Essays on Markov-Switching Dynamic Stochastic General Equilibrium Models

by

Andrew T. Foerster

Department of Economics
Duke University

Date: __________________________

Approved:

Juan Rubio-Ramirez, Supervisor

Craig Burnside

Francesco Bianchi

Barbara Rossi

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Economics in the Graduate School of Duke University
2011
Abstract

Essays on Markov-Switching Dynamic Stochastic General Equilibrium Models

by

Andrew T. Foerster

Department of Economics
Duke University

Date: ______________________

Approved:

Juan Rubio-Ramirez, Supervisor

Craig Burnside

Francesco Bianchi

Barbara Rossi

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Economics in the Graduate School of Duke University
2011
Abstract

This dissertation presents two essays on Markov-Switching dynamic stochastic general equilibrium models.

The first essay is "Perturbation Methods for Markov-Switching Models," which is co-authored with Juan Rubio-Ramirez, Dan Waggoner, and Tao Zha. This essay develops an perturbation-based approach to solving dynamic stochastic general equilibrium models with Markov-Switching, which implies that parameters governing policies or the environment evolve over time in a discrete manner. Our approach has the advantages that it introduces regime switching from first principles, allows for higher-order approximations, shows non-certainty equivalence of first-order approximations, and allows checking the solution for determinacy. We explain the model setup, introduce an iterative procedure to solve the model, and illustrate it using a real business cycle example.

The second essay considers a model with financial frictions and studies the role of expectations and unconventional monetary policy during financial crises. During a financial crisis, the financial sector has reduced ability to provide credit to productive firms, and the central bank may help lessen the magnitude of the downturn by using unconventional monetary policy to inject liquidity into credit markets. The model allows agents in the economy to expect policy changes by allowing parameters to change according to a Markov process, so agents have expectations about the probability of the central bank intervening during a crisis, and also have expecta-
tions about the central bank’s exit strategy post-crisis. Using this Markov Regime Switching specification, the paper addresses three issues. First, it considers the effects of different exit strategies, and shows that, after a crisis, if the central bank sells off its accumulated assets too quickly, the economy can experience a double-dip recession. Second, it analyzes the effects of expectations of intervention policy on pre-crisis behavior. In particular, if the central bank commits to always intervening during crises, there is a loss of output in pre-crisis times relative to if the central bank commits to never intervening. Finally, it considers the welfare implications of committing to intervening during crises, and shows that committing can raise or lower welfare depending upon the exit strategy used, and that committing before a crisis can be welfare decreasing but then welfare increasing once a crisis occurs.
## Contents

Abstract iv  
List of Tables ix  
List of Figures x  
Acknowledgements xi  

1 Introduction 1  

2 Perturbation Methods for Markov Switching Models 3  

2.1 Introduction ................................................. 3  
2.2 The Model .................................................. 4  
2.3 First Order Approximation ................................. 9  
  2.3.1 Solving for Derivatives of \( x \) ....................... 11  
  2.3.2 Solving for Derivatives of \( \varepsilon \) and \( \chi \) .... 11  
  2.3.3 Non-Certainty Equivalence of First-Order Approximation . . . 15  
2.4 The Solution to the Quadratic System ..................... 18  
2.5 An Example ................................................. 23  
  2.5.1 The Model ............................................. 24  
  2.5.2 Solving the Model .................................... 26  
  2.5.3 Solution ............................................... 29  
  2.5.4 Simulations ........................................... 31  
2.6 Conclusion .................................................. 33
3 Financial Crises, Unconventional Monetary Policy Exit Strategies, and Agents Expectations

3.1 Introduction

3.2 Model
   3.2.1 Households
   3.2.2 Financial Intermediaries
   3.2.3 Credit Policy
   3.2.4 Intermediate Goods Firms
   3.2.5 Capital Producing Firms
   3.2.6 Retail Firms
   3.2.7 Government Policy

3.3 Regime Switching and Equilibrium
   3.3.1 Markov Switching in the Capital Quality Process
   3.3.2 Markov Switching in the Taylor Rule
   3.3.3 Markov Switching in Credit Policy
   3.3.4 Transitions and Timing
   3.3.5 Equilibrium and Model Solution

3.4 Crisis Responses
   3.4.1 Single-Period Crisis
   3.4.2 Multi-Period Crisis

3.5 Pre-Crisis: Effects of Expectations
List of Tables

3.1 Markov Switching Parameters ......................... 81
3.2 Markov Switching Probabilities ....................... 81
3.3 Fixed Parameters .................................... 82
3.4 Regime Switching Parameters ....................... 83
3.5 Normal Regime Conditional Mean, Single-Period Crisis ........... 83
3.6 Normal Regime Conditional Mean, Multi-Period Crisis ........... 84
List of Figures

2.1 Decision Rules, Symmetric Case ......................... 38
2.2 Decision Rules, Non-Symmetric Case ..................... 39
2.3 Simulations, Symmetric Case ............................. 40
2.4 Simulations, Non-Symmetric Case ....................... 41
3.1 Federal Reserve Balance Sheet ........................... 84
3.2 Flow Chart of the Economy ............................... 85
3.3 Single Period Crisis - Responses Under Commitment .... 86
3.4 Single Period Crisis - Exit Strategies .................. 87
3.5 Multi-Period Crisis - Responses Under Commitment ..... 88
3.6 Multi-Period Crisis - Exit Strategies .................. 89
3.7 Single Period Crisis - Effects of Expectations on Pre-Crisis Output .... 90
3.8 Multi-Period Crisis - Effects of Expectations on Pre-Crisis Output .... 91
3.9 Single Period Crisis - Welfare Costs .................... 92
3.10 Multi-Period Crisis - Welfare Costs ................... 93
Acknowledgements

I would like to thank my advisor, Juan Rubio-Ramirez, for his positive influence on me the last several years. He has shown a strong commitment to working with students since arriving at Duke, and my development as an economist has greatly benefitted from this commitment. One way he has emphasized students in the last few years is by setting up optional reading groups on a variety of topics. The chapters of my dissertation are a direct result from two of our reading groups on Markov Switching models and financial friction models. The atmosphere he created made learning and research fun and exciting.

I would also like to thank the other members of my committee, Francesco Bianchi, Craig Burnside, and Barbara Rossi for their instruction and comments throughout my Ph.D., starting from excellent classes and going through the job market process. I’d also like to acknowledge the support of my fellow students who made my time at Duke exciting and productive. In particular, I have greatly benefitted from discussions with Tae Bong Kim and Hernan Seoane, who have been good friends.

I’m also grateful for financial support from the Department of Economics and the Graduate School at Duke. In particular, much of my research was accomplished under summer fellowships from both, and a Conference Travel Grant allowed a trip to the Society for Economic Dynamics meetings to present my work.

Finally, I’d like to thank my parents, Mark and Cathy, and my brother, Rick, for their unconditional support, without which none of this would have been possible.
In recent years, dynamic stochastic general equilibrium (DSGE) models with Markov Switching have become an increasingly popular method for addressing policy or environment changes in a manner consistent with rational expectations equilibria. In Markov Switching models, a subset of parameters change over time, and agents are aware of the probability of changes and their optimizing behavior reflects these expectations.

The following chapters are two separate essays on DSGE models with Markov Switching. The first is a methodological essay, describing a perturbation solution method for a wide class of models, and the second is an application of the methodology to a model with financial crises and unconventional monetary policy.

The first essay is "Perturbation Methods for Markov-Switching Models," which is co-authored with Juan Rubio-Ramirez, Dan Waggoner, and Tao Zha. This essay develops an perturbation-based approach to solving dynamic stochastic general equilibrium models with Markov-Switching, which implies that parameters governing policies or the environment evolve over time in a discrete manner. Our approach has the advantages that it introduces regime switching from first principles, allows for
higher-order approximations, shows non-certainty equivalence of first-order approximations, and allows checking the solution for determinacy. We explain the model setup, introduce an iterative procedure to solve the model, and illustrate it using a real business cycle example.

