

A DYNAMIC MODEL OF THE FIRM*†

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In this paper we utilize traditional microeconomic theory and elementary queuing theory to develop a computer simulation model of a single-product, multi-process firm. One of our objectives is to demonstrate that the body of economic theory known as the "theory of the firm" may be used to provide a convenient frame of reference in applying some of the more recently developed analytical tools of operations research and computer technology to the analysis of the behavior of the firm. The static equilibrium model of the firm presented in *Value and Capital* by J. R. Hicks is taken as a point of departure in constructing a simulation model in which (1) the time interval between the arrival of orders at the firm is a stochastic variate with a known probability distribution, (2) each order which the firm receives must pass through n processes before it is transformed into a single unit of output, and (3) the time interval which an order spends at the j th process ($j = 1, \dots, n$) is a stochastic variate with a known probability distribution.

Introduction

Before the advent of the electronic computer and the development of present day mathematical and statistical techniques in the fields of operations research, econometrics, and mathematical economics, the neoclassical model of the firm formulated by J. R. Hicks [5] was probably the most widely accepted mathematical model of the firm among economists. However, over the past twenty-five years the Hicksian model of the firm has been subjected to a continuous stream of criticism (and rebuttal) by both economists and, more recently, operations researchers alike, beginning with the early attacks made by Hall and Hitch [4], as well as Lester [6] in the pre-operations research years and followed by such critics as Cyert and March¹ in more recent years.

It is not the purpose of this paper to provide a complete survey of the arguments for and against the traditional theory of the firm, e.g., the Hicksian model, or to take a position in favor of either the economist's concept of the theory of the firm or the operations researcher's approach to the problem. Instead, we merely want to point out that there is considerable need for a merger of the two alternative approaches to the problem of the firm, i.e., the traditional static

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¹ See [2], Chapter 1.

equilibrium models of the economists and the more sophisticated mathematical tools of the operations researchers. Yet surprisingly, there have been very few attempts to bridge this gap between the operations research and economic theory approaches to the behavior of the firm.² In this paper we intend to go one step further in eliminating the apparent barrier which exists between economists and operations researchers.

We utilize traditional microeconomic theory and elementary queuing theory in this paper to develop a computer simulation model of a single-product, multi-process firm. One of our objectives is to demonstrate that the body of economic theory known as the "theory of the firm" may be used to provide a convenient frame of reference in applying some of the more powerful analytical tools of operations research and computer technology to the analysis of the behavior of the firm. The static equilibrium model formulated by J. R. Hicks is taken as the point of departure in constructing a simulation model in which (1) the time interval between the arrival of orders at the firm is a stochastic variate with a known probability distribution, (2) each order which the firm receives must pass through n processes before it is transformed into a single unit of output, and (3) the time interval which an order spends at the j th process ($j = 1, \dots, n$) is a stochastic variate with a known probability distribution.

However, unlike Cyert and March [2] and other recent critics of the theory of the firm, we do not intend to reject the classical framework, but rather to extend it to include uncertainty. In general, our view of the role of static equilibrium models of the firm coincides closely with the views expressed by Joan Robinson in her book entitled *Economic Philosophy* [9].

The concept of equilibrium, of course, is an indispensable tool of analysis. . . . But to use the equilibrium concept one has to keep it in its place, and its place is strictly in the preliminary stages of an analytical argument, not in the framing of hypotheses to be tested against the facts, for we know perfectly well that we shall not find facts in a state of equilibrium.³

Although static equilibrium models are useful in the preliminary stages of the analysis of the problem of the firm, our simulation model possesses at least two attributes not found in the traditional models of the firm. First, it offers more flexibility in terms of the assumptions upon which it rests than do the neoclassical static equilibrium models. Second, it makes controlled experimentation possible over extended periods of time. That is, it is possible to observe the effects on total profit (or some other measure of effectiveness) of changes in the number of orders received, the price of the product, the number of processes, the level of factor inputs, and the probability distribution of order arrival times and process times. Through computer simulation we can generate time paths reflecting and testing any number of alternative hypotheses concerning the behavior of the firm and its environment. Furthermore, this method of approach can easily be extended to the analysis of multi-product firms whose behavior

² Two notable exceptions to this trend are [1] and [3].

³ See [9], p. 81.

involves interactions which are far more complex than those of the single product firm.

