Poisson Percolation on the Square Lattice

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A senior thesis submitted to the Department of Mathematics for graduating with honors

Duke University, Durham, NC

April 2019

Abstract

In this paper, we examine two versions of inhomogeneous percolation on the 2D lattice, which we will refer to as non-oriented and oriented percolation, and describe the limiting shape of the component containing the origin in both cases. To define the nonoriented percolation process that we study, we consider the square lattice where raindrops fall on an edge with midpoint \( x \) at rate \( \|x\|^{-\alpha} \). The edge becomes open when the first drop falls on it. We call this process ”nonoriented Poisson percolation”. Let \( \rho(x,t) \) be the probability that the edge with midpoint \( x = (x_1, x_2) \) is open at time \( t \) and let \( n(p,t) \) be the distance at which edges are open with probability \( p \) at time \( t \).

We show that with probability tending to 1 as \( t \to \infty \): (i) the cluster containing the origin \( C_0(t) \) is contained in the square of radius \( n(p_c - \varepsilon, t) \), and (ii) the cluster fills the square of radius \( n(p_c + \varepsilon, t) \) with the density of points near \( x \) being close to \( \theta(\rho(x,t)) \) where \( \theta(p) \) is the percolation probability when bonds are open with probability \( p \) on \( \mathbb{Z}^2 \). Results of Nolin suggest that if \( N = n(p_c, t) \) then the boundary fluctuations of \( C_0(t) \) are of size \( N^{4/7} \). In the second part of the paper, we prove similar, yet not-studied-before, results for the asymptotic shape of the cluster containing the origin in the oriented case of Poisson percolation. We show that the density of occupied sites at height \( y \) in the open cluster is close to the percolation probability in the corresponding homogeneous percolation process, and we study the fluctuations of the boundary.
Acknowledgements

I would like to thank my thesis advisors, Professor Richard Durrett and Professor Matthew Junge, for all their invaluable guidance. Ever since taking my first probability course with Professor Durrett during my sophomore year, I have tremendously benefitted from his expertise that he patiently and generously shared with me over the years. I am thankful to Professor Junge and Professor James Nolen for organizing and leading the "Random Fragmentation" summer research project, which reinforced my earlier interest in probability theory and served as the basis for this senior honors thesis.

I would also like to thank Professor David Kraines for warmly welcoming me to the Duke Mathematics Department, as soon as I arrived on campus in the summer of 2015. I remember being a confused incoming first-year student, unsure how to navigate the multitude of opportunities Duke University had to offer and reluctant to venture in advanced mathematics courses "too early". Professor Kraines has kindly offered me support and trust in my mathematical abilities, by strongly encouraging me to take graduate-level mathematics classes and by introducing me to several wonderful paths for pursuing undergraduate mathematics research.

I would like to express my appreciation to the Duke Mathematics Department for providing me with incredible resources for my mathematical and professional growth. I am very thankful for the summer support that I received through the "DOMath" collaborative research project for two years in a row. I am grateful to Professor Anita Layton for being my mathematics major advisor and for her dedicated mentorship in a mathematical biology project on renal blood flow control. At the same time, I would like to thank Professor Leslie Saper for his great academic and career advice. Furthermore, passionate Duke mathematics professors have challenged and inspired me through beautiful classes. All of this helped me have a fantastic experience as a mathematics major.
1 Introduction

Percolation was introduced by Broadbent and Hammersley a little over 60 years ago to model the infiltration of a fluid through a porous medium [4]. In the basic case of nonoriented homogeneous percolation, which serves as the theoretical foundation for our subsequent work, we consider the regular squared lattice $\mathbb{Z}^2$, where each edge is independently open or closed with probability $p$ or $1-p$, respectively. Our main object of interest is the open cluster around the origin, defined to be the set of points that can be reached from the origin through a path of open edges. See Fig. 1 below for a realization of bond percolation on the square lattice. One of the most fundamental questions is whether this cluster, which can also be regarded as a subgraph, contains an infinite component. There is known to be a critical value $p_c$ such that for $p > p_c$ such a component exists almost surely. In particular, in the case of the 2D lattice, $p_c$ was shown to equal $1/2$ [16]. A vast amount of literature is devoted to understanding the geometry of this component for different values of $p$. See Grimmett’s book [16] for a thorough introduction or the article by Beffara and Sidoravicius [2] for a briefer overview.

![Figure 1: Bond percolation on the 2D lattice.](image)

This paper is divided into two connected parts. Both of them extend the homogeneous percolation version introduced in the previous section to nonhomogeneous versions, one occurring on the regular, nonoriented lattice and another on the oriented lattice, respectively. We thus begin our first case, by studying the geometry of the open cluster containing the origin in a nonhomogeneous version of bond percolation on the square lattice that we will refer to as Nonoriented Poisson Percolation (Sections 2–4), then build upon our results by examining the shape of the open cluster in a corresponding process on the oriented 2D lattice (Sections 5–10).

2 Nonoriented Poisson Percolation

To intuitively describe the first process we consider, named "nonoriented Poisson percolation", imagine that each edge of the lattice becomes open after an independent exponentially
distributed time with rate that decreases in the distance from the origin. To mathematically formalize this, we say that an edge of the lattice with midpoint $x$ is assigned an independent Poisson process with rate $\|x\|^{-\alpha}$ where $\|x\| = \max\{|x_1|, |x_2|\}$ is the $L^\infty$ norm. The edge becomes open at the time of the first arrival. Our inspiration comes from the rainstick process. It was introduced by Pitman and Tang in [24] with followup work by Pitman, Tang and Duchamps [7]. In this discrete time process raindrops fall one after the other on the positive integers and sites become wet when landed on. The locations of raindrop landings are independent and identically distributed random variables $X_i$ with a geometric distribution:

$$P(X_i = k) = (1 - p)^{k-1}p$$

for $k \geq 1$. Let $T$ be the first time that the configuration is a single wet component containing 1, and let $K$ be its length. Pitman and Tang observed in [24] that the value of $K$ describes the size of the first block in a family of regenerative permutations.

Understanding block sizes has been useful for studying the structure of random Mallows permutations [1, 15]. In [6] the asymptotic behavior of $T$ and $K$ as $p \to 0$ was studied. They proved that $T \approx \exp(ec/p)$ and $K \approx e^{c/p}$, where $c \approx 1.524$ is a constant defined by an integral. This says that the first block is large, and takes a very large amount of time to form. It turns out that an exponentially decaying tail is needed for the rainstick process to terminate with probability one. Theorem 5 in [6] shows that if raindrops land beyond site $k$ with probability $\exp(-k^\beta)$ for $\beta < 1$ then $T$ is infinite with positive probability.

The nonoriented Poisson percolation we study here is a two dimensional version of the rainstick process. In both processes distant edges are less likely to become open (wet). However, we have a power-law tail rather than a geometric, so it is likely that there is no time at which there is a single component. So, we will instead study the size and density of the wet cluster containing the origin.

To state our results we introduce some notation. Here, we study Poisson percolation only on the two dimensional lattice $\mathbb{Z}^2$. An edge with midpoint $x$ will be open at time $t$ with probability

$$\rho(x, t) = 1 - \exp(-t\|x\|^{-\alpha}).$$

We define the cluster containing the origin at time $t$ to be the set of points $C_0(t)$ that can be reached from the origin by a path of open edges. Let

$$c_{p,\alpha} = (-\log(1-p))^{-1/\alpha}. \quad (1)$$

A little algebra gives

$$n(p, t) = \max\{\|x\| : \rho(x, t) \geq p\} = c_{p,\alpha} t^{1/\alpha}.$$

Notice that $n(p, t)$ is an increasing function of $p$. Our first result gives an upper bound on $C_0(t)$.

**Theorem 1.** Set $p_c = 1/2$. Let $R(0, r) = \{x : \|x\| \leq r\}$ be the square with side length $2r$ centered at 0. For any $\epsilon > 0$, as $n \to \infty$ it holds that

$$P(C_0(t) \subseteq R(0, n(p_c - \epsilon, t))) \to 1.$$

Having shown that $C_0(t)$ is with high probability contained within $R(0, n(p_c - \epsilon, t))$, we would like to describe what it looks like inside $R(0, n(p_c + \epsilon, t))$. To do this we relate it...
to standard bond percolation on \( \mathbb{Z}^2 \). Let \( C_0 \) be the open cluster containing the origin in bond percolation where each edge is open with probability \( p \) and otherwise is closed. Set \( \theta(p) = P_p(|C_0| = \infty) \), where \( P_p \) is the probability measure for bond percolation on \( \mathbb{Z}^2 \). It is well known that

\[
p_c := \inf \{ p : \theta(p) > 0 \} = 1/2
\]

is the critical value for bond percolation on the two-dimensional lattice. For this and other facts we use about percolation, see Grimmett’s book [16].

