Differential Geometry Tools for Data Analysis

by

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Henry D. Pfister

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

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ABSTRACT

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Abstract

The thesis is divided into two parts: the moving anchor parameterization method and the comparative morphology through certain geometric functionals.

In the first part we show that parameterizing points on curves or surfaces or their higher dimensional analogs by their distances to well-chosen anchor points can lead to representations that are much less curved. We then use this feature to construct approximation methods that are very simple and that achieve results comparable in quality to standard higher-order methods.

The second part is motivated by the observation that the Dirichlet normal energy of a 2-dimensional surface embedded in 3D measures, in some sense, the deviation of the surface from the minimal surface for a functional linear in the integrated squared mean and integrated gauss curvatures. A modified version of this functional with different parameters is also of interest in applications; this suggests that their minimal surfaces (with respect to which these modified functionals measure the deviation) are of interest as well. The second part of the thesis concerns the analysis of the associated partial differential equations and the construction procedure for such examples.
Acknowledgements

I will be forever indebted to my advisor, Professor Ingrid Daubechies, for her generosity, support, compassion, understanding and, most importantly, for nurturing me to think independently. I am also indebted to her for giving me the opportunity to study Mathematics with her despite being a Physics student and for a long time bearing with my lack of mathematical rigor and precision both in theory and in the applied domain. I consider myself extremely lucky to have observed and learned from her the art of approaching problems and being able to say a great deal about them through her very powerful yet extremely clear deductions.

I am also indebted to my Co-Advisor Professor Colleen Robles for opening the door of the world of Cartan to me. This is something that was not possible for me to navigate by myself. Colleen guided me with great patience and helped me immensely in making me comfortable with Cartan’s works, whereupon I have understood geometry much better than I used to do. I thank her immensely for also going through my writing and giving great feedback.

I am also extremely thankful to Dr. Barak Sober for highly insightful discussions. Many thanks to Professors Jim Nolen and Henry Pfister for agreeing to be my committee members. Many thanks also to Professor Robert Bryant for an excellent course on Representation Theory, the subject that was unapproachable for me before.

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Contents

Abstract iv
Acknowledgements v
List of Figures x
List of Tables xi
1 Introduction 1
  1.1 Manifold-valued Data .................................. 1
    1.1.1 Contribution of Part 1 of the thesis ............... 3
  1.2 Shape Quantification ................................. 3
    1.2.1 Contribution of Part 2 of the thesis ............... 4
I Anchor Parameterization 5
2 Preliminaries for Part 1 6
  2.1 Geometrical Preliminaries: Review of Cartan’s Method of Moving Frames .................................. 6
  2.2 Review of Manifold Moving Least Squares ............... 17
3 Moving Anchor Parameterization 20
  3.1 Anchor points and other structures ..................... 21
    3.1.1 Computation of the curvature at a point on $\Phi(\mathcal{M})$ ................................. 35
    3.1.2 Effect of Weighting in the Moving Anchor Point Transformation ......................... 59
  3.2 Numerical Examples .................................. 71
    3.2.1 2D curves ........................................ 71
    3.2.2 Denoising ......................................... 74
### Table of Contents

#### 4 Conclusions and Future Work

#### II Functionals for Shape characterization

5 Statement of the Problem

5.1 Motivation for our problem

5.1.1 Moving frame approach to maps between surfaces

5.2 Review of Exterior Differential Systems

6 Dirichlet densities linear in $H^2$ and $K$

6.1 The Question

6.1.1 Notations

6.1.2 Case-A

6.1.3 Case-B

6.1.4 Case-C

6.1.5 Case-D

6.2 Conformal immersion

6.2.1 Case of Conformal Maps

6.3 EDS I for Case A

6.3.1 Analysis of solutions

6.4 EDS II for Case A

6.4.1 Cartan’s Test

6.4.2 Prolongation

7 Conclusions

8 Appendix

8.1 Maple computation in proof of Lemma 6.3.4
8.2 Maple computation in proof of Proposition 6.4.3 . . . . . . . . . . . . 132

Bibliography 133
# List of Figures

2.1 Illustration of the Frenet Frame ........................................ 12

2.2 Illustration of the MMLS .................................................. 18

3.1 Illustration of Anchor Point parameterization .......................... 22

3.2 Illustration of the helix under the moving anchor point transformation 29

3.3 Illustration of the weighting scheme with the helix example. ....... 30

3.4 Illustration of the sine curve under the moving anchor point transformation .................................................. 32

3.5 Illustration of a closed 3D curve under the moving anchor point transformation. .................................................. 33

3.6 Illustration of a sphere under the moving anchor point transformation 34

3.7 Random curves for the example in Subsection 3.2.1 .................... 72

3.8 Histogram of the root-mean square error of fitting a straight line for the chopped components (of the order of $O(10^4)$) the 9000 curves. .. 73

3.9 Number of curves with different RMSEs ............................... 74

3.10 Denoising of the spectrum of a matrix. ................................. 76

3.11 Denoising of a tooth point cloud. ....................................... 77

3.12 Histogram for the reconstructed errors with different noise levels for the tooth point cloud ........................................ 78

3.13 Comparing MMLS and moving APT for helix with uniform noise .. 80

3.14 Comparing MMLS and moving APT for helix with gaussian noise .. 80

4.1 Illustration of the dependence of the location and number of anchors relative to the manifold with the number of chopped parts. ........ 81
List of Tables

2.1 Indices convention for one-forms and vectors corresponding to tangent, normal and ambient spaces of $\mathcal{M}$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

3.1 Indices convention for one-forms and vectors corresponding to tangent, normal and ambient spaces of $\Phi(\mathcal{M})$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
Chapter 1

Introduction

1.1 Manifold-valued Data

To learn from data one typically must assume that the data have some kind of “hidden structure.” In this thesis, we are concerned with data sets for which the underlying structure is a manifold — more precisely, we assume the data lie on or near a low dimensional submanifold in (euclidean) space. The tools we shall use are mostly geometric in nature; i.e. we build methodologies that exploit the geometry of manifolds for dealing with scientific data.

Following [Lee13] we shall use the term “manifold” to mean a smooth manifold, i.e. a manifold with a maximal atlas that is $C^\infty$. [ A smooth manifold is a topological space with a maximal smooth atlas. A topological manifold of dimension $\mu$ is a topological space which is Hausdorff, second-countable and locally euclidean of dimension $\mu$.] Although such a requirement might seem restrictive, it still covers many useful applications. For most datasets from physical processes, the smoothness assumption is reasonable, as the processes often have an exact analytical description. Many datasets from biological processes also satisfy this requirement, even if the underlying process may not have an exact analytical description.

Our goal is also to focus on data manifolds where the intrinsic dimension is clearly known. Such restrictions are justified in many applications. Casts of fossils of bones and teeth collected for anthropological studies are examples of two dimensional manifold-valued data that are typically densely sampled and corrupted by noise. It
is of considerable interest to denoise such surfaces so that one can directly work with the ground truth point cloud information. Other examples of manifold-valued data consist in finding and fitting $O(10^4)$ particle trajectories from a collision event in a particle accelerator experiment. These are one-dimensional curves and there are typically $O(10^5)$ hits from which the tracks have to be identified and then fitted to obtain parameters of interest.

Most of the manifold learning literature concerning investigation of point clouds focuses on dimensionality reduction and attempts to either prioritize the local neighborhood information or find a global embedding of data. Both local and global methods have been found to be efficient in determining the intrinsic dimension but have challenges like sensitivity to noise, introducing distortions etc. There has been a line of work to handle noise with the goal of finding intrinsic dimension of data [LLJM09],[LMR17]. The inspiration of Part 1 of the thesis stems from David Levin’s [Lev98] introduction of a moving least squares scheme to approximate hypersurfaces in euclidean spaces from noisy point cloud data when we know the intrinsic dimension of the data. Sober and Levin [SL19] generalized this method to submanifolds of arbitrary codimension; their method is called Manifold Moving Least Squares Squares (MMLS). It recovers the underlying manifold with an accuracy that is polynomial in the local distance between data points. MMLS also takes as an input the degree of the polynomial to be approximated and works under sufficiently dense sampling assumptions.

There are two ways this method may not be applicable for certain manifold-valued data. In the particle trajectories example, the trajectories are sparsely sampled, since there are typically typically 10-20 detectors with layers where the hits are recorded. On the other hand, there are examples of datasets where the data have a highly polynomial character and a very accurate interpolation is required. Such is the case
with potential energy surfaces of chemical reactions where the hypersurfaces in a high dimensional space have a very high-degree polynomial character and it is not computationally feasible to compute the values at each point in the data grid.

1.1.1 Contribution of Part 1 of the thesis

In Chapter 3, we introduce the moving anchor point parameterization that circumvents these challenges and thus enables the applicability of the approach to manifolds of unknown degree of non-linearity that may be only sparsely sampled. Indeed, in this method one can simply use a linear fit in the embedding space and save higher-order computations in high dimensions. Here we explicitly compute the reduction of curvature through this parameterization and show by examples the effectiveness of the method. Chapter 2 reviews the Cartan’s method of moving frames and a very brief introduction to the MMLS method is presented. Chapter 4 outlines the future work.

1.2 Shape Quantification

The second part of the thesis is inspired by the study in [BBL+11]. The authors in this paper studied a collection of shapes, namely lemur teeth, that have variable curviness and sharpness. The morphological variability is a reflection of variability in the animals’ diets. The question they studied was how one can infer diet from tooth morphology and consequently reconstruct the ecology of extinct organisms and paleoecosystems. They proposed the use of Dirichlet Normal Energy of the surface, which is the bending energy of a surface. The quantity is given by a linear combination of the integrals of the squared mean and gauss curvatures.

If we look at different combinations of these geometric functionals (involving $H^2$
and $K$), they could be interesting, in the sense that the numerical value could measure the deviation of the minimizing surface. We would like to mention that the use of geometric functionals is not new in the study of morphology. There is a considerable body of work in cosmology that began from [MBW93] and computes the so-called Minkowski functionals (integrals of mean, gauss curvatures, surface area and volume) that characterize the topology and geometry of large-scale structures of the universe. The novelty in our treatment is that we would like to use these functionals to measure deviations from ideal minimizing surfaces.

1.2.1 Contribution of Part 2 of the thesis

In Chapter 6, we establish two results. If we fix two Riemannian surfaces $\mathcal{M}$ and $\mathcal{N}$, and a positive function $L$ on $\mathcal{M}$, then local solutions exist for maps having the Dirichlet energy integrand to be $L$ and also satisfying the harmonic map Euler-Lagrange equation. We also establish identities involving $L$ that must be satisfied as integrability conditions. In Chapter 5, we state the problem and give a brief review of the method of exterior differential systems of Cartan that is used to establish the above results. Chapter 7 gives the conclusion and scope for future work.
Part I

Anchor Parameterization
Chapter 2

Preliminaries for Part 1

2.1 Geometrical Preliminaries: Review of Cartan’s Method of Moving Frames

In this chapter, we review the method of moving frames developed by Élie Cartan in the 1930s. An important application of this method is to calculate invariants of submanifolds in a coordinate-free way through an algebraic machinery afforded by differential forms. Before we describe the machinery in detail, we would attempt to intuitively motivate the framework.

The moving frame framework is related to and originated from the study of rigid body dynamics [CCL99]. Consider a free rigid body in 3-dimensional space. This has in total six degrees of freedom: three due to translation and three due to rotation (say, about a point $P$ on it). By a frame at $P$ we mean an orthonormal basis attached to $P$. It is called a body frame if it moves with the two kinds of motions of the body and is not fixed in space. For example, the three basis vectors of this frame can be along the principal axes of the rigid body. If the coordinate of $P$ at a time $t$ is $\vec{r}(t)$, then the velocities measured in a fixed frame and a body frame are related by

$$\left( \frac{d\vec{r}(t)}{dt} \right)_{\text{fixed frame}} = \left( \frac{d\vec{r}(t)}{dt} \right)_{\text{moving frame}} + \text{“fictitious” component} = \vec{v}(t) + \vec{\omega}(t) \times \vec{r}(t)$$

(2.1)

where $\vec{\omega}(t)$ is the angular velocity. The components of $\vec{v}(t)$ and $\vec{\omega}(t)$ determine the motion of the body completely. But due to conservation of linear and angular mo-
menta, their components satisfy a system of PDEs known as Euler equations. To generalize this motion to arbitrary dimensional space, Cartan’s method exploits the homeomorphism between the space of all such body frames and the motion (Lie) group. In the above case the motion group was SO(3). Then these consistency or integrability conditions become the fundamental structure equations of the Lie group, called the Maurer Cartan equations. In what follows, we will first examine the infinitesimal motion of a point in euclidean space and then see the case when the point is constrained to be on a submanifold.

We define a frame $F_r$ at $r \in \mathbb{R}^N$ as the tuple

$$F_r = (r; u) \text{ such that } u = (u_1, ..., u_N) \text{ is an O.N. basis of } T_r \mathbb{R}^N \cong \mathbb{R}^N \text{ at } r$$

and the frame bundle of $\mathbb{R}^N$ by $\mathcal{F}(\mathbb{R}^N)$

$$\mathcal{F}(\mathbb{R}^N) = \{F_r\} \text{ such that } r \in \mathbb{R}^N. \quad (2.2)$$

The triple $\mathcal{F}(\mathbb{R}^N), SO(N), \mathbb{R}^N$ constitutes a principal fiber bundle. More precisely:

- Given any element $g \in SO(N)$, the action of $SO(N)$ acting on $\mathcal{F}(\mathbb{R}^N)$ is given by

$$\mathcal{F}(\mathbb{R}^N) \times SO(N) \to \mathcal{F}(\mathbb{R}^N)$$

$$(r, u) \times g \mapsto (r, ug). \quad (2.4)$$

This group action is free, i.e., the only elements fixed under the action are those of the form $((r, u), \text{Id})$, where Id is the $N$-dimensional identity matrix.

- The projection map $\pi : \mathcal{F}(\mathbb{R}^N) \to \mathbb{R}^N$ given by $(r, u) \mapsto r$ is smooth; given $F = (r, u), F' = (r', u') \in \mathcal{F}(\mathbb{R}^N)$ we have $\pi(F) = \pi(F')$ if and only if $r = r'$, and
there must then exist a \( g \in \text{SO}(N) \) so that \( u' = ug \). Clearly, \( \mathcal{F}(\mathbb{R}^N)/\text{SO}(N) = \mathbb{R}^N \).

- If \( U \in \mathbb{R}^N \) is an open neighborhood then the local triviality of \( \mathcal{F}(\mathbb{R}^N) \) can be seen by considering the map

\[
U \times \text{SO}(N) \rightarrow \pi^{-1}(U)
\]

\[
\left( (x^1, \ldots, x^N), g \right) \mapsto \left( (x^1, \ldots, x^N) \sum_{j=1}^{N} g^j_1 \frac{\partial}{\partial x^j}, \ldots, \sum_{j=1}^{N} g^j_N \frac{\partial}{\partial x^j} \right)
\]

which clearly is a diffeomorphism. Also, if we write the inverse of this map as \( F \mapsto (\pi(F), \phi(F)) \), then it can be seen in a straightforward way that \( \phi(Fg') = \phi(F)g' \) with \( g' \in \text{SO}(N) \).

The third point enables one to consider the frame bundle as a smooth manifold since \( ((x^1, \ldots, x^N), [g_{ij}]) \) can serve as coordinate system on \( \pi^{-1}(U) \). The notation for the projection map \( \pi \) will be henceforth taken to be \( r \) since

\[
r : \mathcal{F}(\mathbb{R}^N) \rightarrow \mathbb{R}^N.
\]

Then given scalar-valued one forms \( \{ \omega^A \}_{A=1}^{N} \) on \( \mathcal{F}(\mathbb{R}^N) \), the push-forward of the above map can be expressed as a linear combination of the basis \( u \), considered as vector fields on \( \mathbb{R}^N \) (for a reference see Ex. 2.40 in ([Cle17])):

\[
dr = \sum_{A=1}^{N} \omega^A u_A.
\]

The \( \{ \omega^A \}_{A=1}^{N} \) are called solder forms. On the other hand, for \( A = 1, \ldots, N \) any vector
$u_A$ in basis $u$ at $r$ can be considered as the map

$$u_A : \mathcal{F}(\mathbb{R}^N) \to T_r\mathbb{R}^N.$$  \hfill (2.8)

Similar to (2.7) the differentials of these maps can be expressed in terms of scalar-valued forms $\omega^A_B$ on $\mathcal{F}(\mathbb{R}^N)$. The $\{\omega^B_A\}_{A,B=1}^N$ are called the connection forms. Due to the orthonormality of the vectors in $u$, for $A \neq B$ we have

$$0 = d\langle u_A, u_B \rangle = \langle du_A, u_B \rangle + \langle u_A, du_B \rangle = \langle \sum_C \omega^C_A u_C, u_B \rangle + \langle u_A, \sum_C \omega^C_B u_C \rangle = \omega^B_A + \omega^A_B.$$  \hfill (2.10)

Thus the connections forms are antisymmetric. Taking the exterior derivative of the first equation in (2.7), we obtain the First structure equation of Cartan

$$d\omega^A = \sum_{B=1}^N \omega^B \wedge \omega^A_B \quad \text{for} \quad A = 1, ..., N.$$  \hfill (2.11)

Similarly, taking the the exterior differentiation of the second equation in (2.9), we get the Second structure equation of Cartan

$$d\omega^B_A = \sum_{C=1}^N \omega^C_A \wedge \omega^B_C.$$  \hfill (2.12)

These structure equations are with respect to a connection defined on $\mathcal{F}(\mathbb{R}^N)$. Now,

---

1The equations (2.7 and 2.9)

$$du_A = \sum_{B=1}^N \omega^B_A u_B \quad \text{where} \quad du_A : T_{(r,u)}\mathcal{F}(\mathbb{R}^N) \to T_{u_A}(T_r\mathbb{R}^N) \cong \mathbb{R}^N$$  \hfill (2.9)

are analogous to the components of velocities $\vec{v}(t)$ and $\vec{\omega}(t)$ in 2.1.

2These are analogous to the Euler equations in the rigid body case.
let $\mathcal{M} \subset \mathbb{R}^N$ be a $\mu$-dimensional smooth submanifold and let $r \in \mathcal{M}$. Then

$$\mathcal{F}_{\mathcal{M}} = \{(r; u) \in \mathcal{F} \text{ s.t. } r \in \mathcal{M}, \text{ and } u \text{ is an } N\text{-dimensional basis for } \mathbb{R}^N\} \subset \mathcal{F}.$$  \hfill (2.13)

Since $r : \mathcal{F}_{\mathcal{M}} \rightarrow \mathcal{M}$, we have $dr \in T_r \mathcal{M}$. Thus the one-forms introduced in (2.7) must satisfy

$$\omega^a = 0 \quad \text{for } a = \mu + 1, \ldots, N. \hfill (2.14)$$

Intuitively, this means that the infinitesimal displacement is along the manifold. To distinguish between quantities corresponding to the tangent and normal spaces of the manifold, we adopt an indexing scheme inspired by [G+77] in Table (2.1).

**Table 2.1**: Indices convention for one-forms and vectors corresponding to tangent, normal and ambient spaces of $\mathcal{M}$. The notations used for both the dual form and connection form on $\mathcal{M}$ is taken to be $\omega$ and the distinction will be clear by the number of indices used. Adapted bases are those of the moving frame.

<table>
<thead>
<tr>
<th>Unique One Forms notation</th>
<th>Tangent Space Index</th>
<th>Normal Space Index</th>
<th>Ambient Space Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}$ $\omega$</td>
<td>$i, j, k, l, m, \ldots$</td>
<td>$a, b, c, \ldots$</td>
<td>$A, B, C, \ldots$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Standard Bases</th>
<th>Adapted Bases</th>
</tr>
</thead>
<tbody>
<tr>
<td>${e_A}$</td>
<td>${u_A} = {u_i}, {u_a}$</td>
</tr>
</tbody>
</table>

Let $U \subset \mathcal{M}$ be an open neighborhood and we fix a section

$$\sigma : U \rightarrow \mathcal{F}_{\mathcal{M}}.$$ \hfill (2.15)

Then $\sigma(r) = (u_1(r), \ldots, u_\mu(r))$ is a local coframing on $U$. If $\omega^1, \ldots, \omega^\mu$ are dual to
$u_1, \ldots, u_\mu$, then it is straightforward to show that

$$\omega^i = \sigma^* \omega^i \quad \text{for} \quad i = 1, \ldots, \mu \quad (2.16)$$

and that the $\{\omega^i\}$ form a coframing on $U$. We have $\omega^a = 0$ from (2.14). We also have

$$\nabla_{u_j} u_k = \sum_i \lambda_{jk}^i u_i \quad \text{with} \quad \lambda_{jk}^i \in C^\infty(U). \quad (2.17)$$

It can be shown that if

$$\omega_B^A = \sigma^* \omega_B^A \quad (2.18)$$

then

$$\omega_j^i = \sum_k \lambda_{jk}^i \omega^k. \quad (2.19)$$

We refer to these coefficients as connection coefficients. From the antisymmetry property in equation (2.10), the $\lambda_{jl}^i$ must satisfy

$$\lambda_{jl}^i = -\lambda_{lj}^i \quad (2.20)$$

With the notation in Table (2.1), equations (2.7) and (2.9) become

$$dr = \sum_{i=1}^\mu \omega^i u_i \quad (2.21)$$

$$du_A = \sum_{A=1}^N \omega_A^B u_B. \quad (2.22)$$

The first structure equation of Cartan in (2.11) becomes

$$d\omega^i = \sum_{j=1}^\mu \omega^j \wedge \omega_j^i \quad \text{for} \quad i = 1, \ldots, \mu \quad (2.23)$$
and the second structure equation in (2.12) can be written in a factorized way:

\[
\begin{align*}
    d\omega^B_A &= \sum_{j=1}^\mu \omega^j_A \wedge \omega^B_j + \sum_{a=\mu+1}^N \omega^a_A \wedge \omega^B_a \\
    &\quad \text{for } A, B = 1, \ldots, N. \quad (2.24)
\end{align*}
\]

**Figure 2.1:** Example of an adapted frame — the “Frenet Frame” on a space curve (in \( \mathbb{R}^3 \)).

In order to introduce the notion of curvature, we now state the Cartan’s lemma, which will be referred to throughout the thesis.

**Lemma 2.1.1.** [Cartan’s Lemma] Let \( \{\alpha_i\}_{i=1}^\mu \) be pointwise linearly dependent 1 forms on a \( \mu \)-dimensional manifold \( \mathcal{M} \). Let \( \{\beta_j\}_{j=1}^{\mu'} \) with \( \mu' \leq \mu \) be 1-forms on \( \mathcal{M} \) such that

\[
\sum_{j=1}^{\mu'} \beta_j \wedge \alpha_j = 0. \quad (2.25)
\]
Then
\[ \beta_j = \sum_i g_{ji} \alpha_i \] (2.26)

with \( g_{ji} = g_{ij} \) being a symmetric function on \( \mathcal{M} \).

With this result in hand, taking exterior derivative of (2.14), we obtain
\[ 0 = d\omega^a = \sum_{j=1}^{\mu} \omega^j \wedge \omega_j^a. \] (2.27)

By Cartan’s Lemma, \( \omega_j^a = \sum_{i=1}^{m} h_{ij}^a \omega^i \) with \( h_{ij}^a = h_{ji}^a \) (2.28)

where the \([h_{ij}^a]\) are the components of the second fundamental form. The vector-valued second fundamental form tensor is defined through
\[ \Pi = \sum_{i,j=1}^{\mu} \sum_{b=\mu+1}^{N} h_{ij}^b \omega^i \otimes \omega^j \otimes u_b. \] (2.29)

The curvature 2-forms are given by
\[ \Omega_j^i = \sum_{a=\mu+1}^{N} \omega_j^a \wedge \omega_a^i. \] (2.30)

The components of the Riemann curvature tensor are defined by
\[ \Omega_j^i = \frac{1}{2} \sum_{k,l=1}^{\mu} \text{Riem}_{ijkl} \omega^k \wedge \omega^l \] (2.31)

with symmetries
\[ \text{Riem}_{ijkl} = -\text{Riem}_{jikl} = \text{Riem}_{jilk}. \] (2.32)
Combining (2.28), (2.30) and (2.31), we get

\[ \text{Riem}_{ijkl} = \sum_{a=\mu+1}^{N} (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a). \] (2.33)

We can also express the exterior derivative \( d\omega^i \) as a linear combination of \( \omega^j \wedge \omega^k \) utilizing (2.19):

\[ d\omega^i = \sum_{j,k=1}^{\mu} \lambda^i_{jk} \omega^j \wedge \omega^k = \frac{1}{2} \sum_{j<k} \left( \lambda^i_{jk} - \lambda^i_{kj} \right) \omega^j \wedge \omega^k : = \frac{1}{2} \sum_{j<k} f^i_{jk} \omega^j \wedge \omega^k. \] (2.34)

Clearly,

\[ f^i_{jk} = -f^i_{kj} \quad \text{and} \quad f^i_{jk} = \lambda^i_{jk} - \lambda^i_{kj}. \] (2.35)

Together with \( \lambda^i_{jl} = -\lambda^j_{il} \), (2.35) then implies

\[ \lambda^i_{jk} = \frac{1}{2} \left( f^i_{ij} - f^i_{ki} - f^i_{jk} \right). \] (2.36)

**Remark 2.1.2.** For the readers familiar with the concept of metric tensor and Christoffel symbols, we would like to distinguish between the terms connection coefficients and Christoffel symbols. For an arbitrary basis \( \{v_i\}_{i=1}^{\mu} \) that is not orthonormal, we have their Lie bracket as

\[ [v_i, v_j] = \sum_k f^k_{ij} v_k. \] (2.37)

Then the connection coefficients are:

\[ \lambda^i_{jk} = \frac{1}{2} \Gamma^i_{jk} + \frac{1}{2} \left( f^i_{ij} - f^i_{ki} - f^i_{jk} \right). \] (2.38)

---

3Since \( \Omega^i_j = \sum_{a=\mu+1}^{N} \omega^i_a \wedge \omega^j_a = \sum_{k,l=1}^{\mu} \sum_{a=\mu+1}^{N} h_{ik}^a h_{jl}^a \omega^k \wedge \omega^l = \frac{1}{2} \sum_{k<l} \sum_{a=\mu+1}^{N} (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a) \omega^k \wedge \omega^l \).
Three interesting cases that can arise are:

- If the basis is orthonormal then $\Gamma^i_{jk} = 0$ for $i \neq j \neq k$.

- If the basis is holonomic, i.e., a coordinate basis, then all the $f^k_{ij}$ vanish.

- If the solder forms are Lie-group valued then the $f^k_{ij}$ are constants known as structure constants of that Lie group.

We recall that the sectional curvature of a tangent 2-plane $\Pi \subset T_r\mathcal{M}$ spanned by two orthonormal vectors $x, y$ is defined as

$$\text{Sec}(\Pi) = \text{Riem}(x, y, y, x). \quad (2.39)$$

For any $k \in C^\infty(\mathcal{M})$ we introduce a notational convention

$$dk = \sum_i k_i \omega^i. \quad (2.40)$$

**Lemma 2.1.3.** The sectional curvature and the connection coefficients introduced in (2.19) are related through

$$\text{Sec}_{ij} = 2 \left( -\left(\lambda^i_{ij}\right)_j - \left(\lambda^i_{jj}\right)_i + \sum_m \lambda^i_{jm} \lambda^m_{ji} - \sum_m \lambda^i_{jm} \lambda^m_{ij} - \sum_m \lambda^m_{ij} \lambda^i_{ji} + \sum_m \lambda^m_{ii} \lambda^m_{jj}\right). \quad (2.41)$$

**Proof.** Putting $A = j$ and $B = i$ in (2.24) we get

$$d\omega^i_j = \sum_m \omega^i_j \wedge \omega^i_m + \sum_a \omega^i_j \wedge \omega^i_a = \sum_m \omega^i_j \wedge \omega^i_m + \sum_{i,k} \frac{1}{2} \text{Riem}_{ijkl} \omega^l \wedge \omega^k. \quad (2.42)$$
We compute

\[
d\omega^i_j = \sum_m d(\lambda^i_jm^m) = \sum_{m,l} (\lambda^i_jm^m) \omega^l \wedge \omega^m + \sum_{m,l} \lambda^i_jm^m \omega^l \wedge \omega^m = \sum_{k,l} (\lambda^i_jk^k) \omega^l \wedge \omega^k + \sum_{m,l,k} \lambda^i_jm^m \lambda^m_lk^k \omega^l \wedge \omega^k.
\]

(2.43)

\[
\sum_m \omega^m_j \wedge \omega^i_m = \sum_{m,k,l} \lambda^m_jl^l \lambda^i_mk^k \omega^l \wedge \omega^k.
\]

(2.44)

These imply

\[
\frac{1}{2} \text{Riem}_{ijkl} = (\lambda^i_jk^k)_l - (\lambda^i_jl^l)_k + \sum_m \lambda^i_jm^m \lambda^m_lk^k - \sum_m \lambda^i_jm^m \lambda^m_kl^l - \sum_m \lambda^i_jl^l \lambda^m_mk^k + \sum_m \lambda^i_jm^m \lambda^m_ml^l.
\]

(2.45)

Hence, taking \(k = i\) and \(l = j\) we get

\[
\frac{1}{2} \text{Sec}_{ij} = \frac{1}{2} \text{Riem}_{ijji} = (\lambda^i_ji^i)_j - (\lambda^i_jj^j)_i + \sum_m \lambda^i_jm^m \lambda^m_i^i - \sum_m \lambda^i_jm^m \lambda^m_j^j - \sum_m \lambda^i_jm^m \lambda^m_i^i + \sum_m \lambda^i_jm^m \lambda^m_ml^l.
\]

(2.46)

and by the antisymmetry properties of the connection coefficients the relation follows.

Since \(M\) is compact, we must have

\[
|\nabla^i \text{Riem}| \leq K_i \quad \text{for } 0 \leq i \leq 2.
\]

(2.47)

Then we have the following result.

**Lemma 2.1.4.** For compact \(M\) the connection coefficients and their derivatives defined on a geodesic ball \(U\) of radius \(r\) are bounded by constants depending only on \(r\)
and the constants $K_0, K_1, K_2$ occurring in (2.47).

**Proof.** Let the Christoffel symbols in normal coordinates on $U$ be denoted by $\hat{\Gamma}$. Thus we can express the corresponding connection coefficients through

$$
\omega^i_j = \sum_m \hat{\Gamma}^m_{ij} dx^m = \sum_l \lambda^l_{ij} \omega^l.
$$

(2.48)

[Eic91] proves that then there exist constants $C_0, C_1$ such that

$$
|\Gamma^m_{ij}| \leq C_0(r, K_0, K_1) \quad |D\Gamma^m_{ij}| \leq C_1(r, K_0, K_1, K_2)
$$

(2.49)

where $r$ is the upper bound of the radius of the normal chart. [Sch01] proves a similar result for manifolds with boundaries. Since the coefficients of the metric tensor in normal coordinates are bounded by constants involving $K_0$ [Eic91], we have

$$
|\lambda^m_{ij}| \leq \lambda_0(r, K_0, K_1) \quad |(\lambda^m_{ij})_p| \leq \lambda_1(r, K_0, K_1, K_2).
$$

(2.50)

\[\square\]

### 2.2 Review of Manifold Moving Least Squares

We now very briefly review a manifold learning algorithm that we will frequently refer to in this thesis. The Manifold Moving Least Squares (MMLS) algorithm [SL19] directly approximates a manifold from scattered, possibly noisy, point cloud data. The input is a manifold embedded in euclidean space with bounded noise. The method assumes the intrinsic dimension of the manifold to be known. Moreover, there are assumptions on the ground truth sampling of the manifold: the fill-distance, density and separation of the point cloud, which the authors refer to as the $h$-$\rho$-$\delta$ set. The
approximation of the point cloud consists of two steps — the entire procedure is depicted in Figure 2.2.

