Stability Analysis of Time Delay Systems Using Spectral Element Method

by

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John Dolbow

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mechanical Engineering and Materials Science in the Graduate School of Duke University 2010
Abstract
(Mechanical Engineering and Materials Science)

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Abstract

The goal of this work is to develop a practical and comprehensive methodology to study the response and the stability of various delay differential equations (DDEs). The development of these new analysis techniques is motivated by the existence of delays in the governing equations of many physical systems such as turning and milling processes.

Delay differential equations appear in many models in science in engineering either as an intrinsic component (e.g. machining dynamics) or as a modeling decision (biology related dynamics). However, the infinite dimensionality of DDEs significantly complicates the resulting analysis from both an analytical and numerical perspective. Since the delay results in an infinite dimensional state-space, it is often necessary to use an approximate procedure to study DDEs and ascertain their stability.

Several approximate techniques appear in the literature to study the stability of DDEs. However, a large number of these techniques—such as D-subdivision, Cluster Treatment of Characteristic Roots and Continuous Time Approximation—are limited to autonomous DDEs. Moreover, the methods that are suitable for non-autonomous DDEs, e.g. the Semi-discretization approach, often result in a very large system of algebraic equations that can cause computational difficulties. Collocation-type methods, such as Chebyshev-collocation approach, have also been used to study DDEs. One major limitation of the conventional Chebyshev collocation approach is
that it is strictly applicable to DDEs with continuous coefficients. An alternative approach that can handle DDEs with piecewise continuous coefficients is the Temporal Finite Element Analysis (TFEA). However, TFEA has only linear rates of convergence and is limited to $h$-convergence schemes. The limited rate of convergence in TFEA has prohibited its application to a wide class of DDEs such as DDEs with complicated coefficients or with distributed and multiple delays.

In this thesis, I develop a spectral element method for the stability analysis of DDEs. The spectral element method is a Galerkin-type approach that discretizes the infinite dimensional DDE into a finite set of algebraic equations (or a dynamic map). The stability of the system is then studied using the eigenvalues of the map.

In contrast to TFEA, the spectral element method was shown to have exponential rates of convergence and $hp$-refinement capabilities. Further, a comparison with the widely-used collocation methods showed that our approach can often yield higher rates of convergence. The higher rates of convergence of the developed approach enabled extending it to DDEs with multiple and distributed delays. I further extended this approach to calculating the periodic orbits of DDEs and their stability.

As an application of the methods developed in this thesis, I studied the stability of turning and milling models. For example, a distributed force model was proposed to characterize cutting forces in turning. The stability of the resulting delay integro-differential equation was studied using the methods developed in this study and they were shown to agree with practical observations. As another example, the stability of a milling process— whose model contains piecewise coefficients—was investigated. The effect of multiple-flute engagement, which contributed to the complexity of the coefficients, was also investigated. The resulting stability charts revealed new stability observations in comparison to typical analysis methods. Specifically, I was able to show that unstable regions appear in what was deemed a stable region by prior analysis techniques.
To my parents for their endless love and support.
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Delay differential equations (DDEs) have been widely used to model physical and biological phenomena. Some examples include machining dynamics [8–10], systems biology [11–13], models of epileptic seizures [14], laser systems [15] and human balance [16, 17]. In some models, for example systems biology related applications, delays are used to avoid modeling certain processes that are known to take a predetermined amount of time but otherwise contribute little to the dynamics. In other models, machining dynamics for example, the delay is intrinsic to the system. Therefore, DDEs require a function segment over a period of time as an initial condition rather than a point value at time zero as with an ordinary differential equation. The infinite dimensionality of DDEs significantly complicates the resulting analysis from both an analytical and numerical perspective [18, 19]. For instance, an important aspect of studying time delay systems is to determine their stability over a range of control parameters. However, since the delay results in an infinite dimensional state-space, it is often necessary to use an approximate procedure to ascertain stability.

Due to the lack of a comprehensive analysis methodology, the time delay effects
are often times neglected or poorly approximated which leads to instabilities and deteriorated performance. For instance, in control, even a small time delay can destabilize an otherwise stable system.

1.1 Difference from ODEs

Although time delays can play an important role in many processes, the associated analysis is far more complicated than systems without delays. The inclusion of delays in the system model changes the governing equations from ordinary differential equations (ODEs) into delay differential equations (DDEs) with infinite-dimensional state space as will be shown in the following sections.

1.1.1 Autonomous Case

Consider the autonomous first order ODE

\[ \dot{x}(t) = ax(t), \]  

(1.1)

where \( a \) is a scalar. The solution for Eq. (1.1) has the form \( x(t) = ce^{\lambda t} \) where \( c \) is a constant and \( \lambda \) is complex. Substituting the assumed solution into Eq. (1.1) gives

\[ \lambda = a, \]  

(1.2)

which implies that Eq. (1.1) is stable only if \( a < 0 \).

On the other hand, consider the DDE

\[ \dot{x}(t) = ax(t - \tau), \]  

(1.3)

where \( \tau \) is the duration of the delay. Assuming a solution form similar to that of Eq. (1.1) yields

\[ \lambda = ae^{-\lambda \tau}. \]  

(1.4)
This is a transcendental equation with an infinite number of roots. The stability
criterion requires that the roots have negative real parts. Therefore, the delay has
increased the dimension of the state space from 1 in Eq. (1.1) to infinity in Eq. (1.3).

These results can be generalized to the higher order system

\[ \dot{x}(t) = Ax(t - \tau), \]  

(1.5)

where \( x \) is a vector and \( A \) is a constant matrix. All the solutions of Eq. (1.5) have
the form

\[ x(t) = \Phi(t)x(0), \]  

(1.6)

where \( \Phi(t) \) is called the fundamental matrix. The matrix \( \Phi(T) \), where \( T = \tau \) for
convenience, is the principal or the transition matrix. Its eigenvalues are called the
characteristic multipliers which can be obtained from

\[ \det(\mu I - \Phi(T)) = 0. \]  

(1.7)

Equation (1.5) is asymptotically stable if all the characteristic multipliers are in
modulus less than one.

In practice, the above analysis is difficult to apply since obtaining the fundamental
matrix can be a difficult task. A more practical approach is to obtain a discrete
solution form that maps the states of the system one delay period ahead according
to

\[ x_n = Ux_{n-1}, \]  

(1.8)

where \( x_n \) and \( x_{n-1} \) are the states of the system in the current and the previous delay
period, respectively, and \( U \) is a finite dimensional approximation of \( \Phi(T) \). The
condition for asymptotic stability requires that the eigenvalues of the monodromy
operator, \( U \), lie within the unit circle in the complex plane, see Fig. 1.1. This
approach is explained in more detail in the next chapter.
1.1.2 Non-autonomous Case

Many models include parametric excitation which requires the analysis of systems with time-periodic coefficients. A classical example of parametrically excited systems is Mathieu’s equation [20]. Another example from the machining processes literature is milling [21]. A prototypical, time-periodic ODE is

\[ \dot{x}(t) = a(t)x(t), \quad (1.9) \]

where \( a(t + T) = a(t) \) is periodic with respect to \( T \), the system period. According to Floquet theory [22], the solution to Eq. (1.9) is

\[ x(t) = v(t)e^{\lambda t}, \quad (1.10) \]

where \( v(t) = v(t + T) \) is \( T \)-periodic. Mapping Eq. (1.10) one system period ahead gives

\[ x(t + T) = e^{\lambda T}x(t). \quad (1.11) \]

Floquet stability theory states that Eq. (1.10) is stable if all the characteristic multipliers \( \mu = e^{\lambda T} \) are within the unit circle in the complex plane, see Fig. 1.1.

Introducing a time delay into Eq. (1.9) yields the nonautonomous DDE

\[ \dot{x}(t) = a(t)x(t - \tau). \quad (1.12) \]

The solution to Eq. (1.12), \( x(t) = v(t)e^{\lambda t} \), has a form analogous to the time periodic ODE case. However, whereas the time periodic ODE will have a finite dimensional Floquet transition matrix, the time periodic DDE will have an infinite dimensional eigenvalue spectrum. Further, the stability condition on the characteristic multipliers of the time periodic DDE is difficult to apply since there is no analytical expression for the Floquet transition matrix. In practice, approximate methods are often used to study the stability of time periodic DDEs via a discrete map similar to Eq. (1.8).

The above discussion also applies to the general, higher order case of time periodic DDEs given by \( \dot{x}(t) = A(t)x(t) \), where \( x \) is a vector and \( A(t) \) is a \( T \)-periodic matrix.
The stability criteria dictates that all the eigenvalues, $\mu$, of the monodromy operator $U$, should lie within the unit circle in the complex plane. Moreover, the manner in which the eigenvalues depart the unit circle produces different bifurcation behavior. For example, an eigenvalue leaving the unit circle through $-1$ or $1$ results in a period doubling bifurcation or a fold bifurcation, respectively, whereas two complex conjugate eigenvalues departing the unit circle results in secondary Hopf bifurcation.

1.2 Research Contributions

The goal of this work is to develop a practical and comprehensive methodology to study the response behavior and the stability of various time delay systems. The contributions of the current work are placed in the context of engineering applications. More specifically, the development of new analysis techniques is motivated by the study of mechanical models appearing in many physical systems.

1.2.1 Machining Dynamics

Machining processes are an important application of delay differential equations. The delays in machining models are inherent to the process since both the current and previous tool oscillations affect the cutting forces. For example, Fig. 1.2 shows
a schematic of a turning process and the relationship between the cutting forces and the chip thickness. The cutting force is a function of the chip thickness which is influenced by current and previous tool marks on the workpiece surface; therefore, the values of the state variables in the current and the previous periods are necessary to model the system.

The stability of the turning process, depicted in Fig. 1.2, has been studied by many authors [8, 23, 24]. However, in these studies the cutting forces were modeled by a point force at the tool tip. Although this representation of cutting forces is satisfactory in the middle and high cutting speed range, stability results at low speeds show substantially different results.

![Figure 1.2: Schematic of a turning process and the force-chip thickness relationship in turning. Graph (a) depicts a turning process and shows the dependence of the chip thickness, \( h \), on the current and previous tool oscillations, while (b) shows the nonlinear relationship between cutting forces and the chip thickness. The process is usually linearized around the nominal chip thickness, \( h_0 \), for stability analysis.](image)

In this work, I present an alternative cutting force model that distributes the forces over the tool chip interface. In contrast to the prototypical point force model which yields a discrete delay, this new approach yields a delay integro-differential equation with both discrete and distributed delays. The study of this equation requires developing a methodology to obtain the stability of distributed delay systems.

I present a technique that can be used to obtain the stability boundaries for systems with distributed delays. This technique could also be used in other application
areas, such as the stability of wheel shimmy [25,26], and population dynamics [27].

Another very common machining process is milling, see Fig. 1.3a. Whereas turning is usually described by autonomous DDEs, milling is modeled by time periodic DDEs. The non-autonomous nature of milling equations complicates the stability analysis even further and gives rise to different bifurcation phenomenon, e.g. period doubling. Similar to turning, milling has also been the topic of many studies in the last decade [28–32]. However, most of these studies considered mills with straight (zero-helix) tooth cutters. More specifically, prior studies on milling only considered tools with straight cutting flutes where all points on the cutting edge enter and exit the cut at the same time.

![Diagram](image)

**Figure 1.3:** Schematic of a helical end mill with multiple flutes. Graph (b) shows a uniform helix mill. Graph (c) shows a variable helix mill.

Recent studies, those that considered mills with non-zero helix angles as show in Fig. 1.3-b and c, have revealed a different stability behavior due to the helix angle effects [1–3]. However, the majority of these publications considered the case of only one tooth engaged in the cut at a time. Specifically, the two main analysis techniques used in this thesis, temporal finite element analysis (TFEA) and Chebyshev collocation, have not been applied to the case of multiple teeth in the cut.
As part of this thesis, I extend these analysis methods to handle the case of multiple teeth in the cut for both zero-helix and helical mills. The differences between the stability behavior of zero-helix and helical mills are then investigated as well as the effect of multiple flute engagement. The importance of studying helical mills stems from their widespread use in milling processes. Besides helical mills, another potential stabilizing strategy is to use a mill with unequal flute angles [4], i.e. an unequal angular pitch spacing between the helical flutes.

The effect of simultaneously engaged flutes of unequal pitch mills leads to a model of multiple discrete delays [5]. This model also occurs in other application areas such as traffic stability [6]. Without any assumptions on the ratio of the different time delays, there is only a limited number of techniques to obtain stability results. Among these techniques, only very few are semi-analytical tools that permit a quick investigation of the full process parameter space. In particular, the analysis techniques used in this study have not been developed to study systems with general discrete delays.

In this work I extend the spectral element method to DDEs with multiple delays. Developing this capability would improve the understanding of a broad class of systems and help gain insight into their stability behavior. Further, adding a technique to handle multiple delays is part of my goal to develop a versatile toolbox for the stability analysis of DDEs.

The above discussion is related to the stability of deterministic, linear DDEs. However, the question of stability is equally important in nonlinear systems. The analysis of nonlinear DDEs is more complex since these systems exhibit different and rich dynamics compared to their linear counterpart. For example, nonlinear systems have multiple equilibria, or attractors, and the interplay between nonlinearity and delays may trigger jumps between these attractors.

In this thesis, I develop a new technique to obtain the stability of nonlinear
DDEs. As a first step, I develop a methodology to obtain the stability of the periodic solution of delayed nonlinear equations. The nonlinear equation is linearized about the periodic solution, and the stability of the linearized DDE is then obtained for arbitrary combinations of the system period and time delays.

1.3 Thesis Organization

This thesis is organized as follows. In chapter 2, the stability analysis techniques used in this study are described. This is followed by mapping considerations and the different types of meshes in Chapter 3. The stability analysis of distributed delays is described in Chapter 4 whereas the analysis of multiple delays is shown in Chapter 5. Chapter 6 develops an approach to calculate the periodic orbits of nonlinear DDEs and to determine their stability. The stability analysis of a turning process with a distributed force model is then performed in Chapter 7. Chapter 8 investigates the stability of a milling process with simultaneously engaged helical flutes while a summary of this thesis is provided in Chapter 9.
Delay differential equations have been widely used to model physical and biological phenomena [33]. Some examples include machining dynamics [9, 29], laser systems [15, 34], coupled delay-line oscillators [35], traffic models [36, 37], and human balance [16, 17]. However, although the inclusion of delays often leads to more realistic models, the resulting infinite dimensional state space significantly complicates the analysis [18, 19]. The interest in studying DDEs often lies in determining the stability of equilibria. However, since delays result in an infinite dimensional state-space, it is often necessary to use approximate procedures to calculate the ascertain stability.

Many analysis methods have appeared in literature for studying delay equations. For example, the stability of DDEs has been studied using D-subdivision [18], Continuous Time Approximation (CTA) [38–40], and the Cluster Treatment of Characteristic Roots methods (CTCR) [41, 42]. The focus in D-subdivision and the CTCR methods is on charting the stability boundaries by investigating the DDE’s characteristic polynomial. Whereas CTCR can find the stability of a wide class of autonomous delay equations, the D-subdivision method is confined to a much smaller class of DDEs, namely systems with no or very small damping [18].
On the other hand, the CTA method converts the DDE into an equivalent high and finite dimensional state space. Therefore, it can yield both stability and equilibria of the underlying DDE. Nevertheless, for some DDEs the CTA method can produce very large matrices leading to significant computational difficulties. In addition, many issues regarding the convergence and numerical stability of CTA have not been addressed yet. A common limitation of all the above methods is that they have only been applicable to autonomous delay equations and alternative methods need to be used for non-autonomous DDEs.

Examples of other methods that can be used for autonomous as well as non-autonomous DDEs include the semi-discretization method [43,44], collocation-based methods [45–49], and temporal finite element analysis [50].

The semi-discretization method divides the time-line into short intervals on which an approximate analytical solution can be obtained. The collection of these expressions can then be used to create a finite dimensional transition matrix which approximates the infinite dimensional monodromy operator of the DDE. However, the semi-discretization method, similar to the closely related CTA method, can result in large matrices to produce a convergent solution. Further, a large number of matrix operations might be necessary—especially for time-periodic systems with incommensurate period and delays [44].

Collocation methods are a type of weighted residual approach with Dirac delta trial functions [51]. These methods typically use a discretization based on Chebyshev or Legendre points to construct a dynamic map that approximates the DDE. The characteristic multipliers, i.e. eigenvalues, of the map are then analyzed to ascertain stability. Collocation methods have been successful in studying a wide class of delay equations and a proof of their convergence can be found in Ref. [47]. However, a recent study concluded that better convergence rates can be achieved if the weighted residual approach is instead combined with high order trial functions (i.e. a high
order Galerkin type approach) [52].

One Galerkin type approach is the Temporal Finite Element Analysis (TFEA) which has been used to study the stability of equilibria in linear DDEs [4,53,54]. In comparison to many other discretization methods, TFEA could easily handle DDEs with piecewise continuous coefficients. In TFEA, the time interval of interest is discretized into a finite number of temporal elements. The original DDE is then transformed into the form of a discrete map whose characteristic multipliers determine stability. The initial formulation of TFEA restricted the approach to at most 2nd order DDEs [7,21,28,53]. However, in Refs. [55,56] the technique was generalized to linear DDEs with a single discrete delay and the improved approach was called state-space TFEA.

Although the state-space TFEA was successful in extending the usefulness of TFEA to a broader class of DDEs, it still suffered from several key limitations. Specifically, state-space TFEA relied mostly on increasing the number of elements to achieve convergence, i.e. $h$-convergence. This limited the approach to the slower linear rates of convergence as opposed to the fast exponential rates observed in some collocation methods. The inability to achieve spectral convergence, or $p$-convergence, was due to the inefficient choice of the trial functions. For historic reasons, these functions were obtained through Hermite interpolation on equidistant nodes, as will be shown in Section 2.3. Due to this choice, increasing the order of the trial functions was impractical and further lead to an ill conditioned system. In addition, the state-space TFEA approach applied analytical integration to generate the weighted residual equations. The integration was performed using a software with symbolic manipulation capabilities; however, this was often computationally expensive especially for systems with complicated time dependent coefficients where closed-form integrals could not be obtained. Relying on analytical integration and $h$-convergence alone prevented the conventional TFEA approach from handling more general delay
This chapter describes a spectral element approach to study the stability of DDEs. In contrast to the conventional state-space TFEA, the developed spectral element approach can have exponential rates of convergence and it admits $hp$-convergence schemes. The described spectral element approach also avoids the limitations of analytical integrations in state-space TFEA by applying highly accurate numerical quadratures—enabling the study of more complicated DDEs. The efficiency of this new approach is compared to well-established methods in literature using various case studies. Specifically, stability results are compared to the conventional state-space TFEA and Legendre collocation methods. Our results reveal that the presented approach can have higher rates of convergence than collocation methods.

Section 2.1 outlines the steps of the stability analysis of DDEs using the spectral element method (SEA) and TFEA. Section 2.5 describes a global collocation approach for the stability of DDEs using two sets of points: Chebyshev and Legendre nodes whereas Section 2.7 describes a piecewise collocation method for DDEs with discontinuous coefficients. Section 2.8 provides case studies to compare the spectral element approach (SEA), TFEA and the collocation methods.

### 2.1 Spectral Element Analysis

The spectral element method for DDEs introduced in [57] was developed as part of this thesis and it is an improvement over the state-space TFEA approach presented in [50]. The advantage of the spectral element approach over the state-space TFEA method is the ability to achieve an $hp$-convergence scheme. This allowed taking full advantage of spectral convergence, which was lacking in the conventional approach, while maintaining the flexibility associated with finite element methods.

Both TFEA and SEA are analysis techniques for DDEs that inherit the flexibility and the robustness of the regular finite element method. In TFEA, the time interval
of interest is discretized into a finite number of temporal elements. The original DDE is then transformed into the form of a discrete map whose characteristic multipliers, i.e. eigenvalues, are analyzed to ascertain stability.

In this section, we describe a spectral element approach to study the stability of DDEs with a single discrete delay. This approach avoids the limitations of the prototypical temporal finite element (TFEA) approach by 1) the use of the extrema of Legendre polynomials as optimum node locations within each element, 2) employing higher-order trial functions obtained through the barycentric equations for Lagrange interpolation through the element nodes, and 3) using a Gauss quadrature rule to evaluate the weighted residual integrals. These features give the current approach the advantage of evaluating the weighted residual integrals and populate the global matrices using only the information from the temporal mesh and the state-space matrices. We also explore the similarity between finite elements and spectral techniques through a Legendre and a Chebyshev collocation approaches and provide case studies to compare TFEA, spectral elements and the collocation approaches.

This section is organized as follows. Section 2.2 describes the general procedure in the TFEA approach without specifying the trial functions. Hermite trial functions, which were used in the conventional TFEA method, are then introduced in section 2.3. Section 2.4 introduces the procedure to obtain higher-order, Lagrange trial functions whereas Section 2.4.1 provides an alternative procedure for evaluating the weighted residual integrals using a Gaussian quadrature. In Section 2.5, two pseudospectral collocation approaches based on Legendre and Chebyshev points are described to study the stability of DDEs. Section 2.8 uses several case studies to compare the conventional TFEA approach, which uses Hermite trial functions and $h$-convergence, to spectral elements which uses Lagrange trial functions and $p$-convergence, and the pseudospectral collocation methods.
2.2 TFEA

This section describes the general procedure to ascertain the stability of DDEs using the state-space TFEA method. A scalar, autonomous DDE is used first in Section 2.2.1 to illustrate the procedure. The approach is then generalized in Section 2.2.2 to higher-order, time-periodic DDEs. To help facilitate the analyses that follow, no specific form is assigned to the trial functions until Sections 2.3 and 2.4.1.

TFEA is a discretization technique that can be used to study the stability of autonomous as well as non-autonomous DDEs. The discretization aims to create a dynamic map over a single delay period. The eigenvalues of the map $\mu$ can then be used to determine the system stability. More specifically, the system is asymptotically stable if all the eigenvalues are within the unit circle in the complex plane as shown in Fig 1.1.

2.2.1 Scalar autonomous DDE

To further explain the procedure, consider the scalar, autonomous DDE described by \[ \dot{x}(t) = \alpha x(t) + \beta x(t - \tau), \] (2.1)

where $\alpha$ and $\beta$ are scalars, and $\tau$ is a constant time delay. In TFEA, the time line is discretized into temporal elements as shown in Fig. 2.1. An approximate solution is then sought in each element according to

\[ x_j(t) = \sum_{i=1}^{n+1} a_{ji}\phi_i(\eta), \] (2.2a)

\[ x_j(t - \tau) = \sum_{i=1}^{n+1} a_{j-E,i}\phi_i(\eta), \] (2.2b)
where $\eta = \sigma/t_j$ is the normalization of the local coordinate $\sigma$ with respect to $t_j$—the length of the $j$th element, $n + 1$ is the number of the polynomial trial functions $\phi(\eta)$ used, and $E$ is the number of elements in each period. The choice of the trial functions has a great impact on the expandability and robustness of the method as will be shown in the subsequent sections. However, regardless of the trial functions used, the method proceeds by substituting the approximate solution into Eq. (2.1); this gives

$$\sum_{i=1}^{n+1} \left( \frac{1}{t_j} a_{ji} \phi'_i(\eta) - \alpha a_{ji} \phi_i(\eta) - \beta a_{j-E,i} \phi_i(\eta) \right) = \text{error},$$

(2.3)

where the prime indicates a derivative with respect to $\eta$ and the error term is associated with the approximation procedure. The error can be minimized by multiplying by a set of test functions and setting the integral over the duration of the element to zero. This is called the method of weighted residuals which also generates enough equations to create the dynamic map. The test functions are required to be independent to obtain a set of linearly independent equations [58]. Further, to keep the resulting matrices square, the number of the test functions is always one less than the number of the trial functions.

Assuming that the index of the test functions $\psi(\eta)$ is $p$, the weighted expression of Eq. (2.3) becomes

$$\int_0^1 \left( \frac{1}{t_j} a_{ji} \phi'_i(\eta) - \alpha a_{ji} \phi_i(\eta) - \beta a_{j-E,i} \phi_i(\eta) \right) \psi_p(\eta) d\eta = 0.$$  

(2.4)

Applying each test function in turn, the resulting equations for each element can be combined into a global matrix equation. For example, assume that 4 elements were used to discretize the current and the previous period (illustrated in Fig. 2.1). The
resulting expression for the global matrix reads

\[
\begin{bmatrix}
N_{31} & N_{32} & N_{33} & 0 & 0 \\
N_{41} & N_{42} & N_{43} & 0 & 0 \\
0 & 0 & N_{41} & N_{42} & N_{43} \\
0 & 0 & N_{41} & N_{42} & N_{43}
\end{bmatrix}
\begin{bmatrix}
a_{31} \\
a_{32} \\
a_{41} \\
a_{42} \\
a_{43}
\end{bmatrix}
= \begin{bmatrix}
P_{11} & P_{12} & P_{13} & 0 & 0 \\
P_{21} & P_{22} & P_{23} & 0 & 0 \\
0 & 0 & P_{21} & P_{22} & P_{23} \\
0 & 0 & P_{21} & P_{22} & P_{23}
\end{bmatrix}
\begin{bmatrix}
a_{11} \\
a_{12} \\
a_{21} \\
a_{22} \\
a_{23}
\end{bmatrix},
\] (2.5)

where the terms inside the matrices of Eq. (2.5) are the following scalar terms:

\[
N_{ji}^p = \int_0^1 \left( \frac{1}{t_j} \phi_i(\eta) - \alpha \phi_i(\eta) \right) \psi_p(\eta) \, d\eta,
\] (2.6a)

\[
P_{ji}^p = \int_0^1 \beta \phi_i(\eta) \psi_p(\eta) \, d\eta.
\] (2.6b)

In Eq. (2.5), it was assumed that three nodes were necessary to interpolate the solution over each element. The trial functions are chosen such that the approximate states of the system are equivalent to the actual states at these nodes. Therefore, the approximation coefficients can be substituted with the actual states at these points, and Eq. (2.5) takes the abbreviated form

\[
Hx_m = Gx_{m-1},
\] (2.7)

where \(m\) and \(m-1\) refer to the current and the previous period, respectively, while the terms \(H\) and \(G\) are defined in Eq. (2.5). Equation (2.7) describes a dynamic map across one delay period of the system in Eq. (2.1). The stability of this system can be determined by the eigenvalues of the monodromy operator \(U = H^{-1}G\).

Alternatively, the inversion of \(H^{-1}\) can be avoided by setting the determinant of \(G - \mu H\) (where \(\mu\) represents the eigenvalues) to zero. The stability of the system is determined by examining the eigenvalues as shown in Fig. 1.1. Another approach to obtain the dynamic map is a collocation approach which will be explained in section 2.5.
2.2.2 Generalization to higher order, time-periodic systems

The state-space TFEA approach presented above can be extended to $q$th order systems with time-periodic coefficients of the form

$$\dot{x}(t) = A(t)x(t) + B(t)x(t - \tau),$$

(2.8)

where $A(t) = A(t + T)$ and $B(t) = B(t + T)$ are $q \times q$, $T$-periodic matrices. For simplicity, only the case $T = \tau$, which appears in many applications such as machining processes [59], will be considered here. The extension of TFEA to this class of equations requires more care in the discretization procedure to ensure the time periodic terms take on the correct values. For example, assume that the time duration of each element is $t_j$, the discretization then requires the substitution $t = \sigma + (j - 1)t_j = (\eta + j - 1)t_j$ into $A(t)$ and $B(t)$ to ensure that they assume the correct values over the entire period. In addition, the expression for the current
and the delayed state variables are written as the vectors

$$x_j(t) = \sum_{i=1}^{n+1} a_{ji} \phi_i(\eta), \quad (2.9a)$$

$$x_j(t - \tau) = \sum_{i=1}^{n+1} a_{j-E,i} \phi_i(\eta). \quad (2.9b)$$

The above approximate expressions are then substituted into Eq. (2.8) and the method of weighted residuals is applied to create a dynamic map—similar to Eq. (2.5). Assuming that each period was discretized with 2 elements, the resulting global matrices read

$$\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
N_{31}^1 & N_{32}^1 & N_{33}^1 & 0 & 0 \\
N_{31}^2 & N_{32}^2 & N_{33}^2 & 0 & 0 \\
0 & 0 & N_{41}^1 & N_{42}^1 & N_{43}^1 \\
0 & 0 & N_{41}^2 & N_{42}^2 & N_{43}^2
\end{bmatrix}
\begin{bmatrix}
a_{31} \\
a_{32} \\
a_{41} \\
a_{42} \\
a_{43}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & I \\
P_{11}^1 & P_{12}^1 & P_{13}^1 & 0 & 0 \\
P_{11}^2 & P_{12}^2 & P_{13}^2 & 0 & 0 \\
0 & 0 & P_{21}^1 & P_{22}^1 & P_{23}^1 \\
0 & 0 & P_{21}^2 & P_{22}^2 & P_{23}^2
\end{bmatrix}
\begin{bmatrix}
a_{11} \\
a_{12} \\
a_{21} \\
a_{22} \\
a_{23}
\end{bmatrix}, \quad (2.10)$$

where $I$ is the $q \times q$ identity matrix. The new expressions for the matrices populating the global matrices are

$$N_{ji}^p = \int_0^1 \left( \frac{1}{t_j} \phi_i'(\eta) - A((\eta + j - 1)t_j) \phi_i(\eta) \right) \psi_p(\eta) \, d\eta, \quad (2.11a)$$

$$P_{ji}^p = \int_0^1 B((\eta + j - 1)t_j) \phi_i(\eta) \psi_p(\eta) \, d\eta, \quad (2.11b)$$

and they both have the same dimension as $A(t)$ and $B(t)$. Equation (2.10) can be used to define a dynamic map that has the same form as Eq. (2.7). The eigenvalues of the monodromy operator can then be used to determine the stability of the system in Eq. (2.8) with the same stability criterion shown in Fig. 1.1.

