

# COMPLEX MONOPOLES I: THE HAYDYS MONOPOLE EQUATION

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**ABSTRACT.** We study complexified Bogomolny monopoles using the complex linear extension of the Hodge star operator, these monopoles can be interpreted as solutions to the Bogomolny equation with a complex gauge group. Alternatively, these equations can be obtained from dimensional reduction of the Haydys instanton equations to 3 dimensions, thus we call them Haydys monopoles.

We find that (under mild hypotheses) the smooth locus of the moduli space of finite energy Haydys monopoles on  $\mathbb{R}^3$  is a hyperkähler manifold in 3-different ways, which contains the ordinary Bogomolny moduli space as a complex Lagrangian submanifold—an (ABA)-brane—with respect to any of these structures. Moreover, using a gluing construction we find an open neighborhood of the normal bundle of this submanifold which is modeled on a neighborhood of the tangent bundle to the Bogomolny moduli space. This is analogous to the case of Higgs bundles over a Riemann surface, where the (co)tangent bundle of holomorphic bundles canonically embeds into the Hitchin moduli space.

These results contrast immensely with the case of finite energy Kapustin–Witten monopoles for which we show a vanishing theorem in the second paper of this series [11]. Both papers in this series are self contained and can be read independently.

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## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Preparation and motivation.** Let  $(M, g)$  be a Riemannian 3-manifold, and  $\Lambda^*M$  its exterior algebra bundle. For any orthogonal vector bundle  $E \rightarrow M$  the Hodge star operator extends to  $E$ -valued differential forms yielding a map  $*$  :  $\Lambda^*M \otimes E \rightarrow \Lambda^{3-*}M \otimes E$ . Fix a principal  $G$ -bundle  $P \rightarrow M$ , where  $G$  is a compact Lie group. A smooth pair  $(\nabla, \Phi)$  consisting of a  $G$ -connection on  $P$  and a section of  $\mathfrak{g}_P = \text{ad}(P)$  (equipped with a  $G$ -invariant inner product), is called a *Bogomolny monopole* if

$$*F_\nabla = d_\nabla\Phi. \tag{1.1}$$

In the situation when  $M = \mathbb{R}^3$  equipped with the Euclidean metric and  $G = \text{SU}(2)$  several things are known about solutions to this equation. For instance, up to the action of the automorphisms of  $P$ , the (finite energy) Bogomolny monopoles form a smooth noncompact moduli space. This can be equipped with the canonical  $L^2$ -metric which turns out to be complete and hyperkähler. For higher rank structure groups, for instance when  $G = \text{SU}(N)$  with  $N > 2$ , less is known, but in many cases (cf. Hypothesis 3.3 and Remark 3.4 later), analogous results hold true. In particular, when the moduli space is smooth at a Bogomolny monopole  $m = (\nabla, \Phi)$  any (gauge fixed) tangent vector  $v = (a, \Psi)$  at  $m$  satisfies the linearized Bogomolny monopole equations:

$$*d_\nabla a - d_\nabla\Psi - [a, \Phi] = 0, \tag{1.2a}$$

$$d_\nabla^*a + [\Psi, \Phi] = 0, \tag{1.2b}$$

with the second equation arising from requiring the tangent vector to be orthogonal to the slice cut out by the action of the gauge group at  $m$ . Moreover, the formal  $L^2$  dual equations

$$*d_\nabla a + d_\nabla\Psi + [a, \Phi] = 0, \tag{1.3a}$$

$$d_\nabla^*a + [\Psi, \Phi] = 0, \tag{1.3b}$$

have no solutions in  $L^2$ , at least under certain standard hypotheses; again, see Hypothesis 3.3. This is in fact the reason why the Implicit Function Theorem can be used to show smoothness of the moduli space of finite energy Bogomolny.

Now we “complexify” the Bogomolny equation (1.1) by considering  $\Lambda_{\mathbb{C}}^*M = \Lambda^*M \otimes_{\mathbb{R}} \mathbb{C}$ ,  $G_{\mathbb{C}}$  the complexification of  $G$ , and  $P_{\mathbb{C}} = P \times_G G_{\mathbb{C}}$  which is the principal  $G_{\mathbb{C}}$ -bundle associated with the standard conjugation action of  $G$  on  $G_{\mathbb{C}}$ . The Hodge star operator  $*$  may now be extended in two inequivalent ways to  $\Lambda_{\mathbb{C}}^*M \otimes \mathfrak{g}_P \simeq \Lambda^*M \otimes \mathfrak{g}_{P_{\mathbb{C}}}$ . This may be either as a complex linear operator, which we still denote  $*$ , or as a conjugate linear one, which we

denote by  $\bar{*}$ .

Depending on which such extension one uses we obtain two different complex monopole equations. In this second paper we shall only consider one of these which is made using  $*$ . Let  $(\mathbf{A}, \Upsilon)$  respectively be a connection on  $P_{\mathbb{C}}$  and a section of  $\mathfrak{g}_{P_{\mathbb{C}}}$ . Then, we have the following complex monopole equation

$$*F_{\mathbf{A}} = d_{\mathbf{A}}\Upsilon, \quad (1.4)$$

**Remark 1.1.** *The equation obtained using  $\bar{*}$  instead is given by*

$$\bar{*}F_{\mathbf{A}} = d_{\mathbf{A}}\Upsilon, \quad (1.5)$$

*and studied in the second paper in this series [11].*

Let the real gauge group be  $\mathcal{G} = \text{Aut}(P)$  and the complex one by  $\mathcal{G}_{\mathbb{C}} = \text{Aut}(P_{\mathbb{C}})$ . Both complex monopole equations (1.4) and (1.5) are invariant under the usual action of  $\mathcal{G}_{\mathbb{C}}$ . In order to work only modulo the action of  $\mathcal{G}$  we shall proceed as follows. Observe that  $\mathbf{A}$  can be uniquely written as  $\mathbf{A} = \nabla + ia$ , with  $\nabla$  a connection on  $P$  and  $a \in \Omega^1 \otimes \mathfrak{g}_P$ . Similarly  $\Upsilon = \frac{1}{\sqrt{2}}(\Phi + i\Psi)$ , with  $\Phi, \Psi \in \Omega^0 \otimes \mathfrak{g}_P$ . A standard procedure in gauge theory “breaks down the gauge symmetry” to the real gauge group  $\mathcal{G}$  by imposing an extra equation of the form

$$d_{\nabla}^*a = i[\Upsilon, \bar{\Upsilon}] \Leftrightarrow d_{\nabla}^*a + [\Psi, \Phi] = 0, \quad (1.6)$$

which is a Coulomb type gauge fixing condition. In particular, this makes the full system of PDE’s elliptic modulo the action of  $\mathcal{G}$ .

**Remark 1.2.** *Alternatively, this extra equation may be motivated by comparison with the Kempf–Ness Theorem in finite dimensional situations. Indeed, equation (1.6) may be interpreted as a moment map equation for an Hamiltonian action of  $\mathcal{G}$  on the space of quadruples  $(\nabla, a, \Psi, \Phi)$  equipped with a natural  $L^2$ -symplectic structure.*

One other point of interest in equations (1.4) to (1.6) is that they may be obtained from dimensional reductions of the instanton equations of Haydys (cf. [4]) and Kapustin–Witten (cf. [9]) respectively. In this paper we shall focus on the first of these. For more on the later we shall refer the reader to second paper in this series [11]. While the Kapustin–Witten equations, and their dimensional reductions, have been dragging an increased interest from the mathematical community (see, for example, [6, 7, 10, 15–19]) the Haydys equation has remained less explored. However, it was pointed out in [4] that its moduli space carries interesting geometric structures which as we shall see have a shadow in the dimensional

reduction we consider here cf. Section 4.2. For completeness and motivation we included the computations corresponding in section 2.

In order to establish notation, recall that the wedge product of two  $\mathfrak{g}_{p^x}$ -valued forms  $a = \sum_{|I|=p} a_I dx^I$  and  $b = \sum_{|J|=q} b_J dx^J$  is given by

$$[a \wedge b] = \sum_{\substack{|I|=p \\ |J|=q}} [a_I, b_J] dx^I \wedge dx^J,$$

and satisfies  $[a \wedge b] = (-1)^{pq+1}[b \wedge a]$ . Using this, a simple computation shows that equations (1.4) and (1.6) are equivalent to

$$*F_{\nabla} - d_{\nabla}\Phi - \frac{1}{2} * [a \wedge a] + [a, \Psi] = 0, \quad (1.7a)$$

$$*d_{\nabla}a - d_{\nabla}\Psi - [a, \Phi] = 0, \quad (1.7b)$$

$$d_{\nabla}^*a + [\Psi, \Phi] = 0. \quad (1.7c)$$

**Remark 1.3.** *A similar computation shows that equations (1.5) and (1.6) are equivalent to*

$$*F_{\nabla} - d_{\nabla}\Phi - \frac{1}{2} * [a \wedge a] + [a, \Psi] = 0, \quad (1.8a)$$

$$*d_{\nabla}a + d_{\nabla}\Psi + [a, \Phi] = 0, \quad (1.8b)$$

$$d_{\nabla}^*a + [\Psi, \Phi] = 0. \quad (1.8c)$$

*Again we point out that these equations will not be dealt with in this paper as they are studied in the second paper of this series [11].*

Given that, as mentioned in the previous paragraph, these equations are obtained from dimensional reduction of the Haydys instanton equation, we name equations (1.7a) to (1.7c) the *Haydys monopole equations* and their solutions *Haydys monopoles*. In the same way, we shall call equations (1.8a) to (1.8c) *Kapustin–Witten monopole equations* and *Kapustin–Witten monopoles* to their solutions. Observe that both these sets of gauge theoretic equations with gauge group  $\mathcal{G}$  (rather than  $\mathcal{G}_{\mathbb{C}}$ ), are elliptic modulo its action.

Furthermore, notice that the equations (1.7a) and (1.8a) are the same, and can be seen as a quadratic (but algebraic) perturbation of the Bogomolny monopole equation (1.1). As for the second and third Haydys monopole equations (1.7b) and (1.7c), these are exactly the tangent space equations (1.2a) and (1.2b) for the Bogomolny moduli space. On the other hand, the second and third Kapustin–Witten monopole equations (1.8b) and (1.8c) are dual equations (1.3a) and (1.3b).

We shall now introduce the relevant energy functional in this complex monopole setting. Denote by  $\|\cdot\|$  the usual  $L^2$  norm for sections of any bundle over  $M$ . Given a quadruple

$(\nabla, \Phi, a, \Psi)$  as before, we define the Yang–Mills–Higgs type energy functional given by

$$\begin{aligned} \mathcal{E}(\nabla, \Phi, a, \Psi) = & \|F_\nabla\|^2 + \|\nabla a\|^2 + \|\nabla\Phi\|^2 + \|\nabla\Psi\|^2 + \frac{1}{4}\|[a \wedge a]\|^2 \\ & + \|[a, \Phi]\|^2 + \|[a, \Psi]\|^2 + \|\Psi, \Phi\|^2. \end{aligned} \quad (1.9)$$

We also point out, without proof, that up to an overall constant and for  $M$  Ricci flat, the energy (1.9) is simply a sum of the  $L^2$  norms of  $F_{\nabla+ia}$  with  $d_{\nabla+ia}(\Phi + i\Psi)$  on  $M$ .

