Optimal Power Generation of a Wave Energy Converter in a Stochastic Environment

by

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Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Civil and Environmental Engineering in the Graduate School of Duke University 2011
ABSTRACT
(Wave Energy Conversion)

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Abstract

In applications of ocean wave energy conversion, it is well known that feedback control can be used to achieve favorable performance. Current techniques include methods such as tuning a device to harvest energy at a narrow band of frequencies, which leads to suboptimal performance, or methods that are anticausal and require the future wave excitation to be known. This thesis demonstrates how to determine the maximum-attainable power generation and corresponding controller for a buoy-type wave energy converter with multiple generators in a stochastic sea environment using a causal dynamic controller. This is accomplished by solving a nonstandard $H_2$ optimal control problem. The performance of the causal controller is compared to the noncausal controller for various cases. This work provides a significant improvement over current control techniques because it involves a causal controller that can absorb a large amount of power over a broader bandwidth than other control techniques, including absorbing power across multiple modes of resonance. The importance of an adaptive control algorithm is also demonstrated.
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List of Abbreviations and Symbols

Symbols

Bolded symbols typically denote vectors or matrices. The few symbols that may be duplicated are put in context where they are used.

\[ a \] Wave amplitude.
\[ C_c \] Added damping matrix.
\[ F_a \] Hydrodynamic force and moment vector.
\[ G_a \] Transfer function relating wave height (input) to voltage (output).
\[ G_i \] Transfer function relating control current (input) to voltage (output).
\[ i \] Control current (input) vector.
\[ M_c \] Added mass matrix.
\[ \bar{P}_{gen} \] Average power generation.
\[ R \] Square matrix with stator resistance values along the diagonal.
\[ S_a \] Wave height power spectrum.
\[ S_p \] Harvester power spectrum.
\[ v \] Voltage (output) vector.

Abbreviations

Below is a non-exhaustive list of abbreviations in this thesis.

JONSWAP Joint North Sea Wave Project (refers to ocean wave spectra).
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LQG</td>
<td>Linear-quadratic-Gaussian.</td>
</tr>
<tr>
<td>RELS</td>
<td>Recursive extended least-squares.</td>
</tr>
<tr>
<td>STR</td>
<td>Self-tuning regulator.</td>
</tr>
<tr>
<td>WEC</td>
<td>Wave energy converter.</td>
</tr>
</tbody>
</table>
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A substantial amount of energy exists in ocean waves. This energy travels over great distances with relatively small energy losses and is viewed as a potential source of renewable energy. The power available in ocean waves varies globally and seasonally. Good locations to harvest energy can provide an average power flux, in terms of power per meter of wave front, between 20 and 70 kW/m. These locations are usually in middle to high latitudes and exist primarily on west facing coasts (Barstow et al. (2008)). It is estimated that each year 2,100 TW-h of wave energy reaches the area immediately off of the United States coast in locations where the average power flux exceeds 10 kW/m, which is considered to be the minimal power flux necessary for wave energy conversion to be worthwhile. Of the total wave energy that reaches U.S. coastline, it is technologically feasible to harvest approximately 260 TWh which is about 6% of national energy demand (Bedard et al. (2007)).

Wave power was thrust into the engineering community’s lexicon when much of the world began looking at energy alternatives in response to the oil crisis of 1973. It became a popular topic after the concept of converting mechanical energy from ocean waves into electricity for utility-scale use was promoted in 1974 by Salter.
This paper proposes a new device that is now commonly referred to as the “Salter duck” and defines the problem of ocean wave energy conversion by stating, “The essential problem is finding a method to convert dispersed, random, alternating forces into concentrated, direct force, using a mechanism which is efficient at low levels and yet robust enough to withstand the worst conditions.”

It can be said that Salter is responsible for creating interest in wave energy conversion, but modern research in the area has its origins with Yoshio Masuda, a former Japanese naval commander. Masuda’s studies date back to the 1940s in Japan where he initiated the scientific pursuit of harvesting energy from ocean waves when he researched and developed ways to power small-scale navigation buoys (Falcão (2010)).

Hydrodynamic theory, the crucial foundation of wave energy conversion, was extensively developed throughout the 20th century due to its relevance in determining the dynamic motions of ships and forces acting on other structures at sea such as oil rig platforms. Techniques used to analyze motions of rigid bodies in the presence of ocean waves include both analytical and numerical methods, the former being used for simpler geometries and the latter being used for more complex ones. An example of an analytical technique is found in Hulme (1982) where the profile of the submerged body is a hemisphere and exists in water of infinite depth. Motion in both the heave and surge degrees of freedom are considered. Numerical methods typically involve dividing the body’s surface into individual panels. An example can be found in Newman (2002). Panel methods are commonly used in the aerospace industry. A semi-analytical method is provided in Kokkinowrachos et al. (1986) that can be applied to arbitrary shaped bodies of revolution with vertical axis. This is accomplished by dividing the body into discrete ring-shaped macroelements of which
each element has an analytical solution.

Since the 1970s, much work has been done to determine the amount of power that exists in ocean waves and many types of wave energy converters (WECs) have been conceived and developed to some extent including oscillating water columns, surface floating attenuators, and floating buoy-type point absorbers. Some of these WECs are now in use, primarily in western Europe (Clément et al. (2002)). But WEC technology has many challenges that have slowed progress towards a day where large-scale wave farms are a common source of sustainable electrical energy. Mechanically speaking, other than developing designs that can efficiently convert mechanical wave energy into electrical energy, there are survivability issues that arise. Survivability is even more important for deep water devices where there is higher energy content in the waves. Despite the demand for environmentally friendly and sustainable sources of energy, economic and political challenges are also present and difficult to overcome. These include numerous state and federal regulatory hurdles (especially in the United States where regulations are outdated and aimed towards conventional hydroelectric plants) and limited funding for research and development operations (Bedard et al. (2007)).

Evans (1981) and McCormick (1981) provide extensive overviews that reflect the state of the art as of the early 1980s when research of WEC technology had somewhat of a boom period. Relatively recent works that provide a good survey of where WEC technology stands today include Falcão (2010), Cruz (2008), Falnes (2002), Salter et al. (2002), Falcão (2004), and Falnes (2007).

1.1 Current technologies

WECs typically fall into one of three categories: oscillating bodies, oscillating water columns (OWCs), and overtopping devices. Oscillating bodies are floating, articulated structures with transducers, also called power take-off (PTO) systems, installed
between degrees of freedom. They absorb wave energy due to dynamic fluid-structure interaction. OWCs and overtopping devices involve structural installations which are static. OWCs absorb energy by coupling the resonant oscillation of water elevation in a submerged air chamber with a gas turbine. Overtopping devices operate on the same principles as conventional hydropower plants, in which the gravitational potential energy of water propagating over a device is made to drive a turbine.

1.1.1 Oscillating bodies

Oscillating bodies themselves can be divided into subcategories: point absorbers, attenuators, and terminators. A point absorber is a device that can absorb wave energy from all directions and its orientation with respect to the direction of wave propagation is irrelevant. An attenuator is an elongated device that is aligned with the direction of wave propagation and typically bends along its length. Contrary to attenuators, terminators are oriented perpendicular to the direction of wave propagation. Fig. 1.1, which is adapted from Thomas (2008), shows the various orientations of oscillating body WECs with respect to the incident waves.

![Figure 1.1: Orientations of oscillating body WECs.](image)

An example of a point absorber is the PowerBuoy developed by Ocean Power Technologies. PowerBuoys are modular and can be deployed in large multi-device wave farms. As waves pass by, they excite the disc-shaped floater portion of the buoy vertically with respect to the relatively fixed portion of the buoy that has a large
damping plate and is also moored to the sea floor by a conventional mooring system. The relative motion of the floater portion moves hydraulic fluid in the buoy that then spins a generator. The PowerBuoy has undergone sea trials and is currently ready for market (Unattributed (2011)). Other examples of point absorber oscillating bodies are the Archimedes Wave Swing (AWS), IPS Buoy, and Wavebob.

An example of an attenuator is the Pelamis WEC developed by Pelamis Wave Power. Like the PowerBuoy, Pelamis WECs can also be deployed in wave farms and involves a power hydraulic PTO. The Pelamis WEC is a long articulated structure that bends at its hinges as waves pass by and this creates pressurized oil that drives the power hydraulic PTO (Yemm (2008)). The Pelamis has also undergone sea trials and is deployed off of Scotland and Portugal.

An example of a terminator is the aforementioned Salter Duck. One manifestation of the Salter Duck technology involves a series of floating bodies known as ducks arranged in a spine and allowed to rotate along the spine independently in pitch. The dynamics at the joints in between each duck allows for the electronic control of stiffness, damping, and yielding bending moment with measurement of bending moment and joint angle (Salter (2008)). This device also has a power hydraulic PTO. Extensive tests were done for the device, but due to the politics at the time, robustness and practicality issues, and controversy over power production calculations, a full scale device has never gone to sea.

1.1.2 Oscillating water columns

Oscillating water columns (OWCs) can either be fixed structures attached to land or the sea floor in some fashion or they can be floating structures that are moored and are relatively static. An OWC consists of a chamber that is filled with air and exposed to the surface of the water. As a wave passes, it increases the water height inside the chamber and compresses the air which in turn is forced out through
a turbine. As the water level goes back down, the air is drawn back in through the same turbine. In order to be efficient, the turbine needs to spin in a constant direction regardless of the direction of airflow. The common solution to this problem is to use a bidirectional turbine that has a symmetrical airfoil known as the Wells turbine. The PTO mechanism in the case of OWCs is a rotational generator driven by a turbine. Examples of OWCs that have had at least prototypes built and tested include the LIMPET, Mighty Whale, Mutriku, Oceanlinx, Pico, and Sperboy.

1.1.3 Control algorithms

Large improvements in WEC buoy power generation are possible when control algorithms are introduced. In regular seas with harmonic waves, it is possible to use simple tuning techniques where the buoy dynamics are “tuned” so that the buoy resonates at the same frequency as the waves. However, ocean waves are not purely harmonic, but rather are stochastic in nature and have a frequency content spread across a wide band of frequencies. Thus, it is more realistic to characterize ocean waves by a power spectrum such as the Joint North Sea Wave Project (JONSWAP) power spectrum (Faltinsen (1990)). Tuning methods are suboptimal because they neglect the power that exists at other frequencies. However, tuning methods are still advantageous in irregular seas as they can be easily tuned to the predominant frequency of the ocean waves at any given time.

Latching is an early technique that is intended to work in stochastic, or irregular seas (Budal and Falnes (1980)). Latching involves locking the oscillating device at particular points in its motion and holding it there temporarily in order to roughly mimic a regular sinusoidal motion. A comparison of different latching control strategies is given in Barbarit et al. (2004).

Active, or dynamic control, involves using feedback to optimally extract, or even inject, power to and from the system to allow for power to be absorbed over a
wider range of frequencies and maximize average power generation. Active control can produce a 1.5–2.8 fold increase in annual energy production (Eidsmoen (1996)). Impedance matching, which is a well known control technique, produces the best possible power production, but it relies on future information and is therefore an anticausal controller requiring either the use of deployed sensors or the use of prediction. Impedance matching is treated in Miller et al. (1990) and MacMartin et al. (1991).

![Conceptual diagram of active control.](image)

Many different types of control algorithms are examined for the AWS WEC in Valério et al. (2008). The AWS is a fully submerged WEC that is an oscillating body type WEC that has been built and tested. Simulation results are given for proportional integral derivative (PID) control, reactive control, phase and amplitude control, latching control, feedback linearization control, internal model control, and switching control.

1.2 Overview of thesis work

This thesis concerns an oscillating body WEC buoy with three tethers that extend downwards to spools that are attached to permanent magnet synchronous generators at the sea floor as shown in Fig. 2.1. This is similar to the buoy proposed by Srokosz (Srokosz (1979)). It is possible to control the force in each tether through controlling the current sent to the generators which have the ability to absorb or introduce
energy to the system to maximize energy production by affecting the dynamics of
the system as a whole.

The main contributions of this thesis are

• to present a general dynamic model for a specific WEC shown in Fig. 2.1;

• to demonstrate how to approximate the infinite-dimensional fluid-structure in-
teraction effects by a finite-dimensional state space;

• to demonstrate how to determine the maximum-attainable power generation
and corresponding causal dynamic controller for a buoy-type WEC device with
multiple generators in an irregular sea through $\mathcal{H}_2$ control theory.