The second essay considers a model with financial frictions and studies the role of expectations and unconventional monetary policy during financial crises. During a financial crisis, the financial sector has reduced ability to provide credit to productive firms, and the central bank may help lessen the magnitude of the downturn by using unconventional monetary policy to inject liquidity into credit markets. The model allows agents in the economy to expect policy changes by allowing parameters to change according to a Markov process, so agents have expectations about the probability of the central bank intervening during a crisis, and also have expectations about the central bank’s exit strategy post-crisis. Using this Markov Regime Switching specification, the paper addresses three issues. First, it considers the effects of different exit strategies, and shows that, after a crisis, if the central bank sells off its accumulated assets too quickly, the economy can experience a double-dip recession. Second, it analyzes the effects of expectations of intervention policy on pre-crisis behavior. In particular, if the central bank commits to always intervening during crises, there is a loss of output in pre-crisis times relative to if the central bank commits to never intervening. Finally, it considers the welfare implications of committing to intervening during crises, and shows that committing can raise or lower welfare depending upon the exit strategy used, and that committing before a crisis can be welfare decreasing but then welfare increasing once a crisis occurs.
2

Perturbation Methods for Markov Switching Models

2.1 Introduction

Recent analysis of macroeconomic models suggests that across a wide range of applications, finding model parameters that do not vary over time presents a difficult challenge. One way to deal with parameter instability is to break samples into subperiods based upon different parameterizations, and consider separate analysis across these subperiods. Examples of this method include Clarida et al. (2000) and Lubik and Schorfheide (2004), but these fail to consider the fact that agents may be able to expect changes in parameters when forming their expectations.

One way to allow for parameters that change over time is to allow these parameters to follow a Markov process. Using reduced form vector auto-regressions (VARs), Hamilton (1989) and Sims and Zha (2006) consider the possibility of parameter switching. While this approach allows for considering models where parameters change rather than sub-sample analysis, the backward-looking nature of VARs does not allow for the formation of expectations that are potentially important aspects of
parameter instability.

Markov switching dynamic stochastic general equilibrium (DSGE) models, on the other hand, allow for agents in the model to form expectations that include the parameter instability. Papers such as Davig and Leeper (2007), Farmer et al. (2008), and Farmer et al. (2009) develop methods for solving DSGE models when parameters switch over time.

This paper contributes to the literature on Markov switching DSGE models by developing a solution algorithm based upon perturbation methods. In contrast to other methodologies, the one developed here allows for much greater degree of transparency in the nature of regime switching by introducing it from first-principles. Perturbation also allows for higher-order approximations, which are important for capturing the effects of variances, as shown by Schmitt-Grohe and Uribe (2004). Our method also demonstrates that first-order approximations are not certainty equivalence, which implies that the other methods based upon log-linearization may be missing an important aspect of the economy. Finally, our method allows a check for determinacy, which is an important concept in DSGE models with Markov switching.

The remainder of the paper is as follows. In section 2.2, we set up a general class of DSGE models with Markov switching. Section 2.3 demonstrates how to solve the first order approximation, which depends mostly on solving a quadratic system, which we discuss in section 2.4. In section 2.5, we illustrate our method using a simple real business cycle model, and section 2.6 concludes. The appendices have additional material: Appendix A contains additional technical material on the methodology, and Appendix B contains figures.

2.2 The Model

Let us consider a dynamic general equilibrium model in which the parameters follow a discrete state Markov chain indexed by $s_t$ with transition matrix $P = (p_{s,s'})$. The
element $p_{s,s'}$ represents the probability that $s_t = s'$ given $s_t = s$ for $s, s' \in \{1, \ldots, n_s\}$ where $n_s$ is the number of regimes and when $s_{t+1} = s$ we say that the model is in regime $s$. The vector of changing parameters $\theta_t$ at time $t$ is of size $n_\theta \times 1$. Given any $x_{t-1}, \varepsilon_t$, and $\theta_t$, the set of equilibrium conditions of a wide variety of this class of models can be written as

$$\mathbb{E}_t f (y_{t+1}, y_t, x_t, \chi \varepsilon_{t+1}, x_{t-1}, \varepsilon_t, \theta_{t+1}, \theta_t) = 0 \tag{2.1}$$

where $\mathbb{E}_t$ denotes the mathematical expectations operator conditional on information available at time $t$. The vector $x_{t-1}$ of predetermined variables (endogenous and exogenous) is of size $n_x \times 1$, the vector $y_t$ of non-predetermined variables is of size $n_y \times 1$, the vector $\varepsilon_t$ of independent innovations to the exogenous predetermined variables with mean equal to zero is of size $n_\varepsilon \times 1$, and $\chi$ is perturbation parameter. The function $f$ maps $\mathbb{R}^{2(n_y+n_x+n_\varepsilon+n_\theta)}$ into $\mathbb{R}^{n_y+n_x}$. Since the parameters, $\theta_t$, in (2.1) depend on the state of the Markov chain, we have $n_s$ sets of equilibrium conditions, one for each value of the Markov chain, instead of the single set of equilibrium conditions that we have in the constant parameter case.

The solution to the model is of the form

$$y_t \equiv g (x_{t-1}, \varepsilon_t, \chi, s_t) \tag{2.2}$$

$$y_{t+1} \equiv g (x_t, \chi \varepsilon_{t+1}, \chi, s_{t+1}) \tag{2.3}$$

and

$$x_t \equiv h (x_{t-1}, \varepsilon_t, \chi, s_t) \tag{2.4}$$

where $g$ maps $\mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots, n_s\}$ into $\mathbb{R}^{n_y}$ and $h$ maps $\mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots, n_s\}$ into $\mathbb{R}^{n_x}$. We wish to find the Taylor expansion of the functions $g$ and $h$ around the steady state. We consider parameters, $\theta_t$, that depend on the regime in the following

---

1 There may also be a set of non-changing parameters that we do not include in $\theta_t$. 

5
way

\[ \theta_t \equiv \theta(\chi, s_t) \]  

(2.5)

where \( \theta \) maps \( R \times \{1, \ldots, n_s\} \) into \( R^{n_\theta} \) and \( \theta(0, s_t) = \bar{\theta} \) for all \( s_t \). A natural form for the dependence of parameters on the regimes is

\[ \theta_t = \bar{\theta} + \chi \hat{\theta}(s_t), \]  

(2.6)

where \( \bar{\theta} \) denotes the unconditional expectation of \( \theta_t \), or the weighted mean based upon the ergodic distribution of the Markov chain \( s_t \), and \( \hat{\theta}(s_t) \) is the deviation from \( \bar{\theta} \) in regime \( s_t \). In any case, it is important to highlight that functional form (2.6) is not necessary but just convenient for our derivations; any other functional form such that \( \theta(0, s_t) = \bar{\theta} \) for all \( s_t \) holds would also work.

Given this dependence of parameters on the regime, we can define the steady state of the model as vectors \( x_{ss} \) and \( y_{ss} \) such that

\[ f \left( y_{ss}, x_{ss}, 0, x_{ss}, 0, \bar{\theta}, \bar{\theta} \right) = 0 \]

and it is the case that

\[ y_{ss} = g \left( x_{ss}, 0, 0, s_t \right) \]

and

\[ x_{ss} = h \left( x_{ss}, 0, 0, s_t \right) \]

for all \( s_t \).

Using (2.2), (2.4), and (2.5) we can re-write the function \( f \) as

\[ F \left( x_{t-1}, \varepsilon_t, \varepsilon_{t+1}, s_{t+1}, \chi, s_t \right) = \]

\[ f \left( g \left( h \left( x_{t-1}, \varepsilon_t, \chi, s_t \right), \chi \varepsilon_{t+1}, \chi, s_{t+1} \right), g \left( x_{t-1}, \varepsilon_t, \chi, s_t \right) \right) \]

\[ , h \left( x_{t-1}, \varepsilon_t, \chi, s_t \right), \chi \varepsilon_{t+1}, x_{t+1}, \varepsilon_t, \theta \left( \chi, s_{t+1} \right), \theta \left( \chi, s_t \right) \) \]

for all \( x_{t-1}, \varepsilon_t, \varepsilon_{t+1}, s_{t+1} \), and \( s_t \). The function \( F \) maps \( \mathbb{R}^{n_x+2n_x+1} \times \{1, \ldots, n_s\} \times \{1, \ldots, n_s\} \) into \( \mathbb{R}^{n_y+n_x} \).
We assume that innovations to the exogenous predetermined variables, \( \varepsilon_t \), are independent of the Markov chain, \( s_t \), hence, we can re-write (2.1) as

\[
G(x_{t-1}, \varepsilon_t, \chi, s_t) = \sum_{s' = 1}^{n_x} p_{st,s'} \int \mathcal{F}(x_{t-1}, \varepsilon_t, \varepsilon', s', \chi, s_t) \mu(\varepsilon') \, d\varepsilon' = 0 \tag{2.7}
\]

for all \( x_{t-1}, \varepsilon_t, \) and \( s_t \) where \( \mu \) is the density of the innovations. The function \( G \) maps \( \mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots n_s\} \) into \( \mathbb{R}^{n_y+n_x} \).

