,q})$ iff $b_{i,q} = b_{i,q-1}$. Let $\mathcal{Q}(\mathbf{b}_i)$ be the set of (non-empty) steps corresponding to bid $\mathbf{b}_i$.

Theorem 2. *Consider any bid* $\mathbf{b}_i \in \mathrm{supp}(\tilde{\mathbf{b}}_i)$. *For each quantity* q, *there exists bounds* $\underline{v}_{i,q}(\mathbf{b}_i), \overline{v}_{i,q}(\mathbf{b}_i)$ *having the following properties.* (a) $\mathbf{v}_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}}$ *implies* $v_{i,q} \in [\underline{v}_{i,q}(\mathbf{b}_i), \overline{v}_{i,q}(\mathbf{b}_i)]$. (b) *As long as* $V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}} \neq \emptyset$, *there exists* $\mathbf{v}'_i, \mathbf{v}''_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}}$, *such that* $v'_{i,q} = \overline{v}_{i,q}$ *and* $v''_{i,q} = \underline{v}_{i,q}$. *In particular, these bounds are*

$$\overline{v}_{i,q}(\mathbf{b}_i) = \min_{x=1,\ldots,q} \{v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i), \hat{v}_{i,x}(\mathbf{b}_i)\} \tag{11}$$

$$\underline{v}_{i,q}(\mathbf{b}_i) = \max_{x=q,\ldots,S} \{v^*_{i,[x,\max Q(b_{i,x})]}(\mathbf{b}_i), \check{v}_{i,x}(\mathbf{b}_i)\} \tag{12}$$

where $\hat{v}_{i,x}(\mathbf{b}_i)$ *and* $\check{v}_{i,x}(\mathbf{b}_i)$ *are defined implicitly by the equations in* Box I.

Discussion of Theorem 2(a). Consider whether bidder i has an incentive on the margin to deviate by slightly raising his price on all quantities in the range $[\min Q(b_{i,q}), q]$ ("upward deviation at q"). Since bidder i has non-increasing marginal values, his marginal value on all quantities in this range is bounded below by his marginal value on quantity q. In particular, if $v_{i,q} > v^*_{i,[\min Q(b_{i,q}),q]}(\mathbf{b}_i)$ for any quantity q, then $v_{i,x} > v^*_{i,[\min Q(b_{i,q}),q]}(\mathbf{b}_i)$ for all $x \in [\min Q(b_{i,q}), q]$. Consequently, by definition of the "indifference level" $v^*_{i,[\min Q(b_{i,q}),q]}(\mathbf{b}_i)$, bidder i must strictly prefer the upward deviation at q, a contradiction of the hypothesis that bid $\mathbf{b}_i$ is a best response. This explains why, for all $\mathbf{v}_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}}$, $v_{i,q}$ is bounded above by $v^*_{i,[\min Q(b_{i,q}),q]}(\mathbf{b}_i)$ for all q. By a similar logic, if $v_{i,q'} < v^*_{i,[q',\min Q(b_{i,q'})]}(\mathbf{b}_i)$, then bidder i must strictly prefer to deviate by slightly lowering his price on all quantities in the range $[q', \max Q(b_{i,q'})]$ ("downward deviation at q'"). See Fig. 1 for a stylized illustration of the upward deviation at q and downward deviation at q'. (In the figure, q, q' belong to the same step, so that $b_{i,q} = b_{i,q'}$.)

Next, observe that NIMV implies that $v_{i,q} \leq v_{i,x} \leq v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i)$ for all $x \in [\min Q(b_{i,x}), q)$. This explains why

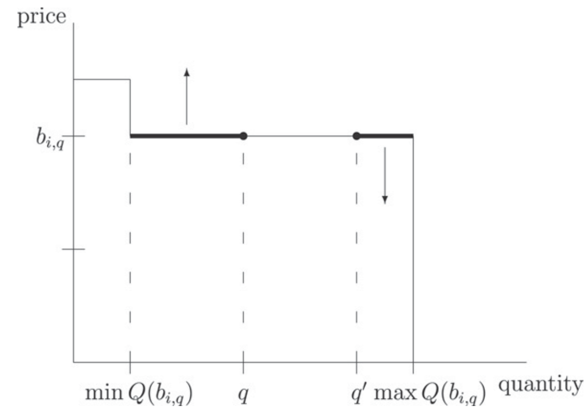

Fig. 1. Upward deviation at quantity q and downward deviation at quantity q'.

$v_{i,q}$ is bounded above by $\min_{x\leq q} v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i)$ for all q and, similarly, why $v_{i,q}$ is bounded below by $\max_{x\geq q} v^*_{i,[x,\max Q(b_{i,x})]}(\mathbf{b}_i)$ for all q.

Less easy to explain in words is the fact that $v_{i,q}$ is bounded above by $\min_{x\leq q} \hat{v}_{i,x}(\mathbf{b}_i)$ and bounded below by $\max_{x\geq q} \check{v}_{i,x}(\mathbf{b}_i)$ for all q. Intuitively, these conditions reflect a relationship between the least upper bound on bidder i's marginal value on quantity q and the greatest lower bound on his marginal value on quantity $q+1$. Namely, these marginal values are mutually constraining, because bidder i must not have any incentive to raise or lower his price on all quantities in the range $Q(b_{i,q}) \cup Q(b_{i,q+1})$. Lemma 2 highlights one consequence of this connection between the upper and lower bounds on bidder i's marginal values. Namely, Lemma 2 allows one to compute the lower bounds directly from the upper bounds, and vice versa.

Lemma 2. *Suppose that* $V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}} \neq \emptyset$. *Then, for all* q,

$$\sum_{x\in[\min Q(b_{i,q}),q]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, \overline{v}_{i,q}(\mathbf{b}_i))}{\mathrm{d}b_{i,x}} + \sum_{x\in[q+1,\max Q(b_{i,q+1})]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, \underline{v}_{i,q+1}(\mathbf{b}_i))}{\mathrm{d}b_{i,x}} = 0. \tag{13}$$

Fig. 2 illustrates some of the general features of the marginal value intervals $[\underline{v}_{i,q}(\mathbf{b}_i), \overline{v}_{i,q}(\mathbf{b}_i)]$. (The bid $\mathbf{b}_i$ is traced by filled dots; it has two steps, $Q(\$2) = \{1, 2, 3\}$ and $Q(\$1) = \{4, 5\}$.) Namely, (i) the upper and lower bounds are weakly decreasing in quantity and (ii) the lower bound on the smallest quantity in each step equals the upper bound on the largest quantity of that step.

Discussion of Theorem 2(b). Theorem 2's bounds on bidder i's marginal values are tight, in the following sense. Suppose that there exists *some* non-increasing marginal value schedule in $V_i^{\mathrm{FOC}}(\mathbf{b}_i)$. Then, for all q, there exists such a schedule in which bidder i's marginal value for the qth unit equals the lower bound $\underline{v}_{i,q}(\mathbf{b}_i)$ and another such schedule in which this marginal value equals the upper bound $\overline{v}_{i,q}(\mathbf{b}_i)$. (However, the marginal schedules $(\underline{v}_{i,1}(\mathbf{b}_i), \ldots, \underline{v}_{i,S}(\mathbf{b}_i))$ and $(\overline{v}_{i,1}(\mathbf{b}_i), \ldots, \overline{v}_{i,S}(\mathbf{b}_i))$ typically do *not* belong to $V_i^{\mathrm{FOC}}(\mathbf{b}_i)$.)

4. Testable restrictions in discriminatory auctions

In Section 3, I derived necessary conditions *on bidder values* for a given bid to be a best response with respect to a given distribution of others' bids, assuming non-increasing marginal values (NIMV). Here, I derive necessary conditions *on the distribution of others' bids* for a given bid to be a best response for *any* non-increasing

$$0 = \sum_{y\in[\min Q(b_{i,x}),x]} \frac{d\Pi_{i,y}(b_{i,y}, \hat{v}_{i,x}(\mathbf{b}_i))}{db_{i,y}} + \sum_{y\in[x+1,\max Q(b_{i,x+1})]} \frac{d\Pi_{i,y}\left(b_{i,y}, \max_{z=x+1,\ldots,\max Q(b_{i,x+1})}\left\{v^*_{i,[z,\max Q(b_{i,x+1})]}(\mathbf{b}_i)\right\}\right)}{db_{i,y}}$$

$$0 = \sum_{y\in[x,\max Q(b_{i,x})]} \frac{d\Pi_{i,y}(b_{i,y}, \check{v}_{i,x}(\mathbf{b}_i))}{db_{i,y}} + \sum_{y\in[\min Q(b_{i,x-1}),x-1]} \frac{d\Pi_{i,y}\left(b_{i,y}, \max_{z=\min Q(b_{i,y}),\ldots,x-1}\left\{v^*_{i,[\min Q(b_{i,x-1}),z]}(\mathbf{b}_i)\right\}\right)}{db_{i,y}}$$

Box I.