The Hicksian Model of the Firm⁴

Since one of the objectives of this paper is to compare our dynamic simulation model of the firm with Hicks' static equilibrium model, it is appropriate that we begin by summarizing the important characteristics of the Hicksian model. The Hicksian model rests upon the following major assumptions:

1. The prices of the firm's factors and products are fixed and known. (That is, perfect competition is assumed.)
2. The objective of the firm is to maximize profit subject to the technical constraints imposed by its production function.
3. A continuous production function exists (with nonzero first and second order partial derivatives) which relates the set of independent factor variables to the set of independent product variables.
4. The exact nature of the firm's production function has been predetermined by a set of technical decisions by the firm's engineers and technicians.
5. The firm's production function is characterized by: a decreasing marginal rate of technical substitution between any two factors; a decreasing marginal product for all factor-product combinations; and an increasing marginal rate of product transformation between any two products.
6. All of the firm's factors and products are perfectly divisible.
7. Neither the factor prices, product prices, nor the parameters which determine the firm's production function will change over the time period which is being considered. (This is a static model.)
8. Neither the factor prices, product prices, nor the parameters which determine the production function are permitted to be random variables. (Complete certainty is assumed.)⁵

The Hicksian model of the multi-product, multi-factor firm is concerned with two different types of decisions for the profit maximizing firm—output decisions and input decisions. The output decisions are concerned with "Which products to produce?" and "In what quantities should these products be produced?" The input decisions are concerned with "Which factors of production to buy?" and "In what quantities should these factors be purchased?" (Needless to say, these are by no means independent decisions.) Given the assumed information about product prices, factor prices, and production technology, the decision-maker for the firm then derives analytically a set of decision rules for maximizing total profit. The decision process for the neoclassical firm is represented in the form of a flow chart in Figure 1. The broken lines represent informational flows while the solid lines denote flows of factors of production and final products.

The Hicksian model may be stated mathematically as follows: The production function for the firm is given by

$$(1) \quad Q(X_1, \dots, X_r, \dots, X_s, Y_1, \dots, Y_s, \dots, Y_w) = 0$$

⁴ See [5], pp. 319-320.

⁵ See [7], p. 38.

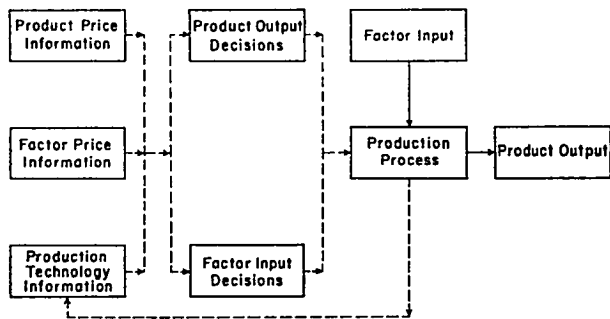


FIG. 1. A flow chart of the decision process for the Hicksian model of the firm

where

$$(2) \quad X_r \geq 0 \text{ are products,} \quad (r = 1, 2, \dots, v)$$

and

$$(3) \quad Y_s \geq 0 \text{ are factors,} \quad (s = 1, 2, \dots, w)$$

For any given set of factors, Y_1, Y_2, \dots, Y_w , there may be several technically feasible sets of products, X_1, X_2, \dots, X_v . Assign arbitrary values to $v - 1$ of these products and determine the largest value of the remaining product which is consistent with equation (1). This will assure a single valued production function. If all the factors and all but one of the products are assigned, then the remaining product is fully determined. Within the domain of definition the production function has continuous partial derivatives of first and second order [7].

If we let P_r denote the price of the r^{th} product and P_s denote the price of the s^{th} factor, then the firm's profit function is defined as

$$(4) \quad \Pi = \sum_{r=1}^v P_r X_r - \sum_{s=1}^w P_s X_s$$

The objective of the firm is to maximize total profit (4) subject to the technical constraints imposed by its production function (1). This constrained maximization problem may be solved analytically by the straightforward Lagrangian differential gradient method which yields the following set of optimum decision rules (necessary conditions for profit maximization) for the firm.

Rule 1. The price ratio of any two products must equal the marginal rate of product transformation between the two products.

Rule 2. The price ratio of any two factors must equal the marginal rate of technical substitution between the two factors.

Rule 3. The price ratio of any factor-product combination must be equal to the marginal product for the particular factor-product combination.

Having described the Hicksian model of the firm, we now turn to a dynamic extension of this model.

