EXTREMAL RAYS AND NULL GEODESICS ON
A COMPLEX CONFORMAL MANIFOLD

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A holomorphic conformal structure on a complex manifold $X$ is an everywhere non-degenerate
section $g \in H^0(S^2\Omega^1_X(N))$ for some line bundle $N$. In this paper, we show that if $X$ is a projective
complex $n$-dimensional manifold with non-numerically effective $K_X$ and admits a holomorphic
conformal structure, then $X \cong \mathbb{Q}^n$. This in particular answers affirmatively a question
of Kobayashi and Ochiai. They asked if the same holds assuming $c_1(X) > 0$. As a consequence, we
also show that any projective conformal manifold with an immersed rational null geodesic is
necessarily a smooth quadric $\mathbb{Q}^n$.

0. Introduction

Let $X$ be a $n$-dimensional complex manifold. A holomorphic conformal structure
on $X$ is an everywhere non-degenerate holomorphic section $g \in H^0(S^2\Omega^1_X(N))$ for some
holomorphic line bundle $N$ on $X$. Locally, $g$ can be thought of as a holomorphic metric
given by $g_a = \sum g_{a\bar{b}}dz^a\overline{dz}^b$ on the coordinate chart $U_a$ such that $\det(g_{a\bar{b}})$ is
everywhere non-zero. On the overlap $U_a \cap U_b$, we have $g_a = f_{ab}\theta_{\theta_\theta}$, where $f_{ab} \in \mathcal{O}_{U_a \cap U_b}$
is an invertible holomorphic function on $U_a \cap U_b$. The set of holomorphic functions
$f_{ab}$'s are the transition functions for the line bundle $N$. We call $N$ the conformal
line bundle of $X$. It is clear that any complex torus admits a holomorphic conformal
structure with trivial conformal line bundle. A more interesting example is a smooth
hyperquadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ (see [8], or Sec. 2 for a description).

About ten years ago, after classifying all compact complex conformal surfaces and
conformal manifolds of any dimension with Kähler-Einstein metrics, Kobayashi and
Ochiai proposed in [8] the following question:

**Question.** Let $X$ be a compact complex manifold of dimension $n$ with a holomorphic
conformal structure. If $c_1(X) > 0$ and $n \geq 3$, is it true that $X \cong \mathbb{Q}^n$?

This question was answered positively in [8] assuming the existence of a Kähler-
Einstein metric on $X$. Their proof was based on Berger's holonomy reduction theorem.

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When $n$ is odd, the answer to this question is trivially positive because of index considerations (see (1.1)).

This paper grew out of an attempt to understand relationships between extremal rays and null geodesics on a complex conformal manifold. As a consequence, we are able to show that the answer to the above question is positive. In fact, we will prove a stronger result. Precisely, we will show:

**Theorem 0.1.** Let $X$ be a $n$-dimensional complex projective manifold with a holomorphic conformal structure. If $n \geq 3$ and $K_X$ is not nef, then $X \cong \mathbb{P}^n$.

Complex conformal geometry plays an important role in the Penrose’s twistor program. The compactified (and complexified) Minkowski space $\mathbb{C}M$ is a smooth four-dimensional hyperquadric $\mathbb{Q}^4$ with a natural conformal structure. Once a conformal structure (in any dimension) is given, we can define the notion of complex null geodesics. In the case of complexified Minkowski space, these correspond to (complex) light rays. In general, null geodesics are holomorphic curves which are both null (this means their tangent vectors are null vectors) and geodesics (with respect to the Levi-Civita connections of local representatives of the conformal metric). The notion of null geodesic is well-defined globally on a conformal manifold since two conformally equivalent metrics have the same null cones and null geodesics (see [9] for a proof). For a precise definitions of these notions, we refer readers to [9], or Sec. 1. LeBrun studied the space of all null geodesics on a general complex conformal manifold. Assuming that the space of null geodesics is globally convex (this means that any two points on the manifold can be connected by at most one null geodesic), he was able to give a pretty good description of the space. For example, he showed that the space has a natural contact structure.

Theorem 0.1 has the following interesting consequence.

**Corollary 1.6.** If $X$ is a $n$-dimensional ($n \geq 3$) complex projective conformal manifold with an immersed rational null geodesic, then $X \cong \mathbb{P}^n$.

The idea of the proof of Theorem 0.1 comes from two seemingly unrelated sources: Mori’s theory of extremal rays and LeBrun’s work on null geodesics. Their relationship is expressed in terms of two algebraic-geometric characterizations of null geodesics (see Corollary 1.3 and Corollary 1.5 below). These algebraic geometric characterizations of null geodesics enable us to show that general minimal degree rational curve of an extremal rational curve is a null geodesic. Consequently, any two general minimal degree rational curves (of an extremal ray) cannot be tangent to each other. This last fact will play a very important role in our proof. Because the space of null geodesics is locally convex, we see that minimal degree rational curves are also locally convex. By local convexity of a family of curves we mean that there is at most one curve in the family connecting two given points in a neighborhood (in classic topology) around any point of $X$. Local convexity of null geodesics is guaranteed by a theorem of Whitehead (see [15]). Although the use of local convexity is an essential part of our proof, it is not enough to prove the theorem. We need also some global properties about minimal degree rational curve. Precisely, we need to show that through two general points on
there is at most one minimal degree rational curve, which, by abuse of conventions, we call the \textit{global convexity property}. This is achieved by some delicate algebraic geometric arguments, which are done in the last section of this paper.

Here is the basic strategy for the proof of Theorem 0.1. We first show that \( X \) is a Fano manifold with Picard number one, and there exists a divisor in \( X \) whose intersecting number with a rational curve is one. This implies that the index of the manifold is \( \dim(X) \), which in turn implies the Theorem 0.1 immediately by virtue of a theorem of Kobayashi and Ochiai [7]. The divisor we will construct is the locus of all the rational curves (or equivalently, null geodesics) with certain numerical properties in \( X \) passing through a fixed point. In the case when \( X \cong \mathbb{P}^n \), this divisor is simply the intersection of \( X \) with its tangent hyperplane at a point.

The paper is organized as follows. There are three sections. In the first section, we study some general properties of a complex conformal manifold. The main goal of that section is to establish a criterion for a curve to be a null geodesic. The second section is devoted to proving Theorem 0.1 assuming that Proposition 2.12 holds. The last section is devoted to the proof of Proposition 2.12.

All the varieties in this paper are defined over \( \mathbb{C} \). Almost all our notations are standard. We will explain them when they occur. But there are a few frequently used notations which we feel that it is necessary to explain at this point. For any coherent sheaf \( \mathcal{F} \), we denote by \( \mathcal{F}^\ast \) its dual. For a line bundle, or an invertible sheaf, \( \mathcal{L} \), both \( 2\mathcal{L} \) and \( \mathcal{L}^2 \) mean \( \mathcal{L} \otimes \mathcal{L} \). If \( E \) is a vector bundle, or a locally free sheaf, then we define \( \mathcal{P}(E) \) to be \( E^\ast \setminus \{0\}/\mathbb{C}^\ast \), instead of \( E \setminus \{0\}/\mathbb{C}^\ast \). The reason for this is that it is more conventional for algebraic geometers. \( S^m E \) \( (m \) is a positive integer) means the \( m \)-th symmetric tensor of the vector bundle \( E \). We simply denote by \( \mathcal{O}(l) \) \( (l \) is an integer) the line bundle \( \mathcal{O}_{\mathbb{P}^1}(l) \) on \( \mathbb{P}^1 \). If \( \mathcal{F} \) is a coherent sheaf on \( \mathbb{P}^1 \), we denote \( \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(l) \) by \( \mathcal{F}(l) \). We denote by \( T_X \) \( ( \) respectively, \( \Omega_X^1 \) \) the holomorphic tangent bundle (respectively, cotangent bundle) of \( X \). We denote by \( \text{Chow}^1(X) \) the Chow variety of 1-cycles whose intersection number with \( -K_{X} \) is \( l \). Throughout this paper we will use the theory of extremal rays freely. We refer readers to [3] and [12] for basic materials about this subject.

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\section{Algebraic Characterizations of Null Geodesics}

To start with, let \( X \) be a complex holomorphic conformal manifold with a conformal
structure $g \in H^0(S^2\Omega^1(N))$. We sometimes call $g$ a (holomorphic) conformal metric on $X$. Let $n$ be the dimension of $X$ throughout this paper. Let $C$ be a smooth complex curve, and $C \to X$ be a holomorphic immersion, i.e., the image $f(C)$ has at worst unramified singularities.

We are going to introduce some definitions. These definitions are explained in details in [9], to which we refer the readers.

Let $v$ be a type $(1,0)$ tangent vector on $X$, $v$ is called null if $g(v, v) = 0$. Since multiplying by a conformal factor does not change the equation $g(v, v) = 0$, the notion of nullity is well-defined globally. At each point $x \in X$, we have a null tangent cone denoted by $Q_x \subset T_{x,x}$. It is the set of all null tangent vectors of $X$ at the point $x$. This was called the sky at $x$. It is the affine cone over a smooth hyperquadric. An immersed holomorphic curve $C \subset X$ is called null if for any point $p \in f(C)$ on the curve, and any holomorphic tangent direction $v_p$ at $p$, $g_p(v_p, v_p) = 0$. This definition can be extended to any map $f : C \to X$, not necessarily unramified. In general, $f(C)$ is called null if the regular part of $f(C)$ is null. This clearly agrees with the definition for immersed curves.

On each local coordinate chart $U_x$, let $g_{a\bar{b}} = \sum g_{ab} dx^a dx^b$ be a local representative of the conformal metric $g$. We can associate to $g_{a\bar{b}}$ a Levi-Civita connection $D_a$ on $U_x$, very much like the case in the real Riemannian geometry. Namely, we have the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} \sum_m g^{im}_a \partial_j g_{akm} + \partial_j g_{akm} - \partial_k g_{akj}$$

where $\partial_j = \partial/\partial z^j_x$, and $[g^{ij}_a] = [g_{ij}]^{-1}$ as matrices. An immersed holomorphic curve $C$ in $U_x$ is a (complex) geodesic if there exists a holomorphic parameter $\xi$ on $C$ such that

$$\frac{d^2 x^i_{\xi}}{d\xi^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j_{\xi}}{d\xi} \frac{dx^k_{\xi}}{d\xi} = 0$$

for all $i = 1, \ldots, n$.

But these connections on various coordinate charts do not match on the overlaps since the conformal metric $g$ is not globally defined. However, since they differ only by some conformal factors on the overlaps, the notion of null geodesic is well-defined globally (see [9] for a proof). For a morphism $f : C \to X$, $f(C)$ is called a null geodesic if it is both null and a geodesic on each coordinate cover $U_x$.

Let $Y = \mathbb{P}(T_X)^{\text{def}} = \Omega^1_X \setminus \{0\}/\mathbb{C}^*$. Using the conformal structure $g$, we obtain a well-defined divisor of $Y$, which we denote by $S$. Precisely,

$$S = \left\{(p, [\omega_p]) \mid p \in X, \omega_p \in \Omega^1_{p,x} \setminus \{0\} \text{ and } g(\omega_p, \omega_p) = 0\right\} \subset Y = \mathbb{P}(T_X)$$
where \( \Omega^1_{g,X} \) is the holomorphic cotangent space of \( X \) at the point \( p \in X \), and \( g_p \) is any representative of the conformal metric \( g \) at the point \( p \). Therefore \( S \) is a smooth quadric bundle over \( X \). Let \( Y \to X \) be the natural projection, and \( S_x = \pi^{-1}(x) \cap S \) for a point \( x \in X \). Then it is clear that for any \( x \in X \), \( S_x \cong \mathbb{Q}^{n-2} \) and the sky \( Q_x \) is an affine cone over \( S_x \).

Denote by \( L \) the tautological line bundle \( \mathcal{O}_Y(1) \) on \( Y \).

**Lemma 1.1.** \( \mathcal{O}_Y(S) \cong 2L \otimes \pi^*N^* \).

**Proof.** Note that the conformal structure induces a bundle isomorphism: \( T_x \cong \Omega^1_x(N) \). Let \( g^{-1} \in H^0(S^2 T_x \otimes N^*) \) be the inverse of \( g \in H^0(S^2 \Omega^1_x(N)) \). In view of the isomorphism

\[
H^0(L^2 \otimes \pi^*N^*) \cong H^0(S^2 T_x \otimes N^*)
\]

\( g^{-1} \) induces a section \( s_y \in H^0(L^2 \otimes \pi^*N^*) \). It is straightforward to check that the divisor \( S \) constructed above is exactly the vanishing locus of \( s_y \), i.e., \( S = \{ s_y \}_0 \). This implies the lemma immediately. \( \square \)

The following fact about a complex conformal manifold will be used frequently in the rest of the paper

\[
2K_X = -nN \tag{1.1}
\]

where \( K_X = \wedge^n \Omega^1_X \) is the canonical bundle of \( X \). (1.1) can be proved by taking the determinant of the isomorphism \( T_x \cong \Omega^1_x(N) \).

