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Stability theory via Liapunov's method: a note on the contribution of Takuma Yasui

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The recent paper by D. W. Hands, 'The role of crucial counterexamples in the growth of economics knowledge: two case studies in the recent history of economic thought,' is the first serious examination, by a historian-methodologist, of the 1940–1960 work on stability of competitive equilibria.

The implicit narrative history, linked to the case studies that Hands extracts, involves Hicks's development of perfect and imperfect stability, and Samuelson's clarifications associated with the introduction of the ideas of dynamic stability to economic argumentation. Indeed, Hands' first case concerned Samuelson's demonstration that Hicksian 'stability' was not necessarily related to true dynamic stability. Hands' second case involved the conjecture "normal preferences and the tatonnement dynamic are associated with stable positions of equilibria." A counterexample to the conjecture, developed initially by Scarf, was based on the then recent stability proofs of Arrow, Block, and Hurwicz.

Those proofs were developed around 1957–1958. The Hicks and Samuelson theorems were presented around 1939–1942. The intervening period was one of the most active in the history of mathematical economics. Why then did stability analysis take nearly twenty years to develop?

One possible answer is that the problems concerned the qualitative behavior of systems of ordinary differential equations. Since the relevant mathematical theory was not developed until the early to middle 1950s, mathematical economists had to wait upon the emerging developments in mathematics. That is, the economic theory required that there be ways to analyze the asymptotic behavior of solutions of certain classes of differential equations. The methods used by Samuelson required either explicit solutions of those equation systems or else analyses of certain approximations to those systems, usually by linearization. In any event, the mathematics was not well enough developed for economists to have available to them the solution techniques they needed.

On this view, it was not until the middle 1950s, when there was a growth

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of interest in Liapunov's[1] theory of stability among mathematicians, that economists began to master the tools needed to prove that the competitive equilibrium was stable.[2] It was not, in other words, until economists understood stability theory, and the second or indirect method of Liapunov, that there could be analysis of stability of the competitive equilibrium.[3]

There are two problems with this view. The first is that Samuelson, in his 1947 book (based on his 1941 article) had virtually rediscovered the indirect method.[4] And second, a distinguished Japanese economist had independently rediscovered Liapunov's original papers and had applied them in print to developing a consistent set of approaches to stability analysis, unifying thereby the approaches of Hicks and Samuelson.

I shall in this note focus on the too little appreciated contributions of Takuma Yasui. Although I hope to treat the historical development of the stability literature in more detail, and with more attention to methodological issues, in another paper, here I want simply to point out that Yasui certainly recognized the importance of Liapunov's stability theory for the analysis of stability of economic equilibrium.[5]

1. The spelling of this mathematician's name is a bother. In Russian it is clear-cut. In English, we have various transliterations. We find "Liapunov," "Liapounov," "Lyapunov," "Lyapounov," "Liapounoff," "Lyapunoff," etc. Following the transliteration rules of the American Mathematical Society, I write "Liapunov." When I deal with quoted material, I reproduce the author's spelling. For another aspect of this weighty matter, see Phillip Davis's troubles with "Tschbecheff" in *The thread: a mathematical yarn.*

2. It is unclear to me the exact mechanism by which mathematicians began to revive this theory. It appears that the first wave was associated with Soviet mathematicians' interest, after the war, in commemorating Liapunov on the hundredth anniversary of his birth. The papers that resulted passed to the West, and it appears that Solomon Lefschetz was instrumental in bringing Liapunov's theory to the attention of the American mathematics community. The Princeton University Press photoprinting of the French translation of Liapunov's 1907 memoir, in 1947, certainly dates the resurgence of mathematical interest in Liapunov and the qualitative theory of differential equations.

3. This result required *tâtonnement* price dynamics (differential equations) and excess demand functions possessing certain properties related to the underlying economic structure of agents' preferences and choices.

4. Theorem 3, p. 438, of Samuelson's *Foundations of economic analysis* is really Liapunov's theory. As Hahn notes, "The problem at issue was what we would now call a gradient process: $\dot{x}_i = F_i(x_1 \ldots x_n)$ where $F_i(\ ) = [\partial F(\ )/\partial x_i]$. Samuelson normalised so that F attained its maximum at $x_i = 0$ all i. Hence for x sufficiently close to zero $\Sigma F_i(0 \ldots 0)x_i < 0$. He then noted that this implied that the function $\frac{1}{2}\Sigma x_i^2$ was monotone declining for $x \neq 0$ and used this to establish stability in the small. Concavity and boundedness could have been used to give stability in the large. Boundedness and using $F(x)$ as a Lyapounov function would have shown convergence to some critical value. . . . [Thus] he used the methods of Lyapounov without naming them as such. He was here within a hair's breadth of a development that did not occur for another ten years" (Hahn, 50–51).