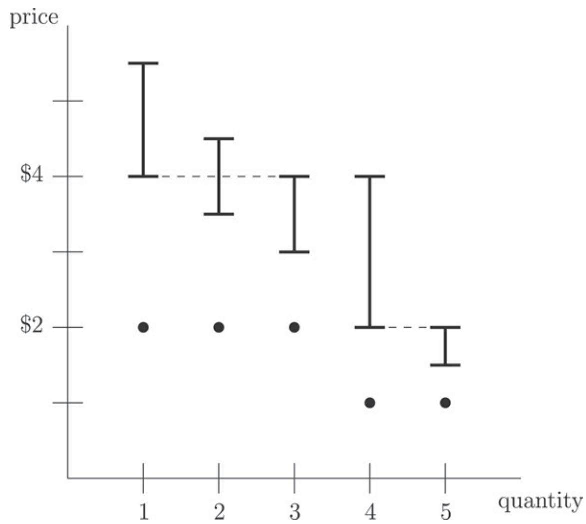

Fig. 2. Bounds on bidder values given NIMV (Theorem 2).

marginal value schedule. In particular, for each bidder i and each bid $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$, I provide a set of novel testable restrictions imposed by the joint hypothesis that (a) this bid was a best response and (b) bidder i had NIMV when he made this bid.[7] If one is willing to assume NIMV, these restrictions can be used to test the best response hypothesis on a bidder-by-bidder (and bid-by-bid) basis; for an example, see Chapman et al. (2006). On the other hand, if one is willing to assume equilibrium bidding, these restrictions can be used to test the non-increasing marginal values hypothesis; for an example, see Section 6.

Throughout this section, consider a fixed bidder i and a fixed bid $\mathbf{b}_i \in \text{supp}(\tilde{\mathbf{b}}_i)$. Before I proceed, some definitions are needed.

Definition 6 (*Upward Deviations*). For every q and $\Delta > 0$, define the "upward deviation at q by Δ", $\mathbf{b}_i^{\uparrow\Delta,q}$, as follows: $b_{i,x}^{\uparrow\Delta,q} = b_{i,x}$ for all $x < \min Q(b_{i,q})$ and all $x > q$; $b_{i,x}^{\uparrow\Delta,q} = b_{i,q} + \Delta$ for all $x \in [\min Q(b_{i,q}), q]$. This deviation is feasible as long as $\Delta \leq b_{i,\min Q(b_{i,q})-1} - b_{i,q}$.

Definition 7 (*Downward Deviations*). For every q and $\Delta > 0$, define the "downward deviation at q by Δ", $\mathbf{b}_i^{\downarrow\Delta,q}$, as follows: $b_{i,x}^{\downarrow\Delta,q} = b_{i,x}$ for all $x < q$ and all $x > \max Q(b_{i,q})$; $b_{i,x}^{\downarrow\Delta,q} = b_{i,q} - \Delta$ for all $x \in [q, \max Q(b_{i,q})]$. This deviation is feasible as long as $\Delta \leq b_{i,q} - b_{i,\max Q(b_{i,q})+1}$.

For the bid $\mathbf{b}_i$ to be a best response, all feasible deviations must be unprofitable. In particular, for every pair of quantities $q_1 \leq q_2$ and every small enough Δ, the upward deviation at q_1 by Δ and the downward deviation at q_2 by Δ must both be unprofitable.

By (1), the incremental profit from these deviations takes the form

$$\Pi_i(\mathbf{b}_i^{\uparrow\Delta,q_1}, \mathbf{v}_i) - \Pi_i(\mathbf{b}_i, \mathbf{v}_i) = \sum_{x\in[\min Q(b_{i,q_1}),q_1]} \Big((v_{i,x} - b_{i,x} - \Delta)(w_{i,x}(b_{i,x} + \Delta) - w_{i,x}(b_{i,x})) - \Delta w_{i,x}(b_{i,x})\Big) \quad (14)$$

$$\Pi_i(\mathbf{b}_i^{\downarrow\Delta,q_2}, \mathbf{v}_i) - \Pi_i(\mathbf{b}_i, \mathbf{v}_i) = \sum_{x\in[q_2,\max Q(b_{i,q_2})]} \Big(\Delta w_{i,x}(b_{i,x} - \Delta) - (v_{i,x} - b_{i,x}) \times (w_{i,x}(b_{i,x}) - w_{i,x}(b_{i,x} - \Delta))\Big). \quad (15)$$

Taking $\Delta \to 0$ yields necessary conditions for $\mathbf{b}_i$ to be a best response:

$$\sum_{x\in[\min Q(b_{i,q_1}),q_1]} (v_{i,x} - b_{i,x})w'_{i,x}(b_{i,x}) \leq \sum_{x\in[\min Q(b_{i,q_1}),q_1]} w_{i,x}(b_{i,x}) \quad (16)$$

$$\sum_{x\in[q_2,\max Q(b_{i,q_2})]} (v_{i,x} - b_{i,x})w'_{i,x}(b_{i,x}) \geq \sum_{x\in[q_2,\max Q(b_{i,q_2})]} w_{i,x}(b_{i,x}). \quad (17)$$

Note that, since bidder i has non-increasing marginal values (NIMV), $v_{i,x_1} \geq v_{i,q_1} \geq v_{i,q_2} \geq v_{i,x_2}$ for all $x_1 \leq q_1$ and $x_2 \geq q_2$. Also note that, by definition of the steps $Q(b_{i,q_1})$ and $Q(b_{i,q_2})$, $b_{i,x} = b_{i,q_1}$ for all $x \in Q(b_{i,q_1})$ and $b_{i,x} = b_{i,q_2}$ for all $x \in Q(b_{i,q_2})$. Thus, after re-arranging, (16) and (17) imply an inequality condition on the distribution of bids that does not depend on bidder values.

$$b_{i,q_1} + \frac{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} w_{i,x}(b_{i,q_1})}{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} w'_{i,x}(b_{i,q_1})} \geq v_{i,q_1} \geq v_{i,q_2} \geq b_{i,q_2} + \frac{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} w_{i,x}(b_{i,q_2})}{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} w'_{i,x}(b_{i,q_2})}. \quad (18)$$

Theorem 3 and its corollary are immediate consequences of (18).

Theorem 3. *Bid* $\mathbf{b}_i$ *can only be a best response for bidder* i *if the distribution of others' bids is such that, for every pair of quantities* $q_1 \leq q_2$,

$$b_{i,q_1} + \frac{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} w_{i,x}(b_{i,q_1})}{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} w'_{i,x}(b_{i,q_1})}$$

[7] Testable restrictions derived in the context of single-object auctions generalize to multi-unit auctions. For example, Chapman et al. (2006) show how the hazard-rate monotonicity restriction derived in Guerre et al. (2000) for first-price auctions extends to discriminatory auctions. This paper focuses on a new set of restrictions that have no analogue in single-object auctions.

$$\geq b_{i,q_2} + \frac{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} w_{i,x}(b_{i,q_2})}{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} w'_{i,x}(b_{i,q_2})}. \tag{19}$$

Corollary 1. *Bid $\mathbf{b}_i$ can only be a best response for bidder i if the distribution of others' bids is such that, for every pair of quantities $q_1 \leq q_2$ such that $b_{i,q_1-1} > b_{i,q_1} = b = b_{i,q_2} > b_{i,q_2+1}$,*

$$\frac{w_{i,q_1}(b)}{w'_{i,q_1}(b)} \geq \frac{w_{i,q_2}(b)}{w'_{i,q_2}(b)}. \tag{20}$$

Discussion: Adapting Theorem 3 to settings with "complexity cost". How should we interpret the failure of one or more of the conditions (19)? One idea, due to Kastl (2005), is to consider an alternative model in which bidders incur some additional cost when submitting a bid having more steps. We can test such a model using this paper's approach, by restricting attention to feasible deviations that do not increase the number of steps. In particular, consider any $q_1 \leq q_2$ such that $1 \notin Q(b_{i,q_1})$ and $S \notin Q(b_{i,q_2})$.[8] The upward deviation at q_1 by $\Delta_1 = b_{i,\min Q(b_{i,q_1})-1} - b_{i,q_1}$ is such a deviation, since it transfers quantities $[\min Q(b_{i,q_1}), q_1]$ from one step to another without creating a new step. Similarly, the downward deviation at q_2 by $\Delta_2 = b_{i,q_2} - b_{i,\max Q(b_{i,q_2})+1}$ does not create a new step. As in the derivation of (18), the fact that these two deviations must be unprofitable, combined with the assumption that bidders have non-increasing marginal values, implies the following inequality restriction on the distribution of bids:

$$b_{i,q_1} + \Delta_1 + \frac{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} w_{i,x}(b_{i,q_1})}{\sum_{x\in[\min Q(b_{i,q_1}),q_1]} \frac{w_{i,x}(b_{i,q_1}+\Delta_1)-w_{i,x}(b_{i,q_1})}{\Delta_1}}$$
$$\geq b_{i,q_2} + \frac{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} w_{i,x}(b_{i,q_2}-\Delta_2)}{\sum_{x\in[q_2,\max Q(b_{i,q_2})]} \frac{w_{i,x}(b_{i,q_2})-w_{i,x}(b_{i,q_2}-\Delta_2)}{\Delta_2}}. \tag{21}$$

Discussion: Implementing a test based on a single restriction of the form (20). Suppose that an econometrician does not know the true distribution of bids but observes a sample of M discriminatory auctions, in which the same bidders have independent private values drawn from the same distribution and play the same equilibrium strategies in each auction.[9] (The econometrician observes all bids in each auction, with bidder identities.)