In short, this work provides a significant improvement over current control tech-
niques for WECs because it involves a controller that can absorb a large amount of
power over a broader bandwidth than other control techniques, including absorbing
power from multiple modes of resonance. The importance of an adaptive control
algorithm that can adapt to changes in the sea state is demonstrated in this thesis.
The work provided in this thesis is applicable to other WECs since most of the work
is general enough to account for any type of WEC model that may be derived.
Model Definition

The first step towards finding an optimal controller for a WEC buoy involves modeling the WEC’s dynamics along with the wave excitation. The first section of this chapter shows the development of the dynamic model for the WEC. This involves the relationship between the wave excitation and the generator voltages and the generator control currents and generator voltages. The following sections describe the hydrodynamic force model that is used in the development of the dynamic model from the first section and how the ocean wave spectra are described. The final section provides several buoy designs to be referenced throughout the thesis.

2.1 Dynamic model development

Consider the free body diagram for an arbitrary rigid body, shown in Fig. 2.1. Note that the superscripts that appear in the figure are suppressed in the development concerning a single tether. The location of the center of mass of the rigid body, relative to the origin, $O$, of inertial reference frame, is the vector $r$. A retractable tether is mounted from an anchor to a fixed point on the body. The location of the anchor, relative to origin $O$, is $a$. The location of the attachment point, relative to
the anchor is $s$. The attachment location on the body, relative to the center of mass, is $b$. $t > 0$ is the tension in the tether. The tether force vector is defined as

$$f = -t \hat{e}_s,$$

where $\hat{e}_s$ denotes the unit vector in the same direction as $s$.

Consider the dynamics of the rigid body for small oscillations about equilibrium. The vector of angular displacements of the body $\theta = [\theta_x \, \theta_y \, \theta_z]^T$ is taken to be relative to the axes as shown, and these angles are assumed to be zero in equilibrium. The “small angle” approximation is also assumed, which allows for the order of angular rotations to be independent, and furthermore that

$$\delta b \approx \theta \times b_0 = B_0 \theta,$$

where

$$B_0 = \begin{bmatrix}
0 & b_{0z} & -b_{0y} \\
-b_{0z} & 0 & b_{0x} \\
b_{0y} & -b_{0x} & 0
\end{bmatrix},$$

and the subscript “0” signifies equilibrium.
Thus, the change in $s$ is

$$\delta s = \delta r + B_0 \theta.$$  \hfill (2.4)

The linearized change in the tether length is

$$\delta \|s\| \approx \hat{e}_{s_0}^T \delta s,$$  \hfill (2.5)

where $\| \cdot \|$ is the Euclidean norm operator. The change in tether force is

$$\delta f = \delta \left( -\frac{t}{\|s\|} s \right)$$  \hfill (2.6)

$$\approx -\hat{e}_{s_0} \delta t - \frac{t_0}{\|s_0\|} \left[ I - \hat{e}_{s_0} \hat{e}_{s_0}^T \right] \left( \delta r + B_0 \theta \right).$$  \hfill (2.7)

It is presumed that the tension varies linearly with the displacement, velocity, and acceleration of $\|s\|$, and also depends on a supplemental tension control force $u$. It is also assumed that there exist constants $\{k, c, m\}$ such that

$$\delta t = u + k \delta \|s\| + c \frac{d}{dt} \delta \|s\| + m \frac{d^2}{dt^2} \delta \|s\|$$  \hfill (2.8)

$$= u + \hat{e}_{s_0}^T (k \delta r + c \dot{r} + m \ddot{r}) + \hat{e}_{s_0}^T B_0 \left( k \theta + c \dot{\theta} + m \ddot{\theta} \right).$$  \hfill (2.9)

The change in the moment $h$ imparted on the rigid body, taken about the center of mass, is

$$\delta h = \delta (b \times f) \approx b_0 \times \delta f - f_0 \times \delta b.$$  \hfill (2.11)

Defining $S_0$ similarly to $B_0$ and substituting in the expression for $\delta f$ results in

$$\delta h = \frac{t_0}{\|s_0\|} B_0 \left[ I - \hat{e}_{s_0} \hat{e}_{s_0}^T \right] \left( \delta r + B_0 \theta \right) + B_0 \hat{e}_{s_0} \hat{e}_{s_0}^T \left( k \delta r + c \dot{r} + m \ddot{r} \right)$$  \hfill (2.12)

$$+ B_0 \hat{e}_{s_0} \hat{e}_{s_0}^T B_0 \left( k \theta + c \dot{\theta} + m \ddot{\theta} \right) + B_0 \hat{e}_{s_0} u - \frac{t_0}{\|s_0\|} S_0 B_0 \theta.$$
\( \delta f \) and \( \delta h \) can be conveniently represented as
\[
\begin{bmatrix}
\delta f \\
\delta h
\end{bmatrix} = G_t u - K_t \begin{bmatrix}
\delta r \\
\delta \theta
\end{bmatrix} - C_t \begin{bmatrix}
\delta \dot{r} \\
\delta \dot{\theta}
\end{bmatrix} - M_t \begin{bmatrix}
\delta \ddot{r} \\
\delta \ddot{\theta}
\end{bmatrix},
\] (2.13)

where, using the fact that \( B_0 = -B_0^T \),
\[
K_t = \begin{bmatrix}
I \\
B_0^T
\end{bmatrix} \left( k \hat{e}_{s0} \hat{e}_{s0}^T + \gamma_0 \left[ I - \hat{e}_{s0} \hat{e}_{s0}^T \right] \right) \begin{bmatrix}
I & B_0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & \gamma_0 S_0 B_0
\end{bmatrix},
\] (2.14)
\[
C_t = c \begin{bmatrix}
I \\
B_0^T
\end{bmatrix} \hat{e}_{s0} \hat{e}_{s0}^T \begin{bmatrix}
I & B_0
\end{bmatrix},
\] (2.15)
\[
M_t = m \begin{bmatrix}
I \\
B_0^T
\end{bmatrix} \hat{e}_{s0} \hat{e}_{s0}^T \begin{bmatrix}
I & B_0
\end{bmatrix},
\] (2.16)
\[
G_t = - \begin{bmatrix}
I \\
B_0^T
\end{bmatrix} \hat{e}_{s0},
\] (2.17)

and where \( \gamma_0 = t_0 / \| s_0 \| \).

Now, consider a buoy with \( N \) tethers. For an arbitrary amount of tethers, the total dynamic component of the force and moment on the buoy is
\[
\begin{bmatrix}
\delta f \\
\delta h
\end{bmatrix} = \begin{bmatrix}
\delta f_w \\
\delta h_w
\end{bmatrix} + \begin{bmatrix}
\delta f^1 + \ldots + \delta f^N \\
\delta h^1 + \ldots + \delta h^N
\end{bmatrix}
\] (2.18)
\[
= \begin{bmatrix}
\delta f_w \\
\delta h_w
\end{bmatrix} - \tilde{K}_t \begin{bmatrix}
\delta r \\
\delta \theta
\end{bmatrix} - \tilde{C}_t \begin{bmatrix}
\delta \dot{r} \\
\delta \dot{\theta}
\end{bmatrix} - \tilde{M}_t \begin{bmatrix}
\delta \ddot{r} \\
\delta \ddot{\theta}
\end{bmatrix} + \begin{bmatrix}
G^1_t & \ldots & G^N_t
\end{bmatrix} u,
\] (2.19)

where
\[
u = [u^1 \ldots u^N]^T,
\] (2.20)

and
\[
\tilde{K}_t = \sum_{i=1}^{N} K^i_t,
\] (2.21)
and $\tilde{C}_t$ and $\tilde{M}_t$ are defined similar to $\tilde{K}_t$. Interestingly, we note that for any vector $\mathbf{q}$

$$
\sum_{i=1}^{N} \gamma_0 S_0^i B_0^i \mathbf{q} = \sum_{i=1}^{N} -(\mathbf{q} \times \mathbf{b}_0^i) \times \mathbf{f}_0^i 
$$

(2.22)

$$
= \sum_{i=1}^{N} -\mathbf{q} \times (\mathbf{b}_0^i \times \mathbf{f}_0^i) 
$$

(2.23)

$$
= -\mathbf{q} \times \left( \sum_{i=1}^{N} \mathbf{b}_0^i \times \mathbf{f}_0^i \right). 
$$

(2.24)

In static equilibrium, the term in the parentheses is the sum of moments acting on the buoy by the tethers. Assuming no other moments act on the buoy static equilibrium, it can be concluded that the above term will be zero for all $\mathbf{q}$, implying that

$$
\sum_{i=1}^{N} \gamma_0 S_0^i B_0^i = 0. 
$$

(2.25)

As such, the $\gamma_0^i S_0^i B_0^i$ terms in each $K_t^i$ cancel out.

Regarding the wave forces and moments, they are presumed to be of the form

$$
\begin{bmatrix}
\delta f_w \\
\delta h_w
\end{bmatrix} = \begin{bmatrix}
\delta f_a \\
\delta h_a
\end{bmatrix} + \begin{bmatrix}
\delta f_b \\
\delta h_b
\end{bmatrix} + \begin{bmatrix}
\delta f_c \\
\delta h_c
\end{bmatrix}, 
$$

(2.26)

where $\{\delta f_a, \delta h_a\}$ are the force and moment imparted on the buoy by the wave, $\{\delta f_b, \delta h_b\}$ are the force and moment due to buoyancy, $\{\delta f_c, \delta h_c\}$ are the hydrodynamic (i.e. added mass and damping) forces.

Let the Fourier transforms of $\{\delta f_a, \delta h_a\}$ be denoted $\mathcal{F}(\delta f_a)$ and $\mathcal{F}(\delta h_a)$. It is presumed that there is a $6 \times 1$ transfer function matrix $F_a(j\omega)$ relating the wave amplitude $a$ to these forces and moments, i.e.

$$
\begin{bmatrix}
\mathcal{F}(\delta f_a) \\
\mathcal{F}(\delta h_a)
\end{bmatrix} = F_a(j\omega) F(a). 
$$

(2.27)
Determining $F_a(j\omega)$ involves solving a frequency dependent partial differential equation for a fluid-structure interaction problem which does not produce a rational transfer function. The resulting transfer function is infinite-dimensional, but it is possible to find series solutions for different shapes and levels of accuracy (Kokkinowrachos et al. (1986)). This is discussed in the following section.

The change in hydrostatic buoyancy force $\delta f_b$ and moment $\delta h_b$ (relative to equilibrium) are always linearly related to the displacements $\delta r$ and $\delta \theta$ via a buoyancy stiffness matrix $K_b$, i.e.

$$\begin{bmatrix} \delta f_b \\ \delta h_b \end{bmatrix} = -K_b \begin{bmatrix} \delta r \\ \delta \theta \end{bmatrix}. \quad (2.28)$$

The particular components of $K_b$ will vary with the buoy shape. $K_b$ can be determined by linearizing the stiffness about the center of mass of the static buoy at equilibrium.

Let the Fourier transforms of $\{\delta f_c, \delta h_c\}$ be denoted $F(\delta f_c)$ and $F(\delta h_c)$. Then it is presumed that

$$\begin{bmatrix} F(\delta f_c) \\ F(\delta h_c) \end{bmatrix} = -[j\omega M_c(\omega) + C_c(\omega)] \begin{bmatrix} F(\dot{r}) \\ F(\dot{\theta}) \end{bmatrix}, \quad (2.29)$$

where $M_c(\omega)$ and $C_c(\omega)$ are the added mass and damping matrices respectively. These are challenging to determine for the same reasons given for $F_a(j\omega)$ and solutions can also be found. This is also discussed in the following section.