In what follows we will be using the following notation

\[
D_G(x_{t-1}, \varepsilon_t, \chi, s_t) = [D_{j}G^{i}(x_{t-1}, \varepsilon_t, \chi, s_t)]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq n_x+n_\varepsilon+1}
\]

to refer the \( (n_y + n_x) \times (n_x + n_\varepsilon + 1) \) matrix of partial derivatives of \( G \) with respect to \( (x_{t-1}, \varepsilon_t, \chi) \) evaluated at \( (x_{t-1}, \varepsilon_t, \chi, s_t) \). Note that we do not take derivatives with respect to \( s_t \) because it is a discrete variable. Equivalently,

\[
D_G(x_{ss}, 0, 0, s_t) = [D_{j}G^{i}(x_{ss}, 0, 0, s_t)]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq n_x+n_\varepsilon+1}
\]

refers to the \( (n_y + n_x) \times (n_x + n_\varepsilon + 1) \) matrix of partial derivatives of \( G \) with respect to \( (x_{t-1}, \varepsilon_t, \chi) \) evaluated at \( (x_{ss}, 0, 0, s_t) \). To simply notation, we will use the following definitions

\[
DG_{ss}(s_t) \equiv DG(x_{ss}, 0, 0, s_t)
\]

and

\[
D_{j}G_{ss}^{i}(s_t) \equiv D_{j}G^{i}(x_{ss}, 0, 0, s_t)
\]

for all \( i \) and \( j \) and all \( s_t \). Thus, we will write

\[
DG_{ss}(s_t) = [D_{j}G^{i}(s_t)]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq n_x+n_\varepsilon+1}
\]

for all \( s_t \).

In the same way,

\[
Df_{ss} = [D_{j}f^{i}(y_{ss}, y_{ss}, x_{ss}, 0, x_{ss}, 0, \bar{\theta}, \bar{\theta})]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq 2(n_y+n_x+n_\theta+n_\varepsilon)}
\]
is the \((n_y + n_x) \times (2(n_y + n_x + n_{\theta} + n_{\chi}))\) matrix of partial derivatives of \(f\) with respect to all its components evaluated at \((y_{ss}, y_{ss}, x_{ss}, 0, x_{ss}, 0, \overline{\theta}, \overline{\vartheta})\),

\[
\mathcal{D}g (x_{ss}, 0, 0, s_t) = [\mathcal{D}j g_i^j (x_{ss}, 0, 0, s_t)]_{1 \leq i \leq n_y, 1 \leq j \leq n_y + n_x + n_{\chi} + 1}
\]

is the \(n_y \times (n_x + n_{\chi} + 1)\) matrix of partial derivatives of \(g\) with respect to \((x_{t-1}, \varepsilon_t, \chi)\) evaluated at \((x_{ss}, 0, 0, s_t)\) for all \(s_t\), and

\[
\mathcal{D}h (x_{ss}, 0, 0, s_t) = [\mathcal{D}j h_i^j (x_{ss}, 0, 0, s_t)]_{1 \leq i \leq n_x, 1 \leq j \leq n_y + n_x + n_{\chi} + 1}
\]

is the \(n_x \times (n_x + n_{\chi} + 1)\) matrix of partial derivatives of \(h\) with respect to \((x_{t-1}, \varepsilon_t, \chi)\) evaluated at \((x_{ss}, 0, 0, s_t)\) for all \(s_t\). To simply notation, we will use the following definitions

\[
\mathcal{D}g_{ss} (s_t) \equiv \mathcal{D}g (x_{ss}, 0, 0, s_t),
\]

\[
\mathcal{D}j g_i^j_{ss} (s_t) \equiv \mathcal{D}j g_i^j (x_{ss}, 0, 0, s_t),
\]

\[
\mathcal{D}h_{ss} (s_t) \equiv \mathcal{D}h (x_{ss}, 0, 0, s_t),
\]

and

\[
\mathcal{D}j h_i^j_{ss} (s_t) \equiv \mathcal{D}j h_i^j (x_{ss}, 0, 0, s_t)
\]

for all \(i\) and \(j\) and all \(s_t\). Thus, we will write

\[
\mathcal{D}g_{ss} (s_t) = [\mathcal{D}j g_i^j_{ss} (s_t)]_{1 \leq i \leq n_y + n_x, 1 \leq j \leq n_y + n_x + n_{\chi} + 1}
\]

and

\[
\mathcal{D}h_{ss} (s_t) = [\mathcal{D}j h_i^j_{ss} (s_t)]_{1 \leq i \leq n_y + n_x, 1 \leq j \leq n_y + n_x + n_{\chi} + 1}
\]

for all \(s_t\).
2.3 First Order Approximation

In this section we concentrate on finding the first order Taylor expansions to $g$ and $h$. Thus, we are looking for expansion to functions $g$ and $h$ around the point $(x_{ss}, 0, 0, s_t)$ of the form

$$g(x_{t-1}, x, x, s_t) - y_{ss} \approx [D_{1}g_{ss}(s_t), \ldots, D_{n_x}g_{ss}(s_t)](x_{t-1} - x_{ss})$$

$$+ [D_{n_x+1}g_{ss}(s_t), \ldots, D_{n_x+n_c}g_{ss}(s_t)]\varepsilon_t + D_{n_x+n_c+1}g_{ss}(s_t)\chi$$

and

$$h(x_{t-1}, x, x, s_t) - x_{ss} \approx [D_{1}h_{ss}(s_t), \ldots, D_{n_x}h_{ss}(s_t)](x_{t-1} - x_{ss})$$

$$+ [D_{n_x+1}h_{ss}(s_t), \ldots, D_{n_x+n_c}h_{ss}(s_t)]\varepsilon_t + D_{n_x+n_c+1}h_{ss}(s_t)\chi$$

for all $s_t$ where $D_jg_{ss}(s_t)$ is the $j^{th}$ column vector of $Dg_{ss}(s_t)$ and $D_jh_{ss}(s_t)$ is the $j^{th}$ column vector of $Dh_{ss}(s_t)$. To simply notation, we will use the following definitions

$$D_{n,m}g_{ss}(s_t) \equiv [D_{n}g_{ss}(s_t), \ldots, D_{m}g_{ss}(s_t)]$$

and

$$D_{n,m}h_{ss}(s_t) \equiv [D_{n}h_{ss}(s_t), \ldots, D_{m}h_{ss}(s_t)]$$

for all $n$ and $m$ and all $s_t$.

Hence, we can re-write the above describe approximations as

$$g(x_{t-1}, x, x, s_t) - y_{ss} \approx$$

$$D_{1,n_x}g_{ss}(s_t)(x_{t-1} - x_{ss}) + D_{n_x+1,n_x+n_c}g_{ss}(s_t)\varepsilon_t + D_{n_x+n_c+1}g_{ss}(s_t)\chi$$

and

$$h(x_{t-1}, x, x, s_t) - x_{ss} \approx$$

$$D_{1,n_x}h_{ss}(s_t)(x_{t-1} - x_{ss}) + D_{n_x+1,n_x+n_c}h_{ss}(s_t)\varepsilon_t + D_{n_x+n_c+1}h_{ss}(s_t)\chi$$

The objective is now to find the coefficients

$$\{D_{1,n_x}g_{ss}(s), D_{1,n_x}h_{ss}(s)\}_{s=1}^{n_s}, \{D_{n_x+1,n_x+n_c}g_{ss}(s), D_{n_x+1,n_x+n_c}h_{ss}(s)\}_{s=1}^{n_s},$$

and $$\{D_{n_x+n_c+1}g_{ss}(s), D_{n_x+n_c+1}h_{ss}(s)\}_{s=1}^{n_s}$$
of the above describe expansions. Notice that we need to find a set of $n_s$ policy functions, one for each possible value of the chain, instead of the single set of policy functions that we need to find in the constant parameter case.