The hazard rates in (20) can be difficult to estimate in practice, so consider the discretized version of this restriction (derived in the same way as (18)):

$$\Delta + \frac{w_{i,q_1}(b)}{\frac{w_{i,q_1}(b+\Delta)-w_{i,q_1}(b)}{\Delta}} \geq \frac{w_{i,q_2}(b-\Delta)}{\frac{w_{i,q_2}(b)-w_{i,q_2}(b-\Delta)}{\Delta}}. \tag{22}$$

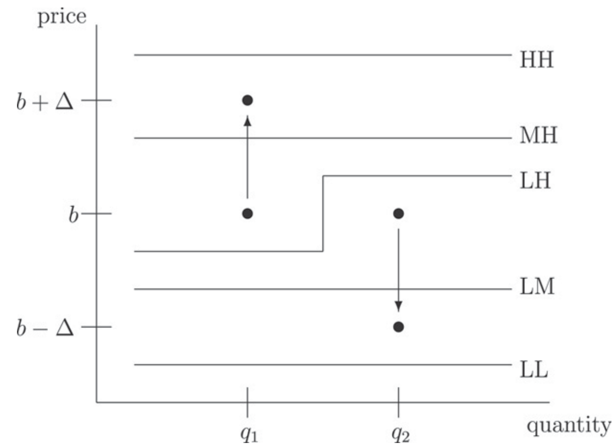

Fig. 3. Five relevant possibilities for bidder i's residual supply.

Testing this inequality requires that one estimates the probabilities of four events, corresponding to whether bidder i's residual inverse supply for quantity q_1 is less than b or in $(b, b+\Delta)$ and whether his residual inverse supply for quantity q_2 is less than $b-\Delta$ or in $(b-\Delta, b)$. Complicating matters is that these events are not disjoint, e.g. $s_{i,q_2} < b-\Delta$ implies that $s_{i,q_1} < b$, since residual inverse supply is non-decreasing.

Consider the following partition of the space of all possible realized residual inverse supply curves $\mathbf{s}_i$ into five disjoint sets: (i) $\mathbf{s}_i \in$ "*HH*" if $s_{i,q_1} > b+\Delta$ and $s_{i,q_2} > b$, (ii) $\mathbf{s}_i \in$ "*MH*" if $s_{i,q_1} \in (b, b+\Delta)$ and $s_{i,q_2} > b$, (iii) $\mathbf{s}_i \in$ "*LH*" if $s_{i,q_1} < b$ and $s_{i,q_2} > b$, (iv) $\mathbf{s}_i \in$ "*LM*" if $s_{i,q_1} < b$ and $s_{i,q_2} \in (b-\Delta, b)$, (v) $\mathbf{s}_i \in$ "*LL*" if $s_{i,q_1} < b$ and $s_{i,q_2} < b-\Delta$. ("*H*", "*M*", and "*L*" are mnemonic for "high", "middle", and "low".) Every realized residual supply curve (up to a zero-measure set) must belong to one of these five sets, since it must be non-decreasing in quantity. Fig. 3 illustrates an idealized curve belonging to each of the five sets.

Let $r_{HH}, \ldots, r_{LL}$ denote the true probabilities of these events. Note that $w_{i,q_1}(b+\Delta) = 1 - r_{HH}$, $w_{i,q_1}(b) = 1 - r_{HH} - r_{MH}$, $w_{i,q_2}(b) = r_{LM} + r_{LL}$, and $w_{i,q_2}(b-\Delta) = r_{LL}$. Thus, restriction (22) translates as $\Delta + \frac{1-r_{HH}-r_{MH}}{r_{MH}/\Delta} \geq \frac{r_{LL}}{r_{LM}/\Delta}$ or, after re-arranging,

$$X(q_1, q_2) = (1-r_{HH})r_{LM} - r_{MH}r_{LL} \geq 0. \qquad (\text{DISCRIM–}(q_1, q_2))$$

Evaluating DISCRIM–(q_1, q_2) given a sample of M identical auctions is equivalent to a standard problem of testing a moment inequality involving multinomial probabilities given M iid draws from the relevant multinomial distribution. Namely, let $\hat{r}^M_{HH}, \ldots, \hat{r}^M_{LL}$ be the empirical probability of each of the five events, and $\widehat{X}^M(q_1, q_2) = (1-\hat{r}_{HH})\hat{r}_{LM} - \hat{r}_{MH}\hat{r}_{LL}$. Then, as $M \to \infty$, $\sqrt{M}(\widehat{X}^M(q_1, q_2) - X(q_1, q_2))$ is asymptotically normal with variance that can also be consistently estimated by standard methods. (See Lehmann and Casella (1998), pp. 106, 193–194.)

Discussion: Implementing a test based on multiple restrictions of the form (19). Any subset of the $S(S-1)/2$ inequality restrictions (19) can (when discretized as in (22)) be translated into an equivalent set of moment inequalities similar to DISCRIM–(q_1, q_2), but a joint test of such conditions is complicated by the fact that the moments are correlated. Any joint test must therefore correct for the possibility that one or more of these inequalities may appear to be violated because of sampling error, even when all such restrictions are truly satisfied.

The simplest and most conservative approach would be to employ the Bonferroni correction, in which the joint hypothesis is rejected iff any of the individual hypotheses can be rejected

[8] The condition $1 \notin Q(b_{i,q_1})$ means that quantity q_1 is not in the first step in bid $\mathbf{b}_i$, while $S \notin Q(b_{i,q_2})$ means that q_2 is not in the last step. In particular, since $q_1 \leq q_2$, such a pair of quantities can only exist if bid $\mathbf{b}_i$ has at least three steps. I suspect that any bid having less than three steps can be rationalized as a best response given some non-increasing marginal values and large enough complexity costs.

[9] The last of these assumptions can be especially troubling in practice, since multi-unit auctions may have multiple equilibria. To address this problem, Hortacsu (2001) and Hortacsu and McAdams (2008) develop techniques to estimate the distribution of bids by re-sampling from the profile of bids observed *in a single auction*. Such techniques require one to assume a certain degree of bidder symmetry.

with sufficient confidence.[10] (See e.g. Miller (1981).) Since the confidence threshold for any individual test under this correction becomes more stringent as the number of tests increases, it may be sensible to choose *a priori* a subset of all pairs of quantities (q_1, q_2) and test the moment inequalities just for those pairs. For example, Chapman et al. (2006) only consider pairs (q_1, q_2) such that $q_1 = q_2$.

5. Extension: Uniform-price auctions

Our analysis of the discriminatory auction in Sections 2–4 applies with very little modification to the uniform-price auction. Here I highlight the few details that differ.

Model. For concreteness, consider the uniform $(S+1)$-st price auction (or simply "uniform price auction").[11] The model is identical to that of Section 2, except that bidder i pays the $(S+1)$-st highest unit-bid $\tilde{b}^{S+1}$ rather than his own unit-bid $b_{i,q}$ for each quantity q when he wins that unit. Consequently, payoffs take the form

$$\Pi_i^U(\mathbf{b}_i, \mathbf{v}_i; \tilde{\mathbf{s}}_i) \equiv \sum_{q=1}^{S} w_{i,q}(b_{i,q}) \left(v_{i,q} - E[\tilde{b}^{S+1} | b_{i,q} > \tilde{s}_{i,q}] \right). \quad (23)$$

Such payoffs decompose as $\Pi_i^U(\mathbf{b}_i, \mathbf{v}_i; \tilde{\mathbf{s}}_i) = \sum_{q=1,\dots,S} \Pi_{i,q}^U(b_{i,q}, v_{i,q}; \tilde{s}_{i,q-1}, \tilde{s}_{i,q})$, where

$$\begin{aligned}\Pi_{i,q}^U(b_{i,q}, v_{i,q}; \tilde{\mathbf{s}}_i) = {} & w_{i,q}(b_{i,q}) \big(v_{i,q} - E[\tilde{s}_{i,q} | b_{i,q} > \tilde{s}_{i,q}] \\ & - (q-1) E[\tilde{s}_{i,q} - \tilde{s}_{i,q-1} | b_{i,q} > \tilde{s}_{i,q}] \big) \\ & - \Pr(\tilde{s}_{i,q} > b_{i,q} > \tilde{s}_{i,q-1})(q-1) \\ & \times E[b_{i,q} - \tilde{s}_{i,q-1} | \tilde{s}_{i,q} > b_{i,q} \geq \tilde{s}_{i,q-1}]. \end{aligned} \quad (24)$$

This decomposition is not obvious from (23), but it accounts for the fact that raising one's bid on unit q may raise the price paid on the first $q-1$ units; see the working-paper version for a derivation. Note that, unlike in the discriminatory auction, the marginal effect of raising bidder i's qth unit-bid on his expected payoff depends on the *joint* distribution of $(\tilde{s}_{i,q-1}, \tilde{s}_{i,q})$. To see why, observe that bidder i's qth unit-bid sets the price only when $\tilde{s}_{i,q} > b_{i,q} > \tilde{s}_{i,q-1}$. (If $b_{i,q} \geq s_{i,q}$, then $b_{i,q}$ is greater than or equal to the Sth highest unit-bid overall, by the definition of $s_{i,q}$. Conversely, $b_{i,q}$ is less than or equal to the $(S+2)$-nd highest unit-bid if $b_{i,q} \leq s_{i,q-1}$.)