Structure

Our dynamic model of the firm is concerned with the behavior of a single-product, multi-process firm. The structure of this model differs from that of the Hicksian model in several ways.

First, although this model can be extended fairly easily to include multi-product firms, for expository purposes we have elected to restrict it to the single-product case only. Hence, the question of optimum product mix no longer exists, for we have assumed away the Hicksian decision concerning, "Which products to produce?"

Second, the Hicksian model is based on the assumption of perfect competition in both the product markets and the factor markets. However, in our model we partially relax the assumption of perfect competition in the product market while retaining perfect competition in the factor markets. The quantity of output which can be sold at a particular price is assumed to be a stochastic variate. That is, the firm cannot say with complete certainty how many units of output it will sell during a particular time interval at a given price. Demand is said to be a stochastic process. This assumption appears to be at least partially borne out in the real world, for the total sales of a firm depend not only on the prevailing market price but also on the effects of advertising and promotional expenditures, the marketing strategies of competitors, the national economy, and other factors over which it may be able to exercise little or no control. However, it has been found that with some firms, it may be possible to find a probability density function describing the behavior of demand, or equivalently a probability density function describing the frequency with which orders are received by the firm at given price levels. In this model we assume the existence of the latter type of probability density function. We define a stochastic variate AT_i , the time interval between the arrival of the i th order and the $(i - 1)$ th order ($i = 1, \dots, m$), with a known probability density function $f(AT)$, expected value ET , and variance VT . But in order to maintain some degree of compatibility with the Hicksian model, it is convenient to assume that the firm has no control over either the density function, expected value, or variance of AT . That is, $f(AT)$, ET , and VT are fixed and known.

Third, the production process of this firm is in reality a series of n processes $P_1, P_2, \dots, P_j, \dots, P_n$. Each unit of final output of the firm is assumed to have passed through all n of these processes in a particular order. Furthermore, each process is assumed to have its own separate production function. The Hicksian model consists of only one process and one production function.

Fourth, the variable time was completely abstracted from the Hicksian model, since the model was entirely static in nature. That is, the rate of factor input, Y_t , and the rate of product output, X_t , which we specified in the Hicksian model were only applicable at a particular instant in time rather than over a

⁶ The original version of this model will appear in Chapter 4 of a forthcoming book by the authors [8].

continuous time interval. However, in a dynamic model of the firm the time dimension must be treated explicitly. The production function for the j^{th} process of this model is given by,

$$(5) \quad Q_j(t) = Q_j(Y_{1j}(t), Y_{2j}(t), \dots, Y_{sj}(t), \dots, Y_{wj}(t), u_j(t)) \quad j = 1, \dots, n$$

where,

$$(6) \quad t = \text{time.}$$

$$(7) \quad Q_j(t) = \text{the rate of output of process } j \text{ at time } t, \quad j = 1, \dots, n.$$

$$(8) \quad Y_{sj}(t) = \text{the rate of input of factor } s \text{ for process } j \text{ at time } t, \\ s = 1, \dots, w, j = 1, \dots, n.$$

$$(9) \quad u_j(t) = \text{a stochastic variate with a known probability distribution,} \\ \text{expected value } EU, \text{ and variance, } VU, \quad j = 1, \dots, n.$$

The production function (5) states that the rate of output of the j^{th} process at time t is a function of the rate of input of factors of production at the j^{th} process at time t and a stochastic element, but independent of the rate of output of process $j - 1$. The stochastic element is an exogenous variable denoting the aggregate effect on the rate of output of the j^{th} process of all factors over which the firm has no control. Again for the sake of compatibility with the Hicksian model, let us assume that for a particular value of t , the production function of the j^{th} process has a decreasing marginal rate of technical substitution between any two factors and a decreasing marginal product for each factor-product combination.