Intuitively, near \( x \in R(0, n(p_c + \epsilon, t)) \) the density of points in \( C_0(t) \) will be close to \( \theta(p(x, t)) \). To state this precisely, let \( n = n(p_c + \epsilon, t) \). Fix \( 1/2 < a < 1 \) and tile the plane with boxes of side length \( n^a \):

\[
R_{i,j} = [in^a, (i+1)n^a] \times [jn^a, (j+1)n^a],
\]

with center \( x_{i,j} \). Let \( D_{i,j} = |C_0(t) \cap R_{i,j}|/n^{2a} \) be the density of points in \( R_{i,j} \) that belong to \( C_0(t) \) and let \( \Lambda(t, \epsilon) = \{(i,j) : R_{i,j} \subset R(0, n(p_c + \epsilon, t))\} \). We prove that, as \( n \to \infty \), the density of \( C_0(t) \) in each of these boxes converges to the density of the infinite component in bond percolation with probability \( \rho(x_{i,j}, t) \) of an edge being open.

**Theorem 2.** For any \( \epsilon, \delta > 0 \), as \( t \to \infty \),

\[
P \left( \sup_{(i,j) \in \Lambda(t, \epsilon)} |D_{i,j}(t) - \theta(\rho(x_{i,j}, t))| > \delta \right) \to 0.
\]

From this we get a result about the size of \( C_0(t) \).

**Corollary 3.** With probability tending to 1 as \( t \to \infty \) it holds that

\[
\left| \frac{|C_0(t)|}{t^{2/a}} - \iint \theta(1 - \exp(-\|x\|^{-a})) \, dx_2 \, dx_1 \right| \to 0.
\]

Our proof of Theorem 2 makes heavy use of the planar graph duality for two dimensional bond percolation. Here we outline the argument that is given in detail in Section 4. Consider bond percolation on the dual lattice \( \mathbb{L} := \mathbb{Z}^2 + (1/2, 1/2) \) with nearest neighbor edges. Every edge \( e \) on \( \mathbb{Z}^2 \) is paired with an edge \( e^* \) on \( \mathbb{L} \) that has the same midpoint. If \( e \) is open (resp. closed), then \( e^* \) is closed (resp. open). The pairing means that if the density on the original lattice is \( p \), then the density on the dual lattice is \( 1 - p \). We use \( P^*_p \) to denote the percolation on the dual lattice. It is known that there is a top-to-bottom open crossing of \([a,b] \times [c,d]\) if and only if there is no left-to-right closed crossing of \([a-1/2, b+1/2] \times [c+1/2, d-1/2]\). Having mentioned the exact size of the rectangles once, we will ignore the 1/2’s in what follows.

Let \( I_n = [-\lfloor n/(C_1 \log n) \rfloor - 1, \lfloor n/(C_1 \log n) \rfloor] \) and for \( j \in I_n \) let

\[
R_j = [jC_1 \log n, (j+1)C_1 \log n] \times [-n, n],
\]

\[
R^j = [-n, n] \times [jC_1 \log n, (j+1)C_1 \log n].
\]
Note for the next step that the limits on \( j \) are chosen so that the first and last strips in each direction lie outside of \( R(0, n(p_c + \epsilon, t)) \). Let \( \text{rad}(C_x) \) be the radius of the cluster that contains \( x \). It is known that in homogeneous percolation

\[
P^*_{p_c-\epsilon}(\text{rad}(C_x) \geq k) \leq Ce^{-\gamma_r k},
\]

for some constants \( C \) and \( \gamma_r \) that depend on \( p_c - \epsilon \). So, if \( n = n(p_c + \epsilon, t) \) and we pick \( C_1 \) large enough then

\[
P^*_{p_c-\epsilon}(\text{rad}(C_x) \geq C_1 \log n) \leq n^{-3},
\]

for all \( x \in \mathbb{L} \). It follows from (3) that, with high probability, for all \( j \in I_n \): (i) there is no left to right dual crossing of any \( R_j \) and hence each \( R_j \) has an open top to bottom crossing; and (ii) there is a left to right open crossing of all of the \( R^j \).

![Figure 2: (i) and (ii) give us a net of interweaving crossings.](image)

Let \( G(x, t) \) be the event that \( \text{rad}(C_x) > 2C_1 \log n \). It is easy to see that if \( \|x - y\| > 4C_1 \log n \) then \( G(x, t) \) and \( G(y, t) \) are independent. Bounding the second moment of \( |C_0(t) \cap R_{i,j}| \) and using Chebyshev’s inequality in conjunction with a union bound over all of the boxes gives the desired result.

After the results mentioned above were proved, we learned about gradient percolation. In 1985, Sapoval, Rosso, and Gouyet [26] considered a model in which a site \( (x, y) \) is occupied with probability

\[
p(y) = 1 - \frac{2}{\pi^{1/2}} \int_{0}^{y/(2t^{1/2})} e^{-u^2} du.
\]

This formula arose from a model in which particles do the simple exclusion process in the upper half-space and the \( x \)-axis is kept occupied by adding particles at empty sites. They looked at the geometry of the boundary of the connected component containing the \( x \)-axis, finding that the front was fractal with dimension \( D_f = 1.76 \pm 0.002 \). This paper has been cited 395 times according to Google Scholar. Proving rigorous result about the boundary was mentioned as an open problem in the survey Beffara and Sidorovicius [2] wrote for the Encyclopedia of Mathematical Physics, a five volume set first published in 2004 by Elsevier.

In 2008 Pierre Nolin [22] proved rigorous results for a related percolation model on the two dimensional honeycomb lattice. In the homogeneous version the plane is tiled with hexagons.
that are black with probability \( p \) and white with probability \( 1 - p \). This is equivalent to site percolation on the triangular lattice. Since the pioneering work of Kesten [19] in the early 1980s, it has been known that the critical value for this model is \( 1/2 \). In 2001 Smirnov and Werner [27] used conformal invariance and work of Kesten [20] on scaling relations to rigorously compute critical values for this model.

![Figure 3: Nolin's parallelogram](image)

Nolin considered percolation in a parallelogram with height \( N \), length \( \ell_N \), and interior angles of 60 and 120 degrees, with sites black with probability \( 1 - y/N \) when \( 0 \leq y \leq N \). As in our result, the boundary of the cluster of black sites containing the \( x \)-axis will be close to the line \( y = N/2 \). Writing \( \approx N^a \) for a quantity that is bounded below by \( N^{a-\delta} \) and above by \( N^{a+\delta} \) for any \( \delta > 0 \), Nolin proved the following result, predicted in [26].

**Theorem 4.** The boundary of the cluster containing the \( x \)-axis remains within \( \approx N^{4/7} \) of the line \( y = N/2 \) and has length \( \approx N^{3/7} \ell_N \).

To connect with the original work in [26], Nolin says “one can expect to observe a nontrivial limit, of fractal dimension 7/4, with an appropriate scaling (in \( N^{4/7} \)) of the axes, but the critical exponents obtained do not correspond to a fractal dimension of the limiting object.”

Since it is expected, but not yet proved, that the critical exponents are the same for bond percolation on the square lattice, we cannot convert Nolin’s result into a theorem about our model. To make the connection between our result and his, let

\[
N = n(p_c, t) = c_{p_c, \alpha} t^{1/\alpha},
\]

where \( c_{p, \alpha} \) is defined in (1). Changing variables

\[
\rho((yN, 0), t) = 1 - \exp(-t(c_{p_c, \alpha} t^{1/\alpha})^{-\alpha}) = 1 - \exp(-y^{-\alpha} \log 2) \equiv f(y).
\]

Near 1 we have \( f(1 + \delta) = 1/2 + f'(1) \delta + o(\delta) \). Theorem 1 and 2 imply that we can confine our attention this region. Only near the corners of the right-edge of \( R(0, N) \) do we notice a difference between a model with probabilities that depend on \( x \) and ours that depend on \( ||x|| \), so it is reasonable to expect that the conclusion of Theorem 4 will hold for our model. Note that the formula for \( f(y) \) tells us that boundary fluctuations will not depend on \( \alpha \) but the density profile of \( C_0(t) \) will.
Figure 4: $C_0$ for $\alpha = 1$ when $n(p_c, t) = 150$ ($t = 104$). The ambient box has radius 150.