Next we will give some algebraic characterizations of null-geodesics on a complex conformal manifold. First let us observe that there is a natural contact structure on \( Y = \mathbb{P}(T_X) \). The associated contact line bundle is \( \mathcal{O}_Y(-1) \), which is the dual of \( L \). Therefore we have the following short exact sequence:

\[
0 \to L^\perp \to \mathcal{O}_Y \xrightarrow{\phi} L \to 0. \tag{1.2}
\]

The contact form is given locally by \( \theta = \sum_{i=1}^n \xi_i dz^i \), where \( z = (z^1, \ldots, z^n) \) is a local coordinate on \( X \), and \( [\xi_1, \ldots, \xi_n] \) a projective coordinate on the fiber of \( \pi: Y \to X \). More precisely, on the chart, say, where \( \xi_i \neq 0 \), the contact form is given by \( \theta_i = dz^i + \sum_{j \neq i} \frac{\xi_j}{\xi_i} d \xi_i \). Therefore the bundle \( L^\perp \) is spanned by all tangent vectors perpendicular to the contact form \( \theta \). The homomorphism \( \phi \) is defined via contraction by the contact form. The one form \( \theta \), in fact, lives naturally on the whole cotangent bundle \( \Omega^1_x \) and \( d\theta = \sum_{i=1}^n d \xi_i \wedge dz^i \) is a natural symplectic form on \( \Omega^1_x \).

Choose a coordinate cover \( \{ U_x \} \) of \( Y \) such that \( \theta \) is the local contact form. Let \( \omega_x = d\theta|_{U_x} \). Then \( \{ \omega_x \} \) define a conformal symplectic structure on the bundle \( L^\perp \). That is, \( \{ \omega_x \} \) may not be glued to give a well-defined symplectic form on \( L^\perp \), but on the overlap \( U_x \cap U_y \), we have: \( \omega_x = h_{xy} \omega_y \), where \( h_{xy} \) is the transition function for
the contact line bundle $\mathcal{C}_Y(-1)$ on $Y$. The equation $\omega_x = h_x \omega_y$ is obtained by differeniating the equation $\theta_x = h_x \theta_y$ (since $\theta_y|_{L^\perp} = 0$ for each $x$).

In terms of these local coordinates, we can represent the conformal structure locally by an invertible symmetric $n \times n$ matrix $(\eta^0)$. And the divisor $S \subset Y$ can be locally defined by the equation \{(cz^i, [z^j]) | \sum_{i,j} \eta^{ij}(z^i z^j) = 0\}. If we restrict the bundle map $\phi$ in (1.2) to $T_S \subset T_Y|_S$, we get a homomorphism of bundles: $T_S \cong L|_S$. It is easy to check that $\phi|_S$ is surjective using local coordinates. Let $T^0_S$ be $\ker(\phi|_S)$. Then $T^0_S$ is a well-defined subbundle of $L^\perp|_S$. The rank of $T^0_S$ is $2n - 3$ since $L^\perp$ is of rank $2n - 2$.

Denote $\omega$ by the collection of $\{\omega_x\}$'s. Restricting $\omega$ to $T^0_S$, we get a bundle map locally on each coordinate chart: $\omega(S): T^0_S \rightarrow T^0_S$. Even though $\omega(S)$ is not well-defined globally, the kernel of $\omega(S)$ is. Let $N^0_S \overset{\text{def}}{=} \ker(\omega(S))$.

We claim that $N^0_S$ is a line subbundle of $T^0_S \subset L^\perp|_S$. This can be shown as follows. First of all, $\omega(S)$ has to degenerate since the rank of $T^0_S$ is $(2n - 3)$, which is odd. Secondly, the rank of $\ker(\omega(S)) \leq \text{codim}(S) = 1$ since each $\omega_x$ is non-degenerate. Hence $\ker(\omega(S))$ is a line subbundle of $T^0_S$.

By the definition of $N^0_S$ and $T^0_S$, we have the following inclusions of subbundles (not just subsheaves):

$$N^0_S \subset T^0_S \subset T_S \subset T_Y|_S; \quad T^0_S = T_S \cap L^\perp$$

Following LeBrun, holomorphic integral curves of $N^0_S$ are called phase-space trajectories of $S$. The importance of this notion is illustrated by the following result of LeBrun (see p. 213 in [9] for a proof).

**Proposition 1.2.** [9] Null geodesics of $X$ are precisely the phase-space trajectories of $S$.

The line bundle $N^0_S \subset T^0_S$ can be identified $N^*_{S/Y}$ (see also [6]), i.e., we have

$$N^0_S = N^*_{S/Y} \quad (1.3)$$

where $N^*_{S/Y}$ is the co-normal bundle of $S$ in $Y$. We can show (1.3) as follows. The contact sequence (1.2) induces the following short exact sequence:

$$0 \rightarrow T^0_S \rightarrow L^\perp|_S \rightarrow N^*_{S/Y} \rightarrow 0. \quad (1.4)$$

Using the conformal symplectic structure $\{\omega_x\}$ on $L^\perp$, we get (1.3) immediately.

Now we are going to translate the above result of LeBrun into the language of algebraic geometry. Let $C \rightarrow X$ be a holomorphic immersion (i.e., unramified) from a smooth algebraic curve $C$. For any point $p \in f(C)$ and any tangent direction $v_p$ of $f(C)$ at $p$, we get a point $(p, v_p) \in \mathbb{P}(T^1_X)$. In this way we construct a lift of the map $f$ to $\mathbb{P}(T^1_X)$. Denote this lift by $f_1: C \rightarrow \mathbb{P}(T^1_X)$. Let $C_1 = f_1(C) \subset \mathbb{P}(T^1_X)$. Then $C_1$ is a smooth algebraic curve since $f$ is an immersion. Since $X$ is conformal, $T_S \otimes N^{-1} \cong \Omega^1_X$. This implies that $\mathbb{P}(T^1_X) \cong \mathbb{P}(T_X) = Y$. Now composing this last isomorphism with the lift $f_1: C \rightarrow \mathbb{P}(T^1_X)$, we get a lift $f': C \rightarrow Y$. Then it is clear that $f'$ maps $C$ isomorphically
onto its image. We still denote by $C$ the image $\hat{f}(C) \subset Y$ when there is no danger of confusion. Therefore the resulting morphism $C \hookrightarrow X$ is nothing but the projection 
$C \overset{\pi}{\to} X$, i.e., $f = \pi|_C$. It is clear from the construction that

$$L|_C \cong \Omega^*_C \otimes \pi^*N|_C$$

(1.5)

since $L \otimes \pi^*N^*$ is the tautological line bundle of $\mathbb{P}(\Omega^*_C)$ under the isomorphism $\mathbb{P}(\Omega^*_C) \cong \mathbb{P}(T_Y)$.

There is an isomorphism similar to (1.5) that holds for a general (not necessarily immersed) morphism $C \overset{\hat{f}}{\to} X$. Let $B$ be the ramification divisor for the morphism $f$. It is the zero divisor of the differential of $f$. Then the following is true:

$$f^*L \cong \Omega^*_C(-B) \otimes f^*N$$

(1.5a)

were $\hat{f} : C \to Y$ is the lift of $f : C \to X$ to $Y$ as constructed above. We can prove (1.5a) as follows. First note that $L \otimes \pi^*N^*$ is the tautological line bundle of $\mathbb{P}(\Omega^*_C)$. Secondly, the lift of $C \overset{\hat{f}}{\to} X$ to $\mathbb{P}(\Omega^*_C)$ is defined by the quotient line bundle $\Omega^*_C(-B)$ of $f^*\Omega^*_C$. Therefore $f^*L \otimes f^*\pi^*N^* \cong \Omega^*_C(-B)$, i.e., $f^*L \otimes f^*N \cong \Omega^*_C(-B)$ since $f = \pi \circ \hat{f}$. This implies (1.5a) immediately. In general, the lift $\tilde{f}$ does not map $C$ isomorphically onto its image due to the presence of ramifications.

Note also that the $f(C) \subset X$ is null if and only if the lift of $C$ to $Y$ is contained in $S$. Now Proposition 1.2 implies the following corollary immediately.

**Corollary 1.3.** Let $C \overset{\hat{f}}{\to} X$ be a holomorphic immersion from a smooth curve $C$ into $X$ such that its image is null. Then $f(C)$ is a null geodesic if and only if the two line subbundles $T_C \hookrightarrow T_S|_C$ and $N^0|_C \hookrightarrow T_S|_C$ are identical.

We have the following criterion for a curve to be null.

**Lemma 1.4.** Let $C \overset{f}{\to} X$ be an holomorphic map (not necessarily an immersion) from a smooth curve $C$ into $X$. Then $f(C)$ is null if $h^0(S^2\Omega^*_C \otimes f^*N) = 0$.

**Proof.** Let $g \in H^0(S^2\Omega^*_C(N))$ be the given conformal structure. Then $f(C)$ is null if and only if $f^*g = 0$ on $C$. Since $f^*g \in H^0(S^2\Omega^*_C \otimes f^*N)$, the lemma is obviously true.

The following corollary will play an essential role in the proof of Theorem 0.1.

**Corollary 1.5.** Let $f : C \to X$ be a null-immersion, i.e., its image is null. If $f(C)$ is a null geodesic, then $T_C \cong f^*N$. Conversely, if $T_C \cong f^*N$ and $h^0(T_S|_C \otimes \Omega^*_C) = 1$, then $f(C)$ is a null geodesic.

**Proof.** Let us first prove that

$$N^0|_C \cong T^0 S \otimes \pi^*N^*|_C.$$  

By Lemma 1.1, we have $C_C(S) \cong L^2 \otimes \pi^*N^*$. Since $N^0 \cong C_C(S)|_S$, (1.5) implies that $N^0|_C \cong (\Omega^*_C)^{\otimes 2} \otimes \pi^*N^*|_C$. Now (1.6) easily follows from (1.3).
If $C$ is a null geodesic, then $T_C \cong N^0_{S|C}$ by Corollary 1.3. By (1.6), $N^0_{S|C} \cong T^{\otimes 2}_C \otimes \pi^*N^*$. Therefore $T_C \cong \pi^*N_{|C}$. However, $f = \pi_{|C}$. Therefore $T_C \cong f^*N$.

If $T_C \cong f^*N$, then $T_C \cong N^0_{S|C}$ by (1.6). If moreover $h^0(T_{S|C} \otimes \Omega_C^1) = 1$, then any line subbundle of $T_{S|C}$ that is isomorphic to $T_C$ has to be identical to $T_C$ as a line subbundle of $T_{S|C}$. By Corollary 1.3, $C$ is a null geodesic of $X$. Hence we are done.

If $f : C \to X$ is not an immersion, there is still an isomorphism similar to (1.6). As before, let $B$ be the ramification locus of $f$. Then

$$f^*N^0_S \cong T^{\otimes 2}_C(2B) \otimes f^*N^* \quad (1.6a)$$

where $f : C \to Y$ is the lift of $f$ to $Y$. This can be easily proved using (1.5a).

**Corollary 1.6.** If $X$ is a $n$-dimensional ($n \geq 3$) complex projective conformal manifold with an immersed rational null geodesic, then $X \cong \mathbb{P}^n$.

**Proof.** Let $\mathbb{P}^1 \subset X$ be an immersed rational null geodesic. Then Corollary 1.5 implies that $T_{\mathbb{P}^1} \cong f^*N$. Therefore $N \cdot f(\mathbb{P}^1) = 2$. By (1.1), $2K_X = -nN$. Therefore $K_X \cdot f(\mathbb{P}^1) < 0$, i.e., $K_X$ is not nef. Theorem 0.1 implies the corollary immediately.

## 2. The Proof of Theorem 0.1

The goal of this section is to prove Theorem 0.1, assuming Proposition 2.12 below, which will be proved in the next section. This section is organized as follows. There are basically two main results in this section. Lemma 2.1--Lemma 2.7 are devoted to showing that the manifold $X$ in Theorem 0.1 is a Fano manifold of Picard number one, and its unique extremal ray $R$ is of length $n$. Lemma 2.8--Lemma 2.10 are devoted to showing Proposition 2.11. That proposition asserts that through a general point, every minimal degree rational curve of $R$ is a null-geodesic, and the restriction of the tangent bundle of $X$ is of the “expected” decomposition.

Throughout this section, we let $X$ be a $n$-dimensional complex projective conformal manifold with $K_X$ being not nef. Recall that we say a line bundle $L$ is nef if $L \cdot C \geq 0$ for any effective curve $C \subset X$. By a celebrated theorem of Mori ([11] and [12]), there is an extremal ray $R$ on $X$. From now on, we fix an extremal ray $R$ on $X$.

We define the *length* of the extremal ray $R$ (denoted by $l(R)$) as:

$$l(R) = \left\{ -K_X \cdot \Gamma \mid [\Gamma] \in R \text{ and } \Gamma \text{ is a rational curve} \right\}$$

where $[\Gamma]$ means the numerical equivalence class of the rational curve $\Gamma$.

By a theorem of Mori [11], $l(R) \leq n + 1$. We say a rational curve $\Gamma$ is a *minimal degree rational curve*, or an *extremal rational curve of $R$* if $[\Gamma] \in R$ and $-K_X \cdot \Gamma = l(R)$.

**Example.** Let $X = \mathbb{P}^n \subset \mathbb{P}^{n+1}$ be a smooth hyperquadric and $R$ be the extremal ray generated by a straight line $\ell \subset X$. Then $l(R) = n$ and all extremal rational curves of this extremal ray are straight lines in $X$. There is a natural conformal structure on $X$, which can be described as follows. Assume, without loss of generality, that
Then the symmetric two form $g = \sum_{i=0}^{n+1} dz_i \wedge d\bar{z}_i$ on $\mathbb{C}^{n+2}$ descends to a conformal structure on $X$ with the conformal bundle $N \cong \mathcal{O}_X(2)$. It is easy to show that this is the only conformal structure on $\mathbb{Q}^n$.

To start with, let $r = 0, 1, 2$. Fix $r$ point(s) $\{t_1, \ldots, t_r\}$ on $\mathbb{P}^1$. Of course, when $r = 0$, the set $\{t_1, \ldots, t_r\}$ is empty. Let $\mathbb{P}^1 \xrightarrow{\pi} X$ be a morphism such that it is birational onto its image. Consider the space $\text{Hom}(\mathbb{P}^1, X; \{t_i, f(t_i)\})$ of all morphisms from $\mathbb{P}^1$ to $X$ fixing the given $r$ point. Let $U_r$ be the irreducible component of $\text{Hom}(\mathbb{P}^1, X; \{t_i, f(t_i)\})$ that contains the point $f$.