Although the traditional production function in economic theory was designed to measure, $Q_j(t)$, the quantity of output per unit time at time t , it is more convenient in this model to use the reciprocal relationship,

$$(10) \quad ST_j = 1/Q_j(t) \quad j = 1, \dots, n$$

where ST_j denotes the time required to produce one unit of output or one production order in the j^{th} process. This transformation permits us to treat ST_j , the time required to produce one unit of output in the j^{th} process as a stochastic variate. Furthermore, the probability density function for ST_j and its parameters are completely determined by the level of factor inputs for process j and the probability density function, expected value, and variance of u_j . But the firm is assumed to know $f(u_j)$, EU , and VU . Hence, for each process ST_j may be treated as a stochastic variate with a known probability density function $f_j(ST_j)$, expected value ET_j , and variance VT_j . In other words, the firm cannot completely control the value of ST_j , but it can affect ET_j or VT_j , or even $f_j(ST_j)$ by altering the rate of factor inputs for process j . For example, such factors as the frequency of machine-breakdowns and variations in employee efficiency over time are likely to affect the process time of the j^{th} process. Although the firm may be able to reduce the frequency of break-downs in process j by acquiring more reliable equipment (or applying preventive maintenance techniques),

as well as more highly skilled employees, there is still a limit to the extent over which it can control the time required to produce one unit of output in process j . Hence, for specified rates of factor input at the j^{th} process, ST_j is a stochastic variate which is not subject to further control by the firm.

Flow Chart

The flow chart which appears in Figure 2 may serve to clarify the structure of our model. However, it should be noted that this flow chart does not include the firm's decision processes. Decision processes will be discussed after we have described the technical structure of the firm in complete detail.

In Figure 2 the block on the left-hand side of the flow chart denotes the arrival of an order to produce one unit of final output. This information is then transmitted to process one where factor inputs are used to produce one unit of process one output. The unit of output from process one then moves on to process two where it undergoes a further transformation before passing on to processes three, four, and five, etc., and finally reaches process N where it is transformed into one unit of completed output. Every order which is received by the firm passes through these N processes in a similar manner.

Mathematical Model

We now turn to the formulation of a mathematical model describing our dynamic multi-process firm. Let,

(11) AT_i = the time interval between the arrival of the i^{th} order and the $(i - 1)^{\text{th}}$ order, where an order is defined as the demand by a customer for the firm to produce one unit of final output and $i = 1, \dots, m$.

(12) $ST_{i,j}$ = the process time for the i^{th} order in the j^{th} process, where $i = 1, \dots, m$ and $j = 1, \dots, n$.

(13) $WT_{i,j}$ = the amount of time which the i^{th} order spends waiting to enter the j^{th} process, where $i = 1, \dots, m$ and $j = 1, \dots, n$.

(14) $IDT_{i,j}$ = the amount of time which the j^{th} process remains idle while waiting for the i^{th} order to arrive, where $i = 1, \dots, m$ and $j = 1, \dots, n$.

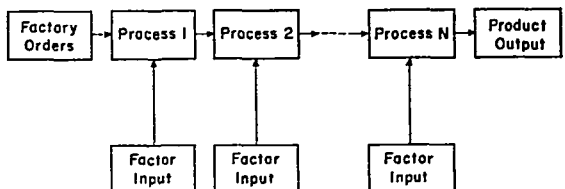


FIG. 2. A flow chart for a dynamic model of the firm

$$(15) \quad T_{ij} = WT_{ij} + ST_{ij}, \quad i = 1, \dots, m, j = 1, \dots, n.$$

= the total time in which the i th order spends at the j th process.

When the first order arrives at the firm, i.e., when $i = 1$, the following equations are assumed to describe the multi-process system

$$(16) \quad AT_1 = 0$$

$$(17) \quad WT_{11} = 0, \quad WT_{12} = 0, \quad \dots, \quad WT_{1n} = 0$$

$$(18) \quad IDT_{11} = 0, \quad IDT_{12} = ST_{11}, \quad \dots, \quad IDT_{1n} = \sum_{j=1}^{n-1} ST_{1j}$$

$$(19) \quad T_{11} = ST_{11}, \quad T_{12} = ST_{12}, \quad \dots, \quad T_{1n} = ST_{1n}$$

For subsequent arrivals, i.e., when $i = 2, 3, \dots, m$, these equations must be modified accordingly. The T -equations become,

$$(20) \quad \begin{aligned} T_{i1} &= WT_{i1} + ST_{i1} && i = 2, \dots, m \\ T_{i2} &= WT_{i2} + ST_{i2} && i = 2, \dots, m \\ &\vdots \\ T_{in} &= WT_{in} + ST_{in} && i = 2, \dots, m \end{aligned}$$