3 Proof of Theorem 1

Proof. Let $N = n(p_c - \epsilon, t)$. Using (3) and the fact that $P$ and $P^*$ are the same (except for being defined on different lattices)

$$P_{p-\epsilon}(\text{rad}(C_x) \geq C_1 \log N) \leq N^{-2}. \quad (4)$$

Let $B_N$ be the event that there is an open path from $\partial R(0, N)$ to $\partial R(0, N + C_1 \log N)$. To bound $P(B_N)$ note that if there is such an open path then there is one that stays entirely in the annulus $R(0, N + C_1 \log N) - R(0, N)$ where all of the bonds are open with probability $p_c - \epsilon$. Using (4) with a union bound gives

$$P(B_N) \leq \sum_{x \in \partial R(0,N)} P_{p_c-\epsilon}(\text{rad}(C_x) \geq C_1 \log N) \leq 8/N \rightarrow 0.$$ 

This implies

$$P(\exists x \in C_0(t) : ||x|| \geq N + C_1 \log N) \rightarrow 0$$

which proves the desired result.

4 Proof of Theorem 2

We fix a time $t$, let $n = n(p_c + \epsilon, t)$ and partition the box $R(0, n)$ into two sets of strips $R_j$ and $R^j$, as described in Section 1. Define the following pair of events:

- $A_j = \{\exists$ a top-to-bottom crossing in $R_j = [jC_1 \log n, (j + 1)C_1 \log n] \times [-n, n]\},$
- $A^j = \{\exists$ a left-to-right crossing in $R^j = [-n, n] \times [jC_1 \log n, (j + 1)C_1 \log n]\}.$
Lemma 5. For \( j \in I_n \), (i) \( P(A_j) \geq 1 - n^2 \) and (ii) \( P(\bigcap_{j \in I_n} A_j \cap A^j) \geq 1 - 2n^{-1} \).

Proof. By symmetry, it suffices to prove (i) for the events \( A^j \). Denote the left and right edges of \( R^j \) by \( \partial^L R^j \) and by \( \partial^R R^j \), respectively, and by using the dual lattice \( \mathbb{L} = \mathbb{Z}^2 + (1/2, 1/2) \) defined in Section 1, the complement of \( A_j \)

\[
A^{j,c} = \bigcup_{x \in \partial^L R^j} \{ \exists \text{ an open path from } x \text{ to } \partial^R R^j \text{ in } \mathbb{L} \}.
\]

Using (3) with a union bound, we have

\[
P(A^{j,c}) \leq |\partial^L R^j| P(\text{rad}(C_{p_c-\epsilon}(x)) > C_1 \log(n)) \leq n \cdot n^{-3} = n^{-2},
\]

proving our first claim.

To prove (ii), note that we have a total of \( \leq 2n \) horizontal and vertical strips and thus

\[
P \left( \bigcup_{j \in I_n} A^j_c \cup A^{j,c} \right) \leq 2n \cdot n^{-2} = 2n^{-1}.
\]

Lemma 5 guarantees that there exists a “net” with mesh-size \( C_1 \log n \) throughout \( R(0, n) \). It is necessary to show that \( C_0 \) is captured by this net.

Lemma 6. \( P(\text{there exists a closed edge in } [-C_1 \log n, C_1 \log n]^2) \to 0 \).

Proof. Let \( R \) denote the square in the lemma statement. Since \( t = cn^\alpha \) for some \( c > 0 \), it follows that \( \max_{x \in R}(1 - \rho(x, t)) \leq \exp(-cn^\alpha/C_1 \log n) \). Using this estimate in a union bound over the \( 4C_1^2 \log 2 \) edges in \( R \) gives the claimed convergence. \( \square \)

We now consider a second partition of our original box \( R(0, n) \), by tiling it with boxes \( R_{i,j} = [in^a, (i+1)n^a] \times [jn^a, (j+1)n^a] \), centered at \( x_{i,j} \), as described in Section 1. We will argue that the density of open sites computed in each rectangle \( R_{i,j} \), is, with high probability, close to the percolation probability when bonds are open with probability \( \rho(x_{i,j}, t) \).

For points \( x \) inside an arbitrary box \( R_{i,j} \), we examine the behavior of \( \theta(\rho(x, t)) \), which is the percolation probability probability measure for bond percolation with parameter \( \rho(x, t) \). The following result shows that, as \( t \to \infty \), \( \theta(\rho(x, t)) \) remains almost constant as \( x \) varies within \( R_{i,j} \).

Lemma 7. Let \( n = n(p_c, t) \) and \( a < 1 \). As \( n \to \infty \)

\[
\sup \{ |\theta(\rho(x, t)) - \theta(\rho(y, t))| : \|x - y\| \leq 2n^a \} \to 0.
\]

Proof. Since \( p \to \theta(p) \) is uniformly continuous, and \( \theta(p) = 0 \) for \( p < p_c \), it suffices to show that

\[
\sup \{ |\rho(x, t) - \rho(y, t)| : \|x - y\| \leq n^a, \|x\|, \|y\| \leq n \} \to 0.
\]
A little algebra gives
\[ \rho(x, t) - \rho(y, t) = e^{-t\|x\|^\alpha}(1 - e^{-t\|y\|^\alpha - \|x\|^\alpha}). \]

Suppose first that \( \|x\| \leq n^b \) where \( a < b < 1 \). The second term is \( \leq 1 \). Since \( n = c(p_\epsilon) t^{1/\alpha} \) the first is \( \leq \exp(-ct^{1-b}) \to 0 \).

If \( \|x\| \geq n^b \) and \( \|x - y\| \leq n^a \) then for large \( n \), \( \|y\| \geq n^b/2 \). Let \( u \) be the point in \( \{x, y\} \) with smaller norm and let \( v \) be the one with larger norm. Notice that
\[ \|u\|^{-\alpha} - \|v\|^{-\alpha} \leq \|u\|^{-\alpha} - (\|u\| + 2n^a)^{-\alpha} = \int_{\|u\|}^{\|u\| + 2n^a} (\alpha x^{-\alpha-1}) \, dx \leq 2n^a \max_{\|x\| \leq \|u\| + 2n^a} (\alpha x^{-(\alpha+1)}). \]

Since \( b > a \) and \( \alpha > 0 \), we have
\[ \|u\|^{-\alpha} - \|v\|^{-\alpha} \leq 2\alpha n^a (n^b/2)^{-(1+\alpha)} \to 0. \]

This completes the proof. \( \square \)

**Lemma 8.** Let \( \theta_{i,j} = \theta(\rho(x_{i,j}, t)) \). For each \( \delta > 0 \), there is a constant \( C_\delta \), independent of \( i, j \in I_n \) and of \( \delta \), so that
\[ P \left( |C_0 \cap R_{i,j} - \theta_{i,j} n^{2a}| > \delta n^{2a} \right) \leq \frac{C_\delta \log n^2}{\delta^2 n^{2a}}. \]

**Proof.** To argue this, we define the following random variable
\[ S_{i,j} = \sum_{y \in R_{i,j}} 1\{\text{rad}(C_y \geq 2C_1 \log n)\}, \]
where \( C_1 \log n \) represents the lengths of the short sides of the rectangles \( R_j \) and \( R^j \). For all \( y \in R_{i,j} \), let \( A_y = \{\text{rad}(C_y \geq 2C_1 \log n)\} \). Recalling that this set of rectangles generates with high probability a net of open horizontal and vertical crossings, we note that \( S_{i,j} = |C_0 \cap R_{i,j}| \).