Then naturally we have a morphism:

$$U_r \times \mathbb{P}^1 \xrightarrow{\pi} X, \quad \Xi(v, t) = v(t)$$

for any $v \in U_r$ and $t \in \mathbb{P}^1$. Let $G_r$ be the subgroup of $\text{Aut}(\mathbb{P}^1)$ fixing the given $r$ point. Then $G_r$ acts on $U_r$ by $g \circ v(t) = v(g^{-1}(t))$ for $g \in G_r, v \in U_r$ and $t \in \mathbb{P}^1$. Therefore $G_r$ acts on $U_r \times \mathbb{P}^1$ and $\Xi$ is a $G_r$-invariant morphism. Let $Z \subset U_r$ be an arbitrary closed subscheme. Then $Z$ is said to be $G_r$-invariant if $G_r(Z) \subset Z$.

The following lemma is a slight variation of Corollary 1.3 of [6].

**Lemma 2.1.** Let $Z \subset U_r$ be an irreducible and $G_r$-invariant subscheme of $U_r$. Let $d$ be the dimension of the image of $\Xi(Z \times \mathbb{P}^1) \subset X$. Then for a general point $v \in Z$, $v^*T_X$ has the following decomposition:

$$v^*T_X \cong \bigoplus_{a_i} O(a_i)$$

such that $a_1 \geq \cdots \geq a_d \geq r$.

**Proof.** It suffices to prove the same thing for the reduced part of $Z$. Hence we can assume that $Z$ is reduced and irreducible. Note that the group $G_r$ acts transitively on the complement of the given $r$ point(s) of $\mathbb{P}^1$. Therefore the image of $\Xi|_{Z \times \{t\}}: Z \times \{t\} \rightarrow X$ also has dimension $d$ for a general $t \in \mathbb{P}^1$ since $G_r(Z) \subset Z$ by our assumptions. By the Generic Smoothness Theorem, for a general $v \in Z$, the differential of $\Xi|_{v \times \{t\}}$ at $v \times \{t\}$ has rank $d$. That is, the following homomorphism

$$T_{v, Z} \xrightarrow{d\Xi|_{\{v\}} \circ v^*T_X|_{\{t\}}} v^*T_X|_{\{t\}}$$

has rank $d$ for a general point $(v, t) \in Z \times \mathbb{P}^1$. Since $Z \subset U_r$, by Proposition 2 in [11], $T_{v, Z} \subset T_{v, t} = H^0(v^*T_X(-r))$. Hence the above homomorphism $d\Xi|_{\{v\}}$ factors through the inclusion $v^*T_X(-r)|_{\{t\}} \hookrightarrow v^*T_X|_{\{t\}}$. Hence the image of the following natural homomorphism

$$H^0(v^*T_X(-r)) \rightarrow v^*T_X(-r)|_{\{t\}}$$

has dimension at least $d$. This is enough to imply the lemma. \qed
Let us fix a morphism $\mathbb{P}^1 \xrightarrow{f} X$ such that $[f(\mathbb{P}^1)]$ belongs to the given extremal ray $R$. We denote by $E(f)$ the image $\Xi(U_0 \times \mathbb{P}^1)$, and $F_x(f)$ by image $\Xi(U_x \times \mathbb{P}^1)$, where $x = f(0)$. Note that $E(f)$ is simply the union of the images of all the deformations of $\mathbb{P}^1 \xrightarrow{f} X$. $F_x(f)$ is the union of the images of all the deformations of $\mathbb{P}^1 \xrightarrow{f} X$ fixing the point $0 \in \mathbb{P}^1$. It is clear to see that $E(f)$ is contained in the exceptional locus $E_R$ of the contraction morphism $\phi_R$ of $R$, and $F_x(f)$ is contained in a fiber of $\phi_R|_{E_R}$. The following lemma was proved in [17].

**Lemma 2.2.** If $-\deg f^*K_X = l(R)$, then

$$\dim E(f) + \dim F_x(f) \geq n + l(R) - 1.$$  \hspace{1cm} (2.2)

For the readers' convenience, we will give a proof below.

**Proof.** There is a natural morphism $\alpha: U_0 \to \text{Chow}^{l(R)}(X)$, the Chow variety of 1-cycles with intersection number $l(R)$ with $-K_X$, such that $\alpha(f) = f(\mathbb{P}^1)$ for any $f \in U_0$. Let $T_0$ be the image of $\alpha(U_0)$. Consider the following morphism:

$$\mathbb{P}^1 \times U_0 \xrightarrow{\psi} T_0 \times X, \quad \Psi(t,f) = (\alpha(f), \Xi(t,f)).$$

Let $M_0 \subset T_0 \times X$ be the image of $\Psi$. Then we have the following diagram:

$$\begin{array}{ccc}
M_0 & \xrightarrow{p} & X \\
\downarrow q & & \\
T_0 & & \\
\end{array}$$

where $p$ and $q$ are two projections.

First note that $E = p(M_0)$. Let $T_0(x) = q(p^{-1}(x))$. Then $F_x(f) = p(q^{-1}T_0(x))$. If we denote still by $p$ its restriction to $q^{-1}T_0(x)$, then $p: q^{-1}T_0(x) \to F_x(f)$ is finite. Otherwise, there would be a non-trivial deformation of $h(\mathbb{P}^1)$ for some $[h] \in U_0$ fixing two different points on $h(\mathbb{P}^1)$. By the breaking-up technique (see [11]), there must be a rational curve $\Gamma \subset X$ such that $[\Gamma] \in R$ and $-K_X \cdot \Gamma < -K_X \cdot h(\mathbb{P}^1) = l(R)$. This is a contradiction by the definition of the length of an extremal ray. Therefore $p: q^{-1}T_0(x) \to F_x(f)$ is finite.

Hence $\dim F_x(f) = \dim q^{-1}T_0(x) = \dim T_0(x) + 1$. Since $p$ maps $p^{-1}(x)$ isomorphically onto $T_0(x)$, we have

$$\dim E \geq \dim M_0 - \dim T_0(x)$$

$$= \dim M_0 - \dim F_x(f) + 1.$$ 

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\[ \dim M_0 = \dim U_0 + 1 - \dim \text{Aut}(\mathbb{P}^1) \]
\[ \geq \chi(f^*T_X) + 1 - 3 \]
\[ = n + l(R) - 2. \]

Here we use Proposition 3 in [11] to conclude that \( \dim U_0 \geq \chi(f^*T_X) \). Combining these inequalities, we obtain the lemma immediately.

**Lemma 2.3.** Let \( X \) be a \( n \)-dimensional projective conformal manifold with non-nef canonical bundle. Then there is an extremal ray \( R \) on \( X \) such that \( l(R) = n \).

**Proof.** By the Cone Theorem (see [12]), there exists an extremal ray \( R \) on \( X \). Since \( 2K_X = -nN \) (by Eq. (1.1)) and \( l(R) \leq n + 1, l(R) \) is either \( n \) or \( \frac{n}{2} \) (\( n \) is necessarily even in this case). If \( l(R) = n \), then we are done. Assume that \( n = 2k \) and \( l(R) = k \). We will derive a contradiction in this case.

Let \( \mathbb{P}^1 \to X \) be a morphism such that \( f^*K_X = \mathcal{O}(-k) \). Then \( N \cdot f(\mathbb{P}^1) = 1 \) by (1.1). For simplicity, let \( E \) be \( E(f) \). Without loss of generality, we can assume that \( x = f(0) \) is a smooth point of \( f(\mathbb{P}^1) \). Denote \( F_x(f) \) by \( F_x \). Then Lemma 2.2 implies that

\[ \dim E \geq 3k - 1 - \dim F_x. \]  
\[ (2.3) \]

**Claim.** \( \dim E \geq 2k - 1. \)

**Proof of the claim.** Let \( d = \dim F_x \). Denote by \( U_i \ni f \) an irreducible component of \( \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \). For a general point \( u \in U_i \), let \( u^*T_X \cong \bigoplus_{i=1}^{2k} \mathcal{O}(a_i) \) such that \( a_1 \geq \cdots \geq a_{2k} \). By Lemma 2.1, we have

\[ a_1 \geq \cdots \geq a_d \geq 1. \]
\[ (2.4) \]

However the conformal structure \( g \) induces an isomorphism \( T_X \cong \Omega^1_X(N) \). This last isomorphism implies:

\[ \bigoplus_{i=1}^{2k} \mathcal{O}(a_i) \cong \bigoplus_{i=1}^{2k} \mathcal{O}(1 - a_i). \]
\[ (2.5) \]

Therefore \( a_i + a_{2k-i+1} = 1 \) for \( 1 \leq i \leq 2k \). Now this last equality and (2.4) easily imply that \( d \leq k \). By (2.3), we have \( \dim E \geq 2k - 1 \) immediately. Hence the claim is proved.

Let \( \mathbb{P}^1 \to X \) be a general point in \( U_0 \). Suppose that \( v^*T_X \cong \bigoplus_{i=1}^{2k} \mathcal{O}(a_i) \) such that \( a_1 \geq \cdots \geq a_{2k} \). The above claim and Lemma 2.1 imply that at least the first \( 2k - 1 \) (if not all) \( a_i \)'s are non-negative. In particular, \( a_{2k-1} \geq 0 \). However, this last inequality contradicts Lemma 2.4 below. Hence we are done, i.e., \( l(R) \) must be \( n \).

The following lemma was used in the proof of the above lemma.
Lemma 2.4. Let $X$ be a $n$-dimensional projective conformal manifold with non-nef canonical line bundle. Suppose that we have a non-trivial morphism $v : \mathbb{P}^1 \to X$ such that $v(\mathbb{P}^1) \cdot N = 1$, or 2. When $v(\mathbb{P}^1) \cdot N = 2$, we assume furthermore that $v : \mathbb{P}^1 \to X$ is ramified. Let $v^*T_X \cong \bigoplus_{i=1}^m \mathcal{O}(a_i)$ be such that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $a_{n-1} < 0$.

Proof. Let $m = v(\mathbb{P}^1) \cdot N$. Then $m = 1$ or 2 by our assumption. The given conformal structure $g$ induces an isomorphism $g : T_X \cong \Omega_X^1(N)$. Hence $a_i + a_{n-i+1} = v(\mathbb{P}^1) \cdot N = m$ for all $i$ with $1 \leq i \leq n$.

Suppose that the lemma does not hold, i.e., $a_{n-1} \geq 0$. We will draw a contradiction. Since $a_i \geq a_{n-i}$ for all $1 \leq i \leq n - 1$, we get $a_i \geq 0$ for all $1 \leq i \leq n - 1$. Hence we have:

$$0 \leq a_{n-1} \leq \cdots \leq a_2 \leq m \leq 2, \quad \text{and} \quad a_n = m - a_1. \quad (2.6)$$

When $m = 1$, then $a_1 \geq 2$ due to the fact that $v : \mathbb{P}^1 \to X$ is a non-trivial morphism. When $m = 2$, we have $a_1 \geq 3$, otherwise $v : \mathbb{P}^1 \to X$ would be unramified by (2.6), which contradicts the assumption that $v$ is ramified when $m = 2$. Therefore in either case, we have $a_1 > m$ assuming the lemma does not hold.

Let $B_v$ be the ramification divisor of the morphism $v$. Then (2.6) implies that $\deg(B_v) = a_1 - 2$, i.e., $\mathcal{O}_{\mathbb{P}^1}(B_v) \cong \mathcal{O}(a_1 - 2)$. When $m = 1$, this last isomorphism is obvious since the image of the tangent bundle homomorphism $T_{\mathbb{P}^1} \cong \mathcal{O}(2)^{2n} v^*T_X$ lies entirely in the first component $\mathcal{O}(a_1)$. When $m = 2$, the image of the homomorphism $T_{\mathbb{P}^1} \cong \mathcal{O}(2)^{2n} v^*T_X$ lies also entirely in the first component $\mathcal{O}(a_1)$ by (2.6) and the assumption that $v : \mathbb{P}^1 \to X$ is ramified. Hence in either case, we have $\mathcal{O}_{\mathbb{P}^1}(B_v) \cong \mathcal{O}(a_1)$ assuming the lemma is not true.

In summary, we have just shown that if we assume that the lemma does not hold, then $a_1 > m$ and $\mathcal{O}(B_v) \cong \mathcal{O}(a_1 - 2)$.

Following the same notations as in the previous section, we denote by $\bar{v} : \mathbb{P}^1 \to \mathbb{P}(T_X)$ the lift of $v$ to $\mathbb{Y}$. Therefore $\bar{v} = \pi \circ \bar{v}$. Let $L = \mathcal{O}_Y(1)$ be the tautological line bundle of $Y$. By (1.5a), $\bar{v}^*L \cong \mathcal{O}(m - a_1)$ since $v^*N \cong \mathcal{O}(m)$. Consider the relative Euler sequence:

$$0 \to \mathcal{O}_Y \to \pi^*\Omega^1_Y \otimes L \to T_{Y/X} \to 0 \quad (2.7)$$

where $T_{Y/X}$ is the relative tangent bundle for $Y = \mathbb{P}(T_X) \to X$. Now (2.7) implies that

$$\bar{v}^*T_{Y/X} \cong \bigoplus_{i=1}^m \mathcal{O}(b_i), \quad \text{where} \quad b_i = m - a_i - a_1 < 0, \quad 1 \leq i \leq n - 1. \quad (2.8)$$

On the one hand, since $a_1 > m$, we have

$$h^0(v^*T_X \otimes \mathcal{O}(2 - a_1)) = 0, \quad (2.9)$$

by (2.6) and (2.8). However $T_{\mathbb{P}^1}$ fits into the following exact sequence

$$0 \to T_{\mathbb{P}^1} \to T_{Y/X} \to \pi^*T_X \to 0. \quad (2.10)$$

Therefore (2.9) implies that $h^0(\bar{v}^*T_{\mathbb{P}^1} \otimes \mathcal{O}(m - 2a_1)) = 0$. 