It is also known that

$$\begin{bmatrix} \mu I \\ 0 \end{bmatrix} \begin{bmatrix} \delta \ddot{r} \\ \delta \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \delta f \end{bmatrix}, \quad (2.30)$$

where $\mu$ is the mass of the buoy and $J$ is the rotational inertia matrix. Thus, the equation of motion becomes

$$\left( \begin{bmatrix} \mu I \\ 0 \end{bmatrix} + \tilde{M}_t \right) \begin{bmatrix} \delta \ddot{r} \\ \delta \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \delta f_w \\ \delta h_w \end{bmatrix} - \tilde{K}_t \begin{bmatrix} \delta r \\ \delta \theta \end{bmatrix} - \tilde{C}_t \begin{bmatrix} \delta \dot{r} \\ \delta \dot{\theta} \end{bmatrix} + \left[ G^1_t \ldots G^N_t \right] u. \quad (2.31)$$
Bringing these definitions into the differential equation of motion gives the full dynamics of the system by the following pair of equations:

\[
\begin{align*}
\left( \begin{bmatrix} \mu I & 0 \\ 0 & J \end{bmatrix} + \tilde{M}_t \right) \delta \ddot{r} + \delta \ddot{\theta} &= \left[ \begin{bmatrix} \delta f_a \\ \delta h_a \end{bmatrix} + \left( K_b + \tilde{K}_t \right) \delta r \right] - \tilde{C}_t \delta \dot{\theta}
+ \left[ G_1^t \ldots G_N^t \right] u,
\end{align*}
\]

Putting everything in the frequency domain and recognizing that \( F(\delta r) = (j\omega)^{-1} F(\delta \dot{r}) \) and \( F(\delta \ddot{r}) = j\omega F(\delta \dot{r}) \) (and similarly for \( F(\theta) \) and \( F(\ddot{\theta}) \)), the above is

\[
\begin{align*}
\left[ \begin{bmatrix} j\omega \left( \begin{bmatrix} \mu I & 0 \\ 0 & J \end{bmatrix} + M_c(\omega) + \tilde{M}_t \right) + \left( C_c(\omega) + \tilde{C}_t \right) \right] \left[ \begin{bmatrix} F(\delta \dot{r}) \\ F(\delta \ddot{r}) \end{bmatrix} \right] &= F_a(j\omega) F(a) + \left[ G_1^t \ldots G_N^t \right] F(u).
\end{align*}
\]

The voltage vector resulting from the tether extension velocity can be defined as

\[
v = K_e \frac{d}{dt} \left( s_1^T \ldots s_N^T \right),
\]

where \( K_e \) is the resulting linear back EMF constant from the generator and pulley. Then from (2.5), it is known that

\[
v = K_t L \left[ \begin{bmatrix} \delta \dot{r} \\ \delta \ddot{\theta} \end{bmatrix} \right],
\]

where

\[
L = - \left[ G_1^t \ldots G_N^t \right]^T.
\]

Multiplying (2.34) through by the inverse of the matrix in the brackets, i.e.

\[
W(j\omega) = j\omega \left( \begin{bmatrix} \mu I & 0 \\ 0 & J \end{bmatrix} + M_c(\omega) + \tilde{M}_t \right) + \left( C_c(\omega) + \tilde{C}_t \right) + \frac{1}{j\omega} \left( K_b + \tilde{K}_t \right),
\]
and multiplying by \( \mathbf{L} \), gives the transfer functions from the wave amplitude, \( a \) and the control input current \( i = -(1/K_e)u \), to the generator voltages, \( v \), which have the form

\[
\mathcal{F}(v) = G_a(j\omega)\mathcal{F}(a) + G_i(j\omega)\mathcal{F}(i),
\]

where

\[
G_a(j\omega) = K_eLW^{-1}(j\omega)F_a(j\omega),
\]
\[
G_i(j\omega) = K_e^2LW^{-1}(j\omega)L^T.
\]

Figs. 2.7 and 2.8 show \( G_a(j\omega) \) and \( G_i(j\omega) \) for the WEC provided in Table 2.2.

### 2.2 Hydrodynamic forces

The buoy model derived in the previous section is not complete without the infinite-dimensional functions, \( F_a(j\omega) \), \( M_c(\omega) \), and \( C_c(\omega) \), which describe the hydrodynamic forces and moments, added mass, and added damping acting in each of the degrees-of-freedom of the buoy in space. The term “added mass” refers to the mass associated with the displaced volume of water as the buoy portion of the WEC moves, as there is a volume of water that must move when the buoy moves. Similarly, the term “added damping” refers to the additional amount of damping that occurs because of the movement of the same volume of water. There are many ways to determine these functions for various body shapes. For simpler shapes there are often analytical solutions, but numerical techniques exist that are able to handle arbitrary shapes and can typically be found in commercial software packages.

A semi-analytical approach relevant to large bodies of revolution with a vertical axis and finite water depth can be found in Kokkinowrachos et al. (1986) and was used for the hydrodynamic analysis of the cylinder portion of the WEC. The method is based on the discretization of the flow field around the structure using coaxial ring elements, which in the case of the cylindrical buoy in this thesis is a single element.
The velocity potential in each element is approximated by a Fourier series and both the so-called diffraction (i.e. concerning the diffraction of the incident wave) and radiation (i.e. concerning the oscillations of the buoy) problems are solved. The cylindrical body of arbitrary dimensions to be considered is shown in Fig. 2.2 and the forces and moments acting on the body, which is oscillating in several degrees-of-freedom, are assumed to be equivalent to the forces that would act on it if it were constrained in its position. Table 2.1 lists the degrees-of-freedom (DOF) of the buoy. Note that the moment \( M_6 \) will be zero since motion in that degree of freedom cannot be induced due to symmetry. Additionally, the components of the added mass and added damping matrices corresponding to the same degree of freedom will be zero. The infinite-dimensional functions, \( F_a(j\omega) \), \( M_c(\omega) \), and \( C_c(\omega) \), which describe the hydrodynamic forces and moments, added mass, and added damping acting in each of the degrees-of-freedom of the buoy in space, can be determined using the material in the following subsections. Parameters relevant to the geometry of the diffraction and radiation problems are shown in Fig. 2.3.

2.2.1 Diffraction problem

The total velocity potential of the flow field around the cylinder is the superposition of the incident and diffracted wave fields, which in cylindrical coordinates is expressed as

\[
\psi(r, \theta, z) = \psi_I(r, \theta, z) + \psi_B(r, \theta, z),
\]  

(2.42)
where the diffracted wave field $\psi_B(r, \theta, z)$ is derived by solving the diffraction problem around the cylinder via the method of matched axissymmetric eigenfunction expansions. Two different solutions, one satisfying the flow field in region I and another satisfying the flow field in region III which are expressed as different series expansions, are matched by continuity requirements on the hydrodynamic pressure and radial velocity along the vertical boundary between the two regions as well as the kinematic conditions of the vertical walls of the body itself. This results in the linear system of equations to be used for the determination of the unknown coefficients.
needed for the representation of the velocity potential in each fluid region.

The total wave field in region I is given by

$$\psi(r, \theta, z) = -j\omega \frac{H}{2} \sum_{m=-\infty}^{\infty} j^m \Psi^I_m(r, z)e^{jm\theta}, \quad (2.43)$$

where $H$ is the wave height and

$$\frac{1}{d}\Psi^I_m(r, z) = \sum_{a=0}^{\infty} \left( Q_{ma} \frac{I_m(\alpha_a r)}{I_m(\alpha_a b)} + F^I_{ma} \frac{K_m(\alpha_a r)}{K_m(\alpha_a b)} \right) Z_a(z), \quad (2.44)$$

where $I_m(\cdot)$ denotes the modified Bessel function of the second kind, $\alpha_a a \in \{1, \ldots\}$ being the real solutions of the transcendental equation

$$\frac{\omega^2}{g} + \alpha_a \tan(\alpha_a d) = 0, \quad (2.45)$$

and $\alpha_0 = -jk$, where $k$ is the solution of the dispersion relationship

$$\frac{\omega^2}{g} + k \tanh(kd) = 0. \quad (2.46)$$

The orthogonal function $Z_a(z)$ in (2.44) is given by

$$Z_a(z) = \frac{1}{2} \left( 1 + \frac{\sin(2\alpha_a d)}{2\alpha_a d} \right)^{-1/2} \cos(\alpha_a z). \quad (2.47)$$

In (2.44), $Q_{ma}$ is the coefficient for the wave field created by the incident wave whereas $F^I_{ma}$ is the coefficient for the diffracted wave field around the body, i.e. region I, and are

$$Q_{ma} = j^m e^{jkl} \cos(\theta_0 - \beta)e^{-jm\beta} dZ_0(d) I_m(\alpha_a b) \delta_{ba}, \quad (2.48)$$

$$F^I_{ma} = E_{-m}^{-1} M_m Q_{ma}, \quad (2.49)$$

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respectively, where $E_m$ and $M_m$ are $a \times a$ matrices defined as

$$E_m = \frac{h}{d} L^T \text{diag}\{e_n D_{mn}^{III}\} L - \text{diag}\{S_{ma}\}, \tag{2.50}$$

$$M_m = \text{diag}\{D_{ma}^I\} - \frac{h}{d} L^T \text{diag}\{e_n D_{mn}^{III}\} L, \tag{2.51}$$

where

$$L_{na} = (-1)^n N_a^{-1/2} \frac{\alpha_a h}{\alpha_a^2 h^2 - n^2 \pi^2} \sin(\alpha_a h), \tag{2.52}$$

$$D_{mn}^{III} = m + \frac{n\pi b}{h} K_{m+1}(\frac{n\pi b}{h}) K_m(\frac{n\pi b}{h}), \tag{2.53}$$

$$S_{ma} = m + \alpha_a b \frac{K_{m+1}(\alpha_a b)}{K_m(\alpha_a b)}, \tag{2.54}$$

$$D_{ma}^I = m - \alpha_a b \frac{I_{m+1}(\alpha_a b)}{I_m(\alpha_a b)}, \tag{2.55}$$

and $e_n$ is the Neumann’s symbol and is

$$e_0 = 1, \quad e_n = 2(n \geq 1) \tag{2.56}$$

The wave field in region III is given by

$$\psi(r, \theta, z) = -j \omega \frac{H}{2} \sum_{m=-\infty}^{\infty} j^m \Psi_m^{III}(r, z)e^{im\theta}. \tag{2.57}$$

In this case, only the diffracted wave fields contribute and

$$\frac{1}{d} \Psi_m^{III}(r, z) = \sum_{n=0}^{\infty} \varepsilon_n F_{ma}^{III} I_m(\frac{n\pi r}{h}) \frac{I_{m+1}(\frac{n\pi b}{h})}{I_m(\frac{n\pi b}{h})} \cos \left( \frac{n\pi z}{h} \right), \tag{2.58}$$

where $\varepsilon_n = 1$ if $n = 0$ and $\varepsilon_n = 2$ otherwise. The solution to the diffraction problem is

$$F_{mn}^{III} = \sum_{a=0}^{\infty} L_{na} (Q_{ma} + F_{ma}^I) \tag{2.59}$$
The hydrodynamic forces and moments acting on the cylindrical body are given by the integration of the pressure field around the body over the mean wetted surface $S$ as

$$F_t = -\int \int_S j\omega \rho \psi e^{-j\omega t} n dS = F e^{-j\omega t}, \quad (2.60)$$

$$M_t = -\int \int_S j\omega \rho \psi e^{-j\omega t} (r \times n) dS = M e^{-j\omega t}, \quad (2.61)$$

where

$$F = -\rho \omega^2 H \frac{H}{2} \int \int_{S_{m=-\infty}}^{\infty} j^m \Psi_m(r, z) e^{jm\theta} n dS, \quad (2.62)$$

$$M = -\rho \omega^2 H \frac{H}{2} \int \int_{S_{m=-\infty}}^{\infty} j^m \Psi_m(r, z) e^{jm\theta} (r \times n) dS, \quad (2.63)$$

where $r$ denotes the position vector extending from the reference point of forces and moments to each point on $S$ and $n$ is the unit normal vector of the body’s wetted surface in average position pointing outwards. Let $F_q \in \{1, 2, 3\}$ and $M_q \in \{4, 5\}$ denote the force and moment, respectively, along the $q^{th}$ degree of freedom and $M^k_q$ $k \in \{1, 2, 3\}$ be the contribution to the moment $M_q$ of the force acting along the $k^{th}$ degree of freedom so that

$$M_5 = M^1_5 + M^3_5, \quad (2.64)$$

$$M_4 = M^2_4 + M^3_4. \quad (2.65)$$
Evaluation of the integrals in (2.62) and (2.63) is straightforward and leads to

\[
F_1 = \rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi_j^m \sum_{a=0}^{\infty} I_{1a}(Q_{ma} + F_{ma}^I),
\]

(2.66)

\[
M_5^1 = -\rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi_j^m \sum_{a=0}^{\infty} I_{5a}^1(Q_{ma} + F_{ma}^I),
\]

(2.67)

\[
F_2 = \rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi_j^{m+1} \sum_{a=0}^{\infty} I_{1a}(Q_{ma} + F_{ma}^I),
\]

(2.68)

\[
M_4^2 = \rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi_j^{m+1} \sum_{a=0}^{\infty} I_{5a}(Q_{ma} + F_{ma}^I),
\]

(2.69)

\[
F_3 = \rho \omega^2 \frac{H}{2} d 2\pi \left( \frac{b}{2} F_{III}^{00} + \sum_{n=0}^{\infty} I_{3n} F_{III}^{0n} \right),
\]

(2.70)

\[
M_5^3 = -\rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi_j^m \left( \frac{1}{4} b^3 F_{III}^{00} + \sum_{n=1}^{\infty} I_{5n}^3 F_{III}^{0n} \right),
\]

(2.71)

\[
M_4^3 = \rho \omega^2 \frac{H}{2} d \sum_{m \in \{-1, 1\}} \pi m_j^{m+1} \left( \frac{1}{2} b^3 F_{III}^{00} + \sum_{n=1}^{\infty} I_{5n}^3 F_{III}^{0n} \right),
\]