We now center the variable \( S_{i,j} \) around its mean and define:
\[ \tilde{S}_{i,j} = S_{i,j} - ES_{i,j} = \sum_{y \in R_{i,j}} (1\{A_y\} - \theta_y), \]
where \( \theta_y = P(A_y) \) for all \( y \). Since \( E(S_k) = 0 \), we have
\[ \text{var} \left( \tilde{S}_{i,j} \right) = E(\tilde{S}_{i,j}^2) = E \left( \sum_{x, y \in R_{i,j}} (1\{A_y\}1\{A_x\} - \theta_x \theta_y) \right). \]

The random variables \( 1\{A_x\} \) and \( 1\{A_y\} \) are independent, if \( |x - y| \geq 4C_1 \log n \). Using this observation and the fact that \( |E(1\{A_x\}1\{A_y\}) - \theta_{x,y}^2| \leq 1 \), we obtain
\[ E(\tilde{S}_{i,j}^2) = \sum_{|x - y| < 2C_1 \log n} \left( E(1\{A_x\}1\{A_y\}) - \theta_{x,y}^2 \right) \]
\[ = |\{(x, y) \in R_{i,j} : \|x - y\| < 4C_1 \log n\}| \leq C_2 n^{2a}\log^2 n. \]
Using Chebyshev’s inequality gives

\[ P(|\bar{S}_{i,j}| > \delta n^{2a}) \leq \frac{C_2 n^{2a} \log^2 n}{\delta^2 (n^{2a})^2} = \frac{C_2 \log^2 n}{\delta^2 n^{2a}}. \]

Since Lemma 7 implies

\[ |R_{i,j}|^{-1} \sum_{y \in R_{i,j}} \theta_y - \theta_{i,j} \to 0 \]

this proves the lemma. \[ \square \]

Given this series of results, we can quickly establish Theorem 2.

**Proof of Theorem 2.** By Lemma 6 \( \mathbb{C}_0 \) connects to the “net” from Lemma 5. Thus, it contains a crossing of every strip \( R^j \) and \( R_j \) for \( j \in I_n \). Next, note that for each \( \delta > 0 \)

\[
P \left( \sup_{(i,j) \in \Lambda(t, \epsilon)} |D_{i,j}(t) - \theta(P(x_{i,j}, t))| > \delta \right) \leq \sum_{(i,j) \in \Lambda(t, \epsilon)} P(|S_{i,j} - \theta_{i,j} n^{2a}| > \delta n^{2a}).
\]

Using Lemma 8 the above is

\[ \leq n^{2-2a} \frac{C_2 \log^2 n}{\delta^2 n^{2a}} \leq n^{2-4a} \frac{C_2 \log^2 n}{\delta^2} \to 0, \]

since \( a > \frac{1}{2} \). \[ \square \]

**Proof of Corollary 3.** Observe that \( |\mathbb{C}_0(t)| = \sum_{i,j \in I_n} |\mathbb{C}_0(t) \cap R_{i,j}| \). Theorem 2 implies with probability tending to 1 we have

\[
\frac{1}{n^2} \left| \sum_{i,j \in I_n} |\mathbb{C}_0(t) \cap R_{i,j}| - \sum_{i,j} \theta_{i,j} \right| \to 0.
\]

Scaling space by \( t^{1/a} = O(n) \) and noting that the squares now have side length \( O(n^{a-1}) \), we have with probability tending to 1

\[
\frac{1}{t^{2/a}} \sum_{i,j} \theta_{i,j} \to \int \int (1 - \exp(-\|x\|^{-a})) dx_2 dx_1,
\]

which completes the proof. \[ \square \]

## 5 Oriented Poisson Percolation

Recall that the subgraph resulting from homogeneous percolation described in Section 1 is static, that is, it could represent a medium that remains uniformly porous over time. In Section 2, we introduced Poisson percolation, which has a stochastically growing set of open edges. This could potentially model a medium that becomes more porous over time. We
continue this latter case in this second part of the paper, and study a process which could model a medium whose pores can only arise in two directions of the lattice.

To formally define our new case of Poisson percolation, consider the oriented lattice \( \mathcal{L} = \{(m, n) \in \mathbb{Z}^2: m + n \) is even\} with oriented edges from \((m, n) \to (m + 1, n + 1)\) and \((m, n) \to (m - 1, n + 1)\). This is \( \mathbb{Z}^2 \) rotated 45° (see Fig. 5). We will again study the size and shape of the cluster, its density, and edge fluctuations. We think of \( m \) as space and of \( n \) as time. To avoid conflict with the parameter \( t \), we think of it as giving the age of the cluster.

\[
\text{Figure 5: The oriented 2D lattice containing the origin}
\]

Fix \( \beta > 0 \). An edge with midpoint \((x, y)\) and \( y > 0 \) is open with probability

\[
\rho(y, t) = 1 - \exp(-ty^{-\beta}).
\] (5)

Note that what we call ‘\( \beta \)’ here was denoted by ‘\( \alpha \)’ in Sections 2–4. Keeping with convention, we will use ‘\( \alpha \)’ to denote the limiting speed of the rightmost edge in oriented percolation.

Let \( n(p, t) = \max\{y: \rho(y, t) \geq p\} \) be the largest height at which edges are open with probability \( \geq p \). A little algebra shows that

\[
n(p, t) \sim c_{p, \beta} t^{1/\beta} \quad \text{where} \quad c_{p, \beta} = (-\log(1-p))^{-1/\beta}.
\] (6)

We write \((x, m) \to (y, n)\) if there is a path of open edges from \((x, m)\) to \((y, n)\). Let

\[
\mathbb{C}_0(t) = \{(x, n): (0, 0) \to (x, n)\},
\]

and let

\[
f(y) = \rho(yt^{1/\beta}, t) = 1 - \exp(-y^{-\beta}).
\] (7)

Define \( y_c \) by \( f(y_c) = p_c \), where \( p_c \approx 0.64470019 \) (see [18, p. 5242]) is the critical value \( \inf\{p: P(|\mathbb{C}_0| = \infty}\} \) where \( \mathbb{C}_0 \) is the cluster containing the origin. Note that \( \mathbb{C}_0 \) is the cluster containing 0 in homogeneous oriented percolation, whereas \( \mathbb{C}_0 \) is for Poisson percolation. The oriented case has fewer symmetries so the shape is more interesting than a square (see Fig. 6).
5.1 Size and shape of the cluster

To define the limiting shape of $C_0$ we need to introduce the right-edge speed in homogeneous percolation. Consider homogeneous percolation with parameter $p$ on $\mathcal{L}$. Following [9], we let

$$r_k = \sup\{x: \exists y \leq 0 \text{ with } (y, 0) \to (x, k)\}$$

be the rightmost site at height $k$ that can be reached from a site in $(-\infty, 0] \times \{0\}$. The subadditive ergodic theorem guarantees the existence of a limiting speed $r_k/k \to \alpha(p)$ for $p \geq p_c$. Obviously $\alpha(1) = 1$. It is known that $\alpha(p_c) = 0$. When $p < p_c$ the system dies out exponentially fast so $\alpha(p) = -\infty$.

Letting $g(0) = 0$ and $g'(y) = \alpha(f(y))$ for $0 \leq y \leq y_c$ we define our limiting shape

$$\Gamma = \{ (x, y): |x| < g(y), 0 \leq y \leq y_c \} \subseteq \mathbb{R}^2.$$ 

Intuitively the shape result is

$$C_0(t)/t^{1/\beta} \to \Gamma.$$ 

To prove the result, it is convenient to work on the unscaled lattice. Let $r^0_t(k) = \max\{x: (x, k) \in C_0(t)\}$ be the right edge of $C_0(t)$ at height $k$, and let $\ell^0_t(k) = \min\{x: (x, k) \in C_0(t)\}$ be the left edge at height $k$. It is convenient to have $g$ defined for all $y > 0$ so we let $g(y) = g(y_c)$ for $y > y_c$. Let

$$\Gamma_t(y) = t^{1/\beta} g(y/t^{1/\beta}).$$

Throughout the paper we will let $N = n(p_c, t)$.

**Theorem 9.** For any $\eta > 0$, as $t \to \infty$,
(i) \( P(\mathcal{C}_0(t) \subset \mathbb{R} \times [0, (1 + \eta)N]) \to 1 \), and

(ii) \( P\left(-\alpha \Gamma_t(k) \leq \ell^0_t(k), \quad r^0_t(k) \leq (1 + \eta)\Gamma_t(k) \text{ for all } k \leq (1 + \eta)N \right) \to 1.\)

Proving Theorem 9 (i) is easy because our percolation process is subcritical when \( y > N \). To prove Theorem 9 (ii) we fix \( m \) (it does not grow with \( t \)) and decompose \( \mathbb{Z} \times [0, (1 + \eta)N] \) into \( m \) strips, \( \mathbb{Z} \times [z_i, z_{i+1}] \), so that \( \alpha_i = \alpha(\rho(z_i, t)) = 1 - i/m \) for \( i < m \). We dominate the process in each strip by using homogeneous percolation with probability \( p_i = \rho(z_i, t) \). Large deviation estimates on the distance of the right edge from \( \alpha \) from \([10]\) allow us to prove that \( \mathcal{C}_0 \) lies to the left of a piecewise linear function whose slope is \( \alpha_i \) in each strip.