Similar to the previous example, the accuracy of the approximation procedure is dependent upon the trial functions. These functions depend on the approximation
order—characterized by the number of the interpolation points—as well as the interpolation scheme. The choice of the interpolation points usually includes the end points of the element as well as a set of internal nodes. The inclusion of the end points allows enforcing the continuity conditions across the elements in addition to enforcing the continuity conditions across two periods. In addition to the end points, the order of the approximation is increased through adding more nodes at the middle of the element. These nodes, along with the end points, form a set of distinct points in each element where the approximate states coincide with the actual states. The nodes used to approximate the states are distinct, have a certain distribution, and they determine the approximation order. For example, requiring the interpolant to agree with the state variable at the $n + 1$ nodes leads to the $n$th Lagrange polynomial. In contrast, if the interpolant is equivalent to the state variable and its derivative, the resulting trial functions are the $2n + 1$ Hermite polynomials. Both of these interpolation schemes are further explained in the sections that follow.

2.3 Hermite Trial Functions

When TFEA was first applied to DDEs in Refs. [7, 21, 28, 53], Hermite polynomials were the trial functions of choice. In that early version, a piecewise approximation of the solution was sought over each temporal element. The assumed solution, as well as its first derivative, had to agree with the exact solution and its first derivative on the interpolation nodes. Therefore, the first version of TFEA was limited to at most 2nd order DDEs where the coefficients of both the assumed solution and its derivative were interpreted as the displacement and the velocity, respectively, at the interpolation nodes. This requirement on the trial functions naturally led to a Hermite interpolation scheme to generate the necessary polynomials. However, this study will prove that this choice of trial functions is inefficient and that using barycentric Lagrange interpolation leads to a more accurate high-order method.
For instance, let the solution of a second order DDE over the $j$th element be $\nu_j(t)$. Assume that this solution is approximated by a basis of Hermite polynomials on $n+1$ nodes. The resulting expressions for the solution and the delayed argument are

$$
\nu_j(t) = \sum_{i=1}^{n+1} \nu_j(\eta_i) H_{i0}(\eta) + \sum_{i=1}^{n+1} \nu'_j(\eta_i) H_{i1}(\eta),
$$

$$
\nu_j(t-\tau) = \sum_{i=1}^{n+1} \nu_j(\eta_i - \tau) H_{i0}(\eta) + \sum_{i=1}^{n+1} \nu'_j(\eta_i - \tau) H_{i1}(\eta).
$$

The Hermite polynomials $H_{i\ell}(\eta)$, where $\ell = 0$ or 1, are constructed to satisfy the property

$$
H_{i\ell}^{(s)}(\eta_k) = \delta_{ik} \delta_{s\ell}, \quad s, \ell = 0, 1, \quad i, k = 1, \ldots, n + 1,
$$

where $\delta_{ik}$ and $\delta_{s\ell}$ are Kronecker deltas, $s = 0$ denotes the undifferentiated Hermite polynomials while $s = 1$ denotes their first derivative with respect to $\eta$, and the indices $i$ and $k$ refer to the $i$th and the $k$th nodes, respectively. The Hermite polynomials can be obtained using Eq. (2.13) at each node to obtain a system of equations that can be solved for the unknown polynomials’ coefficients. Alternatively, the polynomials can be obtained from the corresponding Lagrange polynomials on the $n+1$ nodes as shown in Refs. [60,61]. Another method to calculate the Hermite polynomials is a Hermite barycentric formula—although this approach is considerably more difficult than its Lagrange counterpart [62,63].

Upon obtaining the interpolating Hermite polynomials, the original TFQA approach proceeded by substituting the assumed solution into the DDE and used the method of weighted residuals to minimize the approximation error [7]. A dynamic map described by a monodromy operator was then obtained similar to Eq. (2.7). The eigenvalues of the monodromy operator were then used to determine the stability of the DDE using the criteria shown in Fig. 1.1
The approach was later extended to linear DDEs of arbitrary order expressed in the state-space form \([50, 55]\). However, for historical reasons, only the Hermite polynomials denoted by \(H_{i0}(\eta)\) in Eq. (2.12) were used as trial functions, i.e. \(\phi_i(\eta) = H_{i0}(\eta)\), whereas the polynomials \(H_{i1}(\eta)\) were ignored. For example, Fig. 2.2 shows the 3rd and 5th order Hermite trial functions. The 3rd order Hermite polynomials are obtained by interpolating at only two nodes—one at the beginning and one at the end of each element—whereas adding an additional interim node leads to a 5th order Hermite interpolation. To elaborate, if the interpolation nodes are placed at the beginning, middle, and end of each element, the resulting 5th order Hermite trial functions equivalent to \(H_{i0}(\eta)\) are

\[
\phi_1(\eta) = 1 - 23\eta^2 + 66\eta^3 - 68\eta^4 + 24\eta^5, \tag{2.14a}
\]
\[
\phi_2(\eta) = 16\eta^2 - 32\eta^3 + 16\eta^4, \tag{2.14b}
\]
\[
\phi_3(\eta) = 7\eta^2 - 34\eta^3 + 52\eta^4 - 24\eta^5, \tag{2.14c}
\]

while the remaining 3 polynomials \(H_{i1}(\eta)\)—where \(i = 1, 2, 3\)—are ignored. The above trial functions are constructed such that the coefficients of the assumed solution directly represent the state variables at the beginning \((\eta = 0)\), middle \((\eta = 1/2)\), and end \((\eta = 1)\) of each temporal element. Figure 2.1 illustrates that the coefficients of the assumed solution take on the values of the state variables at specific instances in time.

State space TFEA was successful in the stability studies of a broader class of delay systems such as 3rd order DDEs \([54]\). However, the convergence scheme was mostly based on adding more temporal elements, i.e. \(h\)-convergence, due to the computational difficulty of increasing the order of the Hermite polynomials and of handling the resulting weighted residual integrals. More specifically, using additional interpolation nodes to improve the approximation comes at the cost of significantly
increasing the Hermite interpolant’s order. If \( n + 1 \) interpolation nodes are used, then the order of the resulting Hermite polynomials will be \( 2n + 1 \). Therefore, it becomes computationally prohibitive to generate and manipulate the resulting trial functions as the approximation order is increased to higher values. The restriction on the order of the trial functions prevents taking full advantage of spectral convergence rates, i.e. \( p \)-convergence, and forces an \( h \)-convergence scheme.

2.4 Higher Order Trial Functions-Spectral Elements

To increase the order of approximation, it is essential to choose the location of the interpolation nodes such that a well-conditioned polynomial approximation is obtained. More specifically, except for low values of \( n \), equidistant nodes lead to highly ill-conditioned systems and give rise to the Runge phenomenon [64]. A better
approach is to use a set of asymptotically arcsin-distributed nodes such as the LGL (Legendre-Gauss-Lobatto) points [65]. These sets of nodes are more clustered at the end of the elements with an asymptotic density proportional to $(1 - u^2)^{-1/2}$ as $n \to \infty$ [64]. More specifically, the $n + 1$ LGL points are the roots of the polynomial $(1 - u^2)L'_n(u)$ where $u$ ranges from $-1$ to $1$ and $L_n(u)$ is the Legendre polynomial of order $n$ [66]. These points can be shifted to an arbitrary interval $[a, b]$ through the relation

$$\tilde{u} = \frac{b - a}{2}u + \frac{b + a}{2},$$

(2.15)

where $u \in [-1, 1]$ and $\tilde{u} \in [a, b]$, e.g. if $a = 0$ and $b = 1$, then $\eta = \tilde{u}$.

Having selected an appropriate set of nodes, the higher-order trial functions can be obtained through Lagrange interpolation. In contrast to a Hermite interpolation scheme, the approximate function is required to agree with the actual one at the interpolation nodes whereas the derivative information is not used. This helps in obtaining higher-order trial functions while maintaining manageable polynomials. For example, if $n + 1$ LGL points are used, a Hermite interpolation scheme would result in polynomials of order $2n + 1$ compared to $n$ for a Lagrange scheme. Using the Lagrange formula, the trial functions can be obtained according to

$$\phi_i(\eta) = \frac{n+1}{\prod_{k=1, k \neq i}^{n+1} (\eta - \eta_k)} \prod_{k=1, k \neq i}^{n+1} (\eta_i - \eta_k),$$

(2.16)

where the indices $i$ and $k$ refer to the $i$th and $k$th interpolation nodes, respectively. The Lagrange polynomial $\phi_i$ corresponding to the $i$th node has the property

$$\phi_i(\eta_k) = \begin{cases} 1, & i = k, \\ 0, & \text{otherwise}, \end{cases} j, k = 1, \ldots, n + 1.$$

(2.17)

Figure 2.3 shows the resulting $7$th order trial functions using $8$ nodes that are equally
spaced in graph (a) while the LGL distribution is used in graph (b). These plots are generated over the range $[-1, 1]$, and they match those in Ref. [66].

![Figure 2.3: Lagrange trial functions of order 7 obtained through interpolating 8 points through an element with (a) equidistant nodes, and (b) through the LGL points.]

Equation (2.16) comes from the typical form of Lagrange interpolation which is only recommended for a small number of nodes. Specifically, using this equation with high values for $n$ requires a high number of additions and multiplications and yields the computation numerically unstable [64]. A more effective representation of Lagrange polynomials is provided by the barycentric formula according to [67]

$$
\phi_i(\eta) = \frac{\varpi_i}{\eta - \eta_i} \sum_{k=1}^{n+1} \frac{\varpi_k}{\eta - \eta_k},
$$

(2.18)

where $\varpi_k$ are the barycentric weights given by

$$
\varpi_k = \frac{1}{\prod_{k \neq j} (\eta_j - \eta_k), \quad j = 1, \ldots, n + 1.}
$$

The barycentric formula requires less computational effort and has better numerical stability than the conventional Lagrange representation [64, 67]; therefore, it was used to generate the trial functions in the present study.
Equation (2.18) can also be used to define an interpolation matrix that maps a set of arbitrary but distinct points onto the set of interpolation points (also called the base points). For example, let the vector of base points be \( x_1 \in \mathbb{R}^{n \times 1} \), and let the vector of the arbitrary unique points be \( x_2 \in \mathbb{R}^{m \times 1} \), then the effect of interpolating \( x_2 \) using \( x_1 \) as the base points is described using the linear transformation

\[
x_2 = \Upsilon x_1
\]  

(2.20)

where \( \Upsilon \in \mathbb{R}^{m \times n} \) is called the interpolation matrix. The entries of the interpolation matrix are calculated using Eq. (2.18) according to

\[
\Upsilon_{ji} = \phi_i(\eta_j),
\]  

(2.21)

where \( \eta_j \in x_2 \) and \( i \in 1, 2, \ldots, n + 1 \). Equations (2.20) and (2.21) define a linear transformation useful in mapping unconstrained meshes and multiple time delays as will be shown in chapters 3 and 5.

In addition to being a more efficient tool to generate the trial functions, the barycentric weights can be used to obtain the value of the derivative of the trial functions evaluated at the interpolation nodes according to

\[
\phi'_i(\eta_k) = \begin{cases} 
\frac{\omega_i/\omega_k}{\eta_i - \eta_k}, & i \neq k \\
-\frac{\omega_i/\omega_k}{\eta_i - \eta_k} \sum_{i=0,i\neq k}^{n+1} \frac{\omega_i/\omega_k}{\eta_i - \eta_k}, & i = k
\end{cases}
\]  

(2.22)

These values are useful in evaluating the weighted residual integrals of Eqs. (2.6) and (2.11) using a Legendre-Gauss-Lobatto quadrature as will be shown in Section 2.4.1. Moreover, the values defined in Eq. (2.22) form the entries of the differentiation matrix which describes a linear transformation from the values of the trial functions at the nodes to the values of the derivative of the trial functions at the same nodes. For example, assume that the vector \( z \) contains the values of a function \( g(t) \) evaluated at the \( n + 1 \) LGL points. If the vector \( z' \) contains the derivative of \( g \) evaluated at
the same collocation points, then the effect of differentiating the function at these points can be described by a differentiation matrix \( D \) according to

\[
z' = Dz,
\]  

(2.23)

where the entries of \( D \) are defined from Eq. (2.22) according to

\[
D_{ki} = \phi'_i(\eta_k).
\]  

(2.24)

One of the most important applications of differentiation matrices is in spectral collocation methods for the numerical solutions of differential equations. An example of collocation techniques for obtaining the stability of DDEs is be described in section 2.5.

2.4.1 LGL quadrature

In the conventional TFEA approach, the weighted residual integrals were evaluated analytically using a software with symbolic manipulation capability. This limited the conventional approach to systems where such analytical expressions can be obtained. In contrast, another advantage of using the LGL points in the current study is that evaluating the integrals using a Gauss quadrature becomes straightforward since the quadrature points are already calculated when the trial functions are obtained. Applying a quadrature rule, as opposed to analytical manipulations, allows faster and more robust calculations of the involved integrals. To increase the order of an element by one, another internal node needs to be added. The trial functions and the associating integrals can then be calculated accordingly. This allows increasing the order of the approximation until a convergent solution is obtained, a procedure that is commonly referred to as a spectral element approach.

The LGL quadrature gives the best estimate of an integral through replacing it by a weighted summation of the function values at the LGL points. These points
are identical to the ones used in this section to generate higher-order trial functions. Denoting the integrand resulting from the weighted residual method by \( f(\eta) \), the corresponding expression for the LGL quadrature reads

\[
\int_{0}^{1} f(\eta) d\eta \approx \sum_{k=1}^{n+1} w_k f(\eta_k), \tag{2.25}
\]

where \( \eta_k \) are the Legendre points shifted from the interval \([-1, 1]\) to \([0, 1]\) using Eq. (2.15), \( f(\eta_k) \) are the values of \( f(\eta) \) evaluated at the LGL points [68], and \( w_k \) are the LGL quadrature weights given by the equation

\[
w_k = \begin{cases} \frac{2}{n(n+1)}, & k = 1, n + 1 \\ \frac{2}{n(n+1)(\eta_k)'^2}, & \text{otherwise,} \end{cases} \tag{2.26}
\]

If \( f(\eta) \) is a polynomial with a degree of at most \( 2n + 1 \), then it is sufficient to use \( n + 1 \) points in the Gauss quadrature to yield an exact estimate of the integral [69]. Equations (2.25) and (2.26) can be used to evaluate the integrals in Eqs. (2.6) and (2.11) which can then be used to populate the corresponding global matrices. For example, consider the general case of the weighted residual integrals in Eq. (2.11). Applying the LGL quadrature rule to these integrals yields

\[
N_{ji} = \sum_{k=1}^{n+1} \left( \frac{1}{t_j} I \phi'_i(\eta_k) \psi_p(\eta_k) w_k \right) - \sum_{k=1}^{n+1} \left( A((\eta_k + j - 1)t_j) \phi_i(\eta_k) \psi_p(\eta_k) w_k \right), \tag{2.27a}
\]

\[
P_{ji} = \sum_{k=1}^{n+1} \left( B((\eta_k + j - 1)t_j) \phi_i(\eta_k) \psi_p(\eta_k) w_k \right). \tag{2.27b}
\]

However, recall the property of \( \phi_i(\eta_k) \) given in Eq. (2.17) which can be restated as \( \phi_i(\eta_k) = \delta_{ik} \), where \( \delta_{ik} \) is the Kronecker delta. Consequently, the terms of the series multiplying \( \phi_i(\eta_k) \) vanish except when \( i = k \) in which case \( \phi_i(\eta_k) = 1 \). This property
can be used to simplify the above expressions to obtain

\[ N_{ji}^p = \sum_{k=1}^{n+1} \left( \frac{1}{\eta_j} I \phi'_i(\eta_k) \psi_p(\eta_k) w_k \right) - A((\eta_i + j - 1)t_j) \psi_p(\eta_i) w_i, \]

(2.28a)

\[ P_{ji}^p = B((\eta_i + j - 1)t_j) \psi_p(\eta_i) w_i. \]

(2.28b)

Equation (2.28) shows the advantage of using a quadrature rule, as opposed to analytical calculations, to evaluate the weighted residual integrals; in addition to handling more complex integrals, the evaluation of these integrals is reduced to a weighted summation of matrices. Furthermore, choosing the quadrature points to be identical to the interpolation points allows the property of Eq. (2.17) to simplify the quadrature even further. These features yield a spectral element method that does not require obtaining the expressions of the trial functions, only the barycentric weights associated with them. These weights can be obtained only by knowing the set of interpolation points within each element. In other words, with the current approach it becomes possible to evaluate the weighted residual integrals using only the information from the temporal mesh and the state-space matrices.

2.5 Stability analysis of DDEs using collocation methods

Collocation methods are used to study the stability of DDEs with smooth coefficients such as Eq. (2.8). Therefore, a global approximation is justified and the residual error is guaranteed to vanish as the approximation order is increased. In the conventional collocation methods, the procedure involves choosing a suitable set of collocation points and obtaining the corresponding spectral differentiation matrix. Whereas this section will describe the conventional collocation approach, Section 2.7 will expand the collocation methods for DDEs with discontinuous coefficients using a multi-interval collocation method.
In the collocation methods that are based on Chebyshev and Legendre basis the definition domain of the collocation points is usually the interval \([-1, 1]\). The shift from the definition domain to an arbitrary domain \([a, b]\) is given by Eq. (2.15). The specific computations of the collocation points and the generation of the spectral differentiation matrices is discussed in Sections 2.5.1 and 2.5.2.

Let the order of the state-space equation (2.8) be \(q\). Using a polynomial discretization scheme and evaluating (2.8) at the collocation points on the interval \([0, \tau]\) gives

\[
\hat{D}m_{r+1} = \hat{M}_Am_{r+1} + \hat{M}_Bm_r, \tag{2.29}
\]

where the \(q \times 1\) vector \(m_{r+1}\) contains the values of \(y(t)\) in the interval \(t \in [0, \tau]\) while the \(q \times 1\) vector \(m_r\) contains the values of the initial function \(\psi(t) = y(t - \tau)\) in \(t \in [-\tau, 0]\) according to

\[
m_{r+1} = \begin{bmatrix} y(t_0) \\ y(t_1) \\ \vdots \\ y(t_N) \end{bmatrix}, \quad m_r = \begin{bmatrix} \psi(t_0) \\ \psi(t_1) \\ \vdots \\ \psi(t_N) \end{bmatrix}. \tag{2.30}
\]

In order to describe the matrix \(\hat{D}\), the \(q(N + 1)\) square differential operator \(\mathbb{D}\) is defined first as the Kronecker product

\[
\mathbb{D} = D \otimes I_q \tag{2.31}
\]

where \(D\) depends on the collocation scheme used and its entries will be specified in Sections 2.5.1 and 2.5.2, and \(I_q\) is an identity matrix. The matrix \(\hat{D}\) is then a modified version of the spectral differentiation matrix \(\mathbb{D}\) with the following changes (1) multiplying by \(2/\tau\) to account for the shift \([-1, 1] \rightarrow [0, \tau]\), and (2) changing the last \(q\) rows to \([0_q \; 0_q \ldots I_q]\) where \(0_q\) and \(I_q\) are \(q \times q\) null and identity matrices, respectively. The change in the last \(q\) rows of \(\hat{D}\), (and the last \(q\) rows of \(\hat{M}_A\) and \(\hat{M}_B\) as will be shown shortly), is necessary to enforce the periodicity condition which, for
this particular case, implies that the states are equal at the end of one period and the beginning of the subsequent one.

The \((N + 1)\) square matrix \(\hat{M}_A\) has the entries

\[
\hat{M}_A = \begin{bmatrix}
A(t_0) & A(t_1) & \cdots & A(t_{N-1}) \\
0_q & 0_q & \cdots & 0_q & 0_q
\end{bmatrix}, \tag{2.32}
\]

where \(A(t_i)\) is the value of \(A(t)\) in (2.8) evaluated at the \(i\)th collocation point. Similarly, the square \((N + 1)\) matrix, \(\hat{M}_B\), has the entries

\[
\hat{M}_B = \begin{bmatrix}
B(t_0) & B(t_1) & \cdots & B(t_{N-1}) \\
I_q & 0_q & \cdots & 0_q & 0_q
\end{bmatrix}, \tag{2.33}
\]

where \(B(t_i)\) is the value of \(B(t)\) in (2.8) evaluated at the \(i\)th collocation point.

Equation (2.29) can be rearranged to obtain the dynamic map

\[
m_{r+1} = Um_r, \tag{2.34}
\]

where the monodromy matrix \(U = (\hat{D} - \hat{M}_A)^{-1}\hat{M}_B\) is the finite approximation to the infinite-dimensional monodromy operator. The stability of the system described by (2.8) can then be determined by examining the eigenvalues of \(U\) using Floquet theory and the stability criterion in Fig. 1.1. Alternatively, the inversion of \((\hat{D} - \hat{M}_A)\) can be avoided by setting the determinant \(|\hat{M}_B - \mu(\hat{D} - \hat{M}_A)|\) to zero, where \(\mu\) is the characteristic multiplier.
2.5.1 Chebyshev collocation nodes

The \( N + 1 \) CGL collocation points are the extrema of the \( N \)th order Chebyshev function, and they are found from

\[
z_n = \cos \frac{n\pi}{N} \quad \text{where } n = 0, 1, \ldots, N,
\]

(2.35)

where the change of variables in (2.15) can be used to shift these points from \([-1, 1]\) to any other interval. Evaluating the DDE at the Chebyshev points minimizes the approximation error and allows using the simplified expression for the Chebyshev spectral differentiation matrix [70]. The entries of the Chebyshev differentiation matrix are given by

\[
D_{00} = -D_{NN} = \frac{2N^2 + 1}{6},
\]

(2.36a)

\[
D_{kk} = \frac{-z_k}{2(1 - z_k^2)}, \quad k = 1, \ldots, N - 1
\]

(2.36b)

\[
D_{kn} = \frac{-\bar{c}_k(-1)^{k+n}}{\bar{c}_n(z_k - z_n)}, \quad k \neq n, \ k, n = 0, \ldots, N,
\]

(2.36c)

where \( \bar{c}_{[\cdot]} = 2 \) if \( [\cdot] = 0 \) or \( [\cdot] = N \), and is 1 otherwise. Equation (2.36) gives the expression for the matrix \( \mathbf{D} \) used in (2.31) when using a Chebyshev collocation scheme.

2.5.2 Legendre collocation nodes

Similar to the Chebyshev points, The LGL points also offer a set of well-distributed points suitable for collocation methods [65]. The \( N + 1 \) LGL points are the roots of the polynomial \((1 - u^2)L'_N(u)\) where \( u \) ranges from \(-1\) to \(1\) and \( L_N(u) \) is the Legendre polynomial of order \( N \) [66]. These points can be shifted to any arbitrary interval through the relation given in (2.15).
Using the LGL nodes gives a simplified expression for the entries of $D$ used in (2.31) when a Legendre collocation approach is used. Specifically, the dimension of this matrix is $(N + 1) \times (N + 1)$ and its entries are given by [71]

$$D_{00} = -D_{NN} = -\frac{N(N + 1)}{4},$$

(2.37a)

$$D_{km} = \begin{cases} \frac{L_N(t_k)}{L_N(t_m)} \frac{1}{(t_k - t_m)}, & k \neq m \\ 0, & \text{otherwise} \end{cases}.$$  

(2.37b)

Upon obtaining the collocation points and the corresponding differentiation matrices, the steps outlined in Section 2.5 can be followed to ascertain the stability of the DDE using a Legendre collocation scheme.

2.6 Connection between collocation and spectral element

TFEA and collocation methods differ mainly in the approach they each use to reduce the approximation error, see Eq. (2.3). However, although collocation methods and TFEA may seem different, they are actually closely related [51]. The connection between the methods can be revealed by considering the weighted residual integral

$$\int_0^1 \left( \frac{1}{t_j} a_{ji} \phi_i'(\eta) - \alpha a_{ji} \phi_i(\eta) - \beta a_{j-E,i} \phi_i(\eta) \right) \psi_p(\eta) \, d\eta = 0.$$  

(2.38)

which was introduced in Eq. (2.4). Recall that collocation methods are global approximation techniques whereas TFEA employs piecewise approximation on a number of elements. However, if the number of elements is set to 1, then TFEA and collocation become defined on the same domain and the length of the element become the system period $t_j = T$. Making this assumption in Eq. (2.4), the key difference between collocation and TFEA methods becomes the choice of the test functions $\psi_p(\eta)$.

These test functions are introduced to reduce the error associated with the approximation as well as to generate enough equations to construct a dynamic map.
Since TFEA and collocation methods adopt different strategies to reduce the error, the test functions used in Eq. (2.4) define the resulting analysis technique. Traditionally, there are three common choices for the set of test functions [51,58]:

\[
\psi_p(\eta) \sim \delta(\eta - \eta_p), \quad \text{(Collocation)}
\]

\[
\psi_p(\eta) \subset \phi_i(\eta), \quad \text{(TFEA, Galerkin)}
\]

\[
\psi_p(\eta) \notin \phi_i(\eta), \quad \forall \ p \quad \text{(TFEA, Petrov-Galerkin)}
\]

where \{\eta_p\} is a suitable set of collocation points and \(\delta(\eta - \eta_p)\) is the Dirac delta function. Equation (2.39a) reveals that collocation methods are special cases of the weighted residual approach caused by using Dirac delta test functions.

Analogously, using a Gaussian quadrature rule to evaluate the integral in Eq. (2.38) shows the transformation from a collocation method to TFEA. Specifically, assume \(E = 1\) and that the quadrature points coincide with the interpolation points. The resulting expressions for one element are shown in Eq. (2.28) and they can be rewritten for all the interpolation points within the element as

\[
\begin{bmatrix}
W_{\text{res}} (D \otimes I_q - M_A) \\
I_q \quad 0 \quad \ldots \quad 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{bmatrix}_m
= \begin{bmatrix}
W_{\text{res}} m_B \\
0 \quad \ldots \quad 0 \quad I_q
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{bmatrix}_{m-1}
\]

(2.40)

where the entries of the \((n + 1) \times (n + 1)\) differentiation matrix \(D\) are given by Eq. (2.24) and the last \(q\) rows are added to enforce the continuity condition at \(t = 0\). The diagonals of the \((n + 1)\) square matrices \(M_A\) and \(M_B\) contain, respectively, the matrices \(A(t)\) and \(B(t)\) from Eq. (2.8) evaluated at the \(n + 1\) collocation points. The
matrices $I_q$ and $0_q$ are the $q \times q$ identity and zero matrices, respectively, whereas the $q(n+1)$ transformation matrix $W_{res}$ is given by

$$W_{res} = \left( \begin{bmatrix} w_0 & w_1 & \cdots & w_n \\ w_0 & w_1 & \cdots & w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_0 & w_1 & \cdots & w_n \end{bmatrix} \otimes \begin{bmatrix} \psi_0(t_0) & \cdots & \psi_0(t_n) \\ \psi_1(t_0) & \cdots & \psi_1(t_n) \\ \vdots & \ddots & \vdots \\ \psi_{n-1}(t_0) & \cdots & \psi_{n-1}(t_n) \end{bmatrix} \right) \otimes I_q$$

(2.41)

where $w_k$ are the interpolation weights given by Eq. (2.26) while the symbol $\otimes$ denotes element-wise multiplication. Equation (2.40) is equivalent to the abbreviated form $Hx_m = Gx_{m-1}$ presented in Eq. (2.7). Note that whereas the resulting matrices for $E = 1$ are full, for $E > 1$ they are not. Nevertheless, the transformation in Eq. (2.40) can be expanded to the case of multiple elements.

To elaborate, the expression for $H$ when $E$ elements are used each with $n+1$ discretization points becomes

$$H = \begin{bmatrix} \hat{W}_{res}(D_1 \otimes I_q - M_{A}^{e=1}) \\ \hat{W}_{res}(D_2 \otimes I_q - M_{A}^{e=2}) \\ \vdots \\ \hat{W}_{res}(D_j \otimes I_q - M_{A}^{e=j}) \\ \vdots \\ \hat{W}_{res}(D_E \otimes I_q - M_{A}^{e=E}) \\ 0_q & 0_q & \cdots & \cdots & 0_q & 0_q & I_q \end{bmatrix},$$

(2.42)

where $\hat{W}_{res} = W_{res} \otimes I_q$, $D_j$ is the $(n+1) \times (n+1)$ differentiation matrix on the $j$th element while the $q(n+1)$ block diagonal matrix $M_{A}^{e=j}$ contains the values of $A(t)$ at the $n+1$ collocation points within the $j$th element. All the non-labeled entries of $H$ in Eq. (2.42) are zeros and the resulting structure is a block matrix with any two blocks overlapping only horizontally by $q$ columns. For a large number of elements, most of the entries in $H$ become zeros; hence, the resulting sparse structure can be
exploited to speed up stability computations via sparse matrix eigenvalue solvers [72]. Similarly, the expression for the $G$ matrix can be obtained using the matrices $W_{\text{res}}$, $B(t)$ and the last $q$ rows of Eq. (2.40) to enforce continuity at $t = 0$.

Equations (2.40) and (2.42) suggest that the spectral element approach we present here can be thought of as a linear transformation of collocation methods. Collocation methods are well documented in the literature and a proof of their convergence can be found in Ref. [47]. Further, the entries of the transformation matrix $W_{\text{res}}$ are constituted from continuous independent test functions and Gaussian quadrature weights. Since linear transformations are well behaved with respect to continuity and convergence, the application of this matrix preserves the convergence properties as evidenced by the various computational studies using TFEA found in literature as well as the examples we used within this study [7,10,53,73].

2.7 Multi-interval Collocation (MIC)

The analysis in Section 2.5 was for DDEs with smooth coefficients over the entire period. Therefore, a global approximation using Chebyshev or Legendre points is justified and the residual error is guaranteed to vanish as the approximation order is increased. However, many physical applications involve discontinuities in the coefficients during each system period, see Fig. 2.4. For example, this occurs in milling processes when one of the cutting teeth enters or exits the cut [10]. In this case, the exponential convergence property associated with the conventional collocation techniques is lost.