(2.72)

where

\[
I_{1a} = b \left( \sin(\alpha_a d) - \sin(\alpha_a h) \right) \frac{1}{\alpha_a} N_a^{-1/2},
\]

(2.73)

\[
I_{3n} = 2 \frac{n \pi b I_1}{h I_0} \left( \frac{\frac{\pi b}{h}}{n^2 \pi^2} \right) h^2 \cos(n \pi),
\]

(2.74)

\[
I_{5a}^3 = b \frac{1}{\alpha_a} N_a^{-1/2} \left( \frac{1}{\alpha_a} (\cos(\alpha_a d) - \cos(\alpha_a h)) + (d - p) \sin(\alpha_a d) - (h - p) \sin(\alpha_a h) \right),
\]

(2.75)

\[
I_{5n}^3 = 2b \frac{h^2}{n^2 \pi} \cos(n \pi) \left( m + \frac{n \pi b I_{m+1}}{h I_m} \left( \frac{\frac{\pi b}{h}}{n} \right) \right) - 1,
\]

(2.76)

and \( p \) is the position of the point with respect to which the moments are calculated.
The hydrodynamic force and moment matrix \( F_a(j\omega) \) is constructed as
\[
F_a(j\omega) = \begin{bmatrix} F_1(j\omega) & F_2(j\omega) & F_3(j\omega) & M_4(j\omega) & M_5(j\omega) & 0 \end{bmatrix}^T. \tag{2.77}
\]

### 2.2.2 Radiation problem

The cylinder is now assumed to vibrate in its \( q^{th} \) mode with amplitude \( \xi_q \) in still water of depth \( d \). The velocity potential of the radiated wave due to the vibration of the body in the \( q^{th} \) direction is
\[
\psi_{R,q}(r, \theta, z) = -j\omega \xi_q \sum_{m=-\infty}^{\infty} \Psi_{m,q}(r, z) e^{jm\theta}, \tag{2.78}
\]
where the unknowns \( \Psi_{m,q} \) can be derived by the method of matched eigenfunction expansions previously discussed for the diffraction problem. The solution for region I is expressed by
\[
\frac{1}{\delta_q} \Psi_{qm}^I = \sum_{a=0}^{\infty} F_{q,ma}^I K_m(\alpha_ar) Z_a(z), \tag{2.79}
\]
and the solution for region III is
\[
\frac{1}{\delta_q} \Psi_{qm}^{III}(r, z) = g_{qm}(r, z) + \sum_{n=0}^{\infty} \varepsilon_n F_{q,ma}^{III} I_m\left(\frac{n\pi r}{h}\right) \cos\left(\frac{n\pi z}{h}\right), \tag{2.80}
\]
where \( \delta_q = d \) for \( q \in \{1, 2, 3\} \) and \( \delta_q = d^2 \) for \( q \in \{4, 5\} \) and
\[
F_{q,ma}^I = E_m^{-1} \left( P_{q,ma} - \frac{h}{d} L_{an} \text{diag}\{e_n D_{ma}^{III}\} Q_{j,mn}^T \right), \tag{2.81}
\]
\[
F_{q,ma}^{III} = \sum_{a=0}^{\infty} L_{na}(F_{ma}) + Q_{q,mn}, \tag{2.82}
\]
\[
\delta_{3m}(r, z) = \frac{z^2 - \frac{1}{2}r^2}{2hd}; \quad m = 0, \tag{2.83}
\]
\[
\delta_{5m}(r, z) = -\frac{r^2 z^2 + \frac{1}{4}r^4}{4hd^2}; \quad m \in \{-1, 1\}, \tag{2.84}
\]
\[
\delta_{4m}(r, z) = jmg_{5m}(r, z); \quad m \in \{-1, 1\}. \tag{2.85}
\]
The hydrodynamic forces and moments acting on the body are given by the integration of the pressure field around the body over the mean wetted surface $S$.

\begin{equation}
F_{pq} = -\int \int_S j\omega \rho \psi_q e^{-j\omega t} \mathbf{n}_p dS = f_{pq} e^{-j\omega \xi_q},
\end{equation}

where

\begin{equation}
f_{pq} = -\rho \omega^2 \int \int_S \sum_{m=-\infty}^{\infty} \Psi_{j,m}(r, z) e^{jm\theta} \mathbf{n}_p dS,
\end{equation}

and the generalized normal component $\mathbf{n}_p$ is

\begin{align}
\mathbf{n}_1 &= \mathbf{n}_2 = \mathbf{n}_3 \equiv \mathbf{n}, \\
\mathbf{n}_4 &= \mathbf{n}_5 \equiv \mathbf{r} \times \mathbf{n}.
\end{align}

Force $f_{pq}$ may be expressed in terms of added mass $m_{pq}$ and added damping coefficients $c_{pq}$ as

\begin{equation}
f_{pq}(j\omega) = \omega^2 \left( m_{pq} + \frac{j}{\omega} c_{pq} \right).
\end{equation}

Equating the real and imaginary parts of (2.87) and (2.90) leads to the calculation of $m_{pq}(\omega)$ and $c_{pq}(\omega)$ which are the $(p, q)^{th}$ components of the added mass matrix $M_c(\omega)$ and added damping matrix $C_c(\omega)$ respectively. The integrals in (2.87) lead
to

\begin{align}
    f_{1q} &= \rho \omega^2 \delta_q \sum_{m \in \{-1,1\}} \pi \sum_{a=0}^{\infty} I_{1a} \left( Q_{q,ma} + F_{q,ma}^I \right), \\
    f_{5q}^1 &= -\rho \omega^2 \delta_q \sum_{m \in \{-1,1\}} \pi \sum_{a=0}^{\infty} I_{5a}^1 \left( Q_{q,ma} + F_{q,ma}^I \right), \\
    f_{2q} &= \rho \omega^2 \delta_q \sum_{m \in \{-1,1\}} m \pi j \sum_{a=0}^{\infty} I_{1a} \left( Q_{q,ma} + F_{q,ma}^I \right), \\
    f_{4q}^1 &= \rho \omega^2 \delta_q \sum_{m \in \{-1,1\}} m \pi j \sum_{a=0}^{\infty} I_{5a}^1 \left( Q_{q,ma} + F_{q,ma}^I \right), \\
    f_{3q} &= \rho \omega^2 \delta_q \left( \pi b F_{III,j,00}^{III} + \sum_{n=0}^{\infty} I_{3n} F_{j,0n}^{III} \right), \\
    f_{5q}^3 &= -\rho \omega^2 \delta_q \left( \frac{\pi}{4} b^3 F_{m0}^{III} + \sum_{n=1}^{\infty} I_{5n} F_{q,mn}^{III} \right), \\
    f_{4q}^3 &= \rho \omega^2 \delta_q \left( \frac{\pi}{2} b^3 F_{j,m0}^{III} + \sum_{n=1}^{\infty} I_{5n} F_{q,mn}^{III} \right),
\end{align}

with

\begin{align}
    f_{5q} &= f_{5q}^1 + f_{5q}^3, \\
    f_{4q} &= f_{4q}^2 + f_{4q}^3.
\end{align}

For the analysis in this thesis, the truncation of the infinite series for region I, region III, and the diffraction problem are 10, 25, and 10 respectively. These truncations are done at sufficiently large values where the solution does not change dramatically by including additional terms in the series.

2.3 Sea state characterization

Unlike other examples that can be found in literature concerning WECs that treat ocean waves as being purely harmonic, this thesis models the ocean wave distur-
bance as having frequency content spread across a wide band of frequencies. This is consistent with reality and is important since a good WEC should be able to absorb as much power as possible and if significant power is available over a wide range of frequencies then it should be exploited.

We assume the wave height (i.e. trough to crest height) \( a(t) \) to be a stationary stochastic process with spectral density \( S_a(\omega) \), and with the normalization convention that the standard deviation of \( a(t) \) is

\[
\sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_a(\omega) d\omega, \tag{2.100}
\]

where the frequency \( \omega \) is in rad/s.

In ocean engineering, one of the most widely-used functions for \( S_a(\omega) \) is the JONSWAP power spectrum, parametrized by its mean wave period \( T_1 \), significant wave height \( H_{1/3} \), and sharpness factor \( \gamma \) (Faltinsen (1990)). The spectrum is found, in terms of these parameters, as

\[
S_a(\omega) = \frac{310\pi}{T_1^4\omega^5} \frac{H_{1/3}^2}{\omega^4} \exp \left[ -\frac{944}{T_1^4\omega^4} \right] \gamma^\phi, \tag{2.101}
\]

where

\[
Y = \exp \left[ -\left( \frac{0.191\omega T_1 - 1}{\sqrt{2}\phi} \right)^2 \right], \tag{2.102}
\]

and

\[
\phi = \begin{cases} 
0.07 & : \omega \leq 5.24 \\
0.09 & : \omega > 5.24 
\end{cases} \tag{2.103}
\]

The sharpness factor \( \gamma \) is constrained to be between 1 and 3.3, the former describing what is known as a fully developed sea state, which has a wider bandwidth of frequency content, and the latter providing a spectrum with a high quality factor, or a narrower band of excitation. Some Examples of JONSWAP power spectra are shown in Figs. 2.4 and 2.5.
Figure 2.4: JONSWAP power spectra: varying mean wave period. $H_{1/3} = 1$ m, $\gamma = 1$.

Figure 2.5: JONSWAP power spectra: varying sharpness factor. $H_{1/3} = 1$ m, $T_1 = 7$ s.
2.4 Example WEC design

Fig. 2.6 shows the basic configuration of the WEC buoy to be considered in this thesis. The buoy to be considered has a cylindrical shape with a height $h$ and radius $b$. The buoy is half submerged in equilibrium. The buoy has uniform density and mass $\mu$. There are three tethers that extend from points on the buoy down to a depth $d$ on the sea floor at an angle $\phi$ from the horizontal plane. The tethers are separated by $120^\circ$ in the horizontal plane with one tether extending along the positive x-axis. All tethers point toward the centroid of the buoy when the buoy is in equilibrium and at rest. The incoming waves have a directional heading of $\beta$ relative to the x-axis.

The generators can be described by a linear back EMF constant $K_e$, stator resistance $R_c$, and linear mass, damping, and stiffness terms $m$, $c$, and $k$. $K_e$ and $R_c$ determine the value of the short-circuit damping $C_e$ of the generators, defined as

$$C_e = \frac{K_e^2}{R_c},$$

which is indicative of the size, efficiency, and quality of the machines used for trans-
Table 2.2: WEC design parameters.

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buoy radius, ( r )</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Buoy height, ( h )</td>
<td>6 m</td>
</tr>
<tr>
<td>Buoy mass, ( \mu )</td>
<td>5,300 kg</td>
</tr>
<tr>
<td>Tether angle, ( \phi )</td>
<td>26.5°</td>
</tr>
<tr>
<td>Water depth, ( d )</td>
<td>20 m</td>
</tr>
<tr>
<td>Incident wave angle, ( \beta )</td>
<td>0°</td>
</tr>
<tr>
<td>Generator linear mass, ( m )</td>
<td>50 kg</td>
</tr>
<tr>
<td>Generator linear damping, ( c )</td>
<td>50 N-s/m</td>
</tr>
<tr>
<td>Generator linear stiffness, ( k )</td>
<td>50 N/m</td>
</tr>
</tbody>
</table>

Production. To keep things simple, the value of \( C_e \) will be varied (versus varying \( K_e \) and \( R_e \)) for any analysis as this is what determines the optimal power generation and controller design. The parameters \( m \) and \( c \) can be determined from actual generator characteristics by

\[
m = \frac{J_g}{(r_g^2)},
\]

\[
c = \frac{B_g}{(r_g^2)},
\]

where \( J_g \), \( B_g \), and \( r_g \) are the rotational inertia of the generator, rotational damping coefficient of the generator, and pulley radius respectively. \( k \) is simply the linear stiffness of the springs that anchor the tethers that are wrapped around the pulleys.