As before, Assumption 2 implies that these "unit-payoff functions" are locally differentiable at $(b_{i,q}, v_{i,q})$ for all $v_{i,q}$:

$$\frac{\mathrm{d}\Pi_{i,q}^U(b_{i,q}, v_{i,q}; \tilde{\mathbf{s}}_i)}{\mathrm{d}b_{i,q}} = w'_{i,q}(b_{i,q}) \left(v_{i,q} - b_{i,q} \right) - (q-1) \times \Pr(\tilde{s}_{i,q} > b_{i,q} > \tilde{s}_{i,q-1}) \quad (25)$$

$$\frac{\mathrm{d}^2\Pi_{i,q}^U(b_{i,q}, v_{i,q}; \tilde{\mathbf{s}}_i)}{\mathrm{d}b_{i,q}\mathrm{d}v_{i,q}} = w'_{i,q}(b_{i,q}).$$

Partial identification. All of the results of Section 3 apply to the uniform-price auction without further modification, once we substitute uniform-price auction payoffs for discriminatory auction payoffs in the formulae. Operationally, this amounts to computing the "indifference levels" $v_{i,X}^{*U}(\mathbf{b}_i)$ for the uniform-price auction (see Definition 3). In particular, for each quantity q, (25) implies

$$v_{i,q}^{*U}(\mathbf{b}_i) = b_{i,q} + (q-1) \frac{\Pr(\tilde{s}_{i,q} > b_{i,q} > \tilde{s}_{i,q-1})}{w'_{i,q}(b_{i,q})}. \quad (26)$$

(Indifference levels $v_{i,X}^{*U}(\mathbf{b}_i)$ for each subset $X \subset \{1, \dots, S\}$ can be similarly computed.)

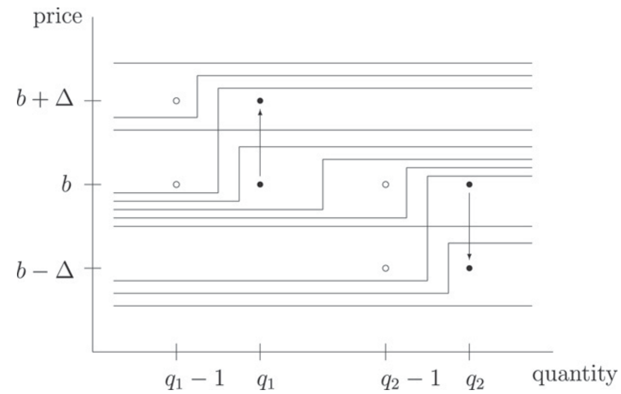

Fig. 4. Eleven relevant possibilities for bidder i's residual supply.

Testable restrictions. For each $q_1 \leq q_2$, we can derive a set of testable restrictions in the uniform-price auction by repeating the argument in Section 4 that applied to the discriminatory auction. For example, consider the incremental profit expressions (14) and (15) associated with a pair of upward and downward deviations. To apply to the uniform-price auction, we need only modify the terms corresponding to the effect of each deviation on expected payment ($-\Delta w_{i,x}(b_{i,x})$ in (14) and $\Delta w_{i,x}(b_{i,x} - \Delta)$ in (15)). For the rest of the derivation of inequality restriction (19), these payment-related terms can be simply carried through, to arrive at a corresponding inequality restriction for the uniform-price auction. (I omit the details since they involve fairly complex formulae that offer little additional insight.)

However, a significant difference arises when it comes to implementing the tests associated with these restrictions. The reason is that, to evaluate the price effect of raising bidder i's bid on quantity q, the econometrician must estimate probabilities associated with the joint distribution of $(\tilde{s}_{i,q-1}, \tilde{s}_{i,q})$. By contrast, in the discriminatory auction, the econometrician must only estimate probabilities associated with the marginal distribution of $\tilde{s}_{i,q}$. For example, to implement a test of the single restriction in the uniform-price auction that corresponds to the single restriction (22) in the discriminatory auction, the econometrician must determine the likelihood that bidder i's residual inverse supply (i) $\tilde{s}_{i,q_1-1}$ is greater than $b+\Delta$, in $(b, b+\Delta)$, or less than b, (ii) $\tilde{s}_{i,q_1}$ is greater than $b+\Delta$, in $(b, b+\Delta)$, or less than b, (iii) $\tilde{s}_{i,q_2-1}$ is greater than b, in $(b-\Delta, b)$, or less than $b-\Delta$, and (iv) $\tilde{s}_{i,q_2}$ is greater than b, in $(b-\Delta, b)$, or less than $b-\Delta$. Consequently, there are eleven "relevant events" for bidder i's realized residual supply in the uniform-price auction, compared to the five relevant events in the discriminatory auction. (Compare Fig. 4 with Fig. 3.)

6. Example

Suppose two units are sold in a uniform third-price auction to two bidders who have independent private values $\mathbf{v}_i = (v_{i,1}, v_{i,2})$ and play a Bayesian Nash equilibrium. Suppose that bidder 1 submits all bids in the triangle $\{\mathbf{b}_1 : 1 \geq b_{1,1} > b_{1,2} \geq 0\}$, each equally likely, and that bidder 2 submits all "flat" bids of the form $\mathbf{b}_2 = (b_2, b_2)$, with $b_2 \sim U[0,1]$. $\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2$ are independent. All of bidder 1's bids can be easily rationalized: each bid $\mathbf{b}_1 \in \mathrm{supp}(\tilde{\mathbf{b}}_i)$ is a best response iff $\mathbf{v}_1 = \mathbf{b}_1$. We shall focus on bidder 2.

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[10] A more satisfying – though much more computationally intensive – approach would be to compute maximum likelihood estimates of the probabilities of all relevant events, with and without the moment inequality constraints, and compare likelihoods.

[11] The analysis can easily be adapted to other uniform-price auctions, such as the Sth price auction (price equals lowest winning unit-bid) and $(S+1/2)$th price auction (price is the average of Sth and $(S+1)$st).

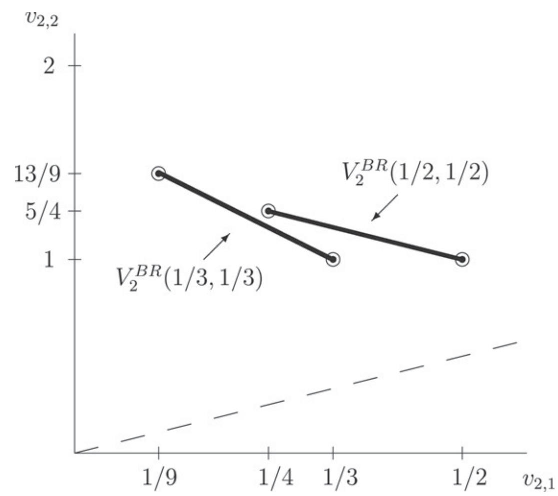

Fig. 5. Sets $V_2^{BR}(\mathbf{b}_2)$ are illustrated for $\mathbf{b}_2 \in \{(1/3, 1/3), (1/2, 1/2)\}$.

Proposition 1. *For all* $b_2 \in (0, 1)$, $V_2^{\text{FOC}}(b_2, b_2) = \{\mathbf{v}_2(\alpha) : \alpha \geq 0\}$ *and* $V_2^{BR}(b_2, b_2) = \{\mathbf{v}_2(\alpha) : \alpha \in [0, 1]\}$, *where* $\mathbf{v}_2(\alpha) = \alpha(b_2, 1) + (1-\alpha)(b_2^2, 1 + (1-b_2)^2)$.

The first part of Proposition 1, characterizing $V_2^{\text{FOC}}(b_2, b_2)$ for all b_2, is an application of Theorem 1. The second part, characterizing the sets of values $V_2^{BR}(\mathbf{b}_2)$ given which (b_2, b_2) is a best response, is possible because of the simplicity of this example. (The second-order conditions of optimal bidding further limit the set of values given which bid (b_2, b_2) might be a best response. Then, one can check by hand that such bids are in fact best responses, as long as the first- and second-order conditions are satisfied.) Fig. 5 illustrates the sets $V_2^{BR}(\mathbf{b}_2)$ given which bid $\mathbf{b}_2$ is a best response, for bids $\mathbf{b}_2 \in \{(1/3, 1/3), (1/2, 1/2)\}$.

Non-increasing marginal values would be rejected for bidder 2. If bidder 2's strategy is a best response, he must have increasing marginal values when submitting any of his bids $\mathbf{b}_2 = (b_2, b_2)$ for $b_2 \in (0, 1)$. Intuitively, bidder 2 always has an incentive to shade his bid on the second unit below his marginal value for that unit, but never to shade his bid on the first unit. (Given increasing marginal values, bidder 2 might want to bid more than his value for the first unit, but he will never bid less than his value for that unit.) Consequently, $v_{2,1} - b_{2,1} < v_{2,2} - b_{2,1}$. Yet bidder 2 is observed bidding the same price on each unit: $b_{2,1} = b_{2,2} = b_2$. We conclude that $v_{2,1} < v_{2,2}$, and reject the hypothesis that bidder 2 has non-increasing marginal values. Fundamentally, the reason that NIMV is rejected here is that bidders have an incentive to engage in "differential bid shading" (Ausubel and Cramton, 1998). If a bidder with such an incentive submits a flat price schedule, he must have increasing marginal values.

Counterfactual: Should the units be bundled? Even though bidder 2's values are not point identified, we can conduct meaningful policy counterfactual analysis. For example, could the auctioneer increase expected revenue by bundling the units together, and selling the bundle using a second-price auction? (The fact that bidder 2 has increasing marginal values is suggestive that bundling may increase revenue, since bidder 2 will no longer face the "exposure problem" of sometimes winning only one unit.)