The coefficients of these policy functions are going to be obtained by using the fact that

$$G(x_{t-1}, \varepsilon_t, \chi, s_t) = 0$$

for all $x_{t-1}, \varepsilon_t, \chi,$ and $s_t$ and, therefore, it must be the case that

$$DG(x_{t-1}, \varepsilon_t, \chi, s_t) = 0$$

for all $x_{t-1}, \varepsilon_t, \chi,$ and $s_t$ and, in particular,

$$DG_{ss} (s_t) = 0$$

for all $s_t$. Thus, we can write that

$$[D_1 G_{ss} (s_t), \ldots, D_{n_s} G_{ss} (s_t)] = 0,$$  \hspace{1cm} (2.8)

$$[D_{n_x+1} G_{ss} (s_t), \ldots, D_{n_x+n_s} G_{ss} (s_t)] = 0,$$

and

$$D_{n_x+n_s+1} G_{ss} (s_t) = 0$$

for all $s_t$ where $D_j G_{ss} (s_t)$ is the $j^{th}$ column vector of $DG_{ss} (s_t)$. Again, notice that we have a set of $n_s$ derivatives of $G$, one for each possible value of $s_t$, instead of the single derivative that we have in the constant parameter case. To simply notation, again, we define

$$D_{n,m} G_{ss} (s_t) \equiv [D_n G_{ss} (s_t), \ldots, D_m G_{ss} (s_t)]$$

therefore, expression (2.8) can be written as

$$D_{1,n_s} G_{ss} (s_t) = 0, \ D_{n_x+1,n_x+n_s} G_{ss} (s_t) = 0, \text{ and } D_{n_x+n_s+1} G_{ss} (s_t) = 0.$$
2.3.1 Solving for Derivatives of $x$

Using (2.7), we can get the following expression for $\mathcal{D}_{1,n_x}g_{ss} (s_t)$

$$\mathcal{D}_{1,n_x}g_{ss} (s_t) =$$

$$\sum_{s'=1}^{n_s} p_{s_t,s'} \left( \frac{\mathcal{D}_{1,n_y}f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s') \mathcal{D}_{1,n_x}h_{ss} (s_t)}{\mathcal{D}_{n_y+1,2n_y+f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s_t)} + \mathcal{D}_{2n_y+1,2n_y+n_x}f_{ss}\mathcal{D}_{1,n_x}h_{ss} (s_t)} + \mathcal{D}_{2n_y+n_x+1,2n_y+n_x+n_x}f_{ss} \right) \mu (\varepsilon') d\varepsilon' = 0$$

for all $s_t$. Next, taking into account that $\int \mu (\varepsilon') d\varepsilon' = 1$, we have that

$$\mathcal{D}_{1,n_x}g_{ss} (s_t) =$$

$$\sum_{s'=1}^{n_s} p_{s_t,s'} \left( \mathcal{D}_{1,n_y}f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s') \mathcal{D}_{1,n_x}h_{ss} (s_t) + \mathcal{D}_{n_y+1,2n_y+f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s_t)} + \mathcal{D}_{2n_y+1,2n_y+n_x}f_{ss}\mathcal{D}_{1,n_x}h_{ss} (s_t) + \mathcal{D}_{2n_y+n_x+1,2n_y+n_x+n_x}f_{ss} \right) = 0$$

for all $s_t$. Now, noting that $\sum_{s'=1}^{n_s} p_{s_t,s'} = 1$, then

$$\mathcal{D}_{1,n_x}g_{ss} (s_t) = \left( \sum_{s'=1}^{n_s} p_{s_t,s'} \mathcal{D}_{1,n_y}f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s') + \mathcal{D}_{2n_y+1,2n_y+n_x}f_{ss} \right) \mathcal{D}_{1,n_x}h_{ss} (s_t)$$

$$+ \mathcal{D}_{n_y+1,2n_y}f_{ss}\mathcal{D}_{1,n_x}g_{ss} (s_t) + \mathcal{D}_{2n_y+n_x+1,2n_y+n_x+n_x}f_{ss} = 0$$

(2.9)

for all $s_t$.

Putting together the $n_s$ versions of (2.9), one for each value of $s_t$, we have a system of $(n_x + n_y) n_s$ quadratic equations on the same number of unknowns given by $\{\mathcal{D}_{1,n_x}g_{ss} (s), \mathcal{D}_{1,n_x}h_{ss} (s)\}_{s=1}^{n_s}$. In section 2.4 we show how to solve this system.

2.3.2 Solving for Derivatives of $\varepsilon$ and $\chi$

We now show that, once we have found $\{\mathcal{D}_{1,n_x}g_{ss} (s), \mathcal{D}_{1,n_x}h_{ss} (s)\}_{s=1}^{n_s}$, finding

$$\{\mathcal{D}_{n_x+1,n_x+1}g_{ss} (s), \mathcal{D}_{n_x+1,n_x+1}h_{ss} (s)\}_{s=1}^{n_s}$$

and

$$\{\mathcal{D}_{n_x+n_x+1}g_{ss} (s), \mathcal{D}_{n_x+n_x+1}h_{ss} (s)\}_{s=1}^{n_s}$$
is simply solving a system of linear equations. Let us first solve for

\[ \{ D_{n_x+1,n_x+n_c}g_{ss}(s), D_{n_x+1,n_x+n_c}h_{ss}(s) \}_{s=1}^{n_s}, \]

then we will find

\[ \{ D_{n_x+n_c+1}g_{ss}(s), D_{n_x+n_c+1}h_{ss}(s) \}_{s=1}^{n_s}. \]

In order to solve for \( \{ D_{n_x+1,n_x+n_c}g_{ss}(s), D_{n_x+1,n_x+n_c}h_{ss}(s) \}_{s=1}^{n_s} \), we obtain the expressions for \( D_{n_x+1,n_x+n_c}G_{ss}(s_t) \)

\[
D_{n_x+1,n_x+n_c}G_{ss}(s_t) = \sum_{s'=1}^{n_s} p_{s_t,s'} \left( \begin{array}{c}
D_{1,n_y}f_{ss}D_{1,n_x}g_{ss}(s')D_{n_x+1,n_x+n_c}h_{ss}(s_t) + \\
D_{n_y+1,2n_y}f_{ss}D_{n_x+1,n_x+n_c}g_{ss}(s_t) + \\
D_{2n_y+1,2n_y+n_c}f_{ss}D_{n_x+1,n_x+n_c}h_{ss}(s_t) + \\
D_{2(n_y+n_c)+n_c+1,2(n_y+n_c)}f_{ss}
\end{array} \right) \mu (\varepsilon') d\varepsilon' = 0
\]

for all \( s_t \). Taking into account that \( \int \mu (\varepsilon') d\varepsilon' = 1 \) and \( \int \varepsilon' \mu (\varepsilon') d\varepsilon' = 0 \), we have that

\[
D_{n_x+1,n_x+n_c}G_{ss}(s_t) = \left( \begin{array}{c}
D_{1,n_y}f_{ss}D_{1,n_x}g_{ss}(s')D_{n_x+1,n_x+n_c}h_{ss}(s_t) + \\
+D_{n_y+1,2n_y+n_c}f_{ss}\end{array} \right) = 0
\]

for all \( s_t \).