**Step 1:** Finding an approximation $H$ of the Tangent Space

For a given point $r \in \mathbb{R}^N$ in the sample set $\{r_i\}$, an affine space $H$ and the approximate projection $q$ of $r$ on to $H$ are obtained by minimizing the functional

$$J(q, H; r) = \sum_{i=1}^{\text{# of samples}} d(r_i, H)^2 \theta_1(||r_i - q||)$$

with $r - q \perp H$ and constraints limiting the search space

$$\text{PCA done about } q$$

(2.51)

where $d(r_i, H)$ is the distance of $r_i$ to the closest point in $H$ and the weight $\theta_1$ is monotonically decaying and compactly supported.

**Step 2:** Polynomial Regression

Step 1 above gives $H$ as well as the projections $x_i$ of each of the points $r_i$. Since a submanifold can be considered as a function from the tangent space to the ambient space, one can write $f(x_i) = r_i$. This step finds a vector-valued polynomial $g$ that
best approximates $f$ by

$$g = \min_{p \in \Pi_2^d} \sum_{i=1}^{N} (p(x_i) - f(x_i))^2 \theta_2(||r_i - q||).$$  \hspace{1cm} (2.52)$$

The polynomial is indicated in green in Figure 2.2. The denoised output is then given by $\hat{r} = g(0)$ and the projection $P$ of $r$ with respect to the point cloud becomes $P(r) = g(0)$. 
Chapter 3

Moving Anchor Parameterization

In this chapter we introduce a map which enables a “warping” of a (sub)manifold embedded in euclidean space. The goal of the warping is to flatten the manifold, so that lower-order (and thus simpler) approximations can be used on the flattened version, which, when “unwarped” again, lead to highly accurate “curved” approximations to the original, with removal of some noise components. This idea is distinct from the Locally Linear Embedding [RS00], which seeks to make locally linear approximations to the manifold in its current embedding. Our map, as we will see, is not local. This does not mean it attempts to flatten the manifold globally: in most cases this is of course impossible. In order to carry out our flattening, we introduce a weighting scheme by means of which we “chop” the manifold adaptively into smaller pieces, each of which is then flattened by our procedure. We will eventually apply this parameterization to study interpolation and denoising of manifold-valued data. This chapter studies the geometry of the parameterization and the effect of the weighting, and applies this approach to datasets which are very sparse but highly accurate (i.e. they have very little noise). In section 3.2.2 we will discuss applications with data that have more noise; when data are noisy one needs to introduce additional assumptions on density of sampling.
3.1 Anchor points and other structures

Let $\mathcal{M}$ be a $\mu$-dimensional compact, connected, smooth Riemannian submanifold of $\mathbb{R}^N$ and $\mathbf{r} = (r^1, \ldots, r^N) \in \mathcal{M}$. With the notations in Table (2.1), we express the solder forms $\{\omega^i\}$ and connection forms $\{\omega^i_j\}$ on $\mathcal{T}_\mathbf{r}(\mathcal{M})$ through

$$dr = \sum_{i=1}^\mu \omega^i u_i \quad d\omega^i_j = \sum_{A=1}^N \omega^A_j \land \omega^i_A$$

(3.1)

for $i, j = 1, \ldots \mu$. We then define anchor points and the associated anchor point transformation as follows.

**Definition 3.1.1 (Anchor Points).** We call a set of points $\{\mathbf{a}_\alpha; \alpha = 1, \ldots, \Lambda\} \subset \mathbb{R}^N$ a set of anchor points for the manifold $\mathcal{M}$ embedded in $\mathbb{R}^N$ if the following hold:

- $\forall \alpha$, $\mathbf{a}_\alpha \notin \mathcal{M}$
- The convex hull of the $\mathbf{a}_\alpha$ is $N$-dimensional, i.e. the $\mathbf{a}_\alpha$ don’t all lie on a lower-dimensional hyperplane
- Denoting by $d_{\mathbf{r}, \alpha}$ the distance between $\mathbf{r} \in \mathcal{M}$ and anchor point $\mathbf{a}_\alpha$,

$$d_{\mathbf{r}, \alpha} = \sqrt{\sum_{A=1}^N (r^A - a^A_\alpha)^2}$$

(3.2)

the $\Lambda$-tuples $\{d_{\mathbf{r}, \alpha}\}_{\alpha=1}^\Lambda$ and $\{d_{\mathbf{r}', \alpha}\}_{\alpha=1}^\Lambda$ have to be different for all $\mathbf{r} \neq \mathbf{r}' \in \mathcal{M}$.

**Definition 3.1.2 (Anchor Point Transformation).** The anchor point transformation associated to the set of anchor points $\{\mathbf{a}_\alpha; \alpha = 1, \ldots, \Lambda\}$ is defined to be the smooth
mapping $\Phi : \mathcal{M} \to \mathbb{R}^\Lambda$ given by

$$
\mathbf{r} \mapsto \begin{pmatrix}
  d^2_{r,1} \\
  d^2_{r,2} \\
  \vdots \\
  d^2_{r,\Lambda}
\end{pmatrix} = \sum_{\alpha=1}^\Lambda d^2_{r,\alpha} \hat{e}_\alpha
$$

(3.3)

where $\{\hat{e}_\alpha\}_{\alpha=1}^\Lambda$ is the standard basis (in column form) in $\mathbb{R}^\Lambda$ and the $d^2_{r,\alpha}$ are given in (3.2). We schematically depict this mapping in Fig. 3.1.

![Figure 3.1: Anchor Point parameterization with $\Lambda$ anchors.](image)

Under the generic assumptions in our definition, the image of $\mathcal{M}$ under the anchor point transformation $\Phi$ in Definition 3.1.6 is a smooth submanifold of dimension $\mu$ in $\mathbb{R}^\Lambda$.

**Lemma 3.1.3.** Let $\mathcal{M}$ be as defined before and let the set $\{a_\alpha\}$ with $\alpha = 1, \ldots, \Lambda$ be a set of anchors satisfying generic assumptions as in Definition 3.1.1. Let $\Phi$ be the corresponding anchor point transformation as defined in (3.3). Then $\tilde{\mathcal{M}} := \Phi(\mathcal{M})$ is a smooth $\mu$-dimensional submanifold of $\mathbb{R}^\Lambda$.

**Proof.** It suffices to show the result for $\mathcal{U} \subset \mathcal{M}$ which is an open neighborhood of
\( r \in \mathcal{M} \). Let \( \mathcal{O} \subset \mathbb{R}^\mu \) be an open subset with coordinates \( \{ x^i \} \) on \( \mathcal{O} \) such that we have

\[
\varphi : \mathcal{O} \to \mathcal{U} \cap \mathcal{M} \\
x = (x^1, \ldots, x^\mu) \mapsto (r^1, \ldots r^N) = r
\]

be a smooth parameterization of \( \mathcal{U} \cap \mathcal{M} \). In order to show \( \Phi(\mathcal{M}) \) is an embedded submanifold of \( \mathbb{R}^{\Lambda} \), we need to prove that \( \Phi \circ \varphi \) is a smooth embedding (see Proposition 5.2 of [Lee13]).

The map \( \Phi \circ \varphi \) is clearly smooth because it is component-wise smooth. The injectivity of \( \Phi \circ \varphi \) follows from the third requirement in Definition 3.1.1. In addition, the Jacobians are non-degenerate by the following argument. The Jacobian i.e. the matrix for the differential of \( \Phi \circ \varphi \) at \( x \) is

\[
\begin{pmatrix}
(r - a_1)^t \\
(r - a_2)^t \\
\vdots \\
(r - a_\Lambda)^t
\end{pmatrix}
\begin{pmatrix}
\frac{\partial r^1}{\partial x^1} & \cdots & \frac{\partial r^1}{\partial x^\mu} \\
\frac{\partial r^2}{\partial x^1} & \cdots & \frac{\partial r^2}{\partial x^\mu} \\
\vdots & \cdots & \vdots \\
\frac{\partial r^N}{\partial x^1} & \cdots & \frac{\partial r^N}{\partial x^\mu}
\end{pmatrix}
\]

(3.5)

Since the differential of a map does not depend on the choice of charts, with a change of coordinate chart to \((\varphi')^{-1} : \mathcal{U} \cap \mathcal{M} \to \mathcal{O}\) we can express the Jacobian as

\[
\mathcal{J}(r) = \begin{pmatrix}
(r - a_1)^t \\
(r - a_2)^t \\
\vdots \\
(r - a_\Lambda)^t
\end{pmatrix}
\begin{pmatrix}
u_1, \ldots, u_\mu
\end{pmatrix} = 
\begin{pmatrix}
\langle r - a_1, u_1 \rangle & \langle r - a_1, u_2 \rangle & \cdots & \langle r - a_1, u_\mu \rangle \\
\langle r - a_2, u_1 \rangle & \langle r - a_2, u_2 \rangle & \cdots & \langle r - a_2, u_\mu \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle r - a_\Lambda, u_1 \rangle & \langle r - a_\Lambda, u_2 \rangle & \cdots & \langle r - a_\Lambda, u_\mu \rangle
\end{pmatrix}
\]

(3.6)

where \( \{ u_i \} \) is an orthonormal basis spanning \( T_r(\mathcal{U} \cap \mathcal{M}) \). To prove that \( \Phi \circ \varphi \) is an immersion, we need to show that the rank of the matrix in (3.6) has constant rank.
μ in the neighborhood. The matrix above is full rank if we can find at least one μth order minor which is non-zero. Without loss of generality we take the block of the first μ rows above and call it \( A \). For an arbitrary choice of anchors, the rows of \( A \) are linearly independent. Thus \( \text{rank}(A) = \mu \) and \( \det(A) \neq 0 \). Since the determinant is a continuous function, the assertion follows.

Since \( \Phi \circ \varphi \) is a smooth, injective immersion and \( \mathcal{O}' \) is compact, \( \Phi \circ \varphi' \) is a smooth embedding (see Proposition 4.22 of [Lee13]); the rank of the Jacobian being \( \mu \), it follows that \( \Phi(\varphi'(\mathcal{O}')) \) is a \( \mu \)-dimensional smooth embedded submanifold of \( \mathbb{R}^A \). □

In the remainder of this chapter, we shall study properties of the anchor point transformation and formulate conditions under which it is particularly useful. In particular, we shall see that the curvature of \( \varphi(M) \) can be significantly smaller than that of \( M \).

Choosing the anchors too far away from the manifold would not allow the embedding to carry optimally useful information about the manifold. Therefore, we will choose the anchors within a tube around the manifold, the definition of which we recall below.

**Definition 3.1.4** (Tube). [Gra12] A tube \( T(M, \tau) \) of radius \( \tau \) around \( M \) is given by

\[
T(M, \tau) = \{ a \in \mathbb{R}^N \text{ such that there is a straight line segment in } \mathbb{R}^N \\
with length \leq \tau, starting from } a \text{ and intersecting } M, meeting } M \\
\text{ orthogonally at the intersection} \}
\]
with the assumption that \( \tau \) is chosen such that the maps
\[
\exp_{N_p}\{(p, v) \in N_pM \mid \|v\| \leq \tau\} \rightarrow T(p, \tau)
\]
\( T(p, \tau) := \{a \in T(M, \tau); \text{the line segment connecting } a \text{ and } p \text{ has length } \leq \tau \) (3.7)
and meets \( \mathcal{M} \) orthogonally\}
defines a diffeomorphism between \( T(M, \tau) \) and
\( \{(p, v); p \in \mathcal{M}, (p, v) \in N_pM, \|v\| \leq \tau\} \).

**Remark 3.1.5.** The diffeomorphism assumption ensures that for each \( a \in T(M, \tau) \), there is only a a single \( p \in \mathcal{M} \) such that \( a \in T(p, \tau) \).

We are now ready to formulate some extra conditions that will be useful to impose on sets of anchor points and introduce the Moving Anchor Point Transformation.

**Definition 3.1.6 (\( \tau \)-tube anchor points).** Let \( \mathcal{M} \) be a compact, connected, smooth embedded submanifold (with or without boundary) of \( \mathbb{R}^N \) and \( T(M, \tau) \) be its tube with radius \( \tau \). We say that a set \( \mathcal{A} = \{a_\alpha \in \mathbb{R}^N; \alpha = 1, ..., \Lambda\} \) of anchor points is a \( \tau \)-tube set of anchor points if

1. \( a_\alpha \in T(M, \tau) \) \( \forall \alpha = 1, \ldots, \Lambda \), i.e.
\[
a_\alpha = p_\alpha + \sum_{b'=\mu+1}^{N} \tau_{b'} u_{b'} \quad \text{where } \tau_{b'} \leq \tau \quad \text{for some } p_\alpha \in \mathcal{M} \quad (3.8)
\]
and \( \{u_{b'}\} \) is an orthonormal basis spanning the normal space to \( \mathcal{M} \) at \( p_\alpha \).

2. If \( a_\alpha = p_\alpha + \sum_{b'} \tau_{b'} u_{b'} \in \mathcal{A} \) then the mirror point also belongs to \( \mathcal{A} \), i.e. \( p_\alpha - \sum_{b} \tau_{b'} u_{b'} \in \mathcal{A} \); for \( \alpha \leq \frac{\Lambda}{2} \), we denote its mirror point by \( a_{\alpha}^{\lambda} \). For \( \alpha \neq \beta \) with \( \alpha, \beta \leq \frac{\Lambda}{2} \), we require \( p_\alpha \neq p_\beta \).
Definition 3.1.7 (Anchor point subset). Let $\mathcal{M}$ be a compact, connected, smooth embedded submanifold (with or without boundary) of $\mathbb{R}^N$ and let $\bar{\mathcal{A}}$ be a $\tau$-tube anchor set for $\mathcal{M}$. Let $A$ be a proper subset of $\bar{\mathcal{A}}$ such that its cardinality $\Lambda$ satisfies $N + 2 \leq \Lambda \leq \bar{\Lambda}$. Then we say that $A$ is an anchor point subset of $\bar{\mathcal{A}}$.

Remark 3.1.8. $\bar{\mathcal{A}}$ is the total set of $\bar{\Lambda}$ anchor points. In practice, the choice of the anchor subset $A$ will be different for different portions of $\mathcal{M}$. Ultimately, the manifold $\mathcal{M}$ will be divided into segments, each of which will be parameterized by using a subset of only $\Lambda < \bar{\Lambda}$ anchor points, avoiding those anchors in $\bar{\mathcal{A}}$ that are either too far from or too close to the segment. To pick those “optimal” anchor points, we shall use a weighting scheme.

Let $r \in \mathcal{M}$ and let $x_\alpha(r)$ be the euclidean distance between $r$ and the projection of $a_\alpha$ on $T_r(\mathcal{M})$ (note that this projection depends on $r$; it is not the same as the root $p_\alpha$ of $a_\alpha$ introduced above, unless $r$ happens to be $p_\alpha$ itself). Now order the values

$$\{x_\alpha(r)^2 \exp(-x_\alpha(r)^2)\}$$

in descending order, resulting in $\{\rho_\gamma\}_{\gamma=1}^{\Lambda}$ with $\rho_\gamma \leq \rho_{\gamma'}$ if $\gamma > \gamma'$, and consider the $r$-dependent restricted anchor point set $A_r$ corresponding to the $\Lambda$ largest values $\rho_\gamma$, $\gamma = 1, \ldots, \Lambda$. Since the positive function defined by $h(u) = u^2 \exp(-u^2)$ is zero for $u = 0$ and tends to zero for $u \to \infty$, $A_r$ consists of anchor points $a_\alpha$ for which the projection onto $T_r(\mathcal{M})$ is neither among the closest to, nor the farthest from $r$, as intended.

The moving anchor point transformation $\Phi_{\text{moving}}$ is then defined as

$$\Phi_{\text{moving}} : \quad r \longrightarrow \left(\|r - a_\alpha\|^2\right) \quad \text{such that} \quad a \in A_r.$$ (3.10)
Then the set of images under this transformation for each \( r \in \mathcal{M} \) is the *moving anchor point transformation* of \( \mathcal{M} \).

**Note:** We note that this definition uses the distances between \( r \) and the projection of the \( a_\alpha \) onto \( T_r \mathcal{M} \) to determine the restricted anchor point set \( A_r \), but the moving anchor point transformation itself uses the distance from \( r \) to the \( a_\alpha \), with \( a_\alpha \in A_r \) (and not the distance between \( r \) and the projection of \( a_\alpha \) onto \( T_r \mathcal{M} \)). We specified that \( \Lambda \) must be at least \( N + 2 \) because we typically need at least \( (N + 2) \) anchors to uniquely determine or solve back for the point in the original space. We will later add extra restrictions to the choice of the \( a_\alpha \) (or the \( p_\alpha \)) that will govern their spacing.

Because of the weighting and thresholding \( \Phi_{\text{moving}}(\mathcal{M}) \) is the union of several disconnected pieces of \( \mathbb{R}^\Lambda \). The image \( \Phi(\mathcal{M}) \subset \mathbb{R}^\Lambda \) corresponds to the full anchor set \( \{a_\alpha\}_{\alpha=1}^\Lambda \). Each constituting segment of \( \Phi_{\text{moving}}(\mathcal{M}) \) is a projection of a connected subset \( S_r \) of \( \Phi(\mathcal{M}) \) for which all points \( r \in \mathcal{M} \) have the same \( A_r \); it follows that each segment is a smooth manifold in its own right by Lemma 3.1.3.

To explain the motivation behind the construction above, we first illustrate it on 2D curves, 3D curves and surfaces. We outline the simplified procedure through the following.

---

**Warping through anchor point transformation**

**Input:** Curve or Surface in 3D, or Curve in 2D

1. Sample points from a tube around the curve or surface \( \mathcal{M} \) satisfying conditions 1-2 in (3.1.6); these sample points constitute the anchor set \( A \).

2. For each \( r \) determine the 3 anchors \( a_\alpha(r) \in A \) for which \( x_\alpha(r)^2 \exp(-x_\alpha(r)^2) \) is largest, where \( x_\alpha(r) \) is the euclidean distance between \( r \) and the projection of \( a_\alpha \) on \( T_r \mathcal{M} \). Call these anchor points \( a_1(r), a_2(r), a_3(r) \).

---
3. Plot the manifold

\[ \left( \| \mathbf{r} - \mathbf{a}_1(\mathbf{r}) \|^2, \| \mathbf{r} - \mathbf{a}_2(\mathbf{r}) \|^2, \| \mathbf{r} - \mathbf{a}_3(\mathbf{r}) \|^2 \right). \]

Open 3D-curve: Helix

We take the helix

\[ (\cos t, \sin t, t) \quad \text{where} \quad \pi < t \leq \pi \]

with its tube having thickness \(-0.5 < \tau < 0.5\)

\[ (\cos t - \tau \cos t \cos s + \frac{\tau}{\sqrt{2}} \sin(t) \sin(s), \sin t - \tau \sin t \cos s - \frac{\tau}{\sqrt{2}} \cos(t) \sin(s), t + \frac{\tau}{2} \sin s). \]
The coordinates for the chosen anchors are as follows.

<table>
<thead>
<tr>
<th>Anchors</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>-0.6000</td>
<td>0.0032</td>
<td>-3.1448</td>
</tr>
<tr>
<td>a₂</td>
<td>-1.4000</td>
<td>-0.0032</td>
<td>-3.1384</td>
</tr>
<tr>
<td>a₃</td>
<td>0.5999</td>
<td>0.0126</td>
<td>0.0126</td>
</tr>
<tr>
<td>a₄</td>
<td>1.3999</td>
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</tr>
<tr>
<td>a₅</td>
<td>-0.6000</td>
<td>-0.0032</td>
<td>3.1448</td>
</tr>
<tr>
<td>a₆</td>
<td>-1.4000</td>
<td>0.0032</td>
<td>3.1384</td>
</tr>
</tbody>
</table>

Figure 3.2: On the left: Helix with the chosen anchors. On the right: Image of the helix under the moving anchor point transformation.

**Open 2D-curve: Sinusoidal Curve**

We take the sinusoidal curve

\[(t, \sin 2t)\] where \(\pi < t \leq \pi\)
Figure 3.3: Illustration of the weighting scheme with the helix example. Different colors indicate the portion of helix with corresponding anchors.
with its tube having thickness $-0.2 < \tau < 0.2$

$\left( t - \frac{2\tau \cos 2s}{\sqrt{1 + 4(\cos 2s)^2}}, \sin 2t + \frac{\tau}{\sqrt{1 + 4(\cos 2s)^2}} \right)$.

The coordinates for the chosen anchors are as follows.

<table>
<thead>
<tr>
<th>Anchors</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>-0.0715</td>
<td>0.4114</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2.7701</td>
<td>-0.1796</td>
</tr>
<tr>
<td>$a_3$</td>
<td>1.6423</td>
<td>0.4114</td>
</tr>
<tr>
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</tr>
<tr>
<td>$a_5$</td>
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</tr>
<tr>
<td>$a_6$</td>
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</tr>
<tr>
<td>$a_7$</td>
<td>3.2131</td>
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</tr>
<tr>
<td>$a_8$</td>
<td>1.1993</td>
<td>0.1796</td>
</tr>
<tr>
<td>$a_9$</td>
<td>-2.7701</td>
<td>0.1796</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>-1.6423</td>
<td>-0.4114</td>
</tr>
</tbody>
</table>

**Closed 3D-curve**

We take the closed curve

$$(\sin t, \cos t, t) \quad \text{where} \quad \pi < t \leq \pi$$
Figure 3.4: Illustration of the sine curve under the moving anchor point transformation. On the left: Sinusoidal curve with the chosen anchors. On the right: Image of the curve under the anchor point transformation with its tube having thickness $-0.3 < \tau < 0.3$

\[
\begin{align*}
\sin t - \tau \cos s \left(16 \cos^4 t \sin t + \sin t\right) &= \frac{4\tau \sin s \sin^3 t}{\sqrt{f(t)}}, \\
\cos t + \frac{-16 \cos t \tau \sin s (\cos^4 t - 2 \cos^2 t + \frac{17}{16})}{g(t)} &= 4\tau \sin s \cos^3 t, \\
t + \frac{\tau \cos s (-8 \cos^2 t + 4)}{f(t)} &= \frac{\tau \sin s}{g(t)},
\end{align*}
\]

where

\[
\begin{align*}
g(t) &= \sqrt{48 \cos^4 t - 48 \cos^2 t + 17} \quad f(t) = \sqrt{1 - 16 \cos^4 t + 16 \cos^2 t}.
\end{align*}
\]
The coordinates for the chosen anchors are as follows.

<table>
<thead>
<tr>
<th>Anchors</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_1</td>
<td>0</td>
<td>-0.9272</td>
<td>0.7090</td>
</tr>
<tr>
<td>a_2</td>
<td>-0.3219</td>
<td>-0.8498</td>
<td>0.1761</td>
</tr>
<tr>
<td>a_3</td>
<td>-0.6606</td>
<td>-0.2408</td>
<td>-0.8418</td>
</tr>
<tr>
<td>a_4</td>
<td>-0.6942</td>
<td>0.4042</td>
<td>-0.9324</td>
</tr>
<tr>
<td>a_5</td>
<td>-0.7380</td>
<td>1.0697</td>
<td>0.3248</td>
</tr>
<tr>
<td>a_6</td>
<td>-0.0353</td>
<td>1.0771</td>
<td>1.2887</td>
</tr>
<tr>
<td>a_7</td>
<td>0.8441</td>
<td>0.7700</td>
<td>0.4625</td>
</tr>
<tr>
<td>a_8</td>
<td>1.2397</td>
<td>0.3836</td>
<td>-0.7724</td>
</tr>
<tr>
<td>a_9</td>
<td>1.2097</td>
<td>-0.2232</td>
<td>-0.6832</td>
</tr>
<tr>
<td>a_{10}</td>
<td>0.4399</td>
<td>-0.5482</td>
<td>0.3195</td>
</tr>
</tbody>
</table>

**Figure 3.5:** Illustration of a closed 3D curve under the moving anchor point transformation. Figures on the top: Closed 3D curve with the chosen anchors. On the bottom: Image of the curve under the anchor point transformation.
Surface

We take the surface

\[(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \quad \text{where} \quad \phi \in [0, \pi), \theta \in [0, 2\pi)\]  \quad (3.11)

with its tube with thickness \(-0.2 \leq \tau \leq 0.2\)

\[(1 + \tau)(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi).\] \quad (3.12)

Figure 3.6: Illustration of a sphere under the moving anchor point transformation.
The coordinates for the chosen anchors are as follows.

<table>
<thead>
<tr>
<th>Anchors</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>-0.3436</td>
<td>0.9021</td>
<td>-0.7128</td>
</tr>
<tr>
<td>a₂</td>
<td>-0.5276</td>
<td>-0.4518</td>
<td>0.9785</td>
</tr>
<tr>
<td>a₃</td>
<td>-0.2291</td>
<td>0.6014</td>
<td>-0.4752</td>
</tr>
<tr>
<td>a₄</td>
<td>-0.3518</td>
<td>-0.3012</td>
<td>0.6523</td>
</tr>
</tbody>
</table>

In each of these cases we see that with these special choice of anchors and weighting scheme, the curves and surfaces are getting “straightened out.” Our next goal will be to study the geometry and the number of “colored segments” in $\mathcal{M}$.

3.1.1 Computation of the curvature at a point on $\Phi(\mathcal{M})$

Motivated by the above toy examples, we would like to determine quantitatively how curved each segment of $\Phi^\text{moving}(\mathcal{M})$ is. The main goal of this section is to obtain explicit upper bounds on the curvature. We first discuss the case where $\mu \geq 2$, for which we compute bounds on the sectional curvature. Next we consider the (simpler) case $\mu = 1$; in that case the curve has a single well-defined curvature, and it is no longer necessary to look at sectional curvatures.

The case $\mu > 1$

The intuition behind the flattening procedure is that “warping” the manifold with the map and then “breaking” it up into segments using the weighting leads to a representation of each segment with a significantly lower curvature. In order to investigate the “warping” we will now compute the curvature at a point $\Phi(\mathbf{r})$ on $\Phi(\mathcal{M})$. Clearly, the notion of weighting is not required to compute the curvature at
a point; it will arise when we seek to bound this curvature. One of the reasons to require that $\Phi(\mathcal{M})$ is a $\mu$-dimensional smooth embedded manifold in Lemma 3.1.3 was to be able to introduce an adapted frame (introduced in Section 2.1) at a point $\Phi(r)$. Using the method of moving frames enables us to compute the curvature in a coordinate-independent way. The notations and indices used for the solder forms, connection forms and adapted bases used below are similar to those in Table (2.1) and are summarized in Table 3.1.

**Table 3.1**: Indices convention for one-forms and vectors corresponding to tangent, normal and ambient spaces of $\Phi(\mathcal{M})$. The notation used for both solder forms and connection forms on $\Phi(\mathcal{M})$ is taken to be $\theta$ and the distinction will be clear by the number of indices used. Adapted bases are those of the moving frame.

<table>
<thead>
<tr>
<th>Unique One Forms Notation</th>
<th>Unique One Forms Notation</th>
<th>Unique One Forms Notation</th>
<th>Unique One Forms Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi(\mathcal{M})$</td>
<td>$\theta$</td>
<td>$i, j, k, l, m, \ldots$</td>
<td>$i', j', k', l', m', \ldots$</td>
</tr>
<tr>
<td>$\Phi(\mathcal{M})$</td>
<td>$\theta$</td>
<td>$\alpha, \beta, \gamma, \ldots$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Unique One Tangent Space Normal Space Ambient Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi(\mathcal{M})$</td>
</tr>
<tr>
<td>$\theta$</td>
</tr>
<tr>
<td>$i, j, k, l, m, \ldots$</td>
</tr>
<tr>
<td>$i', j', k', l', m', \ldots$</td>
</tr>
<tr>
<td>$\alpha, \beta, \gamma, \ldots$</td>
</tr>
</tbody>
</table>

If we pick an arbitrary orthonormal coframing $\{\omega^i\}$ (and respectively $\{\theta^i\}$) on an open subset $\mathcal{U}$ (and respectively $\mathcal{V}$) on $\mathcal{M}$ (and respectively $\Phi(\mathcal{M})$), the pullbacks $\Phi^*(\theta^j)$ can be expressed as a combination of the $\omega^i$. We shall require the coefficients to be smooth, i.e.

$$
\Phi^*(\theta^j) = \sum_i g^j_i \omega^i \quad \text{for some } g^j_i \in C^\infty(\mathcal{M}).
$$

(3.13)

Differentiating the above should in principle allow us to compare curvatures. These $g^j_i$ intuitively contain the information of the distances of the anchors from $r$ and their relative positions with respect to $T_r\mathcal{M}$. But to find expressions of $g^j_i$ will be tedious
for arbitrary coframing, for computing the pull-back we might have to resort to coordinate computations — thus defeating the very purpose of using the Cartan approach. To circumvent this issue we would exploit three basic facts and consequences from linear algebra.

(i) Making use of the abstract manifold corresponding to $\mathcal{M}$:
It will be convenient for certain computations and simplifications to not take into account the embedded nature of $\mathcal{M}$; in other words we identify $\mathcal{M}$ with an abstract Riemannian manifold $\mathcal{M}$ that is embedded isometrically into $\mathbb{R}^N$, i.e.

$$r : \mathcal{M} \to \mathcal{M} \quad \text{such that} \quad r(O) = r. \quad (3.14)$$

Clearly, the pullbacks of the solder forms and connection forms on $\mathcal{T}(\mathcal{M})$ by the pullback map $\mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ equal the solder and connection forms on $\mathcal{T}(\mathcal{M})$. For a reference see [BBG+83]. As is customary, we will practice some abuse of notation and use $\{\omega^i, \omega^j\}$ for the forms defined both on $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}(\mathcal{M})$.