To recover the desired convergence properties, the time of each period is divided into multiple, piecewise-continuous intervals. Inter-boundary points are placed at the points of discontinuities and a collocation method can then be applied to each subinterval. At the boundaries, the appropriate conditions are enforced to obtain the overall approximation over the full period. This approach is called multi interval
Figure 2.4: A representation of the system coefficients over one period. Graph (a) represents smooth coefficients over the full period which allows using a global collocation approach. Graph (b) shows piecewise-continuous coefficients which necessitates a piecewise approximation technique.

collocation (MIC).

For example, let \( t_{p,k} \) be the time at the \( k \)th collocation point within the \( p \)th interval; assume that \( N_p + 1 \) is the number of collocation points used within the \( p \)th interval and that a total of \( E \) subintervals is used in the approximation, i.e. \( p \in \{1, 2, \ldots, E\} \). The expressions for \( m_{r+1} \) and \( m_r \) then become

\[
\begin{align*}
\mathbf{m}_{r+1} &= \begin{bmatrix}
    y(t_{1,0}) \\
    y(t_{1,1}) \\
    \vdots \\
    y(t_{1,N_1}) = y(t_{2,0}) \\
    \vdots \\
    y(t_{j-1,N_{j-1}}) = y(t_{j,0}) \\
    \vdots \\
    y(t_{E,N_E})
\end{bmatrix}, \\
\mathbf{m}_r &= \begin{bmatrix}
    \psi(t_{1,0}) \\
    \psi(t_{1,1}) \\
    \vdots \\
    \psi(t_{1,N_1}) = \psi(t_{2,0}) \\
    \vdots \\
    \psi(t_{j-1,N_{j-1}}) = \psi(t_{j,0}) \\
    \vdots \\
    \psi(t_{E,N_E})
\end{bmatrix}, \\
\text{(2.43)}
\end{align*}
\]

where the length of both column vectors is given by \( \ell q \) where

\[
\ell = 1 + \sum_{p=1}^{E} N_p. \quad (2.44)
\]

In contrast to the typical collocation methods, MIC results in overlapping block diagonal matrices. More specifically, let \( \mathbf{D}_p \) be the rectangular \( q N_p \times q (N_p + 1) \) matrix obtained by omitting the last \( q \) rows of the matrix \( \mathbf{D} \) which was described after Eq. (2.30). A spectral differentiation operator can then be obtained for each element by
scaling $\hat{D}$ by $2/h_p$, where $h_p$ is the length of the $p^{th}$ interval. The individual spectral differentiation operators can then be combined in a global differentiation matrix for all the subintervals according to

$$
\hat{D} = \begin{bmatrix}
\frac{2}{h_1} \hat{D} \\
\frac{2}{h_2} \hat{D} \\
\vdots \\
\frac{2}{h_p} \hat{D} \\
\end{bmatrix}
\begin{bmatrix}
0_q & \cdots & 0_q & \cdots & 0_q & \cdots & I_q \\
\end{bmatrix}, \quad (2.45)
$$

where $I_q$ is a $q \times q$ identity matrix with $q$ representing the order of the underlying DDE. Since any two adjacent intervals share a common node, any two subsequent differentiation operators overlap only horizontally by $q$ columns. Further, the last $q$ rows of the global differentiation matrix are padded with the matrix $[0_q \ 0_q \cdots I_q]$ to enforce the continuity condition as was explained after Eq. (2.29). Hence, the size of the resulting global differentiation matrix is $\ell q \times \ell q$. For example, the resulting global differentiation matrix for $E = 2$, (i.e. using two subintervals), is

$$
\hat{D}_{|E=2} = \begin{bmatrix}
\frac{2}{h_1} \begin{bmatrix}
D_{0,0}I_q & \cdots & D_{0,N_1}I_q \\
\vdots & \ddots & \vdots \\
D_{N_1-1,0}I_q & \cdots & D_{N_1-1,N_1}I_q \\
\end{bmatrix} \\
\frac{2}{h_2} \begin{bmatrix}
D_{0,0}I_q & \cdots & D_{0,N_2}I_q \\
\vdots & \ddots & \vdots \\
D_{N_2-1,0}I_q & \cdots & D_{N_2-1,N_2}I_q \\
\end{bmatrix} \\
0_q & \cdots & 0_q & \cdots & I_q \\
\end{bmatrix}, \quad (2.46)
$$
In the MIC approach, the expression for the matrix $\hat{M}_A$ reads

$$
\hat{M}_A = \begin{bmatrix}
A(t_{1,0}) & \cdots & A(t_{1,N_1}) = A(t_{2,0}) \\
\vdots & \ddots & \vdots \\
A(t_{p-1,N_{p-1}}) = A(t_{p,0}) & \cdots & A(t_{E,N_{E-1}}) \\
0_q & \cdots & 0_q \\
\end{bmatrix},
$$

(2.47)

where the dimension of $\hat{M}_A$ is $\ell q \times \ell q$. Similarly, the matrix $\hat{M}_B$ has the same dimension as $\hat{M}_A$ and it is obtained from

$$
\hat{M}_B = \begin{bmatrix}
B(t_{1,0}) & \cdots & B(t_{1,N_1}) = B(t_{2,0}) \\
\vdots & \ddots & \vdots \\
B(t_{p-1,N_{p-1}}) = B(t_{p,0}) & \cdots & B(t_{E,N_{E-1}}) \\
I_q & 0_q & \cdots & 0_q \\
\end{bmatrix},
$$

(2.48)

The global system can then be written as the dynamic map $m_{r+1} = U m_r$ and the stability can be determined by examining the eigenvalues of the monodromy operator, $U$, see Eq. (2.34) and Fig. 1.1.

### 2.8 Case Studies

To compare the conventional TFEA to the spectral element TFEA and the Legendre collocation approach, the stability of three DDEs was investigated. In these examples, the conventional TFEA approach used the 5th order Hermite polynomials described in Eq. (2.14) and relied on an $h$-convergence scheme. Although the spectral
element approach described in this thesis can use an hp-convergence scheme, in the following examples the number of elements was fixed to 1 and only p-convergence was used to contrast the results with those of the conventional TFEA approach. For the Legendre collocation method, the number of collocation points was increased to obtain spectral convergence.

The first example in Section 2.8.1 is the scalar autonomous DDE of Eq. (2.1). Section 2.8.2 studies the stability of an autonomous third order DDE whereas Section 2.8.3 investigates the stability of an unbalanced rotating shaft with feedback gain; which is described by coupled second order DDEs with time-periodic coefficients. In all these cases, the stability diagram to which all the methods converge is shown. In addition, the convergence of the spectral radius, at a point near the stability boundaries, is compared for all methods.

The reference $\mu^*$ for comparing the spectral radius for all methods was the magnitude of the maximum eigenvalue obtained from the Legendre collocation approach for a high number of collocation points $N = N_{\text{max}}$, i.e. $\mu^* = \max(|\mu(N_{\text{max}})|)$. Specifically, the absolute difference $|\lambda - \mu^*|$, where $\lambda = \max |\mu|$ is the magnitude of the maximum eigenvalue obtained from one of the methods, was plotted either as a function of the number of elements, of the order of the Lagrange trial functions or of the number of collocation points. This provided a common reference for comparing the rates of convergence of the different methods. The common reference was obtained from a Legendre collocation approach since collocation methods are known to have spectral rates of convergence. In all the examples, the maximum number of collocation points and elements as well as the maximum order of Lagrange trial functions was set to 40, i.e. $N_{\text{max}} = n_{\text{max}} = E_{\text{max}} = 40$. 
2.8.1 Scalar example

Consider the scalar system described in Eq. (2.1). If the scalars $\alpha$ and $\beta$ are chosen as the control parameters and $\tau = 1$, then the stability of the system in the control parameter space can be described by the stability diagram in Fig. 2.5-a.

Figure 2.5-b shows the absolute difference $|\lambda - \mu^*|$ at the $(\alpha, \beta)$ point $(-1, 0.0754)$ for the different methods. The integers on the $x$-axis represent the following: (1) the number of elements for the conventional TFEA approach (circles), (2) the number of the collocation points for the Legendre collocation approach (triangles), and (3) the order of the Lagrange polynomial in the spectral element method (dots). It can be seen that the conventional TFEA approach follows a linear convergence scheme whereas both spectral element and the collocation method demonstrate exponential (or spectral) convergence. In addition, the spectral element method converges faster than the collocation approach.

The $hp$ refinement capability of the spectral element approach is shown in Fig. 2.8. Figure 2.8-a corresponds to the error associated with calculating the maximum Floquet multiplier in Eq. (2.1). In each curve, the order of the trial functions was held constant while the number of elements was varied between 1 and 24. It can be seen that a convergent solution can be obtained by either increasing the order of the trial functions or the number of elements. Moreover, it can be seen that increasing both $E$ and $n$ leads to faster convergence.

2.8.2 Third order example

Determining the stability of a third order delay equation would not have been possible with the original version of TFEA [7,21,28,53]. Thus, we consider the following third order delay equation

$$\frac{d^3 x}{dt^3} + \alpha \dot{x} + \beta x(t - \tau) = 0,$$  \hfill (2.49)
with a delay value of $\tau = 1$ and control parameters $\alpha$ and $\beta$. The state space form of Eq. (2.49) is

$$\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -\alpha & 0
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-\beta & 0 & 0
\end{bmatrix} \begin{bmatrix}
y_1(t - \tau) \\
y_2(t - \tau) \\
y_3(t - \tau)
\end{bmatrix},$$  \hspace{1cm} (2.50)

where $y_1 = x$, $y_2 = \dot{x}$, and $y_3 = \ddot{x}$. The $A$ and $B$ matrices of Eq. (2.8) are

$$A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -\alpha & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\beta & 0 & 0
\end{bmatrix}. \hspace{1cm} (2.51)$$

The stability diagram and the convergence plots for the point $(202.04, 28.38)$ are shown in Fig. 2.6. In comparison to the scalar example, it can be seen that for this example $\lambda$ took longer to converge to $\mu^*$ in all the used methods. However, Fig. 2.6-b shows that the conventional TFEA (circles) again converged linearly and had the slowest rate of convergence. In contrast, both the spectral element (dots) and
the collocation approach (triangles) maintained their spectral convergence property. However, similar to the scalar example, the spectral element method converged faster than the collocation approach.

Figure 2.8-b corresponds to the error associated with calculating the maximum Floquet multiplier in Eq. (2.49) as a function of $E$ and $n$ ($hp$ refinement). Each curve represents a constant $n$ whereas $E$ is varied in the horizontal axis between 1 and 24. Similar to Eq. (2.1), convergence can be obtained by either increasing $E$ or $n$. However, faster convergence rates can be obtained if both $E$ and $n$ are increased.

![Figure 2.6: The stability diagram for Eq. (2.49) and the spectral convergence plots for the point (202.04, 28.38) in the parameter space. In graph (a), the stable regions are shaded whereas the unstable regions are left unshaded. Graph (b) shows the convergence of the maximum eigenvalue as a function of: (1) the number of elements in the conventional TFEA approach (circles), (2) the number of collocation points in the Legendre collocation method (triangles), and (3) the order of the Lagrange trial functions (dots).](image)

2.8.3 A non-autonomous, time-periodic system

The spectral element approach can be used to study the stability of non-autonomous systems with time-periodic coefficients. For example, consider the system of coupled
DDEs given by [74]

\[\ddot{x} + 2\zeta \dot{x} + (1 - \delta \cos 2\Omega t)x - \delta \sin (2\Omega t) y = bx(t - T), \quad (2.52a)\]

\[\ddot{y} + 2\zeta \dot{y} + (1 + \delta \cos 2\Omega t)y - \delta \sin (2\Omega t) x = by(t - T). \quad (2.52b)\]

These equations describe the non-dimensionalized deflections of the midspan of an asymmetric shaft where \(\zeta\) is damping, \(\delta\) is a non-dimensional parameter describing the amount of asymmetry in the system, \(b\) is the non-dimensional feedback gain, \(\Omega\) is the non-dimensional rotational frequency, and \(T = \pi/\Omega\) is a constant delay. The delayed term is non-instantaneous or delayed feedback that is used to stabilize the shaft. Equation (2.52) can be written in state-space form as

\[
\dot{y}(t) = C(t)y(t) + D(t)y(t - \tau), \quad (2.53)
\]

where the state matrices are given by

\[
C(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
(-1 + \delta \cos 2\Omega t) & \delta \sin 2\Omega t & -2\zeta & 0 \\
\delta \sin 2\Omega t & (-1 - \delta \cos 2\Omega t) & 0 & -2\zeta
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & b & 0 & 0
\end{bmatrix}. \quad (2.54)
\]

The results of the stability analysis of Eq. (2.53) are shown in Figs. 2.8.3-(a), (b) and (c). In these figures, the unstable region was shaded and the feedback gain was assigned the values \(b = 0\), \(b = 0.02\) and \(b = 0.04\), respectively. It can be seen that even small feedback gains can substantially change the stability boundaries and deteriorate the stable regions. Figure 2.8.3-(d) shows the convergence plots for the magnitude of the maximum Floquet multiplier \(\lambda\) at the \((\Omega, \delta)\) point \((1.16, 0.3)\) for the different methods. Specifically, the convergence is shown as a function of three quantities: (1) the number of elements in the conventional TFEA method (circles),
(2) the number of Legendre collocation points (triangles), and (3) the order of the trial functions in the spectral element approach (dots).

Similar to the previous examples, it can be seen that the spectral element approach and the Legendre collocation approach had a spectral rate of convergence whereas the conventional TFEA converged last with a linear rate of convergence. Furthermore, the spectral element approach converged faster than the other two methods.

Figure 2.8-c corresponds to the error associated with calculating the maximum Floquet multiplier in Eq. (2.52) using $hp$ refinement. In each curve, the order of the trial functions was held constant while the number of elements was varied between 1 and 24. It can be seen that increasing both the order of the trial functions and the number of elements leads to faster convergence. It can also be seen that although system (2.52) has more complicated coefficients than the 3rd order DDE in (2.49), the maximum Floquet multiplier of Eq. (2.52) converged faster. In fact, among all the cases considered here, the convergence rate for the 3rd order example was the slowest.

2.9 Conclusions

This chapter described a spectral element approach to study the stability of DDEs. In contrast to the prototypical Temporal Finite Element Analysis (TFEA), the described spectral element approach had exponential rates of convergence and allowed exploiting $hp$-convergence schemes. The described approach also avoided the limitations of analytical integrations in TFEA through employing highly accurate numerical quadratures—enabling the study of more complicated DDEs. The effectiveness of this new approach was compared to well-established methods in literature using various case studies. Specifically, stability results were compared to the conventional TFEA and Legendre collocation methods.
Figure 2.7: The stability diagram for Eq. (2.52) and the spectral convergence plots for the $(\Omega, \delta)$ point $(1.16, 0.3)$ in the parameter space. The parameters used were $\zeta = 0.02$ and (a) $b = 0$, (b) $b = 0.02$, and (c) $b = 0.04$. In graphs (a)-(c), the unstable regions are shaded whereas the stable regions are left unshaded. Graph (d) shows the convergence of the maximum eigenvalue as a function of: (1) the number of elements in the conventional TFEA approach (circles), (2) the number of collocation points in the Legendre collocation method (triangles), and (3) the order of the Lagrange trial functions (dots).

The developed spectral element approach avoided the limitations of the prototypical temporal finite element (TFEA) approach through 1) the use of the extrema of Legendre polynomials as optimum node locations within each element, 2) employing higher order trial functions obtained through the barycentric equations for Lagrange interpolation through the element nodes, and 3) using a Gauss quadrature rule to evaluate the weighted residual integrals. These features gave the current approach
Figure 2.8: The absolute errors associated with calculating the maximum Floquet multiplier using the spectral element method. Figure (a) corresponds to Eq. (2.1) whereas Figures (b) and (c) correspond to Eqs. (2.49) and (2.52), respectively.

The advantage of evaluating the weighted residual integrals and populate the global matrices using only the information from the temporal mesh and the state-space matrices.

The spectral element approach was also shown to produce block matrices whose structure could be employed in sparse eigenvalue solvers for faster stability calculations. We also explored the connection between the spectral element method and collocation techniques in Section 2.6 and included comments on the convergence of the spectral element approach.

We used a set of case studies to compare the stability results obtained with the spectral element method against the conventional state-space TFEA and Legendre collocation. The resulting stability diagrams to which all the methods converged
were shown in Figs. 2.5-a and 2.6-a as well as Figs. 2.8.3 a through c. The maximum Floquet multiplier at a point near the stability boundary was also calculated through these methods and compared to a reference value obtained using a high order Legendre collocation.

The absolute errors between the calculated and the reference multipliers for each case were plotted on a log-log scale in Figs. 2.5-b, 2.6-b and 2.8.3-d. The results showed that in contrast to the linear rate of convergence in the conventional state-space TFSA, both the spectral element method and the collocation approach had spectral rates of convergence. Further, the spectral element approach had superior rates of convergence in comparison to Legendre collocation method. The $hp$ refinement capability in the spectral element approach was shown in Fig. 2.8 where the absolute error in calculating the maximum Floquet multiplier was plotted as a function of the number of elements and the order of the trial functions.

This chapter also described a multi-interval Chebyshev collocation approach for studying the stability of periodic DDEs with non-smooth coefficients. In contrast to the typical Chebyshev-collocation method, the presented approach gives control over mesh refinement independent of the order used, reduces errors associated with domain stretching, preserves spectral convergence properties, and accommodates the discontinuities in the coefficients via suitable boundary placement.

To summarize, the developed spectral element approach for DDEs is more robust than the conventional TFSA method and it allows $hp$ refinement which was lacking in the conventional approach. Further, our results revealed that the presented approach could have higher rates of convergence than collocation methods. This makes the current approach more suitable for stability studies of a larger class of problems including DDEs with complicated time-dependent coefficients as well as DDEs with multiple and distributed delays.

Moreover, the analysis in this chapter indicate that for the case of periodic DDE
with non-smooth coefficients the MIC approach is more suitable than the conventional Chebyshev approach. Specifically, whereas the MIC approach preserves spectral convergence, the conventional approach loses this property. Moreover, the MIC approach offers more flexibility in terms of selective mesh refinement as well as exploiting $h$-convergence properties in addition to the traditional spectral convergence scheme.
The analysis presented in Chapter 2 was for DDEs with a single discrete delay $\tau$ with either constant coefficients or with periodic coefficients having a period equal to the delay, i.e. $T = \tau$. Under these conditions, the discretization points map exactly onto the previous points eliminating the need for interpolation. A mesh with this property is called a constrained mesh and if it is combined with an appropriate choice of discretization points, such as Legendre-Gauss points, it gives rise to fast convergence as the number of points is increased (called super-convergence, see Ref. [45]). In contrast, the case $T \neq \tau$ gives rise to an unconstrained mesh and necessitates the use of an interpolation matrix (see Eq. (2.20)).

Decoupling the delay and the period and allowing them to vary independently introduces an additional time scale into the DDE. This scenario arises for example in the study of periodic solutions of nonlinear DDEs. The competition between the two time scales $T$ and $\tau$ influences the value of the largest eigenvalue which leads to drastic changes in the stability boundaries [75]. In order to study a DDE with decoupled $T$ and $\tau$, it is necessary to construct the correct mapping for each combination of $T$ and $\tau$. 

3

Constrained and unconstrained meshes

The analysis presented in Chapter 2 was for DDEs with a single discrete delay $\tau$ with either constant coefficients or with periodic coefficients having a period equal to the delay, i.e. $T = \tau$. Under these conditions, the discretization points map exactly onto the previous points eliminating the need for interpolation. A mesh with this property is called a constrained mesh and if it is combined with an appropriate choice of discretization points, such as Legendre-Gauss points, it gives rise to fast convergence as the number of points is increased (called super-convergence, see Ref. [45]). In contrast, the case $T \neq \tau$ gives rise to an unconstrained mesh and necessitates the use of an interpolation matrix (see Eq. (2.20)).

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50
The different mappings associated with different $T$ to $\tau$ ratios are shown in Fig. 3.1b–d. In these figures, the history segments, denoted by the light gray lines, are mapped ahead by one period onto the interval marked by the dark gray lines. Graph (a) shows the mapping of an ODE ($\tau = 0$) with $T$-periodic coefficients associated with applying Floquet theory described in Section 1. Recall that for this case it is sufficient to map the states at any instant in time one period ahead to create a dynamic map and investigate the system stability. However, for $\tau \neq 0$, i.e. Figs. 3.1c,d, it is necessary to map the states described by the history function one period ahead. The goal is to construct a dynamic map in the form

$$Hx_m = Gx_{m-1},$$

where $H$ and $G$ are real matrices whereas $x_m$ and $x_{m-1}$ are the vectors of the states in the current and the previous periods, respectively. The stability of the system is then investigated using the eigenvalues of the map.

This chapter shows the analysis of DDEs with $T \neq \tau$. Section 3.1 provides general considerations for the steps necessary to construct a dynamic map for different $T$. 

Figure 3.1: The different mappings necessary to create an approximate monodromy matrix for (a) $\tau = 0$, (b) $T = \tau$, (c) $T > \tau$, and (d) $T < \tau$. 

$$x(t)$$

(a) $\tau = 0$

(b) $T = \tau$

(c) $T > \tau$

(d) $T < \tau$
to \( \tau \) ratios. Section 3.2 covers the case of constrained meshes \( T = \tau \) presented in chapter 2. Section 3.4 studies the case \( T > \tau \) whereas section 3.3 treats the case \( T < \tau \). Case studies for the different delay and period ratios are presented in Section 3.5 and the chapter ends with the conclusions in section 3.6.

3.1 Mapping considerations

This section describes the general considerations associated with discretizing and mapping a DDE. To elaborate consider the general DDE described by

\[
\dot{x} = f(x(t), x(t - \tau), t) \quad \text{for} \quad t \in [0, T],
\]

\[
x(t) = \varphi(t) \quad \text{for} \quad t \in [-\tau, 0]
\]

where \( x \in \mathbb{R}^n \), \( f \) is smooth, \( T \) is the system period while the scalar \( \tau \) is a positive time delay. The function \( \varphi \) describes the history segment for \( t \in [-\tau, 0] \). Note that for autonomous systems \( T \) is not unique and can be chosen arbitrarily; however, for convenience, it is usually chosen as \( T = \tau \). As was shown in Chapter 2, the first step is to discretize the period \([0, T]\) using a finite number of temporal elements \( E \). Each element is described by the interval

\[
e_j = [t_j^-, t_j^+),
\]

where \( t_j^- \) and \( t_j^+ \) denote the left and right element boundaries, respectively, with the length of the \( j \)th element given by

\[
h_j = t_j^+ - t_j^-.
\]

The barycentric Lagrange formula is then used to obtain an approximate expression for the states over each element using a set of local interpolation nodes normalized by \( h_j \). For example, if \( n + 1 \) interpolation nodes were used then the
approximate states over the \( j \)th element are given according to

\[
x_j(t) = \sum_{i=1}^{n+1} \phi_i(\eta) x_{ji},
\]

(3.6)

where \( x_{ji} = x_j(t_i) \) with \( i \) indicating the \( i \)th interpolation node within the element and \( \phi_i \) are the trial functions (see Chapter 2). Substituting the expression from Eq. (3.5) into Eq. (3.2) gives

\[
\sum_{i=1}^{n+1} \frac{1}{h_j} \phi_i(\eta)x_{ji} - f \left( \sum_{i=1}^{n+1} \phi_i(\eta)x_{ji}, \sum_{i=1}^{n+1} \phi_i(\eta^*)x_{j^*(t^*)i}, t_j^- + \eta h_j \right) = \text{error},
\]

(3.7)

where the residual error is due to the approximation procedure while the time \( t^* \) is defined using modular arithmetics according to

\[
t^* = t_j^- + \eta h_j - \tau \quad (\text{mod } T).
\]

(3.8)

Equation (3.8) is used to define the local normalized time according to

\[
\eta^* = \frac{t^* - t_j^-}{h_j^*},
\]

(3.9)

where \( t_j^- \) is the left boundary of element \( j^* = j^*(t^*) \) while \( h_j^* \) is its length. The function \( j^*(t^*) \) gives the element index to which \( t^* \) belongs and it is given by

\[
j^*(t^*) = \sum_{j=1}^{E} j \chi_{e_j}(t^*),
\]

(3.10)

where the indicator function is defined as

\[
\chi_{e_j}(t^*) = \begin{cases} 1 & \text{if } t^* \in e_j, \\ 0 & \text{otherwise}. \end{cases}
\]

(3.11)

If the elements are uniformly distributed, then Eq. (3.10) reduces to

\[
j^*(t^*) = \left\lfloor \frac{t^*}{h} \right\rfloor
\]

(3.12)
where $\lceil \cdot \rceil$ is the ceiling function and $h$ is the length of each of the uniform elements.

The integer $q$ in Eq. (3.7) is the number of the period to which the delay looks back and it is described by

$$q = \left\lfloor \frac{t - \tau}{T} \right\rfloor,$$

(3.13)

where $\lfloor \cdot \rfloor$ is the floor function and with the understanding that $q = 0$ indicates a mapping onto the interval $[0, T]$.

As an example, consider the mapping shown in Fig. 3.2. In this figure, 3 elements were used to discretize the interval $[0, T]$ and the delay was assumed to map back to at most 2 periods in the past. Note that the same discretization needs to be used in all previous periods up to the maximum period the delay maps back to. Without making any assumptions on the number of interpolation points used within each element, Fig. 3.2 shows the mapping of an arbitrary node $\eta \in \{\eta_i\}_{i=1}^{n+1}$ in the second element at time $t \in [0, T]$. The node is mapped onto a point in the interval $[-2T, -T]$. Since the whole time-line is discretized using the same mesh in each interval of length $T$, Eqs. (3.8) and (3.10) can be used to find the element $j^*$ where the point will map to ($j^* = 3$ in this example). The local time within that element can be found using Eq. (3.9) and, in general, it will not align with one of the interpolation nodes within that element.

When the mapped point does not coincide with any of the interpolation nodes interpolation becomes necessary. The interpolation describes the states at $t^*$ in terms of the states at the interpolation nodes in the $j^*$th element within the $q$th period using Eqs. (2.9) and (2.20). Specifically, this interpolation can be described using the interpolation matrix defined in Eq. (2.20) according to

$$x(\eta^*) = \Upsilon_{(\eta^* \rightarrow \eta)} x^q_{j^*},$$

(3.14)

where $\Upsilon$ is the interpolation matrix from $\eta^*$ onto $\{\eta_i\}_{i=1}^{n+1}$ while $x^q_{j^*}$ is the vector of
states in the \( j^* \) element within the previous \( q \)th period \( (q = 2 \text{ in Fig. 3.2}) \).

\[ -2T \quad -T \quad 0 \quad \eta \quad T \]

\[ t - \tau \quad \text{(mod } T) \quad t^* \]

**Figure 3.2:** An example that shows the steps necessary to construct a dynamic map for DDES with arbitrary \( T \) and \( \tau \) ratios.

To construct a dynamic map, each discretization point in \([0, T]\) is mapped by \( \tau \) using Eq. (3.14) whenever interpolation is needed. The method of weighted residuals is then used (see Chapter 2) and the result is stored in a matrix that describes the evolution of the DDE from \( t \leq 0 \) onto \([0, T]\). The specific type of the map depends on the ratio of \( T \) to \( \tau \) as shown in Fig. 3.1. The following sections will describe in more detail the specific types of mapping and provide some comments on the resulting structure of the matrices.

### 3.2 Case 1: Period = delay

The case of autonomous DDEs or of DDEs with \( T = \tau \) was handled in chapter 2 and is depicted in Fig. 3.1b. This case results in a constrained mesh where each point is mapped exactly onto a corresponding discretization point. To elaborate, let the order of the DDE be \( d \), the size of the discretization mesh be \( n + 1 \) and the number of elements \( E = 3 \). The resulting matrices have the equal sizes \( H \in \mathbb{R}^{d(En+1) \times d(En+1)} \) and \( G \in \mathbb{R}^{d(En+1) \times d(En+1)} \). An example of the resulting structure for a 2nd order DDE with \( n = 4 \) and \( E = 3 \) is shown in Fig. 3.3.
This example is for Eq. (3.16) described further in this chapter. Recall that the stability is determined by the eigenvalues of the monodromy matrix $U = H^{-1}G$ as was described in chapters 1 and 2. Note that for this case the entries are block matrices aligned in a band near the diagonal and all the entries off this band are zeros except the last $d$ rows. These last $d$ rows in both the $H$ and the $G$ matrix contain the $d \times d$ identity matrix multiplying the point $x(0)$ to enforce the continuity condition at $t = 0$.

However, when $T \neq \tau$, it is necessary to interpolate using Eq. (2.20) since, in general, the mesh points do not map exactly onto the corresponding mesh points as will be shown in the following sections.

3.3 Case 2: Period larger than delay

The case $T > \tau$ is shown in Fig. 3.1c. Whereas for $t < 0$ the history function gives the value of the states, the evolution for $t \geq 0$ is governed by the DDE. To obtain the correct mapping for this case, the period $[0, T]$ is discretized and the DDE is used to obtain the mapping from $[-T, 0]$ onto $[0, T]$. Interpolation should be used whenever the mesh points do not line up in the two intervals (see Eq. (2.20)) and
the last $d$ rows are modified to enforce the continuity condition at $t = 0$. The sizes of the resulting matrices will be similar to the case $T = \tau$. However, the structures of the $H$ and the $G$ matrices will be different as shown in Fig. 3.4. This figure shows that non-zero off diagonal entries exist due to the $T$ and $\tau$ ratio which can cause interim mapping as shown in Fig. 3.1c. Note that for this case the delayed term will also modify some terms in the $H$ matrix in addition to modifying the $G$ matrix as shown in Fig. 3.4.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
  \centering
  \includegraphics[width=\textwidth]{H_matrix}
  \caption{$H$}
\end{subfigure}\hfill
\begin{subfigure}{0.45\textwidth}
  \centering
  \includegraphics[width=\textwidth]{G_matrix}
  \caption{$G$}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
  \centering
  \includegraphics[width=\textwidth]{H_matrix}
  \caption{$H$}
\end{subfigure}\hfill
\begin{subfigure}{0.45\textwidth}
  \centering
  \includegraphics[width=\textwidth]{G_matrix}
  \caption{$G$}
\end{subfigure}
\caption{The matrix structures associated with the case $T > \tau$ described in Section 3.3. Non-zero entries are marked by dots while zero entries are unmarked. Figures a,b describe the $H$ and $G$ matrices, respectively, for a rational $T/\tau$ whereas Figs. c,d describe the $H$ and $G$ matrices, respectively for irrational $T/\tau$.}
\end{figure}
3.4 Case 3: Period less than delay

Figure 3.1d shows the case $T < \tau$ where two differences in comparison to the prior cases can be observed: (1) the delay term looks back more than one period up to $N_T = \lceil \tau/T \rceil$ periods where $\lceil \cdot \rceil$ denotes the ceiling function, and (2) mapping the history function one period ahead results in an interval where the history function and the mapped states overlap.