In matrix notation expresion (2.10) can be written as

\[
\begin{bmatrix}
\Theta_{\varepsilon} & \Phi_{\varepsilon}
\end{bmatrix}
\begin{bmatrix}
D_{n_x+1,n_x+n_c}g_{ss}(1) \\
\vdots \\
D_{n_x+1,n_x+n_c}g_{ss}(n_s) \\
D_{n_x+1,n_x+n_c}h_{ss}(1) \\
\vdots \\
D_{n_x+1,n_x+n_c}h_{ss}(n_s)
\end{bmatrix} = \Psi_{\varepsilon}
\]

(2.11)

where

\[
\Theta_{\varepsilon} = \begin{bmatrix}
D_{n_y+1,2n_y}f_{ss} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & D_{n_y+1,2n_y}f_{ss}
\end{bmatrix},
\]

12
expressions for elements of $\Theta$ and $\Psi$ for all $s$ and $t$.

Let us now find $t$ evaluated at $\epsilon \Phi$, that can be solved by inverting $[\Theta_{\epsilon} \Phi_{\epsilon}]$. Thus, given the solution for $\{D_{1,n_x}g_{ss}(s), D_{1,n_x}h_{ss}(s)\}_{s=1}^{n_x}$, we have a system of $(n_x + n_y) n_x n_s$ linear equations, on the same number of unknowns given by the elements of $\{D_{n_x+n_x+n_c}g_{ss}(s), D_{n_x+n_x+n_c}h_{ss}(s)\}_{s=1}^{n_x}$, that can be solved by inverting $[\Theta_{\epsilon} \Phi_{\epsilon}]$.

Let us now find $\{D_{n_x+n_c+1}g_{ss}(s), D_{n_x+n_c+1}h_{ss}(s)\}_{s=1}^{n_x}$. We need to obtain the expressions for $D_{n_x+n_c+1}G_{ss}(s_t) = \sum_{s'=1}^{n_s} p_{s_t,s'} \int D_{1,n_y}f_{ss} \left[ D_{1,n_y}g_{ss}(s') D_{n_x+n_c+1}h_{ss}(s_t) + D_{n_x+n_c+n_c}g_{ss}(s') \zeta' \right] + D_{n_y+n_c+1}f_{ss} D_{n_x+n_c+1}g_{ss}(s_t) + D_{2(n_x+n_c)+1,2(n_y+n_c)+n_c}f_{ss} D_{2(n_x+n_c)+1,2(n_y+n_c)+n_c} D_{n_x+n_c+1}h_{ss}(s_t) + D_{2(n_x+n_c)+1,2(n_y+n_c)+n_c} f_{ss} D_{n_x+n_c+1}h_{ss}(s_t) \right] \mu(\zeta') d\zeta' = 0$

for all $s_t$, where $D \theta (0, s_t)$ is the derivate of $\theta(\chi, s_t)$ with respect to $\chi$

$D \theta (\chi, s_t) = [D_j^{\epsilon} \theta (\chi, s_t)]_{1 \leq i \leq n_y, j=1}$

for all $s_t$ evaluated at $\chi = 0$. 

13
Taking into account that $\int \mu(\varepsilon') d\varepsilon' = 1$ and $\int \varepsilon' \mu(\varepsilon') d\varepsilon' = 0$, we have that

$$D_{n_x+n_x+1} \mathbb{G}_{ss} (s_t) =$$

$$
\left( \sum_{s' = 1}^{n_x} p_{s_t,s'} D_{1,ny} f_{ss} \left\{ D_{1,n_x g_{ss}} (s') D_{n_x+n_x+1} h_{ss} (s_t) + D_{n_x+n_x+1} g_{ss} (s') \right\} \right)
$$

$$D_{n_y+n_y+1} f_{ss} D_{n_x+n_x+1} g_{ss} (s_t) + D_{n_y+n_y+1} f_{ss} D_{n_x+n_x+1} h_{ss} (s_t) +$$

$$\sum_{s' = 1}^{n_x} p_{s_t,s'} D_{2(n_x+n_y+n_x)+1,2(n_x+n_y+n_x)+n_\theta f_{ss} D\theta (0, s')}$$

$$D_{2(n_x+n_y+n_x)+n_\theta+1,2(n_x+n_y+n_x+n_\theta)} f_{ss} D\theta (0, s_t) = 0$$

for all $s_t$.

In matrix notation expression (2.12) can be written as

$$
\begin{bmatrix}
\Theta_X & \Phi_X
\end{bmatrix}
\begin{bmatrix}
D_{n_x+n_x+1} g_{ss} (1) \\
\vdots \\
D_{n_x+n_x+1} g_{ss} (n_x) \\
D_{n_x+n_x+1} h_{ss} (1) \\
\vdots \\
D_{n_x+n_x+1} h_{ss} (n_x)
\end{bmatrix} = \Psi_X
$$

(2.13)

where

$$\Theta_X =$$

$$
\begin{bmatrix}
p_{1,1} D_{1,ny} f_{ss} + D_{n_y+1,2ny} f_{ss} & \cdots & p_{1,n_x} D_{1,ny} f_{ss} \\
\vdots & \ddots & \vdots \\
p_{n_x,1} D_{1,ny} f_{ss} & \cdots & p_{n_x,n_x} D_{1,ny} f_{ss} + D_{n_y+1,2ny} f_{ss}
\end{bmatrix},
$$

$$\Phi_X =$$

$$
\begin{bmatrix}
\sum_{s' = 1}^{n_x} p_{1,s'} D_{1,ny} f_{ss} D_{1,n_x g_{ss}} (s') & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sum_{s' = 1}^{n_x} p_{n_x,s'} D_{1,ny} f_{ss} D_{1,n_x g_{ss}} (s')
\end{bmatrix}
$$

$$+$$

$$\begin{bmatrix}
D_{2ny+1,2ny+n_x} f_{ss} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & D_{2ny+1,2ny+n_x} f_{ss}
\end{bmatrix},
$$

14
Thus, given the solution for \( \{ D_{1,n_{x}g_{ss}}(s), D_{1,n_{x}h_{ss}}(s) \}_{s=1}^{n_{x}} \), we have a system of \((n_{x} + n_{y}) n_{s}\) linear equations, on the same number of unknowns given by the elements of \( \{ D_{n_{x}+n_{c}+1g_{ss}}(s), D_{n_{x}+n_{c}+1h_{ss}}(s) \}_{s=1}^{n_{s}} \), that can be solved by inverting \( \begin{bmatrix} \Theta_{\chi} & \Phi_{\chi} \end{bmatrix} \).

### 2.3.3 Non-Certainty Equivalence of First-Order Approximation

As pointed out by Schmitt-Grohe and Uribe (2004), one important feature of models without Markov switching is certainty equivalence of the first-order approximation. This feature of models implies that first-order approximations are inadequate for analyzing interesting behavior such as response to risk because the approximated decision rules are invariant to changes in volatility. For example, van Binsbergen et al. (2008) and Rudebusch and Swanson (2008) note that at least second-order approximations are needed to analyze some asset pricing implications, such as the yield curve, since second-order approximations are not certainty equivalent, and hence react to changes in volatility. The second-order approximations also imply a degree of difficulty in performing likelihood based estimation, such as Fernández-Villaverde and Rubio-Ramirez (2007) who use the particle filter for estimation. These factors mean that addressing interesting questions with second-order approximations may be necessary but difficult in models without Markov Switching. As we show below, first order approximations to Markov Switching models are not (in general) certain.
equivalence. This nice feature opens the door to analyze risk related behaviors using linearly approximated models.