**1.2. Main results.** Before stating our main results we recall some—by now classic—results on the moduli spaces of Bogomolny monopoles. Let  $M = \mathbb{R}^3$  with the Euclidean metric,  $G$  a compact Lie group, and denote by  $\mathcal{M}_B$  and  $\mathcal{M}_H$  the moduli spaces of Bogomolny and Haydys monopoles. Now  $\mathcal{M}_B$  canonically embeds into  $\mathcal{M}_H$ , as the *real* solutions (that is, with vanishing imaginary parts  $a = 0 = \Psi$ ). Under a certain genericity hypothesis called *maximal symmetry breaking* (see Hypothesis 3.3 and Remark 3.4), the moduli space  $\mathcal{M}_B$  is a smooth and complete hyperkähler manifold. Its tangent bundle  $T\mathcal{M}_B$  is well defined (as a smooth manifold) and  $\mathcal{M}_B$  also embeds into  $T\mathcal{M}_B$ , as the zero section. Under these assumptions, our first main result can be stated as follows:

**Main Theorem 1** (Existence theorem for Haydys monopoles). *There are open neighborhoods  $N_B$  and  $N_H$  of the zero section  $\mathcal{M}_B \subset T\mathcal{M}_B$  and of the real solutions  $\mathcal{M}_B \subset \mathcal{M}_H$  respectively, and a diffeomorphism*

$$\mathcal{F} : N_B \rightarrow N_H.$$

*In particular, there exists finite energy Haydys monopoles that are not Bogomolny monopoles.*

**Remark 1.4.** *This situation contrasts with that of finite energy Kapustin–Witten monopoles on  $\mathbb{R}^3$ . Indeed, while we find that many Haydys monopoles exist which are not simply Bogomolny monopoles, we proved in [11] that any finite energy Kapustin–Witten monopole on  $M = \mathbb{R}^3$  must actually be a Bogomolny monopole.*

The moduli space of finite energy solutions to the Haydys monopole equation on  $M = \mathbb{R}^3$  inherits some interesting geometric structures mirroring the hyperkähler structure on the moduli space of Bogomolny monopoles  $\mathcal{M}_B$ . Before we state our second main theorem, let us remark, that the 4<sup>th</sup> Clifford algebra  $\text{Cl}(4)$  is isomorphic to the algebra of 2-by-2 quaternionic matrices,  $M_{2 \times 2}(\mathbb{H})$ . Now our second main theorem states the following:

**Main Theorem 2.** *The following assertions hold:*

- (a)  $\mathcal{M}_H$  carries an  $\text{Cl}(4)$ -structure. Equivalently, there are 3 different hyperkähler structures,  $(I_1, I_2, I_3)$ ,  $(J_1, J_2, J_3)$ , and  $(K_1, K_2, K_3)$  each of them compatible with the  $L^2$ -metric. Furthermore,  $I_1 = J_1 = K_1$ , and  $I_2, J_2$ , and  $K_2$  pairwise anti-commute. Thus  $e_1 = I_1$ ,  $e_2 = I_2$ ,  $e_3 = J_2$ , and  $e_4 = K_4$  are algebraically independent, anti-commuting complex structures (that generate all other complex structure), thus giving the tangent bundle the structure of a  $\text{Cl}(4)$ -module.
- (b)  $\mathcal{M}_B \hookrightarrow \mathcal{M}_H$  is a complex Lagrangian submanifold with respect to the either of the 3 hyperkähler structures in part (a). More precisely it is complex with respect to complex structures  $I_2, J_2$ , and  $K_2$ , and Lagrangian with respect to the Kähler structures induced by the other complex structures.
- (c) The complex structures  $I_2, J_2$ , and  $K_2$ , together with the  $L^2$ -metric, restrict to  $\mathcal{M}_B \hookrightarrow \mathcal{M}_H$  equipping it with a well defined hyperkähler structure, which is isomorphic to its canonical  $L^2$ -hyperkähler structure.

**Remark 1.5.** In the terminology of [9], part (b) of this theorem is equivalent to saying that  $\mathcal{M}_B \hookrightarrow \mathcal{M}_H$  is a (ABA)-brane with respect to either of the 3 hyperkähler structures in part (a).

**1.3. Organization.** In Section 2, we prove that the Haydys monopole equations (1.7a) to (1.7c) are the dimensional reduction of the 4-dimensional Haydys equation (as in [4]). In Section 3, after introducing the necessary tools we prove Main Theorem 1 whose proof relies on a use of the Banach space contraction mapping principle. In Section 4.2 we study the geometry of the Haydys monopole moduli space, and prove Main Theorem 2.

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## 2. DIMENSIONAL REDUCTION

In this section we prove the dimensional reductions of the 4-dimensional Haydys equation yield the Haydys monopole equations (1.7a) to (1.7c).

Let us start by recalling the notion of complex (anti-)self-duality in dimension 4. Given an oriented, Riemannian 4-manifold  $(X, g_4)$ , let  $\Lambda_{\mathbb{C}}^* X$  be the complexification of its exterior algebra bundle, and let  $*_4$  and  $\bar{*}_4$  be the complex linear and conjugate linear extensions of the Hodge star operator on  $\Lambda_{\mathbb{C}}^* X$ , respectively. Both  $*_4$  and  $\bar{*}_4$  square to the identity on  $\Lambda_{\mathbb{C}}^2 X$

and hence either can be used to define (anti-)self-dual complex 2-forms. In this paper we consider the complex linear case, that is when (anti-)self-duality is defined using  $*_4$ .

Let now  $G$  be a compact Lie group, and  $G_{\mathbb{C}}$  its complex form. Let  $P^X$  be a principal  $G$ -bundle over  $X$ , and define the complexified  $G_{\mathbb{C}}$ -bundle  $P_{\mathbb{C}}^X = P^X \times_G G_{\mathbb{C}}$  as being that associated with respect to the standard conjugation action of  $G$  on  $G_{\mathbb{C}}$ . Let  $\mathfrak{g}_{P^X}$  and  $\mathfrak{g}_{P_{\mathbb{C}}^X}$  be the corresponding adjoint bundles. Note that  $\mathfrak{g}_{P_{\mathbb{C}}^X} \simeq \mathfrak{g}_{P^X} \otimes_{\mathbb{R}} \mathbb{C}$ , and thus

$$\Lambda_{\mathbb{C}}^k X \otimes_{\mathbb{R}} \mathfrak{g}_{P^X} \simeq \Lambda_{\mathbb{C}}^k X \otimes_{\mathbb{C}} \mathfrak{g}_{P_{\mathbb{C}}^X} \simeq (\Lambda^k X \otimes \mathfrak{g}_{P^X}) \oplus i(\Lambda^k X \otimes \mathfrak{g}_{P^X}).$$

Any ‘‘complex’’ connection  $\mathbf{A}$  on  $P_{\mathbb{C}}^X$  decomposes as  $\mathbf{A} = A + iB$ , where  $A$  is a ‘‘real’’ connection on  $P^X$  and  $B \in \Omega^1(X, \mathfrak{g}_{P^X})$ . Then we can decompose the curvature  $F_{\mathbf{A}}$  of  $\mathbf{A}$  as follows

$$F_{\mathbf{A}} = \operatorname{Re}(F_{\mathbf{A}}) + i \operatorname{Im}(F_{\mathbf{A}}).$$

and thus

$$\operatorname{Re}(F_{\mathbf{A}}) = F_A - \frac{1}{2}[B \wedge B],$$

$$\operatorname{Im}(F_{\mathbf{A}}) = d_A B.$$

Let the  $\pm$  superscripts denote the pointwise orthogonal projection from  $\Lambda^2 X \otimes \mathfrak{g}_{P^X}$  onto  $\Lambda_{\pm}^2 X \otimes \mathfrak{g}_{P^X}$ . Now we can say  $\mathbf{A}$  is anti-self-dual with respect to  $*_4$  if

$$*_4 F_{\mathbf{A}} = -F_{\mathbf{A}} \quad \Leftrightarrow \quad \operatorname{Re}(F_{\mathbf{A}})^+ = 0 = \operatorname{Im}(F_{\mathbf{A}})^+. \quad (2.1)$$

Note that when  $\mathbf{A}$  is an  $G$ -connection, that is when  $B = 0$ , then both equations (2.1) reduce to the classical anti-self-duality (instanton) equation on  $X$ .

Supplementing equation (2.1) with  $d_A^* B = 0$  one gets the Haydys equation; cf. [4, Section 4.2].

Assume now that  $X = \mathbb{S}^1 \times M$ , where  $M$  is a Riemannian 3-manifold with metric  $g$ , and  $g_X$  is the product metric. Furthermore, let the orientation of  $X$  be given by the product orientation. The group of orientation preserving isometries of  $X$  has a normal subgroup, which is isomorphic to  $\operatorname{SO}(2)$ , that acts on  $\mathbb{S}^1$  as rotations. Thus, one can look for  $\operatorname{SO}(2)$ -equivariant (‘‘static’’) solutions of the Haydys equation (2.1). It is easy to see, that if  $\mathbf{A}$  is an  $\operatorname{SO}(2)$ -equivariant connection on  $X$ , then there exists a principal  $G$ -bundle  $P \rightarrow M$ , together with an isomorphism between its pullback to  $X$  and  $P^X$ , and a quadruple  $(\nabla, \Phi, a, \Psi)$ , such that  $\nabla$  is a connection on  $P$ ,  $a \in \Omega^1(M, \mathfrak{g}_P)$ , and  $\Phi, \Psi \in \Omega^0(M, \mathfrak{g}_P)$ , with the property that (omitting pullbacks and the isomorphism)

$$\mathbf{A} = \nabla + \Phi dt + i(a + \Psi dt). \quad (2.2)$$

Let  $*$  be the Hodge star operator of  $(M, g)$ . Then we have the following lemma:

**Lemma 2.1** (Dimensional reduction of the Haydys equation). *The complex connection  $\mathbf{A}$  in (2.2) solves the Haydys equation (2.1) and  $d_A^* B = 0$ , if and only if equations (1.7a) to (1.7c) hold.*

*Proof.* Let  $\mathbf{A} = A + iB$ . Recall that the Haydys equations for  $\mathbf{A}$  are

$$\operatorname{Re}(F_{\mathbf{A}})^+ = 0, \quad (2.3a)$$

$$\operatorname{Im}(F_{\mathbf{A}})^+ = 0, \quad (2.3b)$$

$$d_A^{*4} B = 0,$$

where

$$\operatorname{Re}(F_{\mathbf{A}}) = F_A - \frac{1}{2}[B \wedge B], \quad \operatorname{Im}(F_{\mathbf{A}}) = d_A B.$$

Now we further assume that  $\mathbf{A}$  has the form

$$A = \nabla + \Phi dt, \quad B = a + \Psi dt,$$

and the quadrupole  $(\nabla, \Phi, a, \Psi)$  is pulled back from  $M$ . Straightforward computations yield

$$F_A = F_{\nabla} + d_{\nabla} \Phi \wedge dt, \quad \frac{1}{2}[B \wedge B] = \frac{1}{2}[a \wedge a] + [a, \Psi] \wedge dt,$$

thus equation (2.3a) is equivalent to

$$* F_{\nabla} - d_{\nabla} \Phi - \frac{1}{2} * [a \wedge a] + [a, \Psi] = 0,$$

proving equations (1.7a) and (1.8a).

We also have

$$d_A B = d_{\nabla} a + (d_{\nabla} \Psi + [a, \Phi]) \wedge dt,$$

thus for equation (2.3b) we have

$$\operatorname{Im}(F_{\mathbf{A}})^+ = (d_A B)^+ = \frac{1}{2}(d_{\nabla} a - *(d_{\nabla} \Psi + [a, \Phi])) + \frac{1}{2}(*d_{\nabla} a - (d_{\nabla} \Psi + [a, \Phi])) \wedge dt.$$

Thus  $\operatorname{Im}(F_{\mathbf{A}})^+ = 0$  is equivalent to

$$* d_{\nabla} a - d_{\nabla} \Psi - [a, \Phi] = 0,$$

which proves equation (1.7b).

Finally, we have

$$d_A^{*4} B = d_{\nabla + \Phi dt}^{*4} (a + \Psi dt) = d_{\nabla}^* a - [\Phi, \Psi] = d_{\nabla}^* a + [\Psi, \Phi],$$

which proves equation (1.7c). This completes the proof.  $\square$

Before proceeding, let us point out a couple of other possible ways one can interpret the Haydys monopole equations.