A consequence of (1) is that the $G$ matrix in Eq. (3.1) needs to be extended to cover the maximum number of periods the delay looks back. To illustrate, consider Fig. 3.5 which shows the $H$ and $G$ matrices for different $T < \tau$ cases using $n = 4$ and $E = 3$. The size of the $H$ matrix is $d (E n + 1) \times d (E n + 1)$—similar to the above two cases—whereas the size of the $G$ matrix is $d (E n + 1) \times d (N_T E n + 1)$. Figures 3.5a,b are for commensurate $T$ and $\tau$ whereas Figs. 3.5c,d shows the incommensurate case. For this case, note that the $H$ matrix has a size and a structure similar to Fig. 3.5a where the delay terms do not affect the $H$ matrix. Further, Figs. 3.5b,d show how the $G$ matrix is extended to cover the maximum number of periods the delay looks back. For this specific example $N_T = 2$ and for $t > 0$ the delayed term acts on the period $[-2T, -T]$ as shown in Figs. 3.5b,d.

The forms of $H$ and $G$ described above constitute the contribution from the evolution of the DDE. To account for the overlap shown in Fig. 3.1d, and described in point (2) above, additional equations need to be added to the monodromy matrix. Specifically, the monodromy matrix takes the form

$$ U = \begin{bmatrix} 0_{(d(N_T-1)E n)\times d E n} & I_{d(N_T-1)E n} & 0_{(d(N_T-1)E n)\times d} \\ H^{-1}G \end{bmatrix}, \quad (3.15) $$

where $I$ is an identity matrix and $0$ is a matrix of zeros. The eigenvalues of $U$ can then be used to determine the stability of the DDE according to the criteria shown
in Fig. 1.1. 

**Figure 3.5:** The matrix structures associated with the case $T < \tau$ described in Section 3.4. Non-zero entries are marked by dots while zero entries are unmarked. Figures a,b describe the $H$ and $G$ matrices, respectively, for a rational $T/\tau$ whereas Figs. c,d describe the $H$ and $G$ matrices, respectively for irrational $T/\tau$.

### 3.5 Examples

The complication of different ratios of $T$ to $\tau$ will be investigated using the damped delayed Mathieu equation [44]

$$\ddot{x}(t) + \kappa \dot{x}(t) + (\delta + \varepsilon \cos(2\pi/T))x(t) = bx(t - \tau), \quad (3.16)$$

where $\varepsilon$ and $\delta$ are scalars, the time period is $T$, and the constant positive delay is $\tau$.
Figure 3.6 shows the stability diagrams of Eq. (3.16) corresponding to different combinations of $T$ and $\tau$ as well as different values for $\kappa$ and $\epsilon$. The first column corresponds to $\epsilon = 1$ whereas the second row corresponds to $\epsilon = 2$. The results in Fig. 3.6 are in agreement with the results obtained in Ref. [44] using the semi-discretization method.

Figures 3.6a,b show the stability diagram for the case $T = \tau/2 = \pi$ which was obtained using $E = 2$ and $n = 5$. This case corresponds to Section 3.3 where the case $T < \tau$ was described. However, note that since the delay and the period are commensurate, the resulting mesh can be chosen to be constrained.

Figures 3.6c,d show the stability diagram for the case $T = \tau/\sqrt{2} = \sqrt{2}\pi$ which was obtained using $E = 2$ and $n = 10$. This case also corresponds to Section 3.3; however, in contrast to Figs. a,b, this case represents incommensurate period and delay resulting in unconstrained meshes.

The stability chart for the case $T = \tau = 2\pi$ is shown in Figs. e,f where the parameters $E = 2$ and $n = 5$ were used. This case corresponds to the case described in Chapter 2 and Section 3.2.

The case $T = \sqrt{2}\pi$ is shown in Figs. 3.6g,h. This case corresponds to Section 3.3 where $T > \tau$. Note that the resulting mesh is unconstrained since the ratio between the delay and the period is incommensurate. The parameters $E = 2$ and $n = 5$ were used to produce Figs. 3.6g,h.

Finally, the case $T = 2\tau = 4\pi$ is shown in Figs. 3.6i,j. This case also corresponds to Section 3.3 where the inequality $T > \tau$ is true. However, in contrast to Figs. 3.6g,h, the ratio between $T$ and $\tau$ is commensurate and the mesh can be chosen to be constrained.
Figure 3.6: The stability diagrams of Eq. (3.16) using different $T$ to $\tau$ ratios and various values for $\kappa$ and $\epsilon$ (200 $\times$ 200 grid). The first and second columns correspond to $\epsilon = 1$ and $\epsilon = 2$, respectively. $E = 2, n = 5$ in a–f whereas $E = 2, n = 10$ in g–j.
3.6 Conclusions

This chapter described the different types of meshes that result in when discretizing DDEs with periodic coefficients. The associating matrix structures were also described for the different cases of $T$ to $\tau$ ratios. Although for a large class of DDEs the period and the delay are equal, generally they are decoupled and can vary independently. When the delay and the period are different, the DDE will have two competing time scales that will have a great impact on the system stability. Several cases corresponding to different period to delay ratios were discussed.

For example, when the delay and the period are equal, the resulting mesh points are mapped exactly onto previous points and the mesh is called a constrained mesh. Constrained meshes eliminate the need for interpolation and they lead to high rates of convergence called super-convergence. In contrast, when the delay and period are unequal, the resulting unconstrained mesh generally maps some points onto interim regions in the mesh necessitating the use of interpolation. Some exceptions include the cases when the ratio between the period and the delay is rational (see for example Figs. 3.6a,b,i and j. Specifically, choosing the length of the temporal element in these cases to be equal to the length of the smaller time scale can lead to constrained meshes.

To show the effect of different period to delay ratios on stability a set of stability diagrams for the delayed damped Mathieu equation was shown in Fig. 3.6. Figures 3.6a–d corresponded to the case $T < \tau$ described in Section 3.4. Whereas Figs. 3.6a,b described a constrained mesh ($T$ and $\tau$ are commensurate), cases c and d corresponded to an unconstrained mesh since $T$ and $\tau$ were incommensurate. Figures 3.6e–f corresponded to the case $T = \tau$ which was described in Chapter 2 and Section 3.2. This case always produces commensurate meshes and the analysis is simplified by having only one time scale in the DDE.
On the other hand, Figs. 3.6g–j described the case $T > \tau$. Figures 3.6g, h produced unconstrained meshes since the ratio between the period and the delay was irrational. Cases i and j, however, could use a constrained mesh since the ratio between the period and the delay was rational.
In the previous chapters, the delay was a discrete scalar that acts at a single instant of time (called concentrated or discrete delay). However, in some applications the effect of the delay is distributed over a time interval according to the delay integro-differential equation (DIDE)

\[
\dot{x}(t) = f \left( x(t), \int_{0}^{\vartheta} K(t, s)x(t - s) \, ds \right),
\]

(4.1a) \quad x(t) = \varphi(t) \quad \text{for} \quad t \in [-\vartheta, 0],

(4.1b)

where \( x \in \mathbb{R}^n \), \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable, \( \varphi \) is continuous and it represents the history segment, \( K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n} \) is a bounded analytic kernel function while \( \vartheta > 0 \) is the duration of the bounded distributed delay. Examples where several variations of Eq. (4.1) appear include wheel shimmy models [25, 26], traffic models [36, 37] and machining dynamics [54]. A common interest in all these application areas is the stability of equilibria in the governing DIDE.

The stability analysis of DIDEs is more complicated than delay differential equations (DDEs) with discrete delays due to the added complexity of the integral term. This is evidenced by the limited number of studies on DIDEs compared to the more
extensive literature on DDEs with discrete delays. Nevertheless, several methods have been used in literature for studying DIDEs. For instance, the stability conditions of Runge-Kutta type methods for DIDEs was investigated in Refs. [76, 77]. Baker et. al studied the stability of scalar DIDEs with a constant kernel using numerical schemes based on Newton-Cotes formula combined with a linear multi-step method [78].

Luzyanina et al. used a linear multi-step method and a Gauss quadrature to find the stability of scalar DIDEs [79]. Both constant and non-constant kernels comprised of trigonometric and polynomial functions were investigated. Further, computational techniques for discrete DDEs were applied to DIDEs with special kernels in Ref. [80]. Specifically, the kernel functions were either constant or had a gamma-type distribution.

Insperger and Stepan used the semi-discretization method to study the stability of DIDEs for various constant and non-constant kernels [43]. The semi-discretization method divides the time-line into short intervals on which an approximate analytical solution can be obtained. The collection of these expressions can then be used to create a finite dimensional transition matrix which approximates the infinite dimensional monodromy operator of the DDE. However, the semi-discretization method can result in large matrices to produce a convergent solution.

Breda et al. presented an approach to ascertain the stability of DIDEs using the discretization of the infinitesimal generator of the solution operators semigroup [81]. The approximate eigenvalues were then found by solving a large sparse standard eigenvalue problem. However, this approach has been only used for autonomous DIDEs. Further, the size of the resulting matrices can in some cases cause computational difficulties.

The above survey on methods for solving DIDEs revealed that the related research has mostly been focused on stability of numerical schemes for DIDEs with
special types of kernel (mostly constant or gamma-type). Therefore, the focus in this chapter is on extending the spectral element approach to delay equations with distributed delays. Specifically, both autonomous and non-autonomous DIDEs will be investigated for general bounded and smooth kernel functions.

Section 4.1 describes the extension of the spectral element approach to DIDEs. Section 4.2 provides the stability diagrams of some case studies including both autonomous and time-periodic DIDEs. Section 4.3 studies a special case of DIDEs that can be reduced to a DDE with only discrete delay. The chapter ends with the conclusions in Section 4.4.

4.1 Stability analysis of DIDE using spectral element

The stability of DIDEs can be investigated by extending the spectral element approach to this class of DDEs. To illustrate, recall that a Legendre-Gauss-Lobatto (LGL) quadrature rule can be used to approximate integrals according to

\[
\int_{a}^{b} f(t) \, dt \approx b - a - \sum_{k=1}^{m+1} w_{k}^{*} f(t_{k}),
\]

where \( t_{k} \) and \( w_{k}^{*} \) are the LGL quadrature nodes and weights, respectively, and they can be obtained using the equations described in Section 2.4.1. Equation (4.2) yields accurate results for integrands that can be approximated by polynomials of order \( 2m - 1 \) [79]. Throughout this section we will assume that the integrand is well approximated by such polynomial and fix the number of the quadrature points to \( m + 1 \).

Using Eq. (4.2) to approximate the distributed delay integral in Eq. (4.1) gives

\[
\dot{x}(t) - f \left( x(t), \frac{1}{2} \sum_{k=1}^{m+1} w_{k}^{*} K(t, s_{k}) x(t - s_{k}) \right) = 0.
\]
The next step is to divide the interval \([0, \vartheta]\) (for autonomous systems) or \([0, T]\) (for non-autonomous systems) into temporal elements. A local approximate solution is then sought within each element similar to Eq. (2.9). Substituting the approximate solution over the \(j\)th element into Eq. (4.3) yields

\[
\sum_{i=1}^{n+1} \frac{1}{h_j} \phi_i(\eta) x_{ji} - f \left( \sum_{i=1}^{n+1} \phi_i(\eta) x_{ji}, \frac{\partial}{\partial \eta} \sum_{k=1}^{m+1} w^*_k K(t, s_k) \sum_{i=1}^{n+1} \phi_i(\eta_k) x^{q_k}_{ji*(t_k^*)} \right) = \text{error}, \tag{4.4}
\]

where the residual error is due to the approximation procedure while \(h_j\) is the length of the \(j\)th element, see Section 3.1. The time \(t_k^*\) is defined using modular arithmetics according to

\[
t_k^* = t_j^- + \eta h_j - s_k \pmod{T}, \tag{4.5}
\]

where \(t_j^-\) is the left boundary of the \(j\)th element. Equation (4.5) is used to define the local normalized time according to

\[
\eta_k^* = \frac{t_k^* - t_j^-}{h_j^*}, \tag{4.6}
\]

where \(t_j^-\) is the left boundary of the element \(j^* = j^*(t_k^*)\) while \(h_j^*\) is its length. The function \(j^*(t_k^*)\) gives the element index to which \(t_k^*\) belongs and it is obtained by making the substitution \(t^* = t_k^*\) in Eqs. (3.10)–(3.12).

The integer \(q_k\) in Eq. (4.4) is the number of the period to which the delay looks back and it is described by

\[
q_k = \left\lfloor \frac{t - s_k}{T} \right\rfloor, \tag{4.7}
\]

where \(\lfloor \cdot \rfloor\) is the floor function and with the understanding that \(q_k = 0\) indicates a mapping onto the interval \([0, T]\).

Note that for DIDEs each discretization point \(\eta_i\) is mapped back onto \(m+1\) points due to the integral. This mapping of a point into an interval is described in
Fig. 4.1 where it is shown that a point can further get mapped onto more than one element. If any of the mapped points does not coincide the discretization nodes, interpolation using Eq. (3.14) needs to be used.

**Figure 4.1:** The effect of the distributed delay on the mapping of the system states at time \( t \) (black dot). Each point is mapped onto an interval that can span more than one element. In this illustration, the mapping spans two elements marked by \( \overset{1}{\circ} \) and \( \overset{2}{\circ} \), respectively.

Similar to Chapter 3, the mapping is different for the different types of DIDEs. Specifically, whereas it is usually convenient to choose \( T = \vartheta \) for autonomous systems, mapping non-autonomous systems depends on the ratio between the maximum delay \( \vartheta \) and the period \( T \).

**Figure 4.2:** An example mapping of the system states (light gray line) one period ahead (dark gray line) for (a) autonomous as well as (b)–(d) time-periodic DIDEs.

Figure 4.2 shows the different types of mappings of DIDES. Note the similarity
between this figure and Fig. 3.1. Hence, the steps described in Chapter 3 can be used when mapping each of the points in [0, T] to the corresponding \( m + 1 \) points resulting from discretizing the distributed delay integral.

The method of weighted residuals described in Chapter 2 can then be used to reduce the approximation error and create a dynamic map over one period. The eigenvalues of the map can be used to ascertain the system stability using the criteria in Fig. 1.1.

### 4.2 Case studies

The effectiveness of the spectral element method for DIDEs is investigated using a set of case studies. Section 4.2.1 studies the stability of a scalar autonomous DIDE with a constant kernel function whereas Section 4.2.2 investigates a 2nd order time-periodic DIDE using various constant and time-dependent kernel functions.

#### 4.2.1 First order DIDE

As a first example, consider the scalar DIDE

\[
\dot{y}(t) = \alpha y(t) + \beta \int_{0}^{1} y(t - s) \, ds \tag{4.8}
\]

where \( \alpha \) and \( \beta \) are constants [78]. The kernel function here is constant and is equal to 1, i.e. \( K(t, s) = 1 \). Applying the analysis in Section 4.1 using \( E = 2 \) and \( n = 5 \) gives the stability diagram in Fig. 4.3. This diagram is in agreement with the analytical result obtained in Ref. [78].
Figure 4.3: The stability diagrams of Eq. (4.8) for different values of $\epsilon$. The parameters used were $E = 2$, $n = 5$ and $m = 5$ while the grid size was $200 \times 200$. Stable regions are shaded while unstable regions are left unshaded.

4.2.2 Second order DIDE with time-periodic coefficients

As a more complicated example, consider the second order time-periodic DIDE

$$
\ddot{x}(t) + \left(\delta + \epsilon \cos \left(\frac{4\pi t}{T}\right)\right)x(t) = \gamma \int_0^1 K(t, s)x(t-s)ds,
$$

(4.9)

where $\delta$, $\epsilon$ and $\gamma$ are constants while the system period is $T = 0.5$. The stability of this equation was studied in the parameter space $(\delta, \gamma)$ for different values of $\epsilon$ and for various kernel functions $K(s)$. The kernel functions that were used in this study are

$$
K(t, s) = 1, \quad (4.10a)
$$

$$
K(t, s) = \frac{\pi}{2} \sin(\pi s), \quad (4.10b)
$$

$$
K(t, s) = \frac{\pi}{2} \sin(\pi s) + \frac{13\pi}{77} \sin(2\pi s). \quad (4.10c)
$$

The stability diagrams associated with each of these kernels are shown in Figs. 4.4–4.6 and they are all in agreement with the results in Ref. [43].

Figure 4.4 shows the stability diagram for the constant kernel function in Eq. (4.10a).
Figure 4.4: The stability diagrams of Eq. (4.9) using the kernel function in Eq. (4.10a) and different values of $\epsilon$. The parameters used were $E = 3$, $n = 10$ and $m = 15$ while the grid size was $200 \times 200$. Stable regions are shaded while unstable regions are left unshaded.

Note that for $\epsilon = 0$ in Fig. 4.4a the resulting DIDE is autonomous whereas for $\epsilon \neq 0$ in Figs. 4.4b–d the DIDE is time periodic with $T = 0.5$. The parameters used to generate the stability diagrams in Fig. 4.4 are $E = 3$, $n = 10$ and $m = 15$. In contrast to the simple DIDE in Eq. (4.8), Eq. (4.9) required higher $E$, $n$ and $m$ values to converge.

Figure 4.5 shows the stability diagram for the kernel function in Eq. (4.10b). The result for the autonomous version of Eq. (4.9) is shown in Fig. 4.5 whereas the non-autonomous version is depicted in Figs. 4.5b–d. Similar to Fig. 4.5, The parameters
Figure 4.5: The stability diagrams of Eq. (4.9) using the kernel function in Eq. (4.10b) and different values of $\epsilon$. The parameters used were $E = 3$, $n = 10$ and $m = 15$ while the grid size was $200 \times 200$. Stable regions are shaded while unstable regions are left unshaded.

used to generate the stability diagrams in Fig. 4.5 are $E = 3$, $n = 10$ and $m = 15$.

Figure 4.6 shows the stability diagram for the kernel function in Eq. (4.10c). The result for the autonomous version of Eq. (4.9) is shown in Fig. 4.6 whereas the non-autonomous version is depicted in Figs. 4.6b–d. The parameters used to generate the stability diagrams were $E = 3$, $n = 10$ and $m = 15$.

4.3 Special Distribution Functions

For certain choices of the distribution function, namely gamma-type distributions such as exponential, gamma function and sinusoidal distributions, the integro-DDE
Figure 4.6: The stability diagrams of Eq. (4.9) using the kernel function in Eq. (4.10c) and different values of $\epsilon$. The parameters used were $E = 3$, $n = 10$ and $m = 15$ while the grid size was $300 \times 300$. Stable regions are shaded while unstable regions are left unshaded.

can be transformed into a higher order DDE with only discrete delays on the form described in chapter 2. Hence the described techniques can be readily applied to the transformed equation to obtain stability of the original integro-DDE. Fortunately, although this choice of distribution functions is somewhat limiting, it still covers a lot of the common distributions found in physical applications [26,54].

This section uses a scalar DDE to describe the various steps of transforming an integro-DDE into a higher order DDE with only discrete delays. The same steps are used in chapter 7 to study the stability of a turning process at low speeds.

A first-order scalar DDE with both a discrete and a continuous delay is considered
Consider the first order DDE

$$\dot{x}(t) + \alpha x(t) + \beta \int_0^{t_s} x(t - \tau - \hat{t}) w(\hat{t}) d\hat{t} = 0,$$

where $t$ is time, $x$ is a real variable, $\alpha$ and $\beta$ are real scalars, $\tau$ is a discrete delay, $\hat{t}$ is a continuous or short delay, $t_s$ is the duration of the short delay, and $w(\hat{t})$ is a shape or weight function. The integral in Eq. (4.11), as well as the existence of two types of delays, complicates the stability analysis of Eq. (4.11) in its current form. However, with a suitable choice of the weight function $w(\hat{t})$, Eq. (4.11) can be transformed into a solvable form. In fact, a reasonable choice for $w(\hat{t})$ is

$$w(\hat{t}) = e^{-\hat{t} \gamma},$$

where $\gamma = t_s/\tau$ is the ratio of the short delay to the long one and it will be assumed to be constant. This choice of an exponential term for the weight function implies that the effect of the continuous delay disappears gradually as $t_s \to \infty$, see Fig. 4.7. With this choice for $w(\hat{t})$, the continuous delay term in Eq. (4.11) can be eliminated by first differentiating the equation with respect to time to obtain

$$\ddot{x}(t) + \alpha \dot{x}(t) + \beta \int_0^\infty \dot{x}(t - \tau - \hat{t}) e^{-\hat{t} \gamma} d\hat{t} = 0.$$

Note that differentiating the original DIDE does not produce a system that is completely equivalent to the original DIDE [80,83]. Specifically, (i) whereas the zero
solution is the only steady state solution of the original DIDE, the differentiated equation is satisfied for any constant solution. Further, the stability of the zero solution of both systems is not completely equivalent since the characteristic equations of the differentiated equation will always have an additional zero root. Nevertheless, besides the extra root, all the other roots coincide with those of the original DIDE. Therefore, the stability of the original DIDE can practically be investigated by studying the stability of the differentiated equation [80,83].

Therefore, we proceed by integrating the continuous delay term by parts gives

\[
\beta \int_0^\infty \dot{x}(t-\tau-\hat{t})e^{-\gamma \tau \hat{t}} d\hat{t} = -\beta \left[ x(t-\tau-\hat{t})e^{-\gamma \tau \hat{t}} \right]_0^\infty \\
+ \frac{1}{\gamma \tau} \int_0^\infty x(t-\tau-\hat{t})e^{-\gamma \tau \hat{t}} d\hat{t} \\
= \beta x(t-\tau) + \frac{1}{\gamma \tau} [\dot{x}(t) + \alpha x(t)]. \quad (4.14)
\]
Finally, substituting Eq. (4.14) into Eq. (4.13) gives

$$
\ddot{x}(t) + (\alpha + \frac{1}{\gamma \tau}) \dot{x}(t) + \frac{\alpha}{\gamma \tau} \beta x(t - \tau) = 0,
$$

or in state-space form

$$
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\frac{\alpha}{\gamma \tau} & -\left(\alpha + \frac{1}{\gamma \tau}\right)
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
-\beta & 0
\end{bmatrix}
\begin{bmatrix}
y_1(t - \tau) \\
y_2(t - \tau)
\end{bmatrix},
$$

where now Eq. (4.15) represents a transformation of Eq. (4.11) that contains the discrete delay only. The analysis techniques described in chapter 2 can then be used to investigate the stability of Eq. (4.15). Figure 4.8 shows the stability chart for Eq. (4.11) for $\gamma = 0.5$ and $1.5$ assuming a discrete delay $\tau = 1$.

![Stability Chart](image)

**Figure 4.8:** Stability charts of Eq. (4.11) for $\tau = 1$, and (A) $\gamma = 0.5$ and (B) $\gamma = 1.5$. Moreover, a grid of $100 \times 100$ and $E = 10$ were used. Unstable regions are shaded while stable regions are left unshaded.

Therefore, the methods described here successfully transformed the complicated distributed delay problem into a higher-order, simpler DDE that can be handled using the methods described in chapter 2. Although the described transformation works only for gamma-type distribution, these types of distributions appear in a
vast variety of applications including machining dynamics (see chapter 7) and wheel shimmy models.

4.4 Conclusions

This chapter described a spectral element approach for studying the stability of delay integro-differential equations (DIDEs). In comparison to DDEs with a discrete delay, the analysis of DIDEs is more complicated due to the integral term which maps single points onto intervals. The distributed delay term was approximated using a Legendre-Gauss-Lobatto quadrature with \( m + 1 \) mesh points. This approximation caused each discretization node in the interval \([0,T]\) to map onto \( m + 1 \) points. The mapped points were interpolated between the discretization nodes whenever an unconstrained mesh was encountered.

To demonstrate the effectiveness of the developed approach, the stability of several case studies was investigated. The first case study was a scalar DIDE with constant kernel, see Fig. 4.3. The second case study was a time-periodic DIDE with constant as well as time-dependent kernel function. The stability diagrams using a constant kernel are shown in Fig. 4.4 whereas the results obtained using time-periodic kernels are shown in Figs. 4.5, 4.6. The stability diagrams were in agreement with those obtained using well-established methods in literature, namely D-subdivision and the semi-discretization technique. However, it was found that due to the large number of mappings associated with the integral term, the time required to produce the stability diagrams was significantly larger than the discrete delay case.

The case of special kernel functions was investigated in Section 4.3. These constant or gamma-type distribution functions appear in many physical models, such as low speed milling and wheel shimmy, and they allow the DIDE to be transformed into a higher order DDE with only discrete delays. The stability of the original DIDE can then be investigated using the simpler transformed DDE.
Often times, the models of physical systems include more than one delay. Some examples where multiple delays appear include traffic stability models [6, 84–86], laser systems [15, 34] and variable pitch mills [4, 5]. The existence of multiple delays in the system often leads to complicated stability structures in the delays parameter space. Specifically, increasing one of the delays may stabilize the system whereas increasing one of the other delays can destabilize the same system.

The difficulties associated with studying multiple time delay systems MTDS is evidenced by the limited literature on the topic. For example, time integration methods were developed based on either discretizing the solution operator [87–89] or discretizing the infinitesimal generator of the solution operator semigroup [38, 81].

Analytical methods have also been used to study the stability of MTDS. For instance, Hale and Huang studied the stability of autonomous first order MTDS [90] using D-subdivision. Stepan [18] and Niculescu [91] investigated autonomous second order MTDS using D-subdivision and a frequency domain approach, respectively, but without damping terms. Sipahi and Olgac used the Cluster Treatment of Characteristic Roots method to study the stability of autonomous second order MTDS
including damping [92].

The stability of autonomous MTDS was also studied in Ref. [40] using the Continuous Time Approximation method (CTA). However, the CTA method often results in very large matrices which can lead to computational difficulties. Insperger and Stepan investigated the stability of autonomous as well as non-autonomous second order MTDS using the semi-discretization approach [43]. However, the semi-discretization approach often results in large matrices and long computational times.

A collocation method for the stability of MTDS based on piecewise polynomials was analyzed in Refs. [45, 47]. However, as was shown in Chapter 2, the spectral element method can have higher rates of convergence than collocation methods.

This chapter will extend the spectral element approach to study the stability of MTDS. In contrast to prior works on MTDS, the spectral element approach is applicable to both autonomous as well as non-autonomous MTDS. Both MTDS of first order or higher can obtained and systems with or without damping can be investigated. Since the spectral element approach uses efficient interpolation and a set of well-distributed interpolation points, the size of the matrices necessary for convergence is kept small. Further, since the spectral element approach is a semi-analytical procedure, it avoids the tedious time marching calculations.

Section 5.1 describes the spectral element approach for MTDS. The effectiveness of the developed approach is demonstrated in Section 5.2 using a set of case studies. Specifically, Section 5.2.1 investigates the stability of a scalar MTDS [90,93] whereas Section 5.2.2 studies the stability of a second order MTDS with cross talking delays [92]. Section 5.2.3 investigates the stability of a second order MTDS with time-periodic coefficients while Section 5.3 provides concluding remarks.
5.1 Analysis of MTDS using spectral element approach

Differential equations with multiple discrete delays can be described as

\[ \dot{x}(t) = f \left( x(t), \sum_{k=1}^{n_r} x(t - \tau_k) \right), \tag{5.1} \]

where \( x \in \mathbb{R}^n \), \( n_r \) is the number of delays and \( f \) is smooth. Note that Eq. (5.1) is a special case of Eq. (4.1) caused by choosing Dirac delta kernel functions according to

\[ K(t, s) = \sum_{k=1}^{n_r} \delta(s - \tau_k). \tag{5.2} \]

Nevertheless, since the case of multiple delays is of great importance in many physical applications, a separate chapter was dedicated to their analysis.

The first step is to discretize the interval \([0, T]\) for \( T\)-periodic MTDS into a set of temporal elements. If the MTDS is autonomous, then \( T \) is set to \( \max_{1 \leq k \leq n_r} (\tau_k) \) merely for convenience. Within each element, an approximate solution is defined on a set well-distributed nodes such as the set of LGL nodes, see Chapter 2. Substituting the expression for the approximate states over the \( j \)th element into Eq. (5.1) yields

\[ \sum_{i=1}^{n+1} \dot{\phi}_i(\eta) x_{ji} - f \left( \sum_{i=1}^{n+1} \phi_i(\eta) x_{ji}, \sum_{k=1}^{n_r} \sum_{i=1}^{n+1} \phi_i(\eta_k^* x_{ji}^*(t_k^*,i) \right) = \text{error}, \tag{5.3} \]

where the residual error is due to the approximation procedure while \( h_j \) is the length of the \( j \)th element, see Section 3.1. The time \( t_k^* \) is defined using modular arithmetics according to

\[ t_k^* = t_j^- + \eta h_j - \tau_k \pmod{T}. \tag{5.4} \]

where \( t_j^- \) is the left boundary of the \( j \)th element. Equation (5.4) is used to define the local normalized time according to

\[ \eta_k^* = \frac{t_k^* - t_j^-}{h_j^*}, \tag{5.5} \]
where $t_j^* −$ is the left boundary of the element $j^* = j^*(t_k^*)$ while $h_j^*$ is its length. the function $j^*(t_k^*)$ gives the element index to which $t_k^*$ belongs and it is obtained by making the substitution $t_k^* = t_k^*$ in Eqs. (3.10)-(3.12).