Whether waiting time or idle time occurs at a particular process depends on the sign of the following differences, where $i = 2, \dots, m$:

$$(21) \quad \begin{aligned} DIF_1 &= T_{i-1,1} - AT_i \\ DIF_2 &= (T_{i-1,1} + T_{i-1,2}) - (AT_i + WT_{i1} + ST_{i1}) \\ &\vdots \\ DIF_n &= (T_{i-1,1} + T_{i-1,2} + \dots + T_{i-1,n}) \\ &\quad - (AT_i + WT_{i1} + ST_{i1} + \dots + WT_{i,n-1} + ST_{i,n-1}) \end{aligned}$$

If DIF_j is positive for the j th process, then idle time will be zero, and waiting time can be calculated by

$$(22) \quad WT_{ij} = DIF_j \quad i = 2, \dots, m, j = 1, \dots, n$$

If DIF_j is negative for a particular process, then waiting time will be zero, and idle time will be equal to

$$(23) \quad IDT_{ij} = -DIF_j \quad i = 2, \dots, m, j = 1, \dots, n$$

If DIF_j is equal to zero for a particular process, then both waiting time and idle time will be equal to zero for that process.

Furthermore, AT_i is assumed to be a stochastic variate with probability density function $f(AT)$, expected value ET , and variance VT . And for each process, ST_{ij} is assumed to be a stochastic variate with probability density function $f_j(ST_j)$, expected value ET_j , and variance VT_j .

The decision process for our model is somewhat different from that of the Hicksian model, even though the objective of profit maximization has been retained. Although there are a number of alternative ways of viewing the decision process for our firm, we have chosen to assume that at the beginning of each planning period t , the firm decides which factors to purchase for use in planning period t and what quantities of these factors to purchase. That is, the firm is assumed to contract for a set of specific factor input rates over planning period t which can neither be increased nor decreased during the planning period. In other words, the rate of factor inputs is assumed to be fixed over each planning period but completely variable between planning periods. Once the firm specifies the rate of factor inputs for all processes for some planning period t , then the amount of total output produced in planning period t depends entirely on the arrival pattern of orders and the process times for each process in period t , both of which are stochastic variates. To be sure the firm could specify an upper limit on the number of orders it will accept in planning period t , but this would be inconsistent with the objective of profit maximization. Once the firm commits itself to a particular rate of factor inputs over planning period t , then the profit maximization objective dictates that the firm accept all orders which are received during the planning period even though it may not be able to complete production, or even begin production, on all of them during the period. (This statement is based entirely on the assumption that the firm contracts for its factors at the beginning of a planning period and cannot change the rate of factor inputs during the period.) But at the end of each planning period, or equivalently the beginning of the next planning period, the firm can change the rate of input of its various factors of production. Therefore, the decision problem facing our firm is effectively reduced to "Which factors of production to purchase for use in planning period t ?" and "In what quantities should these factors be purchased for each process?" Both of the output decisions of the Hicksian model have been assumed away. Since we are dealing with a single product firm, there is no problem concerning which products to produce. Furthermore, the rate of output is completely determined by the rate of factor inputs and two stochastic variates—order arrival time and process times.

Unfortunately, there is no straightforward analytical technique available which is capable of yielding a set of optimum decision rules for the solution of this problem. Although we can specify the effect which a change in the rate of factor inputs for a particular process will have on the expected value and variance of the process time for that particular process, it is very difficult, if not impossible, to determine analytically what effect, if any, these changes will have on the firm's total profit during a given planning period. Therefore, until some type of computational algorithm (such as, the simplex algorithm of linear programming) is developed which is capable of selecting factor input rates which yield maximum profits for the firm, we must necessarily resort to experimental approaches which permit us to try alternative factor input rates and test the effects that they have on total profit during a planning period. Computer simu-

ation appears to be a reasonably satisfactory alternative for obtaining approximate solutions to this problem.