**Remark 2.2** (Reduction of the Vafa–Witten equations). *The Vafa–Witten equation are another set of 4-dimensional, gauge theoretic PDE’s; cf. [5, Section 4.1] for example. We remark, without proof, that the similar reduction of the Vafa–Witten equations also results in equations (1.7a) to (1.7c). A simple way to see this is to observe that  $\Lambda^1 M \simeq \Lambda_+^2 X$  via the map  $b \mapsto \frac{1}{2}(dt \wedge b + *b)$ .*

**Remark 2.3** (Reduction of the split  $G_2$ -monopole equation). *The Haydys monopole equations (1.7a) to (1.7c) can be obtained as the reduction of the  $G_2$ -monopole equations on  $\mathbb{R}^7$  equipped with the split  $G_2$ -structure of signature (3, 4). See, for example, [12, Section 2].*

### 3. SOLVING THE HAYDYS MONOPOLE EQUATION

The goal of this section is to construct solutions to the Haydys monopole equations (1.7a) to (1.7c) on  $M = \mathbb{R}^3$  and, more concretely, to prove Main Theorem 1. We achieve that as follows: First, we recall the linearization of the Bogomolny equation (1.1) in Section 3.1, then we introduce the relevant function spaces to be used in Section 3.2, and prove a gap theorem for the adjoint of the linearization in Section 3.3. In Section 3.4, we prove a multiplication property of the function spaces introduced in Section 3.2. In Section 3.5, we reinterpret the the Haydys monopole equations (1.7a) to (1.7c) which we supplement in Section 3.6 with the gauge fixing condition. These new set of equations can be viewed as fixed point equation, which we solve using Banach Fixed Point Theorem in Section 3.7. Finally, Section 4.1 contains a computation of the dimension of the moduli space, which reveals that our construction yields, in fact, an open subset of the moduli space.

**3.1. The Bogomolny monopole equation and its linearization.** Let  $M = \mathbb{R}^3$  and  $m_0 = (\nabla_0, \Phi_0) \in \mathcal{A} \times \Omega^0(M, \mathfrak{g})$  be a pair satisfying the Bogomolny monopole equation (1.1), and the finite energy condition, that is  $|F_{\nabla_0}| = |d_{\nabla_0} \Phi_0| \in L^2(\mathbb{R}^3)$ . Furthermore, we make the following two hypotheses on  $m_0$ , which are standard in the literature:

**Hypothesis 3.1.** *There exist a unitary connection,  $\nabla^\infty$ , on  $P_\infty = P|_{\mathbb{S}_\infty^2}$ , and smooth sections  $\Phi_\infty, \kappa \in \Omega^0(\mathbb{S}_\infty^2, \text{ad}(P_\infty))$ , such that  $\Phi_0$  and  $d_{\nabla_0} \Phi_0$  have the asymptotic expansions*

$$\begin{aligned}\Phi_0 &= \pi_\infty^* \Phi_\infty - \frac{1}{2r} \pi_\infty^* \kappa + O(r^{-2}), \\ d_{\nabla_0} \Phi_0 &= \frac{1}{2r^2} \pi_\infty^* \kappa \otimes dr + O(r^{-3}).\end{aligned}$$

Furthermore, we have

$$\nabla^\infty \Phi_\infty = 0, \quad F_{\nabla^\infty} = \frac{1}{2} \kappa \otimes \text{vol}_{\mathbb{S}^2}, \quad [\Phi_\infty, \kappa] = 0.$$

**Remark 3.2.** *Hypothesis 3.1 has been proven in some cases (see [8, Theorem 10.5] for the first example) and, conjecturally, holds for all finite energy monopoles on  $\mathbb{R}^3$ ; cf. [8, Theorem 18.4 & Corollary 18.5]. The authors of this paper, together with Benoit Charbonneau, are currently working on a proof of this conjecture.*

We furthermore impose a more technical hypothesis, which is crucial in the proof of Main Theorem 1.

**Hypothesis 3.3.** *The multiplicity of every eigenvalue of  $\Phi_\infty$  (at every point) is one. A Bogomolny monopole satisfying this hypothesis is said to have maximal symmetry breaking.*

**Remark 3.4.** *It is easy to see that a monopole has maximal symmetry breaking exactly if  $\ker(\text{ad}(\Phi_\infty))$  is Abelian. Note also that any nonflat monopole with structure group  $G = \text{SU}(2)$  has to have maximal symmetry breaking.*

*There exists monopoles without maximal symmetry breaking; see [2, 3].*

*Furthermore, we mention here, without proof, that by adapting the arguments in [1], it is possible to prove that maximal symmetry breaking implies Hypothesis 3.1. The proof of this—among more general claims—is currently being completed by the authors; see Remark 3.2.*

*Finally, we conjecture that Main Theorem 1 holds for monopoles with nonmaximal symmetry breaking as well.*

**Definition 3.5.** *For any  $v_0 = (a_0, \Psi_0) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g})$ , let*

$$d_2(v_0) = *d_{\nabla_0} a_0 - d_{\nabla_0} \Psi_0 - [a_0, \Phi_0],$$

$$d_1^*(v_0) = d_{\nabla_0}^* a_0 - [\Phi_0, \Psi_0],$$

*and  $D = d_2 \oplus d_1^*$ , let  $D^*$  be the formal adjoint of  $D$ . A pair  $v_0 = (a_0, \Psi_0) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g})$  is called a tangent vector to the Bogomolny monopole moduli space at  $m_0$  if it lies in  $\ker_{L^2_1}(D)$ .*

*Moreover, for any  $c_0$ , if we define*

$$(d_{\nabla_0} \Phi_0)^W(c_0) = (*[a_0 \wedge d_{\nabla_0} \Phi_0] - [d_{\nabla_0} \Phi_0, \Psi_0], [\langle d_{\nabla_0} \Phi_0, a_0 \rangle]) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g}).$$

It is easy to see that

$$DD^*c = \nabla_0^* \nabla_0 c - [[c, \Phi_0], \Phi_0],$$

$$D^*Dc = DD^*c + 2(d_{\nabla_0} \Phi_0)^W(c).$$

**Remark 3.6.** *The operator  $D$  is the linearization of the Bogomolny monopole equation (1.1) together with the standard Coulomb type gauge fixing condition  $d_{\nabla_0}^* a_0 = [\Phi_0, \Psi_0]$ . Thus, when  $D : L_1^2 \rightarrow L^2$  is surjective the implicit function theorem can be used to prove that the Bogomolny monopole moduli space is smooth and its tangent space at  $m_0$  can be identified with  $\ker_{L_1^2}(D)$ .*

**3.2. Function spaces.** Now we introduce the various functions spaces that are used in the proof of Main Theorem 1.

**Definition 3.7.** *Let  $\|\cdot\|$  denote  $L^2$ -norms and  $\rho = (1 + |x|^2)^{1/2}$ . Define the Hilbert spaces  $\mathcal{H}_k$  (with  $k = 1, 2$ , or  $3$ ) as the norm-completions of  $C_0^\infty(\mathbb{R}^3, (\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g})$  via the norms*

$$\begin{aligned} \|c\|_{\mathcal{H}_0}^2 &= \|c\|^2, \\ \|c\|_{\mathcal{H}_1}^2 &= \|\nabla_0 c\|^2 + \|\rho^{-1} c\|^2 + \|[\Phi_0, c]\|^2, \\ \|c\|_{\mathcal{H}_2}^2 &= \|\nabla_0^2 c\|^2 + \|\rho^{-1} \nabla_0 c\|^2 + \|[\Phi_0, \nabla_0 c]\|^2 + \|[\Phi_0, [\Phi_0, c]]\|^2 + \|[\Phi_0, \rho^{-1} c]\|^2 + \|\rho^{-2} c\|^2. \end{aligned}$$

The corresponding inner products are denoted by  $\langle \cdot, \cdot \rangle_{\mathcal{H}_k}$ .

**Lemma 3.8** (Sobolev and Hardy's inequalities). *Let  $c \in \mathcal{H}_1$ . Then the Sobolev inequality says*

$$\|\nabla_0 c\| \geq \frac{3}{2} \|c\|_{L^6(\mathbb{R}^3)},$$

and Hardy's inequality says

$$\|\nabla_0 c\| \geq \frac{1}{2} \|\rho^{-1} c\|_{L^2(\mathbb{R}^3)}.$$

**Remark 3.9.** *Let the symbol  $\sim$  denote norm equivalence. Then using Hardy's inequality and Hypothesis 3.1 gives*

$$\begin{aligned} \|c\|_{\mathcal{H}_1}^2 &\sim \|\nabla_0 c\|^2 + \|[\Phi_0, c]\|^2, \\ \|c\|_{\mathcal{H}_2}^2 &\sim \|\nabla_0^2 c\|^2 + \|[\Phi_0, \nabla_0 c]\|^2 + \|\rho^{-2} c\|^2. \end{aligned}$$

**Lemma 3.10** (First order inequalities, partly in [14, Lemma 6.8]). *There are positive constants  $R, C_{m_0}$ , depending only on the monopole  $m_0$ , such that for all  $c_0 \in \mathcal{H}_1$*

$$\begin{aligned} \|c_0\|_{\mathcal{H}_1}^2 &\leq C_{m_0} \|D^* c_0\|_{\mathcal{H}_0}^2, \\ \|c_0\|_{\mathcal{H}_1}^2 &\leq C_{m_0} \left( \|D c_0\|_{\mathcal{H}_0}^2 + \|c_0\|_{L^2(B_R(0))}^2 \right), \\ \|c_0\|_{\mathcal{H}_2}^2 &\leq C_{m_0} \left( \|D^* c_0\|_{\mathcal{H}_1}^2 + \|c_0\|_{L^2(B_R(0))}^2 \right), \\ \|c_0\|_{\mathcal{H}_2}^2 &\leq C_{m_0} \left( \|D c_0\|_{\mathcal{H}_1}^2 + \|c_0\|_{L^2(B_R(0))}^2 \right). \end{aligned}$$

*Proof.* We start by proving the first inequality, which follows from the Weitzenböck type formula  $DD^*c = \nabla_0^* \nabla_0 c - [[c, \Phi_0], \Phi_0]$  as follows

$$\|D^*c\|^2 = \langle c, DD^*c \rangle = \|\nabla_0 c\|^2 + \|[[\Phi_0, c]]\|^2 \sim \|c\|_{\mathcal{H}_1}^2.$$

The method for proving the second inequality is outlined in [14]. For completeness, we shall include here its proof using the strategy outlined in that reference. By Hypothesis 3.1, we have that  $\rho^2 d_{\nabla_0} \Phi_0 \in L^\infty$ , and hence, if  $c \in \mathcal{H}_1$ , then we can integrate by parts and thus

$$\begin{aligned} \|Dc\|_{\mathcal{H}_0}^2 &= \langle c, D^*Dc \rangle = \langle c, \nabla_0^* \nabla_0 c - [[c, \Phi_0], \Phi_0] + 2(d_{\nabla_0} \Phi_0)^W c \rangle \\ &= \|\nabla_0 c\|^2 + \|[[\Phi_0, c]]\|^2 + 2\langle c, (d_{\nabla_0} \Phi_0)^W c \rangle \\ &= \|c\|_{\mathcal{H}_1}^2 + 2\langle c, (d_{\nabla_0} \Phi_0)^W c \rangle. \end{aligned} \tag{3.1}$$

Now, for  $R \gg 1$  let  $\chi_R$  be a smooth bump function supported in  $B_R(0)$  and equal to 1 in  $B_{R-1}(0)$ . Then,

$$\langle c, (d_{\nabla_0} \Phi_0)^W c \rangle = \langle \chi_R c, (d_{\nabla_0} \Phi_0)^W c \rangle + \langle (1 - \chi_R)c, (d_{\nabla_0} \Phi_0)^W c \rangle,$$

with the first terms satisfying

$$|\langle \chi_R c, (d_{\nabla_0} \Phi_0)^W c \rangle| \leq \left( \sup_{x \in B_R(0)} |d_{\nabla_0} \Phi_0(x)| \right) \|c\|_{L^2(B_R(0))}^2.$$

As for the second term, we may use the particular form of  $(d_{\nabla_0} \Phi_0)^W c$  and the Ad-invariance of the inner product to find a bilinear map  $N(\cdot, \cdot)$  so that

$$|\langle (1 - \chi_R)c, (d_{\nabla_0} \Phi_0)^W c \rangle| \lesssim \|(1 - \chi_R)d_{\nabla_0} \Phi_0\|_{L^2}^2 \|N(c, c)\|_{L^2}^2 \lesssim \|(1 - \chi_R)d_{\nabla_0} \Phi_0\|_{L^2}^2 \|c\|_{\mathcal{H}_1}^2,$$

where in the last inequality we have used Lemma 3.16 to be proven later. Now, given that  $d_{\nabla_0} \Phi_0 \in L^2$ , for any positive  $\varepsilon \ll 1$  we may find  $R = R_\varepsilon \gg 1$  so that  $\|(1 - \chi_{R_\varepsilon})d_{\nabla_0} \Phi_0\|_{L^2}^2 \leq \varepsilon$  and so

$$|\langle (1 - \chi_{R_\varepsilon})c, (d_{\nabla_0} \Phi_0)^W c \rangle| \lesssim \varepsilon \|c\|_{\mathcal{H}_1}^2.$$

Inserting these back into equation (3.1) we find

$$\|Dc\|_{\mathcal{H}_0}^2 \gtrsim (1 - \varepsilon) \|c\|_{\mathcal{H}_1}^2 - \left( \sup_{x \in B_{R_\varepsilon}(0)} |d_{\nabla_0} \Phi_0(x)| \right) \|c\|_{L^2(B_{R_\varepsilon}(0))}^2,$$

and rearranging yields the second inequality in the statement.