5. Yasui cited, in his major paper to be discussed below, both Michio Morishima and Sono, a mathematician who wrote on economic dynamics; I have not been able to learn much about Sono's influence, except that his papers were well-known to Yasui. Morishima indeed set up sections of his 'Dynamic economic theory' (1980, from the 1950 Japanese version) with titles like 'Hicks's and Sono's stability conditions.' Unfortunately, the

Yasui is generally regarded as the pregenitor of the tradition of mathematical economics in Japan; the first generation of students he influenced were Morishima and Ichimura,[6] and their students, also under the Yasui influence, included Inada and Negishi. His importance was recognized by Shigeto Tsuru in his 'Survey of economic research in postwar Japan,' *American Economic Review,* June 1964, suppl.), and in its successor article, less accessible but more detailed, 'A survey of economic research in Japan, 1960–1983,' *Economic Review* 35. 4 (Oct. 1984). (This journal is published by the Institute of Economic Research, Hitotsubashi University.)

Tsuru states that after Yasui's graduation from Tokyo University in 1931, he concentrated on Walrasian analysis and published a series of studies in the general equilibrium tradition by 1938. His attempt to deal with dynamic analysis was put off course by his reading of Hicks's *Value and capital,* and he began to take a new interest in stability theory. During the war, "his preoccupation with such abstract theoretical matters was not conducive with his promotion at Tokyo University; and he moved to Tōhoku University, Sendai, in 1944" (Tsuru 1984, 295).

As Yasui notes: "I published in 1948 a paper entitled 'The Dynamic Stability Conditions of Economic Equilibrium' which was written without seeing Samuelson's *Foundations of Economic Analysis.* (At that time Japan was occupied by the U.S. Army and any foreign book could not be imported.)"[7] This paper, which appeared in *Keizai Shichō,* Sept. 1948 (in Japanese), "applied the matrix theory of Frobenius and made an innovative discovery that the stability condition for the difference-equation type reduces itself mathematically to the conditions elucidated by J. Schur and A. Cohn" (Tsuru 1984, 295–96).[8]

Yasui then tackled the larger task of nonlinear stability analysis, and around 1948 began "to consult the library of the department of mathematics everyday, and examined one after another a lot of volumes of mathematical journals, mostly German, French, and English, hoping to come across what I needed. . . . Among the fifty articles which I listed . . . a

English-language edition of 1980 omits all references and bibliographical material.

But see Sono 1944 as it appeared in Morishima's orginal 1950 paper.

6. The influence was not directly personal, since both were students at Kyoto, while Yasui was hundreds of miles away at Tohoku.

7. Yasui here agrees with Tsuru's recollection, however, that "I had a copy which Paul had sent to me; and knowing Yasui's interest at that time, I lent that copy to him immediately after I received it. But by then Yasui's paper was in the printer's hands" (Tsuru, 1986).

8. Tsuru goes on to note: "Yasui wrote a letter to Samuelson on this and received a reply from the latter which acknowledged Yasui's priority and suggested he write on it for *Econometrica.* But Yasui would not do this, believing as he did that the question of priority in such matters—i.e. a rediscovery of a mathematical theorem for application in economic theory—was not very important in any case" (ibid. 296). It is also worth noting that while Yasui had not seen the Samuelson book, he did know Samuelson's 1941 *Econometrica* article on stability.

paper with the title "Problem general de la Stabilité du Mouvement," by A. Liapounoff, in view of its title and content, looked most interesting and fitted to my purpose. So I took up Liapounoff" (Yasui 1985, 2).[9]

Concerning 'A general theory of stability,' Tsuru correctly notes that Yasui "never chose to even have a summary of it published in English. Thus it is generally conceded that priority in the 'rediscovery' of Liapounoff's theory resided in an article by Arrow and Hurwitz in 1958" (Tsuru 1984, 296). The article itself is hard to locate in American libraries: it was published in *Economic Studies Quarterly* 1.1 (Jan. 1950): 13–32, in Japanese as 'Antei no ippan-riron,' *Rironkeizaigaku*.[10]

Yasui began by considering equations of the form

$$dx_i/dt = X_i(x_1, x_2 \ldots, x_n) \quad (i = 1, 2 \ldots, n) \qquad 1.0$$

Such equations have, without loss of generality, equilibrium solutions of the form $x_i = 0$, for all i. Yasui's analysis initially showed the connection between stability of the first and second kinds, as introduced by Samuelson.