To perform this counterfactual experiment, we need to know the distribution of each bidder's value for the bundle, i.e. $\tilde{v}_{1,1} + \tilde{v}_{1,2}$ and $\tilde{v}_{2,1} + \tilde{v}_{2,2}$. Recall that we have inferred bidder 1's value $\tilde{\mathbf{v}}_1 = \tilde{\mathbf{b}}_1$. This identifies the distribution of $\tilde{v}_{1,1} + \tilde{v}_{1,2}$. However, since bidder 2's value is not point identified, there are many possible value distributions consistent with his distribution of bids. Nonetheless, our characterization of the sets $V_i^{BR}(\mathbf{b}_i)$ provide *upper and lower bounds* on $\tilde{v}_{2,1} + \tilde{v}_{2,2}$ conditional on each bid $\tilde{\mathbf{b}}_i$, these bounds being realized at either end-point of the line-segment $V_2^{BR}(\tilde{\mathbf{b}}_2)$ described in Proposition 1:

$$\min\{\tilde{b}_2 + 1, 2(1 - \tilde{b}_2 + \tilde{b}_2^2)\} \leq \tilde{v}_{2,1} + \tilde{v}_{2,2} \leq \max\{\tilde{b}_2 + 1, 2(1 - \tilde{b}_2 + \tilde{b}_2^2)\}.$$

This allows us to identify the "highest" and the "lowest" value distributions that could generate the observed bids in equilibrium, and hence to compute upper and lower bounds for expected revenue in the counter-factual. In particular, expected revenue in the second-price auction when the units are bundled will be *at least* about .955 and *at most* about .99.[12] In the status quo, expected revenue is only .83. Thus, the auctioneer can raise revenue by bundling. □

7. Concluding remarks

This paper develops techniques to infer information about bidders' values from their bids in multi-unit auctions, under the maintained hypotheses of independent private values and equilibrium bidding. The analysis highlighted important differences between single-object auctions and multi-unit auctions. In particular, multi-unit auction models are not point-identified by the first-order conditions of optimal bidding, unless observed bids are strictly downward-sloping. Yet in many multi-unit auctions in practice, bidders are not *allowed* to submit strictly downward-sloping bids. Furthermore, even if bidders were allowed to submit such bids, they may not choose to do so in equilibrium (Anwar, 2007). Consequently, a possibly unbounded set of marginal value schedules can rationalize each bid. To address the issue of boundedness, I provide tight upper and lower bounds on bidders' marginal values under an additional restriction already commonly imposed in practice, namely that bidders have non-increasing marginal values. Novel testable implications of the best response hypothesis are also provided, again under the assumption of non-increasing marginal values.

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Appendix. Proofs

A.1. Proof of Theorem 1.

Preliminaries. Given Assumption 2, we may re-write (5) as:

$$V_i^{\text{FOC}}(\mathbf{b}_i) = \left\{ \mathbf{v}_i : \sum_{q=1}^{S} (\hat{b}_{i,q} - b_{i,q}) \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,q})}{\mathrm{d}b_{i,q}} \leq 0 \text{ for all } \hat{\mathbf{b}}_i \in \mathcal{B} \right\}. \quad (27)$$

Indeed, $V_i^{\text{FOC}}(\mathbf{b}_i)$ may be characterized even more simply in terms of a set of inequalities and equalities, as shown in the following Lemma.

[12] Spreadsheets that implement these computations are available from the author.

Lemma 3. *$\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$ if and only if*

$$\sum_{y\le q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \le 0 \quad \textit{for all } q = 1,\dots,S-1, \quad \textit{and} \tag{28}$$

$$\sum_{y\le \max Q(b)} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} = 0 \quad \textit{for all } Q(b) \in \mathcal{Q}(\mathbf{b}_i) \tag{29}$$

where $\mathcal{Q}(\mathbf{b}_i)$ is the set of "steps" corresponding to bid $\mathbf{b}_i$. (See Definition 5 in the text.)

Throughout the proof to follow, I will consider certain simple deviations from the bid $\mathbf{b}_i$. For any interval of quantities $Q \subset \{1,\dots,S\}$, define $\hat{\mathbf{b}}_i^{+,Q}$ and $\hat{\mathbf{b}}_i^{-,Q}$ as follows:

$$\hat{b}_{i,y}^{+,Q} = b_{i,y} + \epsilon \quad \text{and} \quad \hat{b}_{i,y}^{-,Q} = b_{i,y} - \epsilon \quad \text{for all } y \in Q \tag{30}$$

$$\hat{b}_{i,y}^{+,Q} = \hat{b}_{i,y}^{-,Q} = b_{i,y} \quad \text{for all } y \notin Q. \tag{31}$$

Since $\mathbf{b}_i$ is a feasible (i.e. non-increasing) bid, the deviation $\hat{\mathbf{b}}_i^{+,Q}$ is feasible for small enough ϵ when $b_{i,\min Q} < b_{i,\min Q-1}$ (or $\min Q = 1$), while the deviation $\hat{\mathbf{b}}_i^{-,Q}$ is feasible for small enough ϵ when $b_{i,\max Q} > b_{i,\max Q+1}$ (or $\max Q = S$).

Part I:"$\Rightarrow$". Suppose that $\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$. I need to show that constants $c_1,\dots,c_{S-1} \ge 0$ exist such that

$$v_{i,1} = v_{i,1}^*(\mathbf{b}_i) - c_1 w'_{i,2}(b_{i,2}) \tag{32}$$

$$v_{i,q} = v_{i,q}^*(\mathbf{b}_i) - c_q w'_{i,q+1}(b_{i,q}) + c_{q-1} w'_{i,q-1}(b_{i,q}) \quad \text{for } q = 1,\dots,S-1 \tag{33}$$

$$v_{i,S} = v_{i,S}^*(\mathbf{b}_i) + c_{S-1} w'_{i,S-1}(b_{i,S-1}) \tag{34}$$

and such that $c_q = 0$ whenever $b_{i,q} > b_{i,q+1}$. In particular, consider the constants defined as follows:

$$c_q = \sum_{y=1}^{q} \frac{(v_{i,y}^*(\mathbf{b}_i) - v_{i,y}) w'_{i,y}(b_{i,y})}{w'_{i,q}(b_{i,q}) w'_{i,q+1}(b_{i,q+1})} \quad \text{for all } q = 1,\dots,S-1. \tag{35}$$

Before proceeding further, please note that

$$c_q \gtrless 0 \Leftrightarrow \sum_{y=1}^{q} (v_{i,y}^*(\mathbf{b}_i) - v_{i,y}) w'_{i,y}(b_{i,y}) \gtrless 0$$
$$\Leftrightarrow -\sum_{y=1}^{q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \gtrless 0. \tag{36}$$

The first equivalence holds by definition of c_q, since $w'_{i,q}(b_{i,q}), w'_{i,q+1}(b_{i,q+1}) > 0$ by Assumption 3. The second equivalence holds since $\sum_{y=1}^{q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y}^*(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} = 0$ by (6) the definition of $\mathbf{v}_i^*(\mathbf{b}_i)$ and since $\frac{\mathrm{d}^2\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}\mathrm{d}v_{i,y}} = w'_{i,y}(b_{i,y})$ for all y by (3).

To show that $c_q \ge 0$ for all $q = 1,\dots,S-1$, consider the feasible deviation $\hat{\mathbf{b}}_i^{+,[1,q]}$. By (27), $\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$ implies that $\sum_{y=1}^{q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \le 0$; hence $c_q \ge 0$ by (36). Next, consider any quantity $q \in \{1,\dots,S-1\}$ such that $b_{i,q} > b_{i,q+1}$. For these quantities, the deviation $\hat{\mathbf{b}}_i^{-,[1,q]}$ is also feasible. Since $\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$, $\sum_{y=1}^{q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \ge 0$ and hence $c_q \le 0$. We conclude that $c_q = 0$ for all such quantities.

Finally, we need to verify conditions (32)–(34). (32) and (33) hold by definition of the constants $c_1,\dots,c_{S-1}$,[13] but condition (34) must be proven. Note that (34) can be re-written as $v_{i,S} - v_{i,S}^*(\mathbf{b}_i) = (\sum_{y=1}^{S-1}(v_{i,y}^*(\mathbf{b}_i) - v_{i,y}) w'_{i,y}(b_{i,y}))/w'_{i,S}(b_{i,S})$ or, equivalently,

$$0 = \sum_{y=1}^{S} (v_{i,y}^*(\mathbf{b}_i) - v_{i,y}) w'_{i,y}(b_{i,y}) = \sum_{q=1}^{S} \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,q})}{\mathrm{d}b_{i,q}}.$$

By (27), $\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$ implies $\sum_{q=1}^{S} \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,q})}{\mathrm{d}b_{i,q}} = 0$, since both deviations $\hat{\mathbf{b}}_i^{+,\{1,\dots,S\}}$ and $\hat{\mathbf{b}}_i^{-,\{1,\dots,S\}}$ are feasible. This completes the "only if" part of the proof of Theorem 1.

Part II:"$\Leftarrow$". Suppose that, for some given marginal value schedule $\mathbf{v}_i$, there exist constants $c_1,\dots,c_{S-1} \ge 0$ satisfying (32)–(34) and such that $b_{i,q} > b_{i,q+1}$ implies $c_q = 0$. To complete the proof, I need to show that $\mathbf{v}_i \in V_i^{FOC}(\mathbf{b}_i)$. By Lemma 3, it suffices to verify (28) and (29).