We now turn to proving the second set of inequalities, i.e. the last two ones. Using the first inequality just proved above we compute

$$\begin{aligned}
\|c\|_{\mathcal{H}_2}^2 &\lesssim \|\nabla_0 c\|_{\mathcal{H}_1}^2 + \|[\Phi_0, c]\|_{\mathcal{H}_1}^2 + \|\rho^{-1} c\|_{\mathcal{H}_1}^2 \\
&\lesssim \|D^* \nabla_0 c\|_{\mathcal{H}_0}^2 + \|D^* [\Phi_0, c]\|_{\mathcal{H}_0}^2 + \|D^* \rho^{-1} c\|_{\mathcal{H}_0}^2 \\
&\lesssim \|\nabla_0 D^* c - B(\nabla_0 \Phi_0, c)\|_{\mathcal{H}_0}^2 + \|[\Phi_0, D^* c] + B(\nabla_0 \Phi_0, c) + [\Phi_0, [\Phi_0, c]]\|_{\mathcal{H}_0}^2 \\
&\quad + \|\rho^{-2} c + \rho^{-1} D^* c\|_{\mathcal{H}_0}^2 \\
&\lesssim \|\nabla_0 D^* c\|_{\mathcal{H}_0}^2 + \|[\Phi_0, D^* c]\|_{\mathcal{H}_0}^2 + \|[\Phi_0, [\Phi_0, c]]\|_{\mathcal{H}_0}^2 + \|\rho^{-1} D^* c\|_{\mathcal{H}_0}^2 + \|B(\nabla_0 \Phi_0, c)\|_{\mathcal{H}_0}^2 \\
&\quad + \|\rho^{-2} c\|_{\mathcal{H}_0}^2 \\
&\lesssim \|D^* c\|_{\mathcal{H}_1}^2 + \|[\Phi_0, [\Phi_0, c]]\|_{\mathcal{H}_0}^2 + \|\rho^{-2} c\|_{\mathcal{H}_0}^2, \tag{3.2}
\end{aligned}$$

where  $B(-, -)$  denotes a bilinear operator which is algebraic, and thus continuous. Now, using the definition of the  $\mathcal{H}_1$ -norm we have

$$\|D^* c\|_{\mathcal{H}_1}^2 = \|\nabla_0 D^* c\|^2 + \|[\Phi_0, D^* c]\|^2, \tag{3.3}$$

and we consider each of these separately. For the first term we use the Hardy's inequality together with Young's inequality in the form  $2\langle D^*(\rho^{-1} c), \rho^{-2} d\rho \otimes c \rangle \leq \sqrt{2} \|D^*(\rho^{-1} c)\| + \frac{1}{\sqrt{2}} \|\rho^{-2} c\|^2$

$$\begin{aligned}
\|\nabla_0 D^* c\|^2 &\gtrsim \|\rho^{-1} D^* c\|^2 \gtrsim \|D^*(\rho^{-1} c) - \rho^{-2} d\rho \otimes c\|^2 \\
&\gtrsim (1 - \sqrt{2}) \|D^*(\rho^{-1} c)\|^2 + (1 - \frac{1}{\sqrt{2}}) \|\rho^{-2} \otimes c\|^2 \\
&\gtrsim \frac{3 - \delta - 2\sqrt{2}}{2} \|\rho^{-2} c\|^2 + \delta \|\nabla(\rho^{-1} c)\|^2 + \delta \|[\Phi_0, \rho^{-1} c]\|^2 \\
&\gtrsim \|\rho^{-2} c\|^2 + \|\nabla(\rho^{-1} c)\|^2 + \|[\Phi_0, \rho^{-1} c]\|^2,
\end{aligned}$$

for some fixed  $\delta \in (0, 3 - 2\sqrt{2})$ . As for the second term, we use Hypothesis 3.1, namely that  $d_{\nabla_0} \Phi_0 = O(\rho^{-2})$ , and an argument as that made above to control  $\langle (d_{\nabla_0} \Phi_0)^W c, c \rangle$

$$\begin{aligned}
\|[\Phi_0, D^* c]\|^2 &\gtrsim \|D^* [\Phi_0, c] - (d_{\nabla_0} \Phi_0)^W c\|^2 \\
&\gtrsim \|D^* [\Phi_0, c]\|^2 - \|(d_{\nabla_0} \Phi_0)^W c\|^2 \\
&\gtrsim \|[\Phi_0, [\Phi_0, c]]\|^2 - \langle (d_{\nabla_0} \Phi_0)^W \rho^{-1} c, \rho^{-1} c \rangle \\
&\gtrsim \|[\Phi_0, [\Phi_0, c]]\|^2 - \|c\|_{L^2(B_{R_\varepsilon})}^2 - \varepsilon \|\rho^{-1} c\|_{\mathcal{H}_1}^2.
\end{aligned}$$

We now sum these inequalities, i.e. insert them back into equation (3.3), and recall that  $\|\nabla_0(\rho^{-1}c)\|^2 + \|[\Phi_0, \rho^{-1}c]\|^2 = \|\rho^{-1}c\|_{\mathcal{H}_1}^2$  to obtain

$$\begin{aligned} \|D^*c\|_{\mathcal{H}_1}^2 &\gtrsim \|\rho^{-2}c\|^2 + \|[\Phi_0, [\Phi_0, c]]\|^2 + \|\rho^{-1}c\|_{\mathcal{H}_1}^2 - \|c\|_{L^2(B_{R\varepsilon})}^2 - \varepsilon\|\rho^{-1}c\|_{\mathcal{H}_1}^2 \\ &\gtrsim \|\rho^{-2}c\|^2 + \|[\Phi_0, [\Phi_0, c]]\|^2 + \|\rho^{-1}c\|_{\mathcal{H}_1}^2 - \|c\|_{L^2(B_{R\varepsilon})}^2, \end{aligned}$$

where we have chosen  $\varepsilon > 0$  sufficiently small so it may be absorbed. Then, rearranging we obtain

$$\|\rho^{-2}c\|^2 + \|[\Phi_0, [\Phi_0, c]]\|^2 + \|\rho^{-1}c\|_{\mathcal{H}_1}^2 \lesssim \|D^*c\|_{\mathcal{H}_1}^2 + \|c\|_{L^2(B_R)}^2,$$

for some  $R > 0$ . Then, inserting this into the inequality (3.2) we obtain

$$\|c\|_{\mathcal{H}_2}^2 \lesssim \|D^*c\|_{\mathcal{H}_1}^2 + \|c\|_{L^2(B_R)}^2,$$

which proves the third inequality in the statement. The proof of the last one follows from a very similar computation, which we omit.  $\square$

**Lemma 3.11** (Second order inequalities). *There is  $C > 0$  depending only on the monopole  $m_0$  so that for all  $c_0 \in \mathcal{H}_2$*

$$\|c_0\|_{\mathcal{H}_2}^2 \leq C \left( \|DD^*c_0\|_{\mathcal{H}_0}^2 + \|c_0\|_{L^2_1(B_{R\varepsilon}(0))}^2 \right), \quad (3.4a)$$

$$\|c_0\|_{\mathcal{H}_2}^2 \leq C \left( \|D^*Dc_0\|_{\mathcal{H}_0}^2 + \|c_0\|_{L^2(B_{R\varepsilon}(0))}^2 \right). \quad (3.4b)$$

*Proof.* The last two inequalities in the statement of Lemma 3.10 yield

$$\begin{aligned} \|c_0\|_{\mathcal{H}_2}^2 &\lesssim \|D^*c_0\|_{\mathcal{H}_1}^2 + \|c_0\|_{L^2(B_R(0))}^2 \\ \|c_0\|_{\mathcal{H}_2}^2 &\lesssim \|Dc_0\|_{\mathcal{H}_1}^2 + \|c_0\|_{L^2(B_R(0))}^2. \end{aligned}$$

Composing these with the first two in Lemma 3.10 yields

$$\begin{aligned} \|c_0\|_{\mathcal{H}_2}^2 &\lesssim \|DD^*c_0\|_{\mathcal{H}_0}^2 + \|D^*c_0\|_{L^2(B_R(0))}^2 + \|c_0\|_{L^2(B_R(0))}^2 \\ \|c_0\|_{\mathcal{H}_2}^2 &\lesssim \|D^*Dc_0\|_{\mathcal{H}_0}^2 + \|c_0\|_{L^2(B_R(0))}^2, \end{aligned}$$

which yields the stated inequalities.  $\square$

**Corollary 3.12.** *For  $k = 1, 2$ , the operators  $D, D^* : \mathcal{H}_{k+1} \rightarrow \mathcal{H}_k$  are continuous and Fredholm. In particular  $DD^*, D^*D : \mathcal{H}_2 \rightarrow \mathcal{H}_0$  are also continuous and Fredholm.<sup>1</sup>*

*Proof.* The continuity of these operators is immediate so we focus on the Fredholm property. The inequalities in Lemma 3.10 together with the compactness of the embedding  $\mathcal{H}_1 \hookrightarrow$

<sup>1</sup>When  $D$  and  $D^*$  are considered as maps from  $L^2_1(\mathbb{R}^3)$  to  $L^2(\mathbb{R}^3)$ , Corollary 3.12 does not hold, because of the failure of the Rellich Lemma for unbounded domains.

$L^2(B_{R_e})$ , imply that the operators  $D, D^* : \mathcal{H}_1 \rightarrow \mathcal{H}_0$  have finite dimensional kernel and closed image. Similarly, these inequalities show that both  $\ker_{L^2}(D^*)$  and  $\ker_{L^2}(D)$  are contained in  $\mathcal{H}_1$ . Hence, these can be respectively identified with the cokernel of the operators  $D, D^* : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ , and so their cokernels are also finite dimensional. Putting all these facts together follows that the mentioned first order operators are Fredholm.