(ii) Considering the singular values of the Jacobian

We recall the definition of differential of the map $\Phi \circ r$ at $O \in \mathcal{M}$ to be

$$d(\Phi \circ r)_O : T_O \mathcal{M} \to T_{\Phi \circ r(O)} \mathcal{M}$$

such that

$$d(\Phi \circ r)_O(v)(\widetilde{f}) = v(\widetilde{f} \circ \Phi \circ r) \quad \text{with} \quad v \in T_O \mathcal{M} \quad \text{and} \quad \widetilde{f} \in \mathcal{C}^\infty(\Phi(\mathcal{M})). \quad (3.15)$$

Let $\{v_i\}_{i=1}^\mu$ be an orthonormal base spanning the tangent space $T_O \mathcal{M}$ then the set of

$$u_i = r_* v_i \quad (3.16)$$

is an orthonormal basis spanning $T_r \mathcal{M}$. With this Definition 3.16 of $\{u_i\}$, it is clear
that that the Jacobian matrix of $\Phi \circ r$ at $O$ can be written as
\[
D(\Phi \circ r)(O) = \begin{bmatrix}
\langle r - a_1, u_1 \rangle & \langle r - a_1, u_2 \rangle & \ldots & \langle r - a_1, u_{\mu} \rangle \\
\langle r - a_2, u_1 \rangle & \langle r - a_2, u_2 \rangle & \ldots & \langle r - a_2, u_{\mu} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle r - a_\Lambda, u_1 \rangle & \langle r - a_\Lambda, u_2 \rangle & \ldots & \langle r - a_\Lambda, u_{\mu} \rangle 
\end{bmatrix}.
\tag{3.17}
\]

Noting that $D(\Phi \circ r)(O) \in \mathcal{L}(\mathbb{R}^\mu, \mathbb{R}^\Lambda)$, or more precisely,

$D(\Phi \circ r)(O) \in \mathcal{L}(T_O M, T_{\Phi(r)}(\tilde{M}))$, we consider the singular value decomposition of $D(\Phi \circ r)(O)$. Since rank($D(\Phi \circ r)(O)$) = $\mu$, there exist ordered orthonormal bases

$\{v_1, \ldots, v_\mu\}$ at $T_O M$ and $\{\tilde{u}_1, \ldots, \tilde{u}_{\Lambda}\}$ at $p = \Phi(r) \in \tilde{M}$ such that for all $i = 1, \ldots, \mu$ and $a' = \mu + 1, \ldots, \Lambda$

\[
\tilde{u}_i = \frac{1}{\sigma_i} D(\Phi \circ r)(O) v_i \quad \left( D(\Phi \circ r)(O) \right)^T \tilde{u}_{a'} = 0 \tag{3.18}
\]

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\mu > 0$ are the singular values of the Jacobian matrix.

The singular values are defined by the square-root of the eigenvalues of the following matrix.

\[
\left( D(\Phi \circ r)(O) \right)^T D(\Phi \circ r)(O) v_i = \sigma_i^2 v_i. \tag{3.19}
\]

(iii) Choice of base:

The bases chosen in (3.18) should not affect the computation of curvature at $\Phi(r)$ since intrinsic curvature is basis independent. From a frame bundle point of view, given open sets in $V \subset M$ containing $O$ and in $\tilde{U} \subset \Phi(\tilde{M})$ containing $\Phi(r)$ we have restricted to a particular choice of section or frame

\[
v_i : V \to \mathcal{F}(M) \quad \text{and} \quad \tilde{u}_i : \tilde{U} \to \mathcal{F}(\Phi(\tilde{M})) \tag{3.20}
\]
such that (3.18) holds.

Having now made the choices of bases, let \( \{v_i\} \) and \( \{\bar{u}_i\} \) determine the dual coframes \( \{\omega^i\} \) for \( T^*_p\mathcal{M} \) and \( \{\theta^i\} \) for \( T^*_p\Phi(\mathcal{M}) \) respectively. The differential of the map \( \Phi \circ r \) due to (3.18) can be expressed as

\[
d(\Phi \circ r)(v_i) = \sigma_i \bar{u}_i. \tag{3.21}
\]

Thus the dual linear map or the pullback of \( \Phi \circ r \) is expressed by

\[
\sigma_i \omega^i = (\Phi \circ r)^* \theta^i. \tag{3.22}
\]

Note that there is no summation over \( i \). (This is the reason the Einstein summation convention will not be assumed in the first part of the thesis.) We introduce the notation:

\[
\bar{\theta}^i = (\Phi \circ r)^* \theta^i \quad \text{and} \quad \bar{\theta}^i_j = (\Phi \circ r)^* \theta^i_j. \tag{3.23}
\]

Taking the exterior derivative of (3.22), and using that the exterior derivative commutes with the pull-back, we get

\[
d\sigma_i \wedge \omega^i + \sigma_i d\omega^i = d((\Phi \circ r)^* \theta^i) = (\Phi \circ r)^* d\theta^i. \tag{3.24}
\]

Using Cartan’s first structure equations we obtain

\[
d\sigma_i \wedge \omega^i - \sum_j \sigma_j \omega_j^i \wedge \omega^j = -(\Phi \circ r)^*(\sum_j \theta_j^i \wedge \theta^j) = -\sum_j \bar{\theta}_j^i \wedge \bar{\theta}_j^i \quad \text{by (3.23)} \tag{3.25}
\]

\[
= -\sum_j \bar{\theta}_j^i \wedge \sigma_j \omega^j \quad \text{by (3.22)}.
\]
Rearranging we get

\[ d\sigma_i \wedge \omega^i + \sum_{j \neq i} (\sigma_j \tilde{\theta}_j - \sigma_i \omega^i_j) \wedge \omega^j = 0. \]  

(3.26)

By Cartan’s lemma (Lemma 2.1.1), we have

\[ d\sigma_i = \sum_j A_{ij} \omega^j \]  

(3.27)

for \( i \neq j \)

\[ \tilde{\theta}_j = \frac{\sigma_i}{\sigma_j} \omega^i_j + \frac{1}{\sigma_j} \sum_l B_{ijl} \omega^l \]  

(3.28)

with the symmetries

\[ A_{ij} = B_{iji} \quad \text{for } i \neq j \]  

(3.29)

\[ B_{ijl} = B_{ilj} \quad \text{for } j \neq i \text{ and } l \neq i. \]  

(3.30)

**Remark 3.1.9.** The symmetries above can be seen by the following observation.

Putting (3.27) and (3.28) in (3.26), we get for \( i \neq j \)

\[ 0 = \sum_j A_{ij} \omega^j \wedge \omega^i + \sum_{j,l} B_{ijl} \omega^l \wedge \omega^j = \sum_j (A_{ij} - B_{iji}) \omega^j \wedge \omega^i + \sum_{j,l \neq i} (B_{ijl} - B_{ilj}) \omega^j \wedge \omega^l. \]  

(3.31)

Since for any \( j \neq l \), \( \omega^j \wedge \omega^l \neq 0 \), the symmetry relations (3.29) and (3.30) follow.

From (3.28) we have

\[ \sigma_j \tilde{\theta}_j = \sigma_i \omega^i_j + \sum_l B_{ijl} \omega^l. \]  

(3.32)

Interchanging \( i \) and \( j \) in the above equation and applying the anti-symmetries of
\[ \theta_j = -\theta_i \] and \[ \omega_j = -\omega_i \] we get

\[ -\sigma \theta_j = -\sigma_j \omega_j + \sum_l B_{jil} \omega^l. \] (3.33)

Subtracting (3.33) from (3.32) we get

\[ \theta_j = \omega_j^i + \sum_l \frac{B_{ijl} - B_{jil}}{\sigma_i + \sigma_j} \omega^l. \] (3.34)

Both sides of the above expression are antisymmetric and this is the expression we will use to compare curvatures. We also have from (3.28) using the antisymmetry of \( \theta_j \)

\[ \frac{1}{\sigma_j^2 - \sigma_i^2} \sum_l (\sigma_i B_{ijl} + \sigma_j B_{jil}) \omega^l = \omega_j^i = \sum_l \lambda_{jl}^i \omega^l \] (3.35)

where \( \lambda_{jk}^i \) were the connection coefficients introduced in Chapter (2.1). Thus

\[ \sigma_i B_{ijl} + \sigma_j B_{jil} = (\sigma_j^2 - \sigma_i^2) \lambda_{jl}^i. \] (3.36)

Let \( i \neq j \neq l \). Considering the equation from (3.36) obtained by the permutation \( i \to l, j \to i, l \to j \) we get

\[ \sigma_i B_{ijl} + \sigma_i B_{lij} = (\sigma_i^2 - \sigma_j^2) \lambda_{ij}^l \] applying (3.30) \[ \Rightarrow \sigma_i B_{ijl} + \sigma_i B_{lij} = (\sigma_i^2 - \sigma_j^2) \lambda_{ij}^l \] (3.37)

and considering \( i \to j, j \to l, l \to i \) in (3.36) we get

\[ \sigma_j B_{jil} + \sigma_i B_{lij} = (\sigma_j^2 - \sigma_i^2) \lambda_{il}^j \] applying (3.30) \[ \Rightarrow \sigma_j B_{jil} + \sigma_i B_{lij} = (\sigma_j^2 - \sigma_i^2) \lambda_{il}^j \] (3.38)
Subtracting (3.38) from the sum of (3.36) and (3.37) we get for $i \neq j \neq l$

$$B_{ijl} = \frac{1}{2\sigma_i} \left( \left( \sigma_j^2 - \sigma_i^2 \right) \lambda^i_{jl} - \left( \sigma_i^2 - \sigma_j^2 \right) \lambda^j_{li} \right). \quad (3.39)$$

Thus we can express the $B$’s in terms of the quantities on $\mathcal{M}$ (the connection coefficients, $\lambda^j_{il}$) and the map (the $\sigma_i$’s).

Finally, we note and list the following symmetries of the quantities introduced above. We already have the symmetry

$$B_{ijl} = B_{ilj} \quad \text{for } j, l \neq i. \quad \text{(S1)}$$

Now, $\tilde{\theta}_i = \omega^i_i = 0$ in (3.28) gives

$$B_{iii} = 0. \quad \text{(S2)}$$

Putting $l = i$ in eq. (3.36), using (3.29) and the antisymmetry $\lambda^i_{jk} = -\lambda^j_{ik}$, we have for $i \neq j$:

$$\sigma_i B_{iji} + \sigma_j B_{jii} = \left( \sigma_i^2 - \sigma_j^2 \right) \lambda^j_{li}$$

$$B_{jii} = \frac{\left( \sigma_i^2 - \sigma_j^2 \right) \lambda^j_{li} - \sigma_i A_{ij}}{\sigma_j}. \quad \text{(S3)}$$

It remains to calculate $A_{ij}$ defined by (3.27) which was

$$d\sigma_i = \sum_j A_{ij} \omega^j. \quad (3.40)$$

From $\sigma_i^2 = v^T_i \left( D(\Phi \circ r)(O) \right)^T D(\Phi \circ r)(O) v_i$ in eq. (3.19) and expliciting that $v^T_i v_i = 1$, so that

$$(dv^T_i) v_i + v^T_i dv_i = 0 \quad (3.41)$$

42
we obtain

\[ 2\sigma_i d\sigma_i = 2\sigma_i \sum_j A_{ij} \omega^j = v_i^T d \left( \left( D(\Phi \circ r)(O) \right)^T D(\Phi \circ r)(O) \right) v_i. \tag{3.42} \]

**Note:** The singular values \( \{\sigma_i\} \) and the corresponding singular values of \( D(\Phi \circ r)(O) \) depend smoothly on \( O \) as long as no degeneracies are encountered. We shall assume that we are in this generic situation. The following lemma provides bounds on the \( A_{ij} \) and some of the coefficients of \( dA_{ij} \) written as a linear combination of the \( \{\omega^k\} \).

To derive this bound, we will make use of the observation that

\[ \sum_k \sigma_k^2 = \sum_\alpha ||r - a_\alpha||^2, \tag{3.43} \]

as well as other detailed properties of the construction.

**Lemma 3.1.10.** Let \( U \) be a geodesic ball on \( \mathcal{M} \) and consider its anchor point transformation in (3.3). Set

\[ L := \max_{i=1,...,\mu} \frac{\sqrt{\frac{1}{n} \sum_k \sigma_k^2}}{\sigma_i}. \tag{3.44} \]

Then \( A_{ij} \) and \( |A_{ij}|_p \) are bounded above by the following explicit expressions

\[ |A_{ij}| \leq \mu L \left( 1 + \lambda_0 \sqrt{\frac{\mu \sum_k \sigma_k^2}{\sigma_i}} \right) \]

\[ |A_{ij}|_p \leq \frac{\Lambda + \mu^2 L^2}{\sigma_i} + \mu^\frac{3}{2} \left( \lambda_0 + 2\lambda_0 (\mu)^\frac{3}{2} L^2 \right) L \]

\[ + \mu^\frac{3}{2} \left( \lambda_0 \sqrt{\mu} + \lambda_1 + \lambda_0^2 \mu + \lambda_0^4 \mu^3 L^2 \right) \sqrt{\sum_k \sigma_k^2} \]

where

\[ \max_U |\lambda_{ij}^m| \leq \lambda_0 \quad \max_U |(\lambda_{ij}^m)_p| \leq \lambda_1. \tag{3.46} \]
Proof. For brevity let us denote \((D(\Phi \circ r)(O))^T D(\Phi \circ r)(O)\) by \(J\) and \((r - a_\alpha, u_k)\) by \(\rho_{\alpha,k}\). The \(k\)-\(l\) element of \(J\) is given by

\[
J_{kl} = J_{lk} = \sum_\alpha (r - a_\alpha) \langle (r - a_\alpha, u_k) = \sum_\alpha \rho_{\alpha,l} \rho_{\alpha,k}.
\]

(3.47)

Before finding \(A_{ij}\) and their derivatives, we list here some useful inequalities.

Inequality (1) \[
\sum_\alpha ||r - a_\alpha|| \leq \mu \sqrt{\sum_\alpha ||r - a_\alpha||^2} = \mu \sum_k \sigma_k^2
\]

(3.48)

Inequality (2) \[
\sum_k |v^k_i| \leq \mu \sqrt{\sum_k |v^k_i|^2} = \mu \sigma_k
\]

(3.49)

Inequality (3) \[
\sum_m |\rho_{\alpha,m}| = \sum_\alpha |\langle r - a_\alpha, u_m\rangle| \leq \mu \sqrt{||r - a_\alpha||^2} = \mu ||r - a_\alpha||
\]

(3.50)

Inequality (4) \[
\sum_{l,\alpha} |\rho_{\alpha,l} v^l_i| \leq \mu \sqrt{\sum_\alpha \sum_l |\langle r - a_\alpha, u_l\rangle|^2} = \mu \sum_\alpha ||r - a_\alpha||
\]

(3.51)

Inequality (5) \[
\sum_{\alpha,k,l,m} |\rho_{\alpha,m} \rho_{\alpha,l} v^k_i v^l_i| \leq \mu \sqrt{\sum_\alpha \sum_k |\rho_{\alpha,m} v^k_i|^2} \sum_{\alpha,l,m} |\rho_{\alpha,m} \rho_{\alpha,l} v^l_i| \leq \mu \sum_\alpha ||r - a_\alpha|| ||r - a_\alpha||
\]

(3.52)
We now proceed to find the bound on $A_{ij}$. Taking the derivative of (3.47) we have

\[
d J_{kl} = \sum \langle dr, u_k \rangle \rho_{\alpha,l} + \sum \langle dr, u_l \rangle \rho_{\alpha,k} + \sum \langle \mathbf{r} - \mathbf{a}_\alpha, du_k \rangle \rho_{\alpha,l} + \sum \langle \mathbf{r} - \mathbf{a}_\alpha, du_l \rangle \rho_{\alpha,k}
\]

\[
= \sum \langle \sum \omega^m u_n, u_k \rangle \rho_{\alpha,l} + \sum \langle \sum \omega^m u_n, u_l \rangle \rho_{\alpha,k} + \sum \langle \mathbf{r} - \mathbf{a}_\alpha, \sum \omega^m u_m \rangle \rho_{\alpha,l} + \sum \langle \mathbf{r} - \mathbf{a}_\alpha, \sum \omega^m u_m \rangle \rho_{\alpha,k}
\]

\[
= \sum (\omega^k \rho_{\alpha,l} + \omega^l \rho_{\alpha,k} + \sum \lambda_{kj}^m \omega^j \rho_{\alpha,m} \rho_{\alpha,l} + \sum \lambda_{lj}^m \omega^j \rho_{\alpha,m} \rho_{\alpha,k})
\]

\[
= \sum \sum_j (\delta_{jk} \rho_{\alpha,l} + \delta_{jl} \rho_{\alpha,k} + \sum \lambda_{kj}^m \rho_{\alpha,m} \rho_{\alpha,l} + \sum \lambda_{lj}^m \rho_{\alpha,m} \rho_{\alpha,k}) \omega^j.
\] (3.53)

Thus from $v_i^T d J v_i = \sum_{k,l} d J_{kl} v_i^k v_i^l$ and (3.42)

\[
2 \sigma_i A_{ij} = \sum_{\alpha,k,l} \left( \delta_{jk} \rho_{\alpha,l} + \delta_{jl} \rho_{\alpha,k} + \sum_m \lambda_{kj}^m \rho_{\alpha,m} \rho_{\alpha,l} + \sum_m \lambda_{lj}^m \rho_{\alpha,m} \rho_{\alpha,k} \right) v_i^k v_i^l
\]

which implies

\[
\sigma_i A_{ij} = \sum_{\alpha,k,l} \left( \delta_{jk} \rho_{\alpha,l} + \sum_m \lambda_{kj}^m \rho_{\alpha,m} \rho_{\alpha,l} \right) v_i^k v_i^l.
\] (3.54)

Hence, we have

\[
\sigma_i |A_{ij}| \leq \left| \sum_{\alpha,l} \rho_{\alpha,l} v_i^j v_i^l \right| + \left| \sum_{\alpha,k,l,m} \lambda_{kj}^m \rho_{\alpha,m} \rho_{\alpha,l} v_i^k v_i^l \right|
\]

\[
\leq |v_i^j| \sum_{\alpha,l} |\rho_{\alpha,l} v_i^l| + \lambda_0 \sum_{\alpha,k,l,m} |\rho_{\alpha,m} \rho_{\alpha,l} v_i^k v_i^l|
\]

\[
\leq \sqrt{\mu} \sum_k \sigma_k^2 \left( 1 + \lambda_0 \sqrt{\mu} \sum_k \sigma_k^2 \right)
\] (3.55)

which implies

\[
|A_{ij}| \leq \mu L \left( 1 + \lambda_0 \sqrt{\mu} \sum_k \sigma_k^2 \right).
\] (3.56)

45
Now, we compute $dA_{ij}$. From (3.54) we get

$$\sigma_i A_{ij} = v_i^T C^j v_i$$  \hspace{1cm} (3.57)

where the $C^j$ matrix is given by

$$[C^j]_{kl} = \sum_\alpha \delta_{jk} \rho_{\alpha,l} + \sum_m \lambda^m_{kj} J_{ml} = \sum_\alpha \left( \delta_{jk} \rho_{\alpha,l} + \sum_m \lambda^m_{kj} \rho_{\alpha,m} \rho_{\alpha,l} \right). \hspace{1cm} (3.58)$$

Note that

$$dJ_{ml} = \sum_p C^p_{ml} \omega^p. \hspace{1cm} (3.59)$$

Differentiating (3.57) we have

$$\sum_p A_{ip} A_{ij} \omega^p + \sigma_i dA_{ij} = v_i^T dC^j v_i = \sum_{k,l} dC^j_{kl} v_i^k v_i^l \hspace{1cm} (3.60)$$

and we compute

$$dC^j_{kl}$$

$$= \sum_\alpha \left( \delta_{jk} \sum_p (\delta_{pl} + \sum_q \lambda^q_{lp} \rho_{\alpha,q}) \omega^p + \sum_m (\lambda^m_{kj} \rho_{\alpha,m} \rho_{\alpha,l}) \omega^p \right) + \sum_m \lambda^m_{kj} dJ_{ml} \hspace{1cm} (3.61)$$

$$= \sum_p \left( \sum_\alpha \left( \delta_{jk} (\delta_{pl} + \sum_q \lambda^q_{lp} \rho_{\alpha,q}) + \sum_m (\lambda^m_{kj} \rho_{\alpha,m} \rho_{\alpha,l}) + \sum_m \lambda^m_{kj} C^p_{ml} \right) \omega^p. \right.$$  

Then

$$\sigma_i (A_{ij})_p$$

$$= \sum_{k,l} \left( \sum_\alpha \left( \delta_{jk} (\delta_{pl} + \sum_q \lambda^q_{lp} \rho_{\alpha,q}) + \sum_m (\lambda^m_{kj} \rho_{\alpha,m} \rho_{\alpha,l}) + \sum_m \lambda^m_{kj} C^p_{ml} \right) v_i^k v_i^l - A_{ip} A_{ij}. \right.$$  \hspace{1cm} (3.62)
Then
\[|\sum_{k,l} \sum_{\alpha} \delta_{jk}(\delta_{pl} + \sum_q \lambda_p^q \rho_{\alpha,q})v^k_i v^l_i| = |v^j_i(v^p_i \Lambda + \sum_{l,\alpha,q} \lambda_p^q \rho_{\alpha,q} v^l_i)|\]
\[\leq |\Lambda + v^j_i \sum_{\alpha,l,q} \lambda_p^q \rho_{\alpha,q} v^l_i| \leq \Lambda + \lambda_0 \sum_{\alpha,q} |\rho_{\alpha,q}| \sum_l |v^l_i| \quad (3.63)\]
\[\leq \Lambda + \lambda_0 \sqrt{\mu} \sqrt{\mu} \sum_{\alpha} |r - a_\alpha| \quad \text{by inequalities (3.49) and (3.50)}\]
\[\leq \Lambda + \lambda_0 (\mu)^{\frac{1}{2}} \sum_k \sigma_k^2 \quad \text{by inequality (3.48)}\]

Also,
\[|\sum_{\alpha,m,k,l} (\lambda_{k,j}^m p \rho_{\alpha,m} \rho_{\alpha,l} v^k_i v^l_i)| \leq \lambda_1 |\sum_{\alpha,m,k,l} \rho_{\alpha,m} \rho_{\alpha,l} v^k_i v^l_i| \leq \lambda_1 \mu \sum_k \sigma_k^2, \quad (3.64)\]
by ineq. (3.52)

\[|\sum_{m,k,l} \lambda_{k,j}^m p \rho_{\alpha,m} \rho_{\alpha,l} v^k_i v^l_i| \leq \lambda_0 \sum_{m,k,l} |\sum_{\alpha} (\delta_{pm} \rho_{\alpha,l} + \sum_n \lambda_{\alpha,m,n} \rho_{\alpha,l}) v^k_i v^l_i|\]
\[\leq \lambda_0 \sum_{\alpha,m,k,l} |\sum_n \lambda_{\alpha,m,n} \rho_{\alpha,l} v^k_i v^l_i|\]
\[\leq \lambda_0 \sum_{\alpha,k,l} |\rho_{\alpha,l} v^k_i v^l_i| + \lambda_0^2 \mu \sum_{\alpha,k,n,l} |\rho_{\alpha,n} \rho_{\alpha,l} v^k_i v^l_i|\]
\[\leq \lambda_0 \mu \sqrt{\sum_k \sigma_k^2} + \lambda_0^2 \mu^2 \sum_k \sigma_k^2 = \lambda_0 \mu \sqrt{\sum_k \sigma_k^2} (1 + \lambda_0 \mu \sqrt{\sum_k \sigma_k^2})\]
\[\quad (3.65)\]
\[A_{ip} A_{ij} \leq \mu^2 L^2 \left(1 + \lambda_0 \sqrt{\mu} \sqrt{\sum_k \sigma_k^2}\right)^2. \quad (3.66)\]
Combining the inequalities,

\[
\sigma_i |(A_{ij})_p| \leq \\
\Lambda + \mu^2 L^2 + \left( \lambda_0 \mu + 2 \lambda_0 (\mu)^{\frac{5}{2}} L^2 \right) \sqrt{\sum_k \sigma_k^2 + \left( \lambda_0 (\mu)^{\frac{3}{2}} + \lambda_1 \mu + \lambda_0^2 \mu^2 + \lambda_0^2 \mu^4 L^2 \right) \sum_k \sigma_k^2}.
\]

This implies

\[
|A_{ij}|_p \leq \frac{\Lambda + \mu^2 L^2}{\sigma_i} + \mu^{\frac{3}{2}} \left( \lambda_0 + 2 \lambda_0 (\mu)^{\frac{3}{2}} L^2 \right) L \\
+ \mu^{\frac{3}{2}} \left( \lambda_0 \sqrt{\mu} + \lambda_1 + \lambda_0^2 \mu + \lambda_0^2 \mu^3 L^2 \right) L \sqrt{\sum_k \sigma_k^2}.
\]

Proposition 3.1.11. The absolute value of the sectional curvature at a point on \( \Phi(\mathcal{M}) \)
\[ \Phi(U) \text{ where } U \text{ is a geodesic ball of } \mathcal{M}, \text{ is bounded above by the explicit expression} \]

\[
\left| \sec_{ij} \right| \leq \frac{2\Lambda + 3\mu^2L^2 + \mu^3L^2}{\sigma^4} + \frac{\mu L\lambda_0}{\sigma^3} \left( 2\sqrt{\mu} + 4\mu^2L^2 + 2\mu^2L + 6 + 2\mu + 2\mu^3L \right) \\
+ \frac{2\lambda_0\mu^5/2L}{\sigma^2} + \frac{2\lambda_1\mu^2L}{\sigma^2} + \frac{\lambda_0^2\mu^2L}{\sigma^2} \left( 4\mu + 2\mu^3L^2 + \mu^2L + 6 + \mu^3L \right) \\
+ \left| \sec_{ij} \right| L + \frac{17 \lambda_0^3}{4 \sigma^2 L} + \mathcal{O} \left( \frac{\max_i \Delta \nu_i}{\sigma^2} \right) + \mathcal{O} \left( \frac{\max_i \nu_i}{\sigma^3} \right) 
\]

under the assumption that \( L \) is not too far from 1.

**Proof.** We first derive an expression for the sectional curvature on \( \mathcal{M} \). Differentiating the expression in (3.34) and using Cartan's first structure equations and using the definition of connection coefficients in (2.19), we get

\[
\begin{align*}
\partial_t j_i & = d\omega^i_j + \sum_l d\left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) \wedge \omega^l + \sum_m \frac{B_{ijm} - B_{jim}}{\sigma_i + \sigma_j} d\omega^m \\
& = d\omega^i_j + \sum_{l,k \neq k} d\left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) \wedge \omega^l + \sum_m \frac{B_{ijm} - B_{jim}}{\sigma_i + \sigma_j} \omega^k \wedge \lambda^m_k \omega^l \\
& \quad + d\left( \frac{B_{ij} - B_{jii}}{\sigma_i + \sigma_j} \right) \wedge \omega^i \\
& = d\omega^i_j + \sum_{l,k \neq i} \left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) \omega^k \wedge \omega^l + \sum_m \frac{B_{ijm} - B_{jim}}{\sigma_i + \sigma_j} \omega^k \wedge \lambda^m_k \omega^l. 
\end{align*}
\]

We again can write \( d\left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) \) as a linear combination of the \( \{ \omega^k \} \).

\[
d\left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) = \sum_k \left( \frac{B_{ijkl} - B_{jilk}}{\sigma_i + \sigma_j} \right) \omega^k
\]

(3.71)
We also compute from (3.34)

\[
\sum_{m} \tilde{\theta}^{m} \wedge \tilde{\theta}^{i} = \sum_{m} \left( \omega^{m} + \sum_{k} \frac{B_{mjk} - B_{jmk}}{\sigma_{m} + \sigma_{j}} \omega^{k} \right) \wedge \left( \omega^{i} + \sum_{l} \frac{B_{ml} - B_{mil}}{\sigma_{i} + \sigma_{m}} \omega^{l} \right)
\]

\[
= \sum_{m} \omega^{m} \wedge \omega^{i} + \sum_{m,k,l} \left( \frac{B_{mjk} - B_{jmk}}{\sigma_{m} + \sigma_{j}} \lambda^{k}_{ml} + \frac{B_{ml} - B_{mil}}{\sigma_{i} + \sigma_{m}} \lambda^{m}_{jk} + \frac{(B_{mjk} - B_{jmk})(B_{ml} - B_{mil})}{(\sigma_{m} + \sigma_{j})(\sigma_{i} + \sigma_{m})} \omega^{k} \wedge \omega^{l} \right)
\]

Subtracting (3.72) from (3.70) and using the second structure equation of Cartan, we obtain

\[
\frac{1}{2} \sum_{k,l} \text{Riem}_{ijkl} \tilde{\theta}^{k} \wedge \tilde{\theta}^{l}
\]

\[
= \frac{1}{2} \sum_{k,l} \text{Riem}_{ijkl} \omega^{k} \wedge \omega^{l} + \sum_{l,k} \left( \frac{B_{ijl} - B_{jil}}{\sigma_{i} + \sigma_{j}} \right) \omega^{k} \wedge \omega^{l} + \sum_{m} \frac{B_{ijm} - B_{jim}}{\sigma_{i} + \sigma_{j}} \lambda^{m}_{kl} \omega^{k} \wedge \omega^{l}
\]

\[
- \sum_{m,k,l} \left( \frac{B_{mjk} - B_{jmk}}{\sigma_{m} + \sigma_{j}} \lambda^{k}_{ml} + \frac{B_{ml} - B_{mil}}{\sigma_{i} + \sigma_{m}} \lambda^{m}_{jk} + \frac{(B_{mjk} - B_{jmk})(B_{ml} - B_{mil})}{(\sigma_{m} + \sigma_{j})(\sigma_{i} + \sigma_{m})} \omega^{k} \wedge \omega^{l} \right)
\]

The LHS becomes \( \frac{1}{2} \sum_{k,l} \text{Riem}_{ijkl} \sigma_{k} \sigma_{l} \omega^{k} \wedge \omega^{l} \). Thus equating the coefficients of \( \omega^{k} \wedge \omega^{l} \) in (3.73) we get

\[
\frac{1}{2} \text{Riem}_{ijkl} \sigma_{k} \sigma_{l}
\]

\[
= \frac{1}{2} \text{Riem}_{ijk} + \left( \frac{B_{ijl} - B_{jil}}{\sigma_{i} + \sigma_{j}} \right) + \sum_{m} \left( \frac{B_{ijm} - B_{jim}}{\sigma_{i} + \sigma_{j}} \lambda^{m}_{kl} - \left( \frac{B_{ijk} - B_{jik}}{\sigma_{i} + \sigma_{j}} \right) \right) - \sum_{m} \frac{B_{ijm} - B_{jim}}{\sigma_{i} + \sigma_{j}} \lambda^{m}_{jk}
\]

\[
- \sum_{m} \left( \frac{B_{mjl} - B_{jml}}{\sigma_{m} + \sigma_{j}} \lambda^{m}_{ik} + \frac{B_{imk} - B_{mik}}{\sigma_{i} + \sigma_{m}} \lambda^{m}_{jl} + \frac{(B_{mjl} - B_{jml})(B_{imk} - B_{mik})}{(\sigma_{m} + \sigma_{j})(\sigma_{i} + \sigma_{m})} \right)
\]

\[
(3.74)
\]
Setting $k = j$ and $l = i$ in (3.74) leads to

$$\frac{1}{2} \text{Riem}_{iji} \sigma_i \sigma_j = \frac{1}{2} \text{Sec}_{ij} \sigma_i \sigma_j$$

$$= \frac{1}{2} \text{Riem}_{iji} + (\frac{B_{iji} - B_{jii}}{\sigma_i + \sigma_j})_i + \sum_m \frac{B_{ijm} - B_{jim}}{\sigma_i + \sigma_j} \lambda^m_{ji} + \frac{B_{ij} - B_{ijj}}{\sigma_i + \sigma_j} + \sum_m \frac{B_{jim} - B_{ijm}}{\sigma_i + \sigma_j} \lambda^m_{ij}$$

$$- \sum_m (\frac{B_{jmj} - B_{mjj}}{\sigma_m + \sigma_j} \lambda^m_{ii} + \frac{B_{imi} - B_{mii}}{\sigma_i + \sigma_m} \lambda^m_{jj} + \frac{(B_{mjj} - B_{jmj})(B_{imi} - B_{mii})}{(\sigma_m + \sigma_j)(\sigma_i + \sigma_m)})$$

$$+ \sum_m (\frac{B_{mji} - B_{jmi}}{\sigma_m + \sigma_j} \lambda^m_{ij} + \frac{B_{imj} - B_{mij}}{\sigma_i + \sigma_m} \lambda^m_{ji} + \frac{(B_{mji} - B_{jmi})(B_{imj} - B_{mij})}{(\sigma_m + \sigma_j)(\sigma_i + \sigma_m)})$$