On the other hand, Lemma 14 implies that \( v(P^1) \) is a null curve since \( h^0(S^2 Q, \otimes v^* N) = h^0(\mathcal{O}(m - 4)) = 0 \) (note that \( m \leq 2 \)). Hence the lift curve \( C = \phi^*(P^1) \) is contained in \( S \subset Y = P(T_X) \), the quadric bundle over \( X \) as defined in the previous section. We see in the previous section that \( N^0 \) is a line subbundle (not just a subsheaf) of \( T_X \). Therefore \( \phi^* N^0 \) is a line subbundle of \( \phi^* T_X \). Note that \( \phi^* N^0 \) is isomorphic to \( \mathcal{O}(2a_1 - m) \) by (1.6a). Hence \( h^0(\phi^* T_X \otimes \mathcal{O}(m - 2a_1)) > 0 \). This contradicts what we have just proved in the previous paragraph. Therefore, if we assume that the lemma is not true, when we arrive at a contradiction. Hence we are done.

By Lemma 2.3, the extremal \( R \) on \( X \) length \( n = \dim X \), i.e., \( l(R) = n \). Let us fix a morphism \( P^1 \to X \) such that \( K_X f(P^1) = -n \). From now on, let us fix once and for all an irreducible component \( V \) of \( \text{Hom}(P^1, X) \) such that \( V \ni f \), where \( \text{Hom}(P^1, X) \) is the scheme that parameterizes all the morphism from \( P^1 \) to \( X \) (see [11]). Let \( \text{Hom}(P^1, X; \{0, x\}) \subset \text{Hom}(P^1, X) \) be the closed subscheme such that a morphism \( v \in \text{Hom}(P^1, X; \{0, x\}) \) if and only if \( v(0) = x \).

Let \( x \in X \) be a point in \( X \). Let \( V_x \) be an arbitrary irreducible component of \( V \cap \text{Hom}(P^1, X; \{0, x\}) \). Let \( G_0 \) be the group of automorphisms of \( P^1 \) fixing the origin. Then \( G_0 \) acts on \( V_x \) by: \( g \circ v(t) \to g^{-1} v \) for any \( v \in V_x \) and \( t \in P^1 \).

Consider the natural morphism:

\[
\Xi : P^1 \times V_x \to X, \quad \Xi(t, v) = v(t).
\]

Let \( D_x \) be the closure of \( \Xi(P^1 \times V_x) \). It will become clear from the proof of Lemma 2.10 that \( \Xi(P^1 \times V_x) \) is the fact closed in \( X \), i.e., taking the closure is unnecessary. By Lemma 2.2, it is clear that \( D_x \) is either a divisor, or \( D_x = X \). We claim that \( D_x \) is necessarily a divisor. If \( D_x = X \), then Lemma 2.1 implies that for a general point \( v \in V_x \), \( a_i \geq 1 \) for all \( i \)'s if \( v^* T_X \cong \bigoplus_{a_i \geq 0} \mathcal{O}(a_i) \). Since \( a_i \geq 2 \), this would imply that \( l(R) = -K_X \cdot v(P^1) \geq n + 1 \), which contradicts the fact that \( l(R) = n \). Therefore \( D_x \) is a divisor.

We will show that \( D_x \) is the divisor we are looking for, i.e., \( D_x \cdot f(P^1) = 1 \). First, let us prove the following.

**Lemma 2.5.** Let \( x \in X \) be a point such that there is a morphism \( f : P^1 \to X \) in \( V \) with the property that \( f^* T_X = \mathcal{O}(2) \oplus \mathcal{O}^{*2} \oplus (1) \oplus \mathcal{O} \) and \( f(0) = x \). (By Lemma 2.7 below, a general point \( x \) of \( X \) always has this property.) Let \( V_x \subset V \cap \text{Hom}(P^1, X; \{0, x\}) \) be an irreducible that contains \( f \), and \( D_x \) is the divisor covered by minimal degree rational curves from \( V_x \). Then there exists a neighborhood (in the classic topology) \( W \) of \( x \in X \) such that \( D_x \cap W \) is biholomorphic to a neighborhood of the vertex of the cone over a smooth hyperquadratic \( \mathbb{Q}^{*2} \) with \( x \) corresponding to the vertex. In particular, \( D_x \) is normal at \( x \).

**Proof.** Fix a representative, which is denoted by \( g_x \), of the conformal metric \( g \) on \( X \) locally around \( x \). As it was pointed out in [9] that, by a theorem of Whitehead [15], we can choose an analytic normal coordinate neighborhood \( W \subset X \) around the given point \( x \) such that complex geodesics (in the metric \( g_x \)) are all affine lines in \( \mathbb{C} \). Let \( z = (z^1, \ldots, z^n) \) be a local coordinate. We further assume that \( x \) is the origin in \( W \). Let \( Q = \{ z \in W | g_0(z, z) = 0 \} \), where \( g_0 = g_x(0) \). Since \( f^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{*2} \oplus (1) \oplus \mathcal{O} \),
we have $\nu^*T_x \cong O(2) \oplus O^{n-2} \oplus (1) \oplus O$ for a general point $v \in V_x$. By Lemma 2.8 below, $v(P^1) \subset X$ is a null geodesic through $x$ for a general point $v \in V_x$. Hence each branch of $v(P^1) \cap W$ is contained in $Q$ for a general $v \in V_x$ since, being a null geodesic, each branch of $v(P^1) \cap W$ is a null affine line in $W$ by our choice of $W$. By definition, $D_x$ is the closure (in Zariski topology) of $\bigcup_{v \in V_x} v(P^1)$. Therefore a dense open subset of $D_x \cap W$ is contained in $Q$. Hence $D_x \cap W \subset Q$, too. But both $Q$ and $D_x \cap W$ have dimensions $n-1$, and $Q$ is an irreducible (since $n \geq 3$) analytic subvariety of $W$. Therefore $D_x \cap W = Q$. This implies the lemma immediately. Hence we are done. \(\square\)

**Lemma 2.6.** Let $X$ be as in Theorem 0.1, then $X$ is a Fano manifold with Picard number one, i.e., $\rho(X) = 1$.

**Proof.** By Lemma 2.3 there exists an extremal ray $R$ on $X$ such that $l(R) = n$. In this case, it is easy to show (see Proposition (2.4) in [16] for a proof) that $X$ is either a Fano manifold with Picard number one, or the associated contraction map $\varphi_R : X \to Z$ sends $X$ to a smooth curve $Z$. Moreover in the later case, a general fiber $F$ is a Fano manifold with Picard number one and having an extremal ray $R_{\nu}$ with $l(R_{\nu}) = n = \dim F + 1$.

Now assume to the contrary that the later case occurs. We will derive a contradiction. Let $x$ be a general point in $X$. Let $F$ be a general fiber of the fibration $\varphi_R : X \to Z$ such that $x \in F$. Hence $F$ can be assumed to be irreducible and it is smooth at $x$. Let $\mathbb{P}^1 \hookrightarrow X$ be a non-constant morphism such that $f^*K_X \cong O(-n)$, $f(0) = x$ and $[f^*(\mathbb{P}^1)] \in R$. Without loss of generality, we may assume that $f \in V$, the fixed component of $\text{Hom}(\mathbb{P}^1, X)$. Let $V_x \ni f$ be an irreducible component of $V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\})$. Let $D_x$ be the divisor covered by minimal degree rational curves from $V_x$, i.e., the closure of $\Xi(\mathbb{P}^1 \times V_x)$. Hence $\varphi_R(D_x) = \varphi_R(x) \in Z$ since $\varphi_R$ is a contraction morphism and $[\nu^*(\mathbb{P}^1)] \in R$ for any $v \in V_x$. Therefore $D_x \subset F$. Since $F$ is a general fiber of $\varphi_R$, $F$ is smooth and irreducible. Hence $D_x = F$ since they have the same dimensions. Thus $D_x$ is smooth at $x$. However, since $x$ is a general point of $X$, by Lemma 2.5 above and Lemma 2.7 below, $D_x$ is locally a quadric cone near the point $x$. Hence $D_x$ is singular at $x$ since $n \geq 3$. This is a contradiction. Hence $X$ must be a Fano manifold with $\rho(X) = 1$. We are done. \(\square\)

Lemma 2.7-2.10 below are devoted to establishing Proposition 2.11. Lemma 2.8 below was also used in the proof of Lemma 2.5 above. Recall again that we let $V \subset \text{Hom}(\mathbb{P}^1, X)$ be a fixed component such that for any $v \in V$, we have $-K_X \cdot v(\mathbb{P}^1) = n$, i.e., $v(\mathbb{P}^1)$ is a minimal degree rational curve.

**Lemma 2.7.** Let $x \in X$ be a general point in $X$. Then the following are true.

1. For an arbitrary morphism $v : \mathbb{P}^1 \to X$ in $V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\})$, $\nu^*T_x$ is semiample, i.e., it is generated by global sections.

2. For an arbitrary point $v : \mathbb{P}^1 \to X$ in $V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\})$, $v : \mathbb{P}^1 \to X$ is unramified.

3. For a general morphism $v : \mathbb{P}^1 \to X$ in $V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\})$, we have $\nu^*T_x \cong O(2) \oplus O^{n-2} \oplus (1) \oplus O$. 
Proof. Let $U \subset V$ consist of all morphisms $f : \mathbb{P}^1 \to X$ such that $f^* T_X$ is semiample, i.e., $f^* T_X$ is generated by its global sections. Then $f \in U$ if and only if $h^1(f^* T_X(-1)) = 0$. By Grauert's semi-continuity theorem, $U \subset V$ is a Zariski open subset of $V$. Since minimal degree rational curves cover a Zariski open subset of $X$ and $x$ is a general point of $X$ by our choice, Lemma 2.1 implies that $U \neq \emptyset$. Let $V^1$ be the complement of $U$, i.e., $V^1 = V \setminus U$. For any $x \in X$, let $\text{Hom}(\mathbb{P}^1, X; \{0, x\}) \subset \text{Hom}(\mathbb{P}^1, X)$ be the closed subscheme such that $v \in \text{Hom}(\mathbb{P}^1, X; \{0, x\})$ if and only if $v(0) = x$. Then the first part of the lemma is equivalent to saying that $V^1 \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) = \emptyset$ for a sufficiently general point $x \in X$.

Consider the following morphism $\Xi : V^1 \times \mathbb{P}^1 \to X$, $\Xi(f, t) = f(t)$, where $f \in V^1$, $t \in \mathbb{P}^1$. Suppose that $V^1_x := V^1 \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \neq \emptyset$ for a general point $x \in X$. Then $\Xi$ is dominant. It is clear that $V^1$ is $\text{Aut}(\mathbb{P}^1)$-invariant. Hence Lemma 2.1 above implies that for a general point $v \in V^1$, we have

$$v^* T_X \cong \bigoplus_{i=0}^n \mathcal{O}(a_i), \quad a_0 \geq \cdots \geq a_n \geq 0.$$ 

Hence $v \in U$. However, $v \in V^1 = V \setminus U$, hence $v \notin U$. We get a contradiction. Therefore $V^1_x = \emptyset$ if $x$ is a sufficiently general point of $X$. Thus the first part of the lemma is proved.

Let us fix a general point $x$ such that $V^1_x = \emptyset$. Let $v \in V$ be an arbitrary point such that $v(0) = x$. Then by what we just proved, we have $v^* T_X \cong \bigoplus_{i=0}^n \mathcal{O}(a_i), a_0 \geq \cdots \geq a_n \geq 0$. Using the conformal structure $T_X \cong \Omega_1(N)$, we see that $a_0 + a_{n-i-1} = 2$ for $i = 1, \ldots, n$. Since $a_i \geq 0$ for any $i$, we see that $a_i \leq 2$ for any $i$. Hence the non-zero homomorphism $\phi_* : T_{\mathbb{P}^1} \cong \mathcal{O}(2) \to v^* T_X$ is everywhere non-zero. In particular, $v : \mathbb{P}^1 \to X$ has to be unramified everywhere on $\mathbb{P}^1$.

For the last part of the lemma, let us note that minimal degree rational curves through $x$ cover a divisor in $X$. Hence Lemma 2.1 implies that for a general morphism $v : \mathbb{P}^1 \to X$ such that $-K_x \cdot v(\mathbb{P}^1) = n$ and $v(0) = x$, if $v^* T_X \cong \bigoplus_{i=0}^n \mathcal{O}(a_i)$, then $a_0 \geq \cdots \geq a_{n-1} \geq 1$. Since we just proved that $a_0 \geq 2$ and $a_0 + a_{n-i-1} = 2$ for any $i$, we get $a_0 = 2$, and $a_i = 1$ for $i$ with $2 \leq i \leq n - 1$ and $a_n = 0$ since $\sum_{i=0}^n a_i = n$. Therefore $v^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-2} \oplus (1) \oplus \mathcal{O}$ if $v$ is general enough.

Hence we are done.

Lemma 2.8. Let $v : \mathbb{P}^1 \to X$ be a morphism such that $v^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-2} \oplus (1) \oplus \mathcal{O}$. In particular, $v$ is unramified. Then $v(\mathbb{P}^1) \subset X$ is a null geodesic with respect to the given conformal structure on $X$.