In Part II of the article, Yasui linked the discussion of stability with the theory of quadratic forms, in the manner of Liapunov. He introduced the quadratic form $V(x_1, x_2, \ldots x_n)$, which he called the complementary function to X in equation (1.0). Differentiating V with respect to the variable t and letting V_1 be that derivative function containing terms only of degree two or less, then V_1 is a quadratic form also. Yasui showed that, if V is positive definite, and V_1 is negative definite, then the stationary solutions $x_i = 0$ of (1.0) are stable in both the first and second Samuelson meanings: i.e. they possess stability of both the first and second kind (pp. 18–19). The proofs of this, and related propositions, were rather detailed; they introduced the salient Liapunov theory into the mathematical economics literature,

The next pages of Yasui's article provided examples. He showed that in the constant coefficient case, where $X_i = a_{i1}x_1 + \ldots + a_{in}x_n$, the function V could be taken as $V = \frac{1}{2}x_1^2 + x_2^2 + \ldots + x_n^2)$, a positive definite complementary function; the associated function V_1 is the quadratic form $\Sigma\Sigma a_{ij}x_i x_j$. Thus stability analysis of the constant coefficient equation system is equivalent to a study of the properties of the induced quadratic form; the eigenvalues of the differential equations and the definiteness of the

9. Yasui has written: "I was then entirely unaware how great this Russian mathematician was and what rank he held in his profession. In retrospect, it was as if I happened to have angled for a big fish from the dark sea. . . . Compared with such more recent neat expositions as W. Hahn's or LaSalle-Lefschetz's Liapunov books, my paper was assuredly tortuous as well as clumsy, but it was a little solace that no economist of the day paid attention to Liapounoff" (Yasui 1985, 2).

10. My own discussion of this article is based on a translation prepared by Mr. Suichi Murata, of Duke University, in November 1985.

quadratic form induced by the equation's constant coefficients are equivalent tools to analyse stability of the equilibrium motion of the system.

This example is extended in Part III of Yasui's article to the full treatment of the matrix equation $dX/dt = AX$. Yasui briefly examined issues of repeated eigenvalues, and eigenvalues with zero real parts, by means of the Liapunov theory. Although Yasui tackles no specific economic problem with the theory, he does 'unpack' the concept of stability and provide its modern meaning; this of course was Samuelson's objective in his early 1940s *Econometrica* articles. The Liapunov theory allowed Yasui to present a unified perspective on the dynamic system, its eigenvalues, and its associated quadratic forms. The interrelationships among these elements of stability theory were used by economists like Arrow and Hurwicz to establish stability theorems of the form "if economic assumptions A, B, etc. hold, then the competitive equilibrium is stable when price dynamics are governed by rules α, β, etc."

Yasui was the first economist to call attention to the importance of Liapunov's contribution to stability theory, and Yasui was the first explicitly to 'import' Liapunov theory into the economics literature. The fact that Yasui's paper was written in Japanese, and not readily accessible to English-speaking economists, is of course important, but it should not be allowed to obscure the influence of his interest in dynamic theory work on individuals like Morishima and Negishi, whose own contributions to the literature on competitive dynamics were far more influential and wide-ranging than Yasui's.[11]

Yasui independently 'rediscovered' Liapunov in 1948–1949 while working essentially alone in a Japan isolated from postwar mathematical economics. Why, then, was it the mid-1950s before economists in the United States did the stability analysis of competitive *tâtonnement* dynamics?[12] The usual answer is that the proofs had to wait on the 'acceptance' of Liapunov techniques. But we have shown here that such techniques were available to anyone who was interested as early as 1907, and the question of what to look for was clear by the early 1940s, as Yasui's own work demonstrates. This episode brings to the fore some disturbing questions

11. In fact, Morishima rediscovered some major elements of Liapunov's theory in his 1948 work which was a dynamization of Sono's static theory. That handwritten mimeographed paper, circulated among Japanese theorists, had a title which translates as 'Economic equilibrium and its stability.' "One year later I published an abridged version of the report in a journal, which is 'Static and Dynamic Stability Conditions' (again in Japanese) where I repeated the same argument" (Morishima 1986).

12. In a letter to the author, Samuelson denies that there is any "problem" or "question" to be answered here. He suggests that the delay was attributable to the "inability to find any compelling sufficiency conditions in terms of economic reality" (Samuelson 1986, 1). I disagree, and would suggest that without a coherent mathematical framework for testing such sufficiency conditions, any search would appear daunting. One only looks for what one *can conceivably find.*

about the interaction of mathematical tools and economic reasoning. The story of the introduction of the new dynamic theory into economic analysis is worth an examination more detailed than previous narratives have provided.

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