(3.75)

The above expression is symmetric in $i, j$ on both sides. From (S3) for any $k \neq l$ we get

$$B_{kk} - B_{lk} = (\sigma_k + \sigma_l)(\sigma_l - \sigma_k)(\lambda^l_{kk} + A_{kl})$$

(3.76)

and substitute in (3.75) to evaluate I,II,III,IV,V,VI separately below.
\[ I = \left( -\frac{\lambda_i^j \sigma_i}{\sigma_j} + \lambda_i^j i_j + A_{ji} \right) = (\lambda_i^j)_{ij} + \left( \frac{A_{ij}}{\sigma_j} \right)_{ij} - \left( \frac{\lambda_i^j A_{jj}}{\sigma_j} \right)_{ij} + \frac{\sigma_i}{\sigma_j^2} \lambda_i^j A_{ij} \]

\[ II = \sum_{m \neq i, j} \left( -\frac{\sigma {\lambda_{ij}^m - \lambda_{jm}^m}}{2\sigma_j} + \lambda_{im}^j - \frac{\sigma_i}{2\sigma_j} (\lambda_{ij}^m - \lambda_{jm}^m) + \frac{\sigma_m^2}{2\sigma_i \sigma_j} (\lambda_{ij}^m - \lambda_{jm}^m) \right) \lambda_{ij}^m \]

\[ III = \left( -\frac{\lambda_j^i \sigma_j}{\sigma_i} + \lambda_{ij}^i i_j + A_{ji} \right) = (\lambda_{ij}^i)_{ij} + \left( \frac{A_{ij}}{\sigma_i} \right)_{ij} - \left( \frac{\lambda_j^i A_{ij}}{\sigma_i} \right)_{ij} + \frac{\sigma_j}{\sigma_i^2} \lambda_j^i A_{ji} \]

\[ IV = \sum_{m \neq i, j} \left( \frac{\sigma_j}{2\sigma_i} (\lambda_{ij}^m - \lambda_{mi}^m) - \frac{\sigma_i}{2\sigma_j} (\lambda_{ij}^m - \lambda_{jm}^m) + \frac{\sigma_m^2}{2\sigma_i \sigma_j} (\lambda_{ij}^m - \lambda_{jm}^m) \right) \lambda_{ij}^m \]

\[ V = \sum_{m} \left( \frac{\sigma_j}{\sigma_i^2} \lambda_{mj}^m - \lambda_{iij}^m - A_{ji} \lambda_{mi}^m \right) (\lambda_{ii}^m \frac{\sigma_j}{\sigma_i^2} + A_{im} \lambda_{ij}^m \frac{\sigma_j}{\sigma_i^2} + A_{im} A_{jj}^m) \]

\[ VI = -\sum_{m} \left( \lambda_{ij}^m - \lambda_{jm}^m \right)^2 \frac{\sigma^2}{4\sigma_j \sigma^2_m} - \sum_{m} (\lambda_{im}^j - \lambda_{mi}^j)^2 \frac{\sigma^2}{4\sigma_i \sigma^2_m} + \sum_{m} (\lambda_{ij}^m - \lambda_{jm}^m)^2 \frac{\sigma^2}{4\sigma_i \sigma_j} \]

\[ + \sum_{m} \sigma_i \sigma_j (\lambda_{mi}^j - \lambda_{mj}^j) (\lambda_{jm}^i - \lambda_{mi}^i) + \sum_{m} \lambda_{ij}^m \lambda_{ji}^m \]

(3.77)
Using MAPLE we compute

\[ I + II + III + IV + V + VI \]

\[
= -\frac{\text{Sec}_{ij}}{2} + \left( \frac{A_{ij}}{\sigma_j} \right)_{j} + \left( \frac{A_{ji}}{\sigma_i} \right)_{i} - \frac{2\lambda_{ij}^i A_{ij}}{\sigma_j} - \frac{2\lambda_{jj}^j A_{ji}}{\sigma_i} + \frac{\sigma_i^2 \lambda_{ij}^j A_{jj}}{\sigma_j^2} + \frac{\sigma_j^2 \lambda_{jj}^j A_{ii}}{\sigma_i^2} \\
- \sum_{m} A_{im} \lambda_{jj}^j \frac{\sigma_j}{\sigma_m^2} - \sum_{m} A_{jm} \lambda_{ii}^i \frac{\sigma_i}{\sigma_m^2} + \sum_{m} A_{im} A_{jm} - \frac{\lambda_{ij}^i}{\sigma_j} + \sum_{m} \frac{\sigma_i \sigma_j}{\sigma_m^2} \lambda_{ii}^i \lambda_{jj}^j \\
- (\lambda_{jj}^j)_{i} \frac{\sigma_j}{\sigma_i} - (\lambda_{ii}^i)_{j} \frac{\sigma_i}{\sigma_j} + \sum_{m} \frac{\sigma_i \sigma_j}{\sigma_m^2} \lambda_{ii}^i \lambda_{jj}^j \\
+ \sum_{m} \frac{\sigma_i^3}{\sigma_j \sigma_m^2} (\lambda_{ij}^j - \lambda_{ji}^j)^2 - \sum_{m} \frac{\sigma_j^3}{\sigma_i \sigma_m^2} (\lambda_{jm}^m - \lambda_{mj}^m)^2 \]

(3.78)

Thus

\[ \text{Sec}_{ij} \leq \frac{1}{\sigma_i \sigma_j} \left( \frac{A_{ij}}{\sigma_j} \right)_{j} + \frac{1}{\sigma_i \sigma_j} \left( \frac{A_{ji}}{\sigma_i} \right)_{i} - \frac{2\lambda_{ij}^i A_{ij}}{\sigma_j} - \frac{2\lambda_{jj}^j A_{ji}}{\sigma_i} + \frac{1}{\sigma_i^2} \lambda_{ij}^i A_{jj} + \frac{1}{\sigma_j^2} \lambda_{jj}^j A_{ii} \\
- \sum_{m} A_{im} \lambda_{jj}^j \frac{1}{\sigma_i \sigma_m^2} - \sum_{m} A_{jm} \lambda_{ii}^i \frac{1}{\sigma_j \sigma_m^2} + \sum_{m} A_{im} A_{jm} - \frac{1}{\sigma_i} \lambda_{ii}^i - \frac{1}{\sigma_j} \lambda_{jj}^j \\
+ \sum_{m} \frac{1}{2\sigma_j} (\lambda_{mj}^m - \lambda_{jm}^m)^2 - \sum_{m} \frac{1}{2\sigma_i} (\lambda_{im}^m - \lambda_{mi}^m)^2 + \sum_{m} \frac{1}{2\sigma_j} (\lambda_{jm}^j - \lambda_{mj}^j)^2 - \sum_{m} \frac{1}{2\sigma_i} (\lambda_{jm}^j - \lambda_{mj}^j)^2 \]

(3.79)

It is convenient to introduce the deviation of each \( \sigma_i^2 \) from the average of all the \( \sigma_k^2 \);
more precisely, we define $\nu_i$ by

$$(1 + \nu_i)\sigma_i = \sqrt{\frac{1}{\mu} \sum_k \sigma_k^2} \quad \forall i = 1, \ldots, \mu. \quad (3.80)$$

Note that $\nu_i = 0$ if all the $\sigma_i$ are equal; we also have

$$L = \max_{i=1,\ldots,\mu} (1 + \nu_i). \quad (3.81)$$

We use the shorthand

$$\sigma = \sqrt{\frac{1}{\mu} \sum_k \sigma_k^2} \quad (3.82)$$

and then

$$\frac{1}{\sigma_i} = \frac{1 + \nu_i}{\sigma}. \quad (3.83)$$
In terms of these shorthand notations we obtain

\[
\left| \text{Sec}_{ij} \right| \\
\leq 2 \left( \frac{\Lambda + \mu^2 L^2}{\sigma^4} + (\lambda_0 + 2\lambda_0 \mu^{3/2} L^{3/2} \mu^{3/2} L) \right) + \mu^2 \left( \lambda_0 \sqrt{\mu} + \lambda_1 + \lambda_0^2 \mu + \lambda_0^3 \mu^3 L^2 \right) L \frac{\sigma}{\sigma^2} \\
+ \frac{\mu^2 L^2 (1 + \lambda_0 \mu \sigma^2)}{\sigma^4} + 6 \lambda_0 \mu \frac{L}{\sigma^3} (1 + \lambda_0 \mu \sigma) + 2 \frac{\mu^2 L \lambda_0}{\sigma^3} (1 + \lambda_0 \mu \sigma) + \mu \frac{\mu^2 L^2 (1 + \lambda_0 \mu \sigma)^2}{\sigma^4} \\
+ \left| - (\lambda_{ij}^i) \frac{1}{\sigma^2} - (\lambda_{ij}^j) \frac{1}{\sigma^2} + \sum_m \frac{1}{\sigma^2} \lambda_{ij}^m \lambda_{ij}^m \right| \\
+ \sum_m (\lambda_{ij}^m - \lambda_{ij}^m)^2 \frac{3}{4 \sigma^2} + \sum_m \frac{1}{2 \sigma^2} \left( \lambda_{ij}^m - \lambda_{ij}^m \right) \lambda_{ij}^m L \lambda_{ij}^m \\
+ \left| - (\lambda_{ij}^i) \frac{2 \nu_i}{\sigma^2} - (\lambda_{ij}^j) \frac{2 \nu_j}{\sigma^2} + \sum_m \frac{2 \sigma m}{\sigma^2} \lambda_{ij}^m \lambda_{ij}^m \right| \\
+ \sum_m \frac{\nu_j \lambda_{ij}^m - \lambda_{ij}^m \lambda_{ij}^m}{\sigma^2} + \sum_m \frac{\nu_i \lambda_{ij}^m - \lambda_{ij}^m \lambda_{ij}^m}{\sigma^2} \\
+ \sum_m (\lambda_{ij}^m - \lambda_{ij}^m)^2 \frac{3}{2 \sigma^2} (\nu_i + \nu_j - \nu_m) + \sum_m \frac{\nu_m \lambda_{ij}^m - \lambda_{ij}^m \lambda_{ij}^m}{\sigma^2} \\
+ \sum_m (\lambda_{ij}^m - \lambda_{ij}^m)^2 \frac{3}{2 \sigma^2} (\nu_i + \nu_j - \nu_m) + \sum_m \frac{\nu_m \lambda_{ij}^m - \lambda_{ij}^m \lambda_{ij}^m}{\sigma^2} \\
\leq \frac{2 \Lambda + 3 \mu^2 L^2 + \mu^3 L^2}{\sigma^4} + \frac{\mu^2 L \lambda_0}{\sigma^3} \left( 2 \sqrt{\mu} + 4 \mu L + 2 \mu^2 L + 6 + 2 \mu + 2 \mu^3 L \right) \\
+ \frac{2 \lambda_0 \mu^{5/2} L}{\sigma^2} + \frac{2 \lambda_1 \mu L}{\sigma^2} + \frac{\lambda_0^2 \mu^2 L}{\sigma^2} \left( 4 \mu + 2 \mu^3 L + 2 \mu^2 L + 6 + \mu^3 L \right) \\
+ \frac{\left| \text{Sec}_{ij} \right|}{\sigma^2} (1 + \max_i \nu_i) + \frac{17 \lambda_0^2}{4 \sigma^2} (1 + \max_i \nu_i) + \left( \frac{\max_i \Delta \nu_i}{\sigma^2} \right) + \left( \frac{\max_i \nu_i}{\sigma^3} \right). \\
\text{(3.84)}
\]
Flattening of the image of a geodesic ball under the anchor point mapping

Apart from the term that can be explicitly recognized as $\frac{|\text{Sec}_{ij}|}{\sigma^2} L$, several other terms in the right hand side of this expression contain $\lambda_0$ and $\lambda_1$, which are, respectively, the maximum of the sectional curvature and the derivative of the sectional curvature on a geodesic ball $U$ containing $r$. Note, however, that all these terms contain factors $\frac{L}{\sigma^2}$. If $L$ is not too far from 1 and $\sigma^2$ is sufficiently large, then one can reasonably hope that this results in $\overline{\text{Sec}_{ij}}$ being significantly smaller than $\text{Sec}_{ij}$. We argue in the remaining part of the section that $\Phi(U)$ will be close to the euclidean space for manifolds $\mathcal{M}$ which have non-negative sectional curvature everywhere.

As a first step to make such a comparison with euclidean space, we compute the volume of $\Phi(U)$, which is

$$\int_{\Phi(U)} \theta^1 \wedge \ldots \wedge \theta^\mu.$$  \hspace{1cm} (3.85)

Without the parameterization of $\mathcal{M}$ (which is never given in the applications in which we want to use our analysis) this expression cannot be evaluated directly. Instead, we use the the expression for the volume of a geodesic ball with center $C$ and radius $\rho$, contained in a normal neighborhood of $C$ on a $\mu$-dimensional manifold, derived in [G+74] expressed as a Taylor expansion in terms of coefficients of the powers of $\rho$:

$$\frac{\alpha_\mu \rho^\mu}{\mu} \left( 1 + \sum_i \frac{\overline{\text{Sec}_{ii}}}{6 \mu} \rho^2 + \rho^4 \text{(terms involving Sec}_{ij}) + \Theta(\rho^6) \right)$$  \hspace{1cm} (3.86)

where $\alpha_\mu$ is an appropriate constant. The expression gives a measure of how much the volume of $\Phi(U)$ differs from that of an euclidean ball of the same radius: when the geometry is euclidean, the expansion reduces o the first term, which is exactly the volume of a $\mu$-dimensional euclidean ball of radius $\rho$.

Note that (3.86) applies only to geodesic balls contained within a normal neighbor-
hood such that traveling along a geodesic, in any direction over a geodesic distance \( \rho \) from the ball’s center, one does not encounter a conjugate point of the center. When we have \( \overline{\text{Sec}}_{\Phi(U)} < \text{Sec}_U \), by a corollary of the Rauch Comparison theorem [BC11], if there are two geodesics in \( U \) and \( \Phi(U) \) emanating from \( r \) and \( \Phi(r) \) parameterized by arc length over the same interval, then the first conjugate point of \( r \) must occur before that of \( \Phi(r) \). Thus if \( U \) is a geodesic ball contained in a normal neighborhood of \( r \) then \( \Phi(U) \) is a normal neighborhood of \( \Phi(r) \), and we are free to use (3.86) to estimate the volume of a geodesic ball around \( \Phi(r) \) within \( \Phi(U) \). When \( \overline{\text{Sec}}_{\Phi(U)} << \text{Sec}_U \), one can then argue from (3.86) that the geometry of \( \Phi(\mathcal{M}) \) around \( \Phi(r) \) is much closer to euclidean space than that of \( \mathcal{M} \) around \( r \).

The remaining issue is to have a measure of closeness of \( \Phi(U) \) to the euclidean space. This is not explored in detail, but we outline three ways in which we can answer this through the various geometric comparison theorems.

(i) By Gunther’s volume comparison theorem [Lee18], we know that if \( \overline{\text{Sec}}_{ij} \leq K \) (a constant) and if \( \rho \) is the radius of a ball \( B \) centered in \( \Phi(U) \), then

\[
V_{\Phi(U)}(B) \leq v_k(\rho) \tag{3.87}
\]

where \( v_k(\rho) \) is volume of a geodesic ball with radius \( \rho \) on the \( \mu \)-dimensional sphere of constant curvature \( K \); an equality for any \( \rho \) implies that \( \Phi(U) \) is a ball of radius \( \rho \) of constant curvature \( K \). We note that we can actually compute \( v_K(\rho) \) and not \( V_{\Phi(U)} \) from (3.86) using Proposition (3.1.11) above. The expression \( v_K(\rho) - v_0(\rho) \) would give a measure of the deviation for euclidean space.

(ii) One could also explore how close the manifold is to euclidean space in the sense of Gromov-Hausdorff close. If \( |\text{sec}| < K \) and \( V_{\Phi(U)}(1) \leq V_K(1) \), then by Cheeger’s lemma [PAR06] for \( \mu \geq 2 \) we know that \( \text{inj}(\Phi(\mathcal{M})) > i_0 \) where \( i_0 \) depends on \( K, V_K(1), \mu. \)

57
Then by the convergence theorem of Riemannian geometry there exists $q, r > 0$ such that $||\Phi(U)||_{C, r} \leq q$ where $q$ depends on $i_0, K$ [PAR06].

**The case $\mu = 1$**

We end this section by considering the special case $\mu = 1$, i.e. when the data are assumed to lie on an unknown curve. In this case, the analysis above, carried out for $\mu \geq 2$, has to be adapted somewhat: for curves in euclidean space, there is no concept of intrinsic curvature. Instead, we will compute the absolute curvature of a curve in euclidean space under the anchor point transformation. Let the curve $C$ given by

$$t \mapsto \left(x^1(t), ..., x^N(t)\right) = r(t) \quad (3.88)$$

be parameterized by arc-length. We recall the following facts from [Ste99]. The absolute curvature of $r(t)$ is given by

$$\kappa = \left|\frac{d^2r(t)}{dt^2}\right|. \quad (3.89)$$

The absolute curvature of a curve measures the deviation of a curve from a straight line. The other associated curvatures measure the deviation of the curve from lying in subsequent lower dimensional planes in $\mathbb{R}^N$. Now, the parametrization of the curve $\Phi(C)$ obtained by pushing forward the parametrization of $C$,

$$t \mapsto \left(||r(t) - a_1||^2, ..., ||r(t) - a_L||^2\right) = \Phi(r(t)) \quad (3.90)$$

is no longer an arc-length parametrization through $t$. The unit tangent along $\Phi(C)$ at $r$ is given by

$$\bar{u}_l(t) = \frac{\Phi'(r(t))}{||\Phi'(r(t))||} \quad (3.91)$$
and the absolute curvature becomes

$$\kappa = \frac{||\vec{u}_1'(t)||}{||\Phi'(t)||}. \quad (3.92)$$

We can compute these quantities explicitly.

$$\Phi'(r(t)) = 2 \left( \langle r(t) - a_1, \frac{dr(t)}{dt} \rangle, \ldots, \langle r(t) - a_\Lambda, \frac{dr(t)}{dt} \rangle \right) \quad (3.93)$$

$$||\Phi'(r(t))|| = 2 \sqrt{\sum_\alpha \langle \frac{dr(t)}{dt}, r(t) - a_\alpha \rangle^2} \quad (3.94)$$

$$u_1'(t) = \frac{2}{||\Phi'(r(t))||} \left[ (1, \ldots, 1) + \left( \langle r(t) - a_1, \frac{d^2r(t)}{dt^2} \rangle, \ldots, \langle r(t) - a_\Lambda, \frac{d^2r(t)}{dt^2} \rangle \right) \right] - \frac{2\Phi'(r(t))}{||\Phi'(r(t))||^2} \sum_\alpha \langle \frac{dr(t)}{dt}, r(t) - a_\alpha \rangle \left( \langle \frac{d^2r(t)}{dt^2}, r(t) - a_\alpha \rangle + 1 \right) \quad (3.95)$$

By taking norms of the expressions in LHS and RHS of this last equation (and possibly using Cauchy-Schwarz inequalities), one then obtains bounds similar to those in the more general case.

### 3.1.2 Effect of Weighting in the Moving Anchor Point Transformation

In the previous section, we studied the dependence on the anchor point locations of the sectional curvature of the manifold $\Phi(\mathcal{M})$ (or a piece of this manifold). The bound on the sectional curvature depends on the singular values of a matrix determined by
the region of the manifold under consideration and the anchor point locations. In
order to obtain a small value for this upper bound, these singular values must be
sufficiently large, and the condition number (the ratio of the largest to the smallest)
cannot be too large. By partitioning the manifold into different pieces, for each of
which we select a different subset of anchor points, we can achieve a parameteriza-
tion of the whole manifold that, in each reparameterized piece, has small sectional
curvature. In practice, we carry out the selection of these special anchor points by a
weighting scheme: the anchor points for which the weights are largest are those that
will be selected. As already described in (3.9)), the weights we use are given by

\[ x_\alpha^2(r) \exp(-x_\alpha^2(r)) \] (3.96)

where \( x_\alpha \in \mathbb{R} \) is given by

\[ x_\alpha^2(r) = \sum_{i=1}^{\mu} |\langle r - a_\alpha, u_i \rangle|^2. \] (3.97)

As was the case in the previous section, we distinguish again two cases: \( \mu \geq 2 \) and
\( \mu = 1 \).

In the \( \mu = 1 \) case, i.e. when \( M \) is a curve, the weight is straightforward to calculate
since an approximation for the tangent at a point can be obtained, say by finding
the straight line connecting the adjacent points in the absence of noise or similarly
by determining the best-fitting straight line for the neighboring point cloud in the
presence of noise. Any other weighting scheme can also be chosen as long neither
distant nor close points are considered. We would show the lowering of sectional
curvature for each piece in two steps. First, we investigate under what locations
of the anchors, with respect to the manifold, the bounds on the singular values are
satisfied.

60
We introduce the following notation.

Define for $\beta = 1, \ldots, \mu$ \[ ||r - a_\beta|| = \Delta_\beta. \] (3.98)

Define for $\beta = \mu + 1, \ldots, \Lambda$ \[ ||r - a_\beta|| = s_\beta. \] (3.98)

Lemma 3.1.12. Of the total $\Lambda$ anchors:

- Let $\mu$ anchors with length $\{\Delta_\beta\}$ be chosen such that for $\beta = 1, \ldots, \mu$
  \[ d(a_\beta, T_rM) := \delta_\beta \leq \delta \quad \text{and} \quad \max_{\alpha, \beta \leq \mu} |\Delta_\alpha - \Delta_\beta| \leq \overline{\Delta}. \] (3.99)

- Let $N - \mu$ anchors with length $\{s_\beta\}$ be chosen such that for $\beta = \mu + 1, \ldots, \Lambda$
  \[ \frac{\langle r - a_\beta, u_j \rangle}{||r - a_\beta||} = \cos \angle (r - a_\beta, u_j) \leq \max_{\beta, j} \cos \angle (r - a_\beta, u_j) = \varepsilon << 1. \] (3.100)

Introducing

\[ S := \min_{\beta > \mu} s_\beta \quad \text{and} \quad D := \min_{\beta \leq \mu} \Delta_\beta \] (3.101)

we have for

\[ J = \begin{bmatrix}
\langle r - a_1, u_1 \rangle & \langle r - a_1, u_2 \rangle & \ldots & \langle r - a_1, u_\mu \rangle \\
\langle r - a_2, u_1 \rangle & \langle r - a_2, u_2 \rangle & \ldots & \langle r - a_2, u_\mu \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle r - a_\Lambda, u_1 \rangle & \langle r - a_\Lambda, u_2 \rangle & \ldots & \langle r - a_\Lambda, u_\mu \rangle 
\end{bmatrix} \] (3.102)

the bounds

\[ \sigma = \sqrt{\frac{1}{\mu} \sum_k \sigma_k^2} \geq \sqrt{D^2 + \frac{S}{\mu} \sum_{\alpha > \mu} \sum_l \cos \varphi_{\alpha, l} - \delta^2} \] (3.103)
\[ L^2 = \max_{k=1,...,\mu} \frac{1}{\mu} \sum_i \sigma_i^2 \]

\[ = 1 + 2\mu\sqrt{\mu^3 + 1} + \mu\varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2 + \left( 2 - 4\sqrt{\mu^3 + 1} + 2\mu\varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2 + 8\mu\sqrt{\mu^3 + 1} \right) \frac{\Delta}{D} \]

\[ + \mathcal{O}\left( \frac{(\varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2)^2}{D^2} \right) + \mathcal{O}\left( \frac{\Delta^2}{D^2} \right). \]

(3.104)

Proof. We will provide estimates for the singular values of \( J \) by means of a perturbation argument. We consider the “unperturbed” case to be where all the \( \mathbf{r} - \mathbf{a}_i, \ i = 1, ..., \mu \), line up perfectly with the \( \mathbf{u}_i, \ i = 1, ..., \mu \), and have lengths \( \Delta_i \), i.e.

\[ \langle \mathbf{r} - \mathbf{a}_i, \mathbf{u}_j \rangle = (\Delta_i) \delta_{ij} \]

(3.105)

with \( \Delta_i = ||\mathbf{r} - \mathbf{a}_i|| \) for \( i = 1, ..., \mu \). In addition, we also assume that for \( \beta > \mu \), the \( \mathbf{r} - \mathbf{a}_\beta \) are orthogonal to the tangent space \( T_{\mathbf{r}} \mathcal{M} \), i.e. \( \langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_i \rangle = 0 \) for all \( i \in \{1, ..., \mu\} \) and all \( \beta > \mu \). Then in this ideal case the singular values of \( J \) would have been \( \{\Delta_i\} \). Returning to the non-ideal case, we have

\[ J_{\beta i} = \left( \langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_i \rangle \right)_{i=1,...,\mu} \]

(3.106)
and thus

\[(J^T J)_{ij} = \sum_{\beta=1}^{\Lambda} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle \]

\[= \sum_{\beta \leq \mu} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle + \sum_{\beta > \mu} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle \]

\[= \sum_{\beta > \mu} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle = (\Delta_i)^2 \delta_{i,j} + \sum_{\beta \leq \mu} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle = (\Delta_i)^2 \delta_{i,j} + Q_{ij} \]

\[= (J + P + Q)_{ij}. \]

The eigenvalues of $J$ are given by $\{\Delta_\beta^2\}_{\beta=1}^\mu$. In the more realistic “perturbed” situation, we are considering two different perturbations $P$ and $Q$. We begin with $Q$.

Firstly, we assume that $|\langle r - a_\beta \rangle| \leq ||r - a_\beta|| \varepsilon$ for $\beta > \mu$, with $\varepsilon \ll 1$.

Define $\angle(r - a_\beta, u_j) := \vartheta_{\beta,j}. \quad (3.108)$

Thus for $\beta > \mu$

\[Q_{ij} := \sum_{\beta > \mu} \langle r - a_\beta, u_i \rangle \langle r - a_\beta, u_j \rangle = \sum_{\beta > \mu} (s_\beta)^2 \cos \vartheta_{\beta,i} \cos \vartheta_{\beta,j}. \quad (3.109)\]

Secondly, now let $\beta \leq \mu$. Since the $r - a_\beta$ no longer align perfectly with the $\{u_i\}$s, let $\{u'_i\}$ be a set of unit vectors in $T_r \mathcal{M}$ along the vectors that join $r$ and the projections
of the \( \{ \mathbf{r} - \mathbf{a}_\beta \} \)'s on \( T_r \mathcal{M} \) for \( \beta \leq \mu \). Thus we can write

\[
\mathbf{r} - \mathbf{a}_\beta = \langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_\beta' \rangle \mathbf{u}_\beta' + \text{components perpendicular to } T_r \mathcal{M}
\]

\[
= ||\mathbf{r} - \mathbf{a}_\beta|| \cos \left( \arcsin \frac{\delta_\beta}{\Delta_\beta} \right) \mathbf{u}_\beta' + \text{components perpendicular to } T_r \mathcal{M}
\]

\[
= \Delta_\beta \sqrt{1 - \frac{\delta_\beta^2}{(\Delta_\beta)^2}} \mathbf{u}_\beta' + \text{components perpendicular to } T_r \mathcal{M}.
\]

(3.110)

Then

\[
P_{ij} = \sum_{\beta \leq \mu} \langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_i \rangle \langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_j \rangle - (\Delta_i)^2 \delta_{i,j}
\]

\[
= \sum_{\beta \leq \mu} ||\langle \mathbf{r} - \mathbf{a}_\beta, \mathbf{u}_\beta' \rangle||^2 \langle \mathbf{u}_\beta', \mathbf{u}_i \rangle \langle \mathbf{u}_\beta', \mathbf{u}_j \rangle - (\Delta_i)^2 \delta_{i,j}
\]

(3.111)

\[
= \sum_{\beta \leq \mu} \left( (\Delta_\beta)^2 - \delta_\beta^2 \right) \langle \mathbf{u}_\beta', \mathbf{u}_i \rangle \langle \mathbf{u}_\beta', \mathbf{u}_j \rangle - (\Delta_i)^2 \delta_{i,j}
\]

which implies

\[
|P_{ij} + \Delta_i^2 \delta_{i,j}| = \left| \sum_{\beta \leq \mu} \left( (\Delta_\beta)^2 - \delta_\beta^2 \right) \langle \mathbf{u}_\beta', \mathbf{u}_i \rangle \langle \mathbf{u}_\beta', \mathbf{u}_j \rangle \right|
\]

\[
\leq \sqrt{\sum_{\beta \leq \mu} \left( (\Delta_\beta)^2 - \delta_\beta^2 \right)^2} \sqrt{\sum_{\beta \leq \mu} \langle \mathbf{u}_\beta', \mathbf{u}_i \rangle^2 \langle \mathbf{u}_\beta', \mathbf{u}_j \rangle^2} \quad \text{by Cauchy-Schwarz.}
\]

Thus \( (P_{ij} + \Delta_i^2 \delta_{i,j})^2 = |P_{ij} + \Delta_i^2 \delta_{i,j}|^2 \leq \mu \sum_{\beta \leq \mu} \left( \Delta_\beta^2 - \delta_\beta^2 \right)^2 \)

(3.112)

Thus for \( i \neq j \)

\[
P_{ij} \leq \sqrt{\mu} \sqrt{\sum_{\beta \leq \mu} \left( \Delta_\beta^2 - \delta_\beta^2 \right)^2}
\]

(3.113)
and the diagonal element

\[ P_{ii}^2 \]

\[ \leq \mu \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 - \Delta_i^4 - 2P_{ii} \Delta_i^2 \]

\[ \leq \mu \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 - \Delta_i^4 - 2P_{ii} \Delta_i^2 \]

\[ \leq \mu \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 - \Delta_i^4 - 2\Delta_i^2 \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right) \langle u_{\beta}', u_i \rangle^2 + 2\Delta_i^4 \quad \text{by (3.111)} \]

\[ \leq \mu \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 + \Delta_i^4 + 2\Delta_i^2 \sum_{\beta \leq \mu} \left( \delta_{\beta}^2 - \Delta_{\beta}^2 \right) \langle u_{\beta}', u_i \rangle^2 \quad \text{by construction } \Delta_{\beta} > \delta_{\beta} \]

\[ \leq \mu \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 + \Delta_i^4. \quad (3.114) \]

Hence,

\[ ||P||_F^2 = \sum_i P_{ii}^2 + \sum_{i,j \neq i,j} P_{ij}^2 \leq \mu^3 \sum_{\beta \leq \mu} \left( \Delta_{\beta}^2 - \delta_{\beta}^2 \right)^2 + \sum_i \Delta_i^4 = \sum_i \left( \mu^3 (\Delta_i^2 - \delta_i^2)^2 + \Delta_i^4 \right). \quad (3.115) \]

On the other hand, we have

\[ ||Q||_F = \sqrt{\sum_i \sum_j Q_{ij}^2} \]

\[ = \sqrt{\sum_i \sum_j \left( \sum_{\beta > \mu} (s_{\beta})^2 \cos \vartheta_{\beta,i} \cos \vartheta_{\beta,j} \right)^2} \]

\[ = \sqrt{\sum_{\beta > \mu} (s_{\beta})^2 \sum_{\beta' > \mu} (s_{\beta'})^2 \sum_i \sum_j \left( \cos \vartheta_{\beta,i} \cos \vartheta_{\beta,j} \right)^2} \]

\[ \leq \sum_{\beta > \mu} (s_{\beta})^2 \mu^2 \mu \varepsilon^4 = \mu \varepsilon^2 \sum_{\beta > \mu} (s_{\beta})^2. \quad (3.116) \]