To see the certainty equivalence of the model without Markov switching, consider equation (2.13) with only one regime, so $n_s = 1$. In this case, we have that

$$
\begin{bmatrix}
\Theta \\

\Phi
\end{bmatrix}
\begin{bmatrix}
D_{n_x+n_x+1}g_{ss} (1) \\
D_{n_x+n_x+1}h_{ss} (1)
\end{bmatrix} = \Psi_

(2.14)
$$

where

$$
\begin{bmatrix}
\Theta \\

\Phi
\end{bmatrix} = \begin{bmatrix}
D_{1,n_y}f_{ss}D_{1,n_x}g_{ss} (1) + D_{2n_y+1,2n_y+n_x}f_{ss}D_{1,n_y}f_{ss} + D_{n_y+1,2n_y}f_{ss}
\end{bmatrix}
$$

and

$$
\Psi_

= - \left( D_{2(n_x+n_y+n_x)+1,2(n_x+n_y+n_x)+n_y}f_{ss}D_\theta (0,1) + D_{2(n_x+n_y+n_x)+n_y+1,2(n_x+n_y+n_x+n_y)}f_{ss}D_\theta (0,1) \right).
$$

Clearly, in the fixed regime case we have that $\theta (\chi,1) = \bar{\theta}$. Therefore, it is the case that $D_\theta (0,1) = 0$, which implies $\Psi_

= 0$. Consequently the system (2.14) is a homogenous one. If a unique system exists, then it is given by

$$
D_{n_x+n_x+1}g_{ss} (1) = D_{n_x+n_x+1}h_{ss} (1) = 0.
(2.15)
$$

Since in the fixed-regime case the only source of uncertainty is $\varepsilon_{t+1}$, solution (2.15) implies that the linear approximation is certain equivalent, i.e.

$$
D_{1,n_x}g_{ss} (1) (x_{ss} - x_{ss}) + D_{n_x+1,n_x+n_x}g_{ss} (1) 0 + D_{n_x+n_x+1}g_{ss} (1) 0 = 0
$$

and

$$
D_{1,n_x}h_{ss} (1) (x_{ss} - x_{ss}) + D_{n_x+1,n_x+n_x}h_{ss} (1) 0 + D_{n_x+n_x+1}h_{ss} (1) 0 = 0.
$$

Now, turning to the case of Markov switching, note that if equation (2.13) is a non-homogenous system, i.e. if $\Psi_

\neq 0$, then it will be the case that $D_{n_x+n_x+1}g_{ss} (s) \neq 0$ and $D_{n_x+n_x+1}h_{ss} (1) \neq 0$ if an unique solution exists.
So, let us analyze when $\Psi_{\chi} \neq 0$. Consider the expression for $\Psi_{\chi}$

$$
\Psi_{\chi} = 
\begin{bmatrix}
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss}\sum_{\sigma=1}^{n_s}p_{1,\sigma}D\theta(0, s') \\
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss}\sum_{\sigma=1}^{n_s}p_{n_x, n_y}D\theta(0, s') \\
\vdots \\
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss}\sum_{\sigma=1}^{n_s}p_{n_x, n_y}D\theta(0, n_s)
\end{bmatrix}
$$

Clearly, if $D\theta(0, s) = 0$ for all $s$, then $\Psi_{\chi} = 0$. So a necessary condition for non-certainty equivalence is that $D\theta(0, s) \neq 0$ for some $s$; this condition simply states that all regimes cannot be identical (which would be equivalent to the fixed regime case).

However, the condition that $D\theta(0, s) \neq 0$ for some $s$ is not sufficient for $\Psi_{\chi} \neq 0$. In addition, it must be the case that either

$$
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss} \neq 0 \text{ or } D_2(n_x+n_y+n_c)+n_{q}f_{ss} \neq 0,
$$

which will be true when the switching parameters do not enter the equilibrium conditions multiplicatively with a variable which expected value equals zero in steady state. In particular, since the innovations to the shocks, $\varepsilon_{t+1}$, have an expected value of zero in steady state, switching the variance of shocks will lead to certainty equivalence because

$$
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss} = 0 \text{ and } D_2(n_x+n_y+n_c)+n_{q}f_{ss} = 0.
$$

In summary, the necessary and sufficient conditions for no certainty equivalence are (i) that $D\theta(0, s) \neq 0$ for some $s$ and (ii)

$$
D_2(n_x+n_y+n_c)+1.2(n_x+n_y+n_c)+n_{q}f_{ss} \neq 0 \text{ or } D_2(n_x+n_y+n_c)+n_{q}f_{ss} \neq 0.
$$
2.4 The Solution to the Quadratic System

As mentioned in section 2.3.1, the \( n_s \) versions of (2.9) form a system of \( (n_x + n_y) n_s n_x \) quadratic equations in the elements of \( \{D_{1,n_x} g_{ss} (s), D_{1,n_x} h_{ss} (s)\}_{s=1}^{n_s} \). We now explain how to find the solution to this system.

Note that we can write (2.9) into matrix form to produce

\[
\begin{bmatrix}
D_{2n_y+1,2n_y+n_x} f_{ss} & p_{s,t,1} D_{1,n_y} f_{ss} & \cdots & p_{s,t,n_x} D_{1,n_y} f_{ss}
\end{bmatrix}
\begin{bmatrix}
D_{1,n_x} g_{ss} (1) \\
\vdots \\
D_{1,n_x} g_{ss} (n_s)
\end{bmatrix}
= -\begin{bmatrix}
D_{2n_y+n_x+1,2n_y+n_x+n_z+n_x} f_{ss} & D_{n_y+1,2n_y} f_{ss}
\end{bmatrix}
\begin{bmatrix}
I \\
D_{1,n_x} g_{ss} (s_t)
\end{bmatrix}
\]

for each \( s_t \).

The idea of the procedure is to find the components of

\[
\{D_{1,n_x} g_{ss} (s), D_{1,n_x} h_{ss} (s)\}_{s=1}^{n_s}
\]

in a iterative way. In order to do so, we propose the following transformation of the above described system

\[
\begin{bmatrix}
D_{2n_y+1,2n_y+n_x} f_{ss} + \sum_{s'=1}^{n_s} p_{s,t,s'} D_{1,n_y} f_{ss} D_{1,n_x} g_{ss} (s') & p_{s,t,s} D_{1,n_y} f_{ss}
\end{bmatrix}
\begin{bmatrix}
I \\
D_{1,n_x} g_{ss} (s_t)
\end{bmatrix}
= -\begin{bmatrix}
D_{2n_y+n_x+1,2n_y+n_x+n_z+n_x} f_{ss} & D_{n_y+1,2n_y} f_{ss}
\end{bmatrix}
\begin{bmatrix}
I \\
D_{1,n_x} g_{ss} (s_t)
\end{bmatrix}
\]

or

\[
A \left( s_t, \{D_{1,n_x} g_{ss} (s')\}_{s'=1}^{n_s} \right) \begin{bmatrix}
I \\
D_{1,n_x} g_{ss} (s_t)
\end{bmatrix}
= B \begin{bmatrix}
I \\
D_{1,n_x} g_{ss} (s_t)
\end{bmatrix}
\]

(2.16)
where

\[ A \left( s_t, \{ \mathcal{D}_{1,n_x} g_{ss} (s') \}_{s' = s_t}^{n_x} \right) = \]

\[ \mathcal{D}_{2ny+1,2ny+n_x} f_{ss} + \sum_{s' = 1, s' \neq s_t}^{n_x} p_{s_t,s'} \mathcal{D}_{1,n_y} f_{ss} \mathcal{D}_{1,n_x} g_{ss} (s') \]

\[ p_{s_t,s_t} \mathcal{D}_{1,n_y} f_{ss} \] \[

and

\[ B = -\left[ \mathcal{D}_{2ny+n_x+n_y+1,2ny+n_x+n_y} f_{ss} \quad \mathcal{D}_{n_y+1,2ny} f_{ss} \right]. \]

Hence, we have \( n_s \) systems like (2.16), one for each \( s_t \). We now provide an outline of the iterative procedure to find the solution. In the next subsection we give a more detailed description of the algorithm.

1. We will start with an initial guess of the solution \( \left\{ \mathcal{D}_{1,n_x} g_{ss} (s), \mathcal{D}_{1,n_x} h_{ss} (s) \right\}_{s=1}^{n_s} \)

where the superindex 0 indicates that we are dealing with a initial guess to the policy functions for all possible values of \( s_t \).

2. The next step is to use \( \left\{ \mathcal{D}_{1,n_x} g_{ss} (s), \mathcal{D}_{1,n_x} h_{ss} (s) \right\}_{s=2}^{n_s} \) (the initial guess of policies for values of the regime 2 or higher) and (2.16) for \( s_t = 1 \) to find a new value for \( \left\{ \mathcal{D}_{1,n_x} g_{ss} (1), \mathcal{D}_{1,n_x} h_{ss} (1) \right\} \) (a new policy for regime 1), where the reader should notice that we have moved the superindex from 0 to 1 to highlight that we are getting new values for the derivatives of the policy functions.