In order to prove that the second order operators  $DD^*, D^*D : \mathcal{H}_2 \rightarrow \mathcal{H}_0$  are Fredholm is enough that  $D$  and  $D^*$  also be Fredholm when defined from  $\mathcal{H}_2$  to  $\mathcal{H}_1$ , which can be done through a very similar computation. Alternatively, it follows from the same argument as above, but using inequalities (3.4a) and (3.4b) instead.  $\square$

**3.3. A Gap Theorem.** In the proof of Lemma 3.10 we saw that

$$\|D^*c\|^2 = \|\nabla_0 c\|^2 + \|[\Phi_0, c]\|^2,$$

which implies the operator  $D^* : \mathcal{H}_1 \rightarrow \mathcal{H}_0$  is injective. Indeed, the inequality

$$\|D^*c\|_{\mathcal{H}_0}^2 \gtrsim \|c\|_{\mathcal{H}_1}^2.$$

holds for any  $c \in \mathcal{H}_1$ . Thus,  $DD^* : \mathcal{H}_2 \rightarrow \mathcal{H}_0$  is also injective, and, since it is a formally self-adjoint elliptic operator, its spectrum is gapped.

**Theorem 3.13** (Gap Theorem). *There is a constant  $C > 0$ , possibly depending on the monopole  $m_0 = (\nabla_0, \Phi_0)$ , such that*

$$\|DD^*c\|_{\mathcal{H}_0}^2 \geq C\|c\|_{\mathcal{H}_2}^2. \quad (3.5)$$

*Proof.* The proof of this assertion is a standard argument by contradiction using the fact that  $\ker_{\mathcal{H}_2}(DD^*) \subseteq \ker_{\mathcal{H}_2}(D^*) = \{0\}$ . Indeed, if inequality (3.5) is not true, then there is a sequence  $c_i \in \mathcal{H}_2$  such that  $\|c_i\|_{\mathcal{H}_2} = 1$  and  $\|DD^*c_i\|_{\mathcal{H}_0}^2 \rightarrow 0$ . Since the sequence  $\{c_i\}$  is bounded in  $\mathcal{H}_2$  by assumption, there is a weak  $\mathcal{H}_2$ -limit, say  $c_\infty \in \mathcal{H}_2$ , which satisfies  $DD^*c_\infty = 0$ . As  $DD^*$  has no  $\mathcal{H}_2$ -kernel, we have  $c_\infty = 0$ , that is  $c_i$  converges weakly to 0.

Now, consider any bounded domain  $\Omega \subset \mathbb{R}^3$ , the embedding  $\mathcal{H}_2(\Omega) \hookrightarrow L_1^2(\Omega)$  is compact,  $c_i \rightarrow c_\infty = 0$  strongly in  $L_1^2(\Omega)$ . Putting this together with inequality (3.4a) for  $DD^*$  follows that

$$\|c_i\|_{\mathcal{H}_2}^2 \leq C \left( \|DD^*c_i\|_{\mathcal{H}_0}^2 + \|c_i\|_{L_1^2(B_R(0))}^2 \right).$$

Taking the limit as  $i \rightarrow \infty$ , the right hand side converges to zero, so we also have

$$\lim_{i \rightarrow \infty} \|c_i\|_{\mathcal{H}_2} = 0,$$

which contradicts  $\|c_i\|_{\mathcal{H}_2} = 1$ .  $\square$

Using the Lax–Milgram Theorem, we immediately conclude the following:

**Corollary 3.14** (Green operator of  $DD^*$ ). *There is continuous linear map*

$$G : \mathcal{H}_0 \rightarrow \mathcal{H}_2,$$

such that  $G \circ DD^* = \text{id}_{\mathcal{H}_2}$ .

### 3.4. Multiplication properties of the function spaces.

**Lemma 3.15.** *Let*

$$N_0 : (\Lambda^1 \oplus \Lambda^0) \oplus (\Lambda^1 \oplus \Lambda^0) \rightarrow \mathbb{R},$$

be a bilinear map whose norm is pointwise uniformly bounded, that is there is a positive constant  $C$ , such that for all  $x \in \mathbb{R}^3$ , and all  $\gamma_1, \gamma_2 \in \Lambda_x^1 \oplus \Lambda_x^0$

$$|N_0(\gamma_1, \gamma_2)| \leq C|\gamma_1||\gamma_2|.$$

Then, for any connection  $\nabla$ , there is some other constant  $C'$  such that if now  $c_1, c_2$  are  $L_1^2$  sections, then

$$\|N_0(\gamma_1, \gamma_2)\| \leq C'\|\nabla\gamma_1\|(\|\gamma_2\| + \|\nabla\gamma_2\|).$$

*Proof.* Given that  $\frac{3}{2} < 2 < 3$  and using the Hölder's inequality twice and then the Sobolev inequality from Lemma 3.8, we have

$$\begin{aligned} \|N_0(\gamma_1, \gamma_2)\| &\lesssim \|\gamma_1\| \|\gamma_2\| \\ &\lesssim \|\gamma_1\| \|\gamma_2\|_{L^3} + \|\gamma_1\| \|\gamma_2\|_{L^{3/2}} \\ &\lesssim \|\gamma_1\|_{L^6} \|\gamma_2\|_{L^6} + \|\gamma_1\|_{L^6} \|\gamma_2\|_{L^2} \\ &\lesssim \|\nabla\gamma_1\| \|\nabla\gamma_2\| + \|\nabla\gamma_1\| \|\gamma_2\|, \end{aligned}$$

which concludes the proof.  $\square$

The nonlinearities of the Haydys monopole equations (1.7a) to (1.7c) are quadratic, but come composed with the Lie algebra bracket  $[\cdot, \cdot]$  acting in the  $\mathfrak{g}$ -valued components. Thus, given a quadratic map  $N_0$  as in Lemma 3.15, the maps under consideration are of the form  $N(c_1, c_2)$ , that is if for  $i = 1, 2$ :  $c_i = s_i \otimes \gamma_i$  with  $s_i \in \mathfrak{g}$ ,  $\gamma_i \in \Lambda^0 \oplus \Lambda^1$  we have

$$N(c_1, c_2) = [s_1, s_2] \otimes N_0(\gamma_1, \gamma_2).$$

In that context, and in terms of the  $\mathcal{H}_k$ -norms, the result in the Lemma 3.15 must be rephrased, which requires some preparation. For instance, recall that given a finite energy monopole

$m_0 = (\nabla_0, \Phi_0)$ , there is  $\Phi_\infty : \mathbb{S}^2 \rightarrow \mathfrak{g} - \{0\}$  so that  $\lim_{\rho \rightarrow \infty} \Phi_0|_{\mathbb{S}_\rho^2} = \Phi_\infty$  uniformly. Recall, that a monopole  $m_0$  is said to have maximal symmetry breaking, if  $\ker(\text{ad}(\Phi_\infty))$  is Abelian.

**Lemma 3.16** (Multiplication properties of the  $\mathcal{H}_k$ -spaces). *Let  $m_0 = (\nabla_0, m_0)$  be a monopole with maximal symmetry breaking and  $N$  a quadratic map as above. Then there exists  $C_{m_0} > 0$ , possibly depending on the monopole  $m_0 = (\nabla_0, \Phi_0)$ , such that*

$$\|N(c_1, c_2)\|_{\mathcal{H}_0} \leq C_{m_0} \|c_1\|_{\mathcal{H}_1} \|c_2\|_{\mathcal{H}_1}.$$

*Proof.* We start by proving the claimed inequality inside a bounded domain  $\Omega \subset \mathbb{R}^3$ . Notice that, as  $\Omega$  is bounded and  $\rho$  continuous the  $H^1(\Omega)$ -norm is equivalent to  $L_1^2(\Omega)$  norm. Then, from Lemma 3.15 we immediately obtain

$$\|N(c_1, c_2)\|_{\mathcal{H}_0(\Omega)} \leq C_{m_0} \|c_1\|_{\mathcal{H}_1(\Omega)} \|c_2\|_{\mathcal{H}_1(\Omega)}.$$

We are then left with proving the inequality in the statement outside a compact domain. By Hypotheses 3.1 and 3.3, there exists an  $R > 0$ , such that  $|\Phi_0| > \frac{1}{2}$  and  $\ker(\text{ad}(\Phi_0))$  is Abelian on  $\mathbb{R}^3 - B_R(0)$ . Let  $\underline{\mathfrak{g}}$  denote the trivial  $\text{Lie}(G)$ -bundle, which we identify now with  $\text{ad}(P)$ . Furthermore, let  $\underline{\mathfrak{g}}^\parallel = \ker(\text{ad}(\Phi_0))$ . Then, over  $\mathbb{R}^3 - B_R(0)$ ,  $\underline{\mathfrak{g}}^\parallel$  is a smooth, Abelian Lie-algebra bundle, and

$$\underline{\mathfrak{g}} \simeq \underline{\mathfrak{g}}^\parallel \oplus \underline{\mathfrak{g}}^\perp,$$

with  $\underline{\mathfrak{g}}^\perp = \ker(\text{ad}(\Phi_0))^\perp$ . Sections of  $(\Lambda^0 \oplus \Lambda^2) \otimes \underline{\mathfrak{g}}$  can be then written as  $c = c^\parallel + c^\perp$ . By the maximal symmetry breaking hypothesis again, we have that  $[\underline{\mathfrak{g}}^\parallel, \underline{\mathfrak{g}}^\parallel] = \{0\}$ . Thus

$$[c_1, c_2] = [c_1^\parallel, c_2^\perp] + [c_1^\perp, c_2^\parallel] + [c_1^\perp, c_2^\perp].$$

Thus, it suffices to prove the stated inequality separately to each of these components. Start by noticing that  $\|c^\perp\|_{\mathcal{H}_1} \sim \|c^\perp\|_{L_1^2}$  while  $\|c^\parallel\|_{\mathcal{H}_1} \sim \|\nabla_0 c^\parallel\|_{L^2} + \|\rho^{-1} c^\perp\|_{L^2} \sim \|\nabla_0 c^\parallel\|_{L^2}$ , and so making use of Lemma 3.15 we have

$$\|[c_1^\perp, c_2^\parallel]\| \lesssim (\|c_1^\perp\| + \|\nabla c_1^\perp\|) \|\nabla c_2^\parallel\| \lesssim \|c_1^\perp\|_{\mathcal{H}_1} \|c_2^\parallel\|_{\mathcal{H}_1} \lesssim \|c_1\|_{\mathcal{H}_1} \|c_2\|_{\mathcal{H}_1}.$$

A similar application of Lemma 3.15 with the roles of  $c_1, c_2$  interchanged gives the same bound on the term  $[c_1^\perp, c_2^\parallel]$ , and in any order regarding  $c_1, c_2$  also one for the term  $[c_1^\perp, c_2^\perp]$ .  $\square$

**3.5. Preparation for the proof of the Main Theorem 1.** Let  $\mathcal{D} = \mathcal{A} \oplus \Omega^0(M, \mathfrak{g})$  and  $C = T\mathcal{D} = \mathcal{A} \oplus \Omega^0(M, \mathfrak{g}) \oplus \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g})$  be the configuration space for Haydys monopoles,

that is  $(\nabla, \Phi, a, \Psi) \in \mathcal{C}$ . Furthermore, let  $\mathcal{R} = \Omega^1(M, \mathfrak{g}) \oplus \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g})$ , and

$$\kappa : \mathcal{C} \rightarrow \mathcal{R}; (\nabla, \Phi, a, \Psi) \mapsto \begin{pmatrix} *F_\nabla - d_\nabla\Phi - \frac{1}{2} * [a \wedge a] + [a, \Psi] \\ *d_\nabla a - d_\nabla\Psi - [a, \Phi] \\ d_\nabla^* a + [\Psi, \Phi] \end{pmatrix}.$$

Then we can rewrite the Haydys monopole equations (1.7a) to (1.7c) as

$$\kappa(\nabla, \Phi, a, \Psi) = 0.$$

We call  $\kappa$  the *Haydys map*.