Observe that

$$\sum_{y\le q} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} = \sum_{y\le q} (v_{i,y} - v_{i,y}^*(\mathbf{b}_i)) w'_{i,y}(b_{i,y}) \tag{37}$$

$$= -c_q w'_q(b_{i,q}) w'_{q+1}(b_{i,q+1}) \le 0 \quad \text{for all } q = 1,\dots,S-1 \tag{38}$$

$$= 0 \quad \text{for } q = S. \tag{39}$$

(37) was established in Part I. (38) and (39) follow from the presumption that conditions (32)–(34) are satisfied, as follows. By (32), $(v_{i,1} - v_{i,1}^*(\mathbf{b}_i)) w'_{i,1}(b_{i,1}) = -c_1 w'_1(b_{i,1}) w'_2(b_{i,2})$, so (38) holds for $q = 1$. By (33), $(v_{i,q} - v_{i,q}^*(\mathbf{b}_i)) w'_{i,q}(b_{i,q}) = -c_q w'_q(b_{i,q}) w'_{q+1}(b_{i,q+1}) + c_{q-1} w'_q(b_{i,q}) w'_{q-1}(b_{i,q-1})$ for all $q = 2,\dots,S-1$. This establishes (38) for all $q = 2,\dots,S-1$ by induction on q. Similarly, (39) holds since $\sum_{y\le S-1} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} = -c_{S-1} w'_{S-1}(b_{i,S-1}) w'_S(b_{i,S})$ by (38) while $(v_{i,S} - v_{i,S}^*(\mathbf{b}_i)) w'_{i,S}(b_{i,S}) = c_{S-1} w'_{i,S-1}(b_{i,S-1}) w'_S(b_{i,S})$ by (34).

(28) follows immediately from (38) since $c_q \ge 0$.

Finally, consider any step $Q(b) \in \mathcal{Q}(\mathbf{b}_i)$. If $\max Q(b) = S$, then (29) follows immediately from (39). Otherwise, suppose that $\max Q(b) < S$. Since $\max Q(b)$ is the largest quantity in its step, $b_{i,\max Q(b)} > b_{i,\max Q(b)+1}$. By presumption, then, $c_{\max Q(b)} = 0$. (29) then follows from (38). In either case, we have verified the conditions of Lemma 3, completing the proof. □

A.2. Proof of Theorem 2.

Throughout the proof, let $\mathbf{v}_i$ be an element of $V_i^{FOC}(\mathbf{b}_i) \cap V^{NIMV}$, i.e. a non-increasing marginal value schedule given which all first-order conditions are satisfied. Parts I,II establish Theorem 2(a) while Part III establishes Theorem 2(b).

Part I: $v_{i,q} \le \min_{x=1,\dots,q} v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$ and $v_{i,q} \ge \max_{x=q,\dots,S} v_{i,[x,\max Q(b_{i,x})]}^*(\mathbf{b}_i)$. Suppose for the sake of contradiction that $v_{i,x} > v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$ for some $x \le q$. Since bidder i has NIMV, $v_{i,y} > v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$ for all $y \in [\min Q(b_{i,x}), x]$. Yet then, by definition of the indifference level $v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$, bidder i must strictly prefer to deviate by slightly raising his price on all units in $[\min Q(b_{i,x}), x]$, a contradiction. We conclude that $v_{i,x} \le v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$ for all $x = 1,\dots,q$. Since $v_{i,q} \le v_{i,x}$ for all $x = 1,\dots,q$ by NIMV, $v_{i,q} \le \min_{x=1,\dots,q} v_{i,[\min Q(b_{i,x}),x]}^*(\mathbf{b}_i)$. A symmetric argument establishes that $v_{i,q} \ge \max_{x=q,\dots,S} v_{i,[x,\max Q(b_{i,x})]}^*(\mathbf{b}_i)$.

Part II: $v_{i,q} \le \min_{x=1,\dots,q} \hat{v}_{i,x}(\mathbf{b}_i)$ and $v_{i,q} \ge \max_{x=q,\dots,S} \check{v}_{i,x}(\mathbf{b}_i)$. As in Part I, to prove $v_{i,q} \le \min_{x=1,\dots,q} \hat{v}_{i,x}(\mathbf{b}_i)$, it suffices by NIMV to show that $v_{i,x} \le \hat{v}_{i,x}(\mathbf{b}_i)$ for all $x \le q$.

[13] By (35), $v_{i,1}^*(\mathbf{b}_i) - v_{i,1} = c_1 w'_{i,2}(b_{i,2})$ and $v_{i,q}^*(\mathbf{b}_i) - v_{i,q} = c_q w'_{i,q+1}(b_{i,q+1}) - c_{q-1} w'_{i,q-1}(b_{i,q-1})$ for all $q = 2,\dots,S-1$.

Suppose for the sake of contradiction that $v_{i,x} > \hat{v}_{i,x}(\mathbf{b}_i)$ for some $x \leq q$. By NIMV and the fact that $\frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}}$ is increasing in $v_{i,y}$ for all y,

$$\sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} > \sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, \hat{v}_{i,x}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}}. \tag{40}$$

Next, let $z(x) = \arg\max_{z=x+1,\ldots,\max Q(b_{i,x+1})} \left\{ v^*_{i,[z,\max Q(b_{i,x+1})]}(\mathbf{b}_i) \right\}$. By definition of the indifference level $v^*_{i,[z(x),\max Q(b_{i,x+1})]}(\mathbf{b}_i)$,

$$\sum_{y\in[z(x),\max Q(b_{i,x+1})]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v^*_{i,[z(x),\max Q(b_{i,x+1})]}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} = 0. \tag{41}$$

At the same time, $\mathbf{v}_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i)$ implies

$$\sum_{y\in[z(x),\max Q(b_{i,x+1})]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \geq 0 \tag{42}$$

since otherwise bidder i would strictly prefer to deviate by lowering his price on all quantities in the range $[z(x), \max Q(b_{i,x+1})]$.

Next, Step I implies $v_{i,y} \geq v^*_{i,[z(x),\max Q(b_{i,x+1})]}(\mathbf{b}_i)$ for all $y \leq z(x)$. In particular,

$$\sum_{y\in[x+1,z(x)-1]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} \geq \sum_{y\in[x+1,z(x)-1]} \frac{\mathrm{d}\Pi_{i,y}\left(b_{i,y}, v^*_{i,[z(x),\max Q(b_{i,x+1})]}(\mathbf{b}_i)\right)}{\mathrm{d}b_{i,y}}. \tag{43}$$

Combining the first equation in Box I, and (40)–(43), we conclude that

$$\sum_{y\in[\min Q(b_{i,x}),\max Q(b_{i,x+1})]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,y})}{\mathrm{d}b_{i,y}} > 0.$$

This contradicts the assumption that $\mathbf{v}_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i)$, since bidder i strictly prefers to deviate by raising his price on all quantities in the range $[\min Q(b_{i,x}), \max Q(b_{i,x+1})]$. We conclude that $v_{i,q} \leq \min_{x=1,\ldots,q} \hat{v}_{i,x}(\mathbf{b}_i)$. A symmetric argument establishes $v_{i,q} \geq \max_{x=q,\ldots,S} \check{v}_{i,x}(\mathbf{b}_i)$. This completes the proof of Theorem 2(a).

Part III: Tightness of bounds.

Define $\mathbf{v}'_i$ of Theorem 2(b) as follows: $v'_{i,x} = \overline{v}_{i,q}(\mathbf{b}_i)$ for all $x \in [\min Q(b_{i,q}), q]$, $v'_{i,x} = \underline{v}_{i,q+1}(\mathbf{b}_i)$ for all $x \in [q+1, \max Q(b_{i,q+1})]$, and $v'_{i,x} = v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$ for all $x \notin Q(b_{i,q}) \cup Q(b_{i,q+1})$. I must show that $\mathbf{v}'_i \in V^{\mathrm{NIMV}}$ and that $\mathbf{v}'_i \in V_i^{\mathrm{FOC}}(\mathbf{b}_i)$.

Part III-a: *non-increasing marginal values.* Let $Q(b)$ be any (non-empty) step of quantities all bid at price b. By (12), $\underline{v}_{i,\min Q(b)}(\mathbf{b}_i) \geq v^*_{i,Q(b)}(\mathbf{b}_i)$. Similarly, by (11), $\overline{v}_{i,\max Q(b)}(\mathbf{b}_i) \leq v^*_{i,Q(b)}(\mathbf{b}_i)$. Next, note that $\overline{v}_{i,q}(\mathbf{b}_i)$ and $\underline{v}_{i,q}(\mathbf{b}_i)$ are non-increasing in q by definition. Finally, since $V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}} \neq \emptyset$, Theorem 2(a) implies that $\overline{v}_{i,q}(\mathbf{b}_i) \geq \underline{v}_{i,q}(\mathbf{b}_i)$ for all q. All together, we conclude that

$$v^*_{i,Q(b)}(\mathbf{b}_i) \geq \overline{v}_{i,q} \quad \text{for all } q > \max Q(b) \tag{44}$$

$$\leq \underline{v}_{i,q} \quad \text{for all } q < \min Q(b) \tag{45}$$

$$\in [\underline{v}_{i,q}, \overline{v}_{i,q}] \quad \text{for all } q \in Q(b). \tag{46}$$

As can be easily checked, (44)–(46) together imply that the marginal value schedule $\mathbf{v}'_i$ is non-increasing. (The details are straightforward and omitted.)

Part III-b: *first-order conditions all satisfied.* Repeating an argument provided in the proof of Theorem 1, it suffices to check that

$$\sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v'_{i,y})}{\mathrm{d}b_{i,y}} \leq 0 \quad \text{for all quantities } q \tag{47}$$

$$\sum_{y\in Q(b_{i,x})} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v'_{i,y})}{\mathrm{d}b_{i,y}} = 0 \quad \text{for all steps } Q(b). \tag{48}$$

There are two cases to consider.