The integer $q_k$ in Eq. (5.3) is the number of the period to which the delay looks back and it is described by

$$q_k = \left\lfloor \frac{t - \tau_k}{T} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function and with the understanding that $q_k = 0$ indicates a mapping onto the interval $[0, T]$.

Similar to Chapter 3, the mapping is different for the different types of MTDS. Specifically, whereas it is usually convenient to choose $T = \max_{1 \leq k \leq n_r} (\tau_k)$ for autonomous systems, mapping non-autonomous systems depends on the ratio between the maximum delay $\vartheta$ and the period $T$. In either case, interpolation is necessary whenever the mapped nodes in $[0, T]$ do not line up with any of the prior nodes, see Section 3.1.

The method of weighted residuals described in Chapter 2 can then be used to reduce the approximation error and create a dynamic map over one period. The eigenvalues of the map can be used to ascertain the system stability using the criteria in Fig. 1.1.

5.2 Case studies

This section uses 3 case studies to demonstrate the effectiveness of the stability analysis using spectral element approach. Section 5.2.1 studies an introductory scalar MTDS. Section 5.2.2 investigates the stability of a 2nd order MTDS with a cross talk delay term, i.e. one of the delays is a combination of the other delays. Section 5.2.3 studies a 2nd order MTDS with time-periodic coefficients.
5.2.1 Scalar MTDS

Consider the scalar MTDS

\[ \dot{x}(t) + x(t) + 2x(t - \tau_1) + 2x(t - \tau_2) = 0. \]  

The stable regions for this equation in the \((\tau_1, \tau_2)\) space were studied in Ref. [90] using D-subdivision and in Ref. [93] using the method of Cluster Treatments for Characteristic Roots.

Note that the system is autonomous; therefore, we can choose to discretize the period \([0, \max(\tau_1, \tau_2)]\), i.e. choose \(T = \max(\tau_1, \tau_2)\). Also note that since the coefficients of the delayed term are equal, it is sufficient to obtain the stability boundaries only below (or above) the line \(\tau_1 = \tau_2\) then mirror the boundaries with respect to that line to obtain the full stability diagram.

The stability diagram obtained using the spectral element method is shown in Fig. 5.1a. This diagram was obtained using \(n = 10, E = 3\) and a \(150 \times 150\) grid and it is in agreement with Refs. [90,93].

\[ \text{Figure 5.1: The stability diagram obtained using the spectral element method for (a) Eq. (5.7) and (b) Eq. (5.8). Figure (a) was obtained using } n = 10, E = 3 \text{ and } 150 \times 150 \text{ grid whereas figure (b) was obtained using } n = 15, E = 1 \text{ and } 200 \times 200 \text{ grid.} \]
5.2.2 Second order MTDS with cross talking delays

Consider the second order MTDS

\[ \ddot{x}(t) + 7.1 \dot{x}(t) + 21.1425x(t) + 6 \dot{x}(t) - \tau_1 + 14.8x(t) - \tau_1 + 2 \dot{x}(t) - \tau_2 + 8x(t) - \tau_1 - \tau_2 = 0, \tag{5.8} \]

where \( \tau_1 \) and \( \tau_2 \) are constant positive delays \([92]\). Note that the last term in Eq. (5.8) contains the delay cross-talk component. The stability of this system was investigated in Ref. \([92]\) using the Cluster Treatment of Characteristic Roots method.

Note that since the system is autonomous, it is convenient to choose \( T = \tau_1 + \tau_2 \) so that the cross talk term will map exactly onto the mesh points whereas interpolation is necessary, in general for the delays \( \tau_1 \) and \( \tau_2 \).

The spectral element approach was used to produce the stability diagram for Eq. (5.8) shown in Fig. 5.1b. This diagram is in agreement with Ref. \([92]\).

5.2.3 Second order time-periodic MTDS

Whereas Sections 5.2.1 and 5.2.2 studied the stability of autonomous MTDS, this section studies the stability of a 2nd order MTDS with time-periodic coefficients given by

\[ \ddot{z}(t) + (6 + \epsilon \cos(2\pi t))z(t) = z(t - \tau_1) + z(t - \tau_2). \tag{5.9} \]

Letting \( \mathbf{y}(t) = [y_1(t) \ y_2(t)] \) where \( y_1(t) = z(t), \ y_2(t) = \dot{z}(t) \), Eq. (5.9) can be put into state-space form according to

\[ \dot{\mathbf{y}}(t) = \mathbf{A} \mathbf{y}(t) + \mathbf{B}_1 \mathbf{y}(t - \tau_1) + \mathbf{B}_2 \mathbf{y}(t - \tau_2), \tag{5.10} \]

The stability is then determined by the eigenvalues of the monodromy matrix which can be constructed following the steps in Section 5.1.

The spectral element approach was used to produce the stability diagram for Eq. (5.9) shown in Fig. 5.2. Figure 5.9a corresponds to the choice \( \epsilon = 0 \) in Eq. (5.9)
resulting in a non-autonomous system. For this choice of \( \epsilon \), \( T = \max(\tau_1, \tau_2) \) was chosen and the steps outlined in Section 5.1 were followed to calculate stability. In contrast, choosing \( \epsilon \neq 0 \), as in Fig. 5.2b where \( \epsilon = 6 \), fixes \( T \) and the different cases for \( T \) and \( \tau \) ratios described in Chapter 3 need to be observed. The stability diagrams for both cases were in agreement with the semi-discretization results in Ref. [43]. The parameters \( n = 15, E = 1 \) were used to produce Fig. 5.2a whereas \( n = 30 \) and \( E = 1 \) were used in Fig. 5.2b. Both figures were obtained by discretizing the \((\tau_1, \tau_2)\) space using a uniform 200 \times 200 mesh.

![Figure 5.2](image-url)

**Figure 5.2:** The stability diagrams corresponding to Eq. (5.9) using (a) \( \epsilon = 0 \) and (b) \( \epsilon = 6 \). Figure (a) was obtained using \( n = 15 \) and \( E = 1 \) whereas figure (b) was obtained using \( n = 30 \) and \( E = 1 \). Both figures were obtained using point-by-point evaluation on a 200 \times 200 mesh.

### 5.3 Conclusions

This chapter described the stability analysis of systems with multiple delays using the spectral element approach. It was shown that MTDS are a special case of distributed delay equations when the Dirac delta functions are chosen as the kernel function, see Eq. (5.2). The discretization of the MTDS was shown in Section 5.1 for autonomous as well as non-autonomous MTDS. To show the effectiveness of the presented approach, the stability of a set of case studies was investigated.
Specifically, the stability of a scalar MTDS in the delays parameter space was studied in Section 5.2.1 and the resulting stability diagram was shown in Fig. 5.1a. Sections 5.2.2 and 5.2.3 investigated the stability of 2nd order MTDS. To elaborate, Section 5.2.2 studied the stability of a 2nd order autonomous MTDS that included a term with interacting delays while the resulting stability chart was shown in Fig. 5.1b. On the other hand, Section 5.2.3 studied the stability of a 2nd order MTDS with time periodic coefficients while the resulting stability charts were shown in Fig. 5.2.

For the examples shown in this chapter, all the produced stability charts were found to match the results from well-established methods in the literature, namely, D-subdivision, CTCR and semi-discretization. However, in contrast to D-subdivision and CTCR, the presented approach had the advantage of handling time-periodic systems. Moreover, the size of the matrices necessary to ascertain stability were smaller than their semi-discretization counterparts giving the spectral element approach a clear computational advantage.
Nonlinear Delay Differential Equations

The analysis that was presented in all the previous chapters was restricted to linear DDEs. Nevertheless, the spectral element approach can be extended to nonlinear DDEs as well. Due to the difficulties associated with the analytical aspects of DDEs, especially nonlinear DDEs, there has been significant focus on their numerical solution. The majority of this work has focused on initial value solvers, mostly extending Runge-Kutta solvers to DDEs [94–96], e.g., RADAR5 by Guglielmi and Hairer [97]. These solvers were designed to handle the specific complexities of DDEs, such as the propagation of discontinuities caused by the initial conditions. In this study, however, we are interested in long-time dynamics, namely periodic orbits. The numerical methods for analyzing periodic orbits directly as boundary value problems are much less developed than the corresponding methods for initial value problems. They typically use a collocation-type discretization based on Chebyshev or Gauss-Legendre points [45–47,49]. Stability of the periodic orbit may then be derived from the linearization of the discretized system.

In this chapter, we consider an alternative approach to numerically approximating periodic orbits of nonlinear DDEs based on the spectral element method.
With the spectral element method we seek to study the periodic orbits of general nonlinear DDEs of the form

\[ \frac{dx}{dt} = g(t, x(t), x(t - \tau)), \]

with one constant delay \( \tau > 0 \), and \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuously differentiable function. Although we describe the approach for one delay to keep the presentation clear, the extension to multiple delays is straightforward. The spectral element method that we present here can be used to find both the periodic orbit and its stability. This will be demonstrated with a variety of different examples. Specifically, we will study the periodic orbits of the delayed Mackey-Glass equation, the delayed Van der Pol equation and the delayed Duffing equation.

The organization of this chapter is as follows. Section 6.1 describes the formulation of the boundary value problem (BVP) used to solve for the periodic orbits. Section 6.2 describes the discretization method used to obtain a finite dimensional approximation of the infinite dimensional DDE. Section 6.3 describes a solution method based on the spectral element approach to obtain periodic orbits of Eq. (6.1). Section 6.4 describes obtaining the linearized stability of the periodic orbit using Floquet theory. Section 6.5 provides several case studies to demonstrate the effectiveness of the current approach while Section 6.6 contains concluding remarks.

### 6.1 Problem formulation

The periodic orbits of Eq. (6.1) can be obtained as the solutions of an infinite dimensional boundary value problem (BVP). This BVP has three components:

1. The equation describing the system evolution with time (e.g., see (6.1)).
2. The periodicity condition \( x(s) = x(s + T) \) for \( s \in [-\tau, 0] \).
3. A phase condition which removes any translational invariancy in the system and so ensures a unique solution to the BVP.

These three components are discussed in more detail in Sections 6.1.1, 6.1.2 and 6.1.3.

Since it is not possible to deal directly with the infinite dimensional BVP numerically, it must first be discretized to produce a finite dimensional approximation. The idea is that as the degree of approximation increases, the solution of the finite dimensional problem converges to that of the infinite dimensional problem.

General convergence proofs for different discretizations and solution methods are very sparse in the current literature and typically focus on collocation as the solution method, e.g., [47]. In the collocation method, the evolution equation (6.1) is required to hold exactly at finitely many collocation points. The spectral element method described here instead uses weighted integrals across the temporal domain. Nevertheless, there is a connection between collocation methods and the spectral element approach as will be shown in Section 6.3.

Regardless of the discretization method (Section 6.2) or the solution method (Section 6.3), the end result is a large system of algebraic equations which must then be solved using a general nonlinear root finding method, e.g., a Newton iteration. One additional benefit to using a Newton iteration is that stability information (Floquet multipliers) of the periodic orbit can be determined from the Jacobian used in the last step of the iteration.

6.1.1 The evolution equation

As previously described, any periodic solution of Eq. (6.1) can be found by solving a related two-point BVP. Since, in general, the period of the orbit is an additional unknown, time is rescaled such that \( t = T\hat{t} \); thus, the true period enters into the equations as an explicit variable \( T \) and the time period over which the BVP is posed
becomes simply unity. After dropping tildes and rearranging, the equation describing
the evolution of the DDE for \( t > 0 \) becomes

\[
 f = \frac{dx}{dt} - Tg(x(t), x(t - \tau/T)) = 0, \ t \in [0, 1], \tag{6.2}
\]

where \( g \) is twice continuously differentiable and \( T \) is the unknown period of the orbit.

6.1.2 The periodicity condition

The second component of the two-point BVP is the periodicity condition which
requires that the states at \( t - \tau/T < 0 \) be mapped back to \([0, 1]\). This condition can
be described by

\[
 x(s) = x(s + 1), \ s \in [-\tau/T, 0]. \tag{6.3}
\]

In this study we use the modulo operator for \( t - \tau/T < 0 \) combined with algebraic
equations at \( t = 0 \)—whose number is equal to the number of states—to enforce the
periodicity condition, see Section 6.3.

6.1.3 The phase condition

There are several possible choices for the phase condition in (6.5), for example, fixing
one of the solution components at \( t = 0 \) [98]. Another choice is to use a discrete
orthogonality condition \((x(0) - x_0(0))^T \dot{x}_0(0) = 0\), where \( x_0 \) is the initial solution
and \( x \) is the corrected solution. However, the most commonly used alternative is to
impose the functional orthogonality relationship

\[
 p(x) = \int_0^1 x_0^T(t) \dot{x}(t) dt, \tag{6.4}
\]

where \( \dot{x} := \frac{dx}{dt} \) and the superscript ‘T’ denotes the matrix transpose [99]. This
condition minimizes the difference between \( x_0(t) \) (the initial guess) and \( x(t) \) with
respect to translation.

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Combining Eqs. (6.2)–(6.4), the two-point BVP can be written as

\[
\begin{align*}
\begin{cases}
  f = \frac{dx}{dt} - Tg(x(t), x(t - \tau/T)) = 0, & t \in [0, 1], \quad \text{(evolution)} \\
  x(s) - x(s + 1) = 0, & s \in [-\tau/T, 0], \quad \text{(periodicity)} \\
  p(x) = 0, & \quad \text{(phase)}
\end{cases}
\end{align*}
\]  

(6.5)

6.2 Discretization method

There are several methods to discretize the periodic orbits of Eq. (6.5); they are mostly polynomial approximations of the form

\[
x(t) = \sum_{i=1}^{n+1} x_i \phi_i(t),
\]

(6.6)

where \( t_i \in [0, 1] \) is the set of discretization points, \( x_i = x(t_i) \), and \( \phi_i \) are the basis or trial function which can be defined globally or piecewise. These methods are distinguished by 1) the location of the discretization points, and 2) the method for minimizing the residual error. The location of the discretization points determines the approximation interpolant and its derivatives. The choice of the discretization mesh, and hence the approximation basis, is also dependent on the method used for minimizing the residual error. For example, if a Galerkin type approach is used, the selected mesh needs to accommodate an accurate quadrature rule such as a Gaussian quadrature. For collocation type methods, the mesh nodes need to be optimally placed to guarantee minimal residual error, e.g. the nodes coincide with the roots or extrema of orthogonal polynomials.

Examples of common discretization nodes include the roots or the extrema of Chebyshev, Legendre or Hermite polynomials. Each of these meshes yields a well-conditioned system, minimizes the residual error and accommodates accurate Gauss quadrature rules. Consequently, any of these meshes can be used for collocation-based or Galerkin-based solution methods. Alternatively, a Fourier approach can
also be used to find the periodic solution.

A Fourier approach has the advantage of automatically incorporating a periodicity assumption which eliminates the need for adding a periodicity condition. However, the Fourier approach cannot be used for stability calculations; hence, additional steps are necessary to determine the stability of the DDE [51].

In this study, we use a piecewise polynomial approximation. The time interval $[0, 1]$ is discretized into a finite number of temporal elements $E$. Each element is described by the interval $e_j$ defined in Eqs. (3.4) and (3.5)

A polynomial approximation is then used to obtain an approximate expression for the states over each element. For example, the states over the $j$th element are approximated by

$$x_j(t) = \sum_{i=1}^{n+1} \phi_i(\eta) x_{ji}$$

where $n$ is the order of the interpolating polynomial, $\eta \in [0, 1]$ is the local time in the element normalized by the length of the element, while $x_{ji} = x_j(t_i)$ is the state vector at the $i$th interpolation node within the $j$th element.

Barycentric Lagrange interpolation can be used to obtain the trial functions according to Eqs. (2.18) and (2.19). Equation (2.18) can also be used to define an interpolation matrix that maps a set of arbitrary but distinct points onto the set of interpolation points (also called the base points). For example, let the vector of base points be $x_1 \in \mathbb{R}^{n \times 1}$, and let the vector of the arbitrary unique points be $x_2 \in \mathbb{R}^{m \times 1}$, then the effect of interpolating $x_2$ using $x_1$ as the base points is described using the linear transformation

$$x_2 = \Upsilon x_1$$

where $\Upsilon \in \mathbb{R}^{m \times n}$ is called the interpolation matrix. The entries of the interpolation matrix are calculated using Eqs. (2.18) and (2.21). Equation (6.8) defines a linear
transformation useful in interpolating the delayed states in terms of the discretization nodes.

In addition to being a more efficient tool to generate the trial functions, the barycentric weights can be used to obtain the value of the derivative of the trial functions evaluated at the interpolation nodes as was shown in Eq. (2.22).

These values are useful in evaluating the weighted residual integrals in Galerkin-type methods using Gauss quadrature. Moreover, the values defined in Eq. (2.22) form the entries of the differentiation matrix which describes a linear transformation from the values of the trial functions at the nodes to the values of the derivative of the trial functions at the same nodes as was shown in Eqs. (2.23) and (2.24). Further, the calculation of the differentiation matrix entries for the LGL nodes simplifies to the expressions shown in Eq. (2.37).

6.3 Solution method

The periodic solution of Eq. (6.1) is obtained using Newton iteration on the discretized version of the BVP (6.5). To apply Newton iteration, the discretized version of Eq. (6.5) is written in the form

\[ f_u(u) \Delta u = f(u), \]  

(6.9)

where \( u = \{x_{ji}, T\} \) is the vector of discretized states.

This section derives the expressions that are necessary to apply Newton iteration. Specifically, Section 6.3.1 gives the expression for the residual on the mesh points and the discrete periodicity condition—which are components of the \( f(u) \) term in Eq. (6.9). Section 6.3.2 shows the discrete version of the phase condition while Section 6.3.3 derives the expression \( f_u(u) \), which contains the Jacobian used for periodic solution calculations.
6.3.1 The residual

Substituting Eq. (6.7) into the expression for \( f \) in Eq. (6.5) yields the residual on the \( j \)th element according to

\[
R_j = \sum_{i=1}^{n+1} \frac{1}{h_j} \dot{\phi}_i(\eta)x_{ji} - Tg \left( \sum_{i=1}^{n+1} \phi_i(\eta)x_{ji} \right),
\]

where the local normalized time is given according to

\[
\eta^* = \frac{t^* - t_j^-}{h_j},
\]

while the time \( t^* \) is defined using modular arithmetics according to

\[
t^* = t_j^- + \eta h_j - \tau \pmod{1}.
\]

The function \( j^*(t^*) \) gives the element index to which \( t^* \) belongs and it is given by

\[
j^*(t^*) = \sum_{j=1}^{E} j \chi_{e_j}(t^*),
\]

where the indicator function \( \chi_{e_j}(t^*) \) was defined in Eqs. (3.11) and (3.12).

The positive integer \( q \) in Eq. (6.10) is the number of the period to which the delay looks back and it is described by the absolute value function

\[
q = \left\lfloor \frac{t - \tau}{T} \right\rfloor
\]

where \( \lfloor \cdot \rfloor \) is the floor function and with the understanding that \( q = 0 \) indicates a mapping onto the interval \([0, 1]\).

The values assigned to the index \( q \) in Eq. (6.14) are used in the stability analysis which does not make any assumption on the periodicity of the linearized equations, see Section 6.4. However, since in this section we are seeking a periodic solution of
Eq. (6.1), the delayed term is handled using the modulo operation to map the states onto $[0, 1]$ instead of $[-\tau/T, 1]$. Specifically, for obtaining the periodic solution, the substitution $q = 0$ (indicating a mapping onto $[0, 1]$) is made in Eq. (6.10) resulting in the expression

$$R_j = \sum_{i=1}^{n+1} \frac{1}{h_j} \dot{\phi}_i(\eta)x_{ji} - T g \left( \sum_{i=1}^{n+1} \phi_i(\eta)x_{ji}, \sum_{i=1}^{n+1} \phi_i(\eta^*)x_{j^*}(t^*) \right),$$  \hspace{1cm} (6.15)

where $q = 0$ was omitted to simplify notation. This substitution eliminates the unknowns corresponding to $t < 0$ and implicitly enforces the periodicity condition for $t < 0$. Nevertheless, the periodicity condition at $t = 0$ still needs to be enforced explicitly according to

$$x(0) - x(1) = 0,$$  \hspace{1cm} (6.16)

where for a $d$-dimensional system, Eq. (6.16) gives $d$ equations.

Using the method of weighted residuals on Eq. (6.10) results in

$$\int_0^1 \left( \sum_{i=1}^{n+1} \frac{1}{h_j} \dot{\phi}_i(\eta)x_{ji} - T g \right) \psi_p(\eta) d\eta = 0,$$  \hspace{1cm} (6.17)

where $\psi_p$ are weight functions with $p \in \{1, 2, \ldots, n\}$. A discrete version of the integral in Eq. (6.17) can be obtained using Gaussian quadrature rules. To illustrate, recall from Eq. (4.2) that a Legendre-Gauss-Lobatto (LGL) quadrature rule can be used to approximate integrals. In this study, we chose the quadrature nodes to be identical to the interpolation nodes whereas the quadrature weights were calculated using Eq. (2.26).

Using an LGL quadrature in Eq. (6.17) results in

$$\sum_{k=1}^{n+1} \left( \sum_{i=1}^{n+1} \frac{1}{h_j} \dot{\phi}_i(\eta_k)x_{ji} - T g \right) \psi_p(\eta_k) w_k = 0.$$  \hspace{1cm} (6.18)
Let the value of the initial guess of the states on the interpolation points within the $j$th element be represented by the vector $x_0$. The delayed states $x_{\tau}$ are then obtained using interpolation according to

$$x_{\tau} = \Upsilon_{(t^* \rightarrow t)} x_0,$$  \hspace{1cm} (6.19)

where $\Upsilon$ is the matrix that interpolates the delayed states in terms of the states at the interpolation points, see Eq. (2.21). In addition, let $\hat{D}_j$ be the differentiation matrix on the $j$th element obtained using the Kronecker product

$$\hat{D}_j = \frac{1}{h_j} D \otimes I_d$$  \hspace{1cm} (6.20)

where $D$ is given by Eq. (2.37) while $I_d$ is the $d \times d$ identity matrix and $d$ is the order of the nonlinear DDE. Using the above definitions, Eq. (6.18) can be written as

$$R = W_{\text{res}} \hat{R} = 0,$$  \hspace{1cm} (6.21)

where The matrix $W_{\text{res}} : \mathbb{R}^{d(En+1)} \rightarrow \mathbb{R}^{dEn}$ was given in Eq. (2.41) while $\hat{R}$ is given by

$$\hat{R} = \hat{D}x_0 - Tg \left(x_0, \Upsilon_{(t^* \rightarrow t)} x_0\right),$$  \hspace{1cm} (6.22)

while the $d(En + 1) \times d(En + 1)$ matrix $\hat{D}$ is the global differentiation matrix which contains all the individual element differentiation matrices $\hat{D}_j$.

Convergence can then be obtained by increasing the number of elements $E$ and/or the order of the interpolation polynomial $n$. The method described above which uses weighted residual integral to minimize the error and uses quadratures to approximate integrals is called the spectral element method. This method allows $hp$-refinement schemes and can yield higher rates of convergence (spectral rates of convergence). It is interesting to point out the connection between the spectral element method we present here and the typical collocation methods. This connection is best described
using Eqs. (6.21) and (6.22). In fact, the term \( \hat{R} \) in Eq. (6.21) is similar to the matrices that are generated from a collocation scheme—where a set of algebraic equations is produced by evaluating the DDE at the collocation points. The effect of invoking the spectral element method is therefore described by Eq. (6.22) and it can be thought of as a linear transformation of \( \hat{R} \).

Note that the discretization of \( f \) in Eq. (6.5) using \( E \) elements and \( n + 1 \) interpolation nodes gives rise to \( d(En+1) \) unknown states. Appending the unknown period \( T \) as the last entry in the vector of states \( u = \{x_{ji}, T\} \) further increases the number of unknowns to \( d(En + 1) + 1 \). To solve for the unknowns it is necessary to obtain at least an equal number of equations.

Equation (6.21) provides \( dEn \) equations while the periodicity condition in Eq. (6.16) provides \( d \) more equations. The last additional equation, corresponding to the phase condition, is set to zero. Setting this entry to zero is equivalent to stating that the phase condition is always satisfied. The specific phase condition described in Eq. (6.4) is taken into account in the term \( f_{u}(u) \) of Eq. (6.9) as will be shown in Section 6.3.3. Consequently, \( d(En + 1) + 1 \) equations are obtained to solve for the equal number of unknowns and they are arranged into the column vector

\[
f(u) = \begin{bmatrix} \hat{R} \\ x(0) - x(1) \\ 0 \end{bmatrix}.
\]

Equation (6.23) provides the expression for one of the components in Eq. (6.9) that are necessary to apply Newton iteration. The other components are described in the following sections.

6.3.2 Discretized phase condition

Recall that the phase condition is necessary to ensure a unique solution to Eq. (6.5). Assume that the order of the DDE is \( d \) and that the interval \([0, 1]\) was divided into
$E$ elements each with $n + 1$ interpolation nodes. The phase condition in Eq. (6.4) can be described piecewise over all the elements covering $[0, 1]$ according to

$$ p(x) = \int_0^1 \dot{x}_0^T x \, dx = \sum_{j=1}^{E} \int_{t_j^-}^{t_j^+} \dot{x}_0^T x \, dx. \quad (6.24) $$

If a quadrature rule is defined on the $n + 1$ base nodes within each element, Eq. (6.24) can be approximated by

$$ p(x) \approx \sum_{j=1}^{E} \frac{h_j}{2} \sum_{k=1}^{n+1} w_k \dot{x}_{0k} \, x_{jk}, \quad (6.25) $$

where $w_k$ are the quadrature weights.

Let $x_0 \in \mathbb{R}^{d(En+1)}$ contain the initial guess of the states on the base points. Further, let $x \in \mathbb{R}^{d(En+1)}$ contain the true states on the same mesh.

Choosing the quadrature points to be identical to the discretization points on $[0, 1]$, the discretized phase condition reads

$$ p(x) = (\hat{D}x_0)^T \otimes (w \otimes 1_{1 \times d}) \, x, \quad (6.26) $$

where $\otimes$ indicates element-wise multiplication, $1_{1 \times d}$ is a $1 \times d$ vector of ones while the $1 \times (En + 1)$ vector $w$ is given by

$$ w = \left[ \frac{h_1 w_1}{2} \quad \frac{h_1 w_2}{2} \quad \ldots \quad \frac{h_1 w_{n+1} + h_2 w_1}{2} \quad \ldots \quad \frac{h_E w_1}{2} \quad \frac{h_E w_2}{2} \quad \ldots \quad \frac{h_E w_{n+1}}{2} \right]. \quad (6.27) $$

Note that in $w$ the entries at the beginning and end of any two subsequent elements are added together.

### 6.3.3 The Jacobian

The other necessary component to apply Newton iteration on Eq. (6.15) is the term $f_u(u)$. This term contains the linearized version of Eq. (6.15). The linearization of
Eq. (6.15) can be written in the form $f_u(u)\Delta u = f(u)$ according to

$$
\sum_{i=1}^{n+1} \dot{\phi}_i(\eta) \Delta x_{ji} - TA_0(\alpha) \sum_{i=1}^{n+1} \phi_i(\eta) \Delta x_{ji} - TA_1(\alpha) \sum_{i=1}^{n+1} \phi_i(\eta^*) \Delta x_{j^*,i} - g(\alpha) \Delta T
$$

$$
- \frac{\tau}{T} A_1(\alpha) \sum_{i=1}^{n+1} \dot{\phi}_i(\eta^*) x_{j^*,i} \Delta T = - \sum_{i=1}^{n+1} \dot{\phi}_i(\eta) x_{ji} + Tg(\alpha),
$$

(6.28)

where $\alpha = \left( \sum_{i=1}^{n+1} \phi_i(\eta) x_{ji}, \sum_{i=1}^{n+1} \phi_i(\eta^*) x_{j^*,i} \right)$ and

$$A_0(\zeta, \eta) = \frac{\partial}{\partial \zeta} g(\zeta, \eta),
$$

$$A_1(\zeta, \eta) = \frac{\partial}{\partial \eta} g(\zeta, \eta).
$$

(6.29a) (6.29b)

Evaluating Eq. (6.28) on the $(En+1)$ mesh points and arranging the terms into matrices results in

$$J \begin{bmatrix} \Delta x \\ \Delta T \end{bmatrix} = -\hat{R},
$$

(6.30)

where $\Delta x = x - x_0$, $\hat{R}$ was described in Eq. (6.22) and the matrix $J$ has the dimensions $d(En+1) \times d(En+1) + 1$.

The spectral element method is applied by multiplying both sides of Eq. (6.30) by $W_{\text{res}}$ according to

$$W_{\text{res}}J \begin{bmatrix} \Delta x \\ \Delta T \end{bmatrix} = -W_{\text{res}}\hat{R}.
$$

(6.31)

Define the $dEn \times d(En+1) + 1$ matrix $\hat{J}$ as $\hat{J} = W_{\text{res}}J$. The term $f_u(u)$ in Eq. (6.9) is then obtained by appending the continuity condition at $t = 0$ (i.e., $x^1(1) = x^0(0)$) and the phase condition (Eq. (6.26)) to $\hat{J}$. Specifically, the expression for $f_u(u)$ is
given by
\[
f_u(u) = \begin{bmatrix}
I_d & 0_d & \ldots & 0_d & \hat{J} \\
p(x) & -I_d & 0
\end{bmatrix},
\] (6.32)
where \(p(x)\) is given by Eq. (6.26). Equations (6.32) and (6.23) can then be used in a
Newton iteration algorithm (see Eq. (6.9)) to solve for the states on the mesh points
as well as the period of the orbit.

6.4 Stability calculations

The basic tools to analyze the local stability of the periodic orbits of Eq. (6.1) are
the monodromy matrix and Floquet theory. The monodromy matrix \(Q\) is the dis-
cretization of the time integration operator of the linearized equation [46]. The
discretization is performed without the modulo operation, i.e. the linearized equa-
tion is discretized on \([-\bar{\tau}/T, 1]\). This corresponds to the linearization described in
Eq. (6.10) except the index \(q\) is maintained to keep track of the periods the delay
looks back to.