3. Then, we use

\[ \left\{ \left\{ \mathcal{D}_{1,n_x} g_{ss} (1), \mathcal{D}_{1,n_x} h_{ss} (1) \right\}, \left\{ \mathcal{D}_{1,n_x} g_{ss} (s), \mathcal{D}_{1,n_x} h_{ss} (s) \right\}_{s=3}^{n_s} \right\} \]

(the initial guess of policies for values of the regime 3 or higher and the new policy for regime 1) and (2.16) for \( s_t = 2 \) to find a new value for

\[ \left\{ \mathcal{D}_{1,n_x} g_{ss} (2), \mathcal{D}_{1,n_x} h_{ss} (2) \right\} \]

(a new policy for regime 2).
4. Next, we use 
\[ \left\{ \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{n_s}, \left\{ D_1^{[0]} g_{ss} (s), D_1^{[0]} h_{ss} (s) \right\}_{s=1}^{n_s} \right\} \]
(the initial guess of policies for values of the regime 4 or higher and the new policy for regimes 1 and 2) and (2.16) for \( s_t = 3 \) to find a new value for \( \left\{ D_1^{[i]} g_{ss} (3), D_1^{[i]} h_{ss} (3) \right\} \) (a new policy for regime 3).

5. In general, we use
\[ \left\{ \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{\omega}, \left\{ D_1^{[0]} g_{ss} (s), D_1^{[0]} h_{ss} (s) \right\}_{s=\omega+1}^{n_s} \right\} \]
and (2.16) for \( s_t = \omega \) to find \( \left\{ D_1^{[i]} g_{ss} (3), D_1^{[i]} h_{ss} (3) \right\} \).

6. Once we have gotten \( \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{n_s} \) (new policies for all the regimes), we check if
\[ \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{n_s} \]
is different from
\[ \left\{ D_1^{[0]} g_{ss} (s), D_1^{[0]} h_{ss} (s) \right\}_{s=1}^{n_s} \].

7. If they are equal the algorithm stops. If they are different, we will use
\[ \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{n_s} \]
and the steps described above to find \( \left\{ D_1^{[i]} g_{ss} (s), D_1^{[i]} h_{ss} (s) \right\}_{s=1}^{n_s} \). The algorithm will stop when the new policies are not different from the old ones.

Two issues arise with this procedure. First, how to solve (2.16) and second, how to check for determinancy or uniqueness of the solution. Appendix A explain how to find
the new policy functions using (2.16). In a typical model without Markov switching, determinacy is easily verified by checking whether the number of eigenvalues of the system (2.16) inside the unit circle equals to the number of state variables. In a model with Markov switching, as the one described here, the problem is more subtle. As shown in Farmer, Waggoner, and Zha (2009), it could easily be the case that the number of stable eigenvalues associated with each of the regimes is equal to the number of states but the system, as a whole, does not have a stable solution under several concepts of stability. The good news is that for the case of mean-square stability, as defined in Costa et al. (2004), we can check if the Markov switching model is stable. In particular, we would need to check if the following matrix has its eigenvalues inside the unit circle

\[ T = (P' \otimes I_{n_d}) \text{diag} \left[ \mathcal{D}_{1,n_z} h_{ss} \otimes \mathcal{D}_{1,n_z} h_{ss} \right] \]  

(2.17)

where

\[ \text{diag} \left[ \mathcal{D}_{1,n_z} h_{ss} \otimes \mathcal{D}_{1,n_z} h_{ss} \right] = \]

\[
\begin{bmatrix}
\mathcal{D}_{1,n_z} h_{ss} (1) \otimes \mathcal{D}_{1,n_z} h_{ss} (1) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \mathcal{D}_{1,n_z} h_{ss} (n_s) \otimes \mathcal{D}_{1,n_z} h_{ss} (n_s)
\end{bmatrix}.
\]

Thus, when we consider Markov switching we would need to get policy functions \( \{ \mathcal{D}_{1,n_z} h_{ss} (i) \}^{n_s}_{i=1} \) for all possible allocations of eigenvalues between endogenous predetermined and non-predetermined variables and check how many of them are stable under (2.17). If only one policy function is stable then the model only has one stable solution. If more than one are stable, the model has multiple stable solutions. If none are stable, the model has no stable solutions. A more detailed description of how to proceed with the ordering of eigenvalues in Appendix A.

As mentioned, all possible solutions must be constructed and checked for stability. This is a burden that becomes increasingly problematic as the number of
possible solutions grows. The number of possible solutions grows with the number of regimes and the number of variables in an extremely rapid manner. As noted in the Appendix, for each regime, the problem of constructing solutions depends upon the selection of eigenvalues for construction of the matrices \( \{D_{1,n_x} h_{ss} (s)\}_{s=1}^{n_s} \). Each regime has \( n \) eigenvalues, and \( n_x \) must be used for the matrix \( D_{1,n_x} h_{ss} (s) \). Some eigenvalues are required to be selected: those associated with the exogenous variables must be in \( D_{1,n_x} h_{ss} (s) \) by definition. So if \( n_{exog} \) is the number of exogenous variables, then \( n_{endo} = n - n_{exog} \) is the number of endogenous variables, and \( n'_x \) is the number of non-exogenous predetermined variables, then constructing solutions amounts to choosing \( n_x - n'_x \) eigenvalues (those associated with the endogenous predetermined variables) from \( n_{endo} \) possible selections. And this must be chosen for each regime, meaning a total of

\[
n_s \binom{n_{endo}}{n'_x} = n_s \frac{n_{endo}!}{n'_x!(n_{endo} - n'_x)!}
\]

possible solutions.

To reduce the number of stability checks, which can be immensely time consuming for large systems, we consider the following conjecture.

**Conjecture** Rather than considering all possible solutions, it is sufficient to construct those associated with the "marginal switch" of eigenvalues. To do this, order the eigenvalues in increasing order, with the eigenvalues associated with exogenous variables ordered first. Construct all possible solutions that only involve switching the \( n_x \)-th and \( (n_x + 1) \)-th eigenvalues for each regime. If all of these possible solutions produce only one stable solution, that solution is unique, and if they produce multiple solutions, the solution is not unique, and if they produce no solutions, no solution exists.

This conjecture implies that rather than checking the full set of solutions, the number of checks is reduced to \( 2^n_s \) checks, as the "marginal switch" needs to be
made for each of the regimes.

The Formal Algorithm We now provide a more detailed description of our algorithm. In order to do that we need the following definition

\[
D_{\tau,\omega} \equiv \left\{ D_{1, n_s}^\tau g_{ss} (s), D_{1}^\tau h_{ss} (s) \right\}_{\omega}^{\omega \cdot 1}, \left\{ D_{1, n_s}^{\tau - 1} g_{ss} (s), D_{1}^{\tau - 1} h_{ss} (s) \right\}_{s=0+1}^{n_s}
\]

for \( \omega \in \{0, \ldots, n_s\} \), where the reader should notice that \( D_{1}^{\tau \cdot 0} gh = D_{1}^{\tau + 1.0} gh \) for all \( \tau \).

Algorithm 1 Consider a dynamic general equilibrium model in which the parameters follow a discrete state Markov chain as described by (2.1) for all \( s_t \). Consider the system (2.16) for all \( s_t \).

1. Choose a particular ordering of eigenvalues.

2. Let \( \tau = 1 \).

3. Let \( \omega = 0 \).

4. Guess an initial set of policy functions \( D_{1, n_s}^{\tau, \omega} gh \).

5. Use \( D_{1, n_s}^{\tau, \omega} gh \) and (2.16) when \( s_t = \omega + 1 \) and the procedure described in Appendix 2.7 to solve for \( D_{1, n_s}^{\tau, \omega + 1} gh \).

6. Let \( \omega = \omega + 1 \). If \( \omega > n_s \) continue to 7, otherwise go to 5.

7. If \( \| D_{1}^{\tau + 1.0} gh - D_{1}^{\tau \cdot n_s} gh \| < \varepsilon \) stop, otherwise let \( \tau = \tau + 1 \) and \( \omega = 0 \) and go to 5.