The next result gives a corresponding lower bound.

**Theorem 10.** For any \( \eta > 0 \), as \( t \to \infty \)

\[ P(\ell^0_t(k) \leq -\alpha \Gamma_t(k) \text{ and } (1 - \eta)\Gamma_t(k) \leq r^0_t(k) \text{ for all } k \leq (1 - \eta)N) \to 1. \]

Again we divide space into strips \( \mathbb{Z} \times [z_i, z_{i+1}] \), but now we lower bound the process by using homogeneous percolation with probability \( p_{i+1} = \rho(z_{i+1}, t) \). In each strip we use a block construction to relate our process to a 1-dependent oriented bond percolation on \( \mathbb{Z}^2 \) with parameter \( p = 1 - \epsilon \). On the renormalized lattice the right edge has speed close to 1. When we map the path of the right edge back to the Poisson percolation process, we get a piecewise linear function that serves as a lower bound on the location of \( r^0_t(k) \).

### 5.2 Cluster density

Let \( P_p \) be the probability measure for oriented bond percolation on \( \mathcal{L} \), when edges are open with probability \( p \). Let \( \mathcal{C}_0 \) be the open cluster containing the origin, and let \( \theta(p) = P_p(|\mathcal{C}_0| = \infty) \). Let

\[ G(t, \eta) = \bigcup_{y=0}^{(1-\eta)N} \left[-\alpha \Gamma_t(y), (1 - \eta)\Gamma_t(y)\right] \times \{y\}. \]

Intuitively, our next result says that near \((x, y) \in G(t, \eta)\) the density of points in \( \mathcal{C}_0(t) \) will be close to \( \theta(\rho(y, t)) \). To state this precisely, fix \( 1/2 < a < 1 \) and tile the plane with boxes of side length \( N^a \):

\[ R_{i,j} = [iN^a, (i + 1)N^a] \times [jN^a, (j + 1)N^a], \]

and let \((x_{i,j}, y_{i,j})\) be the center of \( R_{i,j} \). Let \( D_{i,j} = |\mathcal{C}_0(t) \cap R_{i,j}|/N^{2a} \) be the fraction of points in \( R_{i,j} \) that belong to \( \mathcal{C}_0(t) \) and let \( \Lambda(t, \eta) = \{(i, j) : R_{i,j} \subset G(t, \eta)\} \) be the indices of boxes that fit inside \( G(t, \eta) \).

**Theorem 11.** For any \( \eta, \delta > 0 \), as \( t \to \infty \),

\[ P\left(\sup_{(i,j) \in \Lambda(t, \eta)} |D_{i,j}(t) - \theta(\rho(y_{i,j}, t))| > \delta\right) \to 0. \]
5.3 Boundary fluctuations

The first three results were laws of large numbers. We will now consider the fluctuations of the right edge \( r_t(k) \). When

\[
C^k_0(t) := C_0(t) \cap (\mathbb{Z} \times \{k\}) \neq \emptyset
\]

we have \( r^0_t(k) = r_t(k) \). \( r_t(k) \) has the advantage that it is well defined even if \( C^k_0(t) = \emptyset \). In the homogeneous case, Galves and Presutti [13] were the first to prove such a central limit theorem for the supercritical contact process. Letting \( r_t \) be the rightmost occupied site at time \( t \) in the contact process with birth rate \( \lambda \) when initially all points \( y \leq 0 \) are occupied, they showed that

\[
\frac{r_{ns} - \alpha(\lambda) n s}{\sigma(\lambda) \sqrt{n}} \Rightarrow B_s.
\]

Here \( B_s \) standard Brownian motion and \( \Rightarrow \) is weak convergence of stochastic processes. Their proof also applies to oriented percolation. It implies that, if we start with the nonpositive integers occupied, then there is a constant \( \sigma(p) \) so that as \( n \to \infty \)

\[
\frac{r_{[ns]} - \alpha(p) n s}{\sigma(p) \sqrt{n}} \Rightarrow B_s.
\]

Two years later Kuczak [21] simplified the proof by introducing what he called break points: times \( T_i \) at which the right-most particle starts an oriented percolation that does not die out. In this case, for \( i \geq 1 \), the increments \((r_{T_{i+1}} - r_{T_i}, T_{i+1} - T_i)\) are i.i.d. Using his method, we prove the analogue for Poisson percolation.

**Theorem 12.** As \( t \to \infty \),

\[
\frac{r_t([Nu]) - \int_0^{Nu} \alpha(p(y,t))dy}{N^{1/2}} \Rightarrow W_u,
\]

where \( W_u, 0 \leq u < 1 \) is a Gaussian process with independent increments. It holds that \( EW_u = 0 \) and

\[
EW_u^2 = \frac{1}{N} \int_0^{Nu} \sigma^2(p(y,t))dy.
\]

Given the result for the homogeneous case, this conclusion is what one would expect to hold; if we divide the space into a large number of thin strips we have a sequence of homogeneous oriented percolation processes.

Very little is known rigorously about critical exponents for oriented percolation, so we are not able to prove a mathematical analogue of Nolin’s result. However, we can give a physicist style derivation of the following:

**Conjecture 13.** Fluctuations in the height of \( C_0(t) \) are of order \( N^{0.634} \).

First, we recall that oriented percolation has two correlation lengths. The correlation length in time, \( L_\parallel \) can be defined simultaneously for the subcritical and supercritical cases by

\[
\gamma_\parallel(p) = - \lim_{n \to \infty} \frac{1}{n} \log P_p(n \leq \tau^0 < \infty) \quad L_\parallel(p) = 1/\gamma_\parallel(p),
\]
where $\tau^0$ is the extinction time of the process starting from only 0 occupied. The correlation length in space $L_\perp$ has two different definitions for $p < p_c$ and $p > p_c$. Let $\xi_n^0$ be the the set of occupied sites at time $n$, and define $R_n^0 = \sup\{x : (x, n) \in \xi_n^0 \text{ for some } n\}$. Also define

$$
\gamma_\perp(p) = - \lim_{n \to \infty} \frac{1}{n} \log P_p(R_0^\perp \geq n) \quad L_\perp(p) = 1/\gamma_\perp(p) \quad \text{for } p < p_c,
$$

$$
\gamma_\perp(p) = - \lim_{n \to \infty} \frac{1}{n} \log P_p(\tau^{\{-n, \ldots, n\}} < \infty) \quad L_\perp(p) = 1/\gamma_\perp(p) \quad \text{for } p > p_c.
$$

Here $\tau^{\{-n, \ldots, n\}}$ is the extinction time of the process starting from $\{-n, \ldots, n\}$ occupied. These limits exist due to supermultiplicativity (i.e. $P(R_0^\perp \geq m + n) \geq P(R_0^\perp \geq m)P(R_0^\perp > n)$). See [12] for more details, and some other definitions.

The corresponding critical exponents are defined by

$$
L_\parallel(p) \approx |p - p_c|^{-\gamma_\parallel} \quad L_\perp(p) \approx |p - p_c|^{-\gamma_\perp}.
$$

Here $\approx$ could be something as precise as $\sim C|p - p_c|^{-\gamma}$ or $\log L(p)/\log |p - p_c| \to -\gamma$. Numerically, see [18, equation (15)]

$$
\gamma_\parallel = 1.733847 \quad 2\gamma_\perp = 2.193708.
$$

Nolin gives the following “hand-waving” argument for his result [22, page 1756]. If we are at distance $N^b$ behind the front, then $p - p_c = O(N^{b-1})$ and the correlation length is

$$
|p - p_c|^{-\nu_\parallel} = O(N^{(1-b)/\nu_\parallel}).
$$

If $b = (1-b)\nu_\parallel$, i.e., $b = \nu_\parallel/(1+\nu_\parallel)$, then the correlation length matches the distance behind the front. In this case the physical interpretation of the correlation length implies that the percolation process will look like the critical case. Nolin’s proof of the localization of the front, see [22, Theorem 6], is not long, but it is based on properties of sponge crossing, which will be difficult to generalize to the oriented case. However, there has been some recent work in that direction by Duminil-Copin, Tassion, and Teixera [8], as well as Sakai [25].