The linearized equations for stability analysis will have a form similar to Eq. (6.28)
with two main differences: (1) the substitution \(x_{j^*,i} = x_{j^*,i}^q\) is made to ensure correct
mapping and (2) the partial derivatives with respect to \(T\) are omitted. The mon-
odromy matrix represents a linear map from the states in the segment \([-\bar{\tau}/T, 0]\) onto
the states in the segment \([-\bar{\tau}/T + 1, 1]\) according to
\[
u_T = Q u_0, \quad (6.33)
\]
where \(u_T \in [-\bar{\tau}/T + 1, 1]\) and \(u_0 \in [-\bar{\tau}/T, 0]\). The stability of the periodic solutions
is found from calculating the eigenvalues (Floquet multipliers) of the monodromy
matrix. Besides a trivial +1 Floquet multiplier for autonomous systems, the system
is stable if the remaining multipliers are within the unit circle in the complex plane,
see Fig. 6.1. The computed eigenvalues of $Q$ form approximations to the eigenvalues of the integration operator.

As the discretization is refined by increasing the number of the LGL nodes, and, consequently, the order of the trial functions, more Floquet multipliers are better approximated.

![Stability Criteria Diagram](image)

**Figure 6.1**: The stability criteria dictates that all the eigenvalues $\mu$ of the monodromy operator $Q$ should lie within the unit circle in the complex plane. Moreover, the manner in which the eigenvalues depart the unit circle produces different bifurcation behavior as shown.

### 6.5 Examples

To demonstrate the effectiveness of the current approach, we calculate the periodic orbits and their stability for several case studies. We also calculate the errors associated with the approximate periodic orbits using the error norms described in Section 6.5.1. Section 6.5.2 studies the Mackey-Glass equation. Sections 6.5.3 and 6.5.4 study the delayed Van der Pol equation and the delayed Duffing equation, respectively. In each case, the periodic orbit is calculated and plotted and its stability is ascertained using the procedure described in Section 6.4.
6.5.1 Error analysis

This section describes the error norms used in this study. The error is calculated between the spectral element solution and a reference solution for different mesh sizes. This then enables the calculation of the rate of convergence of the Spectral element solution.

The norms typically used to quantify the error can be categorized into continuous and discrete norms. Continuous norms are defined over the whole period whereas discrete norms are only defined at the representation points. However, although discrete norms show the super-convergence effects more clearly, they do not always provide a good indication of the quality of the overall solution [49]. The two most common norms are the $L_2$ and the $L_\infty$ norms defined as

$$
||x||_2 = \left( \int_0^1 (x(t) - \tilde{x}(t))^2 dt \right)^{\frac{1}{2}} 
$$

($L_2$ continuous norm) (6.34a)

$$
||x||_\infty = \max_{t \in [0,1]} (x(t) - \tilde{x}(t)) 
$$

($L_\infty$ continuous norm) (6.34b)

$$
||x||_2 = \left( \sum_{k=1}^{n+1} (x_k - \tilde{x}_k)^2 \right)^{\frac{1}{2}} 
$$

($\ell_2$ discrete norm) (6.34c)

$$
||x||_\infty = \max_k |x_k - \tilde{x}_k| 
$$

($\ell_\infty$ discrete norm) (6.34d)

where $\tilde{x}$ is the reference solution. Because of its extensive use in other numerical continuation packages such as AUTO [100], we will use the continuous $L_2$ norm in this study. Further, the discrete $L_2$ and the discrete $L_\infty$ will also be used to get a better idea about the errors.

Equations (6.34) makes use of a reference solution $\tilde{x}$ which ideally would be an exact analytical solution. In general, exact solutions of DDEs are not known and instead a high accuracy solution using an approximation technique is used. The steps used in refining the mesh and calculating the error are as follows:
1. Generate a high precision solution $\tilde{x}$, e.g. using 1000 mesh points.

2. Generate the lower resolution initial solution $x_{\text{ini}}$ from $\tilde{x}$ using interpolation. This is the initial solution that is used in the next step. Generating the initial guess for the coarse solution from the reference solution ensures that there is no phase shift between the two solutions.

3. Refine the initial solution $x_{\text{ini}}$ from the previous step using spectral element approach to obtain the lower resolution solution $x$.

4. Use the selected norm from Eq. (6.34) to calculate the error between $x$ and $\tilde{x}$.

5. Increase the resolution and repeat the above steps to calculate the error for various meshes.

6.5.2 Mackey-Glass equation

The Mackey-Glass equation models the regeneration of white blood cells [101, 102]. It is one of the classical examples for using nonlinear DDEs in characterizing physiological phenomena. The Mackey-Glass equation is given by

$$\frac{dx(t)}{dt} = ax(t) + b \frac{x(t-\tau)}{1 + x^c(t-\tau)},$$

(6.35)

where $a$, $b$, and $c$ are scalars. For $a = 1$, $b = 1.5$, $c = 10$ and $\tau = 2$, there is a stable periodic solution, shown in Fig. 6.2, with period $T \approx 20.08$.

The results of the error analysis associated with using the spectral element method are also shown in Fig. 6.2. The reference solution shown in these figures was obtained using the spectral element method with $n = 1000$ and $E = 1$ and it matched the solution obtained from numerical simulation. For each error norm, three curves corresponding to $E \in \{1,2,3\}$ are plotted as a function of $n$ where $2 \leq n \leq 50$. For each value of $E$ and $n$, an initial solution was constructed from the
Figure 6.2: The periodic solution and the error norms for Eq. (6.35) (where $T \approx 20.08$). The reference solution was obtained using $n = 1000$ and $E = 1$ while the error norms were calculated using Eq. (6.34a), (6.34c) and (6.34d). In the error plots, the values $E = 1, 2, 3$ were used whereas $n$ was varied in the range $2 \leq n \leq 50$.

reference solution to eliminate phase shifts between the reference solution and the final solution. The initial solution was then refined using Newton iteration. It can be seen that the method converges exponentially as evidenced by all the considered error norms. For each error norm there is a saturation point where the error is no longer decreased with increasing $n$ and $E$ indicating a close match with the reference solution. Using the stability analysis described in Section 6.4, the periodic orbit was found to be stable.

Note that in Fig. 6.2 the convergence rate improved by increasing the number of elements and consequently the number of the mesh points. However, with the spectral element method there are two techniques that can be used to increase the size
of the mesh: (1) fix the number of elements and increase the order of the interpolation
polynomial \((p\text{-refinement})\) and (2) fix the order of the polynomial and increase the
number of elements \((h\text{-refinement})\). Whereas the former technique was investigated
in Fig. 6.2, the latter technique was studied in Fig. 6.7.

In Fig. 6.7, the \(L_2\) norm is plotted against the length of the uniform elements
\(h = 1/E\). The \(L_2\) norm was chosen in this figure since it describes the quality of
the overall solution. The order of the interpolation polynomial was held constant
at either 3, 4 or 5 while the length of the uniform elements was varied between
\(1/500 \leq h \leq 1\), i.e., \(1 \leq E \leq 500\). Figure 6.7a shows the plot that corresponds to
the \(L_2\) error norm of Eq. (6.35). It is shown that as \(n\) was held constant and \(h\) was
decreased, the error norm decreased linearly until it reached a saturation point where
any further reduction in the element length did not not influence convergence. From
the numerical data in Tab. 6.4, the \(h\)-refinement rate of convergence for Eq. (6.35)
was approximately \(O(h^n)\), where \(n\) is the order of the interpolation polynomial.

Table 6.1: A comparison of the error values for Eq. (6.35) using different combinations
of \(E\) and \(n\) that yield meshes with the same number of points.

<table>
<thead>
<tr>
<th>(E = 1)</th>
<th>(E = 2)</th>
<th>(E = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(L_2)</td>
<td>(L_3)</td>
</tr>
<tr>
<td>6</td>
<td>9e-3</td>
<td>1.58e-2</td>
</tr>
<tr>
<td>12</td>
<td>1.48e-4</td>
<td>1.88e-4</td>
</tr>
<tr>
<td>18</td>
<td>6.23e-6</td>
<td>7.69e-6</td>
</tr>
<tr>
<td>24</td>
<td>3.66e-7</td>
<td>4.48e-7</td>
</tr>
<tr>
<td>30</td>
<td>5.09e-8</td>
<td>8.77e-8</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>42</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>48</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Recall that the size of the mesh depends on the order of the interpolation polyno-

mial \(n\) and the number of elements \(E\). Assume that uniformly distributed elements
are used, then two meshes with equal size can be produced if the values of \(n\) and \(E\)
for these meshes are chosen according to \(n_1 E_1 = n_2 E_2\). A comparison of the error
norms for Eq. (6.35) associated with using different meshes with equal sizes is shown
in Tab. 6.1. Each \((n, E)\) combination in any one row results in different meshes with equal sizes. Therefore, the corresponding error norms in each row can be compared to evaluate—for the same mesh size—the effect of increasing \(n\) versus increasing \(E\). Table 6.1 shows that for this example it is more advantageous to hold \(E\) fixed while \(n\) is increased. We stress that the above discussion is only valid for equally distributed elements. Indeed, we suspect that using an adaptive mesh where the elements are clustered near regions of sharp solution changes will improve convergence. However, adaptive meshes are outside the scope of this study and therefore we will only study equally distributed elements.

6.5.3 Delayed Van der Pol equation

The van der Pol equation was introduced in the 1920s as a model to describe the oscillations in the vacuum tube triode circuit. This equation became a classical example in nonlinear dynamics and has been used widely to model systems with limit cycle oscillations. Adding a feedback term, which could be the result of a delayed feedback controller, yields a delayed Van der Pol equation according to [103,104]

\[
\ddot{x}(t) + \epsilon(x^2 - 1)x(t - \tau) + x(t) = 0. \tag{6.36}
\]

For \(\epsilon = 0.1\) and \(\tau = 4.6\), a stable periodic solution is shown in Fig. 6.3 with period \(T \approx 6.38\) while the first derivative is shown in Fig. 6.4.

The results of the error analysis associated with using the spectral element method to obtain the solution are shown in Fig. 6.2. The reference solution shown in these figures was obtained using the spectral element method with \(n = 1000\) and \(E = 1\) and it matched the solution obtained from numerical simulation. For each error norm, three curves corresponding to \(E \in \{1, 2, 3\}\) are plotted as a function of \(n\) where \(2 \leq n \leq 25\). For each value of \(E\) and \(n\), an initial solution was constructed from the reference solution and then was corrected using Newton iteration.
Figure 6.3: The periodic solution and the error norms for Eq. (6.36) (where $T \approx 6.38$). The reference solution was obtained using $n = 1000$ and $E = 1$ while the error norms were calculated using Eq. (6.34a), (6.34c) and (6.34d). In the error plots, the values $E = 1, 2, 3$ were used whereas $n$ was varied in the range $2 \leq n \leq 25$.

It was noticed that increasing $n$ beyond 25 yielded 0 error norms indicating a match between the calculated solution and the reference solution.

Similar to Fig. 6.2, it can be seen that the method converges exponentially as evidenced by all the considered error norms. Further, the convergence rate is improved as the number of mesh points is increased via increasing $E$ from 1 to 3. Similar conclusions can be drawn from the analysis of the derivative of the periodic solution shown in Fig. 6.4. Applying the stability analysis described in Section 6.4 confirmed the stability of the calculated periodic orbit.

Figure 6.7b shows the $||x||_2$ norm as a function of $h$, the uniform element length. The order of the interpolation polynomial was held constant at either 3, 4 or 5.
Figure 6.4: The first derivative of the periodic solution to Eq. (6.36) and the corresponding error norms.

while the length of the uniform elements was varied between $1/500 \leq h \leq 1$, i.e., $1 \leq E \leq 500$. It is shown that for $n = 3$, the error norm decreased linearly as $h$ was decreased. For $n = 5$, the error norm also decreased linearly except at $E \in \{5, 6, 7\}$ where the error dropped from $4.70e - 3$ at $E = 4$ to $4.81e - 6$ at $E = 5$ before going back up at $E = 6, 7$ to continue its linear trend. Table 6.4 shows the approximate rate of convergence for $n = 3$ and $n = 5$ which was found to be approximately $O(h^n)$. However, the error plot for $n = 4$ showed a different trend. For this case, although the first part of the error plot was linear with a convergence rate of approximately $O(h^4)$ (see Tab. 6.4), it dropped from $2.67e - 5$ at $E = 16$ to $2.39e - 13$ at $E = 17$ where it reached a saturation region where the errors were not practically reduced as $h$ was decreased. Similar observation can be seen for the $||\dot{x}||_2$ results in Fig. 6.7c.
A comparison of the error norms for Eq. (6.36) associated with using different meshes with equal sizes is shown in Tab. 6.2. Each \((n, E)\) combination results in different meshes with equal sizes. Table 6.2 shows that for this example it is more advantageous to hold \(E\) fixed while \(n\) is increased. We reiterate though that the above discussion is only valid for equally distributed elements and that adaptive meshes can often yield more favorable results.

Table 6.2: A comparison of the error values for Eq. (6.36) using different combinations of \(E\) and \(n\) that yield meshes with the same number of points.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(L_2)</th>
<th>(L_\infty)</th>
<th>(E = 1)</th>
<th>(L_2)</th>
<th>(L_\infty)</th>
<th>(E = 2)</th>
<th>(L_2)</th>
<th>(L_\infty)</th>
<th>(E = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3.89 e-1</td>
<td>5.40 e-1</td>
<td>7.19 e-1</td>
<td>3</td>
<td>1.56 e-1</td>
<td>2.79 e-1</td>
<td>2.71 e-1</td>
<td>2</td>
<td>2.10 e-1</td>
</tr>
<tr>
<td>12</td>
<td>4.85 e-4</td>
<td>6.02 e-4</td>
<td>1.10 e-3</td>
<td>6</td>
<td>1.30 e-2</td>
<td>2.38 e-2</td>
<td>2.91 e-2</td>
<td>4</td>
<td>2.93 e-2</td>
</tr>
<tr>
<td>18</td>
<td>6.88 e-7</td>
<td>8.66 e-7</td>
<td>1.51 e-6</td>
<td>9</td>
<td>1.68 e-5</td>
<td>2.93 e-5</td>
<td>3.27 e-5</td>
<td>6</td>
<td>9.47 e-4</td>
</tr>
<tr>
<td>24</td>
<td>1.91 e-8</td>
<td>2.34 e-8</td>
<td>3.60 e-8</td>
<td>12</td>
<td>7.65 e-5</td>
<td>1.30 e-4</td>
<td>1.76 e-4</td>
<td>8</td>
<td>9.99 e-5</td>
</tr>
</tbody>
</table>

\(6.5.4\) Delayed Duffing equation

The Duffing equation is another classical example of nonlinear equations. It appears in the models of many dynamical systems such as magnet-based nonlinear energy harvesters [105]. The Duffing equation was studied extensively in literature. For example, the harmonically forced delayed Duffing oscillator was investigated in Ref. [106]. However, the analysis in that study used the method of multiple scales which was confined to the case of small damping, weak nonlinearity, weak feedback and soft excitation. In the current investigation, we study the autonomous version of the delayed Duffing oscillator; however, we relax the restricting assumptions imposed in Ref. [106]. Specifically, a duffing oscillator with state feedback can be described
by the equation

\[ \ddot{x}(t) + 2\zeta \dot{x}(t) + x(t) + 3\mu x^3(t) = 2ux(t - \tau) + 2v\dot{x}(t - \tau) \quad (6.37) \]

where \( \zeta, \mu, u \) and \( v \) are scalars and \( \tau > 0 \) is a constant time delay. If the parameters \( \zeta = \mu = u = 0.05, v = -0.05 \) and \( \tau = \pi \) are used in Eq. (6.37), then the resulting periodic solution (with \( T \approx 4.51 \)) and its derivative can be described by Figs. 6.5 and 6.6, respectively.

\[ \begin{align*}
\text{Figure 6.5: The periodic solution and the error norms for Eq. (6.37) (where } T \approx 4.51). \text{ The reference solution was obtained using } E = 500 \text{ and } n = 1000 \text{ while the error norms were calculated using Eqs. (6.34a), (6.34c) and (6.34d). In the error plots, the values } E = 1, 2, 5 \text{ were used whereas } n \text{ was varied in the range } 2 \leq n \leq 40. 
\end{align*} \]

The results of the error analysis associated with using the spectral element method to obtain the solution are also shown in these two figures. The reference solution was obtained using 500 elements each with a 5th order interpolating poly-
For each error norm, three curves corresponding to $E \in \{1, 2, 5\}$ are plotted as a function of $n$ where $2 \leq n \leq 40$. For each value of $E$ and $n$, an initial solution was constructed from the reference solution and then was corrected using Newton iteration. It was noticed that the spectral element solution converged to the reference solution (zero error norms) with exponential convergence rate using the following $(E,n)$ pairs: $(1,39)$, $(2,18)$ and $(5,12)$. Faster convergence was achieved as $E$ was increased from 1 to 5 and lower values for $n$ were needed to obtain zero error norms. Similar conclusions can be drawn from the analysis of the derivative of the periodic solution shown in Fig. 6.6. Applying the stability analysis described in Section 6.4 confirmed the stability of the calculated periodic orbit.

Figure 6.6: The first derivative of the periodic solution to Eq. (6.37) and the corresponding error norms.

Figure 6.7d shows the $||x||_2$ norm as a function of $h$—the uniform element length.
The order of the interpolation polynomial was held constant at either 3, 4 or 5 while the length of the uniform elements was varied between $1/500 \leq h \leq 1$, i.e., $1 \leq E \leq 500$. It is shown that for $n = 3$, the error norm decreased linearly as $h$ was decreased. For $n = 5$, the error norm also decreased linearly except at one point. Specifically, the error dropped from $6.87e - 4$ at $E = 7$ to $1.42e - 13$ at $E = 8$ before going back to $2.37e - 4$ at $E = 9$ and continuing the initial linear trend for $E \geq 9$. Ignoring the anomaly at $E = 7$, the numerical data in Tab. 6.4 indicate that both $n = 3$ and $n = 5$ converged at a rate of approximately $O(h^n)$.

On the other hand, the case $n = 4$ showed a different behavior. For this case, the convergence rate did not follow a linear trend. In fact, after initially lagging behind the error for $n = 5$ case, the error for $n = 4$ dropped from $2.93e - 4$ at $E = 20$ to $5.27e - 14$ at $E = 21$ and remained in a saturation region where further decrease of $h$ did not improve convergence. Actually, below a certain $h$, the errors started growing as $h$ was decreased. This is attributed to the limitations of the double precision arithmetic that was used. To elaborate, as the errors in the solution of the spectral element method become very small, the errors in the linear solve step of the Newton iteration becomes relatively large [49]. Similar observation can be seen for the $||\dot{x}||_2$ results in Fig. 6.7e.

A comparison of the error norms for Eq. (6.37) associated with using different meshes with equal sizes is shown in Tab. 6.3. Each $(n, E)$ combination in any one row results in different meshes with equal sizes. Table 6.3 shows that for this example it was more advantageous to hold $E$ fixed while $n$ was increased. However, using an adaptive mesh can often yield more favorable results especially with sharply-changing solutions.
Table 6.3: A comparison of the error values for Eq. (6.37) using different combinations of $E$ and $n$ that yield meshes with the same number of points.

<table>
<thead>
<tr>
<th>$x$ norms</th>
<th>$E = 1$</th>
<th>$E = 2$</th>
<th>$E = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$L_2$</td>
<td>$l_2$</td>
<td>$l_\infty$</td>
</tr>
<tr>
<td>10</td>
<td>1.34e-2</td>
<td>1.73e-2</td>
<td>3.50e-2</td>
</tr>
<tr>
<td>20</td>
<td>4.14e-5</td>
<td>5.89e-5</td>
<td>9.10e-5</td>
</tr>
<tr>
<td>30</td>
<td>1.20e-7</td>
<td>1.53e-7</td>
<td>2.61e-7</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$ norms</th>
<th>$E = 1$</th>
<th>$E = 2$</th>
<th>$E = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$L_2$</td>
<td>$l_2$</td>
<td>$l_\infty$</td>
</tr>
<tr>
<td>10</td>
<td>5.27e-2</td>
<td>6.46e-2</td>
<td>1.20e-2</td>
</tr>
<tr>
<td>20</td>
<td>2.77e-4</td>
<td>3.43e-4</td>
<td>6.61e-4</td>
</tr>
<tr>
<td>30</td>
<td>1.26e-6</td>
<td>1.55e-6</td>
<td>2.99e-6</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.4: Numerically computed orders of convergence for cubic, quartic and quintic polynomials with the solution of Eq. (6.35), the solutions and the derivative of Eq. (6.36), and the solution and the derivative of Eq. (6.37). Entries marked with a ‘*’ represent the order of convergence of the linear part of the error norm.

<table>
<thead>
<tr>
<th></th>
<th>Eq. (6.35)</th>
<th>Eq. (6.36)</th>
<th>Eq. (6.37)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>3</td>
<td>2.78</td>
<td>2.95</td>
<td>2.94</td>
</tr>
<tr>
<td>4</td>
<td>4.00</td>
<td>3.99*</td>
<td>3.94*</td>
</tr>
<tr>
<td>5</td>
<td>5.15</td>
<td>5.00</td>
<td>5.02</td>
</tr>
</tbody>
</table>

6.6 Conclusions

This chapter considered an alternative approach to numerically approximating periodic orbits of nonlinear DDEs based on the spectral element method. This method is a modification of the spatial spectral element method which has been widely used in simulating the partial differential equations arising in models of fluids and structures [66, 107]. The temporal spectral element method can also be viewed as an evolution of the state-space temporal finite element method (state-space TFEM) which has been used to study the stability of equilibria of linear DDEs [4, 53, 54].

The spectral element method was shown to successfully produce the periodic solutions of nonlinear DDEs. The considered case studies were obtained by intro-
Figure 6.7: The $L_2$ error plotted against the mesh size $h = 1/E$ for Eqs. (6.35) (Fig. a), (6.36) (Figs. b,c), and (6.37) (Fig. d,e) using cubic (+), quartic (◦) and quintic (×) polynomials. The first column shows the error of the solution whereas the second column shows the error of the derivative.

Producing delays to three standard equations from the nonlinear dynamics literature: the Mackey-Glass equation (scalar), the Van der Pol equation (2nd order), and the Duffing equation (2nd order). The periodic solutions obtained with the spectral element method converged to the reference solutions as was shown in Figs. 6.2–6.6. The error norm plots showed that as $n$ was increased, exponential rates of convergence were observed in all the considered examples.

In fact, Figs. 6.3–6.6 show that the zero error norms can be obtained for several
combinations of $(E,n)$. These figures also showed that as the size of the mesh was increased by increasing the number of elements, the solution converged faster to the reference solution. This demonstrates the $hp$-refinement capability of the current approach where 2 methods can be used to speed up convergence: either increase the number of elements $E$ or increase the order of the interpolating polynomials $n$.

The $L_2$ error norm associated with increasing $E$ while holding $n$ constant was shown in Figs. 6.4. For the Mackey-Glass equation, it was found that the spectral element method converged at a rate of approximately $O(h^n)$ as was shown in Tab. 6.4. Similarly, for the delayed Van der Pol and the delayed Duffing equation, it was found that the error norms for 3rd and 5th order polynomials typically followed a linear trend to convergence at a rate of approximately $O(h^n)$, see Tab. 6.4. However, for the delayed Van der Pol equation, it was found that a 4th order polynomial would initially follow a linear convergence rate before reaching a critical $h$ value where the solution would nonlinear converge with a rate that is even faster than the 5th order polynomial rate.

Further, for the delayed Duffing equation, it was found that using a 4th order polynomial would yield nonlinear rates of convergence that would exceed those of the corresponding 5th order approximation. In addition, using a 5th order polynomial in the Duffing equation showed a linear rate of convergence except at a critical $h$ value where the solution would seem to converge before the error increased again and continued to follow a linear trend as $h$ was decreased. Explaining the above anomalies is a topic of future research; however, the authors speculate that they might be associated with the accuracy of the differentiation matrix in representing the real differential operator at these $n$ and $h$ values.

In addition, the stability of the periodic solutions was substantiated using the concepts described in Section 6.4. The results of this study establish the spectral element approach as a useful technique for studying nonlinear delay equations and
open up a wider range of applications where this technique can be used.
The relative vibrations between the tool and the workpiece is a normal phenomenon associated with cutting operations. When these vibrations become unstable, they are commonly referred to as chatter which may result in inferior part surfaces and increased tool wear. Chatter can also damage the workpiece, the fixture and/or the machine spindle. Mapping the areas of stability as a function of the machining parameters, namely the depth of cut and the spindle speed, not only helps avoid these detrimental effects of chatter but also increases the efficiency of the cutting process. Predictive models that can generate stability regions for a wide combination of speeds and cutting depths eliminate the costly and time-consuming trial and error alternative.

In order to generate the stability charts for a cutting operation, it is necessary to model the system dynamics through suitable equations of motion, descriptive force models and apply proper solution techniques. Models describing cutting tool vibrations began to appear in literature about half a century ago [108–111]. Several models have been proposed to characterize the cutting forces as a function of the cutting parameters, such as the depth of cut and the instantaneous chip thickness.
whose product forms the instantaneous chip area. These models treated cutting forces as a point force acting at the tool tip. This conventional approximation of the cutting forces has been verified experimentally in the middle range of cutting speeds. However, actual observations of the cutting process at low speeds show improved stability when compared to those obtained from theoretical predictions, (e.g. see Fig. 7.1).

![Stability Charts](image)

**Figure 7.1:** Qualitative stability charts shown to illustrate the phenomenological increase in stability at relatively low cutting speeds. Graph (a) shows the stability boundaries for a typical point force model and graph (b) provides a qualitative representation of the commonly observed stability increase at low speeds.

There are many machining operations that can be performed only at low speeds. For instance, harder to machine materials, like stainless steel and titanium, are used extensively in medical tool manufacturing and in the aerospace industry. Titanium, for example, has very high strength to weight ratio and excellent corrosion resistance, which makes it ideal for aerospace applications [112]. Titanium also has great biocompatibility so it is widely used in medical implants [113]. However, titanium is difficult to machine owing to its inherent properties of high strength combined with poor thermal conductivity. This requires titanium to be machined only at low speeds.

The improved stability behavior at low speeds has been attributed to an energy dissipation mechanism commonly called process damping. Process damping plays a key role in stability determination in machining processes [114–116]. Tobias and
Fishwick [108], Tlusty [117] and Minis et al. [23] tried to account for process damping by including the displacement variable and its derivative in the cutting force model. Other models recognized process damping as the result of the interference between the cutting tool flank face and the undulated machined surface [114] where the amplitude of the force was assumed to be proportional to the material volume displaced due to interference [118–122]. Chiou and Liang [123] and later Clancy and Shin [124] used this model to account for process damping to capture the effect of tool wear on stability in turning. The same model was implemented by Chandiramani and Pothala [125] in their study of regenerative chatter in a two degree of freedom model for turning.

An alternative explanation for the increased stability at lower speeds was recently introduced by Stepan [18,126]. Instead of modeling the cutting forces as a single point force, a continuous or distributed time delay model was introduced to capture the force distribution over the tool-chip interface (see Fig. 7.2).

This chapter investigates the influence of the distributed force model on the stability behavior of continuous and interrupted turning. In particular, an ap-

\[
\begin{align*}
\xi(t) & \quad \xi(t-\tau) \\
h \quad & m
\end{align*}
\]

Figure 7.2: Schematic diagram of a: (a) distributed force model and (b) conventional point force model. Case (a) uses a stress distribution over the tool rake face and applies a finite time for the chip to travel along the tool-chip interface. Case (b) is the conventional modeling approach of using a point force and a discrete delay model.

This chapter investigates the influence of the distributed force model on the stability behavior of continuous and interrupted turning. In particular, an ap-
proach is described to transform the distributed-delay equations into a discrete-delay system [54, 82]. Theoretical stability investigations are performed using a state-space TFEA (temporal finite element analysis) technique [56]. Finally, different continuous-to-discrete delay ratios are used to elucidate parameter regimes where the phenomenological increased stability behavior is adequately captured.

7.1 Turning Process Models

This section starts by deriving a variational equation of motion for a single degree of freedom turning process. In section 7.1.2, the conventional point force model is described and the corresponding variational equation of motion is obtained. The resulting equation is then non-dimensionalized and is written into its state-space form which is used for stability determination. In section 7.1.3, the distributed force model is introduced and the corresponding variational equation of motion is derived. This equation includes the discrete time delay, due to the tool passage period, as well as a continuous time delay due to the chip sliding over the tool-chip interface.

7.1.1 Equation of Motion

The governing equation of motion for a rigid workpiece and a tool compliant in one direction, \( z(t) \), is

\[
m \ddot{z} + c \dot{z} + k z = -F_z(A),
\]  

(7.1)

where \( m \), \( c \) and \( k \) are the modal mass, stiffness and damping respectively, and \( F_z(A) \) is the cutting force component along the \( z \) direction as a function of the instantaneous chip area, \( A(t) \). The instantaneous chip area is the product of the depth of cut, \( b \), and the instantaneous chip thickness, \( h(t) \), i.e. \( A(t) = b h(t) \).

Let \( z^* \) be the equilibrium solution for the system, which corresponds to cutting at the nominal depth of cut \( h_o \). Under these conditions, the system is in equilibrium with
the static cutting force, $f_o$. Then for any other solution, $z(t)$, one can write [127]

$$
z(t) = z^* + \xi(t),$$

(7.2)

where $\xi(t)$ represents a perturbation of the equilibrium solution. The growth or decay of such small perturbations determines the stability of the original system, Eq. (7.1). Similarly, the cutting force can be described as the summation of a static component and a dynamic variation due to oscillations

$$
F_z(A) \approx f_o + \Delta F_z(A) \quad (7.3a)
$$

$$
\approx -kz^* + \Delta F_z(A), \quad (7.3b)
$$

Equations (7.2) and (7.3) are substituted into Eq. (7.1) to obtain the variational equation

$$
m\ddot{\xi}(t) + c\dot{\xi}(t) + k\xi(t) \approx -\Delta F_z(A).$$

(7.4)

Dropping the approximation sign and dividing by the mass, $m$, Eq. (7.4) becomes

$$
\ddot{\xi}(t) + 2\zeta\omega_n \dot{\xi}(t) + \omega_n^2 \xi(t) = -\frac{1}{m} \Delta F_z(A),
$$

(7.5)

where $\omega_n = \sqrt{k/m}$ is the angular natural frequency, $\zeta = c/(2m\omega_n)$ is the damping ratio, and $\Delta F_z(A)$ is the dynamic force variation. The expression for the force variation depends on the adopted force model and the type of cutting, i.e. continuous or interrupted. In the next section, the conventional point force model will be used in conjunction with Eq. (7.5) to produce an equation that can be used to determine the stability regions.