In addition to \( C_e \), the mean wave period \( T_1 \) and sharpness factor \( \gamma \) of the sea state will be varied in the analysis. Note that the significant wave height \( H_{1/3} \) will be equal to 1 m for all of the analysis. Power generation is proportional to the square of the significant wave height and there is no need to vary \( H_{1/3} \). WEC design parameters to be used in this thesis are shown in Table 2.2. Figs. 2.7 and 2.8 show the frequency response plots for \( G_a \) and \( G_i \) corresponding to this specific design. Note that in this case \( K_e = 500 \text{ V-s/m} \).
2.5 Power generation

The purpose of developing a model of the WEC is to test its performance in terms of power generation, specifically when it uses an active control law. In general, the WEC should be able to harvest more power on average when it has better generator efficiency, or larger values of $C_e$. A WEC with large values of $C_e$ will be able to apply larger currents to the generators without experiencing large losses due to internal resistances, otherwise known as “$i^2R$” losses. In addition to values for $C_e$, the characteristics of the ocean wave power spectrum will affect power generation greatly. As previously mentioned, power generation will scale up proportionally to $H_{1/3}^2$. Unlike in most WECs which absorb power from a small band of frequencies around a single mode of resonance, power generation should not be as sensitive to $T_1$ since the active controller should be able to absorb power over multiple modes of
resonance. As is evident from the frequency response plots of $G_a$ and $G_i$, there are three distinct modes of resonance for this WEC. By looking at the relative phases between different generator voltages which are proportional to the velocity of the tethers, it is possible to tell that the modes of resonance are ordered as surge/sway, pitch/roll, and heave in order from lowest to highest frequency.

A general diagram of a WEC with a control algorithm is shown in Fig. 1.3. A more specific diagram showing the WEC with an active control law to be examined in this thesis is shown in Fig. 2.9. The controller is to be designed to maximize the average power generation over time.
Figure 2.9: Feedback control for WEC model.
Optimal Noncausal Performance

Knowing past, current, and future wave disturbances, it is possible to control a WEC in a way that maximizes energy production using a linear controller. However, since the future wave disturbance can be used to determine the optimal control law, the control law is not necessarily causal. A control law that uses information about future wave disturbances is impractical because this requires a way for buoys to sense its surroundings and use some form of prediction algorithm. However, an optimal noncausal controller and its performance is important to the analysis provided in this thesis because the performance of optimal causal control laws can be compared to it.

With the system model defined for the WEC buoy and a characterization for the stochastic ocean wave environment, the optimum power generation can be determined through classical impedance matching. To begin, both $G_a(s)$ and $G_i(s)$ are presumed to be strictly proper. If $G_a(s)$ is strictly proper, then the harvester will have a finite bandwidth. If $G_i(s)$ is not strictly proper, then it may not approach zero as $s \to j\infty$ (a property of all physical systems) and that creates the potential for an unrealistic controller. We will also assume that the transfer function $G_i(s)$
has an equivalent circuit comprised of time-invariant resistors, capacitors, and
inductors. Therefore, $G_i(s)$ is presumed to be weakly strictly positive real (WSPR)
(Brogliato et al. (2007)).

It is assumed that the power electronics are controlled by a linear controller $Y$, which establishes an effective admittance between $v$ and $i$; i.e.,

$$\hat{i}(s) = -Y(s)\hat{v}(s), \quad (3.1)$$

and that the transmission losses in the electronics can be approximated as resistive; i.e.,

$$P_{loss} = i^T(t)Ri(t), \quad (3.2)$$

where $R$ is positive definite and diagonal, with components equal to the stator re-
sistances of the machines.

The objective is to maximize the average (i.e., expected) total power generation, equal to the power extracted minus the losses; i.e.,

$$\bar{P}_{gen} = E\{ -v^T i - i^T Ri \} \quad (3.3)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_p(\omega) d\omega, \quad (3.4)$$

and $E(\cdot)$ denotes the stationary expectation. $S_p(\omega)$ is the spectral density of generated power in the frequency domain, found as

$$S_p(\omega) = G_a(j\omega)^* [I + G_i(j\omega)Y(j\omega)]^{-*} [\Re \{ Y(j\omega) \} - Y(j\omega)^*RY(j\omega)] \times [I + G_i(j\omega)Y(j\omega)]^{-1} G_a(j\omega)S_a(\omega), \quad (3.5)$$

where $(\cdot)^*$ denotes the matrix adjoint and $\Re\{\cdot\}$ denotes the real part of complex data. The optimal anticausal controller in the frequency domain, $Y_0(j\omega)$, is found by first finding the $Y(j\omega)$ which maximizes $S_p(\omega)$ at each frequency, i.e.,

$$Y_0(j\omega) = [G_i^T(-j\omega) + 2R]^{-1}. \quad (3.6)$$
The Laplace-domain representation of $Y_0(s)$ is just the analytic continuation of the controller above (MacMartin et al. (1991)).

Note that at no point in the determination of the optimal anticausal controller, is it necessary to know $S_a(\omega)$. Rather, information about the sea state is only necessary to compute the optimal control performance. Plots showing the power spectra for the impedance matched controller for various sea states and values of $C_e$ are provided in Figs. 3.1, 3.2, and 3.3. In these figures, $C_e$ is given as $2.5 \times 10^5 \text{ kg/s}$, $2.5 \times 10^4 \text{ kg/s}$, and $2.5 \times 10^2 \text{ kg/s}$. These values of short-circuit damping can be considered to be the result of a combination of a linear back EMF of $K_e = 500 \text{ V-s/m}$ and stator resistance values of $R_c = 1 \Omega$, $10 \Omega$, and $1 \text{k}\Omega$ respectively.

### 3.1 Impedance Matched Controller Synthesis

With the system model defined for the WEC buoy and a characterization for the stochastic ocean wave environment, the optimum power absorption can be determined through classical impedance matching. $G_a(j\omega)$ is the matrix transfer function
Figure 3.2: Anticausal harvester power spectra: $T_1 = 7$ s. $H_{1/3} = 1$ m, $\gamma = 1$.

Figure 3.3: Anticausal harvester power spectra: $T_1 = 5$ s. $H_{1/3} = 1$ m, $\gamma = 1$. 
from the excitation to voltage, \( \mathbf{G}_i(j\omega) \) is the matrix transfer function from the control input to voltage, \( \mathbf{R} \) is the matrix that characterizes resistance, and \( \mathbf{Y}(j\omega) \) is the controller. All matrices have dimension \( N \times N \) except \( \mathbf{G}_a(j\omega) \) which has dimension \( N \times 1 \). Both \( \mathbf{G}_a(s) \) and \( \mathbf{G}_i(s) \) are assumed to be strictly proper. Additionally, \( \mathbf{G}_i(s) \) is assumed to be weakly strictly positive real (WSPR) (Brogliato et al. (2007)).

It is assumed that the power electronics are controlled by a linear controller \( \hat{\mathbf{Y}} \) where

\[
\hat{\mathbf{i}}(s) = -\mathbf{Y}(s)\hat{\mathbf{v}}(s),
\]

and that the losses in the electronics can be approximated as

\[
P_{\text{loss}} = \mathbf{i}^T(t)\mathbf{R}\mathbf{i}(t),
\]

where \( \mathbf{R} \) is positive definite and corresponds to the resistive values for the stators of the generators.

The objective is to maximize the average net power generation which includes the power extracted minus the losses, i.e.

\[
\bar{P}_{\text{gen}} = E\left[\mathbf{-v}^T\mathbf{i} - \mathbf{i}^T\mathbf{R}\mathbf{i}\right]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_p(\omega)d\omega,
\]

Since \( \mathbf{v} \) and \( \mathbf{i} \) in the frequency domain are

\[
\mathbf{v}(j\omega) = [\mathbf{I} + \mathbf{G}_i(j\omega)\mathbf{Y}(j\omega)]^{-1}\mathbf{G}_a(j\omega)a(j\omega),
\]

\[
\mathbf{i}(j\omega) = -\mathbf{Y}(j\omega)[\mathbf{I} + \mathbf{G}_i(j\omega)\mathbf{Y}(j\omega)]^{-1}\mathbf{G}_a(j\omega)a(j\omega),
\]

the power spectrum for power absorbed by the harvester becomes

\[
S_p(\omega) = \mathbf{G}_a(j\omega)^*[\mathbf{I} + \mathbf{G}_i(j\omega)\mathbf{Y}(j\omega)]^{-*}[\Re\{\mathbf{Y}(j\omega)\} - \mathbf{Y}(j\omega)^*\mathbf{R}\mathbf{Y}(j\omega)]
\]

\[
\times [\mathbf{I} + \mathbf{G}_i(j\omega)\mathbf{Y}(j\omega)]^{-1}\mathbf{G}_a(j\omega)S_a(\omega),
\]

where \( S_a(\omega) \) is the power spectrum of the excitation \( a \). We want to find the \( \mathbf{Y}(j\omega) \) that maximizes \( S_p(\omega) \) at all frequencies. The power spectrum can be written as a
function of $Y^{-1}$ as

$$S_p(Y^{-1}) = G_u^*[Y^{-1} + G_i]^{-*}[\Re\{Y^{-1}\} - R][Y^{-1} + G_i]^{-1}G_uS_u. \quad (3.14)$$

It is clear that maximizing this function is equivalent to maximizing

$$P(Y^{-1}) = \text{tr}\{[Y^{-1} + G_i]^{-*}[\Re\{Y^{-1}\} - R][Y^{-1} + G_i]^{-1}\}. \quad (3.15)$$

We can define the complex matrices $G_i$, and $Y$ as

$$Y^{-1} = A + B j, \quad (3.16)$$

$$G_i = C + D j, \quad (3.17)$$

and $P(Y^{-1})$ becomes

$$P(Y^{-1}) = \text{tr}\{[(A + C) + (B + D)j]^{-*}[A - R][(A + C) + (B + D)j]^{-1}\}. \quad (3.18)$$

To find what $Y^{-1}$ maximizes $P(Y^{-1})$, we want to find the $A$ and $B$ that make $\partial P/\partial A$ and $\partial P/\partial B$ equal to zero. We can make the definition

$$P(Y^{-1}) = \text{tr}\{Z(Y^{-1})^*X(Y^{-1})Z(Y^{-1)}\}, \quad (3.19)$$

where

$$Z = [(A + C) + (B + D)j]^{-1}, \quad (3.20)$$

$$X = [A - R], \quad (3.21)$$

and

$$\frac{\partial Z}{\partial A_{kl}} = -[(A + C) + (B + D)j]^{-1}\hat{e}_k\hat{e}_l^T[(A + C) + (B + D)j]^{-1}, \quad (3.22)$$

$$\frac{\partial X}{\partial A_{kl}} = \hat{e}_k\hat{e}_l^T. \quad (3.23)$$

It can be shown that

$$\frac{\partial}{\partial A_{kl}} [Z^*XZ] = \frac{\partial Z^*}{\partial A_{kl}}XZ + Z^*\frac{\partial X}{\partial A_{kl}}Z + Z^*X\frac{\partial Z}{\partial A_{kl}}. \quad (3.24)$$
Therefore

\[ \frac{\partial}{\partial A_{kl}}[P] = \text{tr}\{[\hat{e}_k]^{-2}e_k\hat{e}_l^T\}^* [A - R][A + C] + (B + D)j \}

\[ + [A + C] + (B + D)j \}

\[ - [A - R][A + C] + (B + D)j \}

\[ = -2\text{tr}\{[A + C] + (B + D)j \}

\[ + \text{tr}\{[A + C] + (B + D)j \}

\[ - 2[A - R][A + C] + (B + D)j \}

\[ = \hat{e}_k^T [A + C] + (B + D)j \}

\[ - 2[A + C] + (B + D)j \}

\[ = \hat{e}_k^T [A + C] + (B + D)j \}

\[ - 2[A - R][A + C] + (B + D)j \}

\[ = 0 = [(A + C) + (B + D)j] - 2[A - R]

\[ = -A + 2R + C + (B + D)j.\]

From here we can get the condition for the optimal controller which is

\[ 0 = [(A + C) + (B + D)j] - 2[A - R]

\[ = -A + 2R + C + (B + D)j. \]

Since both A and B have to be real valued matrices, we do not need to get a second equation to solve for both unknowns and it is clear that

\[ A = 2R + C, \]

\[ B = -D. \]
Therefore, the controller that maximizes $S_p(\omega)$ at all frequencies is

$$Y_0(j\omega) = \left[ G_i(j\omega)^* + 2R \right]^{-1}, \quad (3.35)$$

or in the Laplace domain

$$Y_0(s) = \left[ G_i^T(-s) + 2R \right]^{-1}. \quad (3.36)$$

(MacMartin et al. (1991)) Note that the controller is anticausal because if $G_i(s)$ is WSPR, then the parahermitian conjugate of $Y_0$ is

$$Y_0^T(-s) = \left[ G_i(s) + 2R \right]^{-1}, \quad (3.37)$$

and this function is strictly positive real for any $R > 0$. This implies that all of its poles are in the open left-half plane and all of the poles of $Y_0(s)$ are in the open right-half plane. Thus, its dynamics are always anticausal. This means that the controller is anticipatory and requires future information.
In order to determine the optimal causal performance it is necessary to have finite-dimensional state space models that characterize the transfer functions $G_a$ and $G_i$. It is also necessary to have a finite-dimensional state space model that describes the wave height process, i.e. a filter that has an equivalent power spectrum to the desired JONSWAP power spectrum that is subjected to white noise of unit spectral intensity. It should be noted that these finite-dimensional state space models will be inexact because the hydrodynamic functions $F_a(\omega)$, $M_c(\omega)$, and $C_c(\omega)$ from Ch. 2 are infinite-dimensional and the JONSWAP power spectrum is characterized by a piecewise function that is not rational.