**Remark 3.17.** *Note that if  $m_0 = (\nabla_0, \Phi_0)$  is a Bogomolny monopole on  $M$ , then for  $v_0 = (a_0, \Psi_0) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g})$  the last two components of  $\kappa(m_0, v_0)$  are exactly the linearization of the Bogomolny equation (1.1) together with the usual Coulomb type gauge fixing condition which we have been writing as  $D(v_0) = (d_2 \oplus d_1^*)(v_0)$ ; see Section 3.1. Let now  $v_0$  be a tangent vector to the Bogomolny monopole moduli space at  $m_0$  as in Definition 3.5. Then the last two components and the terms involving  $m_0$  in the first component of  $\kappa(m_0, v_0)$  vanish, but the (quadratic) terms depending on  $v_0$  need not, in general. So  $(m_0, c_0)$  fails to solve the Haydys monopole equations (1.7a) to (1.7c) for  $v_0 \in \ker(D) - \{0\}$ , but the error is of order  $O(|v_0|^2)$  pointwise.*

**3.6. Linearization and gauge fixing.** In this section, we look for solutions of the Haydys monopole equations (1.7a) to (1.7c) as follows: Let  $m_0$  be a (finite energy) Bogomolny monopole, and  $v_0 = (a_0, \Psi_0)$  be a tangent vector to the Bogomolny monopole moduli space at  $m_0$  with unit  $L^2$ -norm. Then consider  $c_0 = (m_0, tv_0) = (\nabla_0, \Phi_0, ta_0, t\Psi_0)$ , for small  $t$ , which is  $O(t^2)$  away from being a Haydys monopole, as in Remark 3.17. Since  $\mathcal{C}$  is an affine space over  $\mathcal{V} = (\Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \mathfrak{g}))^{\oplus 2}$ , we have that  $T_{c_0}\mathcal{C} \simeq \mathcal{V}$ . Let then  $\delta c_0 = (b_1, \phi, b_2, \psi) \in \mathcal{V}$ , and we search for a solution which is of the form  $c = (m, v) = (m_0, tv_0) + \delta c_0$ . As we are interested in solutions up to gauge equivalence only, it is convenient to work on the orthogonal complement of a slice of the gauge action. For that reason, we add the condition that  $\delta c_0$  is orthogonal to the gauge slice passing through  $(m_0, tv_0)$ , which is equivalent to

$$g_{c_0}(\delta c_0) = d_1^*(b_1, \phi) - *[a_0 \wedge *b_2] - [\Psi, \psi] = 0.$$

We can further restrict the form of  $\delta c_0$ : write  $\delta c_0 = (\delta m_0, \delta v_0)$ , and require that both  $\delta m_0$  and  $\delta v_0$  are perpendicular to the kernel of  $D$ , that is to all tangent vector to the Bogomolny monopole moduli space at  $m_0$ . In other words,  $\delta c_0 \in \text{Im}(D^* \oplus D^*)$ . We do not lose any generality due to this requirement in what follows. Indeed, a tangential component in  $\delta m_0$

would just “redefine”  $m_0$ , and—even more clearly—a tangential component in  $\delta v_0$  could be absorbed in  $v_0$ .

### 3.7. Solving the Haydys monopole equation.

*Proof of Main Theorem 1.* From the discussion in the previous section, the gauge fixed Haydys monopole equations (1.7a) to (1.7c) around  $c_0 \in \mathcal{C}$  is encoded in a map

$$\widehat{\kappa}_{c_0} : \mathcal{V} \rightarrow \mathcal{R} \oplus \Omega^0(M, \mathfrak{g}); \delta c_0 \mapsto (\kappa(c_0 + \delta c_0), g_{c_0}(\delta c_0)).$$

Now let  $c_0 = (m_0, tv_0)$ , where again  $v_0$  is a tangent vector to the Bogomolny monopole moduli space at  $m_0$  with unit  $L^2$ -norm, and  $t$  is to be specified later. Let  $\delta c_0 = s\delta c$ , where  $\delta c = (\delta m, \delta v)$  and  $s$  a *small* parameter also to be specified later. By Remark 3.17, the gauge fixed Haydys monopole equations (1.7a) to (1.7c) become  $\widehat{\kappa}_{c_0}(s\delta c) = 0$ . Since  $\widehat{\kappa}_{c_0}$  is quadratic, the equation becomes

$$0 = \widehat{\kappa}_{c_0}(s\delta c) = \widehat{\kappa}_{c_0}(0) + s d\widehat{\kappa}_{c_0}(\delta c) + s^2 Q_{c_0}(\delta c, \delta c), \quad (3.6)$$

where  $Q_{c_0}$  is the continuous quadratic remainder in the Taylor expansion of  $\widehat{\kappa}$  around  $(c_0, 0)$ , and there are no higher order terms. As noted in Remark 3.17, we have

$$\widehat{\kappa}_{c_0}(0) = (\kappa(m_0, tv_0), 0) = t^2 (\kappa(m_0, v_0), 0) = O(t^2 |v_0|^2).$$

Short computation shows that in the direction  $(b_1, \phi, b_2, \psi) \in \mathcal{V} \simeq T_{c_0}\mathcal{C}$ , the linearization of  $\kappa$  at  $c_0 = (\nabla_0, \Phi_0, a_0, \Psi_0)$  is

$$d\kappa_{c_0}(b_1, \phi, b_2, \psi) = \begin{pmatrix} d_2(b_1, \phi) - ([a_0 \wedge b_2] + [*b_2, \Psi_0] + [*a_0, \psi]) \\ d_2(b_2, \psi) + ([b_1 \wedge a_0] - [*b_1, \Psi_0] - [*a_0, \phi]) \\ d_1^*(b_2, \psi) - (*[b_1 \wedge *a_0] + [\phi, \Psi_0]) \end{pmatrix},$$

which combined with the linearization of  $g$ , yields

$$d\widehat{\kappa}_{c_0}(\delta c) = d\widehat{\kappa}_{c_0}(\delta m, \delta v) = (D\delta m, D\delta v) + t L_{c_0}(\delta c)$$

for some continuous, linear reminder term  $L_{c_0}$ , which is algebraic (not a differential operator). Since  $\delta c = (\delta m, \delta v) \in \text{Im}(D^* \oplus D^*)$ , we can write  $(\delta m, \delta v) = (D^*u_1, D^*u_2) = \mathbf{D}^*u$ , where  $\mathbf{D}^* = D^* \oplus D^*$  and  $u = (u_1, u_2) \in \mathcal{H}_2^{\oplus 2}$ . Thus we have

$$d\widehat{\kappa}_{c_0}(\delta c) = (DD^*u_1, DD^*u_2) + t L_{c_0}(\mathbf{D}^*u) = \mathbf{D}\mathbf{D}^*u + t L_{c_0}(\mathbf{D}^*u).$$

Hence the gauge fixed Haydys equation (3.6)—now in terms of  $u$ —becomes

$$0 = t^2 (\kappa(m_0, v_0), 0) + s \mathbf{D}\mathbf{D}^*u + st L_{c_0}(\mathbf{D}^*u) + s^2 Q_{c_0}(\mathbf{D}^*u, \mathbf{D}^*u), \quad (3.7)$$

By Corollary 3.14, the operator  $DD^* : \mathcal{H}_2 \rightarrow \mathcal{H}_0$  admits a continuous Green's operator  $G$ . Let  $\mathbf{G} = G \oplus G$ , thus  $\mathbf{G} \circ \mathbf{D}\mathbf{D}^* = \text{id}_{\mathcal{H}_2^{\oplus 2}}$ . Thus, applying  $\mathbf{G}$  to equation (3.7) yields

$$\begin{aligned} 0 &= \mathbf{G}\left(t^2 (\kappa(m_0, v_0), 0) + s \mathbf{D}\mathbf{D}^*u + st L_{c_0}(\mathbf{D}^*u) + s^2 Q_{c_0}(\mathbf{D}^*u, \mathbf{D}^*u)\right) \\ &= t^2 \mathbf{G}(\kappa(m_0, v_0), 0) + s u + st \mathbf{G}(L_{c_0}(\mathbf{D}^*u)) + s^2 \mathbf{G}(Q_{c_0}(\mathbf{D}^*u, \mathbf{D}^*u)). \end{aligned} \quad (3.8)$$

Note, that equation (3.8) can be rewritten as a fixed point equation on  $u$  as

$$u = F(u) = -\frac{t^2}{s} \mathbf{G}(\kappa(m_0, v_0), 0) - t \mathbf{G}(L_{c_0}(\mathbf{D}^*u)) - s \mathbf{G}(Q_{c_0}(\mathbf{D}^*u, \mathbf{D}^*u)). \quad (3.9)$$

In what follows, given an operator  $X$ , let  $\|X\|$  denote its norm as an operator. In order to use the Banach Fixed Point Theorem, we now prove that for  $t$  sufficiently small and  $s$  chosen appropriately,  $F$  is a contraction from  $B_1(0) \subset \mathcal{H}_2^{\oplus 2}$  to itself. First, we prove it maps  $B_1(0) \subset \mathcal{H}_2^{\oplus 2}$  to itself. Indeed, using Lemma 3.16 and Corollary 3.12 together with  $\|v_0\|_{\mathcal{H}_0} = 1$ , we obtain that for  $u \in B_1(0) \subset \mathcal{H}_2^{\oplus 2}$

$$\begin{aligned} \|F(u)\|_{\mathcal{H}_2^{\oplus 2}} &\leq \frac{t^2}{s} \|\mathbf{G}(\kappa(m_0, v_0), 0)\|_{\mathcal{H}_2^{\oplus 2}} + t \|\mathbf{G}(L_{c_0}(\mathbf{D}^*u))\|_{\mathcal{H}_2^{\oplus 2}} + s \|\mathbf{G}(Q_{c_0}(\mathbf{D}^*u, \mathbf{D}^*u))\|_{\mathcal{H}_2^{\oplus 2}} \\ &\leq C_{m_0} \|\mathbf{G}\| \left( \frac{t^2}{s} + t \|\mathbf{D}^*\| + s \|\mathbf{D}^*\|^2 \right). \end{aligned}$$

For a fix  $t > 0$ , and varying, but positive  $s$ , the term in the parentheses is minimized when  $s(t) = \frac{t}{\|\mathbf{D}^*\|}$ , in which case we get

$$\|F(u)\|_{\mathcal{H}_2^{\oplus 2}} \leq 3C_{m_0} \|\mathbf{G}\| \|\mathbf{D}^*\| t \leq 6C_{m_0} \|G\| \|D^*\| t.$$

Hence, for  $t \leq t_{\max}(m_0) = (6C_{m_0} \|G\| \|D^*\|)^{-1}$ ,  $F$  definitely maps the ball  $B_1(0) \subset \mathcal{H}_2^{\oplus 2}$  to itself. Now we show that in this case  $F$  is also a contraction. Let  $u, v \in B_1(0)$ . Then, by Lemma 3.16 and Corollary 3.12, we have for  $s = \frac{t}{\|\mathbf{D}^*\|}$

$$\begin{aligned} \|F(u) - F(v)\|_{\mathcal{H}_2^{\oplus 2}} &\leq t \|\mathbf{G}(L_{c_0}(\mathbf{D}^*(u - v)))\|_{\mathcal{H}_2^{\oplus 2}} \\ &\quad + \frac{t}{\|\mathbf{D}^*\|} \|(G \circ Q_{c_0})(\mathbf{D}^*u, \mathbf{D}^*u) - (G \circ Q_{c_0})(\mathbf{D}^*v, \mathbf{D}^*v)\|_{\mathcal{H}_2^{\oplus 2}} \\ &\leq t C_{m_0} \|\mathbf{G}\| \|\mathbf{D}^*\| \|u - v\|_{\mathcal{H}_2^{\oplus 2}} \\ &\quad + \frac{t}{\|\mathbf{D}\|} \|\mathbf{G}\| \|Q_{c_0}(\mathbf{D}^*(u + v), \mathbf{D}^*(u - v))\|_{\mathcal{H}_0^{\oplus 2}} \\ &\leq t C_{m_0} \|\mathbf{G}\| \left( \|\mathbf{D}^*\| \|u - v\|_{\mathcal{H}_2^{\oplus 2}} + \frac{1}{\|\mathbf{D}^*\|} \|\mathbf{D}^*(u + v)\|_{\mathcal{H}_1^{\oplus 2}} \|\mathbf{D}^*(u - v)\|_{\mathcal{H}_1^{\oplus 2}} \right) \\ &\leq t C_{m_0} \|\mathbf{G}\| \left( \|\mathbf{D}^*\| \|u - v\|_{\mathcal{H}_2^{\oplus 2}} + \frac{1}{\|\mathbf{D}^*\|} \|\mathbf{D}^*\|^2 \|u + v\|_{\mathcal{H}_1^{\oplus 2}} \|u - v\|_{\mathcal{H}_1^{\oplus 2}} \right) \\ &\leq t (3C_{m_0} \|\mathbf{G}\| \|\mathbf{D}^*\|) \|u - v\|_{\mathcal{H}_2^{\oplus 2}} \\ &\leq \frac{t}{t_{\max}(m_0)} \|u - v\|_{\mathcal{H}_2^{\oplus 2}}. \end{aligned}$$