### 6 Oriented percolation toolbox

Here we state additional definitions and facts that we will need in the proofs of our theorems. The first is a simple observation that percolation is monotonic in the parameter.

**Fact 1.** Let $G_p \subseteq \mathcal{L}$ be the random subgraph obtained in homogeneous oriented percolation with parameter $p$. There exists a coupling such that if $p < p'$, then $G_p \subseteq G_{p'}$.

This follows by coupling the Bernoulli random variables on each each edge in the canonical way. A similar statement holds in Poisson percolation.

**Fact 2.** Let $G(t)$ be the set of all open edges at time $t$ in Poisson percolation. Fix a subset of edges $H$ in $\mathcal{L}$ and let

$$
p^- = \min\{\rho(x,t) : x \in H\}, \quad p^+ = \max\{\rho(x,t) : x \in H\}.
$$

There exists a coupling such that $G_{p^-} \cap H \subseteq G(t) \cap H \subseteq G_{p^+} \cap H$. 


The estimate in [10, (1) Section 7] bounds the probability that there is a path from 0 to $\mathbb{Z} \times \{k\}$.

**Fact 3.** For any $\delta > 0$, there is a constant $\gamma = \gamma(\delta) > 0$ so that

$$P_{p-\delta}(\xi^0_0 \neq 0) \leq e^{-\gamma n}.$$ 

We are also interested in the speed of the rightmost particle in supercritical homogeneous percolation where we assume all edges in $(-\infty, 0] \times \{0\}$ are open. Recall that $\alpha(p) = \lim_{k \to \infty} r_k/k$ is the limiting speed. [10, (3) Section 11 and Section 14] gives that

**Fact 4.** $\alpha(\cdot)$ is a continuous, strictly increasing function of $p$ for $p$ in $[p_c, 1]$. Moreover, for any $\delta > 0$, the function $\theta(p) = P_p(|C_0| = \infty)$ is Lipschitz continuous on $[p_c + \delta, 1]$.

[10, (2) Section 7] also gives the following estimate.

**Fact 5.** If $p > p_c$ and $\beta > \alpha(p)$, then there are constants $0 < \gamma, C < \infty$ that depend on $p$, and $\beta$ so that

$$P_p(r_n > \beta n) \leq Ce^{-\gamma n}.$$ 

Results in [10, Section 12] imply that

**Fact 6.** If $p \neq p_c$ there exists $\gamma > 0, C < \infty$ such that $P(n \leq \tau^0 < \infty) \leq Ce^{-\gamma n}$.

We will make use of the dual process to oriented homogeneous percolation when proving Theorem 11. This is the process obtained by keeping the same edges open, but reversing the orientation so that edges point southwest and southeast. Note that the dual process has the same law as usual oriented percolation. Thus, Fact 6 also holds for the survival time of a component started at $w$ in the dual percolation.

Supercritical percolation on $L$ almost surely contains an infinite component. Translation invariance of the lattice ensures that an individual edge has probability $\theta(p)$ of belonging to this component. Let $\tau^H$ denote the length of the longest surviving path started from a site in $H$. This is proven in [10, Section 10].

**Fact 7.** There exists $0 < \gamma, C < \infty$ that depend on $p > p_c$ such that for any $A \subseteq L$ it holds that

$$P(\tau^A < \infty) \leq Ce^{-\gamma |A|}.$$ 

Some of our proofs involve comparison with one-dependent oriented percolation. One-dependence means that the values on edges that share a common vertex are correlated, but edges without a common vertex are independent. This type of percolation is analyzed in [10]. Consider one-dependent oriented percolation in which the marginal distribution for each edge is such that it is open with probability at least $1 - \epsilon$. Let $C = \{w : \text{ for some } x \leq 0, (x, 0) \to w\}$, and let $s_k = \sup\{x : (x, k) \in C\}$. According to [10, Theorem 2; Section 11],

**Fact 8.** If $0 < q < 1$ and $\epsilon < 3^{-36/(1-q)}$, then there are constants $0 < \gamma, C < \infty$ so that

$$P(s_k \leq qk) \leq Ce^{-\gamma k}.$$
7 Proof of the Theorem 9

We start by proving (i). Let $\delta > 0$ be small. For $i = 1, 2$, let $y_i = n(p_c - i\delta, t)$. On $(y_1, \infty)$ we use Fact 1 to dominate Poisson percolation by homogeneous percolation in which bonds are open with probability $p_c - \delta$. We have $y_i \sim c_i t^{1/\beta}$ for constants $c_1 < c_2$. Let $k = y_2 - y_1$. Note that at height $y_1$, all the $x$-coordinates of points of $C_0(t)$ are in $[-y_1, y_1]$. It follows from Fact 3 that for large $t$,

$$P(C_0(t) \cap (\mathbb{Z} \times \{y_2\}) \neq \emptyset) \leq 2c_1 t^{1/\beta} \exp(-\gamma(c_2 - c_1) t^{1/\alpha}) \to 0. \quad (8)$$

If $\delta$ is small, then $y_2 < (1 + \eta)N$ and we have the desired upper bound on the height.

Theorem 9 (ii) follows from the following two lemmas. We subdivide time by introducing probabilities $p_i$, $1 \leq i \leq m - 1$ so that $\alpha(p_i) = 1 - i/m$, and let $p_0 = 1$, $p_m = p_c - 2\delta$. Note that these values are well defined by Fact 4. We will choose the value of $m$ appropriately for $\eta$ in just moment. Let $z_0 = 0$ and $z_i = n(p_i, t)$ for $i \geq 1$. The last interval $(z_{m-1}, z_m]$ is longer so that $z_m = y_2$. When $z_i < y \leq z_{i+1}$, we use Fact 2 to bound our system from above by oriented percolation with probability $p_i$, which has edge speed $= 1 - i/m$.

We define sequences $u_i, v_i$ for $0 \leq i \leq m - 1$ inductively by $u_0 = \delta$

$$v_i = u_i + (1 - i/m)(z_{i+1} - z_i), \quad u_{i+1} = v_i + \delta.$$ 

Now define a function $h_t(x)$ to be linear on $[z_i, z_{i+1})$, with $h_t(z_i) = u_i$ and

$$\lim_{y \uparrow z_{i+1}} h_t(y) := h_t(z_{i+1}-) = v_i.$$

Figure 7: Region defined by $h_t(k)$ when $m = 4$. Notice that the slopes of the segments $(u_i, v_i)$ are 1, 4/3, 2, and 4, i.e., 1 over the maximum edge speed in the interval.
Lemma 14. As \( t \to \infty \), \( P(r_k^0(t) \leq h_t(k) \) for all \( k \leq z_m \) \( \to 1 \).

**Proof.** Let \( 1 \leq i < m \). Suppose that \( r_{z_i}(t) \leq v_{i-1} \). To prove the result, it is enough to show that as \( t \to \infty \)

\[
P(r_k(t) \leq h_t(k) \) for \( z_i \leq k < z_{i+1} \) \( \to 1 \). \tag{9}
\]

When \( i = 0 \), the dominating process has \( p_0 = 1 \) so

\[
P(r_k^0(t) \leq h_t(k) \) for \( z_0 \leq k < z_1 \) \( = 1 \).
\]

Now suppose \( i > 0 \). When \( k \in [z_i, z_i + u_i - v_{i-1}) \), it is impossible for the process to reach \( h_t(k) \) since the \( x \)-coordinate of the right-most particle can increase by at most 1 on each step. In order to get from \( v_{i-1} \) to \( v_i \) in time \( z_{i+1} - z_i \), the right edge would have to travel at an average speed of more than \( 1 - (i - 1)/m \). Using Fact 5, and summing over \( k \in [z_i + u_i - v_{i-1}, z_{i+1}] \) proves (9).

Lemma 15. Let \( \eta > 0 \). If we take \( m \) large enough and \( \delta \) small, then \( h_t(y) \leq (1 + \eta)\Gamma_t(y) \) for all \( y \in [0, (1 + \eta)N] \).

**Proof.** We begin by noting that Fact 4 implies that \( \alpha(f(z)) \) is decreasing while \( f(z) > p_c \). If \( m \) is large enough then \( \alpha(p_1) - \alpha(p_{i-1}) < \eta/2 \) for \( 1 \leq i < m \) and \( \alpha(p_{m-1}) = 1/m < \eta/2 \).