The approximated state spaces for $G_a$ and $G_i$ were found using the subspace techniques originally proposed in McKelvey et al. (1996) for determining finite-dimensional approximations for infinite-dimensional systems characterized by frequency response data. Although this algorithm does not give optimal estimates of a given order, it does identify the balanced truncations of infinite dimensional systems in discrete-time from frequency-domain data. The approximated state space corresponding to the JONSWAP spectrum was found using a simple iterative non-linear search algo-
4.1 JONSWAP power spectrum approximation

Finding a finite-dimensional transfer function with a power spectrum that is close to the JONSWAP spectrum can be done by minimizing the sum of the squared-error between the actual JONSWAP spectrum and the absolute value of the candidate continuous-time transfer function squared,

$$\epsilon = \sum_{i=1}^{M} \left(\|\hat{H}_J(i)\|^2 - S_a(i)\right)^2$$  (4.1)

where $M$ is the number of samples in the sampled version ($M$ samples linearly spaced on $\omega \in [0, \omega_f]$) of $S_a$ and the candidate transfer function which is of the form

$$\hat{H}_J(s) = \frac{b_1 s^{N-1} + b_2 s^N + \ldots + b_{N-1} s}{a_1 s^N + a_2 s^{N-1} + \ldots + a_{N+1}},$$  (4.2)

$N$ is the model order, and $\omega_f$ is a frequency value where $S_a$ has converged to near zero. The poles of the system are constrained to be in the left-half-plane in order to ensure stability. This is done by flipping any unstable poles across the imaginary axis and this does not affect the fit since the it does not affect the magnitude of the transfer function. Note that the DC gain of the transfer function is zero. Without this property, large errors can be seen at low frequencies during the later analysis since it could appear that there is a nontrivial amount of power at those frequencies.

The minimization was accomplished through the use of a simplex algorithm. The transfer function can be converted to a continuous-time state space realization that has the form

$$\dot{x}_J = A_J x_J + B_J w,$$  (4.3)

$$a = C_J x_J,$$  (4.4)
where \( w(t) \) is a scalar white noise process with spectral intensity equal to 1. The techniques described above are similar to what is done in Spanos (1986). In this analysis, it was found that a state space of order 4 was sufficient to render close matching to JONSWAP spectra for reasonable sharpness factors. Some examples of JONSWAP power spectra and their finite-dimensional approximations are shown in Fig. 4.1. It was also observed that \( M = 1000 \) and \( \omega_f = 6 \text{ rad/s} \) were sufficient values.

### 4.2 State-space model identification from frequency domain data

Given the frequency domain data for the infinite dimensional transfer functions \( G_a \) and \( G_i \) it is possible to find their corresponding approximate finite-dimensional state space models. The first step to the identification algorithm is to expand uniformly spaced experimental frequency-response data of the single-input-single-output (SISO) system, or SISO subsystem of a larger system, being considered according to

\[
G(e^{-j\theta}) = G^*(e^{j\theta}), \quad 0 \leq \theta \leq \pi, \tag{4.5}
\]

where \( \theta \) is the discrete-time frequency as

\[
G_{M+k} = G_{M-k}^*, \quad k = 1, \ldots, M-1, \tag{4.6}
\]

and perform the \( 2M \)-point inverse discrete-time Fourier transform (DFT) on the expanded data

\[
\hat{g}_i \equiv \frac{1}{2M} \sum_{k=0}^{2M-1} G_k e^{j2\pi ik/2M}, \quad i = 0, \ldots, q+r-1, \tag{4.7}
\]

to determine the estimates of the impulse-response coefficients of \( g_i \).

The next step is to construct the \( q \times r \)-block Hankel matrix

\[
\hat{H}_{qr} \equiv \begin{bmatrix}
\hat{g}_1 & \cdots & \hat{g}_r \\
\vdots & \ddots & \vdots \\
\hat{g}_q & \cdots & \hat{g}_{q+r-1}
\end{bmatrix}, \tag{4.8}
\]
Figure 4.1: Finite-dimensional approximations for $S_a$. (a) $H_{1/3} = 1$ m, $T_1 = 12$ s, $\gamma = 1$; (b) $H_{1/3} = 1$ m, $T_1 = 7$ s, $\gamma = 1$; (c) $H_{1/3} = 1$ m, $T_1 = 5$ s, $\gamma = 1$. 
and perform a singular value decomposition for $\tilde{H}_{qr}$ as follows

$$
\tilde{H}_{qr} \equiv [\hat{U}_1 \quad \hat{U}_2] \begin{bmatrix} \hat{\Sigma}_1 & 0 \\ 0 & \hat{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \hat{V}^T_1 \\ \hat{V}^T_2 \end{bmatrix},
$$

(4.9)

where $\hat{\Sigma}_1$ contains the $n$ dominant singular values on the diagonal.

The system matrices are then estimated as

$$
\hat{A} \equiv (J^q_1 \hat{U}_1)^+ J^q_2 \hat{U}_1,
$$

(4.10)

$$
\hat{C} \equiv J^q_3 \hat{U}_1,
$$

(4.11)

$$
\hat{B} \equiv (I - \hat{A}^{2M}) \hat{\Sigma} \hat{V}^T_1 J^q_4,
$$

(4.12)

$$
\hat{D} \equiv \hat{g}_0 - \hat{C} \hat{A}^{2M-1} (I - \hat{A}^{2M})^{-1} \hat{B},
$$

(4.13)

where

$$
J^q_1 \equiv \begin{bmatrix} I_{(q-1)p} & 0_{(q-1)p \times p} \end{bmatrix},
$$

(4.14)

$$
J^q_2 \equiv \begin{bmatrix} 0_{(q-1)p \times p} & I_{(q-1)p} \end{bmatrix},
$$

(4.15)

$$
J^q_3 \equiv \begin{bmatrix} I_p \\ 0_{p \times (q-1)p} \end{bmatrix},
$$

(4.16)

$$
J^q_4 \equiv \begin{bmatrix} I_m \\ 0_{(r-1)m \times m} \end{bmatrix}.
$$

(4.17)

Stability of the system can be ensured by executing the following routine that projects unstable poles back inside of the unit circle.

- Transform $\hat{A}$ to the diagonal form with the eigenvalues $\lambda_i$ on the diagonal.

- Project any diagonal elements, i.e. eigenvalues, satisfying $1 < |\lambda_i| \leq 2$, into the unit disc by the rule $\lambda'_i \equiv \lambda_i \left( 2 \left| \frac{1}{\lambda_i} \right| - 1 \right)$.

- Set any eigenvalues where $|\lambda_i| > 2$ to zero.

- Move eigenvalues that are on the unit circle inside the unit disc by changing the magnitude of the eigenvalue to $1 - \epsilon$ for some small positive $\epsilon$, i.e. $\lambda'_i \equiv \lambda_i (1 - \epsilon)$. 


• Transform $\hat{A}$ back to its original form.

• Go on to determine $\hat{B}$ and $\hat{D}$ as in (4.12) and (4.13).

A model order of 12 was found to be sufficient for each component of $G_a$ and a model order of 6 was found to be sufficient for each component of $G_i$. A model order was considered sufficient if it managed to capture all of the details of the magnitude and phase data of a subsystem.

4.3 Augmented state-spaces

The algorithm presented in the previous section was run on each SISO component in the discrete-time form of the matrix transfer functions $G_a$ and $G_i$ and the corresponding state-space models were converted from discrete-time to continuous-time models using a standard bilinear approximation. The individual state spaces could then be augmented together to create finite-dimensional state spaces of the form

\[ G_a \sim \begin{bmatrix} A_a & B_a \\ C_a & 0 \end{bmatrix} \]  

\[ G_i \sim \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}. \]

Figs. 4.2 and 4.3 show the frequency domain data for $G_a$ and $G_i$ and their finite-dimensional approximations. The quality of the fits are typical and in this case the model order for each component of $G_a$ and $G_i$ are 12 and 6 respectively. The fits also reflect a final balanced reduction augmented system shown above where the final model order of the finite-dimensional approximations for $G_a$ and $G_i$ are 19 and 10 respectively. Note that the plots are in log scale and thus what appear to be large visible errors around low magnitude data are actually relatively unimportant. Additionally, errors in phase around areas with low magnitude are unimportant.

It should be noted that although they are identified, the feedthrough terms are neglected. They are left out because they are very small and the transfer functions
that they are approximating are identically zero at $\omega = 0$, i.e. DC, and this ensures that property of the transfer functions is preserved. These systems were then augmented together to describe the entire harvester system as

$$\dot{x}_h = Ax_h + Bi + Ea,$$  \hspace{1cm} (4.20)

$$v = Cx_h,$$ \hspace{1cm} (4.21)

where

$$A = \begin{bmatrix} A_a & 0 \\ 0 & A_i \end{bmatrix},$$ \hspace{1cm} (4.22)

$$B = \begin{bmatrix} 0 & B^T_i \end{bmatrix}^T,$$ \hspace{1cm} (4.23)

$$E = \begin{bmatrix} B^T_a & 0 \end{bmatrix}^T,$$ \hspace{1cm} (4.24)

$$C = \begin{bmatrix} C_a & C_i \end{bmatrix},$$ \hspace{1cm} (4.25)
and the augmented state vector is

\[ x_h = \begin{bmatrix} x_a^T \\ x_i^T \end{bmatrix}^T. \]  

(4.26)

A balanced truncation was performed to remove redundancy in the dynamics of the two spaces. \((A, C)\) is observable and \((A, [B \ E])\) is controllable. The final augmented state space that includes the harvester and noise model is

\[ \dot{x} = Ax + Bi + Ew, \]  

(4.27)

\[ v = Cx, \]  

(4.28)
where the augmented matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{C}$ are

$$
\mathbf{A} = \begin{bmatrix} \mathbf{A} & \mathbf{E} \mathbf{C}_J \\ 0 & \mathbf{A}_J \end{bmatrix},
$$

(4.29)

$$
\mathbf{B} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}^T,
$$

(4.30)

$$
\mathbf{E} = \begin{bmatrix} 0 & \mathbf{B}_J \end{bmatrix}^T,
$$

(4.31)

$$
\mathbf{C} = \begin{bmatrix} \mathbf{C} \\ 0 \end{bmatrix},
$$

(4.32)

and the augmented state vector is

$$
\mathbf{x} = \begin{bmatrix} \mathbf{x}_h^T \\ \mathbf{x}_f^T \end{bmatrix}^T.
$$

(4.33)

4.4 Additional identification techniques

In an attempt to find more accurately identified systems, a couple of techniques were developed. The first such technique is to find the $\mathbf{G}_a$ that characterizes the relationship between the wave amplitude, which is several meters in front of the buoy in the direction of the oncoming waves, and the output voltage. This is important because the hydrodynamic forces acting on the buoy are affected by waves that have not yet passed by the buoy and is similar to instituting a time delay, but simpler for this application. Wave amplitudes that are increasingly further away from the buoy have a dimishing effect. It was found that the wave amplitude approximately 5 m out in front of the buoy provides enough spatial delay to accurately identify $\mathbf{G}_a$ while not sacrificing too much in terms of the increased model order needed to model such a delay.