Hence, if  $t < t_{\max}(m_0)$  the hypotheses of the Banach Fixed Point Theorem apply, and so there is a unique solution to the fixed point equation (3.9), which in turn provides a (unique) solution to the Haydys monopole equations (1.7a) to (1.7c) of the form

$$\mathcal{F}(m_0, t v_0) = (m_0 + s \delta m, t v_0 + s(t) \delta v) = \left( m_0 + t \frac{D^* u_1}{\|D^*\|}, t v_0 + t \frac{D^* u_2}{\|D^*\|} \right).$$

This concludes the proof of Main Theorem 1.  $\square$

**Remark 3.18.** *In our construction the neighborhood of the monopole moduli space we constructed is an open ball-bundle, with, a priori, varying radius  $t_{\max}(m_0)$ . In particular the normal bundle of  $\mathcal{M}_B \subset \mathcal{M}_H$  is canonically isomorphic to the tangent bundle  $T\mathcal{M}_B$ .*

*Note furthermore that the situation is similar to that of the Higgs bundle moduli space over a Riemann surface: every holomorphic bundle over a closed Riemann surface can be viewed as a Higgs bundle with vanishing Higgs field, and thus defines the a submanifold of the Hitchin moduli space. Moreover the tangent bundle and normal bundles of this submanifold is isomorphic.*

*We see the same picture with the Riemann surface replaced by  $\mathbb{R}^3$ , holomorphic bundles replaced by monopoles, and Higgs bundles replaced by Haydys monopoles.*

#### 4. ON THE GEOMETRY OF THE HAYDYS MONOPOLE MODULI SPACE

**4.1. Dimension of the Haydys moduli space.** Let us consider the  $G = \text{SU}(2)$  case first. Let  $(\nabla, \Phi)$  a finite energy  $\text{SU}(2)$  Bogomolny monopole. By Hypothesis 3.1,  $\Phi_\infty$  is a nonzero,  $\nabla^\infty$ -parallel section of  $\text{ad}(P_\infty)$ , which is a real, oriented, rank-3 vector bundle over  $\mathbb{S}_\infty^2$ . Hence  $\Phi_\infty$  has a degree, hence considered a map from the 2-sphere to the 2-sphere. Let us denote by  $\mathcal{M}_H^k$  the part of the moduli space of Haydys monopoles with structure group  $\text{SU}(2)$  that contains the Bogomolny monopoles of degree  $k$ . We now prove that  $\dim_{\mathbb{R}}(\mathcal{M}_H^k) = 8k$ . Indeed, the linearization of the gauge fixed Haydys monopole map,  $\widehat{\kappa} : \mathcal{C} \rightarrow \mathbb{R}$  is the linear operator  $D \oplus D$ . As in the case of monopoles, from the analysis in [14, Proposition 9.2] follows that any tangent vector  $v$  to  $\mathcal{M}_H^k$  at Haydys monopole  $(\nabla, \Phi, a, \Psi)$  must be such that

$$\|\nabla c\|^2 + \|[\Phi, c]\|^2 < \infty.$$

Thus,  $\dim_{\mathbb{R}}(\mathcal{M}_H^k) = \dim(\ker_{\mathcal{H}_1}(D \oplus D)) = 2 \dim(\ker_{\mathcal{H}_1}(D))$ , which, in fact, coincides with the index of  $D \oplus D : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ . Finally, this index can then be computed, for  $k > 0$ , as in [14, Proposition 9.1],  $\dim_{\mathbb{R}}(\mathcal{M}_H^k) = 8k$ . Thus, we have constructed an open subset of  $\mathcal{M}_H^k$ .

**Remark 4.1.** Also for all  $G$  we construct an open subset of the connected component of  $\mathcal{M}_H$  containing  $\mathcal{M}_B$ . This follows immediately from the fact that

$$\dim_{\mathbb{R}}(\mathcal{M}_H) = \dim(\ker_{\mathcal{H}_1}(D \oplus D)) = 2 \dim(\ker_{\mathcal{H}_1}(D)) = 2 \dim_{\mathbb{R}}(\mathcal{M}_B).$$

For  $G = \text{SU}(N)$  the latest have been computed in [13, Theorem 4.3.9] as being 4 times the sum of the magnetic weights.

## 4.2. Geometric structures on the Haydys monopole moduli space.

4.2.1. *Linear model.* Let  $\mathfrak{g}$  be a real semisimple Lie algebra and consider the quaternionic vector space  $V = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$ . Writing an element of  $V$  as  $A = A_0 + iA_1 + jA_2 + kA_3$  may be equipped with a  $G$ -bi-invariant metric obtained by  $\langle A, A' \rangle = \sum_{i=0}^3 \langle A_i, A'_i \rangle$ , where in the rightmost term we use the Killing form on  $V$ . The quaternionic structure determines 3 symplectic structures  $\omega_I, \omega_J, \omega_K$  with respect to which adjoint action of  $G$  is tri-Hamiltonian. The 3 moment maps associated with these respectively be written as

$$v_i(A) = [A_0, A_i] + [A_j, A_k]$$

where  $i = 1, 2, 3$  respectively and  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ .

This whole setup may be complexified by considering  $\mathfrak{g}_{\mathbb{C}}$  rather than  $\mathfrak{g}$ . Then, we define

$$V_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{H} = (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}) \oplus i(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}) \cong V \oplus V$$

and use the rightmost term to extend the inner product from  $V$  to  $V_{\mathbb{C}}$ . Furthermore, we use this to consider the 3 quaternionic structures  $V_{\mathbb{C}}$  given by

$$I_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

and similarly for  $J$  and  $K$ . Notice in particular that  $I_1 = J_1 = K_1$  but that all these complex structures together do *not* form a compatible octonionic structure. Further notice that for example  $I_2 \circ J_2 \neq K_2$  and  $I_3 \circ J_3 \neq K_3$ , in fact we have

$$I_2 \circ J_2 \circ K_2 = I_3 \circ J_3 \circ K_3 = \text{diag}(1, -1).$$

Together with the inner product  $\langle \cdot, \cdot \rangle$  these complex structures give rise to 3 sets of hyperkähler structures with respect to which  $G$  acts in an Hamiltonian fashion. The associated moment maps can be written as

$$\mu_{I_1}(A, B) = \sum_{i=0}^3 [A_i, B_i] \tag{4.1}$$

with  $\mu_{J_1} = \mu_{K_1}$  being given by same formula, while

$$\mu_{I_2}(A, B) = ([A_0, A_1] + [A_2, A_3]) - ([B_0, B_1] + [B_2, B_3]), \quad (4.2)$$

and  $\mu_{J_2}, \mu_{K_2}$  are given by a similar formula obtained by cyclic permutations of  $(1, 2, 3)$ . Finally, we have

$$\mu_{I_3}(A, B) = ([A_0, B_1] + [A_2, B_3]) - ([A_1, B_0] + [A_3, B_2]), \quad (4.3)$$

with again  $\mu_{J_3}$  and  $\mu_{K_3}$  being obtain from a cyclic permutation of  $(1, 2, 3)$ .

**Remark 4.2.** *There is one further quite natural hyperkähler structure on  $V_{\mathbb{C}} \cong V \oplus V$  which given by  $\text{diag}(I, I)$ ,  $\text{diag}(J, J)$  and  $\text{diag}(K, K)$ . Using these we can still obtain the moment maps  $\mu_{I_2}, \mu_{J_2}, \mu_{K_2}$  as follows. Instead of considering a Riemannian metric on  $V_{\mathbb{C}}$  we use the indefinite pairing*

$$b((A, B), (A', B')) = \langle A, A' \rangle - \langle B, B' \rangle.$$

*Using it and the quaternionic structure above we define 3 symplectic forms with respect to which we can define moment maps very much in the same manner. These coincide with the moment maps  $\mu_{I_2}, \mu_{J_2}, \mu_{K_2}$ .*

We now consider the joint moment maps

$$\mu_I = (\mu_{I_1}, \mu_{I_2}, \mu_{I_3}) : V_{\mathbb{C}} \rightarrow \mathbb{R}^3,$$

together with  $\mu_J$  and  $\mu_K$ . As the complex structure  $I_1 = J_1 = K_1$  is common to the three triples, its is the only one which immediately restricts to

$$Q = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0).$$

We must now check that all the other ones equally do restrict to  $Q$ . The first observation which is relevant for our analysis is the fact that the zero level set of moment maps  $\mu_I, \mu_J, \mu_K$  are invariant under the map  $\iota : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$  given by  $\iota(A, B) = (A, -B)$  as can be immediately seen from the equations (4.1) to (4.3). We state this as

**Lemma 4.3.**  *$Q \subset V_{\mathbb{C}}$  is invariant under the map  $\iota$ .*

We shall now use this to shortcut the proof of the following statement.

**Proposition 4.4.** *The tangent spaces to  $Q$  are invariant under all the complex structures  $I_i, J_i, K_i$  for  $i = 1, 2, 3$ .*

*Proof.* Let  $x \in \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ , we shall show that  $T_x = T_x(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0))$  is invariant by these complex structures by instead showing that its orthogonal complement

$T_x^\perp$  is itself invariant. To do this notice that  $T_x = \cap_{I=I,J,K} \cap_{i=1}^3 \ker(d\mu_{I_i})$  and so

$$T_x^\perp = \{\nabla\mu_{I_i}, \nabla\mu_{J_i}, \nabla\mu_{K_i}, i = 1, 2, 3\}.$$

However, recall that for  $L = I, J, K$  and  $i = 1, 2, 3$  we have that for any  $\xi \in \mathfrak{g}$

$$d\langle\xi, \mu_{L_i}\rangle(\cdot) = \omega_{L_i}(\xi_*, \cdot) = \langle L_i \xi_*, \cdot \rangle,$$

where  $\xi_*$  denotes the vector field in  $V_{\mathbb{C}}$  obtained through via the infinitesimal action of  $\xi \in \mathfrak{g}$ . Thus, from this formula and the definition of the gradient we find that  $\nabla(\langle\xi, \mu_{L_i}\rangle) = L_i \xi_*$ . Thus, we find

$$T_x^\perp = \{I_i \xi_*, J_i \xi_*, K_i \xi_*, \text{ for } i = 1, 2, 3, \xi \in \mathfrak{g}\},$$

and must show this is invariant under the complex structures  $I_i, J_i, K_i$ . This would be immediate if these complex structures formed a closed algebra as that of the octonions for example. That is only true modulo  $\iota$ , indeed we have  $I_2 \circ J_2 = \iota \circ K_2, I_3 \circ J_3 = \iota \circ K_3$  and similar formulas for other compositions. The result then follows from Lemma 4.3.  $\square$

**Remark 4.5.** *Alternatively, we can explicitly check that  $\mu_{I_2} \circ J_2 = -\mu_{I_2}, \mu_{I_2} \circ J_3 = \mu_{I_2}$  while  $\mu_{I_3} \circ J_2 = \mu_{I_3}, \mu_{I_3} \circ J_3 = -\mu_{I_3}$  with similar formulas for the other moment maps and complex structures, we conclude that these preserve the common zero level set.*

**Remark 4.6.** *Given that  $I_1 = J_1 = K_1$  we have  $\dim(Q) = \dim(\mathfrak{g})$  if  $\mathfrak{g}$  is finite dimensional.*

Notice that the fixed point locus of the involution  $\iota$  is given by  $\text{Fix}(\iota) = V \oplus 0 \subset V_{\mathbb{C}}$  is a complex submanifold with respect to  $I_2, J_2, K_2$  which restrict to  $\text{Fix}(\iota)$  as  $I, J, K$  respectively, thus inducing an hyperkähler structure there. On the other hand,  $\text{Fix}(\iota)$  is totally real with respect to all the remaining complex structures. In fact, it is complex-Lagrangian with respect to the respectively induced complex symplectic structure. This may be trivially checked this by noticing that it is complex with respect to  $\mathcal{I}_2$  and Lagrangian with respect to  $\omega_{\mathcal{I}_3} + i\omega_{\mathcal{I}_1}$  for all  $\mathcal{I} = I, J, K$ . In summary we have the following.