First case: $x \notin Q(b_{i,q}) \cup Q(b_{i,q+1})$. In this case, $v'_{i,y} = v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$ for all $y \in Q(b_{i,x})$. Recall that, by definition of $v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$

$$\sum_{y\in Q(b_{i,x})} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v^*_{i,Q(b_{i,x})}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} = 0. \tag{49}$$

Thus, it suffices to show $\sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v^*_{i,Q(b_{i,x})}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} \leq 0$ for $x = \min Q(b), \ldots, \max Q(b) - 1$. Suppose for the sake of contradiction that

$$\sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v^*_{i,Q(b_{i,x})}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} > 0. \tag{50}$$

By (3) and the Definition 3 of the indifference levels $v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i)$, $v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i) < v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$. Also, by the definition of the upper bound $\overline{v}_{i,x}(\mathbf{b}_i)$, $\overline{v}_{i,x}(\mathbf{b}_i) \leq v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i)$. We conclude that $\overline{v}_{i,x}(\mathbf{b}_i) < v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$. Next, note that $x = \min Q(b), \ldots, \max Q(b) - 1$ and (49) and (50) imply $\sum_{y\in[x+1,\max Q(b_{i,x})]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v^*_{i,Q(b_{i,x})}(\mathbf{b}_i))}{\mathrm{d}b_{i,y}} < 0$. By a symmetric argument, we conclude that $\underline{v}_{i,x+1}(\mathbf{b}_i) > v^*_{i,Q(b_{i,x})}(\mathbf{b}_i)$. All together, we conclude that $\underline{v}_{i,x+1}(\mathbf{b}_i) > \overline{v}_{i,x}(\mathbf{b}_i)$. Yet this contradicts the presumption that $V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}} \neq \emptyset$ since, for any element $\mathbf{v}_i$ of this set, $v_{i,x} \leq \overline{v}_{i,x}(\mathbf{b}_i)$, $v_{i,x+1} \geq \underline{v}_{i,x+1}(\mathbf{b}_i)$, and $v_{i,x} \geq v_{i,x+1}$ is only possible if $\underline{v}_{i,x+1}(\mathbf{b}_i) \leq \overline{v}_{i,x}(\mathbf{b}_i)$.

Second case: $x \in Q(b_{i,q}) \cup Q(b_{i,q+1})$. Without loss, suppose that $x \in Q(b_{i,q})$. (The argument for $x \in Q(b_{i,q+1})$ is symmetric.) By definition of the upper bound, $\overline{v}_{i,x}(\mathbf{b}_i) \leq v^*_{i,[\min Q(b_{i,q}),x]}(\mathbf{b}_i)$. Since $v'_{i,q} = \overline{v}_{i,x}(\mathbf{b}_i)$ for all $q \in [\min Q(b_{i,x}), x]$,

$$\sum_{y\in[\min Q(b_{i,x}),x]} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v'_{i,q})}{\mathrm{d}b_{i,y}} \leq 0 \tag{51}$$

and it suffices to show $\sum_{y\in Q(b_{i,x})} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v'_{i,y})}{\mathrm{d}b_{i,y}} = 0$. If $x \neq \max Q(b_{i,x})$, the desired result follows immediately from Lemma 2, since $Q(b_{i,x}) = Q(b_{i,x+1})$. So suppose that $x = \max Q(b_{i,x})$. The desired result now follows from (51) *unless*

$$\sum_{y\in Q(b_{i,x})} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v'_{i,q})}{\mathrm{d}b_{i,y}} < 0. \tag{52}$$

Yet this contradicts the presumption that $V_i^{\mathrm{FOC}}(\mathbf{b}_i) \cap V^{\mathrm{NIMV}} \neq \emptyset$ since, for any element $\mathbf{v}_i$ of this set, $v_{i,y} \leq \overline{v}_{i,y}(\mathbf{b}_i)$ for all $y \in Q(b_{i,x})$ and $\mathbf{v}_i$ non-increasing implies $v_{i,y} \leq \overline{v}_{i,\max Q(b_{i,x})}(\mathbf{b}_i) = v'_{i,y}$ for all $y \in Q(b_{i,x})$. By (3) and (52), this implies that $\sum_{y\in Q(b_{i,x})} \frac{\mathrm{d}\Pi_{i,y}(b_{i,y}, v_{i,q})}{\mathrm{d}b_{i,y}} < 0$, so that bidder i strictly prefers to deviate given values $\mathbf{v}_i$ by lowering his price on all quantities in the step $Q(b_{i,x})$. This completes the proof. □

A.3. Proof of Lemma 2.

Step 1: Simplifying the definition of $\overline{v}_{i,q}(\mathbf{b}_i)$ and $\underline{v}_{i,q+1}(\mathbf{b}_i)$. Let $Q(b)$ be any (non-empty) step. $\overline{v}_{i,\min Q(b)} \le v^*_{i,Q(b)}(\mathbf{b}_i)$ by (11). Suppose for the moment that $\overline{v}_{i,\min Q(b)} < v^*_{i,Q(b)}(\mathbf{b}_i)$. Then $v_{i,x} < \underline{v}_{i,\max Q(b)}$ for all $x \in Q(b)$, in which case bidder i strictly prefers to deviate by lowering his price on all quantities in $Q(b)$, a contradiction of the presumption that $V_i^{\text{FOC}}(\mathbf{b}_i) \cap V^{\text{NIMV}} \neq \emptyset$. We conclude that $\overline{v}_{i,\min Q(b)} = v^*_{i,Q(b)}(\mathbf{b}_i)$, while a symmetric argument establishes $\underline{v}_{i,\min Q(b)} = v^*_{i,Q(b)}(\mathbf{b}_i)$. By (11) and (12), it must be that

$$\min_{x=1,\dots,\min Q(b)-1} \left\{ v^*_{i,[\min Q(b_{i,x}),x]}(\mathbf{b}_i),\ \hat{v}_{i,x}(\mathbf{b}_i) \right\} \ge v^*_{i,Q(b)}(\mathbf{b}_i) \ge \max_{x=\max Q(b)+1,S} \left\{ v^*_{i,[x,\max Q(b_{i,x})]}(\mathbf{b}_i),\ \check{v}_{i,x}(\mathbf{b}_i) \right\}.$$

In particular, (11) and (12) as applied to quantities q, $q+1$ reduce to

$$\overline{v}_{i,q}(\mathbf{b}_i) = \min_{x=\min Q(b_{i,q}),\dots,q} \{ v^*_{i,[\min Q(b_{i,q}),x]}(\mathbf{b}_i),\ \hat{v}_{i,x}(\mathbf{b}_i) \} \quad (53)$$

$$\underline{v}_{i,q+1}(\mathbf{b}_i) = \max_{x=q+1,\dots,\max Q(b_{i,q+1})} \{ v^*_{i,[x,\max Q(b_{i,q+1})]}(\mathbf{b}_i),\ \check{v}_{i,x}(\mathbf{b}_i) \}. \quad (54)$$

Step 2: Complete the proof. Define terms

$$X^1 = \min_{x=\min Q(b_{i,q}),\dots,q} \left\{ \sum_{y=\min Q(b_{i,q})}^{q} \frac{d\Pi_{i,y}(b_{i,y}, v^*_{i,[\min Q(b_{i,q}),x]}(\mathbf{b}_i))}{db_{i,y}} \right\}$$

$$X^2 = \max_{x=q+1,\dots,\max Q(b_{i,q+1})} \times \left\{ \sum_{y=q+1}^{\max Q(b_{i,q+1})} \frac{d\Pi_{i,y}(b_{i,y}, v^*_{i,[x,\max Q(b_{i,q+1})]}(\mathbf{b}_i))}{db_{i,y}} \right\}.$$

By Box I, (53) and (54),

$$\sum_{y=\min Q(b_{i,q})}^{q} \frac{d\Pi_{i,y}(b_{i,y}, \overline{v}_{i,q}(\mathbf{b}_i))}{db_{i,y}} = \min\{X^1, -X^2\}$$

$$\sum_{y=q+1}^{\max Q(b_{i,q+1})} \frac{d\Pi_{i,y}(b_{i,y}, \underline{v}_{i,q+1}(\mathbf{b}_i))}{db_{i,y}} = \max\{-X^1, X^2\}.$$

Consequently,

$$\sum_{y=\min Q(b_{i,q})}^{q} \frac{d\Pi_{i,y}(b_{i,y}, \overline{v}_{i,q}(\mathbf{b}_i))}{db_{i,y}} + \sum_{y=q+1}^{\max Q(b_{i,q+1})} \frac{d\Pi_{i,y}(b_{i,y}, \underline{v}_{i,q+1}(\mathbf{b}_i))}{db_{i,y}} = 0.$$

This completes the proof. □

A.4. Proof of Lemma 3.

The proof that $\mathbf{v}_i \in V_i^{\text{FOC}}(\mathbf{b}_i)$ implies (28) and (29) is nearly immediate: (28) follows from (27) when we consider the feasible deviation $\hat{\mathbf{b}}_i^{+,[1,q]}$ while (29) follows from (27) when we consider the two feasible deviations $\hat{\mathbf{b}}_i^{+,[1,\max Q(b)]}$ and $\hat{\mathbf{b}}_i^{-,[1,\max Q(b)]}$. (See (30) and (31) for definitions and the surrounding text for why these deviations are feasible.) Somewhat less obvious is the fact that (28) and (29) are *sufficient* conditions for $\mathbf{v}_i \in V_i^{\text{FOC}}$.

Suppose that $\mathbf{v}_i \notin V_i^{\text{FOC}}(\mathbf{b}_i)$, i.e. there exists some feasible deviation $\hat{\mathbf{b}}_i$ such that $\sum_{q=1}^{S} (\hat{b}_{i,q} - b_{i,q}) \frac{d\Pi_{i,q}(b_{i,q}, v_{i,q})}{db_{i,q}} > 0$. To complete the proof, I will show that either (28) must fail for some quantity $q = 1, \dots, S-1$ or (29) must fail for some step $Q(b) \in \mathcal{Q}(\mathbf{b}_i)$.