The second technique is to add a static admittance to the system in order to bring down the peaks of the transfer functions around the modes of resonance so that the identification algorithm will be less likely to sacrifice accuracy over some ranges of frequency that might ultimately prove to be important. This was intended to overcome the shortcomings from previously explored identification routines such as
the one found in Bayard (1994) and the built-in routines in MATLAB, but ultimately, it was not used since the identification routine in McKelvey et al. (1996) works very well. Details for this technique are provided in Appendix A.
Consider the augmented continuous-time finite dimensional state space for the WEC given in (4.27) and (4.28) that includes the sea state as a filtered noise process within it. Now it is possible to use $\mathcal{H}_2$ control theory to determine the optimal causal power generation and the corresponding controller for both the full-state feedback case and the more realistic output feedback case with measurement noise. It is much more practical to have a causal controller that relies only on the current and past output from the system, which are in the form of voltages measured at the generator terminals, instead of a noncausal controller that relies on knowledge of future wave disturbances.
5.1 Full-state feedback

The objective is to find the full-state feedback law \( K : x \rightarrow i \) which maximizes \( P_{\text{gen}} \), or equivalently, which minimizes

\[
-\bar{P}_{\text{gen}} = E \{ v^T i + i^T R i \} \tag{5.1}
\]

\[
= \frac{1}{2} E \left\{ x_h \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x_h^T \\ i \end{bmatrix} \right\}. \tag{5.2}
\]

This constitutes an LQG optimal control problem, but is non-standard because the performance functional is sign-indeterminate. It is a standard result from Aström (1970) that the optimal control solution, if it exists, is

\[
i = Kx = -\frac{1}{2} R^{-1} [B^T P + C] x, \tag{5.3}
\]

where \( P \) is the stabilizing solution to the algebraic Riccati equation

\[
0 = A^T P + PA - \frac{1}{2} [B^T P + C]^T R^{-1} [B^T P + C]. \tag{5.4}
\]

The expression for the optimal causal power spectrum with full-state feedback is

\[
S_p(\omega) = -\mathbb{Re} \left[ H_i^*(j\omega) H_v(j\omega) \right] - H_i^*(j\omega) R H_i(j\omega), \tag{5.5}
\]

where

\[
H_i(j\omega) = K \left[ j\omega I - A - B K \right]^{-1} E, \tag{5.6}
\]

\[
H_v(j\omega) = C \left[ j\omega I - A - B K \right]^{-1} E. \tag{5.7}
\]

and a closed-form expression for the optimal \( \bar{P}_{\text{gen}} \) is

\[
\bar{P}_{\text{gen}}^{\text{opt}} = -\frac{1}{2} E^T P E. \tag{5.8}
\]

Given a WSPR system, \( \bar{P}_{\text{gen}}^{\text{opt}} \) is always positive and the optimal controller is always stabilizing (Scruggs (2010)).
5.2 Output feedback

In practice, the state vector $\mathbf{x}$ will need to be estimated from $\mathbf{v}$. This can be done via a Kalman-Bucy filter that keeps a running estimate of the system states according to the differential equation

$$\frac{d}{dt} \hat{\mathbf{x}} = A\hat{\mathbf{x}} + B\mathbf{i} + F(\mathbf{C}\hat{\mathbf{x}} - \mathbf{v}),$$

(5.9)

where $\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$. Assuming white noise in the measurement channels for $\mathbf{v}$, with spectral intensity matrix $\nu \mathbf{I}$, the optimal estimates are thus obtained via the standard Kalman gain; i.e.,

$$F = -\frac{1}{\nu} \mathbf{S} \mathbf{C}^T,$$

(5.10)

where $\mathbf{S}$ is the solution to the matrix Riccati equation

$$0 = AS + S \mathbf{A}^T - \frac{1}{\nu} \mathbf{S} \mathbf{C}^T \mathbf{C} \mathbf{S} + \mathbf{E} \mathbf{E}^T.$$

(5.11)

With $\mathbf{K}$ defined in (5.3), the augmented closed-loop system becomes

$$\dot{\mathbf{x}}_c = A_c \mathbf{x}_c + \mathbf{E}_c \mathbf{w},$$

(5.12)

$$\mathbf{v} = \mathbf{C}_c \mathbf{x}_c,$$

(5.13)

where

$$A_c = \begin{bmatrix} \mathbf{A} & \mathbf{B} \mathbf{K} \\ -\mathbf{F} \mathbf{C} & (\mathbf{A} + \mathbf{F} \mathbf{C} + \mathbf{B} \mathbf{K}) \end{bmatrix},$$

(5.14)

$$\mathbf{E}_c = \begin{bmatrix} \mathbf{E} \\ \mathbf{0} \end{bmatrix},$$

(5.15)

$$\mathbf{C}_c = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix},$$

(5.16)

and

$$\mathbf{x}_c = [\mathbf{x}^T \quad \hat{\mathbf{x}}^T]^T.$$

(5.17)

Defining the augmented controller as

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{0} & \mathbf{K} \end{bmatrix},$$

(5.18)
the power spectrum resulting from the optimal causal system is

\[ S_p^o(\omega) = -\Re \left[ H_i^o(j\omega)H_v^o(j\omega) \right] - H_i^o(j\omega)RH_v^o(j\omega), \]  

(5.19)

where

\[ H_i^o(j\omega) = K_c [j\omega I - A_c]^{-1} E_c, \]  

(5.20)

\[ H_v^o(j\omega) = C_c [j\omega I - A_c]^{-1} E_c. \]  

(5.21)

and the optimal power generated is

\[ \bar{P}_{gen}^{opt} = -\frac{1}{2} P^T E - \text{tr}\{K_{opt} S K_{opt}^T R\}, \]  

(5.22)

It is interesting to note that in the equation above, the inability to accurately measure the output voltage impedes the ability of the WEC to generate power. In Scruggs (2010), the asymptotic case as \( \nu \to 0 \) is investigated. If \( G_a \) is minimum-phase, this asymptotic case converges to the full-state performance, i.e., (5.8). However, when it is not, the asymptotic limit in the expression above is lower, by a finite amount. However, it can be proven that irrespective of the size of \( \nu \), or the particular parameters of the problem, (5.22) is always positive.

Figs. 5.1, 5.2, and 5.3 show harvester power spectra for various values of mean wave period. Both the causal power spectra resulting from \( \mathcal{H}_2 \) control theory and noncausal power spectra resulting from impedance matching are shown. Table 5.1 provides values of average power generation and correspond directly to those figures.

It can be seen that the optimal causal controllers make the harvester absorb negative amounts of power over significant bands of frequencies. It can also be seen that significant amounts of energy is absorbed over the first two modes of resonance, in this case the modes of resonance related to surge/ sway and pitch/roll. As expected, the anticausal controller always outperforms the causal controller in terms of average power generation and average power generation decreases as \( C_e \) decreases. For really poor generator quality, the anticausal and causal controllers converge.
Figure 5.1: Harvester power spectra: $T_1 = 7$ s, $H_{1/3} = 1$ m, $\gamma = 1$, $\nu = 1 \times 10^{-4}$ V²s. (a) $C_e = 2.5 \times 10^5$ kg/s; (b) $C_e = 2.5 \times 10^4$ kg/s; (c) $C_e = 2.5 \times 10^2$ kg/s.
Figure 5.2: Harvester power spectra: $T_1 = 7$ s, $H_{1/3} = 1$ m, $\gamma = 1$, $\nu = 1 \times 10^{-4}$ $\text{V}^2$ s. (a) $C_e = 2.5 \times 10^5 \text{ kg/s}$; (b) $C_e = 2.5 \times 10^4 \text{ kg/s}$; (c) $C_e = 2.5 \times 10^2 \text{ kg/s}$. 
Figure 5.3: Harvester power spectra: $T_1 = 5$ s, $H_{1/3} = 1$ m, $\gamma = 1$, $\nu = 1 \times 10^{-4}$ V$^2$s. (a) $C_e = 2.5 \times 10^5$ kg/s; (b) $C_e = 2.5 \times 10^4$ kg/s; (c) $C_e = 2.5 \times 10^2$ kg/s.
Table 5.1: Average power generation.

<table>
<thead>
<tr>
<th></th>
<th>$T_1 = 12s$</th>
<th>$T_1 = 7s$</th>
<th>$T_1 = 5s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_e = 2.5 \times 10^5$ kg/s Noncausal</td>
<td>32.1 kW</td>
<td>26.7 kW</td>
<td>20.8 kW</td>
</tr>
<tr>
<td>Causal</td>
<td>24.6 kW</td>
<td>21.9 kW</td>
<td>13.9 kW</td>
</tr>
<tr>
<td>$C_e = 2.5 \times 10^4$ kg/s Noncausal</td>
<td>13.8 kW</td>
<td>5.76 kW</td>
<td>5.26 kW</td>
</tr>
<tr>
<td>Causal</td>
<td>7.45 kW</td>
<td>4.14 kW</td>
<td>3.10 kW</td>
</tr>
<tr>
<td>$C_e = 2.5 \times 10^2$ kg/s Noncausal</td>
<td>1.44 kW</td>
<td>0.243 kW</td>
<td>0.377 kW</td>
</tr>
<tr>
<td>Causal</td>
<td>1.22 kW</td>
<td>0.232 kW</td>
<td>0.277 kW</td>
</tr>
</tbody>
</table>

5.3 Simulation

In order to provide additional insight into the relationship between control current, voltage, and power generation, a simulation was performed using the optimal causal controller and observer for a specific example where $C_e = 2.5 \times 10^5$ kg/s, $H_{1/3} = 1$ m, $T_1 = 10$ s, $\gamma = 1$, and $\nu = 10$ V$^2$-s. Figs. 5.4 and 5.5 show the voltage and current for each generator over 100 s of the simulation. Figs. 5.6 and 5.7 show the power generation and cumulative power generation (energy) over an entire hour of simulated time. For this case the analytical average power generation is about 26 kW, which is consistent with the simulation.

These plots illustrate how LQG control works, but they do not provide a realistic solution. At some samples within the simulation power generation for the harvester exceeds 4 MW. In comparison, the average power generation is merely 26 kW. This is likely due to the fact that the stator resistance is modeled as being frequency-independent, when in reality it should increase at higher frequencies. In addition to including a frequency-dependent stator resistance, it is possible to implement a clipped optimal control strategy (Dyke et al. (1996)) in order to limit the amount of current that the controller can send to the generators at any given time.
Figure 5.4: Simulated voltage output.

Figure 5.5: Simulated control current.
Figure 5.6: Simulated power generation.

Figure 5.7: Simulated energy accumulation.
6

Adaptive Control

The causal controller described in the previous chapter assumes that the characteristics of the sea state (i.e., $\gamma$, $T_1$, and $H_{1/3}$), along with the harvester dynamics are known. However, in practice the sea state slowly varies over time and the WEC needs a way to adapt in order to retain near optimal performance. This can be accomplished either by remotely observing the sea state and having the controller changed accordingly, or automatically by including a real-time system identification algorithm to implement a self-tuning adaptive controller.

6.1 Adaptive control performance

If the controller is not made to adapt to changes in sea state, these changes will result in sub-optimal performance. Considering only voltage feedback, the average power generation is

$$\bar{P}_{gen} = E\{ -v^T i - i^T R i \}, \quad (6.1)$$

$$= E\{ -x_c^T C_{c} K_c x_c - x_c^T K_c^T R K_c x_c \}, \quad (6.2)$$

$$= -\text{tr}\{ C_{c} K_c \Phi + K_c^T R K_c \Phi \}, \quad (6.3)$$
where the covariance matrix $\Phi$ is the solution to the Lyapunov equation

$$A_c \Phi + \Phi A_c^T + E_c E_c^T = 0, \quad (6.4)$$

and

$$A_c = \begin{bmatrix} A & B K \\ -F C & (A + F C + B K) \end{bmatrix}, \quad (6.5)$$

$$E_c = \begin{bmatrix} E \\ 0 \sqrt{\nu} F \end{bmatrix}, \quad (6.6)$$

$$C_c = \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad (6.7)$$

$$K_c = \begin{bmatrix} 0 & K \end{bmatrix}. \quad (6.8)$$

This can be used to illustrate the difference between the performance of two controllers at various sea states: one that adapts to be optimal at the current sea state and another one that does not adapt and is designed to be optimal just at one sea state. Fig. 6.1 shows this comparison when the non-adaptive controller is designed specifically for a sea state described by the parameters $H_{1/3} = 1$ m, $T_1 = 10$ s, and $\gamma = 1$ and measurement noise $\nu = 10$ V$^2$s. $T_1$ is then varied between 5 and 15 s. This comparison is done for multiple values of $C_e$. It is interesting to note that as $C_e$ decreases, the controller becomes more robust (at least in the sense that it will generate positive power over a larger range of mean wave periods) and the difference in power generation is less sensitive to changes in the mean wave period. This is expected, since the controller loses the ability to function very actively as it cannot control the system very well. It is also interesting to note that for high quality generators with larger values of $C_e$, a non-adaptive controller for this particular WEC can do very well at mean wave periods that are significantly lower than the mean wave period that it is designed for. Additionally, a moderate increase in mean wave period can yield a controller that has negative average power generation. These observations suggest that there is some utility in the use of adaptive control in wave
Figure 6.1: Adaptive vs. non-adaptive control. $H_{1/3} = 1$ m, $\gamma = 1$. (a) $C_e = 2.5 \times 10^5$ kg/s; (b) $C_e = 2.5 \times 10^4$ kg/s; (c) $C_e = 2.5 \times 10^2$ kg/s.
6.2 Future work

The need for an adaptive control algorithm has been demonstrated in the previous section by comparing the performance of a controller that knows the sea state exactly and is designed accordingly and one that is designed for a single sea state. There are many adaptive control algorithms that can be applied to this problem. One such control algorithm is known as an indirect self-tuning regulator (STR) and is shown in Fig. 6.2. An indirect STR involves a recursive estimation of the system and subsequent controller and observer gain design as if the system estimate is the same as the true system. This is known as the *certainty equivalence principle*. It is proposed that such a recursive estimation can be accomplished by the use of the recursive extended least-squares (RELS) algorithm. Information on STRs (and other adaptive control algorithms) and the RELS algorithm can be found in Aström and Wittenmark (1995).
6.2.1 Recursive extended least-squares

Some systems, such as the augmented system for the WEC in this thesis, have a non-deterministic input in addition to deterministic ones. In the case of the WEC, the wave height, which is a fixed distance in front of the buoy, is non-deterministic while the control current sent to the generators is deterministic. These systems require a way of treating the non-deterministic input as filtered white noise, if that is a reasonable way to model such an input. This can be done using the RELS algorithm.