**Lemma 4.7.**  *$\text{Fix}(\iota)$  is a complex submanifold of  $V_{\mathbb{C}}$  with respect to  $I_2, J_2, K_2$ , which equip  $\text{Fix}(\iota)$  with an hyperKähler structure. In fact,  $\text{Fix}(\iota)$  is a complex-Lagrangian submanifold of  $V_{\mathbb{C}}$  with respect to the hyperkähler structures induced by  $(I_2, I_3, I_1), (J_2, J_3, J_1), (K_2, K_3, K_1)$  on  $V_{\mathbb{C}}$ .*

**Remark 4.8.** *Using the nomenclature of [9], we may state this by saying that  $\text{Fix}(\iota)$  is a (ABA)-brane with respect to the 3 hyperkähler structures on  $V_{\mathbb{C}}$  induced by  $(I_1, I_2, I_3), (J_1, J_2, J_3), (K_1, K_2, K_3)$ .*

Notice also that the involution  $\iota$  is non (anti)-symplectic or (anti)-holomorphic and so the construction above does not use the standard use of involutions.

This linear model will serve as the model for a more general construction which we will perform to obtain some interesting geometric structures in the moduli space of solutions to the Haydys equation. In the next section we briefly outline the finite dimensional curved version of that construction.

4.2.2. *Curved model.* Let  $X$  be a smooth manifold equipped with 3 different hyperkähler structures as those of the previous section. This means that they are all compatible with the same Riemannian metric  $h$  and the complex structures  $(I_1, I_2, I_3)$ ,  $(J_1, J_2, J_3)$ ,  $(K_1, K_2, K_3)$  satisfy  $I_1 = J_1 = K_1$  and

$$I_2 \circ J_2 \circ K_2 = I_3 \circ J_3 \circ K_3,$$

with each of these sides squaring to the identity.<sup>2</sup> Further, we suppose that there is a Lie group action  $G$  which acts on  $X$  in a tri-Hamiltonian fashion, with respect to all 3 hyperkähler structures. In order to perform a meaningful reduction we seek to require the structure which makes Proposition 4.4 work. For that we must imitate the existence of a bundle map  $\chi$  (imitating the role of  $\iota$  in the proof of Proposition 4.4) which closes the algebra of the complex structures and preserves the moment map equation. This is summarized as follows.

**Proposition 4.9.** *Let  $Q = \bigcap_{I=I,J,K} \bigcap_{i=1,2,3} \mu_{I_i}^{-1}(0)$  and suppose there is  $\chi \in \text{End}(TX|_Q)$  satisfying  $\chi(TQ) \subset TQ$  and such that  $\{I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, \chi\}$  forms a closed algebra. Then, all the hyperkähler structures given by  $h$  and either  $(I_1, I_2, I_3)$ , or  $(J_1, J_2, J_3)$ , or  $(K_1, K_2, K_3)$  restricts to  $Q$  and descend to the quotient  $Q/G$  which then is an hyperkähler manifold of real dimension  $\dim(X) - 8 \dim(G)$  in three different ways.*

4.2.3. *The monopole equations.* By fixing a connection we may regard the space of connections on  $P$  as  $\Omega^1(M, \mathfrak{g}_P)$  and we shall now do the construction of Section 4.2.1 with the quaternionic Lie algebra  $V$  replaced by  $\Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$  for  $M = \mathbb{R}^3$ . We shall now recall the flat hyperkähler structure on  $\Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$ . This is obtained by first fixing the usual  $L^2$ -metric

$$h_B((\dot{\nabla}, \dot{\Phi}), (\dot{\nabla}', \dot{\Phi}')) = \int_M (\langle \dot{\nabla}, \dot{\nabla}' \rangle + \langle \dot{\Phi}, \dot{\Phi}' \rangle) \text{vol},$$

also used to define the metric on the moduli space of Bogomolny monopoles. Then, we consider the complex structures  $I_\nu$ , parametrized by  $\nu \in \mathbb{S}^2 \subset \mathbb{R}^3$  acting on  $(c, \psi) \in \Omega^1(M, \mathfrak{g}_P) \oplus$

<sup>2</sup>It is unclear whether this condition could be dropped in some cases of interest.

$\Omega^0(M, \mathfrak{g}_P)$  as follows. Identify  $(c, \psi)$  with  $c + \psi dt \in \Omega^1(M \times \mathbb{R}_t, \mathfrak{g}_P)$ , then use the identifications  $M \times \mathbb{R}_t \cong \nu^\perp \oplus (\nu\mathbb{R} \times \mathbb{R}_t) \cong \mathbb{C}^2$  to define a complex structure on  $M \times \mathbb{R}_t$  and define  $I_\nu$  as its action by pullback on  $\Omega^1(M \times \mathbb{R}_t, \mathfrak{g}_P)$ . Using  $e_1, e_2, e_3$  as the standard basis of  $\mathbb{R}^3$  we shall write  $I = I_{e_1}, J = I_{e_2}, K = I_{e_3}$ .

Finally, we turn to our version of  $V_C$  which is the configuration space

$$C = (\mathcal{A} \oplus \Omega^0(M, \mathfrak{g}_P)) \oplus (\Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)),$$

equipped the constant coefficient metric  $h$  given by

$$h((\dot{\nabla}, \dot{\Phi}, \dot{a}, \dot{\Psi}), (\dot{\nabla}', \dot{\Phi}', \dot{a}', \dot{\Psi}')) = h_B((\dot{\nabla}, \dot{\Phi}), (\dot{\nabla}', \dot{\Phi}')) + h_B((\dot{a}, \dot{\Psi}), (\dot{a}', \dot{\Psi}')),$$

with  $h_B$  as above. Then, we may equip  $C$  with the 3 quaternionic structures  $(I_1, I_2, I_3)$ , or  $(J_1, J_2, J_3)$ , or  $(K_1, K_2, K_3)$  as in Section 4.2.1. As in there, the gauge group  $\mathcal{G}$  of automorphisms of  $P$  acts on  $C$  by conjugation and so we obtain the moment maps which for convenience we shall organize here as

$$(\mu_{I_i}, \mu_{J_i}, \mu_{K_i}) : C \rightarrow \mathbb{R}^3 \otimes \Omega^0(M\mathfrak{g}_P) \cong \Omega^1(M\mathfrak{g}_P),$$

for  $i = 1, 2, 3$ .<sup>3</sup> A straightforward computation shows that the equation

$$*F_\nabla - d_\nabla\Phi - \frac{1}{2} * [a \wedge a] + [a, \Psi] = 0,$$

can be identified with  $(\mu_{I_2}, \mu_{J_2}, \mu_{K_2}) = 0$ . In the same way, we have

$$*d_\nabla a - d_\nabla\Psi - [a, \Phi] = 0,$$

which can be identified with  $(\mu_{I_3}, \mu_{J_3}, \mu_{K_3}) = 0$ . Finally the last equation

$$d_\nabla^* a + [\Psi, \Phi] = 0,$$

corresponds to  $\mu_{I_1} = \mu_{J_1} = \mu_{K_1} = 0$  which recall is only one equation as  $I_1 = J_1 = K_1$ .

Formally, the same argument as that we used in Proposition 4.4, shows that the all complex structures we are considering restrict to the locus  $C_H \subset C$  cut out by the Haydys equations. Thus the three hyperkähler structures descend to the quotient

$$\mathcal{M}_H = C_H/\mathcal{G},$$

which can be identified with the moduli space of solutions to the Haydys equation. On  $C$  we have an involution  $\iota$  sending  $c = (\nabla, \Phi, a, \Psi)$  to  $\iota(c) = (\nabla, \Phi, -a, -\Psi)$  which trivially preserves  $C_H$ . Thus, by Lemma 4.7,  $\text{Fix}(\iota) \subset C$  and so is a complex Lagrangian submanifold

<sup>3</sup>Note that here we are organizing the moment maps in a nonstandard way. Indeed, the complex structures  $I_i, J_i, K_i$  do not follow the quaternionic relations

of  $\mathcal{C}$  with respect to the whole 3 hyperkähler structures. The points of  $\text{Fix}(\iota)$  correspond to those  $c$  of the form  $c = (\nabla, \Phi, 0, 0)$ . In particular, for  $c \in \text{Fix}(\iota) \cap \mathcal{C}_H$  we find that  $(\nabla, \Phi)$  is actually a Bogomolny monopole. Thus, down in the quotient  $\mathcal{M}_H$  we find that the moduli subspace of Bogomolny monopoles  $\mathcal{M}_B = (\text{Fix}(\iota) \cap \mathcal{C}_H)/\mathcal{G}$  is a complex Lagrangian submanifold of  $\mathcal{M}_H$  with respect to the 3 different hyperkähler structures. This is the main result of this section which we shall state as follows.

**Lemma 4.10.** *Fix( $\iota$ ) is a complex submanifold of  $V_{\mathbb{C}}$  with respect to  $I_2, J_2, K_2$ , which equip  $\text{Fix}(\iota)$  with an hyperKähler structure. In fact,  $\text{Fix}(\iota)$  is a complex-Lagrangian submanifold of  $V_{\mathbb{C}}$  with respect to the hyperkähler structures induced by  $(I_1, I_2, I_3), (J_1, J_2, J_3), (K_1, K_2, K_3)$  on  $V_{\mathbb{C}}$ .*

This finally implies Main Theorem 2.

**Remark 4.11.** *Using the nomenclature of [9] we may state this by saying that  $\text{Fix}(\iota)$  is a (ABA)-brane with respect to the 3 hyperkähler structures on  $V_{\mathbb{C}}$  induced by  $(I_1, I_2, I_3), (J_1, J_2, J_3), (K_1, K_2, K_3)$ .*

*Notice also that the involution  $\iota$  is non (anti)-symplectic or (anti)-holomorphic and so the construction above does not the standard use of involutions.*

**Remark 4.12.** *The structure alluded in part (a) of Main Theorem 2 is similar to a construction which may be done in the (co)tangent bundle of an hyperkähler manifold  $X^{4n}$ . Indeed, consider the induced metric on  $TX$  from the hyperkähler metric on  $X$  and the octonionic structure given as follows. Let  $\pi : TX \rightarrow X$  be the projection and use the Levi-Civita connection of the hyperkähler metric to get a splitting  $T_p(TX) \simeq \mathcal{H}_p \oplus \ker((\pi_p)_*)$ , where  $\ker((\pi_p)_*) = T_{\pi(p)}X$  and  $\mathcal{H}_p \simeq T_{\pi(p)}X$ .*

*However, in our situation it is not clear whether  $\mathcal{M}_H$  is diffeomorphic to  $T\mathcal{M}_B$  so the construction carried above is not a particular instance of this construction. It does rely, however, on the fact that we can canonically write the tangent space to any  $c \in \mathcal{M}_H$  as  $T_c\mathcal{M}_H \simeq V_c \oplus V_c$  with  $V_c$  some quaternionic vector space.*

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