Proof. Let $C_v = v(\mathbb{P}^1) \subset X$ be the image of $v$ and $C \subset Y$ be the lift of $C_v$ to $Y = \mathbb{P}(T_X)$ as constructed in Sec. 1. Since $N : C_v = 2$, $S^2 \Omega_Y \otimes v^* N \cong \mathcal{O}(-2)$, $C_v$ is a null curve by Lemma 1.4. Hence $C \subset S \subset Y$, the quadric bundle over $X$.

Claim. We have

$$h^0(T_X \otimes \Omega_Y^2) = 1. \quad (2.11)$$

Proof of the claim. Consider the relative Euler sequence:
\[ 0 \to \mathcal{O}_Y \to \pi^*\Omega^1_X \otimes L \to T_{Y/X} \to 0 \] (2.12)

where \( T_{Y/X} \) is the relative tangent bundle for \( \pi: Y = \mathbb{P}(T_X) \to X \), and \( L = \mathcal{O}_Y(1) \) is the tautological line bundle of \( Y \). Note that \( L|_C \cong \mathcal{O} \) by (1.3). Now (2.12) implies that

\[ T_{Y/X}|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(-1). \] (2.13)

Therefore \( h^0(T_{Y/X}|_C \otimes \Omega^1_X) = 1 \) since \( C \cong \mathbb{P}^1 \) and \( T_C \) fits in the following exact sequence:

\[ 0 \to T_{Y/X} \to T_Y \to \pi^*T_X \to 0. \] (2.14)

Since \( C \subset S = T_C \subset T_{Y/X} \). So we get \( h^0(T_{Y/X}|_C \otimes \Omega^1_X) \geq 1 \). However, \( T_S \subset T_{Y/X} \) is a subbundle. Hence \( h^0(T_{Y/X}|_S \otimes \Omega^1_X) \leq h^0(T_{Y/X}|_C \otimes \Omega^1_X) \leq 1 \). Therefore, we get that

\[ h^0(T_{Y/X}|_S \otimes \Omega^1_X) = 1. \] (2.15)

This proves the claim.

Since \( T_{\pi}: \cong \mathcal{O}(2) \cong v^*N \), the above claim and Corollary 1.5 imply the lemma immediately. \( \square \)

Let \( x \in X \) be any point. We use \( V_x \subset V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \) to denote an irreducible component. From now on, we denote by \( V_x^0 \subset V_x \) the open subset of \( V_x \) such that \( v \in V_x^0 \) if and only if \( v^*T_x \cong \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \). Lemma 2.7 implies that for a general point \( x \) and for an arbitrary component \( V_x \), we have \( V_x^0 \neq \emptyset \). In Lemma 2.10 below, we will strengthen Lemma 2.7. Namely, we will show that for a general point \( x \in X, V_x^0 = V_x \). To achieve this, let us first note the following easy corollary of Lemma 2.8.

**Corollary 2.9.** Let \( x_1 \) and \( x_2 \) be two general points in \( X \). Let \( v_i \in V_x^0 \) be a point for \( i = 1, 2 \). If \( v_1(\mathbb{P}^1) \neq v_2(\mathbb{P}^1) \) and \( v_1(\mathbb{P}^1) \cap v_2(\mathbb{P}^1) \neq \emptyset \), then they cannot be tangent to each other at their point(s) of intersection.

**Proof.** Since \( v_i \in V_x^0 \), \( v_i(\mathbb{P}^1) \) is a null geodesic through \( x_i \) for \( i = 1, 2 \) by Lemma 2.8. Hence, if \( v_1(\mathbb{P}^1) \cap v_2(\mathbb{P}^1) \neq \emptyset \), then \( v_1(\mathbb{P}^1) \) and \( v_2(\mathbb{P}^1) \) cannot be tangent to each other at their point(s) of intersection by the uniqueness of the solution of the geodesic equation. \( \square \)

**Lemma 2.10.** Let \( x \in X \) be a general point. Let \( V_x \subset V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \) be an arbitrary component. Let as above \( V_x^0 \subset V_x \) be the open subset of \( V_x \) such that \( v \in V_x^0 \) if and only if \( v^*T_x \cong \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \). Then \( V_x^0 = V_x \).

**Proof.** Let \( v \in V_x \) be an arbitrary point. Then Lemma 2.7 implies that \( v^*T_x \) is semi-ample since \( x \) is a general point of \( X \). Hence \( H^2(v^*T_x(-1)) = 0 \). Therefore Proposition 2 in [11] implies that \( V_x \) is smooth of dimension \( \chi(v^*T_x(-1)) = n \).

Consider the natural morphism \( \alpha: V_x \to \text{Chow}(X) \) such that \( \alpha_x(v) = v(\mathbb{P}^1) \) as an 1-cycle. Let \( T_x = \alpha_x(V_x) \) with the reduced scheme structure. Since \( V_x \) is smooth, \( \alpha_x \)
induces a morphism (which we still denote by $\alpha_x$) $\alpha_x : V_x \to T_x$. Let $Y_x \to T_x$ be the normalization of $T_x$ in the function field of $T_x$. Then $\alpha_x$ induces a morphism $\gamma_x : V_x \to Y_x$.

Claim 1. $\gamma_x : V_x \to Y_x$ is a geometric quotient for the $G_0$-action on $V_x$, where $G_0$ is the group of automorphisms of $\mathbb{P}^1$ that fix the origin $0 \in \mathbb{P}^1$.

Proof of claim 1. This can be proved in essentially the same way as in the proof of Lemma 9 ii) in [11]. However, the difference is that in [11], $Y_x$ is the normalization of $T_x$ in the larger field $\mathbb{C}(V_x)^{G_0} \subset \mathbb{C}(T_x)$, where $\mathbb{C}(V_x)^{G_0}$ is the field of $G_0$-invariant rational functions on $V_x$. What we are claiming here is that $\mathbb{C}(V_x)^{G_0} = \mathbb{C}(T_x)$. The reason for this is that the characteristic of our base field, which is $\mathbb{C}$, is zero. This last statement follows from Proposition 0.2 in [14].

Let us sketch a proof below. First of all, we can assume that there is a point $v \in V_x$ such that $v(\mathbb{P}^1)$ is smooth at $x$ since $x$ is a general point of $X$. Since smoothness is an open condition, for a general point $u \in V_x$, $u(\mathbb{P}^1)$ is also smooth at $x$. Following the argument of [11], we see that every fiber of $\gamma_x : V_x \to Y_x$ consists of precisely one $G_0$-orbit of $V_x$ because $Y_x$ is normal. Hence Proposition 0.2 of [15] implies that $\gamma_x : V_x \to \gamma_x(V_x) \subset Y_x$ is a geometric $G_0$-quotient. Since $v(\mathbb{P}^1)$ is a minimal degree rational curve for every $v \in V_x$, the same argument in [11] implies that $\gamma_x(V_x) = Y_x$, i.e., $\gamma_x$ is surjective. Hence $\gamma_x : V_x \to Y_x$ is a geometric $G_0$-quotient. Thus Claim 1 is proved.

For any point $v \in V_x$, we use $[v] \in Y_x$ to denote the $G_0$-orbit that contains $v$. Hence $\gamma_x(v) = [v]$. Since $Y_x \to T_x$ is the normalization of $T_x$ in the function field of $T_x$, generic two distinct points $y_1$ and $y_2$ in $Y_x$ represent two different minimal degree rational curves through $x$. Consider the morphism

$$\Phi : V_x \to \mathbb{P}(T_x^*) \cong \mathbb{P}^{n-1}, \quad \Phi(v) = (dv)_*(\frac{dt}{dt}), \quad v \in V_x$$

where $t$ is any parameter of $\mathbb{P}^1$ at $0 \in \mathbb{P}^1$. Note that $\Phi$ is well-defined by Lemma 2.7 above. By Lemma 1.4, $\Phi(V_x) \subset Q^{n-2}_x \subset \mathbb{P}^{n-1}$, where $Q^{n-2}_x$ is the (projectivized) null-tangent cone of $X$ at $x$. It is clear that $\Phi$ is a $G_0$-invariant morphism. Hence $\Phi$ induces a morphism $\Psi : Y_x \to Q^{n-2}_x$. By Corollary 2.9 and the fact that two distinct general points of $Y_x$ represent two different minimal degree rational curves, $\Psi$ is birational.

Claim 2. $\Psi : Y_x \to Q^{n-2}_x$ is finite.

Proof of claim 2. This was essentially proved on p. 202 in [10]. If $\Psi$ is not finite, then there is a curve $C \subset Y_x$ such that $\Psi(C)$ is a point. Now let $C_1 \to C$ be the normalization of the completion of $C$. By the usual construction, we have a $\mathbb{P}^1$-bundle $\sigma : E \to C_1$ with a section $S \subset E$. Also we have a morphism $p : E \to X$ such that $p(S) = x$. Then for any $z \in C_1$, $p(\sigma^{-1}(z)) \subset X$ is a minimal degree rational curve through $x$. By Lemma 2.7 above, for any $z \in C_1$, $p(\sigma^{-1}(z))$ is unramified at $x$ since $x$ is a general point in $X$. Since $\Psi(C_1)$ is a point, we see that for all $z \in C_1$, $p(\sigma^{-1}(z))$ share a common non-zero tangent direction at $x \in X$. Let $v_z \in T_{x,x} \setminus \{0\}$ be the non-zero
common tangent direction for this family of minimal degree rational curves. Then \( p^*(v_x) \) is a no-where vanishing section for the normal bundle \( N_{S,E} \). This implies that \( N_{S,E} \cong \mathcal{O}_S \). Hence \( S^2 = 0 \). However, \( S \) is contracted down under \( p : E \to X \). Hence \( S^2 < 0 \). Thus we get a contradiction. Hence \( \Psi : Y_x \to \mathbb{Q}^{s-2}_x \) is finite.

The above Claim 2 implies that \( \Psi : Y_x \to \mathbb{Q}^{s-2}_x \) is finite and birational. Now Zariski’s main theorem implies that \( \Psi \) is isomorphic.

Let \( \mathcal{I}_0 \) be the ideal sheaf of the origin \( 0 \in \mathbb{P}^1 \). Let \( v_x : T_{\mathbb{P}^1} \cong \mathbb{O}(2) \to v^*T_X \) be the natural homomorphism. Then \( v \in V_x \setminus V_x^0 \) if and only if \( H^0(v^*T_X \otimes \mathcal{I}_0^2)/v_xH^0(T_{\mathbb{P}^1} \otimes \mathcal{I}_0^2) \neq 0 \). Suppose that \( V_x \setminus V_x^0 \neq 0 \). Then there is a point \( f \in V_x \) such that \( H^0(f^*T_X \otimes \mathcal{I}_0^2)/f_xH^0(T_{\mathbb{P}^1} \otimes \mathcal{I}_0^2) \neq 0 \). As shown in the proof of Lemma 2.4 in [10], \( \Psi \) is ramified at such a point \( y_x(f) \in Y_x \). Hence \( \Psi \) cannot be isomorphic. Thus \( V_x \setminus V_x^0 = \emptyset \), i.e., \( V_x^0 = V_x \). We are done.

In summary, Lemma 2.7–Lemma 2.10 imply the following important proposition.

**Proposition 2.11.** There is a proper and closed subvariety \( B \subset X \) such that for any point \( x \in X \setminus B \) and any point \( v \in V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \), the following statements hold.

1. \( v^*T_X \cong \mathbb{O}(2) \oplus \mathcal{O}^{s-2} \oplus \mathcal{O} \). In particular, \( v \) is unramified.
2. \( v(\mathbb{P}^1) \subset X \) is a null geodesic.
3. \( V_x \subset V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \) be an arbitrary component. Let \( Y_x \) be the geometric quotient of \( V_x \) under the group action \( G_0 \), then \( Y_x \cong \mathbb{Q}^{s-2}_x \).

Now, we are going to prove Theorem 0.1 assuming Proposition 2.12 below, which will be proved in the next section.

Let \( x \in X \) be an arbitrary point such that \( x \notin B \), where \( B \) is the closed and proper subvariety in \( X \) (see Proposition 2.11). Fix an arbitrary component \( V_x \) of \( V \cap \text{Hom}(\mathbb{P}^1, X; \{0, x\}) \). Consider the \( G_0 \)-invariant morphism

\[
F : \mathbb{P}^1 \times V_x \to Y_x \times X \quad F(t, v) = ([v], v(t)).
\]

Following Mori [11], let us define \( Z_x = \text{Spec}_{X \times X}((F_0 \circ \iota_x \times \pi_x)^{G_0}) \), i.e., \( Z_x = (\mathbb{P}^1 \times V_x)/G_0 \), the geometric quotient of \( \mathbb{P}^1 \times V_x \) under the group \( G_0 \) action. Then \( Z_x \) is a \( \mathbb{P}^1 \)-bundle (in Zariski topology) over \( Y_x \) with a section \( S_x = \{(v), 0\} \subset \mathcal{Z}_x \), where \([v]\) means the \( G_0 \)-orbit of \( v \). Let \( \phi : Z_x \to Y_x \) be the morphism that makes \( Z_x \) into a \( \mathbb{P}^1 \)-bundle over \( Y_x \). As we see in the proof of Lemma 2.10, \( Y_x \cong \mathbb{Q}^{s-2}_x \). Hence \( Z_x \) is also smooth since \( Y_x \) is. The above morphism \( F \) induces a morphism \( \Pi : Z_x \to Y_x \times X \) since \( F \) is \( G_0 \)-invariant. Let \( \psi = \pi_2 \circ \Pi \), where \( \pi_2 \) is the second projection for \( Y_x \times X \). Then \( \psi : Z_x \to D_x \subset X \) is surjective and the section \( S_x \) is contracted to the point \( x \). By the breaking-up technique, \( \psi \) is finite away from the section \( S_x \). It is easy to see that \( \psi \) is unramified away from \( S_x \) by the first part of Proposition 2.11.