7.1.2 Point Force Model

In the conventional point force model, cutting forces are characterized by a single force vector acting at a single point—the tool tip. This force is assumed to be a function of the instantaneous chip area, $A(t)$, (see Fig. 7.3).
The dynamic force variation is written as a Taylor series expansion around the nominal uncut chip area $A_o$

$$\Delta F_z(A) = u(t) \left( F_z(A) - f_o \right) = u(t) \sum_{j=1}^{p} k_j (\Delta A)^j$$

(7.6)

where for continuous turning the tool is assumed to be in the cut all the time and $u(t)$ assumes a value of one, i.e. $u(t) = 1$. In contrast to continuous turning, interrupted turning is a piece-wise continuous system where the tool is not always cutting but enters and exits the cut, Fig 7.4.

When the tool is in the cut, a force proportional to the uncut chip area acts on the tool, however, when the tool exits the cut, the system will undergo free vibration,
(i.e. for interrupted turning, \(u(t)\) is a switching function: its value is 1 when the tool is cutting, and 0 when the tool exits the cut). The chip area variation is

\[
\Delta A = A - A_o = b (h - h_o).
\] (7.7)

The coefficients of the power series in Eq. (7.6) come from

\[
k_j = \frac{1}{j!} \frac{d^j F_z(A_o)}{dA^j}, \quad j = 1, 2, \ldots
\] (7.8a)

\[
\Rightarrow \{k_j\} = \{k_1, k_2, k_3, \ldots\} = \left\{ \frac{df(A_o)}{dA}, \frac{1}{2!} \frac{d^2 f(A_o)}{dA^2}, \frac{1}{3!} \frac{d^3 f(A_o)}{dA^3}, \ldots \right\}. \quad (7.8b)
\]

The first coefficient, \(k_1\), in the above expression is referred to as the cutting coefficient and is usually determined experimentally. \(k_1\) represents the linear approximation of the cutting force variation, while the higher order terms, \(k_2 - k_\infty\), provide higher order approximations.

The linear approximation of the cutting force variation is inserted into Eq. (7.5) to obtain

\[
\ddot{\xi}(t) + 2\zeta \omega_n \dot{\xi}(t) + \omega_n^2 \xi(t) = -u(t) \frac{k_1 b}{m} (h(t) - h_o).
\] (7.9)

Substituting \(h(t) = h_o + \xi(t) - \xi(t - \tau)\) into Eq. (7.9) gives

\[
\ddot{\xi}(t) + 2\zeta \omega_n \dot{\xi}(t) + \omega_n^2 \xi(t) = -u(t) \frac{k_1 b}{m} (\xi(t) - \xi(t - \tau))
\] (7.10)

which represents a delay differential equation, DDE, with constant coefficients.

A set of dimensionless parameters for time, spindle speed, time delay and depth of cut is defined as \[8\]

\[
\tilde{t} = \omega_n t,
\] (7.11a)

\[
\tilde{\Omega} = \frac{\Omega}{\omega_n},
\] (7.11b)

\[
\tilde{\tau} = \omega_n \tau,
\] (7.11c)

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The substitutions result in the dimensionless characteristic frequency $\tilde{\omega}_n = 1$. The non-dimensionalized version of Eq. (7.10) then reads

$$\ddot{\xi}(\tilde{t}) + 2\zeta \dot{\xi}(\tilde{t}) + \xi(\tilde{t}) = -u(t) \tilde{b}(\xi(\tilde{t}) - \xi(\tilde{t} - \tilde{\tau})).$$

Equation (7.12) can be written in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(u(t) \tilde{b} + 1) & -2\zeta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ u(t) \tilde{b} & 0 \end{bmatrix} \begin{bmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{bmatrix},$$

where the tilde was dropped from $t$ and $\tau$ for convenience. To help explain the analysis that follows, the above state-space equation is written in a more compact form

$$\dot{x}(t) = A(t)x(t) + B(t)x(t - \tau).$$

Equation (7.14) describes a turning operation subject to the conventional point force model. The stability analysis of equations having the same form as Eq. (7.14) will be explained using the state-space temporal finite element approach, TFEA, in section 7.2, (see [56]). In the next section an analogous equation will be formulated for the distributed force model.

7.1.3 Distributed Force Model

The distributed force model reflects a more realistic representation of the physical cutting forces in turning. Instead of concentrating the cutting forces at a single point, these forces are assumed to have a distribution per unit length, $P_z$, with varying magnitudes along the tool-chip interface. The cutting forces can be described in terms of this distribution by

$$F_z(A) = \int_0^t P_z(A, s)ds,$$
where \( s \) is a local coordinate whose origin is fixed to the tool tip. This local coordinate describes the contact distance between the sliding chip and the active face of the tool. The range of values for \( s \) is from 0 to the length which represents the location where the chip separates from the tool, \( l \). One approximation for \( P_z \) combines the Taylor approximation of the cutting force with an estimated shape function \( W \) (with units of \( 1/m \))

\[
P_z(A, s) = F_z(A)W(s), \quad s \in [0, l]. \tag{7.16}
\]

In order to ensure that this new model maintains the mechanics of the system; substituting \( A = A_o \), which corresponds to cutting under stationary conditions, should yield a value of \( f_o = F_z(A_o) \). Imposing this constraint on equation (7.15) gives

\[
F_z(A_o) = \int_0^l P_z(A_o, s)\,ds = F_z(A_o) \int_0^l W(s)\,ds,
\]

\[
\Rightarrow \int_0^l W(s)\,ds = 1. \tag{7.17}
\]

Equation (7.17) provides a fundamental condition which has to be satisfied by any chosen shape function. Let \( \tau \) be the discrete time delay associated with the tool passage period, and let the short time delay \( t_s \) be the time the chip is in contact with the tool. These two different delays are given by

\[
\tau = \frac{d_o \pi}{v} = \frac{2\pi}{\Omega}, \quad t_s = \frac{l}{v_c} = \frac{l}{r_c v}, \tag{7.18}
\]

where \( d_o \pi \) is the circumference of the cylindrical workpiece, \( \Omega \) is the angular velocity of the workpiece in units of rad/s, \( r_c \) is the chip thickness ratio and \( v_c = r_c v \) is the chip speed over the rake face [24]. It follows that the ratio of the short time delay \( t_s \)}
to the long one \( \tau \) is constant

\[
\frac{r}{\tau} = \frac{t_s}{\tau} = \frac{l}{r_c d_0 \pi}
\]

The shape of the stress distribution can be expressed in the time domain by introducing a local time \( \hat{t} = s/v_c \). Substituting \( \hat{t} \) into Eq. (7.17) yields the following condition on the shape function in the local time domain

\[
\int_0^t W(s) ds = 1,
\]

\[
\Rightarrow \int_0^{t_s} v_c W(v_c \hat{t}) d\hat{t} = 1, \text{ let } w(\hat{t}) = v_c W(v_c \hat{t}),
\]

\[
\Rightarrow \int_0^{t_s} w(\hat{t}) d\hat{t} = 1.
\] (7.19)

This local time, \( \hat{t} \in [0, t_s] \), gives how much earlier in time a certain segment of the chip flowing over the active tool face was being cut at the tip of the tool as shown in Fig. 7.5.

![Figure 7.5: Short time delay embedding into the force model. The area of a certain segment of the chip flowing over the rake face in (b) can be described by the area of the same segment as it was being cut at the tool tip in (a) at time \( t - \hat{t} \).](image)

The assumption that the chip flows at a speed, \( v_c = r_c v \), means that the area of a
section of the chip above the tool tip at a local time \( \hat{t} \) is equal to the area of the same chip section as it was being cut at the tool tip at \( t - \hat{t} \) time units ago, or

\[
A(t, \hat{t}) = b(h_o + \xi(t - \hat{t}) - \xi(t - \tau - \hat{t})), \quad t \in [t_o, \infty), \quad \hat{t} \in [0, t_s], \tag{7.20}
\]

where \( A(t, \hat{t}) \) defines the chip area at time \( t \) and local time \( \hat{t} \). At \( \hat{t} = 0 \), Eq. (7.20) gives back the regular expression for the chip area at the tool tip

\[
A(t, 0) = b(h_o + \xi(t) - \xi(t - \tau)).
\]

The cutting force distribution can now be expressed in the \( z \) direction in the time domain using the global time \( t \) and the local one \( \hat{t} \)

\[
p_z(t, \hat{t}) = P_z(A(t, \hat{t}), v_c\hat{t}) = F_z(A(t, \hat{t})) \frac{1}{v_c} w(\hat{t}), \quad t \in [t_o, \infty), \quad \hat{t} \in [0, t_s]. \tag{7.21}
\]

Substituting the above results; Eqs. (7.15), (7.19), (7.21), the power series from Eq. (7.6) and also the chip area from Eq. (7.20), the cutting force variation in the \( z \) direction reads

\[
\Delta F_z(A(t, \hat{t})) \approx u(t) \int_0^{t_s} \sum_{j=1}^p k_j b^j (\xi(t - \hat{t}) - \xi(t - \tau - \hat{t}))^j w(\hat{t}) d\hat{t} \tag{7.22}
\]

\[
\approx u(t) k_1 b \int_0^{t_s} (\xi(t - \hat{t}) - \xi(t - \tau - \hat{t})) w(\hat{t}) d\hat{t}, \tag{7.23}
\]

where equation (7.23) is the expression for the linearized cutting force variation. For continuous turning \( u(t) = 1 \); thus, substituting Eq. (7.23) into the variational equation of motion Eq. (7.5) yields

\[
\ddot{\xi} + 2\zeta \omega_n \dot{\xi} + \omega_n^2 \xi = - \frac{k_1 b}{m} \int_0^{t_s} (\xi(t - \hat{t}) - \xi(t - \tau - \hat{t})) w(\hat{t}) d\hat{t}. \tag{7.24}
\]
The integro-differential form of Eq. (7.24) as well as the short time delay complicate the stability analysis of the system. However, Eq. (7.24) can still be solved through a suitable choice of the weight function. The solution technique involves increasing the order of the system by one and writing the short delay in terms of the discrete one as will be shown in Section 7.2.

7.2 Stability Analysis

In this section, Eq. (7.24) is converted into a solvable form through using an exponential weight function. The stability analysis of the resulting state-space equations is then carried out using state-space TFEA. The analysis is shown only for the distributed delay system, and a similar analysis can be used for the point force model system, Eq. (7.14). In sections 7.2.1 and 7.2.2, stability boundaries are plotted for continuous and interrupted turning, respectively. In each of these sections, the results are shown for both: the point force model and the distributed force model with different continuous-to-discrete delay ratios.

Assume that the shape of the distributed cutting force system, \( W(s) \), is approximated by the exponential function

\[
W(s) = \frac{1}{l} \exp \left( -\frac{s}{l} \right) \Rightarrow w(\hat{t}) = \frac{v_c}{l} \exp \left( -\frac{v_c}{l} \hat{t} \right) = \frac{1}{r \tau} \exp \left( -\frac{\hat{t}}{r \tau} \right), \hat{t} \in [0, \infty). \quad (7.25)
\]

This choice of the weight function agrees with several studies which showed that the normal forces over the rake face vary exponentially \[24,128–131\]. Further, the analysis of the DDE described in (7.24) is simplified by using this exponential weight function since the distributed delay only increases the system dimension by one. The procedure starts with differentiating (7.24) with respect to the time \( t \) to calculate

\[
^{(3)} \xi(t) + 2\zeta \omega \ddot{\xi}(t) + \omega_n^2 \xi(t) = -\frac{k_1 b}{m r \tau} \int_{0}^{\infty} (\dot{\xi}(t - \hat{t}) - \dot{\xi}(t - \tau - \hat{t})) \exp \left( -\frac{\hat{t}}{r \tau} \right) d\hat{t}.
\]

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However, to account for the discontinuous cutting forces during interrupted cutting, the right hand side is multiplied by a switching function, \( u(t) \), which gives

\[
\xi(t) + 2\zeta \omega \ddot{\xi}(t) + \omega_n^2 \dot{\xi}(t) = -u(t) \frac{k_1 b}{mr \tau} \int_0^\infty (\dot{\xi}(t - \hat{t}) - \dot{\xi}(t - \tau - \hat{t})) \exp\left(-\frac{\hat{t}}{r \tau}\right) d\hat{t}, \tag{7.26}
\]

where \( u(t) \) is 1 for continuous turning and it switches between 0 and 1 for interrupted cutting. Next, the right hand side of Eq. (7.26) is integrated by parts

\[
-u(t) \frac{k_1 b}{mr \tau} \int_0^\infty (\dot{\xi}(t - \hat{t}) - \dot{\xi}(t - \tau - \hat{t})) \exp\left(-\frac{\hat{t}}{r \tau}\right) d\hat{t} =
\]

\[
= u(t) \frac{k_1 b}{mr \tau} \left[ (\dot{\xi}(t - \hat{t}) - \dot{\xi}(t - \tau - \hat{t})) \exp\left(-\frac{\hat{t}}{r \tau}\right) \right]_0^\infty
\]

\[
+ u(t) \frac{k_1 b}{mr \tau} \frac{1}{r \tau} \int_0^\infty (\dot{\xi}(t - \hat{t}) - \dot{\xi}(t - \tau - \hat{t})) \exp\left(-\frac{\hat{t}}{r \tau}\right) d\hat{t}
\]

\[
= -u(t) \frac{k_1 b}{mr \tau} (\dot{\xi}(t) - \dot{\xi}(t - \tau)) - \frac{1}{r \tau} (\ddot{\xi}(t) + 2\zeta \omega_n \dot{\xi}(t) + \omega_n^2 \xi(t)). \tag{7.27}
\]

Substituting the force expression (7.27) back into equation (7.24) gives

\[
(\frac{3}{r \tau} + 2\zeta \omega \ddot{\xi}(t) + \frac{2\zeta \omega_n}{r \tau} \dot{\xi}(t) + \frac{1}{r \tau} (\omega_n^2 + \frac{u(\tilde{t}) k_1 b}{mr}) \xi(t) - u(\tilde{t}) \frac{k_1 b}{mr \tau} \xi(t - \tau) = 0,
\tag{7.28}
\]

and the equation of motion has been transformed into a third order system without a continuous time delay but still having the discrete delay \( \tau \). Eq. (7.28) can be non-dimensionalized using the same parameters introduced in Eq. (7.11). This results in the following non-dimensional third order DDE

\[
(\frac{3}{r \tilde{\tau}} + 2\zeta \ddot{\xi}(\tilde{t}) + (\frac{2\zeta}{r \tilde{\tau}} + 1) \dot{\xi}(\tilde{t}) + \frac{1}{r \tilde{\tau}} (1 + u(\tilde{t}) \tilde{b}) \xi(\tilde{t}) - u(\tilde{t}) \frac{\tilde{b}}{r \tilde{\tau}} \xi(\tilde{t} - \tilde{\tau}) = 0, \tag{7.29}
\]
which can be written in state-space form as

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\frac{1}{r^2}(1 + u(t) \, \bar{b}) & -(\frac{2r}{r^2} + 1) & -(\frac{1}{r^2} + 2\zeta) \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_1(t - \tau) \\
y_2(t - \tau) \\
y_3(t - \tau) \\
\end{bmatrix},
\]

where the tilde was dropped from \(t\) and \(\tau\) for convenience. The state-space equation could be written in an equivalent form as

\[
\dot{\mathbf{y}}(t) = \mathbf{C}(t)\mathbf{y}(t) + \mathbf{D}(t)\mathbf{y}(t - \tau).
\]

The stability analysis of Eq. (7.31) can be carried out using the state-space TFEA approach described in section 2.1.

### 7.2.1 Continuous Turning Stability

Figure 7.6 shows two stability charts that were obtained using the point force model. Two cases were considered: one with a relatively high damping ratio, \(\zeta = 0.02\), and another with a relatively low damping ratio, \(\zeta = 0.0038\). The region below the boundary line represents stable cutting conditions, while that above the boundary line corresponds to an unstable cutting process. It can be seen that at low speeds, this model does not capture the improved stability encountered in practice. For all continuous turning stability plots, a square grid of \(600 \times 600\) points was used. Figure 7.7 shows two stability plots that were obtained with the distributed force model. These plots were generated with a very-low value of the delay ratio, \(r = 0.001\), and they can be compared directly to their counterparts for a point force model in Fig. 7.6. It can be seen that the distributed force model gives results similar to the point force model when the delay ratio is very small, i.e. when the short delay is negligible.
7.2.2 Interrupted Turning Stability

For the point force model, Fig. 7.9 shows the cases when $\rho = 0.05$, 0.10, and 0.20, where $\rho$ represents the fraction of the workpiece revolution that the tool is cutting.
Figure 7.8: Stability charts for the distributed force model of continuous turning plotted as a function of the non-dimensionalized cutting speed and depth of cut. The damping ratio used is $\zeta = 0.0038$, and the delay ratios used are (a) $r = 0.03$, (b) $r = 0.05$ and (c) $r = 0.10$, (unstable regions shaded).

A fine mesh of $2400 \times 600$ was used to generate all the stability plots of interrupted turning. These results are in agreement with the results obtained in reference [8] for interrupted turning. Figures 7.10 and 7.11 are stability charts for interrupted cutting which use the distributed force model. In Fig. 7.10, cases (a), (b) and (c) correspond to a value of $r = 0.001$. This low value of $r$ yields similar results to those for the point force model, (see Fig. 7.9). On the other hand, as the value of $r$ increases to 0.03, as in cases (d), (e) and (f), the stability at low speeds is improved for the different values of $\rho$. This trend is even more prominent in Fig. 7.11 where $r$ was set to 0.05 and 0.10.
Figure 7.9: Stability charts for the point force model of interrupted turning plotted as a function of the non-dimensionalized cutting speed and depth of cut. The damping ratio used is $\zeta = 0.0038$, and the cases for different fractions of the workpiece revolution that the tool is cutting are (a) $\rho = 0.05$, (b) $\rho = 0.10$ and (c) $\rho = 0.20$, (unstable regions shaded).

7.3 Discussion of Results

This chapter investigates the increased stability behavior commonly observed at low cutting speeds. In literature, this improved stability has been attributed to the interference between the workpiece and the tool relief face. The energy dissipation through this interference mechanism has been called process damping. On the other hand, the distributed delay used in this chapter has been an un-modeled param-
Figure 7.10: Stability charts for the distributed force model of interrupted turning plotted as a function of the non-dimensionalized cutting speed and depth of cut. The damping ratio used is $\zeta = 0.0038$, and the cases for different fractions of the workpiece revolution that the tool is cutting and different delay ratios, respectively, are (a) $\rho = 0.05$ and $r = 0.001$, (b) $\rho = 0.10$ and $r = 0.001$, (c) $\rho = 0.20$ and $r = 0.001$, (d) $\rho = 0.05$ and $r = 0.03$, (e) $\rho = 0.10$ and $r = 0.03$, and (f) $\rho = 0.20$ and $r = 0.03$, (unstable regions shaded).
Figure 7.11: Stability charts for the distributed force model of interrupted turning plotted as a function of the non-dimensionalized cutting speed and depth of cut. The damping ratio used is $\zeta = 0.0038$, and the cases for different fractions of the workpiece revolution that the tool is cutting and different delay ratios, respectively, are (a) $\rho = 0.05$ and $r = 0.05$, (b) $\rho = 0.10$ and $r = 0.05$, (c) $\rho = 0.20$ and $r = 0.05$, (d) $\rho = 0.05$ and $r = 0.10$, (e) $\rho = 0.10$ and $r = 0.10$, and (f) $\rho = 0.20$ and $r = 0.10$, (unstable regions shaded).
eter that was shown to have a great impact on low-speed cutting stability. More specifically, an exponential shape function is proposed to approximate the force distribution over the tool-chip contact length, and a constant delay ratio is introduced to describe the ratio between the distributed and the discrete delays.

The distributed force model results in a more complicated governing equation, a second order delayed integro-differential equation, that involves both a discrete and distributed delay. The distributed delay results from the finite time it takes the chip to slide along the rake face of the tool while the discrete delay is from the period between consecutive passages of the cutting tooth. An approach to transform the governing equation of motion into a third order discrete system is described and the state-space representation of the new system is obtained. The state-space TFEA technique is then used to chart the stability boundaries for continuous and interrupted turning. Different delay ratios are used for generating stability charts to study the effect of the distributed delay on stability. For comparison purposes, the point force model is also used to obtain the conventional stability charts for continuous and interrupted turning. It was found that for a small value of the delay ratio, i.e. when the continuous delay is negligible in comparison to the discrete one, the predicted stability boundaries were similar to those of the point force model. However, for larger values of the delay ratio, the distributed force model showed an improved stability behavior at lower speeds when compared to the point force model. The stable parameter space continued to increase with increases in the delay ratio which confirms the stabilizing effect of the distributed delay.

A specific contribution of the current paper is the introduction of a more realistic method to represent cutting forces. More specifically, the present manuscript has described and investigated an alternative physical explanation for process damping where a distributed cutting force model, along with an exponential distribution over the tool-chip interface, is assumed. Although a distributed force model is more realis-
tic, this idea contrasts the standard approach of using a point force. The distributed force model also provides an alternative explanation for the improved stability at low speeds while still allowing analytical stability analysis. The current approach further averts the complications of previous works which either use an ad-hoc damping term that is inversely proportional to spindle speed or a displaced volume relationship that must be numerically and experimentally calibrated. However, although the stability diagrams from the present study agree qualitatively with the observed low speed stability results, the authors believe that only experimental investigations can reveal which models or combination of models will most accurately capture the process damping effects.

Possible tasks for future research include experimental verification and/or a comparison study with historical models of process damping. The success of the current approach in handling short delays also suggests that a similar analysis can be performed to accommodate interference forces on the relief face. This will still allow for an analytical stability analysis in the parameter space while capturing the physical phenomena of the process. Finally, another task is to construct a solution strategy that can accommodate different shape functions for the force distribution, and to expand these solution techniques to other cutting operations.
Self-excited vibrations in a delay oscillator: Application to milling

Self-excited vibrations can occur when the forces exciting a system are coupled to the state variables. This coupling can be the result of a feedback mechanism where the excitation forces are a function of current and past state variables. The equations of motion incorporating such mechanism are typically delay differential equations (DDE) which are either autonomous or time-varying. One physical application for DDEs is found in metal removal processes. More specifically, many studies have investigated the occurrence of self-excited vibrations in machining, commonly known as chatter.

Chatter is typically characterized by large amplitude oscillations that deteriorate the surface finish and can also damage the tool, machine spindle, and the workpiece. For a limited class of DDEs, e.g. continuous turning, the stability boundaries can be obtained in a closed-form [18]. However, the stability analysis of more general DDEs requires using approximation techniques, such as semi-discretization [44], Chebyshev-based methods [48,132,133], collocation methods [45], temporal finite ele-
ment analysis (TFEA) [21,134–136], as well as frequency domain techniques [30,137]. Numerical simulation is also used to study machining stability [138–141]; however, semi-analytical predictions of stability can quickly and accurately give stability regions over the process parameter space of interest making them superior to tedious numerical simulations [142–145].

The presence of piecewise continuous coefficients in the DDE considerably increases the level of complexity. Milling, for instance, is a common metal removal process which is commonly approximated as a periodic, piecewise continuous system. Therefore, approximation schemes, such as TFEA [21,28,29,73,136,146] and collocation methods [9,10], are typically necessary to determine milling stability. The stability of milling processes is typically reported through stability diagrams which chart the boundaries between stable and unstable cuts as a function of the spindle speed and depth of cut [110, 111, 147]. A milling process is stable if it is chatter-free, whereas at the onset of chatter, the process becomes unstable. These diagrams enhance efficiency and reduce costs by eliminating the need for trial and error. In addition to solving for the system stability, semi-analytical techniques can determine the type of bifurcation associated with instabilities.

For example, recent studies have shown that a new bifurcation phenomena can occur in highly intermittent cutting. Besides Neimark-Sacker or secondary Hopf bifurcations, period-doubling bifurcations have been analytically predicted in Refs. [7, 141, 143, 148] and confirmed experimentally in Refs. [73,149–151]; a specific outcome from these works was that they found period-doubling at low radial immersions when only a single tooth was cutting.

A different stability behavior for helical mills was reported and verified experimentally in Refs. [1, 2,136,152]. Closed boundaries, or islands, of unstable period-doubling were shown to appear in the stability diagram due to the helical flutes of the tool. In these works, the period-doubling islands were shown only at rela-
tively low radial immersions. Further, Refs. [136, 152] studied the case of only one flute cutting at a time, see Fig. 8.1-(a), whereas Refs. [1, 2] investigated the case of multiple flutes simultaneously cutting when using a high depth of cut, see Fig. 8.1-(b). Reference [10] studied the case of multiple engaged flutes with zero-helix angle; however, the stability analysis for the cases of multiple non-zero helix flutes simultaneously cutting due to a high step-over distance, (see Fig. 8.1-(c)), or due to a combination of a high depth of cut and a high step-over distance, see Fig. 8.1-(d), has not received much attention.

In this chapter, the state-space TFEA method and the Chebyshev collocation approach are used to study the stability of a milling process. The added complexity of simultaneously engaged flutes for both zero- and non-zero helix tools is considered. In contrast to prior works, multiple flute engagement due to both high depth of cut and high step-over distance are investigated. A particular outcome of this study is the demonstration of a different stability behavior at the loss of stability in comparison to prior results. To elaborate, period-doubling regions are shown to appear at relatively high radial immersions when multiple flutes with either a zero or non-zero-helix angle are cutting simultaneously. We also elucidate the sensitivity of the period-doubling regions to the radial immersion and reveal the strong influence of the parity in the number of flutes—especially at full radial immersion—on the stability behavior. We also present upmilling stability charts where, in contrast to typical results in milling literature, the helical tools induce a strong waviness along the depth of cut direction in the Hopf lobes.

8.1 Mechanical Model

The equation of motion (EOM) for a single-mode helical mill compliant only in the $y$ direction such as the one shown in Fig. 8.2, (or similarly for a workpiece compliant
Figure 8.1: The different cases associated with a helical milling tool. The gray area represents the cutting zone and in graph (a), only a single flute is cutting at any instant, while in graphs (b) and (c) multiple flutes are cutting due to a high depth of cut, and a high radial step-over distance, respectively. Graph (d) shows the case of a high depth of cut combined with a high radial step-over distance. The variables shown in the figure are defined in Section 8.1.

in the $y$ direction), is described by

\[
\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{1}{m_y}F_y(z, t, \tau),
\]

(8.1)

where $m_y$, $\omega_n$, and $\zeta$ are the modal mass, natural frequency and damping ratio, respectively. The term $F_y(z, t, \tau)$ describes the cutting forces in the $y$ direction while the time delay $\tau = 2\pi/N\Omega$ is the tooth passage period for an $N$-tooth cutter rotating at a spindle speed $\Omega$ (rad/sec). An analytical expression for $F_y$ can be obtained by first introducing the differential forces, shown in Fig. 8.3, in the tangential and radial
Figure 8.2: Illustrations of (a) upmilling, and (b) downmilling.

directions [24] according to

\[
\begin{align*}
\frac{df}{dt} &= K_t w(\theta_n(t, z), \tau) \; dz, \\
\frac{df}{dr} &= K_r w(\theta_n(t, z), \tau) \; dz,
\end{align*}
\]

(8.2a) (8.2b)

where \(K_t\) and \(K_r\) are the tangential and radial specific cutting force coefficients, respectively, and the variable \(z\) varies along the axial direction of the tool from 0 at the tip to \(b\)—the depth of cut. The angle \(\theta_n(t, z)\) describes the rotation angle of the \(n\)th flute from the vertical reference shown in Fig. 8.2, and is described by

\[
\theta_n(t, z) = \Omega t - (n - 1)\theta_p - \kappa z, \quad \text{where} \; n = 1, 2, \ldots, N
\]

(8.3)

where \(\theta_p\) is the tool pitch angle, i.e. \(\theta_p = 2\pi/N\) for a tool with uniformly spaced flutes, and \(\kappa = 2\tan\beta/D\) is a helix parameter. The radial chip thickness for the reference cutting tooth can be found by applying the circular tool path assumption which yields [153,154]

\[
w(\theta_n(t, z)) = h \sin(\theta_n(t, z)) + [y(t) - y(t - \tau)] \cos(\theta_n(t, z)),
\]

(8.4)

where \(h\) is the feed per tooth. The total force in the \(y\) direction is found by integrating Eqs. (8.2a) and (8.2b) with respect to the differential axial depth \(dz\) which gives

\[
F_y = g_n(t) \int_{z_n(t)}^{z_n(t)} \left[ \frac{df}{dz} \sin\theta_n(t, z) - \frac{df}{dz} \cos\theta_n(t, z) \right] dz,
\]

(8.5)
where \( g_n(t) \) is a switching function: its value is 1 if the \( n \)th tooth is cutting, and 0 if the tooth exits the cut. The functions \( z_a(n,t) \) and \( z_b(n,t) \) describe the lower and upper limits of integration, respectively, and they are graphed in Fig. 8.4 where they are shown to vary according to

\[
\begin{align*}
    z_a(n,t) &= \begin{cases} 
        0 & \text{if } \theta_{st} \leq \theta_n \leq \theta_{st} + \theta_{tip}, \\
        \frac{\theta_n - (\theta_{st} + \theta_{tip})}{\kappa} & \text{if } \theta_{st} + \theta_{tip} < \theta_n,
    \end{cases} \\
    z_b(n,t) &= \begin{cases} 
        \frac{\theta_n - \theta_{st} - \kappa}{b} & \text{if } \theta_{st} \leq \theta_n \leq \theta_{st} + \theta_{lag}, \\
        b & \text{if } \theta_{st} + \theta_{lag} < \theta_n,
    \end{cases}
\end{align*}
\tag{8.6a}\tag{8.6b}
\]

where \( \theta_{tip} \) is the angular distance over which the tool is cutting at \( z = 0 \), \( \theta_{st} \) is the entry angle which is 0 for upmilling and \( \pi - \theta_{tip} \) for downmilling, while \( \theta_{lag} = \kappa b \) is the angular distance defined in Fig. 8.3.