Simulation of a Dynamic Model of the Firm⁷

Although the simulation method described in this paper was not designed to yield precise or optimum solutions to the firm's input decision problem, it will provide a technique for testing the effects on total profit of alternative factor purchasing policies. The principal advantage of this approach is that it permits the firm to experiment with alternative decision rules within the confines of a tightly controlled laboratory without interrupting actual operations of the real system. The simulated firm can be observed either in real time, compressed time or slow time. The firm can experiment with different levels of factor input for each process and observe the effects on idle time, waiting time, and total profit. In other words, we can specify at the outset a particular set of factor input rates for each process and hold these input rates constant over an extended time period, i.e., over a continuum of planning periods. By specifying the factor combinations for the j^{th} process, we automatically determine $f_j(ST_j)$, ET_j , and VT_j for the j^{th} process. We then use the computer to generate the time paths of all of the variables included in our mathematical model (equations 11-23). The computer is also used to keep a record of waiting times, idle times, total process times, and total production output. Total profit can then be easily computed for each simulated planning period, for the product price, factor prices, and factor inputs are all given, and total output per planning period, as well as total output for the entire simulation run can be obtained from the simulation itself. Therefore, total profit for the simulation run is equal to the sum of the profits in each simulated planning period. The computer enables us to experiment with a very large number of different factor combinations for each process. Although we may not be able to determine the "optimum" factor combination for all N processes, we may be able to come very close, depending on the number of factor combinations we try and the length of each computer run. Needless to say, as is the case in most sampling experiments, the more combinations of factors we try and the longer the simulation runs, the more likely we are to obtain an optimum solution. A second advantage of simulation is simplicity. Most of the computational procedures utilized here require little or no mathematical sophistication.

Two alternative methods have been developed for simulating dynamic models of the sort presented here—fixed time increment methods and variable time increment methods. The fixed time increment method contains a simulated time piece known as a "clock" which is used as a measure of the progress of the system being simulated over time. The "clock" controls the beginning and completion of all activities in the system. All time units are treated as integers by this method. Although experimental results have failed to lead to any generalizations

⁷ Several variations of this simulation model have been programmed in FORTRAN and run on the I.B.M. 1410 computer at Tulane University and the I.B.M. 7072 computer at Duke University. These programs are available.

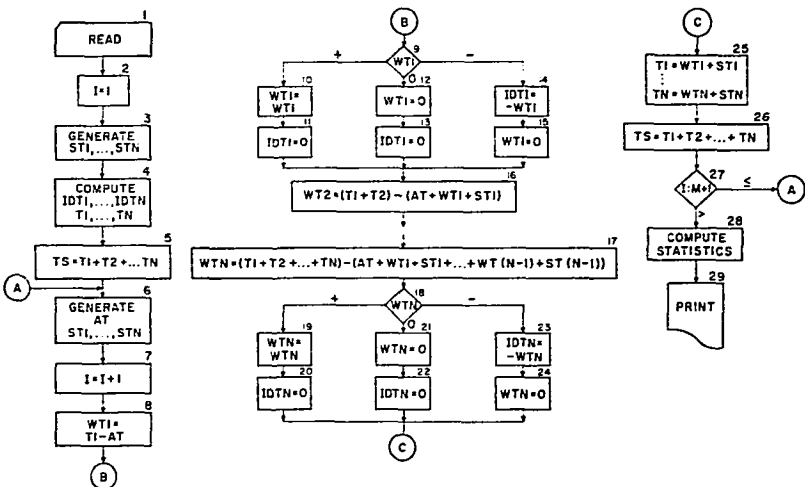


FIG. 3. A flow chart of a simulation model of a multi-process firm

as to which of these two methods requires less computational time, we will consider only the variable time increment method in this paper.⁸

Figure 3 contains a detailed computer block diagram of the logical structure of the variable time increment method as applied to our dynamic model of the firm. This block diagram assumes the availability of computer subroutines for generating stochastic variates having known probability distributions. Given the values of the parameters describing a particular distribution as input data, the subroutine will generate variates which are distributed accordingly. (This is not a particularly strong assumption for computer subroutines are available for most standard probability distributions and can usually be modified to generate variates of less common distributions.)⁹

In block 1 of Figure 3 the parameters M (total number of orders to be considered), N (total number of processes), ET (expected order arrival time), VT (variance of order arrival time), ET_1, ET_2, \dots, ET_N (expected process times), and VT_1, VT_2, \dots, VT_N (variance of process times) are all read into the computer as input data. The choice of a value for M is completely arbitrary and rests primarily on statistical considerations. The total number of orders to be generated by the simulation experiment will be entirely determined by the degree of precision required in estimating such parameters as expected waiting

⁸ We are indebted to Geoffrey Gordon of the I.B.M. Advanced Systems Development Division, Yorktown Heights, New York, for providing us with a GPSS (General Purpose Systems Simulator) program for our model, as well as the results obtained from two trial runs of the program by I.B.M. GPSS uses a fixed time increment method to simulate dynamic models.

⁹ A collection of FORTRAN programs for generating stochastic variates on a computer for most of the standard probability distributions may be found in Chapter 3 of [8].

time, expected idle time, expected output per planning period, and expected profit per planning period. The degree of statistical precision required will, of course, be dictated by the use which is to be made of the estimates.