**Proposition 2.12.** In terms of the above notations, for a general point \( x \in X \), \( \psi : Z_x \to D_x \subset X \) is isomorphic away from the section \( S_x \subset Z_x \).

We will prove this proposition in the next section. Assuming it at this moment, we prove Theorem 0.1. At this point, there are several ways to prove Theorem 0.1. One
way is to use Lefschetz's Hyperplane Theorem as pointed out by Kollár. In what follows, we take a different approach.

**Proof of Theorem 0.1.** Let us recall that the index of $X$ (denoted by index($X$)) is the largest positive integer that divides $K_X$ in the Picard group Pic($X$). By the theorem of Kobayashi-Ochiai [7], if index($X$) = $n$, then $X \cong \mathbb{Q}^n$. By Lemma 2.3 and Lemma 2.6, $X$ is a Fano manifold of Picard number one, and the unique extremal ray $R$ has length $n$, i.e., $l(R) = n$. Therefore to prove Theorem 0.1, it suffices to show that $D_x \cdot v(\mathbb{P}^1) = 1$ for a $v \in V_x$. This is what we are going to prove.

Let $\sigma : \tilde{X} \to X$ be the blowing-up of $X$ at $x$. Let $E$ be the exceptional locus of $\sigma$. By Proposition 2.12 the morphism $\psi : Z_x \to D_x \subset X$ is isomorphic away from $S_x$. Hence we can lift $\psi$ to a morphism $\tilde{\psi} : Z_x \to \tilde{X}$. Moreover, $\tilde{\psi}$ maps $Z_x$ isomorphically onto its image. It is clear that $\tilde{\psi}(Z_x)$ is the proper transform $\tilde{D}_x$ of $D_x$ under the blowing-up $\sigma : \tilde{X} \to X$.

Now choose an arbitrary point $v \in V_x$ and let $C = v(\mathbb{P}^1)$. Let $\tilde{v} : \mathbb{P}^1 \to \tilde{X}$ be the lift of $v$ to $\tilde{X}$, and denote $\tilde{v}(\mathbb{P}^1)$ by $\tilde{C}$. Then both $\tilde{C}$ and $C$ are smooth since $\psi : Z_x \to D_x \subset X$ is isomorphic away from the section $S_x$. Note also that $\tilde{C}$ is the proper transform of the curve $C$. Since $\tilde{D}_x$ is isomorphic to $Z_x$, we have

$$N_{\tilde{C}/\tilde{D}_x} \cong \bigoplus^{n-2} \mathcal{O}.$$  

(2.16)

It is easy to see that the following sequence is exact:

$$0 \to T_{\tilde{X}} \xrightarrow{d_{\tilde{X}}} \sigma^* T_X \to \bigoplus^{n-1} \mathcal{O}_E(-E) \to 0.$$  

This last exact sequence implies that

$$N_{\tilde{C}/\tilde{X}} \cong \sigma^* N_{C/X}(-1)$$  

(2.17)

because $E \cdot \tilde{C} = 1$ ($C$ is smooth!). By Proposition 2.11, $v^* T_X \cong \mathcal{O}(2) \oplus \bigoplus^{n-2} \mathcal{O}(1) \oplus \mathcal{O}$. Therefore (2.17) implies that

$$N_{\tilde{C}/\tilde{X}} \cong \bigoplus^{n-2} \mathcal{O} \oplus \mathcal{O}(-1).$$  

(2.18)

Now let us consider the following exact sequence:

$$0 \to N_{\tilde{C}/\tilde{D}_x} \to N_{\tilde{C}/\tilde{X}} \to N_{\tilde{D}_x/\tilde{X}} \to 0.$$  

(2.19)

In view of (2.16) and (2.18), (2.19) implies that $N_{\tilde{D}_x/\tilde{X}} \cong \mathcal{O}(-1)$. Hence

$$\tilde{D}_x \cdot \tilde{C} = -1.$$  

(2.20)

Since $D_x$ is locally isomorphic to a quadric cone by Lemma 2.5, its multiplicity at $x$ is exactly two. Hence:
\[ \sigma^* D_x = \tilde{D}_x + 2E. \]  

(2.21)

Since \( C \) is smooth at \( x \), \( E \cdot \tilde{C} = 1 \). Now (2.20) and (2.21) imply that \( \sigma^* D_x \cdot \tilde{C} = 1 \), hence \( D_x \cdot C = 1 \) by Projection Formula. Since \( \rho(X) = 1 \) by Lemma 2.6, \( D_x \) must be the positive generator for \( \text{Pic}(X) \). Let \( K_X = -mD_x \) for some \( m \in \mathbb{Z}_+ \). Since \( K_X \cdot C = -n \) by Lemma 2.3, \( m = n \). Hence \( K_X = -nD_x \), i.e., index\((X) = n \) since index\((X) \leq n + 1 \). Now by a theorem of Kobayashi-Ochiai [7], \( X \cong \mathbb{Q}^n \). Hence we complete the proof of Theorem 0.1.

3. The Proof of Proposition 2.12

This section is devoted to showing Proposition 2.12 above. As we showed at the end of the previous section that this is enough to prove Theorem 0.1. We will adopt the same notations as those in the previous sections. In particular, \( X \) is a projective conformal manifold with non-nef canonical bundle, and \( V \subset \text{Hom}(\mathbb{P}^1, X) \) is a fixed component.

To start with, let us introduce some definitions.

Definitions. Let \( f : \mathbb{P}^1 \to X \) be a morphism such that \( f(\mathbb{P}^1) \) is a minimal degree rational curve. \( f(\mathbb{P}^1) \) is called a cuspidal curve if \( f \) is ramified. \( f(\mathbb{P}^1) \) is called a nodal curve if it is singular and \( f \) is unramified.

When there is no danger of confusion, we simply shorten nodal (resp. cuspidal) minimal degree rational curve of \( X \) to nodal (resp. cuspidal) curve.

The essential part of the proof of Proposition 2.12 is to show that there is no nodal minimal degree rational curve through a general point of \( X \). Once we prove this last statement, Proposition 2.12 follows easily (see Proposition 3.2 below). The strategy for ruling out nodal curves through a general point of \( X \) is as follow:

Step I. By Lemma 2.4, cuspidal curves cover a codimension at least two subset of \( X \). This essential step in our proof requires the existence of conformal structure, as we see clearly from the proof of Lemma 2.4.

Step II. If there is a nodal curve through a general point of \( X \), then we can find a point \( x \in X \) and a minimal degree rational curve \( f_0 : \mathbb{P}^1 \to X \) such that \( f_0^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-2} \oplus (1) \oplus \mathcal{O} \), and \( x = f_0(0) \) is a nodal point of \( f_0(\mathbb{P}^1) \). Furthermore, no cuspidal curve through \( x \). This step is achieved by Step I and Proposition 2.11.

Step III. Once Step II is finished, by studying the deformations of \( f_0 \) and using ideas in the proof of Proposition 3.2, we can find infinitely many nodal curves through \( x \). An easy lemma implies that we also have a cuspidal curve through \( x \), which contradicts the choice of \( x \).

Here is the basic logical relationships among the main results of this section:

\[ \text{Proposition 3.2} \quad + \quad \text{Proposition 3.7} \quad \Rightarrow \quad \text{Proposition 2.12} \quad \Rightarrow \quad \text{Theorem 0.1} \]
A key technical idea in the following proofs is to study the intersection between a correspondence subvariety in $\mathbb{Q}^{n-2} \times \mathbb{Q}^{n-2}$ and the diagonal of $\mathbb{Q}^{n-2} \times \mathbb{Q}^{n-2}$.

To start with, let us prove the following lemma.

**Lemma 3.1.** Let $\mathbb{Q}^k \subset \mathbb{P}^{k+1}$ be a smooth hyperquadric in $\mathbb{P}^{k+1}$ $(k \geq 2)$. Let $\Gamma \subset \mathbb{Q}^k \times \mathbb{Q}^k$ be a subvariety such that $p_1|\Gamma : \Gamma \to \mathbb{Q}^k$ is surjective, where $p_1 : \mathbb{Q}^k \times \mathbb{Q}^k \to \mathbb{Q}^k$ is the projection onto the first factor. Let $\Delta \subset \mathbb{Q}^k \times \mathbb{Q}^k$ be the diagonal. Then $\Gamma \cap \Delta \neq \emptyset$.

**Proof.** We will prove the lemma when $k$ is even, which is the case that we need later. When $k$ is odd, the same proof works, except for some minor changes.

Let $m = \dim \Gamma$. Then by our assumption on $\Gamma$, $m \geq k$. Let $\{H_1, \ldots, H_{m-k}\}$ be a set of general sections of $p_2^*O_{\mathbb{Q}^k}(1)$. Let $\Gamma_H := \Gamma \cap H_1 \cap \cdots \cap H_{m-k}$. Then $\dim \Gamma_H = k$. To prove the lemma, it suffices to show that $\Gamma_H$ has non-empty intersection with $\Delta$, i.e., $\Gamma_H \cap \Delta \neq \emptyset$. By choosing $\{H_1, \ldots, H_{m-k}\}$ to be general enough, we can assume that $p_1|\Gamma_H : \Gamma_H \to \mathbb{Q}^k$ is also surjective since $p_1|\Gamma : \Gamma \to \mathbb{Q}^k$ is surjective. Since $\dim \Gamma_H = k$, $p_1|\Gamma_H : \Gamma_H \to \mathbb{Q}^k$ is surjective and generically finite.

Let $[h] \in H^2(\mathbb{Q}^k, \mathbb{Z})$ be the class of hyperplane section of $\mathbb{Q}^k \subset \mathbb{P}^{k+1}$. By Lefschetz's Hyperplane Theorem, we see that $H^i(\mathbb{Q}^k, \mathbb{Z}) \cong \mathbb{Z}[h^{1/2}]$ for $i$ even and $i \neq k$, and it is zero for $i = odd$. By a computation on page 518 of [1], $H^k(\mathbb{Q}^k, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. It is well-known that when $k$ is even, $\mathbb{Q}^k$ has two families of isotropic $\mathbb{P}^{k/2}$s. Let $F_1$ and $F_2$ be two isotropic $\mathbb{P}^{k/2}$s belonging to different families. It is clear that (see Chapter 6 of [5]) either $F_1^2 = F_2^2 = 0$ and $F_1 \cdot F_2 = 1$, or else $F_1^2 = F_2^2 = 1$ and $F_1 \cdot F_2 = 0$ depending on $k/2$ being odd or even. In either case, we have $H^k(\mathbb{Q}^k, \mathbb{Z}) \cong \mathbb{Z} [F_1] \oplus \mathbb{Z} [F_2]$. Hence for $0 \leq i \leq 2k$, we have

$$H^i(\mathbb{Q}^k, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[h^{1/2}], & \text{for } i = even, i \neq k. \\ \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2], & \text{for } i = k. \end{cases}$$

(3.1)

By Lefschetz's Fixed Point Formula (see page 306 in [4]) and (3.1), we have

$$\Delta \cdot \Gamma_H = \sum_{i=0}^{2k} (-1)^i \text{Tr}(\gamma^*|_{H^i(\mathbb{Q}^k, \mathbb{Z})}) = \sum_{i=0}^{2k} \text{Tr}(\gamma^*|_{H^i(\mathbb{Q}^k, \mathbb{Z})})$$

(3.2)

where $\gamma(x) = p_1(x) x^* \Gamma_H$ for any $x \in H^i(\mathbb{Q}^k, \mathbb{Z})$ and any $i$. Since the tangent bundle of $\mathbb{Q}^k$ is generated by global sections, so is the tangent bundle of $\mathbb{Q}^k \times \mathbb{Q}^k$. By Theorems 12.2 in Fulton's book [4], the intersection of two effective algebraic cycles of $\mathbb{Q}^k \times \mathbb{Q}^k$ is still an effective algebraic cycle. By (3.1), the group $H^i(\mathbb{Q}^k, \mathbb{Z})$ is generated by effective algebraic cycles for any $i$. Therefore, Fulton's Theorem implies that

$$\text{Tr}(\gamma^*|_{H^i(\mathbb{Q}^k, \mathbb{Z})}) \geq 0 \quad \text{for any } i.$$  

(3.3)

It is clear that $\text{Tr}(\gamma^*|_{H^k(\mathbb{Q}^k, \mathbb{Z})}) = \deg(p_1|\Gamma_H) > 0$. Hence (3.2) and (3.3) imply that $\Delta \cdot \Gamma_H > 0$. Thus $\Delta \cap \Gamma_H \neq \emptyset$. Therefore $\Gamma \cap \Delta \neq \emptyset$ also since $\Gamma_H \subset \Gamma$. Hence we are done. 

\hfill $\square$
Proposition 3.2. Let $B \subset X$ be the proper and closed subvariety in Proposition 2.11. If there is a point $x \in X \setminus B$ such that there is no singular minimal degree rational curve through $x$, then Proposition 2.12 is true.

Proof. We will use the same notations as those in the previous section. In particular, let $V_x = V \cap \text{Hom}(P^1, X; \{0, x\})$ be an irreducible component, where $V$ is the fixed component of $\text{Hom}(P^1, X)$. By our assumption, $v(P^1)$ is smooth for any $v \in V_x$. Let, as before, $Y_x = V_x/G_0$ be the geometric quotient. Then as we see in the proof of Lemma 2.10, $Y_x \cong Q^k$, where $k = n - 2$. As we remarked at the beginning of this paper, we can assume that $n$ is even. Hence $n \geq 4$, and $k$ is also even and $k \geq 2$.