Now, we bound the effect of the perturbation using a consequence of the Weyl’s Inequalities for symmetric matrix perturbation [HJ12]. Recalling \( J^T J = J + P + Q \),

65
we have for each $k = 1, \ldots, \mu$

$$|\sigma_k^2 - \Delta_k^2| \leq \|P + Q\|_2 \leq \|P + Q\|_F \leq \|P\|_F + \|Q\|_F \leq \mu \varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2 + \sqrt{\sum_i \left(\mu^3 (\Delta_i^2 - \delta_i^2)^2 + \Delta_i^4\right)}.$$  \hspace{1cm} (3.117)

Since the sum of the eigenvalues of a matrix is equal to the trace of the matrix,

$$\sum_l \sigma_l^2 = \sum_l (I + P + Q)_{ll}$$

$$= \sum_l \Delta_l^2 + \sum_l \sum_k \left(\Delta_k^2 - \delta_k^2\right) (u'_k, u_l)^2 - \sum_l \Delta_l^2 + \sum_l \sum_{\alpha > \mu} (s_\alpha)^2 \cos^2 \vartheta_{\alpha,l}$$

$$= \sum_k \left(\Delta_k^2 - \delta_k^2\right) + \sum_l \sum_{\alpha > \mu} (s_\alpha)^2 \cos^2 \vartheta_{\alpha,l}.$$  \hspace{1cm} (3.118)

This implies \( \frac{1}{\mu} \sum_l \sigma_l^2 = \frac{1}{\mu} \sum_k \Delta_k^2 - \frac{1}{\mu} \sum_k \delta_k^2 + \frac{1}{\mu} \sum_l \sum_{\alpha > \mu} (s_\alpha)^2 \cos^2 \vartheta_{\alpha,l} \)

$$\geq \frac{1}{\mu} \sum_k \Delta_k^2 \cos^2 \left(\arcsin \frac{\delta_k}{\Delta_k}\right) + \frac{S}{\mu} \sum_{\alpha > \mu} \sum_l \cos \vartheta_{\alpha,l}^2.$$  \hspace{1cm} (3.119)

$$\geq \frac{D^2}{\mu} \sum_k \cos^2 \left(\arcsin \frac{\delta_k}{\Delta_k}\right) + \frac{S}{\mu} \sum_{\alpha > \mu} \sum_l \cos \vartheta_{\alpha,l}^2.$$
\[
\begin{align*}
\frac{1}{\sigma_k^2} &= \frac{1}{\Delta_k^2 + \sigma_k^2 - \Delta_k^2} = \frac{1}{\Delta_k^2} \left( 1 + \frac{\sigma_k^2 - \Delta_k^2}{\Delta_k^2} \right)^{-1} \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + 2 \frac{\sigma_k^2 - \Delta_k^2}{\Delta_k^2} \right) \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + \frac{2\mu^2}{\Delta_k^2} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2}{\Delta_k^2} \sum_i \left( \mu^3 (\Delta_i^2 - \delta_i^2)^2 + \Delta_i^4 \right) \right) \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + \frac{2\mu^2}{\Delta_k^2} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2}{\Delta_k^2} \sum_i \left( \mu^3 + 1 \right) \Delta_i^4 \right) \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + \frac{2\mu^2}{\Delta_k^2} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2\sqrt{\mu^3 + 1}}{\Delta_k^2} \sum_i \Delta_i^2 \right) \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + \frac{2\mu^2}{\Delta_k^2} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2\mu\sqrt{\mu^3 + 1}}{\Delta_k^2} \sum_i \left( \Delta_i^2 + 2\Delta_k \Delta + \Delta^2 \right) \right) \\
&\leq \frac{1}{\Delta_k^2} \left( 1 + \frac{2\mu^2}{\Delta_k^2} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2\mu\sqrt{\mu^3 + 1}}{\Delta_k^2} \Delta_k^2 + 4 (\mu - 1) \sqrt{\mu^3 + 1} \Delta_k \Delta + 2 (\mu - 1) \frac{\sqrt{\mu^3 + 1}}{\Delta_k} \Delta^2 \right) \\
&\leq \frac{1}{D^2} + \frac{\mu^2}{D^4} \sum_{\beta > \mu} (s_\beta)^2 + \frac{2\mu\sqrt{\mu^3 + 1}}{D^2} + 4 (\mu - 1) \frac{\sqrt{\mu^3 + 1}}{D^3} \Delta + 2 (\mu - 1) \frac{\mu^3 + 1}{D^4} \Delta^2.
\end{align*}
\]

using (3.117)

Also,

\[
\begin{align*}
\frac{1}{\mu} \sum_l \sigma_l^2 &= \sum_l \frac{1}{\mu} \Delta_l^2 + \frac{1}{\mu} \sum_l \sum_{\alpha > \mu} (s_\alpha)^2 \cos^2 \vartheta_{\alpha,l} - \frac{1}{\mu} \sum_l \delta_l^2 \\
&\leq D^2 + \frac{2D}{\mu} \sum_l (\Delta_l - D) + \sum_l \frac{(\Delta_l - D)^2}{\mu} + \epsilon^2 \sum_{\alpha > \mu} (s_\alpha)^2 \\
&\leq D^2 + 2D\Delta + \Delta^2 + \epsilon^2 \sum_{\alpha > \mu} (s_\alpha)^2.
\end{align*}
\]
Then we compute

\[
\sum_l \frac{\sigma_l^2}{\mu \sigma_k^2} \leq 1 + 2\mu \sqrt{\mu^3 + 1} + \mu \varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2 + \left(2 - 4\sqrt{\mu^3 + 1} + 2\mu \varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2 + 8\mu \sqrt{\mu^3 + 1}\right) \frac{\overline{\Delta}}{D} + O\left(\frac{\varepsilon^2 \sum_{\beta > \mu} (s_\beta)^2}{D^2}\right) + O\left(\frac{\overline{\Delta}^2}{D^2}\right).
\]

(3.122)

\[\square\]

We just established that \( \sigma \) can be made large and \( L \) can be made smaller if \( D, \delta, \overline{\Delta}, s_\alpha, \varepsilon \) can be controlled. Precisely, the contributions from \( D \) should be large and those from \( \delta, \overline{\Delta}, s_\alpha, \varepsilon, \vartheta \) should be small. This means that \( \mu \) of them must be in the tangent space of \( r \) such that it is sufficiently far away from it and close to each other. The rest of the \( N - \mu \) should be either close to \( r \) or normal to the \( T_r M \). These quantities are in turn controlled by the weight used. In this section, we shall see that under certain conditions on the location of the anchor points (to which we shall come back), having a collection of anchor points with sufficiently large values of the weights \( w_\alpha(r) \), uniformly through an open region \( R \) of \( M \), guarantees that the sectional curvature of \( \Phi(R) \), parameterized using only those anchor points, is indeed small. This is the content of Proposition 3.1.18. We begin with introducing some definitions and facts.

We recall

**Proposition 3.1.13.** [NSW08] If \( \tau \) is the reach of \( M \subset \mathbb{R}^N \) then

1. \(-2/\tau^2 \leq \sec_{\mathbb{R}} \leq 1/\tau^2\).

2. If \( \phi \) is the angle between two points \( s, t \in M \) then

\[
\cos \phi = \min_{u' \in T_r M} \max_{u'' \in T_r M} |\langle u', u'' \rangle| \geq 1 - \frac{1}{\tau} d_{\mathbb{R}}(s, t).
\]

(3.123)
3. [BLW19] For two points $s, t \in M$, we have

$$\angle_{T_s M, T_t M} \leq \frac{d_M(s, t)}{\tau} \sin\left(\frac{\angle_{T_s M, T_t M}}{2}\right) \leq \frac{||s - t||}{2\tau}. \quad (3.124)$$

With a view to Proposition 3.1.13 item (1), we introduce the following definition.

**Definition 3.1.14.** Fix $0 < \epsilon \leq 1$. Let $q \in M$, such that

$$\sec_M(q) \geq \frac{\epsilon}{\tau^2} \quad \text{or} \quad \sec_M(q) \leq -\frac{2\epsilon}{\tau^2}. \quad (3.125)$$

Then a curvature $\epsilon$-bump can be defined as

$$\mathcal{R}^{h.c.}_q = \text{Connected component of } \pi_{\mathcal{F}(\mathcal{M})} \circ \pi_{Gr_{N,2}(\mathcal{M})} \circ \sec^{-1}\left(\left[\frac{\epsilon}{\tau^2}, \infty\right)\right) \text{ that contains } q,$$

or respectively

$$\pi_{\mathcal{F}(\mathcal{M})} \circ \pi_{Gr_{N,2}(\mathcal{M})} \circ \sec^{-1}\left(\left[-\frac{2\epsilon}{\tau^2}, \infty\right)\right).$$

**Remark 3.1.15.** Note that the sectional curvature is a $C^\infty$ function on the Grassmann bundle of $M$ and not on $M$. The definition above is inspired from the construction in [CLN06]. Note that although we use the same names, the definitions have been altered to fit our purpose. This is a characterization of high-curvature points.

**Definition 3.1.16.** The region $\mathcal{M} \setminus \sqcup \mathcal{R}^{h.c.}_q$ is a disjoint union of open sets each of which will be called a manifold-flat-piece $\mathcal{R}^{mfp}$.

**Definition 3.1.17.** Consider $p, q \in \mathcal{R}$ with $q \in \mathcal{R}^{h.c.}$. Then $p$ is said to be $d$-remote from $\mathcal{R}^{h.c.}_q$ with $d > 0$ if $d_{\mathcal{M}}(p, \mathcal{R}^{h.c.}_q) \geq d \text{diam}(\mathcal{R}^{h.c.}_q)$ where $d_{\mathcal{M}}(p, \mathcal{R}^{h.c.}_q)$ is the geodesic distance between $p$ and the closest point of $\mathcal{R}^{h.c.}_q$ with respect to $p$.

In this section we shall demonstrate the effectiveness of the proposed selection criterium for one manifold-piece. The weighting selection scheme results in anchor
points that are typically well-chosen with respect to \( r \). We shall illustrate this by a numerical experiment in Subsection 3.2.1. The theoretical argument makes use of the derivations of the previous section to prove rigorously, under fairly restrictive conditions (not always observed in practice by examples of interest), that the absolute curvature (for curves) or sectional curvature (for the case \( \mu \geq 2 \)) is smaller after reparametrization than before.

**Proposition 3.1.18.** Let \( \mathcal{M} \) be a submanifold of positive sectional curvature and dimension greater than 1 and consisting of \( d \)-remote points, have positive reach with one curvature \( \epsilon \)-bump. For each \( \mathcal{R}^{\text{mfp}} \) choose \( \frac{n}{2} \) points within an open set in \( \mathcal{R}^{\text{mfp}} \) of diameter \( \rho \). Choose \( \frac{\lambda - \mu}{2} \) points anywhere in \( \mathcal{R}^{\text{h.c.}} \) such that for each \( r \in \mathcal{M} \), we have at least \( \Lambda \) anchors \( \{a_{n}\} \) for which \( ||r - a_{n}||^2 \exp(-||r - a_{n}||^2/d) \) is finite. Consider the \( \Lambda \) highest values of \( ||r - a_{n}||^2 \) and \( \Lambda - \mu \) lowest values \( ||r - a_{n}||^2 \). Then the quantities \( D, \delta, \Delta, \epsilon, S \) are bounded by \( \tau, \epsilon, d, \rho \) for \( h \in \mathcal{R}^{\text{h.c.}} \).

**Proof.** We introduce the notation for the two sets of anchors for \( h \)

\[
\{a_{i}^T\}_{i=1}^{\mu} \quad \{a_{i}^N\}_{i=\mu+1}^{N}. \tag{3.127}
\]

By Bonnet-Myer’s theorem,

\[
diam(\mathcal{R}^{\text{h.c.}}) \leq \frac{\pi \tau}{\sqrt{\epsilon}}. \tag{3.128}
\]

For the bound of \( \Delta \), we have

\[
|\Delta_i - \Delta_j| \leq \left| ||h - a_i^T|| - ||h - a_j^T|| \right| \leq 2\tau + d \frac{\pi \tau}{\sqrt{\epsilon}} := \bar{\Delta}. \tag{3.129}
\]

Moreover, we have \( D = O(d \frac{\pi \tau}{\sqrt{\epsilon}}) \) and \( S = O(\tau) \). Let \( p \in \mathcal{M} \) such that \( ||p - a_i^T|| \leq \tau \).
From Proposition 3.1.13 item 3 we have

$$\angle(T_p\mathcal{M}, T_h\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p, h)}{\tau} \leq \frac{d\pi}{\sqrt{\epsilon}}. \quad (3.130)$$

If $\delta_p$ is the distance between $p$ and the projection of $p$ on $T_r\mathcal{M}$, then

$$\frac{\delta_p}{\sin \angle(T_p\mathcal{M}, T_h\mathcal{M})} \leq ||h - p|| \leq d(h, p) \quad (3.131)$$

and if $\delta_i$ is the distance between $a_i$ and the projection of $a_i$ on $T_r\mathcal{M}$,

$$|\delta_i| \leq |\delta_p| + \tau + \rho + \frac{d\pi\tau}{\sqrt{\epsilon}} \leq 2d\frac{\pi\tau}{\sqrt{\epsilon}} + \tau + \rho := \delta. \quad (3.132)$$

3.2 Numerical Examples

3.2.1 2D curves

This example demonstrates the utility of the weight chosen and the choice of the anchors close to the inflection points of the curve. The approximate inflection point regions were obtained by considering convex hulls of the curves at various points. We generate 9000 curves in MATLAB [rh] where each curve has the form

$$r(t) = 1 + 10 \sum_{i=1}^{10} A_i \sin(it + \phi_i) \quad (3.133)$$

where $A_i$ is sampled as a 10 - dimensional random vector formed by uniformly spacing the log-interval from [-0.5,-2.5] and then multiplying each component by a randomly generated number. Similarly, $\phi_i$'s are generated as 10 - dimensional random vector.
multiplied by $2\pi$.

![Figure 3.7: Some of the generated curves in Subsection 3.2.1](image)

We choose the anchors close to the inflection points and use the moving anchor point transformation. Each curve in the experiment gets chopped into 3-4 pieces and we discard pieces if they have less than 3 points. Removing the outliers, we obtain the histogram in Figure 3.8 depicting the RMSE for fitting linearly all the chopped components of the 9000 curves.
Figure 3.8: Histogram of the root-mean square error of fitting a straight line for all the chopped components (of the order of $O(10^4)$) the 9000 curves. The figure below is zoomed in.

We then take the mean of the RMSE for each curve and plot the number of curves for various intervals of the errors from 0.005 to 0.035 and greater in the bar graph in Figure 3.9.
3.2.2 Denoising

In each of the denoising examples below, we employ the moving anchor point transformation and then use Step 1 of MMLS (i.e. finding the approximate tangent space to a noisy point cloud) to find the reconstructed points in the new space. We solve back for the points using the Levenberg-Marquardt algorithm of MATLAB.

Denoising non-linear surfaces of very high codimension.

We consider the example in [Sob19] of denoising the spectrum of symmetric semi-positive matrices i.e. matrices of the form

$$ A = U D U^t $$

(3.134)
where $U$ is an orthogonal matrix and $D$ is a diagonal matrix. The matrix is sampled in the following way. Let

$$D = \text{diag}(\lambda_1, \lambda_2, 0, ..., 0) \in \mathbb{R}^{10 \times 10}$$

(3.135)

and the eigenvalues lie in the interval $\lambda_1 \in [2, 3]$ and $\lambda_2 \in [3, 4]$. We take $U$ to be a constant matrix. Thus $A$ is a 2-dimensional linear manifold in $\mathbb{R}^{10 \times 10}$. We consider column-stacked matrices of the 10-dimensional symmetric matrix $UDU^t$ and each point on the manifold becomes a vector in $\mathbb{R}^{100}$. The manifold is linear since the matrices form a vector space. Let $x$ be such a column stacked matrix and we cube each coordinate of $x$, that is to say, we consider the transformation

$$x = (x^1, ..., x^{100}) \mapsto ((x^1)^3, ..., (x^{100})^3).$$

(3.136)

This makes the new manifold a 2-dimensional non-linear submanifold in $\mathbb{R}^{100}$. We take 44 equi-spaced points in each of the the interval $[2, 3]$ and $[3, 4]$. This gives 1936 data points in the cloud. We sample this point cloud with zero mean additive uniform noise of 0.5. Our goal is to denoise this cloud. In this particular case the noise is very high compared to the original eigenvalues and Step 2 of the MMLS runs into numerical issues concerning non-stable matrix inversions. Figure 3.10 shows the denoising of the spectra using anchor point transformation and using only Step 1 of MMLS for denoising. We use the same metrics for evaluation of the quality of the denoising as in [SL19]. The percentage error of estimating the leading eigenvalue was calculated to be

$$\text{Percentage Error} = \frac{||\lambda_1^{\text{approx}} - \lambda_1^{\text{truth}}||_2}{||\lambda_1^{\text{truth}}||_2} = 0.034\%$$

(3.137)
and the tail energy which is a measure of the residual eigenvalues defined by was

\[
\text{Tail energy} = \sqrt{\frac{\lambda_3^2 + ... + \lambda_{10}^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + ... + \lambda_{10}^2}} = 0.0028. \tag{3.138}
\]

**Denoising surfaces in 3D**

The following Figure 3.11 shows the denoising of a point cloud of around 5000 points sampled from a lemur tooth where the data is perturbed by uniform noise \( U(-\sigma, \sigma) \) with \( \sigma \) going from 0.01 to 0.03 in steps of 0.005.
Figure 3.11: Denoising of a tooth point cloud.

Figure 3.12 shows the histogram of the square of the difference between reconstructed and ground truth values.
Denoising Helices

We take the example of a helix \((\sin t, \cos t, t)\) from where we would sample 400 points by adding

1. Uniform noise distributed as \(U(-0.2, 0.2)\)
2. Gaussian noise 0.3.

**Uniform Noise:** The reconstruction diagrams for the MMLS method are in first two panels of Figure 3.13 and that for the APT approach are in the fifth and sixth panels of Figure 3.13. For a typical iteration, the reconstruction errors for MMLS and the APT method are close: 0.0019 and 0.0017 respectively. On a visual inspection of the diagrams, the MMLS method approximates accurately at places, however, it gets distorted thus not being able to preserve the shape. Whereas with the APT method, the reconstructed version has slightly shrunk with the shape almost intact. To make this precise, the histogram shows a greater number of accurately projected points for MMLS, whereas much lesser overall spread for the APT. Finally, the error plot in the fourth and eighth panels of Figure 3.13 shows that the APT method has almost no outliers. Moreover, in spite of the fact that the error value for MMLS is close to zero for many points, the APT method keeps a low average value for most points, thus preserving the overall curvature.

**Gaussian Noise:** The reconstruction diagrams for the MMLS method are in the first and second panels of Figure 3.14 and that for the APT method are in the fifth and sixth panels. We have the reconstructions error for a typical iteration for MMLS and the APT method to be 0.0070 and 0.0061 respectively, which are very close. The performance for the approximation is similar to that for the uniform noise.
Figure 3.13: Comparison with MMLS (blue), with moving APT (orange) for uniform noise U(0.2, 0.2)

Figure 3.14: Comparison with MMLS (blue), with moving APT (orange) for gaussian noise 0.3
Chapter 4

Conclusions and Future Work

In this part of the thesis, we showed the effectiveness of a novel parameterization that enables lower degree polynomial approximation of polynomial surfaces. We also studied the warping property of this transformation.

![Image](a) Anchor points both close to high and low curvature points  
(b) Image under the anchor point transformation  

![Image](c) Anchor points close to lower curvature points  
(d) Fewer straighter pieces compared to (b)

Figure 4.1: Illustration of the dependence of the location and number of anchors relative to the manifold with the number of chopped parts.

The next question we would like to study is: given the same sampling assumptions as in MMLS, can we prove rigorously if the moving anchor point transformation...
performs the same or outperforms the MMLS? Another direction would be to study the interrelationship of the weight chosen and the number of chopped parts (see Figure 4.1). We cannot have too many chopped parts especially if the noise is high. We would also want to investigate robustness to noise with different weighting schemes. Moreover, it will be interesting to know the computational complexity of the method within some error tolerance.
Part II

Functionals for Shape characterization
Chapter 5

Statement of the Problem

5.1 Motivation for our problem

Definition 5.1.1. The Dirichlet functional of a smooth map $\phi$ between two Riemannian manifolds, $(\mathcal{M}, g_M)$ and $(\mathcal{N}, g_N)$ is given by

$$L[\phi] = \frac{1}{2} \int_{\mathcal{M}} \text{Tr}(\phi^* g_N) d\mathcal{M}. \quad (5.1)$$

We say $\phi$ is a harmonic map if it is a critical point of the Dirichlet functional.

In coordinates this integrand depends on the first derivatives of $\phi$. In the following example we will see that the the characterization of constant mean curvature (CMC) surfaces is related to the Dirichlet functional with the integrand involving the mean and gauss curvatures of $\mathcal{M}$.

Example 5.1.2. An oriented surface $(\mathcal{M}, g)$ immersed in $\mathbb{R}^3$ is constant mean curvature if and only if its gauss map is harmonic [RV70]. In this case $\mu_h = 2, \mu_k = 1$.

The example above is a special case of the Ruh-Vilms’ theorem. We are interested in harmonic maps $\phi : \mathcal{M}^2 \to \mathcal{N}^2$ whose Dirichlet functional is of the form

$$L[\phi] = \int_{\mathcal{M}} (\mu_h H^2 - \mu_k K) d\mathcal{M}. \quad (5.2)$$

Proof of Example 5.1.2. We give an elementary proof. If $\nu : \mathcal{M} \to \mathbb{S}^2$ denotes the
gauss map of \( \mathcal{M} \), then we know that its trace-Laplacian operator is given by:

\[
\Delta_M \nu = -2 \text{grad}(H) - 4(H^2 - \frac{1}{2} K)\nu.
\] (5.3)

Let \( T\mathcal{M} \) and \( N\mathcal{M} \) be the tangent and normal bundles of \( \mathcal{M} \). Without loss of generality \( N \) is isometrically embedded in a euclidean space. Then the Euler Lagrange equation [Jos06] for a harmonic gauss map is

\[
\Delta_M \nu + \nu \langle d\nu, d\nu \rangle_{T^*\mathcal{M} \otimes f^*TN} = 0
\] (5.4)

and from which we obtain

\[
\Delta_M \nu + 2\nu \mathcal{L}_\nu = 0.
\] (5.5)

Now, the pull-back of the metric on \( \mathbb{S}^2 \) by the gauss map is given by the third fundamental form on \( \mathcal{M} \) [O+68]. Hence, by (5.1),

\[
\mathcal{L}_\nu = \frac{1}{2} g^{ij} III_{ij} = \frac{1}{2} g^{ij}(2H h_{ij} - Kg_{ij}) = 2H^2 - K.
\] (5.6)

Thus comparing (5.3), (5.5) and (5.6) we obtain \( \nabla H = 0 \). \( \square \)

Given a smooth surface \( \mathcal{M} \), any immersion \( \varphi : \mathcal{M} \rightarrow \mathbb{R}^3 \) equips \( \mathcal{M} \) with a metric \( g_\varphi \), a volume \( d\text{vol}_\varphi \), and gauss and mean curvatures \( K_\varphi \) and \( H_\varphi \).

**Definition 5.1.3** ([Bry84]). The *Willmore functional* of an immersion \( \varphi : \mathcal{M} \rightarrow \mathbb{R}^3 \) is

\[
\int_\mathcal{M} H^2_\varphi d\text{vol}_\varphi.
\] (5.7)

**Remark 5.1.4.** The immersion \( \varphi \) is an extremal of the Willmore functional if and
only if the Euler-Lagrange equation
\[ \Delta \phi H \phi + 2H \phi (H^2 \phi - K \phi) = 0 \] (5.8)
holds.

**Remark 5.1.5.** By the Gauss-Bonnet theorem,
\[ \int_M K \phi d\text{vol} \phi = 2\pi \chi_M \]
is a topological invariant of \( M \) and is independent of \( \phi \). In particular, \( \phi \) is an extremal of the Willmore functional if and only if it is an extremal of the functional
\[ \int_M (H^3 \phi - K \phi) d\text{vol} \phi. \]

**Example 5.1.6.** The Conformal gauss map which is \( S^{4,1} \)-valued is harmonic if and only if the surface is Willmore. In this case the lagrangian density is of the form as in (5.2) with \( \mu_h = \mu_k = 1 \) [Bry84].

We are especially interested in examples of maps whose lagrangian has a certain form (5.2) and the harmonicity of which are characterized by conditions on the curvatures of \( M \), as in the above examples.

**5.1.1 Moving frame approach to maps between surfaces**

We now turn to defining the energy functional and tension field of a map in the language of moving frames. Let \( \mathcal{U} \subset M \) and \( \mathcal{V} \subset N \) be open subsets. Assume that \( \mathcal{U} \) (and \( \mathcal{V} \)) admit orthonormal coframes \( \omega^i, \omega^2 \in \mathcal{A}^1(U) \) (and \( \theta^i, \theta^2 \) respectively \( \in \mathcal{A}^1(V) \)). Indices for quantities defined on \( \mathcal{U} \) will be \( 1 \leq i, j, k \leq 2 \) and those on \( \mathcal{V} \)
will be $1 \leq \alpha, \beta, \gamma \leq 2$. The Levi-Civita form satisfies

$$d\omega^1 = \omega^2 \wedge \omega^1_2 \quad d\omega^2 = -\omega^1 \wedge \omega^1_2$$  \hspace{1cm} (5.9)$$

and the gauss curvature $K$ of $\mathcal{M}$ satisfies

$$d\omega^1_2 = -K \omega^1 \wedge \omega^2. \hspace{1cm} (5.10)$$

Similarly, if $\overline{K}$ is the gauss curvature of $\mathcal{N}$ then

$$d\theta^1 = \theta^2 \wedge \theta^1_2 \quad d\theta^2 = -\theta^1 \wedge \theta^1_2 \quad d\theta^1_2 = -\overline{K} \theta^1 \wedge \theta^2.$$ \hspace{1cm} (5.11)$$

Given a map $\phi: \mathcal{M} \to \mathcal{N}$, with the property $\phi(U) \subset V$, we may define functions $\{f_i^\alpha\}_{i, \alpha = 1}^2$ on $\mathcal{U}$, by

$$\overline{\theta}^i := \phi^* \theta^\alpha = f_i^\alpha \omega^i. \hspace{1cm} (5.12)$$

Then the lagrangian can be expressed as

$$\mathcal{L}_\phi := \frac{1}{2} f_i^\alpha f_i^\alpha.$$

The exterior derivative of (5.12) and an application of Cartan’s lemma (Lemma 2.1.1) yields $f_i^\alpha = f_j^\alpha \in C^\infty(\mathcal{U})$ such that

$$f_i^\alpha \omega^j = df_i^\alpha - f_k^\alpha \omega^k_i + f_i^\beta \overline{\theta}_{\beta}^j \hspace{1cm} (5.14)$$

where

$$\overline{\theta}_{\beta}^i := \phi^* \theta^\alpha_{\beta}.$$ \hspace{1cm} (5.15)$$
The Euler Lagrange equation obtained by extremizing (5.13) with respect to variations of \( \phi \) is given by
\[
\tau^\alpha = f^\alpha_{ii} = 0.
\] (5.16)

The quantity \( \tau^\alpha u_\alpha \) is called the tension field of \( \phi \).

5.2 Review of Exterior Differential Systems

In the next chapter we will study overdetermined partial differential equations. We will state such a system of overdetermined partial differential equations in the language of the method of moving frames by Cartan. The issue that will be investigated in this thesis is the space of solutions of these PDE’s and examples of solutions in specific cases. In particular, the Cauchy-Kovalevskaya theorem is not equipped to handle overdetermined systems nor does it automatically reveal integrability conditions of a determined system. Herein comes the motivation for our use of exterior differential systems. Such systems require the PDEs to be first expressed as an ideal of differential forms. We will give a review of this method by an illustrative situation namely, harmonic maps \( \phi : \mathcal{M} \rightarrow \mathcal{N} \) exist locally. We recall that harmonic maps are also defined as those maps which satisfy
\[
f^\alpha_{ii} = 0.
\] (5.17)

We recall the definition

**Definition 5.2.1** ([IL03],[BCG+13]). Let \( \Sigma \) be a manifold of dimension \( n \) and \( \mathcal{I} \subset \Omega^*(\Sigma) \) be an ideal that is closed under taking exterior derivatives. Let \( \Omega \in \Omega^p(\Sigma) \) with \( p \leq n \) be a decomposable \( p \)-form that is non-zero at every point of \( \Sigma \). This condition is referred to as an independence condition on \( \Sigma \). Then the tuple \((\mathcal{I}, \Omega)\)
constitute an exterior differential system (EDS) with independence condition defined on Σ.

**Example 5.2.2.** In order to translate (5.17) in the language of EDS, we define

\[ \Sigma = \mathcal{F}_M \times \mathcal{F}_N \times \text{GL}_2(\mathbb{R}) \times V \]  

(5.18)

where \( f^\alpha_i \) are local coordinates on \( \text{GL}_2(\mathbb{R}) \) and

\[ V = \{ f^\alpha_{ij} \in \mathbb{R}^8 \text{ s.t. } f^\alpha_{ij} = f^\alpha_{ji}, \; f^\alpha_{ii} = 0 \} \cong \mathbb{R}^4. \]  

(5.19)

Defining the covariant derivative of \( f^\alpha_i \):

\[ \pi^\alpha_i = df^\alpha_i - f^\alpha_j \omega^j_i + f^\beta_i \theta^\alpha_{\beta i} \]  

(5.20)

we set

\[ \mathcal{I} = \{ \xi^\alpha := \theta^\alpha - f^\alpha_i \omega^j_i, \quad \xi^\alpha_i := \pi^\alpha_i - f^\alpha_{ij} \omega^j \}. \]  

(5.21)

The independence condition is given by \( \omega^1 \wedge \omega^2 \wedge \omega^1 \neq 0 \).

**Definition 5.2.3 ([IL03],[BCG+13]).** An integral manifold of an EDS \((\mathcal{I}, \Omega)\) is an immersion \( \iota : \tilde{\Sigma} \to \Sigma \), such that for all \( \varphi \in \mathcal{I} \) we have \( \iota^* \varphi = 0 \) and \( \iota^* \Omega \neq 0 \) at each point of \( \Sigma \) and \( \tilde{\Sigma} \) is of dimension \( p \).

**Definition 5.2.4 ([IL03],[BCG+13]).** An integral element \( E \) of dimension \( p \) of an EDS \((\mathcal{I}, \Omega)\) is an element of the Grassmannian \( \text{Gr}(p, T_\sigma \Sigma) \) where \( \sigma \in \Sigma \) such that \( \Omega|_E \neq 0 \) and for all \( \varphi \in \mathcal{I} \)

\[ \varphi|_E = 0. \]  

(5.22)

The Cartan-Kähler theorem generalizes the Cauchy-Kovalevskaya Theorem. It
gives the necessary condition for an integral element to be the tangent space of an integral manifold (the reverse is always true) and gives the data necessary to specify uniquely the integral manifold.