In block 2 the index, I , is set equal to one denoting that the first order has been received by the firm. In block 3 process times are generated by the appropriate subroutines. Next we compute idle times and total times for each process for the initial order according to equations 18 and 19. Of course equations 16 and 17 must also be satisfied at this point. The total process times for all N processes are then totaled in block 5.

The arrival of a second order is indicated in block 6 by the generation of an inter-order time AT_2 , i.e., the time which has elapsed between the arrival of order 1 and the arrival of order 2. Since a new order has been received, N additional process times must also be generated. Block 7 indicates that the total number of orders should be incremented by one. In block 8 the difference indicated by the equation system 21 is obtained for order 2, process 1. If $WT1$ is positive, the idle time will be equal to zero ($IDT1 = 0$) and waiting time will be equal to $WT1$. If $WT1$ is equal to zero, then both waiting time and idle time are equal to zero. If $WT1$ is negative, waiting time will be equal to zero and idle time will be equal to $-WT1$. In block 16 the difference indicated by equation system 21 is obtained for order 2, process 2. $WT2$ is then subjected to a test similar to the test applied to $WT1$. Sign tests must also be applied to $WT3$, $WT4$, \dots , WTN . (The sign tests for $WT2$, $WT3$, \dots , $WT(N - 1)$ have not been included in the flow chart.)

Total process times are computed in block 25 according to the formulae given by the equation system 20. These total process times are then added together in block 26, yielding the total time, TS , (including waiting time) for an order to pass through all N processes. Finally, a test is applied to index- I . If I is less than or equal to $M + 1$, then the procedure must be repeated. That is, we must return to block 6 and generate another inter-order time and a set of N service times. However, if I exceeds $M + 1$, in block 27, this implies that M order arrivals have been simulated. The computer can then compute required statistics such as expected waiting time, expected idle time, total idle time, and expected output per planning period.¹⁰ Since factor prices and factor inputs are known, as well as the price of the product output, we can then calculate total profit on the basis the number of output units generated per planning period. This calculation may either be done by hand or by the computer.

This procedure may be repeated for as many different factor input combinations as we wish to try, each time adjusting the parameters of the probability distributions describing process times accordingly. If we try a fairly large number of different factor combinations and then generate the total profits of the firm for each combination on the computer, we may be able to gradually develop

¹⁰ The statistic "expected output per planning period" cannot be generated directly from our flow chart without modifying it slightly. However, it can be computed quite easily by simple hand calculation techniques. To obtain this statistic on a computer we must use a simulated "clock" to keep track of the number of output units completed during each planning period within a simulation run.

an acceptable factor purchasing policy and answer the questions, "Which factors should be purchased?" and "In what quantities should they be purchased?"

A Special Case

At the outset we stated that one of our objectives was to develop a dynamic simulation model of the firm which would be as nearly compatible with the Hicksian model as possible. In order to convince the skeptical reader that we have not abandoned this objective, we now return to it. In fact, we will show that the single-product, multi-factor version of the Hicksian model is merely a special case of our model.

By assumption both factor prices and product prices are given in the Hicksian model and in our model. Furthermore, under complete certainty and static conditions, the production functions for each of our N processes have, by assumption, the exact same characteristics as the Hicksian production function. [See equation (5).] Therefore, if we assume (1) complete certainty, (2) a single-process firm, and (3) a completely static firm, i.e., if we view the firm at a particular point in time, then our model is transformed into the Hicksian model. Under complete certainty the rate of output (or its reciprocal, process time per unit of output) for a single-process firm (at a particular point in time) is completely determined by the rate and combination of factor inputs. Furthermore, under perfect competition orders are assumed to arrive at a constant rate which is greater than the production rate. Idle time is logically impossible in this special case, and for all practical purposes the waiting line is infinitely long. This special case of our model, which is in reality the Hicksian model can be analyzed either by the Lagrangian differential gradient method discussed previously or by computer simulation. However, since we have assumed away uncertainty in both arrival times and process times and have eliminated idle time altogether, most of the features included in the flow chart found in Figure 3 are no longer relevant. Hence, to use simulation as a vehicle of analysis on this somewhat trivial model requires a drastic simplification in our simulation methodology.