Let $Z_x = (P^1 \times V_x)/G_0$ be the geometric quotient, and $\phi: Z_x \to Y_x$ be the natural $P^1$-bundle structure. Let $\psi: Z_x \to D_x \subset X$ be the natural morphism such that the section $S_x \subset Z_x$ is contracted to $x$. For any point $\xi \in Y_x$, let $\xi_x := \psi(\phi^{-1}(\xi)) \subset X$ be the minimal degree rational curve represented by $\xi \in Y_x$.

Consider the following correspondence

$$
\Gamma := \left\{ (\xi, \eta) \bigg| \xi_x \cap \iota_\eta \supseteq \{x\} \right\} \subset Y_x \times Y_x \cong Q^k \times Q^k.
$$

It is clear that $\Gamma \supset \Delta$, the diagonal of $Y_x \times Y_x \cong Q^k \times Q^k$.

Claim 1. Suppose that $\psi: Z_x \to D_x \subset X$ is not isomorphic away from the section $S_x$. Then there is an irreducible component $\Gamma_1 \subset \Gamma$ such that $\Gamma_1 \neq \Delta$ and $p_1|_{\Gamma_1}: \Gamma_1 \to Y_x \cong Q^k$ is surjective.

Proof of Claim 1. If $\psi$ is not isomorphic away from the section $S_x$, then we have two cases: either $\deg(\psi) \geq 2$, or else $\psi$ is birational but not isomorphic away from $S_x$. In the first case, i.e., $\deg(\psi) \geq 2$, Claim 1 is obvious since each fiber of $\phi$ is mapped birationally onto its image. In the second case, let $\text{cond}_x$ be the conductor for $\psi$. Then certainly $\text{cond}_x \supseteq S_x$ since $\psi$ is not isomorphic away from $S_x$. Let $\Sigma \subset \text{cond}_x$ be the union of components of $\text{cond}_x$ that are different from $S_x$. Hence $\text{cond}_x = S_x \cup \Sigma$ and $\Sigma$ is non-empty of pure codimension one in $Z_x$. By Lemma 2.5, $D_x$ is normal at $x$. By Zariski's Main Theorem, we have a Zariski open neighborhood $U \supset S_x$ such that $\psi|_{U \setminus S_x}$ is isomorphic onto its image. Hence $\Sigma \cap S_x = \emptyset$. Therefore $\phi(\Sigma) = Y_x$ since $\Sigma$ is of codimension one in $Z_x$. Note that $\psi|_{U}: \Sigma \to \phi(\Sigma) \subset D_x$ is finite and unramified, but not isomorphic since $\Sigma \subset \text{cond}_x$. Hence $\#(\psi^{-1}(\phi(\sigma)) \geq 2$ for any $\sigma \in \Sigma$, where $\#(\psi^{-1}(\phi(\sigma))$ means the number of points in $\psi^{-1}(\phi(\sigma))$. For any $\sigma \in \Sigma$, we consider the set $M_\sigma := \phi(\psi^{-1}(\phi(\sigma))) \subset Y_x$. Then it is clear that $\# M_\sigma = 1$ if and only if $\iota_\eta \phi(\sigma)$ is a nodal curve through $x$. Since there is no singular curve through $x$ by our assumption, for any point $\sigma \in \Sigma$, $\# M_\sigma \geq 2$. Since $\phi(\Sigma) = Y_x$, we have that for any point $\xi \in Y_x$, there is a point $\eta \in Y_x$ such that $\eta \neq \xi$, and $\xi_x \cap \iota_\eta \supseteq \{x\}$, i.e., $(\xi, \eta) \in \Gamma \setminus \Delta$. This implies that $p_1|_{\Gamma \setminus \Delta}: \Gamma \setminus \Delta \to Y_x$ is surjective. This is enough to give Claim 1. Thus Claim 1 is proved.

Now by Lemma 3.1, we have that $\Gamma_1 \cap \Delta \neq \emptyset$.

Claim 2. Let $(\eta, \eta) \in \Gamma_1 \cap \Delta$ be an arbitrary point. The minimal degree rational curve $\iota_\eta$ is singular.
Note that Claim 2 implies the proposition immediately since there is no singular minimal degree rational curve through $x$ by our assumption. Therefore, it suffices to prove Claim 2.

**Proof of Claim 2.** Since for any $v \in V_s$, $v^*T_X \cong \mathcal{O}(2) \oplus \oplus^{n-2} \mathcal{O}(1) \oplus \mathcal{O}$, $\psi: Z_x \rightarrow D_x$ is finite and unramified away from the section $S_x$. Hence $\psi$ is locally isomorphic (away from $S_x$) in the classic topology. Suppose that $\ell_x$ is a smooth curve, i.e., $\psi|_{\phi^{-1}(\eta)}: \phi^{-1}(\eta) \rightarrow \ell_x$ is isomorphic. Then there exists a sufficiently small classic open neighborhood $U_\eta \subset \eta$ in $Y_x$ such that $\psi|_{\phi^{-1}(U_\eta)}$ is one-to-one onto its image away from $S_x \cap \phi^{-1}(U_\eta)$. Therefore, by the definition of $\Gamma, (\Gamma \setminus \Delta) \cap (U_\eta \times U_\eta) = \emptyset$. Since $\Gamma_1$ is a component of $\Gamma$ and $\Gamma_1 \neq \Delta, (\eta, \eta) \notin \Gamma_1 \cap \Delta$. This is a contradiction since we assume that $(\eta, \eta) \in \Gamma_1 \cap \Delta$. Hence $\ell_x$ has to be a singular minimal degree rational curve through $x$. Thus Claim 2 is proved, hence Proposition 3.2 is also proved.

Therefore to prove Proposition 2.12, it is enough to show that through a general point in $X$, every minimal degree rational curve is smooth. By Proposition 2.11, every minimal degree rational curve through $x$ is unramified. Hence we have to show that through a general point of $X$, there is no nodal minimal degree rational curve. Since the limit of a family of nodal curves through a given point is a cuspidal curve (see Lemma 3.6 below), we need to control the dimension of cuspidal curves. For this, we have the following lemma, which is an immediate consequence of Lemma 2.4.

**Lemma 3.3.** Let $B_0 \subset X$ be the locus of cuspidal minimal degree rational curves in $X$. Then the codimension of $B_0$ is at least two.

**Proof.** This is an easy consequence of Lemma 2.4 above. Suppose that the codimension of $B_0$ is at most one. Then by Lemma 2.1, for a general cuspidal minimal degree rational curve $C \subset X$, we have a ramified morphism $v: \mathbb{P}^1 \rightarrow X$ such that $v(\mathbb{P}^1) = C$, and $v^*T_X \cong \oplus_{i=1}^n \mathcal{O}(a_i)$ with $a_1 \geq \cdots \geq a_{n-1} \geq 0$. In particular, $a_{n-1} \geq 0$. Since $v(\mathbb{P}^1) \cdot N = 2$, we get a contradiction by Lemma 2.4. So $B_0$ has to be of codimension at least two.

**Corollary 3.4.** Suppose that through a general point in $X$, there is a nodal minimal degree rational curve. Then there is a point $x \in X$ with the following properties:

1. There is a minimal degree rational curve $f_0: \mathbb{P}^1 \rightarrow X$ such that $f_0(0) = x$ and $f_0^*T_X \cong \mathcal{O}(2) \oplus \oplus^{n-2} \mathcal{O}(1) \oplus \mathcal{O}$.
2. $f_0(\mathbb{P}^1)$ has a nodal singularity at $x$, and through $x$, there is no cuspidal minimal degree rational curve.

**Proof.** By assumption, there is at least one nodal minimal degree rational curve through a general point of $X$. Hence the set of nodal curves cover a Zariski open subset of $X$. By Lemma 3.5 below, there are only finitely many minimal degree rational curves which have nodal singularity at a given point. An easy dimension count yields that the set of singular points of all nodal minimal degree rational curves is of codimension at most one. Let us denote this set by $B_1$, i.e., $B_1$ is the set of nodal points of all nodal minimal degree rational curves. Then the codimension of $B_1$ is at most one in $X$. Recall that $B_0$ is the locus of cuspidal minimal degree rational curves on $X$. By Lemma 3.3
above, $B_0$ is of codimension at least two in $X$. Hence $B_1 \setminus B_0 \neq \emptyset$. By our assumptions, for a general point $x_1 \in X$, there is a minimal degree rational curve $f_{x_1} : \mathbb{P}^1 \to X$ such that $x_1 \in f_{x_1}(\mathbb{P}^1)$ and $f_{x_1}(\mathbb{P}^1)$ has a nodal singularity, i.e., $f_{x_1}(\mathbb{P}^1) \cap B_1 \neq \emptyset$ by the definition of $B_1$. If we choose $x_1$ to be sufficiently general, then we have $f_{x_1}(\mathbb{P}^1) \cap (B_1 \setminus B_0) \neq \emptyset$.

Now fix a general point $x_1 \in X$ such that there is a minimal degree rational curve $f_{x_1} : \mathbb{P}^1 \to X$ such that $f_{x_1}(\mathbb{P}^1) \cap (B_1 \setminus B_0) \neq \emptyset$. Let $x \in f_{x_1}(\mathbb{P}^1) \cap (B_1 \setminus B_0)$ be an arbitrary point. We denote $f_{x_1}$ by $f_0$, and, up to an automorphism of $\mathbb{P}^1$, we can assume that $f_0(\mathbb{P}^1) \ni x_1$ and $x_1$ is a sufficiently general point of $X$, we have $f_0^* T_x \cong \mathcal{O}(2) \oplus \mathcal{O}^{s-2} \oplus \mathcal{O}(1) \oplus \mathcal{O}$ by Proposition 2.11. Since $x \in f_0(\mathbb{P}^1) \cap (B_1 \setminus B_0)$, $f_0(\mathbb{P}^1)$ has a nodal singularity at $x$, and there is no cuspidal minimal degree rational curve through $x$. Thus the corollary is proved.

Before we go on, we need the following two easy lemmas, which are also proved in [2].

**Lemma 3.5.** Let $x \in X$ be an arbitrary point. There are only finitely many minimal degree rational curves with nodal singularities at $x$.

**Proof.** Suppose that there are infinitely many minimal degree rational curves through $x$ and have nodal singularities at $x$. Then by the usual construction, we arrive at the following situation. We have a $\mathbb{P}^1$-bundle $\pi : W \to C$ over a smooth complete curve $C$ and a morphism $p : W \to X$ such that $p(W)$ is a surface and $p(S) = x$, where $S \subset W$ is a section for $\pi$. Let $D = p^{-1}(x)$. Then $D$ is a closed subvariety of $W$, and $D \supset S$. Let $D_1 = D \setminus S$. Then $D_1$ cannot contain any fiber of $\pi$ since each fiber is mapped to a minimal degree rational curve through $x$. Note also that $D_1$ intersects a general fiber of $\pi$ since for a general point $t \in C$, $p(\pi^{-1}(t))$ has a nodal singularity at $x$. Hence $p(D_1) = C$. In particular, $D_1$ contains a horizontal curve, say $D_2$. It is clear that $D_2 \neq S$ and $p(D_2) = x$. By passing to a base change of $C$ (if necessary), we can assume that we have two different sections $S$ and $S_1$ such that they are both contracted to $x$ under $p : W \to X$. Hence $S^2 < 0$ and $S_1^2 < 0$. Therefore $(S - S_1)^2 = S^2 - 2S \cdot S_1 + S_1^2 < 0$ since $S \cdot S_1 \geq 0$ (remember $S \neq S_1$). However, $S - S_1$ is numerically equivalent to a multiple of a fiber of $\pi$. Hence $(S - S_1)^2 = 0$, which contradicts the fact that $(S - S_1)^2 < 0$. Hence the lemma is proved.

We will also need the following lemma.

**Lemma 3.6.** Let $x \in X$ be an arbitrary point. Suppose that there are infinitely many nodal minimal degree rational curves through $x$. Then there is also a cuspidal minimal degree rational curve through $x$.

**Proof.** By the usual construction as we did in the proof of Lemma 3.5 above, we have a $\mathbb{P}^1$-bundle $\pi : W \to C$ over a smooth complete curve $C$ and a morphism $p : W \to X$ such that $p(W)$ is a surface and $p(S) = x$, where $S \subset W$ is a section for $\pi$. By our assumption, for a general point $t \in C$, $p(\pi^{-1}(t))$ is a nodal minimal degree rational curve through $x$. Therefore, after passing to a base change of $C$, we can assume that there are two different sections $S_1$ and $S_2$ of $\pi : W \to C$ such that $p(S_1 \cap \pi^{-1}(t)) = \emptyset$. By our assumption, for a general point $t \in C$, $p(\pi^{-1}(t))$ is a nodal minimal degree rational curve through $x$. Therefore, after passing to a base change of $C$, we can assume that there are two different sections $S_1$ and $S_2$ of $\pi : W \to C$ such that $p(S_1 \cap \pi^{-1}(t)) = \emptyset$. By our assumption, for a general point $t \in C$, $p(\pi^{-1}(t))$ is a nodal minimal degree rational curve through $x$. Therefore, after passing to a base change of $C$, we can assume that there are two different sections $S_1$ and $S_2$ of $\pi : W \to C$ such that $p(S_1 \cap \pi^{-1}(t)) = \emptyset$. By our assumption, for a general point $t \in C
For any point $t \in C$, $p(S_t \cap \pi^{-1}(t))$ is a nodal point of $p(\pi^{-1}(t))$ for a general point $t \in C$. By Lemma 3.5 above, there are only finitely many minimal degree rational curves with a node at $x$, we conclude that $S_t \neq S$ for $i = 1, 2$. Hence $S_t \cap S \geq 0$ for $i = 1, 2$. Let $F$ be any fiber of $\pi$. Then numerically, we have $S_t = S + a_t F$. Since $S_t \cdot S \geq 0$, we have that $a_t \geq -S^2$ for $i = 1, 2$. Hence $S_1 \cdot S_2 = S^2 + a_1 + a_2 \geq -S^2 > 0$ since $S$ is contracted down under $p: W \to X$. Therefore $S_1 \cap S_2 \neq \emptyset$.