First, there must exist some step $Q(b) \in \mathcal{Q}(\mathbf{b}_i)$ such that $\sum_{q\in Q(b)} (\hat{b}_{i,q} - b) \frac{d\Pi_{i,q}(b, v_{i,q})}{db_{i,q}} > 0$. (Recall that, when referring to steps, I use b to denote the price bid on all units in step $Q(b)$, i.e. $b_{i,q} = b$ for all $q \in Q(b)$.) Since $\hat{\mathbf{b}}_i$ is feasible, it must be non-decreasing, hence the vector of differences $(\hat{b}_{i,\min Q(b)} - b, \dots, \hat{b}_{i,\max Q(b)} - b)$ is non-decreasing. Consequently, this vector of differences has a decomposition of the form $(\hat{b}_{i,\max Q(b)} - b) * (1,\dots,1) + (\hat{b}_{i,\max Q(b)-1} - \hat{b}_{i,\max Q(b)}) * (1,\dots,1,0) + \dots + (\hat{b}_{i,\min Q(b)} - \hat{b}_{i,\min Q(b)+1}) * (1,0,\dots,0)$. We may therefore re-write

$$\sum_{q\in Q(b)} (\hat{b}_{i,q} - b) \frac{d\Pi_{i,q}(b, v_{i,q})}{db_{i,q}} = (\hat{b}_{i,\max Q(b)} - b) \sum_{q\in Q(b)} \frac{d\Pi_{i,q}(b, v_{i,q})}{db_{i,q}} + \sum_{q=\min Q(b)}^{\max Q(b)-1} (\hat{b}_{i,q} - \hat{b}_{i,q+1}) \sum_{y=\min Q(b)}^{q} \frac{d\Pi_{i,y}(b, v_{i,y})}{db_{i,y}}. \quad (55)$$

Since the left-hand-side of (55) is positive, it must be that either (i) $\sum_{q\in Q(b)} \frac{d\Pi_{i,q}(b, v_{i,q})}{db_{i,q}} > 0$ or (ii) $\sum_{y=\min Q(b)}^{q} \frac{d\Pi_{i,y}(b, v_{i,y})}{db_{i,y}} > 0$ for some $q \in [\min Q(b), \max Q(b) - 1]$.

By presumption, (29) is satisfied for all steps having prices *higher* than b, so that $\sum_{q=1}^{\min Q(b)-1} \frac{d\Pi_{i,q}(b_{i,q}, v_{i,q})}{db_{i,q}} = 0$. Thus, (i) implies $\sum_{q\le \max Q(b)} \frac{d\Pi_{i,q}(b_{i,q}, v_{i,q})}{db_{i,q}} > 0$, a violation of (29) for step $Q(b)$. Similarly, (ii) implies $\sum_{y\le q} \frac{d\Pi_{i,y}(b_{i,q}, v_{i,y})}{db_{i,q}} > 0$ for some quantity q, a violation of (28). This completes the contradiction. □

A.5. Proof of Proposition 1.

Proof. For all $b_2 \in [0,1]$, observe that: $w_{2,1}(b_2) = \Pr(b_2 > b_{1,2}) = b_2(2 - b_2)$ and $w'_{2,1}(b_2) = 2 - 2b_2$; $w_{2,2}(b_2) = \Pr(b_2 > b_{1,1}) = b_2^2$ and $w'_{2,2}(b_2) = 2b_2$; and $p_{2,2}(b_2) = \Pr(b_{1,1} > b_2 > b_{1,2}) = 2b_2(1 - b_2)$. By (26), $\mathbf{v}_2^{*,U}(b_2, b_2) = (b_2, 1)$ for all $b_2 \in [0,1]$. Thus,

$$V_i^{\text{FOC}}(b_2, b_2) = \{(b_2, 1) + c_1(-2b_2, 2 - 2b_2) \text{ for } c_1 \ge 0\}.$$

We must also check whether bidder 2 has any profitable global deviations.

Second-order conditions. First of all, it must be that $\Pi_{2,1}(b_{2,1}, v_{2,1}) + \Pi_{2,2}(b_{2,2}, v_{2,2})$ is weakly concave, otherwise bidder 2 must prefer to either raise or lower both unit-bids from b_2. For all $b_2 \in [0,1]$ routine calculations show $E[b_{1,1} | b_2 > b_{1,1}] = 2b_2/3$, $E[b_{1,2} | b_2 > b_{1,2}] = \frac{b_2^2(1 - 2b_2/3)}{b_2(2 - b_2)}$, $E[b_{1,2} | b_{1,1} > b_2 > b_{1,2}] = b_2/2$, and $E[b_{1,1} - b_{1,2} | b_2 > b_{1,1}] = b_2/3$. Substituting into (24) and simplifying yields,

$$\Pi_{2,1}(b_{2,1}, v_{2,1}) = \left(v_{2,1} - \frac{b_{2,1}^2(1 - 2b_{2,1}/3)}{b_{2,1}(2 - b_{2,1})} \right) b_{2,1}(2 - b_{2,1}) = b_{2,1}(2 - b_{2,1})v_{2,1} - b_{2,1}^2(1 - 2b_{2,1}/3)$$

$$\frac{d^2\Pi_{2,1}(b_{2,1}, v_{2,1})}{db_{2,1}^2} = 4b_{2,1} - 2(1 + v_{2,1}) \quad (56)$$

$$\Pi_{2,2}(b_{2,2}, v_{2,2}) = (v_{2,2} - 2b_{2,2}/3)b_{2,2}^2 - (b_{2,2}/3)b_{2,2}^2 - (b_{2,2} - b_{2,2}/2)2b_{2,2}(1 - b_{2,2}) = b_{2,2}^2(v_{2,2} - 1)$$

$$\frac{d^2\Pi_{2,2}(b_{2,2}, v_{2,2})}{db_{2,1}^2} = 2(v_{2,2} - 1). \quad (57)$$

The second-order condition that $\frac{d^2 \Pi_{2,1}(b_{2,1}, b_2 - c_1 2b_2)}{db_{2,1}^2} + \frac{d^2 \Pi_{2,2}(b_{2,2}, 1 + c_1(2-2b_2))}{db_{2,1}^2} \leq 0$ is satisfied for all $c \in [0, (1-b_2)/2]$. Thus, $V_i^{BR}(b_2, b_2) \subset \{(b_2, 1) + c_1(-2b_2, 2-2b_2)$ for $c_1 \in [0, (1-b_2)/2]\}$. Finally, it is possible to show (checking by hand – details omitted) that bidder i has no profitable global deviations given any of these values, i.e. $V_i^{BR}(b_2, b_2) = \{(b_2, 1) + c_1(-2b_2, 2-2b_2)$ for $c_1 \in [0, (1-b_2)/2]\}$. Note that $(b_2, 1) + (1-b_2)/2(-2b_2, 2-2b_2) = (b_2^2, 1-b_2)$. This completes the proof. □

References

Anwar, A., 2007. Equilibria in multi-unit discriminatory auctions. B.E. Journal of Theoretical Economics: Topics 7 (1).

Athey, S., Haile, P., 2005. Nonparametric approaches to auctions. In: Handbook of Econometrics, vol. 6 (in press).

Ausubel, L., Cramton, P., 1998. Demand reduction and inefficiency in multi-unit auctions. Working Paper. Maryland.

Chapman, J., McAdams, D., Paarsch, H., 2006. Bounding best response violations in discriminatory auctions with private values. Mimeo. Iowa.

Chapman, J., McAdams, D., Paarsch, H., 2007. Bounding revenue comparisons across multi-unit auction formats under epsilon-best response. American Economic Review: Papers and Proceedings 97 (2), 455–458.

Fevrier, P., Preget, R., Visser, M., 2001. Econometrics of share auctions. Working Paper. CREST.

Guerre, E., Perrigne, I., Vuong, Q., 2000. Optimal nonparametric estimation of first-price auctions. Econometrica 68 (3), 525–574.

Haile, P., Tamer, E., 2003. Inference with an incomplete model of english auctions. Journal of Political Economy 11 (1), 1–51.

Hortacsu, A., 2001. Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the turkish treasury auction market. Working Paper. University of Chicago.

Hortacsu, A., McAdams, D., 2008. Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the turkish treasury auction market. Working Paper.

Hortacsu, A., Puller, S., 2008. Understanding strategic bidding in multi-unit auctions: A case study of the Texas electricity spot market. RAND Journal of Economics 39 (1), 86–114.

Kang, B.-S., Puller, S.L., 2008. The effect of auction format on efficiency and revenue in divisible goods auctions: A test using Korean treasury auctions. Journal of Industrial Economics 56 (2) 290–332.

Kastl, J., 2005. Discrete bids and empirical inference in divisible good auctions. Working Paper. Northwestern.

Lehmann, E., Casella, G., 1998. Theory of Point Estimation, 2nd ed. Springer, New York.

Manski, C., 1995. Identification Problems in the Social Sciences. Harvard University Press.

McAdams, D., 2006. Monotone equilibrium in multi-unit auctions. Review of Economic Studies 73 (4), 1039–1056.

McAdams, D., 2007. Partial identification and testable restrictions in multi-unit auctions. Working Paper. MIT.

Miller, R., 1981. Simultaneous Statistical Inference, 2nd ed. Springer, New York.