A RELS algorithm is given in Aström and Wittenmark (1995) for a system with one deterministic input, one non-deterministic filtered white noise input, and one output. This can be expanded into a system with an arbitrary amount of all three.

Consider a system of order $n$ with $N$ deterministic inputs and $M$ outputs. The relationship between the $i^{th}$ output $y_i$, all of the inputs $\{u_1, \ldots, u_N\}$, and the $i^{th}$ (or only) white noise disturbance $e_i$ is

$$A_i(q)y_i(t) = \sum_{j=1}^{N} B_{ij}(q)u_j(t) + C_i(q)e_i(t),$$

(6.9)

where $A_i(q)$, $B_{ij}(q)$, and $C_i(q)$ are polynomials in the forward shift operator $q$, i.e.

$$A_i(q)y_i(t) = y_i(t) + a_{i1}y_i(t-1) + \ldots + a_{in}y(t-n),$$

(6.10)

$$B_{ij}(q)u_j(t) = b_{ij1}u_j(t-1) + b_{ij2}u_j(t-2) + \ldots + b_{ijn}u_j(t-n),$$

(6.11)

$$C_i(q)e_i(t) = c_{i1}e_i(t-1) + c_{i2}e_i(t-2) + \ldots + c_{in}e_i(t-n),$$

(6.12)

and $\{e_i(t)\}$ is white noise. The parameters of the polynomial $C_i$ describe the correlation of the disturbance for the output $y_i$. Note that in this case, the polynomials $B(q)$ and $C(q)$ are of order $n-1$ while $A(q)$ is order $n$. However, in a more general formulation, all three may be of different order.
To begin the algorithm we describe the prediction error, or residual, as

$$
\varepsilon_i(t) = y_i(t) - \Phi_i^T(t - 1) \hat{\Theta}_i(t - 1),
$$

where

$$
\Theta_i^T = [a_{i1} \cdots a_{in} b_{i11} \cdots b_{i1n} \cdots b_{iN1} \cdots b_{iNn} c_{i1} \cdots c_{in}],
$$

$$
\Phi_i^T = [-y_i(t - 1) \cdots - y_i(t - n) u_1(t - 1) \cdots u_1(t - n) \cdots u_N(t - 1) \cdots u_N(t - n) \varepsilon_i(t - 1) \cdots \varepsilon_i(t - n)].
$$

The equations for updating the estimates are given by

$$
\hat{\Theta}_i(t) = \hat{\Theta}_i(t - 1) + P_i^{-1}(t) \Phi_i(t - 1) \varepsilon_i(t),
$$

$$
P_i^{-1}(t) = P_i^{-1}(t - 1) + \Phi_i(t - 1) \Phi_i^T(t - 1) \lambda.
$$

where $\lambda \in (0, 1]$ is the exponential forgetting factor that controls how the past data is weighted in determining the current parameter estimates. A forgetting factor of $\lambda = 1$ weighs all of the data for all time equally. As $\lambda$ decreases, the parameter estimates will change more rapidly, but this may not be desirable since a lower $\lambda$ means that potentially important past data is being severely underweighted. The model can then be approximated by

$$
y_i(t) = \Phi_i^T(t - 1) \hat{\Theta}_i + e_i(t).
$$

It should also be noted that it can be advantageous to replace the residual defined in (6.13) with the posterior residual defined as

$$
\varepsilon_i^p(t) = y_i(t) - \Phi_i^T(t - 1) \hat{\Theta}_i(t).
$$

It should be noted that in this development each output was treated to have its own white noise source. However, the algorithm does not change if there is a single white noise source for multiple outputs, as is the case for the WEC discussed in this thesis. This just means that the components are identified independently.
The model above can be converted directly into a discrete-time matrix transfer function and subsequently changed into other forms such as discrete or continuous-time state space models.

The RELS algorithm allows for the entire system estimate to be updated at each sampling period, but it is not recommended that the controller and observer gain be redesigned at each sampling period. This would require a high level of computation as the discrete-time system estimate needs a lot of processing prior to being used in the methods described in this theory. Even then, matrix Riccati equations must be solved. It is recommended that the controller and observer gains are redesigned only periodically, either at a set amount of time, or triggered by changes in performance or system estimate that exceed some threshold. Also, the subsystem $G_i$ may not need to be recursively identified if it is assumed that it is time-invariant. This is a fair assumption unless the WEC is designed to adapt its configuration in addition to its control and observer gains.

There are additional problems that can arise through the use of an indirect STR in conjunction with an RELS algorithm. These include the use of sufficiently large model orders for the system estimates, and also the avoidance of overparameterization. Additionally, simultaneous estimation and control may pose problems as it is possible that a control input can create an output that makes it hard to observe the effects of ocean waves. It may prove to be more practical to only recursively identify the system when while the control input is turned off over relatively short periods of time.
Conclusions

This thesis demonstrated how a WEC can be modeled as a finite-dimensional linear state space model and how that model can then be used to determine the optimal causal linear feedback control strategy for both the full-state feedback and voltage feedback cases. Additionally, the voltage feedback case was compared to the optimal noncausal control found through the use of classical impedance matching, which in a realistic (i.e., stochastic) wave environment, results in an anticausal controller. Such a controller is impractical, since it is anticipatory and requires information regarding future wave excitation to be known.

It has been shown that the determination of the optimal causal controller for wave energy applications distills to a nonstandard $\mathcal{H}_2$ optimal control problem. This problem can be solved provided that the hydrodynamic interaction model can be approximated by a finite-dimensional state space. The example provided in the thesis illustrates that constraining the controller to be causal modifies the power flow for the WEC and generally results in frequency bands where average power generation is negative, i.e., the controller is injecting power into the system. When high quality machines are used, these bands can be quite significant.
Additionally, it was shown that the optimal controller is sensitive to changes in the sea state, specifically the mean wave period. This is the most pronounced for WECs with high quality generators. This provides motivation for the use of adaptive control algorithms. An adaptive control algorithm known as an indirect self-tuning regulator (STR) is proposed.
Appendix A

Static Controller

Following the same assumptions from Section 3.1, let a static controller for the harvester consist only of elements along its diagonal and let those elements have a constant value that does not depend on frequency

\[
Y_s(j\omega) = \begin{bmatrix}
Y_1 & 0 & \cdots & 0 \\
0 & Y_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Y_N
\end{bmatrix}.
\]  

(A.1)

We want to maximize the average power generated which is

\[
\bar{P}_{\text{gen}} = E \left[ -v^T i - i^T R i \right]
\]

(A.2)

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_p(\omega) d\omega,
\]

(A.3)

where the power spectrum is

\[
S_p(\omega) = \mathbf{G}_a(j\omega)^* [\mathbf{I} + \mathbf{G}_i(j\omega) \mathbf{Y}_s]^{-*} [\Re \{\mathbf{Y}_s\} - \mathbf{Y}_s^* \mathbf{R} \mathbf{Y}_s] \\
\times [\mathbf{I} + \mathbf{G}_i(j\omega) \mathbf{Y}_s]^{-1} \mathbf{G}_a(j\omega) \mathbf{S}_a(\omega).
\]

(A.4)

For small enough \(N\) it is simple to just create a mesh of the possible values for \(Y_1, \ldots, Y_N\) and iterate over that mesh to find the values that maximize (A.3). It is
also possible to take advantage of symmetry in the harvester to reduce the dimension of the mesh when it is clear that two or more of the diagonal elements in $Y_s$ will have to be the same due to the resulting structures of $G_a$ and $G_i$. We want to maximize $\bar{P}_{gen}$ because the static controller that accomplishes this should help to bring down the magnitude of the transfer functions for $G_a$ and $G_i$ when added to the system. This should make the finite-dimensional approximation more accurate since any sharp peaks should be mostly eliminated.

It is possible to find the optimal noncausal controller for a system that already includes a static controller $Y_s$. We know that

$$v = G_a a + G_i i.$$  \hfill (A.5)

We have the relationship

$$i = q - Y_s v,$$  \hfill (A.6)

where $q$ is the control input. Therefore

$$v = G_a a + G_i (q - Y_s v).$$ \hfill (A.7)

Rearranging this equation we get

$$[I + G_i Y_s]v = G_a a + G_i q,$$ \hfill (A.8)

which leads to the equation

$$v = [I + G_i Y_s]^{-1}G_a a + [I + G_i Y_s]^{-1}G_i q$$ \hfill (A.9)

$$= G_a f a + G_q q.$$ \hfill (A.10)

For this description of the system, the expression for the average power generation
\[ P_{\text{gen}} = E\{-v^T i - i^T Ri\} \]  
\[ = E\{-v^T (q - Y_s) - (q - Y_s)^T R (q - Y_s)\} \]  
\[ = E\{v^T [Y_s - Y_s^T R Y_s] v - v^T [I - 2 Y_s R] q - q^T R q\}. \]

In this case, the power spectrum for the harvester becomes
\[ S_p = v^* [Y_s - Y_s^T R Y_s] v - v^* [I - 2 Y_s R] q - q^* R q \]
\[ = G_{af}^* [I + G_q F]^{-*} [Y_s - Y_s^T R Y_s] [I + G_q F]^{-1} G_{af} S_a \]
\[ + G_{af}^* [I + G_q F]^{-*} [I - 2 Y_s R] F [I + G_q F]^{-1} G_{af} S_a \]
\[ - G_{af}^* [I + G_q F]^{-*} F^* R F [I + G_q F]^{-1} G_{af} S_a \]
\[ = G_{af}^* [I + G_q F]^{-*} [Y_s - Y_s^T R Y_s + F - 2 Y_s R F - F^* R F] [I + G_q F]^{-1} G_{af} S_a \]
\[ = G_{af}^* [I + G_q F]^{-*} [(F + Y_s) - (F + Y_s)^* R (F + Y_s)] [I + G_q F]^{-1} G_{af} S_a. \]

Following the same development as the derivation for the anticausal impedance matched controller in Section 3.1, we know that the optimal controller is
\[ F_0(j\omega) = [G_q(j\omega)^* + 2 R]^{-1} - Y_s. \]

The transfer functions \( G_{af} \) and \( G_q \) can be converted back to \( G_a \) and \( G_i \) using the equations
\[ G_a = (I + G_q [I - Y_s G_q]^{-1} Y_s) G_{af}, \]
\[ G_i = G_q [I - Y_s G_q]^{-1}. \]

Finite-dimensional approximations for \( G_{af} \) and \( G_q \) can be identified in the same
manner as \( G_a \) and \( G_i \) were in Ch. 4 and can be described by

\[
G_{af} \sim \begin{bmatrix}
A_{af} & B_{af} \\
C_a & 0
\end{bmatrix}
\]  
(A.22)

\[
G_q \sim \begin{bmatrix}
A_q & B_q \\
C_q & 0
\end{bmatrix}.
\]  
(A.23)

In this case the harvester system that is analogous to (4.20) is

\[
\dot{x}_h = Ax_h + Bq + Ea,
\]  
(A.24)

\[
v = Cx_h,
\]  
(A.25)

where

\[
A = \begin{bmatrix}
A_{af} & 0 \\
B_q Y_s C_{af} & A_q + B_q Y_s C_q
\end{bmatrix},
\]  
(A.26)

\[
B = \begin{bmatrix}
0 & B_q^T
\end{bmatrix}^T,
\]  
(A.27)

\[
E = \begin{bmatrix}
B_{af}^T & 0
\end{bmatrix}^T,
\]  
(A.28)

\[
C = \begin{bmatrix}
C_{af} & C_q
\end{bmatrix},
\]  
(A.29)

and the augmented state vector is

\[
x = \begin{bmatrix}
x_a^T & x_i^T
\end{bmatrix}^T.
\]  
(A.30)
Bibliography