**Claim.** Let $w \in S_1 \cap S_2$ be an arbitrary point, and $t_w = \pi(w) \in C$. Then $p(\pi^{-1}(t_w))$ has a cuspidal singularity at $p(w)$.

**Proof of the claim.** This is local question around $w$. Let us choose a local analytic coordinate neighborhood $w \in U \subset W$ with coordinate $(u, v)$ so that $w = (0, 0)$. Assume that the restriction $\pi|_U: U \to \pi(U) \subset C$ sends $(u, v)$ to $u$. Since $S_i$ is a section for $\pi$, we can assume that $S_i \cap U = \{v = f_i(u)\}$ for $i = 1, 2$, and an analytic function $f_i(u)$ of $u$. Since general minimal degree rational curves in this family have nodal singularity, (by choosing $U$ small enough) we may assume that $f_1(u) \neq f_2(u)$ unless $u = 0$. In terms of these coordinates, the morphism $p: W \to X$ locally around $w$ can be thought of as a vector-valued analytic function of $(u, v)$. Since $p(S_1 \cap \pi^{-1}(t_w)) = p(S_2 \cap \pi^{-1}(t_w))$ for any point $t \in C$, we conclude that $p(u, f_1(u)) = p(u, f_2(u))$ for any $u$. This implies that

$$
\frac{\partial p}{\partial v}(0, 0) = \lim_{u \to 0} \frac{p(u, f_1(u)) - p(u, f_2(u))}{f_1(u) - f_2(u)} = 0. \quad (3.4)
$$

Recall that $t_w = \pi(w) \in C$. Since $t_w$ corresponds to $u = 0$ in these local coordinates, (3.4) above implies that $p|_{\pi^{-1}(t_w)}$ is ramified at $w$, i.e., $p(\pi^{-1}(t_w))$ has a cuspidal singularity at $p(w)$. The claim is proved.

Now the claim implies the lemma immediately. We are done.

Next, we show that conditions in Proposition 3.2 are always satisfied. Therefore Proposition 2.12, hence Theorem 0.1, is proved provided that the following lemma is established. The essential fact that makes the following proof possible is that $D_x$ is normal at $x$ (by Lemma 2.5 above) if there is a minimal degree rational curve through $x$ with the "expected" tangent bundle decomposition, i.e., $\mathcal{O}(2) \oplus \mathcal{O}^{-1} \oplus \mathcal{O}(1) \oplus \mathcal{O}$. Once we have normality of $D_x$ at $x$, we can use Zariski’s theorem to conclude that $D_x$ is topologically unibranch at $x$.

**Proposition 3.7.** Through a general point of $X$, there is no singular minimal degree rational curve.

In view of our comments above, this proposition together with Proposition 3.2 implies Proposition 2.12, which in turn implies Theorem 0.1.

**Proof.** In view of Proposition 2.11, the lemma is equivalent to saying that through a general point of $X$, there is no nodal minimal degree rational curve. Suppose that through a general point of $X$ there exists a nodal minimal degree rational curve. Then Corollary 3.4 implies that there is a point $x \in X$ such that the following two conditions hold:
(1) There is a minimal degree rational curve \( f_0 : \mathbb{P}^1 \to X \) such that \( f_0(0) = x \) and \( f_0^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{-2} \oplus \mathcal{O}(1) \oplus \mathcal{O}. \)

(2) \( f_0(\mathbb{P}^1) \) has a nodal singularity at \( x \), and through \( x \), there is no cuspidal minimal degree rational curve.

The basic strategy for the following proof is this. We will show that there are infinitely many nodal minimal degree rational curves through the chosen point \( x \). Thus Lemma 3.6 implies that there must be a cuspidal curve through \( x \), which contradicts our choice of \( x \).

Let \( V_x \ni f_0 \) be an irreducible component of \( \text{Hom}(\mathbb{P}^1, X ; \{0, x\}) \). We should note that we do not know if for any point \( v \in V_x \), \( v^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{-2} \oplus \mathcal{O}(1) \oplus \mathcal{O} \). In other words, we do not know if \( x \) belongs to \( X \setminus B \), where \( B \) is the closed subvariety of \( X \) defined in Proposition 2.11. However, we believe that the same argument in Lemma 2.10 should conclude that every minimal degree rational curve has the expected tangent bundle decomposition. But we do not need this to complete the following proof.

As before, let \( V_x^0 \subset V_x \) be the Zariski open subset of \( V_x \) such that \( v \in V_x^0 \) if and only if \( v^* T_X \cong \mathcal{O}(2) \oplus \mathcal{O}^{-2} \oplus \mathcal{O}(1) \oplus \mathcal{O} \). Then \( V_x^0 \neq \emptyset \) since \( f_0 \in V_x^0 \). Lemma 3.5 implies that for a general point \( f \in V_x \), \( f(\mathbb{P}^1) \) is smooth at \( x \). As we did in the proof of Lemma 2.10, the geometric quotients \( V_x^0/G_0 \) and \( (\mathbb{P}^1 \times V_x^0)/G_0 \) both exist. Let us denote \( V_x^0/G_0 \) by \( X_x^0 \) and \( (\mathbb{P}^1 \times V_x^0)/G_0 \) by \( Z_x^0 \). Then \( X_x^0 \subset \mathbb{P}^n \) is a Zariski open subset. We also have two morphisms \( \phi : Z_x^0 \to X_x^0 \), which is a \( \mathbb{P}^1 \)-bundle, and a morphism \( \psi : Z_x^0 \to D_x \subset X \), which contracts a section \( S_x^0 \) to \( x \). Recall that \( D_x \) is the union of all minimal degree rational curves through \( x \). It is clear that \( \psi : Z_x^0 \to D_x \) is dominant. By the definition of \( V_x^0, \psi \) is finite and unramified away from the section \( S_x^0 \).

As in the proof of Proposition 3.2, for any point \( \xi \in X_x^0 \), we let \( \ell_x = \psi(\phi^{-1}(\xi)) \) be the minimal degree rational curve represented by \( \xi \). Consider the correspondence.

\[
\Gamma^0 := \left\{ (\xi, \eta) \mid \ell_x \cap \ell_\eta \supseteq \{x\} \right\} \subset Y_x^0 \times Y_x^0 \subset \mathbb{Q}^k \times \mathbb{Q}^k \quad (3.5)
\]

where, for simplicity, we let \( k = n - 2 \). It is clear that \( \Gamma^0 \) contains a Zariski open subset of the diagonal \( \Delta \subset \mathbb{Q}^k \times \mathbb{Q}^k \). We claim that:

**Claim.** There is a component \( \Gamma^0_1 \subset \Gamma^0 \) different from \( \Delta \cap \Gamma^0 \) such that \( ([f_0],[f_0]) \in \Gamma^0_1 \cap \Delta \) and \( \dim_{\{f_0,1,f_0\}} \Gamma^0 \geq k + 1 \).

Assuming this claim, let us see how it implies the proposition. Since \( \Delta \subset \mathbb{Q}^k \times \mathbb{Q}^k \) is smooth of codimension \( k \) in the smooth variety \( \mathbb{Q}^k \times \mathbb{Q}^k \), we see that \( \dim_{\{f_0,1,f_0\}} (\Gamma^0_1 \cap \Delta) \geq 1 \). Hence \( \Gamma^0_1 \cap \Delta \) contains a (possibly open) curve \( C^0 \) \( \ni ([f_0],[f_0]) \). By Claim 2 in the proof of Proposition 3.2, we see that for every point \( (\eta, \eta) \in \Gamma^0_1 \cap \Delta, \ell_\eta \) is a nodal minimal degree rational curve through \( x \). Thus we get infinitely many nodal minimal degree rational curves through \( x \). By Lemma 3.6 above, there is a cuspidal minimal degree rational curve through \( x \). However, this contradicts the assumption that there is no cuspidal minimal degree rational curve through \( x \).

Therefore, to finish proving the proposition, it suffices to show the claim.
Proof of the Claim. Since \( f_0(\mathbb{P}^1) \) has a nodal singularity at \( x \), there are two points \( t_1 \neq t_2 \) on \( \phi^{-1}([f_0]) \) such that \( \psi(t_1) = \psi(t_2) = x \). Assume, without loss of generality, that \( t_2 \not\in S_0 \). Then \( t_2 \) is a smooth isolated point in \( \psi^{-1} \psi(t_2) \) since \( \psi : Z_0^0 \to D_x \) is finite and unramified away from \( S_0 \). By Lemma 2.5, \( \psi(t_2) = x \) is a normal point of \( D_x \), hence \( D_x \) is topologically unibranch at \( x \) by a theorem of Zariski (see page 52 in [13]). By the Fundamental Openness Principle (see page 43 in [13]), \( \psi : Z_0^0 \to D_x \) is open at \( t_2 \) in the classic topology. Choose a sufficiently small classic open neighborhood \( W_2 \subset Z_0^0 \) around \( t_2 \). Then \( \psi(W_2) \ni x \) is an open neighborhood (in the classic topology) of \( x \) in \( D_x \). Since \( \psi(t_1) = x \) and \( \psi \) is continuous in the classic topology, there is a classic open subset \( W_1 \ni t_1 \) such that \( W_1 \cap W_2 = \emptyset \) and \( \psi(W_1) \subset \psi(W_2) \).

Let \( U_i = \psi(W_i) \subset Y_0^0 \) for \( i = 1, 2 \). Then for \( i = 1 \) and \( 2 \), \( [f_0] \in U_i \subset Y_0^0 \) and \( U_i \) is a classic open neighborhood of \( [f_0] \in Y_0^0 \) since \( \phi : Z_0^0 \to Y_0^0 \), being a \( \mathbb{P}^1 \)-bundle, is open (in the classic topology). By choosing \( W_1 \) and \( W_2 \) small enough, we can assume that \( U_1 \) and \( U_2 \) are sufficiently small neighborhoods (in the classic topology) of \( [f_0] \).

Since \( \psi(W_1) \subset \psi(W_2) \) for an arbitrary point \( \xi \in U_1 \), and any point \( t \in \phi^{-1}(\xi) \cap W_1 \), there is a point \( \xi(\xi, t) \in W_2 \) such that \( \psi(t) = \psi(\xi(\xi, t)) \). Let \( \eta(\xi, t) = \phi(\xi(\xi, t)) \in U_2 \). Then we have \( \{x, \eta(t)\} \subset \xi(\xi, t) \cap \phi^{-1}(\xi) \). It is clear that \( \psi(t) \neq x \) for a general point \( t \), i.e., except for some finitely many points. \( t \in \phi^{-1}(\xi) \cap W_1 \). By the definition of \( \Gamma_0^0 \), \( \xi(\xi, t) \in \Gamma_0^0 \cap (U_1 \times U_2) \) for a general point \( i.e., except for some finitely many points \) \( t \in \phi^{-1}(\xi) \cap W_1 \). Since \( W_1 \cap W_2 = \emptyset \), for any \( \xi \in U_1 \), we can choose \( \xi(\xi, t) \in W_2 \) such that \( \eta(\xi, t) \not\in \xi \) for a general point \( t \in \phi^{-1}(\xi) \cap W_1 \).

Since two different minimal degree rational curves can intersect at at most finitely many points, for a fixed \( \xi \in U_1 \), and general two points \( s_1 \) and \( s_2 \) on \( \phi^{-1}(\xi) \cap W_1 \), we can find \( \eta(\xi, s_1) \) and \( \eta(\xi, s_2) \) in \( U_2 \) such that \( \eta(\xi, s_1) \neq \eta(\xi, s_2) \). Since for an arbitrary \( \xi \in U_1 \), the set \( \phi^{-1}(\xi) \cap W_1 \) contains infinitely many points, once we fix an arbitrary point \( \xi \in U_1 \), we have infinitely many \( \eta \)‘s in \( U_2 \) such that \( (\xi, \eta) \in \Gamma_0^0 \cap (U_1 \times U_2) \). Therefore \( q_1 := p_1|_{\Gamma_0^0 \cap (U_1 \times U_2)} : \Gamma_0^0 \cap (U_1 \times U_2) \to U_1 \) is surjective and any fiber of \( q_1 \) has dimension at least one, where \( p_1 : \mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^k \) is the first projection. Hence we get:

\[
\dim(\Gamma_0^0 \cap (U_1 \times U_2)) \geq \dim U_1 + 1 = k + 1. \tag{3.6}
\]

Since \( U_1 \) and \( U_2 \) are two sufficiently small open neighborhoods (in the classic topology) of \( [f_0] \) and \( \Delta \) is of dimension \( k \), we conclude that there is a component \( \Gamma_0^0 \subset \Gamma_0^0 \) different from \( \Delta \cap \Gamma_0^0 \) such that \( ([f_0], [f_0]) \in \Gamma_0^0 \cap \Delta \) and \( \dim([f_0], [f_0]) \Gamma_0^0 \geq k + 1 \). Thus the claim is proved, so is the proposition. \( \square \)

References

2. K. Cho and E. Sato, Fano manifolds of large length without rational and cuspidal curve of minimal degree, preprint.