Below we give a review of this algorithmic approach. Let \( \mathcal{V}_p(\mathcal{I}) \) be the set of all \( p \)-dimensional integral elements of \( \mathcal{I} \). Clearly, \( \mathcal{V}_p(\mathcal{I}) \subset \text{Gr}_p(T\Sigma) \). When an independence condition is contained in the EDS we will be interested in the set \( \mathcal{V}_p(\mathcal{I}, \Omega) = \mathcal{V}_p(\mathcal{I}) \cap \text{Gr}_p(T\Sigma, \Omega) \).

**Definition 5.2.5** ([IL03],[BCG+13]). A \( k \)-dimensional integral element \( E \) is \( \text{Kähler-ordinary} \) if \( \mathcal{V}_k(\mathcal{I}, \Omega) \) is a smooth submanifold of \( \text{Gr}_k(T\Sigma, \Omega) \) near \( E \).

**Definition 5.2.6** ([IL03],[BCG+13]). Fix \( E \in \mathcal{V}_k(\mathcal{I}, \Omega) \) and a basis \( \{e_1^i\}_{i=1}^k \) of \( E \). The polar space of \( E \) is given by

\[
H(E) := \{ w \in T_{\sigma}\Sigma \mid \psi(w, e_1, \ldots, e_k) = 0 \quad \forall \psi \in \mathcal{I}^{k+1} \}. \tag{5.23}
\]

**Definition 5.2.7.** An integral flag of length \( p \) is a nested sequence

\[
E_1 \subset E_2 \subset \ldots \subset E_p \tag{5.24}
\]

with \( \text{dim } E_i = i \) and \( E_p \in \mathcal{V}_p(\mathcal{I}, \Omega) \).

**Definition 5.2.8** ([IL03],[BCG+13]). A Kähler-ordinary integral element \( E \) is \( \text{Kähler-regular} \) if

\[
\text{codim}(H(\tilde{E})) = \text{codim}(H(E)) \tag{5.25}
\]

for all \( \tilde{E} \) in a neighborhood of \( E \) in \( \mathcal{V}_n(\mathcal{I}) \).

We say \( E_p \) is an ordinary integral element and the flag (5.24) is ordinary if \( E_i \) is Kähler regular for all \( 1 \leq i \leq p - 1 \).
Theorem 5.2.9 (Cartan-Kähler [IL03],[BCG+13]). Fix an integral flag as in (5.24). Set $E = E_p$ and assume $E_k$ is Kähler-regular for $k = 0, \ldots, p - 1$. Then there is a smooth $p$-dimensional submanifold $\tilde{\Sigma}$ with $\iota_* T_{\sigma} \tilde{\Sigma} = E_p$. Fix local coordinates $x^1, \ldots, x^p, z^1, \ldots, z^{n-p}$ at $\sigma \in \Sigma$ with $\dim \Sigma = n$ such that $E_k$ is spanned by $\{ \frac{\partial}{\partial x^k} \}$, and $H(E_k)$ is annihilated by $dz^1, \ldots, dz^{c_k}$, where $c_k$ is the codimension of $H(E_k)$. Then in a neighborhood of $\sigma$, $\Sigma$ is a graph $z^a = F^a(x^1, \ldots, x^p)$ with $1 \leq a \leq (n - p)$ and $\tilde{\Sigma}$ is determined by the initial conditions

\begin{align*}
F^a(x^1, 0 \ldots 0) & \quad \text{with } c_0 < a \leq c_1 \\
F^a(x^1, x^2, 0 \ldots 0) & \quad \text{with } c_1 < a \leq c_2 \\
& \quad \vdots \\
F^a(x^1, \ldots, x^p, 0 \ldots 0) & \quad \text{with } c_{p-1} < a \leq c_p.
\end{align*}

(5.26)

The Cartan’s test gives the necessary and sufficient condition for a flag to be ordinary as in the Cartan-Kähler theorem. If the Cartan’s test is not satisfied, the EDS needs to be “prolonged” i.e. more derivatives need to be added to find possible compatibility conditions. The Cartan-Kuranishi prolongation theorem [BCG+13] guaranties that only a finite number of prolongations is needed before we can infer whether the system has a solution or not.

Theorem 5.2.10 (Cartan’s Test [IL03],[BCG+13]). Given any flag of the form (5.24) we have

\[
\text{codim}_{E_p} \mathcal{Y}_p(\mathcal{I}) \geq c_0 + \ldots + c_{p-1} \quad \text{where } c_i = \text{codim } H(E_i) \text{ in } T_{\sigma} \Sigma.
\]

(5.27)

The equality holds if and only if the flag is ordinary or in other words $E_p$ is ordinary.

The Cartan-Kähler theorem and the Cartan’s test have a simpler form for linear
Pfaffian systems such as that in Example 5.2.2.

**Definition 5.2.11 ([BCG+13])**. A Pfaffian system is an EDS with an independence condition and generated by functions and/or one-forms only.

Let $I \subset J$ be subbundles of the cotangent bundle $T^*\Sigma$. Let $\mathcal{I}$ be the ideal generated algebraically by sections of $I$, and let $\mathcal{J}$ be the ideal generated by sections of $J$. Then

**Proposition 5.2.12.** ([BCG+13]) $(\mathcal{I}, \Omega)$ is a linear Pfaffian system if and only if

$$d\mathcal{I} \equiv 0 \mod \mathcal{J}. \quad (5.28)$$

Let $p$ denote the corank of $I$ in $J$, then if $E$ is a $p$-dimensional integral element of $I$, then there is a well-defined linear map

$$J/I \to E^*. \quad (5.29)$$

The independence condition is the requirement that this map be an isomorphism. Now, choose $\{\pi^\epsilon\}$ one-forms with $1 \leq \epsilon \leq \text{corank of } J$ in $T^*\Sigma$, such that $\{\Xi^\alpha\}, \{\omega^i\}, \{\pi^\epsilon\}$ span $T^*\Sigma$ locally. Here $\{\Xi^\alpha\}$ as a local framing of $I$, $\{\Xi, \omega\}$ a local framing of $J$, and $\{\Xi^\alpha, \omega^i, \pi^\epsilon\}$ a local framing of $T^*\Sigma$. In the case of the illustrative EDS in Example 5.2.2, we have $\{\Xi^\alpha\} = \{\xi^1, \xi^2, \xi_1^1, \xi_1^2, \xi_2^1, \xi_2^2\}$ and $\{\omega^i\} = \{\omega^1, \omega^2, \omega_3^1\}$. It can be shown [BCG+13] that Proposition 5.28 is equivalent to the following expression

$$d\Xi^\alpha \equiv A^\alpha_{\epsilon i} \pi^\epsilon \wedge \omega^i \frac{1}{2} T^\alpha_{ij} \omega^j \wedge \omega^i \mod \{I\} \quad (5.30)$$

in local coordinates for some $A^\alpha_{\epsilon i}$ and $T^\alpha_{ij}$. For the EDS in Example 5.2.2, we compute
\[ d\xi^\alpha \equiv 0 \]  
\[ d\xi_i^\alpha \equiv -\pi^\alpha_{ij} \wedge \omega^j - (K\delta_{ik}f^\alpha_j - \overline{K}f^\beta_if^\alpha_jf^\beta_k)\omega^j \wedge \omega^k. \]  
(5.31)

If \( \sigma \in \Sigma \) is a generic point, now define \( V^* = (J/I)_\sigma \) and \( W^* = I_\sigma \). Then

**Definition 5.2.13 ([IL03])**. The tableau at \( \sigma \in \Sigma \) is defined to be

\[ A = A_\sigma := \{ A^\alpha_{\epsilon i} \omega^i_\alpha \otimes w_\alpha \mid 1 \leq \epsilon \leq \dim \Sigma - p - \text{cardinality}(I) \} \]  
(5.32)

where \( \{ w_\alpha \} \) are the dual basis vectors to \( \Xi^\alpha \).

Then we have the Cartan-Kähler theorem for linear Pfaffian systems:

**Theorem 5.2.14 ([BCG+13])**. A linear Pfaffian system \((\mathcal{I}, \Omega)\) is involutive at \( \sigma \in \Sigma \) if

1. torsion \([T]\) vanishes in a neighborhood of \( \sigma \)
2. tableau \( A_\sigma \) is involutive.

Condition 1 in Theorem 5.2.14 means that either the term \( T^\alpha_{ij} \) in (5.30) vanishes if a different choice of \( \pi^\epsilon \) is found (apparent torsion) or if there is no such choice of base then we must restrict to a subset (if it exists) of \( \Sigma \) where the \( T^\alpha_{ij} \) vanish identically (essential torsion).

For the ideal in Example 5.2.2, we can absorb the torsion in (5.31) by defining

\[ \overline{\pi}^\alpha_{ij} := \pi^\alpha_{ij} - K(\delta_{ik}f^\alpha_j + \delta_{jk}f^\alpha_i)\omega^k + \overline{K}(f^\beta_if^\alpha_jf^\beta_k + f^\alpha_if^\beta_jf^\beta_k)\omega^k \]  
(5.33)

and then observing that

\[ d\xi^\alpha_i \equiv -\overline{\pi}^\alpha_{ij} \wedge \omega^j \mod \mathcal{I}. \]  
(5.34)
The involutivity of the tableau is defined in the following.

**Definition 5.2.15 ([IL03]).** If the torsion vanishes then let $A_k = A \cap (\text{span}\{\omega_{\sigma}^{k+1}, ..., \omega_{\sigma}^{p}\} \otimes W)$ and $A^{(1)} = (A \otimes V^*) \cap (W \otimes S^2V^*)$.

**Theorem 5.2.16 ([IL03]).** It holds that

$$\dim A^{(1)} \leq s_1 + 2s_2 + \ldots + ps_p$$

with $s_k = \dim A - \dim A_k$ and the tableau $A$ is involutive if equality holds in (5.35).

Thus in the case of the EDS in Example 5.2.2, $s_1 = 4, s_2 = 0$. We can compute $\dim A^{(1)}$ as follows. Let us define $f^\alpha_{ijk}$ by $\tilde{\pi}^\alpha_{ij} = f^\alpha_{ijk} \omega^k$ such that $f^\alpha_{ijk}$ is symmetric in $i,j,k$ and $f^0_{11k} + f^0_{22k} = 0$. This means that there are $8 - 4 = 4$ degrees of freedom in $f^0_{ijk}$. Hence, the Cartan’s Test is passed and the EDS in Example 5.2.2 is involutive and we conclude that harmonic maps $\phi : M \rightarrow N$ exist locally.
Chapter 6

Dirichlet densities linear in $H^2$ and $K$

6.1 The Question

In light of the examples introduced in Chapter 5, we now pose the question we want to study:

**Question 6.1.1.** Let $(\mathcal{M}, g_\mathcal{M})$ be a Riemannian surface with mean and gauss curvatures $H$ and $K$ respectively and $\phi : \mathcal{M} \to \mathcal{N}$ be a map to a Riemannian surface $(\mathcal{N}, g_\mathcal{N})$ with lagrangian

$$L[\phi] := \mathcal{L}_\phi d\mathcal{M} := \frac{1}{2} \int (f_1^a)^2 d\mathcal{M}. \quad (6.1)$$

Let $\mathcal{P} : \mathcal{M} \to \mathbb{R}$ depend functionally only on $H, K$ and their derivatives. Can we construct examples of $\phi$ and $(\mathcal{N}, g_\mathcal{N})$ where $\mathcal{L}_\phi$ takes the form $\mu_h H^2 - \mu_k K$ and is minimized iff $\mathcal{P}$ is constant?

6.1.1 Notations

We first define the geometric quantities on $\mathcal{M}, \mathcal{N}$ of interest in a coordinate independent way. On the way we will resort to several notations and definitions introduced in Chapter 2.1. For this we need the following definitions.

The group of euclidean motions in $\mathbb{R}^3$:

$$\text{ASO}(3) = \{(x, e_1, e_2, e_3) : x \in \mathbb{R}^3, [e_1 \quad e_2 \quad e_3] \in \text{SO}(3)\}$$

with tautological forms $\omega^1, \omega^2, \omega^3$ and Levi-Civita connection forms $\omega_1^i, \omega_2^i, \omega_3^i$. 
The frame-bundle of $\mathcal{M}$:

$$\mathcal{F}_\mathcal{M} = \{(x,e_1,e_2,e_3) \in \text{ASO}(3) : x \in \mathcal{M}; e_1, e_2 \in T_x\mathcal{M}\}.$$ On $\mathcal{F}_\mathcal{M}$ we have

$$\omega^3 = 0 \quad \Rightarrow \quad \begin{bmatrix} \omega^3_1 \\ \omega^3_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

(6.2)

where $h_{11}, h_{12} = h_{21}, h_{22} \in C^\infty(\mathcal{F}_\mathcal{M})$. On $\mathcal{F}_\mathcal{M}$ only the one forms $\omega^1, \omega^2, \omega^3$ form a coframing.

The gauss and mean curvatures $K, H : \mathcal{F}_\mathcal{M} \to \mathbb{R}$:

$$K = h_{11}h_{22} - h_{12}^2 \quad H = \frac{h_{11} + h_{22}}{2}. \quad (6.3)$$

We note that the above definition is a pullback of $K, H$ defined on $\mathcal{M}$ i.e. pullback of the function $K : \mathcal{M} \to \mathbb{R}, H : \mathcal{M} \to \mathbb{R}$ to the frame bundle $\mathcal{F}_\mathcal{M}$.

Question 6.1.1 is posed in a very general way. We will investigate one particular case.

Define

$$\mathcal{L} = \mu_hH^2 - \mu_kK. \quad (6.4)$$

### 6.1.2 Case-A

Maps between surfaces having Dirichlet energy linear in $H^2$ and $K$ and being harmonic

$$\begin{align*}
(i) \quad (f_{i}^o)^2 &= 2\mathcal{L} \\
(ii) \quad f_{ii}^o &= 0.
\end{align*} \quad (6.5)$$

Examples include (5.1.2) in Chapter 5. The novelty in our treatment is that the lagrangian is prescribed as in (6.4).

Other cases of interest include the three below:
6.1.3 Case-B

Let $\mathcal{P} : \mathcal{U} \to \mathbb{R}$ be a function on the jet space of $H, K$. Then

$$d\mathcal{P} = \mathcal{P}_1 \omega^1 + \mathcal{P}_2 \omega^2. \quad (6.6)$$

Maps between surfaces having Willmore-type Dirichlet energy and satisfying a second order condition

$$(f^\alpha_i)^2 = 2\mathcal{L} \quad \text{and} \quad f^\alpha_{ii} = f^\alpha_i \mathcal{P}_i. \quad (6.7)$$

To the best of our knowledge this case has not been studied in the literature. This example is motivated by the proof of the Example 5.1.2.

6.1.4 Case-C

Fix $V : \mathcal{V} \to \mathbb{R}$. We consider maps between surfaces having Willmore-type lagrangian and being a critical point of the lagrangian of the form

$$\frac{1}{2}(f^\alpha_i)^2 - V = \mathcal{L} \quad \text{and} \quad f^\alpha_{ii} \bar{u}_\alpha = -\nabla V. \quad (6.8)$$

Here $\bar{u}_\alpha$ is the orthonormal frame dual to $\theta^\alpha$. Harmonic maps in potential were first introduced in [FR97]. The novelty of Case C is that $\mathcal{L}$ has the form (6.4); these examples have not been studied in the literature.

6.1.5 Case-D

Fix a Lie subgroup $G \subset GL(n, \mathbb{R})$ and a smooth function $V : \mathcal{V} \to \mathbb{R}$. Consider maps from surface to $n$-dimensional Lie group having Willmore-type lagrangian and
being a critical point of the lagrangian with

$$\frac{1}{2}(f_i^\alpha)^2 - V(y^1, y^2) = \mathcal{L} \quad \text{and} \quad \mathcal{N} = G. \quad (6.9)$$

The last case is motivated by example (5.1.6) of Willmore surfaces whose Conformal gauss map to the de-Sitter group is harmonic.

## 6.2 Conformal immersion

A special case of maps for Case A can be considered by considering the following result.

**Lemma 6.2.1** ([BWW+03]). *Every conformal map $\phi: \mathcal{M} \to \mathcal{N}$ between two dimensional surfaces is harmonic.*

There are some “folklore results” on the implication of harmonic to conformal maps:

- Any harmonic map from a genus zero closed surface to an embedded euclidean submanifold is conformal. [SY97]
- No non-trivial harmonic map exists from a a genus zero closed surface to any surface $\mathcal{N}$ of positive genus. [SY97]
- If $\phi$ is harmonic and $\mathcal{N} = S^2$, then $\phi$ is conformal. [WY20]

**Proof.** Let $\phi: (\mathcal{M}^2, g) \to (\mathcal{N}^2, h)$ be a conformal map with conformal factor $\lambda$ such that

$$\phi^* h = \lambda g. \quad (6.10)$$
Then the Dirichlet energy functional of this map is given by

\[ |d\phi|^2 = \langle g, \phi^* h \rangle = 2\lambda. \]  

(6.11)

The stress energy tensor then becomes zero, since

\[ S(\phi) = \phi^* h - \frac{1}{2}|d\phi|^2 g = \phi^* h - \lambda g = 0. \]  

(6.12)

Since div\( S(\phi) \) is the tangential component of the tension field, \( \tau \), of \( \phi \), then \( \tau^T = 0 \). The normal component of the tension field is given by (see Theorem 3.3.25 in [BWW+03])

\[ \tau^N = \dim(\mathcal{M})\lambda^2 \times \text{(mean curvature of } \mathcal{M} \text{ in } \mathcal{N}). \]  

(6.13)

But the mean curvature is trivially zero for \( \dim(\mathcal{M}) = \dim(\mathcal{N}) \). Thus \( \tau = 0 \) and this direction of the claim follows. \( \square \)

### 6.2.1 Case of Conformal Maps

**Lemma 6.2.2.** We may choose the coframings \( \{\omega^1, \omega^2\} \) on \( U \) and \( \{\theta^1, \theta^2\} \) on \( V \), so that the functions \( f^\alpha_1 : U \to \mathbb{R} \) defined by (6.5) satisfy

\[ f^\alpha_1 = \sqrt{D}\delta^\alpha_1. \]  

(6.14)

**Proof.** From Lemma 6.2.1 we know conformal maps between surfaces are harmonic. Since the tension field of harmonic maps vanish, we get

\[ f^\alpha_{11} + f^\alpha_{22} = 0. \]  

(6.15)
From (6.11) and recalling that $|d\phi|^2/2$ has been prescribed to be $\mathcal{L}$, we know that the conformal factor of a conformal map becomes $\mathcal{L} \in C^\infty(\mathcal{M}, \mathbb{R}^+)$. On the other hand, maps $\phi : \mathcal{U} \to \mathcal{V}$ being conformal implies that

$$
\mathcal{L}
\left((\omega^1)^2 + (\omega^2)^2\right)
= (\tilde{\theta}^1)^2 + (\tilde{\theta}^2)^2
= \left((f^1_1)^2 + (f^2_1)^2\right)(\omega^1)^2 + \left((f^1_2)^2 + (f^2_2)^2\right)(\omega^2)^2 + 2(f^1_1 f^2_1 + f^1_2 f^2_2)\omega^1 \omega^2.
$$

Thus the $f^\alpha_i$ must satisfy

$$
(f^1_1)^2 + (f^2_1)^2 = \mathcal{L} = (f^1_2)^2 + (f^2_2)^2 \quad \text{and} \quad f^1_1 f^2_1 + f^1_2 f^2_2 = 0. \tag{6.17}
$$

Substituting the second equation into the first, we get

$$
f^1_1 = \pm f^2_2 \quad \text{and} \quad f^2_1 = \mp f^1_2. \tag{6.18}
$$

Since the choice of orthonormal frame at a point is unique up to rotation and reflection, we may normalize our framings $\{u_1, u_2\}$ on $\mathcal{U}$ and $\{v_1, v_2\}$ on $\mathcal{V}$ such that

$$
f^1_1 = f^2_2 = \sqrt{\mathcal{F}} \quad f^1_2 = f^2_1 = 0. \tag{6.19}
$$

Then

$$
\theta^1 = \sqrt{\mathcal{F}} \omega^1 \quad \theta^2 = \sqrt{\mathcal{F}} \omega^2. \tag{6.20}
$$
Let us now define
\[
\xi^\alpha = \theta^\alpha - \sqrt{\mathcal{L}} \omega^\alpha \quad \alpha = 1, 2
\]
\[
\xi^3 = \theta_2^1 - \omega_2^1 - \left( \log \sqrt{\mathcal{L}} \right)_2 \omega^1 + \left( \log \sqrt{\mathcal{L}} \right)_1 \omega^2
\]
(6.21)
and define \( p, q : \mathcal{U} \rightarrow \mathbb{R} \), we have
\[
\omega_2^1 = p \omega^1 + q \omega^2.
\]
(6.22)
Next we consider the ambient space
\[
\mathcal{U} \times \mathcal{V}
\]
(6.23)
with the EDS
\[
\{ \xi^\alpha = \theta^\alpha - \sqrt{\mathcal{L}} \omega^\alpha, \quad \xi^3 = \theta_2^1 - \omega_2^1 - \left( \log \sqrt{\mathcal{L}} \right)_2 \omega^1 + \left( \log \sqrt{\mathcal{L}} \right)_1 \omega^2 \}. \quad (6.24)
\]
\textbf{Lemma 6.2.3.} The Pfaffian system (6.24) is completely integrable if
\[
-\overline{K} \mathcal{L} + K + \frac{1}{2} \Delta \log \mathcal{L} - \frac{1}{2} \mathcal{L}_1 q + \frac{1}{2} \mathcal{L}_2 p = 0.
\]
(6.25)
\textbf{Proof.} Differentiating we get
\[
d \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} \equiv \left( -\overline{K} \mathcal{L} + K + \frac{1}{2} \Delta \log \mathcal{L} - \frac{1}{2} \mathcal{L}_1 q + \frac{1}{2} \mathcal{L}_2 p \right) \omega^1 \wedge \omega^2.
\]
(6.26)
The complete integrability now follows from Frobenius theorem. \qed

101
Let us define $a, b, c$- the second fundamental form for $N$ by

$$\theta_1^3 = a\theta^1 + b\theta^2 = a\sqrt{F}\omega^1 + b\sqrt{F}\omega^2$$  \hspace{1cm} (6.27)

$$\theta_2^3 = b\theta^1 + c\theta^2 = b\sqrt{F}\omega^1 + c\sqrt{F}\omega^2$$  \hspace{1cm} (6.28)

Using Cartan’s structure equations (5.9) on $N$ we get

$$a_2\sqrt{F} - (c - a)(\sqrt{F})_2 - (c - a)\sqrt{F}p + 2b\sqrt{F}q - 2b(\sqrt{F})_1 - b_1\sqrt{F} = 0$$

$$c_1\sqrt{F} + (c - a)(\sqrt{F})_1 - (c - a)\sqrt{F}q - 2b\sqrt{F}p - 2b(\sqrt{F})_2 - b_2\sqrt{F} = 0.$$  \hspace{1cm} (6.29)

which will be used to generate examples.

**Example 6.2.4.** Let $M$ be a surface of revolution $(g(x) \sin y, g(x) \cos y, h(x))$ parameterized by arc-length

$$(g'(x))^2 + (h'(x))^2 = 1.$$  \hspace{1cm} (6.30)

Assume gauss curvature $K$ of $M$ is a positive constant. Then

$$K = -\frac{g''(x)}{g(x)}$$  \hspace{1cm} (6.31)

which implies

$$g(x) = A\sin(\sqrt{K}x + x_0).$$  \hspace{1cm} (6.32)

Let $N$ be the unit sphere with the standard metric $ds^2 = du^2 + \sin^2 udv^2$. Consider the map

$$\phi : (x, y) \mapsto (u, v) \text{ such that } u = \sqrt{K}x + x_0 \quad v = \sqrt{K}Ay + y_0$$  \hspace{1cm} (6.33)
where $x_0, y_0$ are constants. We then have

$\omega^1 = dx$ and $\omega^2 = A \sin(\sqrt{K}x + x_0)dy$

$\theta^1 = du$ which implies $\tilde{\theta}^1 = \sqrt{K} dx$ and $f^1_1 = \sqrt{K}, f^1_2 = 0$  \hspace{1cm} (6.34)

$\theta^2 = \sin u dv$ which implies $\tilde{\theta}^2 = \sqrt{K}A \sin(\sqrt{K}x + x_0)dy$

and $f^2_2 = \sqrt{K}, f^2_1 = 0$.

The second equation in (6.5) follows by using Lemma 6.2.1. Since we have

$$ (f^\alpha_i)^2 = 2K \quad (\tilde{\theta}^1)^2 + (\tilde{\theta}^2)^2 = K((\omega^1)^2 + (\omega^2)^2) $$  \hspace{1cm} (6.35)

from the second equation of (6.35) we deduce that $\phi$ is conformal, and then from Lemma 6.2.1 we conclude that $\phi$ is harmonic.

Further examples will be elucidated in a subsequent publication.

### 6.3 EDS I for Case A

We fix Riemannian surfaces $\mathcal{M}, \mathcal{N}$ and a smooth $\mathcal{L} : \mathcal{M} \rightarrow \mathbb{R}^+$. Our goal is to study the existence of maps $\phi : \mathcal{M} \rightarrow \mathcal{N}$ satisfying the PDE system

$$ (f^\alpha_i)^2 = 2\mathcal{L} \quad \text{and} \quad f^\alpha_{ii} = 0. $$  \hspace{1cm} (6.36)

We define

$$ \mathcal{F} = \mathcal{M} \times \mathcal{N} \times \mathbb{R}^4 \backslash \{0\}. $$  \hspace{1cm} (6.37)

Let $(f^\alpha_i) = (f^1_1, f^1_2, f^2_1, f^2_2)$ be coordinates on $\mathbb{R}^4$. Assume $\mathcal{M}$ and $\mathcal{N}$ admit global, orthonormal coframings $\{\omega^1, \omega^2\}$ and $\{\theta^1, \theta^2\}$ respectively. (Such coframings always
exist locally.) The first order condition in (6.36) imposes the constraint

\[
\left( \frac{f_1^1}{\sqrt{2\mathcal{L}}} \right)^2 + \left( \frac{f_1^2}{\sqrt{2\mathcal{L}}} \right)^2 + \left( \frac{f_2^1}{\sqrt{2\mathcal{L}}} \right)^2 + \left( \frac{f_2^2}{\sqrt{2\mathcal{L}}} \right)^2 = 1. \tag{6.38}
\]

Define

\[
\Sigma = \{(x^i, y^\alpha, f_i^\alpha) \in \mathcal{F} \mid (f_i^\alpha)^2 = 2\mathcal{L}\}. \tag{6.39}
\]

Recall that \{\omega^1, \omega^2\} is a coframing on \(\mathcal{M}\), and \{\theta^1, \theta^2\} is a coframing on \(\mathcal{N}\). So, \{\omega^1, \omega^2, \theta^1, \theta^2, df^1, df^2, df^3, df^3\} is a coframing on \(\mathcal{F}\). Define

\[
\eta^1 = f_1^1 df^2_1 - f_2^1 df^1_1 + f_1^2 df^2_2 - f_2^2 df^1_2 \quad \frac{2\mathcal{L}}{2\mathcal{L}}
\]

\[
\eta^2 = f_1^1 df^1_2 - f_2^1 df^1_2 + f_2^2 df^2_1 - f_1^2 df^2_2 \quad \frac{2\mathcal{L}}{2\mathcal{L}}
\]

\[
\eta^3 = f_1^2 df^2_2 - f_2^2 df^2_1 + f_2^3 df^3_1 - f_1^3 df^3_2 \quad \frac{2\mathcal{L}}{2\mathcal{L}}.
\]

Then \{\omega^1, \omega^2, \theta^1, \theta^2, \eta^1, \eta^2, \eta^3\} is a coframing on \(\Sigma\). Finally we define

\[
\xi^\alpha = \theta^\alpha - f_i^\alpha \omega^i. \tag{6.41}
\]

Then \{\omega^1, \omega^2, \xi^1, \xi^2, \eta^1, \eta^2, \eta^3\} is a framing of \(\Sigma\) that is adapted to the PDE.

The covariant derivative of \(f_i^\alpha\) is given by

\[
\pi_i^\alpha = df_i^\alpha - f_k^\alpha \omega^i_k + f_i^\beta \theta_i^\alpha. \tag{6.42}
\]

Define

\[
\varpi^\alpha = \pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1. \tag{6.43}
\]
Define $\mathcal{I} \in \Omega^{*}(\Sigma)$ to be the differential ideal generated by

$$
\mathcal{I} = \{\xi^\alpha, \varpi^\alpha\}_{\text{diff}} = \{\xi^\alpha, \pi^\alpha_1 \wedge \omega^1 + \pi^\alpha_2 \wedge \omega^2, \varpi^\alpha, d\varpi^\alpha\}_{\text{alg}}.
$$

(6.44)

We also impose the independence condition that $\omega^1 \wedge \omega^2$ be nowhere zero. The first order condition in (6.5) has been incorporated into the coframing and the second order condition as a 2-form in the ideal.

**Remark 6.3.1.** To see how the solution to the EDS in (6.44) is locally the solution to the PDE in (6.36), we refer the reader to the general construction of this correspondence in Chapter III of [BCG+13].

### 6.3.1 Analysis of solutions

We will prove in this section

**Proposition 6.3.2.** *Local solutions to (6.36) exist for generic choice of $f_i^\alpha$.*

In order to establish that we prove Proposition 6.3.3 and Lemma 6.3.4 first.

**Proposition 6.3.3.** The subvariety

$$
\mathcal{V}_2(\mathcal{I}) \subset Gr(2, T\Sigma)
$$

(6.45)

is a smooth manifold of codimension 8 where $\mathcal{V}_2(\mathcal{I})$ is the set of all 2-dimensional integral elements of $\mathcal{I}$.