To prove the result, now note that if \( i < m \) then

\[
\Gamma_t(z_i) - \Gamma_t(z_{i-1}) = \int_{z_{i-1}}^{z_i} \alpha(f(y)) \, dy
\]

\[
h(z_i) - h(z_{i-1}) = \alpha(f(z_{i-1}))(z_i - z_{i-1}).
\]

So, by the choices we have made above,

\[
h(z_i) - h(z_{i-1}) < (1 + \eta/2)(\Gamma_t(z_i) - \Gamma_t(z_{i-1})).
\]

Now, if \( \delta \) is small enough \( h(y) < (1 + \eta/2)\Gamma_t(y) \) for \( y < z_{m-1} \). On \([z_{m-1}, z_m]\),

\[
h(y) - h(z_{m-1}) < (\eta/2)(y - z_{m-1}).
\]

If \( \delta \) is small enough we have \( h(y) < (1 + \eta)\Gamma_t(y) \) for \( y < (1 + \eta)N \). \( \square \)

8 Proof of Theorem 10

To get the cluster at 0 started, we observe that it with high probability contains all possible sites within distance \( t^{b/\beta} \) with \( 0 < b < 1 \).

Lemma 16. Let \( \mathcal{K}(n) = \{(x, y) : 0 \leq y \leq n, \text{and } |x| \leq y \} \). For any \( 0 < b < 1 \), as \( t \to \infty \)

\[
P(\mathcal{K}(t^{b/\beta}) \subseteq C_0) \to 1.
\]

**Proof.** By (5), each edge in \( \mathcal{K}(t^{b/\beta}) \) is closed with probability \( \leq \exp(-t^{1-b}) \). Since there are fewer than \( t^{2b/\beta} \) edges, the result follows from a union bound. \( \square \)
8.1 Constructing the renormalized lattice

The next step is renormalizing the lattice to compare Poisson percolation with 1-dependent oriented percolation with parameter $1 - \epsilon$. As in the previous section, we introduce probabilities $p_i, 1 \leq i \leq m - 1$ so that $\alpha(p_i) = 1 - i/m$. We let $z_0 = t^{b/\beta}$ and for $1 \leq i \leq m - 1$ let $z_i = n(p_i, t)$. The key ingredient for describing the density is to show that the rightmost edge of $C_0$ stays to the right of $(1 - \eta)\Gamma$. When $1 \leq i \leq m - 1$ and $z_{i-1} < y \leq z_i$, we bound our system from below by oriented percolation in which edges are open with probability $p_i$, and the edge speed is $\alpha_i = 1 - i/m$.

To lower bound the process in which each edge is open with probability $p_i$ we will use a block construction. So that the lattices associated with different strips will fit together nicely, the $x$ coordinates of the sites in the renormalized lattice will always be at integer multiples of some fixed constant $L$, and we will vary the heights of the blocks. In the $i$th strip $z_{i-1} < y \leq z_i$, we let

$A_i^{m,n} = (c_i^m, d_i^n) = (mL, T_i + n(1 + \delta)L/\alpha_i)$

and let $B_i^{0,0} = -A_i^{0,0}$.

To begin to define the renormalized lattice, we let $T_1 = z_0$. In the $i$th strip, the points in the renormalized lattice are

$$(c_i^m, d_i^n) = (mL, T_i + n(1 + \delta)L/\alpha_i)$$

where $m$ and $n$ are integers so that $m + n$ is even, and $T_i + n(1 + 3\delta)L/\alpha_i \leq z_i$. The last condition is to guarantee that all the edges we consider in the $i$th part of the construction are open with probability at least $p_i$. Note that in each strip the vertical index $n$ begins at 0.

To continue the construction when $i < m - 1$ we let

$$T_{i+1} = \max\{T_i + n(1 + \delta)L/\alpha_i + n(1 + 3\delta)L/\alpha_i \leq z_i\}.$$

Let $A_i^{m,n} = (c_i^m, d_i^n) + A_i^{0,0}$, let $B_i^{m,n} = (c_i^m, d_i^n) + B_i^{0,0}$ and let $I_i^m = c_i^m + (-0.5\delta L, 0.5\delta L)$. The parallelograms are designed so that (see Figure 8)

(i) at height $d_{n+1}^i = d_n^i + (1 + \delta L)/\alpha$, $A_i^{m,n}$ fits in $I_{m+1}^i$.

(ii) at height $d_n^i + (1 + 3\delta L)/\alpha$ the $x$ component of the left edge of $A_i^{m,n}$ is the same as that of the right edge of $B_{m+1,n+1}^i$.

We say that the good event $G_{i,0}^i$ occurs if

(I) in $A_{0,0}^i$ there is a path from the bottom edge to the top edge.

(II) in $B_{0,0}^i$ there is a path from the bottom edge to the top edge.
Note that the existence of the paths in (I) and (II) are determined by the edges in $A^i_{0,0}$ and $B^i_{0,0}$ respectively. The parallelograms are constructed to overlap in such a way (see Figure 8) that, if there is a path in $A^i_{m,n}$, and there are paths in $B^i_{m+1,n+1}$ and $A^i_{m+1,n+1}$, then there is a path from the bottom edge of $A^i_{m,n}$ to the top edges of $A^i_{m+1,n+1}$ and $B^i_{m+1,n+1}$.

We define $G^i_{m,n}$ by translation. In [10, Section 9] it was shown that, given $\epsilon > 0$, for $L \geq L_i$ it holds that $P(G^i_{0,0}) \geq 1 - \epsilon$. Let $\bar{L} = \max_{1 \leq i \leq m-1} L_i$. Suppose $\delta < 0.01$, let $R^i_{0,0} = [-1.5L, 1.5L] \times [0, (1 + 3\delta)L/\alpha_i]$, and let

$$R^i_{m,n} = (c^i_m, d^i_n) + R^i_{0,0}.$$ 

The existence of paths in parallelograms that do not overlap is independent. The box $R^i_{0,0}$ intersects $R^i_{2,1}$, $R^i_{2,-1}$, $R^i_{1,0}$, $R^i_{1,-1}$, and $R^i_{-2,-1}$, so the construction is one dependent (as described after Fact 7).

### 8.2 Lower bound for the right-most particle

To facilitate comparison with oriented percolation, we will renumber the rows of renormalized sites with $z_0 \leq y \leq z_{m-1}$ by the nonnegative integers $0, 1, 2, \ldots, M$ and let $\tau_0, \tau_1, \ldots, \tau_M = \inf\{k : z_k \geq (1 - \eta)N\}$ be the corresponding heights in Poisson percolation on the usual lattice. In our construction, we will pick $L$ large and then let $t \to \infty$, so there are constants $C_1$ and $C_2$ so that $C_1 t^{1/\beta} \leq M \leq C_2 t^{1/\beta}$. Also, fix $0 < b < 1$ and let $K = K(t, b) = \max\{j : \tau_j \leq t^{b/\beta}\}$ be the last parallelogram below height $t^{b/\beta}$. Note also that $K \to \infty$ since $L$ is fixed.
Consider 1-dependent oriented percolation in which edges are open with probability $1 - \epsilon$. Fix two numbers $0 < q < q' < 1$. Define the set of edges $E_K = [q' K, K] \times \{0\}$, and

$$s'_k = \max\{x: \text{there exists } w \in E_K \text{ with } w \to (x, k)\}$$

to be the rightmost edge at height $k$ accessible from $E_K$. By Lemma 16 we know that $E_K$ will have all edges open with probability going to 1. Moreover, we claim that as $t \to \infty$,

$$P(s'_k \geq qk \text{ for all } k \geq 0) \to 1. \tag{10}$$

Fact 7 guarantees that the probability $E_K$ contains a path to infinity goes to 1 as $t \to \infty$. Since a path can displace at most one unit to the left at each height, the first time we could have $s'_k < qk$ is at height $(q' - q)K/2$. Applying the bound from Fact 8 to the rightmost edge started from $E_K$, we then have

$$P(s'_k \leq qk \text{ for some } k \geq 0) \leq \sum_{k=(q' - q)K/2}^{M} Ce^{-\gamma k} \to 0,$$

since $K \to \infty$.