The following simple example should serve to illustrate how one might go about treating the Hicksian model as though it were a special case of our more complicated simulation model. Consider a single-product, single-process firm operating in a world of complete certainty at a particular point in time. This firm may utilize two factors of production, X and Y , to produce its single product, Q . The prices of X , Y , and Q are given and are denoted by the symbols P_x , P_y , and P , respectively. The production function for the single-process firm is given by

$$(24) \quad Q = Q(X, Y)$$

and is characterized by a decreasing marginal rate of technical substitution between X and Y and a decreasing marginal product for both X and Y . Total profit is determined by

$$(25) \quad \Pi = PQ - P_x X - P_y Y$$

Once we specify particular values of X and Y , then total output and total profit are completely determined. Hence, the only two decisions required of the firm

are, "How much of factor X to purchase?" and "How much of factor Y to purchase?"

Two alternative methods may be used to solve this special case of our dynamic model—the Lagrangian differential gradient method or computer simulation. It can be shown quite easily by the Lagrangian differential gradient method that the optimum factor purchasing policy for the firm is a policy which leads to the purchase of quantities of X and Y which satisfy all three of the following conditions:

1. The price ratio between X and Y must equal the marginal rate of technical substitution between X and Y .
2. The price of X must be equal to the value of the marginal product of X .
3. The price of Y must be equal to the value of the marginal product of Y .

Although it is certainly possible to solve this special case of our model by computer simulation methods, the Lagrangian differential gradient method appears to be a much more suitable approach. To obtain an optimum solution to this model on a computer we would employ a standard technique of numerical analysis-optimization of a multi-variate function. That is, the computer would use a trial-and-error procedure for testing a large number of alternative values of X and Y to determine which combination yields maximum profits. Although the solution obtained by simulation would be identical to the analytical solution, the Lagrangian differential gradient method would be by far the simpler of the two methods. Computer simulation should only be applied when there is no convenient analytical method available, such as the Lagrangian method to solve a particular problem. The principal value of simulation lies not in its ability to analyze the behavior of a firm operating under the very special static equilibrium conditions of the Hicksian model, but rather in analyzing the behavior of a more complex, dynamic, multi-process firm operating under uncertainty.

Some Possible Extensions

Although we have gone through the painstaking task of attempting to show that our model may be considered as merely an extension of the Hicksian model rather than as a complete rejection of the Hicksian model, there are indeed a number of other equally important ways in which we may look at our model, i.e., ways which depart considerably from the Hicksian perfectly competitive, static equilibrium model. We shall now briefly summarize several possible extensions of our dynamic model which take us completely outside of the Hicksian framework.

1. The firm may be assumed to possess some degree of monopolistic power which enables it to at least partially control the arrival pattern of orders. That is, by changing either price or advertising and promotional expenditures the firm may be able to affect either the probability distribution or the expected value of order arrival time. We can, therefore, use our simulation model as a means of testing the effects of alternative pricing and advertising policies on total profit. For example, if we know the approximate relationship between price and expected order arrival time, then we can test the effect of price changes on profit (assuming fixed factor prices and fixed rates of factor input) by simu-

lating the behavior of the firm while operating under a number of different pricing policies.

2. We can also experiment with changes in the number of processes used by the firm, testing the effects on profit of adding a new process, eliminating a process, or combining two or more processes.

3. Experiments can also be made with changes in the nature of the production function for a particular process. There is no inherent reason whatsoever for restricting ourselves to the Hicksian concave production function.¹¹ Any number of other assumptions, e.g., fixed proportions, indivisibility, etc., are equally acceptable, and may be included in our model. In fact, we may even want to test the sensitivity of our model to changes in the assumptions underlying our production functions.

4. In other words, we may change any parameter or function in our model that we wish (while holding other parameters constant) and test the effects of these changes by using a computer to generate the time paths of the important variables describing the behavior of the firm. The obvious advantage of this approach is that it permits us to test a very large number of alternative hypotheses and decision rules concerning the behavior of the firm in a relatively short period of time without interrupting the actual operations of the firm.

Conclusions

To be sure some economists will object to the dynamic model proposed herein, for there is indeed "an irresistible attraction about the concept of equilibrium—the almost silent hum of a perfectly running machine; the apparent stillness of the exact balance of counteracting pressures; the automatic smooth recovery from a chance disturbance,"¹² which is conspicuously absent in this model. To this charge we plead nolo contendere.

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¹¹ The Lagrangian differential gradient method requires that the production function be strictly concave.

¹² See [9], p. 81.