**Proof.** We first show that $\mathcal{V}_2(\mathcal{I})$ is Kähler-ordinary (Definition 5.2.8). Let $\{W_i, C_\alpha, H_b\}$ be the basis of $T_p\Sigma$ that is dual to $\{\omega^i, \xi^\alpha, \eta^b\}$ where $i = 1, 2, \alpha = 1, 2, b = 1, 2, 3$. Let
$E$ be a 2-plane satisfying the independence condition. Thus $E = \text{span}\{v_1, v_2\}$ where

$$v_1 = W_1 + p_1^\alpha C_\alpha + q_1^b H_b \quad v_2 = W_2 + p_2^\alpha C_\alpha + q_2^b H_b.$$  \hspace{1cm} (6.46)

Then we have $E \in \mathcal{V}_2(\mathcal{F})$ if and only if

$$\xi^\alpha(v_i) = p_i^\alpha = 0 \quad i = 1, 2$$ \hspace{1cm} (6.47a)

$$(\pi_1^\alpha \wedge \omega^1 + \pi_2^\alpha \wedge \omega^2)(v_1 \wedge v_2) = -\pi_1^\alpha(v_2) + \pi_2^\alpha(v_1) = 0$$ \hspace{1cm} (6.47b)

$$(\pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1)(v_1 \wedge v_2) = \pi_1^\alpha(v_1) + \pi_2^\alpha(v_2) = 0.$$ \hspace{1cm} (6.47c)

Define smooth functions $R_{ij}^\alpha, P_{i\beta}^\alpha, Q_{ib}^\alpha : \Sigma \rightarrow \mathbb{R}$ by

$$\pi_i^\alpha = R_{ij}^\alpha \omega^j + P_{i\beta}^\alpha \xi^\beta + Q_{ib}^\alpha \eta^b.$$ \hspace{1cm} (6.48)

Thus we have $p_i^\alpha = 0$ and can solve (6.47b)-(6.47c) for four of the six $q_i^b$. We express (6.47b)-(6.47c) as

$$\begin{bmatrix}
Q_{21}^1 & Q_{22}^1 & Q_{23}^1 & -Q_{11}^1 & -Q_{12}^1 & -Q_{13}^1 \\
Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & -Q_{11}^2 & -Q_{12}^2 & -Q_{13}^2 \\
Q_{11}^1 & Q_{12}^1 & Q_{13}^1 & Q_{21}^1 & Q_{22}^1 & Q_{23}^1 \\
Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & Q_{21}^2 & Q_{22}^2 & Q_{23}^2
\end{bmatrix}\begin{bmatrix}
q_1^1 \\
q_1^2 \\
q_1^3 \\
q_2^1 \\
q_2^2 \\
q_2^3
\end{bmatrix} + \begin{bmatrix}
R_{21}^1 & -R_{11}^1 \\
R_{21}^2 & -R_{11}^2 \\
R_{11}^1 & R_{12}^1 \\
R_{11}^2 & R_{22}^1 \\
R_{11}^2 & R_{22}^2
\end{bmatrix} = 0.$$ \hspace{1cm} (6.49)
The coefficient matrix above containing the $Q'$s is expressed in terms of the $f_i^\alpha$'s as
\[
\begin{bmatrix}
    f_1^1 & f_2^2 & -f_1^2 & -f_1^1 & -f_2^2 & -f_2^1 \\
    -f_1^2 & f_2^1 & f_1^2 & -f_2^1 & f_1^1 & -f_2^2 \\
    f_2^1 & f_1^2 & f_2^2 & f_1^1 & f_2^1 & -f_1^2 \\
    -f_2^1 & f_2^1 & f_1^2 & f_1^1 & f_2^1 & -f_1^2 \\
    f_1^2 & f_2^1 & f_1^1 & f_2^1 & -f_1^2 & f_2^1 \\
    -f_2^1 & f_1^1 & f_2^1 & f_1^1 & -f_1^2 & f_2^1
\end{bmatrix}.
\] (6.50)

The second, third, fifth and sixth columns are pairwise orthogonal of the matrix in (6.50) is pairwise orthogonal. Thus the matrix will have rank 4. Hence, $\text{codim}(V_2(\mathcal{F})) = 8$ at $E$.

\begin{proof}
Let $e_1$ span $E_1$ and $(e_1, e_2)$ span $E_2$. We have
\[
H(E_0) = \{v \in T\Sigma \mid \xi^\alpha(v) = 0\}
\Rightarrow \quad c_0 = 2.
\] (6.52)

Next
\[
H(E_1) = \{v \in H(E_0) \mid d\xi^\alpha(v, e_1) = 0, \quad \varpi^\alpha(v, e_1) = 0\}.
\] (6.53)

Any $v \in T\Sigma$ may be expressed as
\[
v = x^iW_i + y^\alpha C_\alpha + z^b H_b.
\] (6.54)
Since \( v \in H(E_0) \), we have \( y^\alpha = 0 \). Likewise

\[
e_1 = k^i W_i + s^\alpha C_\alpha + t^b H_b,
\]

and again as \( E_1 \) is an integral element, we have \( s^\alpha = 0 \). The polar equations are

\[
(\pi_1^\alpha \wedge \omega^i)(v, e_1)
\]

\[
= \pi_1^\alpha (v) \omega^i (e_1) - \pi_1^\alpha (e_1) \omega^i (v)
\]

\[
= (R_{ij}^\alpha x^j + Q_{ib}^\alpha z^b) k^i - (R_{ij}^\alpha k^j + Q_{ib}^\alpha t^b) x^i
\]

\[
(\pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1)(v, e_1)
\]

\[
= \pi_1^\alpha (v) \omega^2 (e_1) - \pi_1^\alpha (e_1) \omega^2 (v) - \pi_2^\alpha (v) \omega^1 (e_1) + \pi_2^\alpha (e_1) \omega^1 (v)
\]

\[
= (R_{ij}^\alpha x^j + Q_{ib}^\alpha z^b) k^2 - (R_{ij}^\alpha k^j + Q_{ib}^\alpha t^b) x^2
\]

\[
- (R_{ij}^\alpha x^j + Q_{ib}^\alpha z^b) k^1 - (R_{ij}^\alpha k^j + Q_{ib}^\alpha t^b) x^1.
\]

Combining (6.56) and (6.57), we have

\[
M \begin{bmatrix} x^1 & x^2 & z^1 & z^2 & z^3 \end{bmatrix}^T = 0
\]

where \( M \) is a \( 4 \times 5 \) matrix. If we denote the five columns by \( M_1, M_2, M_3, M_4, M_5 \) then

\[
M_1 = \begin{bmatrix} -(R_{12}^1 - R_{21}^1) k^2 + Q_{1b}^1 t^b \\ -(R_{12}^2 - R_{21}^2) k^2 + Q_{1b}^2 t^b \\ R_{11}^1 k^1 - R_{21}^1 k^1 + (R_{12}^1 k^j + Q_{1b}^1 t^b) \\ R_{11}^2 k^2 - R_{21}^2 k^1 + (R_{22}^2 k^j + Q_{2b}^2 t^b) \end{bmatrix}
\]

108
\[ M_2 = \begin{bmatrix} (R^1_{12} - R^1_{21})k^1 + Q^1_{2b}t_b \\ (R^2_{12} - R^2_{21})k^1 + Q^2_{2b}t_b \\ R^1_{12}k^2 - R^1_{22}k^1 - (R^1_{1j}k^j + Q^1_{1b}t^b) \\ R^2_{12}k^2 - R^2_{22}k^1 - (R^2_{1j}k^j + Q^2_{1b}t^b) \end{bmatrix}, \quad (6.60) \]

\[ M_b = \begin{bmatrix} Q^1_{1b}k^1 + Q^1_{2b}k^2 \\ Q^2_{1b}k^1 + Q^2_{2b}k^2 \\ Q^1_{1b}k^2 - Q^1_{2b}k^1 \\ Q^2_{1b}k^2 - Q^2_{2b}k^1 \end{bmatrix} \quad \text{for } b = 3, 4, 5. \quad (6.61) \]

Notice that \( c_1 = 2 + \text{rank}(M) \). The matrix \( M \) is a function of \( \mathcal{V}_1(\mathcal{F}) \). For generic choice of \((x, y, f; e_1) \in \mathcal{V}_1(\mathcal{F})\) this matrix has rank 4. Let \( N \) be a \( 4 \times 4 \) submatrix obtained by removing the second column from \( M \). Then a Maple computation (see Appendix Section 8.1) shows that \( \det N \) is non-zero for generic choice of \( f^\alpha_i \) and \( k^\alpha_i \) and \( t^b \). Hence,

\[ c_1 = 6 \quad (6.62) \]

and the lemma follows.

**Proof of Proposition 6.3.2.** Combining Proposition 6.3.3 and Lemma 6.3.4, the equality in the Cartan’s test (Theorem 5.27) is satisfied and consequently

\[ \text{codim}(\mathcal{V}_2(\mathcal{F})) = c_0 + c_1 = 8. \quad (6.63) \]

Now, \( \text{codim}_{\mathcal{V}_2}(E) = 10 - 2 = 8 \). Hence, by the Cartan-Kähler’s theorem (Theorem 5.2.9) the solution depends on \( c_0 = 2 \) constants, \( c_1 - c_0 = 4 \) functions of one variable and \( \text{codim}(E) - c_2 = 2 \) functions of two variables.

\[ \square \]
6.4 EDS II for Case A

In order to obtain conditions on the lagrangian or its derivative, which involves second or third derivatives of the map, we consider a more general ambient space involving the frame bundles. Harmonic maps satisfy a local formula: the Bochner identity involving the Laplacian of the energy density. We will recover two integrability conditions – one a Bochner-type formula and another condition involving the lagrangian density in Lemma 6.4.5. To do this we define

$$\mathcal{F} = \mathcal{F}_M \times \mathcal{F}_N \times \text{GL}_2(\mathbb{R}).$$

Define

$$\Sigma = \{ (x, y, f_i^\alpha) \in \mathcal{F} \mid (f_i^\alpha)^2 = 2 \mathcal{L} \}. \quad (6.65)$$

The exterior derivatives of (6.40) are

$$d\eta^\alpha = \frac{\epsilon^{abc}}{\mathcal{L}} \eta^b \wedge \eta^c + d(\log \mathcal{L}) \wedge \eta^a. \quad (6.66)$$

The $$\omega^1, \omega^2, \omega^3, \xi^1, \xi^2, \theta_1^1, \theta_2^1, df_i^\alpha$$ give a coframing of $$\mathcal{F}$$, and the restrictions of $$\omega^1, \omega^2, \omega^3, \xi^1, \xi^2, \theta_1^1, \eta^1, \eta^2, \eta^3$$ give a coframing of $$\Sigma$$. Then

$$\begin{align*}
\pi_1^1 &= \frac{1}{2\mathcal{L}} (-f_2^1 \eta^1 - f_1^2 \eta^2 - f_2^2 \eta^3) + f_1^1 d(\log \sqrt{\mathcal{L}}) + f_2^1 \omega_2^1 + f_2^2 \theta_2^1 \\
\pi_1^2 &= \frac{1}{2\mathcal{L}} (f_1^1 \eta^1 + f_2^2 \eta^2 - f_2^3 \eta^3) + f_2^1 d(\log \sqrt{\mathcal{L}}) - f_1^1 \omega_2^1 + f_2^3 \theta_1^1 \\
\pi_2^1 &= \frac{1}{2\mathcal{L}} (-f_2^1 \eta^1 + f_1^2 \eta^2 + f_2^3 \eta^3) + f_2^1 d(\log \sqrt{\mathcal{L}}) + f_2^2 \omega_2^1 - f_1^1 \theta_2^1 \\
\pi_2^2 &= \frac{1}{2\mathcal{L}} (f_2^2 \eta^1 - f_2^1 \eta^2 + f_1^3 \eta^3) + f_2^2 d(\log \sqrt{\mathcal{L}}) - f_2^1 \omega_2^1 - f_2^1 \theta_1^1.
\end{align*} \quad (6.67)$$

Define

$$\varpi^\alpha = \pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1. \quad (6.68)$$
Define \( J \in \Omega^*(\Sigma) \) to be the differential ideal generated by

\[
J = \{ \xi^\alpha, \varpi^\alpha \}_{\text{diff}} = \{ \xi^\alpha, \pi^\alpha_1 \wedge \omega^1 + \pi^\alpha_2 \wedge \omega^2, \varpi^\alpha, d\varpi^\alpha \}_{\text{alg}}
\]  

(6.69)

and the independence condition \( \Omega = \omega^1 \wedge \omega^2 \wedge \omega^1_2 \). The exterior differential system \((J, \Omega)\) on \( \Sigma \) consists of the differential ideal \( J \subset \Omega^*(\Sigma) \).

**Lemma 6.4.1.** The exterior derivative \( d\varpi^\alpha \) is contained in the ideal algebraically generated by \( \xi^\alpha, d\xi^\alpha, \varpi^\alpha \).

**Proof.** We compute modulo \( J \)

\[
d\varpi^1 = d\pi^1_1 \wedge \omega^2 + \pi^1_1 \wedge \omega^1 \wedge \omega^1_2 - d\pi^1_2 \wedge \omega^1 + \pi^1_2 \wedge \omega^2 \wedge \omega^1_2 \\
\equiv df^1_2 \wedge \omega^2_1 \wedge \omega^2 + df^2_1 \wedge \theta^1_2 \wedge \omega^2 + df^1_1 \wedge \omega^1_2 \wedge \omega^1 - df^2_2 \wedge \theta^1_2 \wedge \omega^1 \\
\equiv (\pi^1_2 - f^2_2\theta^1_2) \wedge \omega^1_2 \wedge \omega^2 + (\pi^2_1 + f^2_2\omega^2_1) \wedge \theta^1_2 \wedge \omega^2 \\
+ (\pi^1_1 - f^2_1\theta^1_2) \wedge \omega^1_2 \wedge \omega^1 - (\pi^2_1 + f^2_1\omega^1_2) \wedge \theta^1_2 \wedge \omega^1 \\
\equiv - (\pi^1_2 \wedge \omega^2 + \pi^1_1 \wedge \omega^1) \wedge \omega^1_2 - (\pi^2_1 \wedge \omega^2 - \pi^2_2 \wedge \omega^1) \wedge \theta^1_2 \equiv 0.
\]  

Similarly,

\[
d\varpi^2 \equiv 0 \mod J.
\]  

(6.71)

\[ \square \]

6.4.1 Cartan’s Test

In this section we establish the following corollary:

**Corollary 6.4.2.** A three-dimensional integral element of the EDS in (6.69) does not pass the Cartan’s test.
Proposition 6.4.3. The subvariety

\[ \mathcal{V}_3(\mathcal{I}) \subset \text{Gr}(3, T\Sigma) \]  \hspace{1cm} (6.72)

is a smooth manifold of codimension 13.

Proof. We first show that \( \mathcal{V}_3(\mathcal{I}) \) is Kähler-ordinary (Definition 5.2.8). Let \( \{W_1, W_2, W_3, C_1, C_2, C_3, H_b\} \) be the coframing of \( T_p \Sigma \) that is dual to \( \{\omega^1, \omega^2, \omega^3, \xi^1, \xi^2, \theta^1, \eta^b\} \) where \( b = 1, 2, 3 \). Let \( E \) be a 3-plane satisfying the independence condition \( \omega^1 \wedge \omega^2 \wedge \omega^3 \neq 0 \). Thus \( E = \text{span}\{v_1, v_2, v_3\} \) is an element in the \( (3 \cdot (9 - 3) + 9 = 18 + 9 = 27) \)-dimensional \( \text{Gr}_3(T\Sigma) \) near \( E = \text{span}\{v_i'\}_{i'=1}^3 \) where

\[ v_i' = W_i' + p_i'^{\alpha'} C_{\alpha'} + q_i'^b H_b \quad \alpha' = 1, 2, 3 \text{ and } i' = 1, 2, 3. \]  \hspace{1cm} (6.73)

Then we have \( E \in \mathcal{V}_3(\mathcal{I}) \) if and only if

\[ \xi^\alpha(v_{i'}) = p_i'^\alpha = 0 \quad \text{for } \alpha = 1, 2 \quad i' = 1, 2, 3 \]  \hspace{1cm} (6.74a)

\[ (\pi_1^\alpha \wedge \omega^1 + \pi_2^\alpha \wedge \omega^2)(v_1, v_2) = -\pi_1^\alpha(v_2) + \pi_2^\alpha(v_1) = 0 \]  \hspace{1cm} (6.74b)

\[ (\pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1)(v_1, v_2) = \pi_1^\alpha(v_1) + \pi_2^\alpha(v_2) = 0 \]  \hspace{1cm} (6.74c)

\[ (\pi_1^\alpha \wedge \omega^1 + \pi_2^\alpha \wedge \omega^2)(v_2, v_3) = -\pi_2^\alpha(v_3) = 0 \]  \hspace{1cm} (6.74d)

\[ (\pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1)(v_2, v_3) = -\pi_1^\alpha(v_3) = 0 \]  \hspace{1cm} (6.74e)

\[ (\pi_1^\alpha \wedge \omega^1 + \pi_2^\alpha \wedge \omega^2)(v_3, v_1) = \pi_1^\alpha(v_3) \]  \hspace{1cm} (6.74f)

\[ (\pi_1^\alpha \wedge \omega^2 - \pi_2^\alpha \wedge \omega^1)(v_3, v_1) = -\pi_2^\alpha(v_3) = 0. \]  \hspace{1cm} (6.74g)

Equations (6.74b) to (6.74c) become four equations when indices are expanded and
are given through

\[
0 \equiv -\pi_1^1(v_2) + \pi_2^1(v_1) = \frac{1}{2 L} (f_1^1 q_1^1 + f_1^2 q_2^1 + f_2^3 q_3^3) - f_1^1 (\log L)_2 - f_1^2 p_2^3 \\
+ \frac{1}{2 L} (f_1^1 q_1^1 + f_2^2 q_1^2 - f_1^2 q_1^3) + f_2^1 (\log L)_1 + f_2^2 p_1^3
\]  
(6.75)

\[
0 \equiv -\pi_1^2(v_2) + \pi_2^2(v_1) = \frac{1}{2 L} (f_2^2 q_2^1 - f_1^1 q_2^1 - f_2^3 q_3^3) - f_1^2 (\log L)_2 + f_1^1 p_2^3 \\
+ \frac{1}{2 L} (f_1^1 q_1^1 - f_1^2 q_1^2 + f_1^1 q_1^3) + f_2^1 (\log L)_1 - f_2^1 p_1^3
\]  
(6.76)

\[
0 \equiv \pi_1^1(v_1) + \pi_2^1(v_2) = \frac{1}{2 L} (-f_2^1 q_1^1 - f_1^2 q_2^1 - f_2^3 q_3^3) + f_1^1 (\log L)_1 + f_2^3 p_1^3 \\
+ \frac{1}{2 L} (f_1^1 q_1^1 + f_1^2 q_2^1 - f_2^3 q_3^3) + f_2^1 (\log L)_2 + f_2^2 p_2^3
\]  
(6.77)

\[
0 \equiv \pi_1^2(v_1) + \pi_2^2(v_2) = \frac{1}{2 L} (-f_2^1 q_1^1 + f_1^2 q_2^1 + f_2^3 q_3^3) + f_1^1 (\log L)_1 - f_1^1 p_1^3 \\
+ \frac{1}{2 L} (f_1^1 q_1^1 - f_1^2 q_2^1 + f_1^3 q_3^3) + f_2^1 (\log L)_2 - f_1^3 p_2^3.
\]  
(6.78)

Equations (6.74d) to (6.74g) become four distinct equations when indices are expanded and are given through

\[
0 \equiv \pi_1^2(v_3) = \frac{1}{2 L} (f_1^1 q_3^1 + f_2^3 q_3^3 - f_1^3 q_3^3) - f_1^1 + f_2^3 p_3^3 
\]  
(6.79)

\[
0 = \pi_2^2(v_3) = \frac{1}{2 L} (f_1^1 q_3^1 - f_2^3 q_3^3 + f_1^3 q_3^3) - f_1^2 + f_2^3 p_3^3
\]  
(6.80)

\[
0 \equiv \pi_1^1(v_3) = \frac{1}{2 L} (-f_1^1 q_3^1 - f_2^3 q_3^3 - f_2^3 q_3^3) + f_2^1 + f_1^3 p_3^3
\]  
(6.81)

\[
0 \equiv \pi_2^1(v_3) = \frac{1}{2 L} (-f_1^1 q_3^1 + f_1^3 q_3^3 + f_1^3 q_3^3) + f_2^1 - f_1^3 p_3^3.
\]  
(6.82)

The coefficient matrix of the q’s when equation (6.79)-(6.82) is expressed as a linear system of equations has rank at most 3 (see Appendix Section 8.2 for a Maple computation). Thus we can solve for \( q_3^1, q_3^2, q_3^3 \) as functions of \( p_3^3 \) from (6.79)-(6.82).
Moreover, from (6.75)-(6.78) we can solve for any one of the four variables —

\[
\begin{align*}
\{q_1^1, q_1^2, q_1^3, q_2^3\} & \quad \text{or} \quad \{q_1^1, q_2^1, q_1^3, q_2^3\} \\
\{q_1^1, q_1^2, q_2^2, q_2^3\} & \quad \text{or} \quad \{q_1^1, q_2^1, q_2^2, q_2^3\}.
\end{align*}
\] (6.83)

This can be seen by expressing (6.75)-(6.78) as a linear system of equations and by noting that the rank of the coefficient matrix of any of the set of variables in (6.83) is at most 4 for a generic choice of the \( f_i^\alpha \)s (see Appendix Section 8.2 for a Maple computation).

Now, \( \dim \Sigma = 3 + 3 + 3 = 9 \) which implies \( \dim \Gr(3, T_p \Sigma) = 3 \cdot (9 - 3) = 18 \) and \( \dim \Gr(3, T \Sigma) = 9 + 18 = 27 \). This implies

\[
\dim(V_3(\mathcal{H})) = 9 + 1 + 4 = 14
\] (6.84)

and

\[
\codim(V_3(\mathcal{H})) = 27 - 14 = 13.
\] (6.85)

\[\square\]

**Lemma 6.4.4.** Let \( E_1 \subset E_2 \subset E_3 \) be a generic choice of an integral flag. Then the sum of the codimensions of the polar spaces

\[
codimH(E_0) + codimH(E_1) + codimH(E_2)
\] (6.86)

is even.

**Proof.** Let \( e_1 \) span \( E_1 \) and \( \{e_1, e_2\} \) span \( E_2 \) and \( \{e_1, e_2, e_3\} \) span \( E_3 \). We have

\[
H(E_0) = \{v \in T_\Sigma \mid \xi^\alpha(v) = 0\}
\] (6.87)
which implies

\[ c_0 = 2. \quad (6.88) \]

Now, due to Lemma 6.4.1

\[ c_2 = 2 + \text{rank}(M) = c_1. \quad (6.89) \]

\[ \square \]

**Proof of Corollary 6.4.2.** Since \( c_0 + c_1 + c_2 = 6 + 2 \cdot \text{rank}(M) \) cannot be equal to 13 which is the codim\( \mathcal{V}_3(\mathcal{F}) \), the Cartan’s test is not satisfied. Thus we need to prolong the system. \[ \square \]

### 6.4.2 Prolongation

In order to prolong the system we choose a coframing different from the one used above and one that is more adapted to the problem at hand. With the identification \( \mathbb{R}^4 \cong \mathbb{H} \), we can define a vector

\[ \phi_0 = [f^\alpha_i] = (f^1_1, f^1_2, f^2_1, f^2_2) \in \mathbb{H} \quad (6.90) \]

and further define

\[ i\phi_0 =: \phi_1, j\phi_0 =: \phi_2, k\phi_0 =: \phi_3. \quad (6.91) \]

Then \( \mathbb{R}^4 \) is the span of \( \{\phi_0, \phi_1, \phi_2, \phi_3\} \). The latter is a coframing if and only if \( \phi_0 \) is non-zero. Fix \( j \) and define \( c^s_j \) for \( s = 0, ..., 3 \) by

\[ [f^\alpha_{ij}] = (f^1_{1j}, f^1_{2j}, f^2_{1j}, f^2_{2j}) = c^s_j \phi_s. \quad (6.92) \]

115
The relations
\[ f_{ij}^\alpha = f_{ji}^\alpha \quad f_{11}^\alpha + f_{22}^\alpha = 0 \] \tag{6.93}
can be re-expressed as constraints in the coefficients \( c_i^s \) as follows by considering
\[ 0 = [f_{i1}^\alpha] - i[f_{i2}^\alpha] = c_1^i \phi_s - ic_2^i \phi_s = c_1^i \phi_s - c_2^i \phi_s \]
\[ = (c_1^0 + c_2^1) \phi_0 + (c_1^1 - c_2^0) \phi_1 + (c_1^2 + c_2^3) \phi_2 + (c_1^3 - c_2^2) \phi_3. \] \tag{6.94}
Thus
\[ c_2^1 = -c_1^0 \quad c_1^1 = c_2^0 \quad c_2^2 = c_1^3 \quad c_2^3 = -c_1^2. \] \tag{6.95}

Since, \( 2\mathcal{L} = \phi_0 \cdot \phi_0 \) we have from (6.67)
\[ \phi_0 \cdot \nabla \phi_0 = d\mathcal{L}. \] \tag{6.96}

Since \( \{\phi_s\}_{s=0}^3 \) spans \( \mathbb{R}^4 \), we can define 1-forms \( \gamma^s \) by
\[ \nabla \phi_0 = \gamma^s \phi_s = [\pi_1^\alpha] \] \tag{6.97}
with
\[ \gamma^0 = \frac{d\mathcal{L}}{2\mathcal{L}}. \] \tag{6.98}
where the above equation is afforded by (6.96). Since \( \pi_i^\alpha = f_{ij}^\alpha \omega^j \), we have
\[ \nabla \phi_0 = \nabla[f_i^\alpha] = [f_i^{\alpha_1} \omega^1 + f_i^{\alpha_2} \omega^2] = c_1^i \phi_s \omega^1 + c_2^i \phi_s \omega^2 = (c_1^i \omega^1 + c_2^i \omega^2) \phi_s. \] \tag{6.99}
Thus
\[ \gamma^s = c_i^s \omega^i. \] \tag{6.100}
Define Maurer-Cartan forms \( \{ \beta^r_s \} \) by

\[
d\phi_s =: \beta^r_s \phi_r. \tag{6.101}
\]

Then (6.91) implies

\[
(\beta^*_r) = \begin{bmatrix}
\beta_0^0 & -\beta_1^0 & -\beta_2^0 & -\beta_3^0 \\
\beta_0^1 & \beta_0^0 & \beta_0^3 & -\beta_0^2 \\
\beta_0^2 & -\beta_0^3 & \beta_0^0 & \beta_0^1 \\
\beta_0^3 & \beta_0^2 & -\beta_0^1 & \beta_0^0
\end{bmatrix}. \tag{6.102}
\]

It can be checked that

\[
[\beta^r_1] = i[\beta^r_0] \quad [\beta^r_2] = j[\beta^r_0] \quad [\beta^r_3] = k[\beta^r_0]. \tag{6.103}
\]

Differentiating (6.101) we have

\[
d\beta^r_s = \beta^r_p \wedge \beta^r_p. \tag{6.104}
\]

In particular,

\[
d\beta^0_0 = 0 \quad d\beta^1_0 = \beta^2_0 \wedge \beta^3_0 \quad d\beta^2_0 = \beta^3_0 \wedge \beta^1_0 \quad d\beta^3_0 = \beta^1_0 \wedge \beta^2_0. \tag{6.105}
\]

The first equation also follows from \( \phi_0 \cdot d\phi_0 = d\mathcal{L} \), which implies

\[
\beta^0_0 = \frac{1}{2} d\log \mathcal{L}. \tag{6.106}
\]

We also have

\[
\gamma^0 = \frac{1}{2} d\log \mathcal{L} = c^0_i \omega^i. \tag{6.107}
\]
Define
\[ \Sigma = \{(x, y, f, c) \in \mathcal{F}_M \times \mathcal{F}_N \times \mathbb{R}^4 \times \mathbb{R}^8 \} \text{ such that } \]
\[ (f^\alpha_i)^2 = 2L \quad [c_1^s] = i[c_2^s] \quad c_0^i = \frac{1}{2L} \mathcal{L}_i \}. \tag{6.108} \]
The forms \( \{\omega^1, \omega^2, \xi^1, \xi^2, \theta^1_2, \eta^1, \eta^2, \eta^3, dc^2_1, dc^3_1\} \) give a coframing on \( \Sigma \). Let \( \mathcal{J} \) be the ideal generated by
\[ \{\xi^\alpha, \eta^1 := \gamma^1 - c^1_i \omega^i, \eta^2 := \gamma^2 - c^2_i \omega^i, \eta^3 := \gamma^3 - c^3_i \omega^i\} \]. \tag{6.109} \]

We would want to express \( d\gamma^1, d\gamma^2, d\gamma^3 \) in terms of the coframing. Since \( \pi^\alpha_i = df^\alpha_i - f^\alpha_j \omega^j_i + f^\beta_i \theta^\alpha_\beta \),
\[ \gamma^s \phi = \pi = \beta^s_0 \phi - \omega^1_2(i \phi_0) + \theta^1_2(\phi_0 \phi) = \beta^s_0 \phi - \omega^1_2 \phi + \theta^1_2(\phi_0 \phi). \tag{6.110} \]
Differentiating (6.110), we get
\[ d\gamma^r \phi_r + \gamma^s \wedge d\phi_s \]
\[ = d\beta^s_0 \phi_r + \beta^s_0 \wedge d\phi_s - \omega^1_2 \wedge d\phi + \theta^1_2 \wedge d(\phi_0 \phi) \]
\[ + K \omega^1 \wedge \omega^2 \phi_1 - \overline{\theta^1_0} \wedge \theta^2(\phi_0 \phi). \tag{6.111} \]
Defining \( \phi, \phi_0 := J^s_c \phi \), we get
\[ d\gamma^r = - \gamma^s \wedge \beta^r_s + d\beta^r_0 + \beta^r_0 \wedge \beta^r_s + K \delta^r_1 \omega^1 \wedge \omega^2 - \omega^1_2 \wedge \beta^r_1 \]
\[ - \overline{K} (\det f) J^r_0 \omega^1 \wedge \omega^2 + \theta^1_2 \wedge \beta^r_0 \phi^r_c. \tag{6.112} \]
Equating coefficients of $\gamma^s$ in (6.110),

$$\gamma^s = \beta^s_0 - \delta_1 \omega^1_2 + J^s_0 \lambda^1_2.$$  \hfill (6.113)

For shorthand, let

$$\lambda = \frac{1}{2} \log \mathcal{L}.$$  \hfill (6.114)

Then mod $\mathcal{F}$, we compute

$$\beta^1_0 \wedge \beta^0_0 \equiv (\lambda_1 c^2_1 - \lambda_2 c^3_1) \omega^1 \wedge \omega^2 + (J^0_0 \lambda_2 - J^1_0 c^2_1) \theta^1_2 \wedge \omega^1 + (J^1_0 c^3_1 - \lambda_1 J^2_0) \theta^1_2 \wedge \omega^2$$

$$+ c^3_1 \omega^1_2 \wedge \omega^1 - c^3_1 \omega^1_2 \wedge \omega^2 - J^2_0 \omega^1 \wedge \theta^1_2$$

$$\beta^2_0 \wedge \beta^0_0 \equiv ((c^1_2)^2 + (c^3_1)^2) \omega^1 \wedge \omega^2 + (J^0_0 c^2_1 - J^2_0 c^3_1) \theta^1_2 \wedge \omega^1 - (J^3_0 c^3_1 + J^0_0 c^1_1) \theta^1_2 \wedge \omega^2$$

$$\beta^3_0 \wedge \beta^1_0 \equiv -(c^3_1 \lambda_1 + c^2_1 \lambda_2) \omega^1 \wedge \omega^2 + (c^1_3 J^1_0 - J^3_0 \lambda_2) \theta^1_2 \wedge \omega^1 + (J^3_0 \lambda_1 + c^2_1 J^1_0) \theta^1_2 \wedge \omega^2$$

$$+ c^3_1 \omega^1_2 \wedge \omega^1 - c^3_1 \omega^1_2 \wedge \omega^2 + J^3_0 \omega^1 \wedge \theta^1_2$$

$$\beta^0_0 \wedge \beta^1_0 \equiv -((\lambda_1)^2 + (\lambda_2)^2) \omega^1 \wedge \omega^2 + d \lambda \wedge \omega^2 - J^1_0 d \lambda \wedge \theta^1_2$$

$$\beta^0_0 \wedge \beta^2_0 \equiv -(\lambda_2 c^2_1 + \lambda_1 c^3_1) \omega^1 \wedge \omega^2 - J^0_0 d \lambda \wedge \theta^1_2$$

$$\beta^0_0 \wedge \beta^3_0 \equiv (\lambda_1 c^2_1 - \lambda_2 c^3_1) \omega^1 \wedge \omega^2 - J^0_0 d \lambda \wedge \theta^1_2$$

$$\beta^0_0 \wedge \omega^i \equiv d \lambda \wedge \omega^i$$

$$\beta^a_0 \wedge \omega^i \equiv c^a_k \omega^k \wedge \omega^i + \delta_{1 a} \omega^1_2 \wedge \omega^i - J^0_0 \theta^1_2 \wedge \omega^i \quad \text{for } a = 1, 2, 3.$$  \hfill (6.115)