![Figure 8.3: Schematic of a helical end mill with multiple flutes as well as the differential cutting forces in the axial, radial, and tangential directions.](image)

The angular distance, \( \theta_{tip} \), depends on the ratio of the radial step-over distance to the tool diameter, which is often called the radial immersion (RI), according to

\[
\theta_{tip} = \cos^{-1}(1 - 2\text{RI}).
\tag{8.7}
\]

Substituting the expression for the total force in the \( y \) direction, Eq. (8.5), into Eq.
The equation of motion now reads

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = \frac{K_c(t,b)}{m_y} (y(t) - y(t - \tau)),$$

where only the dynamic chip thickness was considered, i.e. by ignoring the term $h \sin \theta_n(t,z)$ in Eq. (8.4) which does not affect the stability analysis. The directional force coefficient $K_c(t,b)$ is $\tau$-periodic and is a function of time and the depth of cut $b$ [2,152]. This coefficient is defined using Eqs. (8.2a), (8.2b) and (8.4) as follows

$$K_c(t,b) = \sum_{n=1}^{N_t} g_n(t) \int_{z_a(n,t)}^{z_b(n,t)} (K_t \sin \theta_n \cos \theta_n - K_r \cos^2 \theta_n) \, dz,$$

where the summation is over the total number of simultaneously engaged flutes $N_t = \lceil (\theta_{\text{tip}} + \theta_{\text{lag}})/\theta_p \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.

It is necessary here to make a distinction between the tool pitch angle $\theta_p$ and the mill helix pitch $p$ which are related according to $\theta_p = \kappa p$. Whereas the pitch angle describes the angular distance between two consecutive teeth, the mill helix pitch describes the distance between two adjacent flutes along the axis of the tool, as shown in Fig. 8.3. Further, the mill helix pitch is a geometric property of helixes, and it is given mathematically by [1]

$$p = \frac{D \pi}{N \tan \beta},$$

143
where $D$ is the diameter of the tool and $\beta$ is the helix angle defined in Fig. 8.3. Both $\theta_p$ and $p$ play an important role in determining whether multiple flutes are cutting or not. Specifically, the condition for multiple flutes in contact is

$$\left(\frac{\theta_{\text{tip}} + \theta_{\text{lag}}}{\theta_p}\right) > 1.$$  \hfill (8.11)

This condition captures cases (b), (c) and (d) in Fig. 8.1.

In order to use the state-space TFEA method or the Chebyshev collocation approach, Eq. (8.8) is first re-written in its state-space form

$$\frac{dy(t)}{dt} = A(t)y(t) + B(t)y(t - \tau),$$  \hfill (8.12)

where $A(t + T) = A(t)$ and $B(t + T) = B(t)$ are time-periodic with the period $T$ equal to the time delay $\tau$.

The state-space TFEA method [3, 55, 56] or the Chebyshev collocation method [9, 10] can then be used to transform Eq. (8.12) into a dynamic map of the form

$$y_{n+1} = U y_n,$$  \hfill (8.13)

where $U$ is a finite dimensional approximation of the infinite dimensional monodromy operator for time-periodic DDEs. Equation (8.13) represents a discrete solution form for Eq. (8.8) that maps the states of the system over one delay period $\tau$. The eigenvalues of $U$ determine the asymptotic stability of the DDE according to the condition shown in Fig. 1.1

8.2 Discussion

The results from the stability analysis for the milling process, described by Eq. (8.12), are given in this section. The cases studied are for multi-flute cutters with zero- and non-zero helix angle. Both upmilling and downmilling cases were investigated using state-space TFEA and Chebyshev collocation method. The two analysis techniques
produced identical stability charts; hence, we only show one set of converged stability charts for each case study. The interest here is in the unstable period-doubling islands that appear in the stability diagrams when using helical flutes.

Note that the zero-helix case can also produce period-doubling islands; however, these islands appear due to the parametric nature of the forces in milling and not due to the helix angle. The islands that appear in the zero-helix case are called parametrically induced islands whereas the islands associated with the helical tool are called helix-induced islands [1,8]. Figure 8.5 shows an example of a parametrically induced island for downmilling with a 3-flute cutter at 0.05 radial immersion. The stability of several points is also noted on the stability chart using the notation of Fig. 1.1. Dots denote stable regions whereas triangles denote an unstable period-doubling region. Since we are primarily interested in the island phenomena associated with helical mills, the zero-helix cases are only given for reference.

The parameters used to obtain the stability plots in this study were introduced in Refs. [1,2] and are shown in Tab. 8.1. The results for downmilling are shown in Figs. 8.6 and 8.7 whereas upmilling results are shown in Fig. 8.9. For the downmilling results, both a zero-helix mill (light line), and a helical mill with $\beta = 30^\circ$ (dark line) are reported in each diagram. The horizontal dotted line in each figure marks the helix pitch and the type of bifurcation is also marked on each lobe with “H” indicating a Neimark-Sacker or Hopf bifurcation, and “P” indicating period-doubling lobes. The authors acknowledge that changing the geometry of the tool would in general change the specific cutting coefficients $K_r$ and $K_t$. Therefore, a one-to-one stability comparison between a zero-helix tool and helical one would require accounting for these changes.

In Fig. 8.6, a radial immersion of 0.05 and $N = 4$ produced identical results to those obtained through a frequency domain stability analysis in Ref. [2]. This case corresponds to graphs (a) and (b) in Fig. 8.1, where incrementing the depth of cut to
Figure 8.5: A parametrically induced period-doubling island appears when down-milling at 0.05 radial immersion with a zero-helix 3-flute cutter. The stability of several points is also noted using a notation consistent with the one used in Fig. 1.1; circles were used to denote a stable region whereas triangles were used to denote unstable period-doubling regions.

higher values, i.e. higher than helix pitch $p$, results in multiple-flute engagement (see Fig. 8.1-(b)). It can be seen that the period-doubling boundary which corresponds to the zero-helix case transforms into closed, bounded regions of period-doubling that are often called islands. The depths of cut at the period-doubling free zones between any two islands correspond to the integer multiples of the helix pitch. This is because at these values of cutting depths the helical flutes average the time-periodic forces and the system becomes autonomous [1].

The period-doubling islands for the 4-flute cutter still appeared as the radial immersion was increased to 0.25 and 0.50 in the first column of Fig. 8.6; the latter is the borderline case for only one flute in the cut when using a zero-helix mill and 4 cutting flutes. The period-doubling lobes for the zero-helix case and the islands for the helical tool case were still apparent at these radial immersions. However, in contrast to prior works on a single flute in the cut for cutters with more than 2 flutes, a further increase in the radial immersion in the first column of Fig. 8.6 to 0.75 causes
multiple flute engagement for the zero-helix case and produces cases similar to those depicted in Fig. 8.1-(c) and (d) for the helical tool with 4 flutes. The appearance of period-doubling lobes at this relatively high radial immersion diverges from the stability behavior for a single engaged flute. The period-doubling regions disappear at full radial immersion, see the last row of Fig. 8.6, and the results for zero-helix and helical tools become identical under the assumption of constant specific cutting coefficients.

Figure 8.6: Downmilling stability diagrams of Eq. (8.8) for 4-, 6- and 8-flute cutters. The light line represents the zero-helix case while the thick line represents a helical tool with $\beta = 30^\circ$. The parameters used to generate the plots are shown in Tab. 8.1 and the radial immersions used were 0.05, 0.25, 0.50, 0.75, and 1.00. “H” indicates Hopf bifurcations, while “P” indicates period-doubling bifurcations.
The case for a 6-flute cutter demonstrated a different behavior as shown in Fig. 8.6. For example, in Fig. 8.6 for a 6-flute cutter, period-doubling regions still appeared for the zero-helix case at when only one flute was cutting, i.e. at RI = 0.05 and RI = 0.25, as well as the cases with higher radial immersions and a zero-helix mill in the subsequent rows (except at RI = 1.00). However, when a helical mill was used, the period-doubling lobes disappeared and only Neimark-Sacker lobes persisted for RI = 0.25 and 0.50. Nevertheless, period-doubling lobes reappeared as the radial immersion was increased to 0.75. This shows the sensitivity of the period-doubling islands to changes in the radial immersion and emphasizes the importance of predicting their reappearance even at higher radial immersions.

In contrast, for an 8-flute cutter; whereas period-doubling lobes appeared for the zero-helix case, only Neimark-Sacker lobes appeared for the helical tool cases. This is because as the number of teeth is increased, the spacing between the helical flutes gets smaller—decreasing both the pitch angle and the helix pitch. The forces are therefore averaged out around the tool helix over the full range of cutting speeds and depths of cut. The smoothing effect of the helix angle transforms a discontinuous milling process into a continuous one as the number of flutes is increased. Hence, when a sufficient number of cutting edges is used, only the main Hopf instability region appears.

One common feature of all the cases, shown in Fig. 8.6, is the absence of period-doubling regions at full radial immersion (RI = 1.00). At this value of radial immersion only Neimark-Sacker lobes appear and the stability diagram for zero-helix and helical flutes become almost identical for cutters with an even number of flutes. However, a different effect was revealed when cutters with an odd number of flutes were studied.

For example, Fig. 8.7 shows the downmilling stability charts for several combinations of radial immersions and various cutters with an odd number of flutes. It
Figure 8.7: Downmilling stability diagrams for cutters with $\beta = 30^\circ$ and various odd number of flutes at various radial immersion levels.
can be seen that at full radial immersion, all the studied cases showed an area of period-doubling for the zero-helix case. Further, all the cases except the 9-flute cutter resulted in period-doubling islands at full radial immersion. This influence of the parity in the number of flutes on stability at full radial immersion has important practical implications: not only the depth of cut, radial immersion, and the helix angle need to be optimized in milling, but also the parity of the number of flutes needs to be considered. An even number of flutes results in a constant number of engaged flutes throughout any cutting period. In contrast, an odd number of flutes leads to a situation where the number of engaged teeth is reduced by one at half the cutting period.

Figure 8.8 explains the change in the number of engaged flutes at full radial immersion for two cutters with opposite parities using a top view of a milling schematic. As an example of a cutter with even parity, Fig. 8.8-(a) shows a 4-flute cutter during the beginning, middle and end of one cutting period. It can be seen that 2 flutes remain engaged throughout the cutting period. The distribution of the even number of flutes ensures that a new flute enters the cut as soon as one of the flutes exits. Similar diagrams can be obtained for 6, 8 or any other even number of flutes. In contrast, Fig. 8.8-(b) depicts a 5-flute cutter at full radial immersion. It can be seen that at the beginning of the cut, 3 flutes are engaged simultaneously. However, in the middle of the period, i.e. \( \theta = \theta_p/2 \) or equivalently \( t = T/2 \), the leading flute exits the cut and only two flutes remain. The change in the number of cutting flutes leads to abrupt changes in the directional force coefficient \( K_c(t, b) \) in Eq. (8.9) causing period-doubling regions to appear [10]. However, as the number of odd cutting flutes is increased, e.g. to 9 in Fig. 8.7, the helical flutes smooth out these changes and only Hopf lobes appear in the stability chart.

The stability diagrams reported in Figs. 8.5, 8.6 and 8.7 were for a downmilling model; nevertheless, the stability diagrams of upmilling also demonstrated interesting
features, especially when varying the helix angle and the radial immersion. For example, in Fig. 8.9, a set of stability plots for upmilling with 4 flutes are given for different combinations of radial immersion and helix angle values. The radial immersion values are 0.10, 0.20, 0.75, and 0.80 and vary from left to right whereas the helix angle values of 1, 13, 23, 25, 45, 60, and 63 degrees vary from the bottom to the top. Therefore, each row corresponds to a constant helix angle while each column corresponds to a constant value of the radial immersion. The values of the helix angle and the radial immersion were chosen to capture the qualitative changes in the upmilling stability plots. Note that the used values of the depth of cut were exaggerated beyond practical limits in order to elucidate the changes in the stability properties of the non-zero helix model.

As the radial immersion and helix angle were varied in Fig. 8.9, several trends of the changes in the stability properties appeared. For example, when the radial immersion was changed from 0.10 to 0.20 and from 0.75 to 0.80, the change in the mutual orientation of islands and the secondary Hopf stability lobes was observed. This transition can be explained by the sign change of the cutting force coefficient averaged over one period as was shown in Refs. [9, 10]. This phenomenon was discovered for downmilling in [9] for a model with compliance in the $x$ direction (see Fig. 8.2). However, unlike the model in Ref. [9], this analysis used a $y$ compliance model which explains the occurrence of the same phenomenon in upmilling.

By producing more stability charts between $RI = 0.1$ and $RI = 0.2$, it can be shown that this transition happens gradually: the major Hopf lobe to the left of the “island chain” in the stability diagram for $(RI, \beta) = (0.1, 13)$ at first completely disappears and then reappears back on the other side of the “island chain” while changing its incline to the horizontal axis. Similar transitions happen for other pairs of stability charts, for instance, for $(RI, \beta) = (0.75, 23)$ and $(0.8, 23)$, except that the mutual orientation of Hopf lobes and period-doubling islands changes in the opposite
Further, it should be noted that the radial immersions at which the transitions in the stability characteristics occur correspond to the two minima of the lower boundary of the shaded region in Fig. 8.10-(a) at the radial immersions approximately equal to 0.15 and 0.8. As the value of the helix angle increases, the sizes and vertical positions of the islands change. The common tendency is that the islands shrink and eventually disappear as $\beta$ is increased, however, for lower radial immersions this disappearance happens faster, while for higher radial immersions it is slower.

**Figure 8.8:** Snap shots of two helical mills at full radial immersion with (a) 4 flutes and (b) 5 flutes. The snap shots illustrate the top view for an axial slice of the cutter during the beginning, middle, and end of one cutting period.

In addition, it is seen that when the islands are to the right of Hopf lobes, they shrink and disappear while moving downward (see the RI = 0.1 and RI = 0.8 columns in Fig. 8.9), while in the opposite case they move upward, which is seen in the RI = 0.2 and RI = 0.75 columns. Another observation is that the waviness of the Hopf lobe sections adjacent to the islands is present, especially for higher immersions, and the authors believe that this is a feature that has not been observed in the earlier works on the stability analysis of helical milling tools. The shape of the wavy section of a Hopf lobe changes along with the upward or downward motion of
islands as the helix angle is increased. The waviness becomes insignificant once the islands become sufficiently small.

The disappearance and reappearance of the islands in the stability diagrams of both upmilling and downmilling was investigated in Fig. 8.10 for a 4 flute cutter. Diagrams (a) and (b) show at which combinations of radial immersions and helix angles the islands occurred in upmilling and downmilling, respectively. When the combination of radial immersions and helix angles falls into a shaded region, at least one island is present in the corresponding stability chart while no islands exist for the parameter combinations in the unshaded region. The diagram in Fig. 8.10-(a) qualitatively describes the upmilling stability charts in Fig. 8.9 while Fig. 8.10-(b) is added for completeness. The diagrams in Fig. 8.10 were produced in the grid of radial immersions varying from 0.05 to 1 with the step of 0.05 and the helix angle values varying from 2 to 70 degrees with the step of 2 degrees, and this fact contributed to the coarseness of the plots.

8.3 Conclusions

This chapter investigated the stability of a milling process with simultaneously engaged flutes using the state-space TFEA and Chebyshev collocation methods. In contrast to prior works, multiple flute engagement due to both high depth of cut and high step-over distance was considered, see Fig. 8.1-(c) and (d).

A particular outcome of this study was the demonstration of different stability behavior in comparison to prior works. To elaborate, period-doubling regions were shown to appear at relatively high radial immersions when multiple flutes with either a zero or non-zero helix angle were simultaneously cutting. This was shown using a set of stability charts for downmilling that compared a zero-helix mill to a helical tool under the assumption of constant specific cutting coefficients. It was shown that the TFEA and the Chebyshev collocation results agreed with the frequency domain
stability results for the low radial immersion cases found in literature, see Fig. 8.1-(a) and (b). However, in contrast to prior works, period-doubling regions appeared at relatively high radial immersions when multiple flutes were simultaneously cutting.

These regions appeared as lobes for zero-helix tools, while closed islands characterized the period-doubling regions for helical tools. However, as the number of cutting edges was increased, e.g. to 6 and 8 in Fig. 8.6, the helical flutes smoothed out the force discontinuities and eliminated period-doubling bifurcations. Additionally, in agreement with prior works, period-doubling bifurcations were shown to cease to exist at depths of cut equal to integer multiples of the mill helix pitch.

We also demonstrated stability differences that occur due to the parity in the number of flutes as shown in Figs. 8.8, 8.6 and 8.7. In particular, we showed that whereas islands disappeared at full radial immersion for helical cutters with an even parity, cutters with odd parity produced islands even at full radial immersion as shown in the last row of Fig. 8.7 for cutters with 3, 5 and 7 flutes. This effect in cutters with odd parity is a result of the abrupt changes in the directional force coefficient as the leading flute starts exiting the cut midway through the period, see Fig. 8.8. In contrast, at full radial immersion, cutters with even parity maintained a constant number of flutes in the cut throughout the period and only produced Hopf lobes as shown in the last row of Fig. 8.6.

The study of the stability in upmilling revealed more interesting features for helical mills. For instance, Fig. 8.9 showed that certain combinations of radial immersion and helix angle gave rise to Hopf lobes with pronounced waviness along the depth of cut direction. This contrasts the Hopf lobes usually reported in milling literature and it reflects the strong effect of the helical flutes on the shown cases. Figure 8.9 also showed the change in the mutual orientation of the islands with respect to the secondary Hopf lobe when varying the helix angle and the radial immersion. This transition was found to occur gradually with increasing the radial immersion where
the islands would first shrink slowly, then disappear before they finally reappear on the other side of the Hopf lobe. This transition of the lobes is attributed to the sign change of the cutting force coefficient averaged over one period. Further, we found that the radial immersions at which the transitions in the stability characteristics occurred corresponded to the two minima of the lower boundary of the shaded region in Fig. 8.10-(a). Although a similar transition in the stability lobes was demonstrated for zero-helix tools in Refs. [9, 10], this study is the first to confirm that a similar transition occurs for helical tools as well.

Another observation from Figs. 8.6, 8.7 and 8.9 is the sensitivity of the period-doubling islands to the radial immersion. This sensitivity to changes in the radial immersion was described qualitatively in Fig. 8.10 for both upmilling and downmilling. It was shown that even small changes of the radial immersion can cause the islands to disappear (or reappear) in the stability diagrams. Figure 8.10 also re-iterates the prior conclusion that no period-doubling regions appear for cutters with an even number of flutes.

Finally, although only Hopf and period-doubling bifurcations can occur for tools with constant helix angle and constant pitch, a recent study has found cyclic-fold bifurcations at low radial immersions for both variable pitch and variable helix tools [4]. However, expanding the approach of this chapter to the cases of variable pitch and variable helix tools remains a topic for future research.

Table 8.1: The parameters used to generate the stability plots in this study.

<table>
<thead>
<tr>
<th>$m$ (Kg)</th>
<th>$\omega_0$ (Hz)</th>
<th>$\zeta$</th>
<th>$K_t$ (N/mm²)</th>
<th>$K_r$ (N/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.364</td>
<td>319.375</td>
<td>0.0196</td>
<td>804.3</td>
<td>331</td>
</tr>
</tbody>
</table>
Figure 8.9: Upmilling stability charts for the 4 cutting flutes and for the radial immersion values of 0.10, 0.20, 0.75 and 0.80 (columns) and the helix angle values of 1, 13, 23, 25, 45, 60, and 63 degrees (rows).
Figure 8.10: Diagrams showing the combinations of radial immersion and helix angle where islands occur (shaded region). Results are for 4 cutting flutes and (a) upmilling and (b) downmilling.
This research aimed at advancing modern techniques for studying DDEs and applying them to various dynamical systems that include delays. We developed a systematic methodology to analyze delay equations which arise in many applications. More specifically, although DDEs are frequently encountered in the analysis of physical systems, there is a lack of a comprehensive analysis methodology. This is due to the difficulties associated with studying the infinite spectrum of DDEs.

To motivate the interest in DDEs, Chapter 1 used examples from current literature of physical systems admitting delays. Chapter 1 also provided an introduction to time delay systems and highlighted the added complexity of DDEs compared to ODEs. The stability criteria for DDEs as well as the contributions of this thesis are also described in that chapter.

Chapter 2 described the analysis techniques that were used in this thesis, namely TFEA and Chebyshev collocation. These two state-of-the art techniques have been successfully used to study various time delay systems. Nevertheless, this work extended these techniques to broader types of DDEs such as DDEs with discontinuous coefficients (Section 2.7), with arbitrary delay and period (Chapter 3), with integro-
differential terms (Chapter 4), with multiple time delays (Chapter 5), and nonlinear DDEs (Chapter 6).

Chapter 3 described the different types of meshes that result in when discretizing DDEs with periodic coefficient. The period and the delay were allowed to vary independently—introducing and additional time scale into the DDE—and the associating matrix structures were described. Several cases corresponding to different ratios between the period and delay were discussed to show the complexity associated with the competing time scales.

Chapter 4 described a spectral element approach for studying the stability of delay integro-differential equations (DIDEs). In comparison to DDEs with a discrete delay, the analysis of DIDEs is more complicated due to the integral term which maps single points onto intervals. The only restriction assumed on the bounded kernel function was smoothness. This chapter also described the stability analysis of DIDEs with special distribution functions (gamma-type distribution functions) via converting the DIDE into a higher order DDE with only discrete delays. This case of DIDEs is of practical importance, e.g., in modeling wheel shimmy phenomenon.

Chapter 5 studied DDEs with multiple scalar delays using the spectral element method. This problem was shown to be closely related to the analysis of DIDEs. Nevertheless, due to their importance in many physical systems, a separate chapter was dedicated to DDEs with multiple delays. The existence of multiple delays was shown to cause some difficulties especially for incommensurate delays. This is due to the resulting mapping which in general maps some mesh points onto interim regions rather than mapping them exactly onto other mesh points. This necessitated using interpolation to create the dynamic map and determine the system stability.

Chapter 6 described an approach for calculating the periodic orbits of nonlinear DDEs and their stability. Using the spectral element method, a new approach to numerically approximating periodic orbits of nonlinear DDEs was developed. Finding
the periodic solutions of a DDE was formulated in terms of a boundary value problem which was then discretized using the spectral element method. The discretized equations were then solved using Newton’s method to find the periodic solutions. Further, the linearized stability of the nonlinear DDEs was ascertained using the Jacobian and Floquet theory. Two examples were used to demonstrate the method: Mackey-Glass equation and a delayed Van der Pol equation. The calculated periodic solutions matched the ones obtained from simulation confirming the efficiency of the developed technique.

The analytical techniques developed here were used to study physical systems that included delays. To elaborate more, in Chapter 7 a new distributed force model was proposed to describe forces in turning. The distributed force model led to an autonomous integro-DDE which complicated the stability analysis considerably. However, a method to convert the integro-DDE to a higher order DDE with only discrete delays was described and the resulting stability boundaries were charted. It was found that, in agreement with practical observations, the proposed model captured the increased stability behavior at low speeds.

As another application, Chapter 8 studied the stability of a milling process with simultaneously engaged flutes. This study revealed that in contrast to common assumptions, period doubling instabilities do appear even at high radial immersions. This means that we found unstable areas in a range where other stability studies predicted a stable operation. A practical outcome of this chapter is to alert machinists to the existence of these dangerous unstable regions.
Appendix A

Interrupted Turning-Free Vibration

In order to establish the stability of an interrupted turning process, it is necessary to obtain the state transition matrix describing the system states during free vibration. The third order system derived in the analysis of the distributed force model in Eq. (7.26) will be considered first since the transition matrix for the second order system can be obtained easily from the third order case. For a non-dimensionalized third order system, the equation of motion during free vibration is

\[
(3) \quad \xi(t) + 2\zeta \ddot{\xi}(t) + \dot{\xi}(t) = 0,
\]

which has the solution \( \xi(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 \), where \( \lambda_{1,2} = \zeta \pm \sqrt{\zeta^2 - 1} \). The solutions for the different states of the system can be expressed in matrix form as

\[
\begin{bmatrix}
\xi(t) \\
\dot{\xi}(t) \\
\ddot{\xi}(t)
\end{bmatrix} =
\begin{bmatrix}
e^{\lambda_1 t} & e^{\lambda_2 t} & 1 \\
\lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & 0 \\
\lambda_1^2 e^{\lambda_1 t} & \lambda_2^2 e^{\lambda_2 t} & 0
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix},
\]

or equivalently,

\[
\xi(t) = L c.
\]
Setting $t = t_c$ at the beginning of free vibration, and solving simultaneously for the set of initial conditions forming a $3 \times 3$ identity matrix, Eq. (A.2) reads

$$
I = L C.
$$

(A.4)

where the square matrix of coefficients, $C$, can be found by inverting $L$:

$$
C = L^{-1} = \begin{bmatrix}
0 & -\frac{\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)}e^{-\lambda_1 t_c} & \frac{1}{\lambda_1(\lambda_1 - \lambda_2)}e^{-\lambda_1 t_c} \\
0 & -\frac{\lambda_1}{\lambda_2(\lambda_1 - \lambda_2)}e^{-\lambda_2 t_c} & -\frac{1}{\lambda_2(\lambda_1 - \lambda_2)}e^{-\lambda_2 t_c} \\
1 & -\frac{(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} & \frac{1}{\lambda_1 \lambda_2}
\end{bmatrix}.
$$

(A.5)

Let $t_f$ be the duration of free vibration, then a state transition matrix can be obtained that relates the state of the tool as it exits from the cut, at $t = t_c$, to the state of the tool as re-enters into the cut, at $t = t_c + t_f$:

$$
\begin{bmatrix}
\xi(t_c + t_f) \\
\dot{\xi}(t_c + t_f) \\
\ddot{\xi}(t_c + t_f)
\end{bmatrix} = \begin{bmatrix}
e^{\lambda_1(t_c+t_f)} & e^{\lambda_2(t_c+t_f)} & 1 \\
\lambda_1 e^{\lambda_1(t_c+t_f)} & \lambda_2 e^{\lambda_2(t_c+t_f)} & 0 \\
\lambda_1^2 e^{\lambda_1(t_c+t_f)} & \lambda_2^2 e^{\lambda_2(t_c+t_f)} & 0
\end{bmatrix} \times
\begin{bmatrix}
0 & 0 & \frac{1}{\lambda_1(\lambda_1 - \lambda_2)}e^{-\lambda_1 t_c} \\
0 & 0 & -\frac{1}{\lambda_2(\lambda_1 - \lambda_2)}e^{-\lambda_2 t_c} \\
1 & 1 & \frac{1}{\lambda_1 \lambda_2}
\end{bmatrix}
\begin{bmatrix}
\xi(t_c) \\
\dot{\xi}(t_c) \\
\ddot{\xi}(t_c)
\end{bmatrix}.
$$

(A.6)

\[\Rightarrow \begin{bmatrix}
\xi(t_c + t_f) \\
\dot{\xi}(t_c + t_f) \\
\ddot{\xi}(t_c + t_f)
\end{bmatrix} = \Phi \begin{bmatrix}
\xi(t_c) \\
\dot{\xi}(t_c) \\
\ddot{\xi}(t_c)
\end{bmatrix},\]

where the state transition matrix $\Phi$ is

$$
\Phi = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
(\lambda_1 - \lambda_2) & -\frac{\lambda_2}{\lambda_1}e^{\lambda_1 t_f} + \frac{\lambda_1}{\lambda_2}e^{\lambda_2 t_f} - \frac{(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2}e^{\lambda_1 t_f} - \frac{1}{\lambda_2}e^{\lambda_2 t_f} + \frac{(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2} \\
0 & \lambda_1 e^{\lambda_2 t_f} - \lambda_2 e^{\lambda_1 t_f} \\
0 & \lambda_1 \lambda_2 e^{\lambda_2 t_f} - \lambda_1 \lambda_2 e^{\lambda_1 t_f}
\end{bmatrix}.
$$

(A.7)
The $2 \times 2$ state transition matrix for the second order system can be obtained by eliminating the first row and the first column in Eq. (A.7) to obtain

$$
\Phi = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
\lambda_1 e^{\lambda_2 t_f} - \lambda_2 e^{\lambda_1 t_f} & e^{\lambda_1 t_f} - e^{\lambda_2 t_f} \\
\lambda_1 \lambda_2 e^{\lambda_2 t_f} - \lambda_1 e^{\lambda_1 t_f} & \lambda_1 e^{\lambda_1 t_f} - \lambda_2 e^{\lambda_2 t_f}
\end{bmatrix}.
$$

(A.8)

which agrees with the result obtained in [53].
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Biography

Firas Khasawneh was born on December 1, 1981 in Amman, Jordan. He obtained his high school diploma in 1999 from the Yarmouk University Model School in Irbid, Jordan. He then received his Bachelor degree in Mechanical Engineering from Jordan University for Science and Technology in 2004. Seeking higher education, Firas traveled to the United States in 2004 where he obtained his Masters degree from the University of Missouri-Columbia in 2006. The title of his Masters thesis was “Characterization of Drillability of Sandwich Structure of Carbon Fiber Reinforced Epoxy Composite Over Titanium” and it was supervised by professor A. Sherif El-Gizawy.

In 2007, Firas joined the PhD program in Mechanical Engineering at Duke under the supervision of Brian Mann. His research interests include nonlinear behavior, time delay systems, and stochastic effects on dynamical systems.