To get a lower bound on the right-edge in the Poisson percolation process, we consider the mapping $(s'_k, k) \to (Ls'_k, \tau_k)$ from the renormalized lattice back to the original lattice. Because of (10), we consider the image of the line $y = qk$ under this map. It is given by a piecewise linear function with

- $h(0) = 0$, and $h(t) = qk$ for $k \in [0, z_0]$, and
- $h(k) = h(z_{i-1}) + qa_i(k - z_{i-1})$ for $k \in [z_{i-1}, z_i]$ with $1 \leq i \leq m - 1$.

The renormalized sites that make up the right edge will map to the right of this curve. The paths that connect them will lie in the associated parallelogram from Section 8.1, so they cannot go further than $(1 + 3\delta)L/\alpha_i$ to the left of $h$. It follows that

$$P(r_t(k) \geq h(k) - (1 + 3\delta)L/\alpha_i \text{ for all } z_{i-1} \leq t \leq z_i) \to 1.$$

On $[z_{i-1}, z_i]$, $h$ has slope $qa_i$ while $\Gamma_t$ increases at rate $\leq \alpha_{i-1} = \alpha_i + 1/m$. If $m$ is large enough then $\alpha_i \geq (1 - \eta/2)\alpha_{i-1}$ for $1 \leq i \leq M$. It follows that if $q$ is chosen close enough to 1 then $h(k) - (1 + 3\delta)L/\alpha_i \geq (1 - \eta)\Gamma_t(k)$ for all $z_{i-1} \leq k \leq z_i$ and $1 \leq i \leq m - 1$. The proof for the left edge is similar.

### 9 Proof of Theorem 11

Consider the site $w = (x, y)$ with $z_i \leq y < z_{i+1}$, so that it is in the $i$th strip of the unscaled lattice. Fact 6 implies

$(\star)$ if $n_i = (1/\gamma_i) \log(C_i N^4)$ and the dual process started from $w$ survives for $n_i$ units of time then the probability $w \not\in C_0$ is $\leq 1/N^4$. 

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This says that $C_0$ is closely approximated by the points whose dual survives for time $n_i$. Let

$$R_{j,k} = [jN^a, (j+1)N^a] \times [kN^a, (k+1)N^a]$$

and suppose that all the points in $R_{j,k}$ are in the $i$th strip.

Let $A_w = \{ \tau^w \geq k_i \}$, and count the number of points in $R_{j,k}$ with a long-surviving dual with

$$S_{j,k} = \sum_{w \in R_{j,k}} 1\{A_w\}.$$ 

Since $|R_{j,k}| = N^{2a}$, ($\star$) ensures that

$$P(S_{j,k} \neq |C_0 \cap R_{j,k}|) \leq N^{2a-4}.$$ 

Since there are no more than $N^{2-2a}$ boxes with high probability this holds for all of them.

So, it suffices to study $S_{j,k}$. We start by centering it. Let $\theta_w = P(A_w)$, and define

$$\bar{S}_{j,k} = S_{j,k} - ES_{j,k} = \sum_{w \in R_{j,k}} 1\{A_w\} - \theta_w.$$ 

The advantage of considering $A_w$ is that if $w = (x,y)$, then the event $A_w$ is determined by edges in $[x-n_i, x+n_i] \times [y-n_i, y]$. So if $\|w-w'\| > 2n_i$, then the indicator random variables are independent. Using the bound

$$|1\{A_w \cap A_{w'}\} - P(A_w)P(A_{w'})| \leq 1$$

when $\|w-w'\| \leq 2n_i$, we obtain

$$E\bar{S}_{j,k}^2 \leq N^{2a} \cdot 4n_i^2 \leq \frac{4}{\gamma_i^2} N^{2a} \log^2 C_i N^4.$$ 

(11)

Using (11) with Chebyshev’s inequality gives for $\delta > 0$ and some $C'_i > 0$

$$P(\bar{S}_{j,k} > \delta N^{2a}) \leq \frac{C'_i N^{2a} \log^2 N}{\delta^2 N^{4a}} = O(N^{-2a} \log^2 N).$$ 

(12)

Since there are $O(N^{2-2a})$ many different boxes $R_{j,k}$, it follows from (12) that

$$P\left( \sup_{(j,k) \in \Lambda(\eta,\eta)} |\bar{S}_{j,k}| > \delta N^{2a} \right) = O(N^{2-4a} \log^2 N).$$

The right term is $o(1)$ since $a > 1/2$. To relate this back to $C_0$ we note that $f(y)$ defined in (7) is Lipschitz continuous, and by Fact 4 so is $\theta(p)$ on $[p_c + \delta, 1]$. Thus,

$$\sup\{ |\theta(p(w, t)) - \theta(p(w', t))| : w, w' \in R_{j,k} \} \leq CN^{a-1} \rightarrow 0.$$ 

Using this and Fact 6, we can replace the $P(A_w)$ terms in $S_{j,k}$ with a representative $\theta_{j,k} = \theta(p((x_j, y_k), t))$, and Theorem 11 follows.
10 Proof of Theorem 12

Recall $N = n(p_c, t)$. In our process, the right edge particle cannot be part of an infinite cluster, so we define renewals to be times at which the rightmost particle lives for time at least $\log^2 N$. This is motivated by the bound from Fact 6. To get started, if $b < 1$ then the state at time $t^{b/\beta}$ is an interval and the rightmost particle survives for $\log^2 N$ with probability $\rightarrow 1$ by Lemma 16. Suppose $t_i$ is the time of the $i$th renewal and let $p_i$ be the probability bonds are open at that time. On $[t_i, t_i + 2 \log^2 N]$ bonds are open with probability $\geq p_i - c(\log^2 N)/N$. The 2 is to allow us to find the renewal point and then verify it works. The bonds of interest are in a triangle with point at $(r_i, 0)$, and height $2 \log^2 N$ so we can by Fact 2 couple the inhomogeneous system with a system with probabilities $p_i$ so that with high probability there are no errors.

Unfortunately the increments in the right-edge defined in this way are not independent. If $r_i - r_{i-1}$ is large then the $p$ for the next increment will be smaller. To fix this we will again divide $[0, N]$ into strips by choosing $\alpha(p_i) = 1 - i/m$ and $z_i = n(p_i, t)$ but now we will use $m = N^{0.6}$ strips. For renewals that begin in the strip $z_i < y < z_{i+1}$ we will upper bound the movement of the right edge by using $p = p_i$ and lower bound by using $p = p_{i+1}$. The large number of strips guarantees that the difference between the upper and lower bounds on $E(r_k - r_{k-1})$ will be $N^{-0.6}$ so when we sum $N$ of these terms the result is $O(N^{0.4}) = o(N^{0.5})$

Kuczak [21] has shown that when $p$ is fixed $r_i - r_{i-1}$ has an exponential tail, so using the Lindberg-Feller theorem, see e.g., Theorem 3.4.5 in [11], on the upper bound and on the lower bound

$$\frac{\sum_{k=1}^{n} (r_k - r_{k-1}) - E(r_k - r_{k-1})}{\sqrt{\sum_{k=1}^{n} \text{var}(r_k - r_{k-1})}} \Rightarrow \chi \quad (13)$$

where $\chi$ is standard normal. To convert this to continuous time note that for homogeneous percolation

$$E(r_i - r_{i-1}) = \alpha(p)E_p(t_i - t_{i-1})$$

because $Er(t)/t \rightarrow \alpha(p)$,

$$\text{var}(r_i - r_{i-1}) = \sigma^2(p)E_p(t_i - t_{i-1})$$

because $\text{var} r(t)/t \rightarrow \sigma^2(p)$.

Let $M(s)$ be the number of renewals needed to get to height $s$. Replacing $n$ by $M(s)$ in (13), the result is

$$\frac{r_i(s) - \int_{0}^{s} \alpha(p(y, t)) \, dy}{\int_{0}^{s} \sigma^2(p(y, t)) \, dy} \Rightarrow \chi.$$ 

Taking $s = Nu$ and replacing the denominator by $\sqrt{N}$, we have convergence of the one dimensional distributions to the desired limit. Since the increments of the limit process are independent, convergence of finite dimensional distributions follows easily. Since

$$\sum_{k=1}^{n} (r_k - r_{k-1}) - E(r_k - r_{k-1})$$

is a square integrable martingale, it is not hard to use the $L^2$ maximal inequality to check that the tightness criteria that can be found for example in [3, Section 8]. Alternatively one can invoke [17, Theorem 4.13].
References