Computing,

$$d \eta^1 \equiv \theta^1_2 \wedge (J^1_0 \sum_k c^1_k \omega^k + J^2_0 \sum_k c^2_k \omega^k + J^3_0 \sum_k c^3_k \omega^k + (J^3_0 c^2_1 - J^2_0 c^3_1) \omega^1 - (J^3_0 c^3_1 + J^0_0 c^1_1) \omega^2)$$

$$+ (J^1_0 \theta^1_2 - d \lambda) \wedge \omega^2 + ((c^1_2)^2 + (c^3_1)^2 + K - \overline{K}(\det f) J^0_0 + (\lambda_{11} + \lambda_{22})) \omega^1 \wedge \omega^2$$

\hfill (6.115)
\[ d\eta^2 \equiv -dc_1^2 \land \omega^1 + dc_1^2 \land \omega^2 + c_1^2 \omega^2 \land \omega_2 + c_1^2 \omega^1 \land \omega_1 + (-c_1^2 \lambda_1 - c_1^2 \lambda_2) \omega_1 \land \omega^2 \\
+ 2(c_1^3 J_0^3 - J_0^3 \lambda_2) \theta_2^1 \land \omega_1 + 2(J_0^3 \lambda_1 + c_1^2 J_1^1) \theta_2^1 \land \omega^2 + 2J_0^3 \omega_2^1 \land \theta_2^1 \\
- \overline{K}(\text{det } f)J_0^3 \omega^1 \land \omega^2 + \theta_2^1 \land (\beta_0^0 J_1^0 + \beta_0^2 J_2^0 + \beta_0^3 J_3^0) \]

(6.116)

\[ d\eta^3 \equiv -dc_1^3 \land \omega^1 - dc_1^2 \land \omega^2 + c_1^2 \omega^2 \land \omega_2 - c_1^2 \omega^1 \land \omega_1 \\
+ (\lambda_1 c_1^3 - \lambda_2 c_1^3) \omega^1 \land \omega^2 + 2(J_0^3 \lambda_2 - J_0^3 c_1) \theta_2^1 \land \omega^1 + 2(J_0^3 \lambda_1 J_0^3 \lambda_2) \theta_2^1 \land \omega^2 \\
- 2J_0^3 \omega_2^1 \land \theta_2^1 - \overline{K}(\text{det } f)J_0^3 \omega^1 \land \omega^2 \\
+ \theta_2^1 \land (\beta_0^1 J_0^1 + \beta_0^2 J_2^0 + \beta_0^3 J_3^0). \]

(6.117)

We can compute

\[
2\mathcal{L} J_0^0 = \phi_0 \mathbf{j} \cdot \phi_0 = 0 \\
2\mathcal{L} J_0^1 = \phi_0 \mathbf{j} \cdot \phi_1 = 2(f_1^1 f_1^2 - f_1^1 f_2^2) \\
2\mathcal{L} J_0^2 = \phi_0 \mathbf{j} \cdot \phi_2 = (J_1^3)^2 - (J_1^1)^2 + (f_1^3)^2 - (f_1^1)^2 \\
2\mathcal{L} J_0^3 = \phi_0 \mathbf{j} \cdot \phi_3 = 2(f_1^3 f_1^1 + f_2^1 f_1^1). 
\]

Then

\[
\gamma^0 = \beta_0^0 = \frac{1}{2} d \log \mathcal{L} \\
\gamma^1 = \beta_0^1 - \omega_2^1 + 2(f_1^1 f_2^2 - f_1^1 f_2^2) \theta_2^1 \\
\gamma^2 = \beta_0^2 + ((f_1^1)^2 - (f_1^2)^2 + (f_1^2)^2 - (f_1^1)^2) \theta_2^1 \\
\gamma^3 = \beta_0^3 + 2(f_2^1 f_2^2 + f_1^1 f_1^1) \theta_2^1. \]

(6.119)

It is straightforward to check

\[
J_1^1 = 0 \quad J_2^1 = J_0^3 \quad J_3^1 = -J_0^2 \\
J_1^2 = -J_0^3 \quad J_2^2 = 0 \quad J_3^2 = J_0^1 \\
J_1^3 = J_0^2 \quad J_2^3 = -J_0^1 \quad J_3^3 = 0. 
\]

(6.120)
In summary,

\[ d\eta^1 \equiv (2J_0^3 c_1^2 - 2J_0^2 c_1^3)\theta_2^1 \wedge \omega^1 - 2(J_0^3 c_1^3 + J_0^2 c_1^2)\theta_2^1 \wedge \omega^2 \]
\[ - d\lambda \wedge \omega_2^1 + (c_1^2)^2 + (c_1^3)^2 + K - \overline{K}(\det f)J_0^3 \]

\[ d\eta^2 \equiv -(dc_1^3 - c_1^2 \omega_1^2) \wedge \omega^1 + (dc_1^2 - c_1^2 \omega_1^2) \wedge \omega^2 - (c_1^3 \lambda_1 + c_1^2 \lambda_2 + \overline{K}(\det f)J_0^2)\omega^1 \wedge \omega^2 \]
\[ + 3(c_1^3 J_0^1 - J_0^3 \lambda_2)\theta_2^1 \wedge \omega^1 + 3(J_0^3 \lambda_1 + c_1^2 J_0^1)\theta_2^1 \wedge \omega^2 + 3J_0^3 \omega_2^2 \wedge \theta_2^1 \]

\[ d\eta^3 \equiv -(dc_1^3 - c_1^2 \omega_2^1) \wedge \omega^1 - (dc_1^2 - c_1^2 \omega_1^2) \wedge \omega^2 + (\lambda_1 c_1^2 - \lambda_2 c_1^3 - \overline{K}(\det f)J_0^3)\omega^1 \wedge \omega^2 \]
\[ + 3(J_0^3 \lambda_2 - J_0^1 c_1^2)\theta_2^1 \wedge \omega^1 - 3J_0^2 \omega_2^2 \wedge \theta_2^2 + 3(J_0^3 c_1^3 - \lambda_1 J_0^2)\theta_2^1 \wedge \omega^2. \]

(6.121)

We can write (6.121) in a compact way for convenience

\[ d\eta^1 \equiv P\theta_2^1 \wedge \omega^1 + Q\theta_2^1 \wedge \omega^2 - \lambda_1 \omega^1 \wedge \omega_2^1 - \lambda_2 \omega^2 \wedge \omega_2^1 + T\omega^1 \wedge \omega^2 \]

\[ d\eta^2 \equiv -(dc_1^3 + c_1^2 \omega_2^1 - 3J_0^1 c_1^2 \theta_2^1) \wedge \omega^1 + (dc_1^2 + c_1^2 \omega_2^1 + 3J_0^1 c_1^2 \theta_2^1) \wedge \omega^2 \]
\[ - M\omega^1 \wedge \omega^2 + N\theta_2^1 \wedge \omega^1 + L\theta_2^1 \wedge \omega^2 + S\omega_2^1 \wedge \theta_2^1 \]

(6.122)

\[ d\eta^3 \equiv -(dc_1^3 - c_1^2 \omega_2^1 + 3J_0^1 c_2 \theta_2^1) \wedge \omega^1 - (dc_1^2 + c_1^2 \omega_2^1 - 3J_0^1 c_2 \theta_2^1) \wedge \omega^2 \]
\[ + U\omega^1 \wedge \omega^2 + V\theta_2^1 \wedge \omega^1 + W\theta_2^1 \wedge \omega^2 + X\omega_2^1 \wedge \theta_2^1 \]

with

\[ P = 2(J_0^3 c_1^2 - J_0^2 c_1^3) \quad Q = -2(J_0^3 c_1^3 + J_0^2 c_1^2) \]
\[ M = c_1^2 \lambda_1 + c_1^2 \lambda_2 + \overline{K} \det f J_0^2 \quad U = \lambda_1 c_1^2 - \lambda_2 c_1^3 - \overline{K} \det f J_0^3 \]
\[ N = -3J_0^3 \lambda_2 \quad L = 3J_0^2 \lambda_1 \quad V = 3J_0^2 \lambda_2 \quad W = -3\lambda_1 J_0^2 \quad S = 3J_0^3 \quad X = -3J_0^2. \]

(6.123)

**Lemma 6.4.5.** Necessary conditions for the torsion in (6.122) to be absorbed are

\[ \lambda_2 = \lambda_1 \frac{J_0^3 c_1^3 + J_0^2 c_1^2}{J_0^3 c_1^3 - J_0^2 c_1^2} \]

(6.124)
\[ \lambda_1^2 + \lambda_2^2 = (c_1^2)^2 + (c_3^2)^2 + K - \bar{K}(\det f)J_0^1 + (\lambda_{11} + \lambda_{22}). \] (6.125)

**Proof.** Making the substitutions in (6.122)

\[ \begin{align*}
\theta_2^1 & \rightarrow \alpha^1 + A_1^1\omega^1 + A_2^1\omega^2 + A_3^1\omega_2 \\
\omega_1^2 & \rightarrow \alpha^2 + A_2^2\omega^1 + A_3^2\omega^2 + A_3^2\omega_2 - c_1^2\omega_2 + 3J_0^1c_1^3\theta_2^1 \\
\omega_1^3 & \rightarrow \alpha^3 + A_3^3\omega^1 + A_3^3\omega^2 + A_3^3\omega_2 + c_1^2\omega_2 - 3J_0^1c_1^2\theta_2^1,
\end{align*} \] (6.126)

we have

\[ - A_2^1P + A_3^1Q + T = 0 \] (6.127)
\[ A_3^1P + \lambda_1 = 0 \] (6.128)
\[ A_3^1Q + \lambda_2 = 0 \] (6.129)
\[ A_2^1 + A_3^3 - M - NA_2^1 + LA_2^1 = -3J_0^1c_1^3A_2^1 - 3J_0^1c_1^2A_1^1 \] (6.130)
\[ - A_2^3 + NA_3^1 + SA_1^1 = 3J_0^1c_1^3A_3^1 \] (6.131)
\[ A_3^3 + LA_3^1 + SA_2^1 = -3J_0^1c_1^2A_3^1 \] (6.132)
\[ - A_1^2 + A_3^3 + U - VA_2^1 + WA_1^1 = -3J_0^1c_1^2A_2^1 - 3J_0^1c_1^3A_1^1 \] (6.133)
\[ - A_3^3 + VA_3^1 + XA_1^1 = 3J_0^1c_1^2A_3^1 \] (6.134)
\[ - A_3^3 + WA_3^1 + XA_2^1 = -3J_0^1c_1^3A_3^1. \] (6.135)

From (6.131) and from (6.135), we get

\[ (N - W)A_3^1 + SA_1^1 - XA_2^1 = 0. \] (6.136)

From (6.132) and (6.134)

\[ (V + L)A_3^1 + XA_1^1 + SA_2^1 = 0. \] (6.137)
Thus solving the above two equations assuming $S \neq 0$ or $X \neq 0$ we get

$$A_1^1 = -A_3^1 \frac{(L + 3J_0^1 c_1^2)X + (N - 3J_0^1 c_1^2)S - S(W + 3J_0^1 c_1^2) + (V - 3J_0^1 c_1^2)X}{S^2 + X^2}$$

$$A_2^1 = -A_3^1 \frac{(L + 3J_0^1 c_1^2)S - (N - 3J_0^1 c_1^2)X + S(V - 3J_0^1 c_1^2) + (W + 3J_0^1 c_1^2)X}{S^2 + X^2}.$$  

(6.138)

Plugging in the above equations in (6.128) and (6.129), we obtain

$$A_3^1 = -\frac{\lambda_2}{Q} = -\frac{\lambda_1}{P}.$$  

(6.139)

From this we obtain a relation between $\lambda_1$ and $\lambda_2$:

$$\lambda_2 = \lambda_1 \frac{Q}{P} = \lambda_1 \frac{J_0^3 c_1^3 + J_0^2 c_1^2}{J_0^2 c_1^3 - J_0^3 c_1^2} \Rightarrow \frac{c_1^3}{c_1^2} = \frac{J_0^3 \lambda_2 + J_0^2 \lambda_1}{J_0^2 \lambda_2 - J_0^3 \lambda_1}.$$  

(6.140)

Finally, we get from (6.127) a relation between $T, \lambda_1, \lambda_2$:

$$T = \lambda_1^2 + \lambda_2^2 = (c_1^2)^2 + (c_1^3)^2 + K - \overline{K}(\text{det } f)J_0^1 + (\lambda_{11} + \lambda_{22}).$$  

(6.141)
Chapter 7

Conclusions

In this part of the thesis, we investigated the existence of solutions and integrability conditions for Case-A in subsection 6.1.2. The next direction would be to investigate similar questions for Cases-B, C, D in subsections 6.1.3, 6.1.4, 6.1.5. Since harmonic maps can be computed robustly between surfaces represented by point clouds and curvature functionals can also be calculated robustly, the ultimate goal would be to construct examples for specific morphological application and build a shape-discriminating as well as a shape-finding tool based on these cases.

A completely different, entirely theoretical further direction is the following. It is known that Willmore surfaces are characterized by the harmonicity of the so-called central sphere congruence map (to the de Sitter space) [Bry84]. This case corresponds to $\mu_h = \mu_k = 1$. Thus it will be interesting to see the cases when the target space is a Lie group, symmetric space or a grassmanian with the Dirichlet energy as in (6.4). The expectation is that different values of these constants in the functional will be related to the spectral parameter through the zero curvature representation. One can hope that this will lead to some “geometrical quantization” and a transformation that can lead from old to new non-trivial solutions.
Chapter 8

Appendix

We attach here the maple files for Lemma 6.3.3 and Proposition 6.4.3.
8.1 Maple computation in proof of Lemma 6.3.4

```maple
restart

# Computation for Lemma 6.3.4. Below lambda[i]=1/2*(dlogL)_i,
# omega^1_2=r_1omega^1+r_2omega^2, theta^1_2=s_1theta^1+s_2theta^2

Q[1, 1, 1] := -f[1, 1]; Q[1, 1, 2] := -f[2, 2]; Q[1, 1, 3] := -f[2, 2]
Q[1, 2, 1] := f[1, 1]; Q[1, 2, 2] := f[2, 2]; Q[1, 2, 3] := -f[2, 1]
Q[2, 1, 1] := -f[2, 1]; Q[2, 1, 2] := f[1, 1]; Q[2, 1, 3] := -f[2, 2]
Q[2, 2, 1] := -f[1, 2]; Q[2, 2, 2] := f[1, 1]; Q[2, 2, 3] := f[1, 1]
sum(beta[i], i = 0 .. 2);
```
\[ QO_1 := \text{sum}(Q[1, 1, b] \cdot t[b], b = 1..3) \]
\[ QO_1 := -f_{1, 2} t_1 - f_{2, 1} t_2 - f_{2, 2} t_3 \]  

\[ \alpha := z \mapsto Q_{1, 1, z} k_1 + Q_{1, 2, z} k_2 \]
\[ \beta := z \mapsto Q_{2, 1, z} k_1 + Q_{2, 2, z} k_2 \]
\[ \gamma := Q[1, 1, z] \cdot k[1] + Q[2, 2, z] \cdot k[2] \]
\[ \delta := z \mapsto Q_{1, 1, z} k_2 - Q_{2, 2, z} k_1 \]

\[ M_1 := \text{Matrix}([[-(R[1, 1, 2] - R[1, 2, 1]) \cdot k[2] + QO_1, (R[1, 1, 2] - R[1, 2, 1]) \cdot k[1] + \text{sum}(Q[1, 2, b] \cdot t[b], b = 1..3), \alpha(1), \alpha(2), \alpha(3)]]) \]
\[ M_1 := \left[ \begin{array}{c} -f_{1, 1} f_{2, 2} s_1 + f_{1, 2} f_{2, 1} s_1 + f_{1, 1} \lambda_2 + f_{1, 1} r_1 - f_{1, 2} \lambda_1 + f_{1, 2} r_2 \end{array} \right] k_2 - f_{1, 2} t_1 - f_{2, 1} t_2 \]
\[ -f_{2, 2} t_3, \left( -f_{1, 1} f_{2, 2} s_1 + f_{1, 2} f_{2, 1} s_1 + f_{1, 1} \lambda_2 + f_{1, 1} r_1 - f_{1, 2} \lambda_1 + f_{1, 2} r_2 \right) k_1 + f_{1, 1} t_1 - f_{2, 1} t_3 \]
\[ f_{2, 2} t_2, -f_{1, 2} k_1, -f_{2, 1} k_2, -f_{2, 2} k_1 \]  

\[ M_2 := \text{Matrix}([[-(R[2, 1, 2] - R[2, 2, 1]) \cdot k[2] + \text{sum}(Q[2, 2, b] \cdot t[b], b = 1..3), (R[2, 1, 2] - R[2, 2, 1]) \cdot k[1] + \text{sum}(Q[2, 2, b] \cdot t[b], b = 1..3), \beta(1), \beta(2), \beta(3)]]) \]
\[ M_2 := \left[ \begin{array}{c} -f_{1, 1} f_{2, 2} s_2 + f_{1, 2} f_{2, 1} s_2 + f_{1, 2} \lambda_2 + f_{1, 1} r_1 - f_{2, 2} \lambda_1 + f_{1, 2} r_2 \end{array} \right] k_2 + f_{1, 1} t_2 - f_{1, 2} t_3 \]
\[ -f_{2, 2} t_1, \left( -f_{1, 1} f_{2, 2} s_2 + f_{1, 2} f_{2, 1} s_2 + f_{1, 2} \lambda_2 + f_{1, 2} r_1 - f_{2, 2} \lambda_1 + f_{1, 2} r_2 \right) k_1 + f_{1, 1} t_3 - f_{1, 2} t_2 \]
\[ f_{2, 2} t_2, f_{1, 1} k_1 - f_{1, 2} k_2, f_{1, 1} k_2 - f_{1, 2} k_1 \]  

\[ M_3 := \text{Matrix}([[(R[1, 1, 2] - R[1, 2, 1]) \cdot k[2] + \text{sum}(R[1, 2, j] \cdot k[j], j = 1..2) + \text{sum}(Q[1, 2, b] \cdot t[b], b = 1..3), R[1, 1, 2] \cdot k[2] - R[1, 2, 1] \cdot k[1] - \text{sum}(R[1, 1, j] \cdot k[j], j = 1..2) - \text{sum}(Q[1, 2, b] \cdot t[b], b = 1..3), \gamma(1), \gamma(2), \gamma(3)]]) \]
\[ M_3 := \left[ \begin{array}{c} f_{1, 1, 1} s_1 + f_{1, 1, 2} \lambda_1 + f_{1, 1, 2} r_1 \end{array} \right] k_2 + f_{1, 2, 2} s_1 + f_{1, 2, 2} \lambda_1 + f_{1, 2, 2} r_2 - f_{1, 1, 2} \lambda_2 \right] k_2 \]
\[ f_{1, 1, 1} t_1 - f_{1, 1, 2} t_3 + f_{2, 2} t_2, -\left( f_{1, 2, 2} s_1 + f_{1, 2, 2} \lambda_1 + f_{1, 2, 2} r_2 - f_{1, 2, 2} \lambda_2 \right) k_1 - f_{1, 1, 1} s_1 + \]
\[ f_{2, 2} s_2 + f_{1, 1, 1} \lambda_1 + f_{1, 2, 2} r_1 \right] k_1 + f_{1, 2, 2} t_1 + f_{2, 2, 2} t_2 + f_{2, 2, 2} t_3, -f_{1, 1, 1} k_1 - f_{1, 2, 2} k_2, -f_{2, 2} k_1 - f_{2, 2} k_2 \]  

\[ M_4 := \text{Matrix}([[(R[2, 1, 2] - R[2, 2, 1]) \cdot k[2] + \text{sum}(R[2, 2, j] \cdot k[j], j = 1..2) + \text{sum}(Q[2, 2, b] \cdot t[b], b = 1..3), R[2, 1, 2] \cdot k[2] - R[2, 2, 1] \cdot k[1] - \text{sum}(R[2, 1, j] \cdot k[j], j = 1..2) - \text{sum}(Q[2, 2, b] \cdot t[b], b = 1..3), \delta(1), \delta(2), \delta(3)]]) \]
\[ M_4 := \left[ \begin{array}{c} -f_{1, 1} s_1 + f_{1, 2, 2} \lambda_1 - f_{1, 2, 2} r_1 \end{array} \right] k_2 + \left( f_{1, 2, 2} s_1 + f_{1, 1, 1} \lambda_1 + f_{1, 2, 2} r_2 - f_{1, 1, 1} r_2 \right) k_2 \]
This is the Matrix M

\[
M := \text{Matrix([[[M1], [M2], [M3], [M4]]])}
\]

\[
M := \left[\begin{array}{c}
\left(-f_{1,1}^2 s_1 + f_{1,2}^2 s_2 + f_{2,1}^2 s_1 + f_{1,1} \lambda_2 - f_{1,2} \lambda_1 - f_{1,3}^2 r_2 + f_{1,2} \lambda_2\right) k_2 - f_{1,2} t_1 - f_{2,1} t_2 \\
\left(-f_{1,2}^2 s_1 + f_{1,2} f_{2,1} s_1 + f_{1,1} \lambda_2 + f_{1,1} r_1 - f_{1,2} \lambda_1 + f_{1,2} r_2\right) k_1 - f_{1,1} t_1 - f_{2,1} t_3 \\
\left(-f_{1,2}^2 s_1 + f_{1,2} f_{2,1} s_1 + f_{1,1} \lambda_2 + f_{1,1} r_1 - f_{1,2} \lambda_1 + f_{1,2} r_2\right) k_1 + f_{1,1} t_1 - f_{2,1} t_3 \\
\left(-f_{1,2}^2 s_1 + f_{1,2} f_{2,1} s_1 + f_{1,1} \lambda_2 + f_{1,1} r_1 - f_{1,2} \lambda_1 + f_{1,2} r_2\right) k_1 - f_{1,1} t_1 - f_{2,1} t_3 \\
\end{array}\right] 
\]

\[
[\begin{array}{c}
\left(f_{1,1} f_{2,1} s_1 + f_{1,2}^2 s_2 + f_{1,1} \lambda_1 + f_{1,2} r_1\right) k_2 + \left(f_{1,2} f_{2,1} s_1 + f_{1,2}^2 s_2 - f_{1,1} r_2 + f_{1,2} \lambda_2\right) k_2 \\
\left(f_{1,1} f_{2,1} s_1 + f_{1,2}^2 s_2 + f_{1,1} \lambda_1 + f_{1,2} r_1\right) k_2 + \left(f_{1,2} f_{2,1} s_1 + f_{1,2}^2 s_2 - f_{1,1} r_2 + f_{1,2} \lambda_2\right) k_2 \\
\left(f_{1,1} f_{2,1} s_1 + f_{1,2}^2 s_2 + f_{1,1} \lambda_1 + f_{1,2} r_1\right) k_2 + \left(f_{1,2} f_{2,1} s_1 + f_{1,2}^2 s_2 - f_{1,1} r_2 + f_{1,2} \lambda_2\right) k_2 \\
\left(f_{1,1} f_{2,1} s_1 + f_{1,2}^2 s_2 + f_{1,1} \lambda_1 + f_{1,2} r_1\right) k_2 + \left(f_{1,2} f_{2,1} s_1 + f_{1,2}^2 s_2 - f_{1,1} r_2 + f_{1,2} \lambda_2\right) k_2 \\
\end{array}\right] 
\]

\[
0
\]

\[
\text{with(LinearAlgebra)}:\n\]

\[
\text{Rank(M)} = 4
\]

\[
\text{Column}(M, [1 .. 5])
\]

\[
[\begin{array}{c}
\left(-f_{1,1} f_{2,2} s_1 + f_{1,2} f_{2,1} s_1 + f_{1,1} \lambda_2 + f_{1,1} r_1 - f_{1,2} \lambda_1 + f_{1,2} r_2\right) k_2 - f_{1,2} t_1 - f_{2,1} t_2 - f_{2,2} t_3 \\
\end{array}\]
$$N := \left\langle -\begin{pmatrix} -f_{2,2} t_2 & f_{2,1} k_2 - f_{1,2} k_1 & -f_{1,1} k_2 - f_{2,2} k_1 \\ f_{2,1} k_2 - f_{1,2} k_1 & -f_{2,1} t_2 & f_{2,1} k_2 - f_{1,2} k_1 \\ -f_{1,1} k_2 - f_{2,2} k_1 & f_{2,1} k_2 - f_{1,2} k_1 & -f_{1,1} k_2 - f_{2,2} k_1 \end{pmatrix} \right\rangle$$

$$N := \left\langle \begin{pmatrix} -f_{1,1} t_2 - f_{2,2} t_2 \\
 f_{1,1} k_2 - f_{2,2} k_1 \\
 f_{1,1} k_2 - f_{2,2} k_1 \\
 -f_{1,1} k_2 - f_{2,2} k_1 \end{pmatrix} \right\rangle$$
Above is the Matrix $N$

```latex
\begin{align*}
-2 \left[ & \left( f_{1,1} s_2 + \frac{\lambda_1}{2} - \frac{r_2}{2} \right) f_{2,2}^4 + f_{1,2} \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2}^3 + \left( f_{1,1} s_2 + \left( \lambda_1 - r_2 \right) \right) f_{1,1}^2 \right. \\
& \left. + f_{1,2} \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2}^2 + \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2} f_{1,1} \right] \\
& \left[ \left( f_{1,1} s_1 + f_{2,1} s_2 + f_{1,1} \lambda_1 + f_{2,1} r_1 \right) f_2 + \left( f_{1,2} s_1 + f_{2,1} s_2 - f_{2,1} r_1 + f_{1,1} \lambda_2 \right) f_2 \right. \\
& \left. + f_{1,1} t_1 - f_{2,1} t_3 + f_{2,1} t_2 - f_{2,1} r_2 + f_{1,1} \lambda_2 \right) f_2 \\
& \left[ \left( -f_{2,1}^2 s_1 - f_{1,1} s_2 + f_{2,1} \lambda_1 + f_{2,1} r_1 \right) f_2 + \left( -f_{2,1}^2 s_1 - f_{1,1} s_2 + f_{2,1} \lambda_2 \right) f_2 \right. \\
& \left. + f_{1,1} t_3 - f_{2,1} t_2 + f_{2,1} t_1 - f_{2,1} k_1 - f_{2,1} k_2 + f_{1,1} k_1 - f_{1,1} \lambda_2 \right] \\
& \left. \left[ f_{1,1} t_1 - f_{2,1} t_3 \right] \right].
\end{align*}
```

(simplify(Determinant($N$)))

\begin{align*}
-2 \left[ \left( f_{1,1} s_2 + \frac{\lambda_1}{2} - \frac{r_2}{2} \right) f_{2,2}^4 + f_{1,2} \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2}^3 + \left( f_{1,1} s_2 + \left( \lambda_1 - r_2 \right) \right) f_{1,1}^2 \right. \\
& \left. + f_{1,2} \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2}^2 + \left( f_{1,1} s_1 - 2 f_{2,1} s_2 \right) f_{2,2} f_{1,1} \right] \\
& \left[ \left( f_{1,1} s_1 + f_{2,1} s_2 + f_{1,1} \lambda_1 + f_{2,1} r_1 \right) f_2 + \left( f_{1,2} s_1 + f_{2,1} s_2 - f_{2,1} r_1 + f_{1,1} \lambda_2 \right) f_2 \right. \\
& \left. + f_{1,1} t_1 - f_{2,1} t_3 + f_{2,1} t_2 - f_{2,1} r_2 + f_{1,1} \lambda_2 \right) f_2 \\
& \left[ \left( -f_{2,1}^2 s_1 - f_{1,1} s_2 + f_{2,1} \lambda_1 + f_{2,1} r_1 \right) f_2 + \left( -f_{2,1}^2 s_1 - f_{1,1} s_2 + f_{2,1} \lambda_2 \right) f_2 \right. \\
& \left. + f_{1,1} t_3 - f_{2,1} t_2 + f_{2,1} t_1 - f_{2,1} k_1 - f_{2,1} k_2 + f_{1,1} k_1 - f_{1,1} \lambda_2 \right] \\
& \left. \left[ f_{1,1} t_1 - f_{2,1} t_3 \right] \right].
\end{align*}

\text{(27)}
\[(f_{2,1}^2 \bar{t}_1) f_{1,1} + f_{2,1} t_3 (f_{1,2}^2 + f_{2,1}^2) f_{1,2}) (k_1^2 + k_2^2)\]
8.2 Maple computation in proof of Proposition 6.4.3

```maple
restart

with(LinearAlgebra)

matrix1 := (f[1, 1], f[2, 2], -f[2, 1], f[2, 1], -f[1, 2], f[1, 1], -f[1, 2], f[1, 2], -f[2, 1], f[1, 1], f[2, 2])

\[
\text{matrix1} := \\
\begin{bmatrix}
  f_{1, 1} & f_{2, 2} & -f_{2, 1} \\
  f_{2, 1} & -f_{1, 2} & f_{1, 1} \\
  -f_{1, 2} & -f_{2, 1} & -f_{2, 2} \\
  -f_{2, 2} & f_{1, 1} & f_{1, 2}
\end{bmatrix}
\]

(1)

Rank(matrix1) = 3

Determinant( (f[1, 1], f[2, 2], -f[2, 1], f[2, 1], -f[1, 2], f[1, 1], -f[1, 2], f[1, 2], -f[2, 1], f[1, 1], f[2, 2]) )

\[
f_{2, 1} f_{1, 1}^2 + f_{1, 2}^2 f_{2, 1} + f_{2, 1} f_{1, 2}^2 + f_{2, 1} f_{2, 2}^2
\]

(2)

This is the coefficient matrix if we want to solve for q^1_1, q^2_1, q^3_1, q^1_2

matrix2 := (f[1, 1], f[2, 2], -f[2, 1], f[2, 1], -f[1, 2], f[1, 1], -f[1, 2], f[1, 2], -f[2, 1], f[1, 1], f[2, 2])

\[
\text{matrix2} := \\
\begin{bmatrix}
  f_{1, 1} & f_{2, 2} & -f_{2, 1} & f_{1, 2} \\
  f_{2, 1} & -f_{1, 2} & f_{1, 1} & f_{2, 2} \\
  -f_{1, 2} & -f_{2, 1} & -f_{2, 2} & f_{1, 1} \\
  -f_{2, 2} & f_{1, 1} & f_{1, 2} & f_{2, 1}
\end{bmatrix}
\]

(4)

Rank(matrix2) = 4

Determinant(matrix2)

\[
f_{1, 1} f_{1, 2}^2 + 2 f_{1, 1}^2 f_{1, 2} + 2 f_{1, 1} f_{2, 1}^2 + f_{1, 2} f_{2, 1}^2 + f_{1, 2}^2 f_{2, 1} + 2 f_{1, 1} f_{2, 1} f_{2, 2} + f_{1, 2}^2 + f_{2, 1} f_{2, 2}^2 + f_{2, 1}^2 + 2 f_{1, 1} f_{2, 2} + f_{2, 2}^2
\]

(6)
```
Bibliography