2.5 An Example

In this section we present a simple exercise to illustrate the theoretical framework at hand. The perfect vehicle for such pedagogical effort is the real business cycle model. There are two reasons for it.
First, the stochastic neoclassical growth model is the foundation of modern macroeconomics. Even the more complicated New Keynesian models, such as those in Woodford (2003) or Christiano, Eichenbaum, and Evans (2005), are built around the core of the neoclassical growth model augmented with nominal and real rigidities. Thus, once we understood how to deal with Markov switching in our prototype economy, it will be rather straightforward to extend it to richer environments such as the ones commonly used for policy analysis. Second, the model is so well known, its working so well understood, and its computation so thoroughly explored that the role of time-varying volatility in it will be staggeringly transparent.

2.5.1 The Model

We will consider a real business cycle model where growth in total factor productivity follows a Markov process with only two regimes. In particular, the TFP process will follow a random walk in logs with drift that takes one of two levels, which we can think of as being high and low, so the economy experiences high or low growth. The random walk specification helps simplify the number of variables considered in a stationary equilibrium, and is hence the most parsimonious illustrative example. The specification of two regimes will allow a succinct discussion of the methodology, but, as mentioned above, more regimes can be handled easily within the framework.

To get into the substantive questions as soon as possible, our description of the standard features of our prototype economy will be limited to fix notation. There is a representative household in the economy, whose preferences over stochastic sequences of consumption, $c_t$, are representable by a utility function:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$
where $\beta \in (0, 1)$, $\gamma > 0$, $\gamma \neq 1$. The resource constraint is

$$c_t + k_t = z_t k_{t-1}^\alpha + (1 - \delta) k_{t-1}$$

where $k_t$ is capital and the technological change, $z_t$, proceeds according to a random walk in logs with drift where the Markov switching is in the drift, i.e.

$$\log z_t = \mu_t + \log z_{t-1} + \sigma_t \varepsilon_t$$

where the drift can take two values

$$\mu_t = \mu(s_t), \; s_t \in \{1, 2\}$$

and the transition matrix is

$$P = [p_{i,j}]$$

where

$$p_{i,j} = \text{Pr} (s_t = j | s_{t-1} = i) .$$

For this model it is natural to work with the solution to the social planner’s problem. The optimality conditions are standard. Hence, we have

$$c_t^{-\gamma} = \beta \mathbb{E}_t c_{t+1}^{-\gamma} \left( \alpha z_{t+1} k_{t+1}^\alpha + 1 - \delta \right)$$

and

$$c_t + k_t = z_t k_{t-1}^\alpha + (1 - \delta) k_{t-1}.$$ 

Due to the unit root we need to rescale the economy. Thus, we define $\omega_t = \frac{1}{z_{t-1}^\alpha}$, and let $\tilde{c}_t = \frac{c_t}{\omega_t}$, $\tilde{k}_{t-1} = \frac{k_{t-1}}{\omega_t}$, $\tilde{z}_t = \frac{z_t}{z_{t-1}}$. Then the rescaled equilibrium conditions are

$$\tilde{c}_t^{-\gamma} = \beta \mathbb{E}_t \tilde{c}_{t+1}^{-\gamma} \tilde{z}_{t+1}^{\frac{-\gamma}{1-\alpha}} \left( \alpha \tilde{z}_{t+1} \tilde{k}_{t+1}^\alpha + 1 - \delta \right),$$

$$\tilde{c}_t + \tilde{k}_t \tilde{z}_t^{\frac{1}{1-\alpha}} = \tilde{z}_t \tilde{k}_{t-1}^\alpha + (1 - \delta) \tilde{k}_{t-1}.$$
and, 

\[ \log \tilde{z}_t = \mu_t + \sigma \varepsilon_t. \]

Substituting the expression for \( \tilde{z}_t \), the conditions are then 

\[ \tilde{c}_t^{-\gamma} = \beta E_t \tilde{c}_{t+1}^{-\gamma} e^{-\gamma (\mu_t + \sigma \varepsilon_t)} \left( \alpha e^{\mu_{t+1} + \sigma \varepsilon_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - \delta \right) \]

and 

\[ \tilde{c}_t + \tilde{k}_t e^{\mu_t + \sigma \varepsilon_t} = e^{\mu_t + \sigma \varepsilon_t} \tilde{k}_{t-1}^{\alpha} + (1 - \delta) \tilde{k}_{t-1}. \]

Using the notation in section 2.2, in this example we have, \( x_{t-1} = \tilde{k}_{t-1} \), \( y_t = \tilde{c}_t \), and \( \theta_t = \mu_t \), so 

\[ f \left( y_{t+1}, y_t, x_t, \chi \varepsilon_{t+1}, x_{t-1}, \varepsilon_t, \theta_{t+1}, \theta_t \right) = \]

\[ \begin{bmatrix} \tilde{c}_t^{-\gamma} - \beta \tilde{c}_{t+1}^{-\gamma} e^{-\gamma (\mu_t + \sigma \varepsilon_t)} \left( \alpha e^{\mu_{t+1} + \sigma \varepsilon_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - \delta \right) \\ \tilde{c}_t + \tilde{k}_t e^{\mu_t + \sigma \varepsilon_t} - e^{\mu_t + \sigma \varepsilon_t} \tilde{k}_{t-1}^{\alpha} - (1 - \delta) \tilde{k}_{t-1} \end{bmatrix}. \]

Clearly, we have that 

\[ \tilde{c}_t = g \left( \tilde{k}_{t-1}, \varepsilon_t, \chi, s_t \right), \]

\[ \tilde{c}_{t+1} = g \left( \tilde{k}_t, \chi \varepsilon_{t+1}, \chi, s_{t+1} \right), \]

\[ \tilde{k}_t = h \left( \tilde{k}_{t-1}, \varepsilon_t, \chi, s_t \right), \]

and 

\[ \mu_t = \mu \left( \chi, s_t \right) = \mu + \chi \tilde{\mu} (s_t) \]

2.5.2 Solving the Model

In this subsection we show how to solve the model using a first order approximation. First, we need to find the steady state and second we will define the matrices in expression (2.16) that are necessary to search for the policy functions. Finally, we will solve the model.
**Steady State**

In order to calculate steady state, we set $\chi = 0$. Therefore,

$$c_t = c_{t+1} = c_{ss},$$

$$k_{t-1} = k_t = k_{ss},$$

and

$$\mu_{t+1} = \mu_t = \bar{\mu}.$$

So the equilibrium conditions in steady state are

$$\begin{bmatrix} c_{ss} - \beta c_{ss} e^{\frac{\alpha k_{ss}}{1-\alpha}} (\alpha e^\beta k_{ss}^{-1} - 1 - \delta) \\ c_{ss} - k_{ss} e^{\frac{\beta k_{ss}}{1-\alpha}} - e^\beta k_{ss}^{-1} - (1 - \delta) k_{ss} \end{bmatrix} = 0$$

and so the steady state values are

$$k_{ss} = \left( \frac{1}{\alpha e^\beta} \left( \frac{1}{\beta e^{\frac{\alpha}{1-\alpha}}} - 1 + \delta \right) \right)^{\frac{1}{\alpha-1}}$$

and

$$c_{ss} = e^\beta k_{ss}^\alpha + (1 - \delta) k_{ss} - k_{ss} e^{\frac{\mu}{1-\alpha}}.$$

**The Matrices**

The next step is to define the matrices in expression (2.16). To do that we need the derive the derivatives of the function $f$ evaluated at the steady state. Recall in this example that $n_y = 1$, $n_x = 1$, $n_\varepsilon = 1$, and $n_\theta = 1$. The necessary matrices are

$$D_1 f_{ss} = \begin{bmatrix} \gamma c_{ss}^{-1-\gamma} \\ 0 \end{bmatrix}, \quad D_2 f_{ss} = \begin{bmatrix} -\gamma c_{ss}^{-1-\gamma} \\ 1 \end{bmatrix},$$

$$D_3 f_{ss} = \begin{bmatrix} (1 - \alpha) \alpha \beta c_{ss}^{\gamma} e^{\frac{\alpha-1-\gamma}{\alpha-1} k_{ss}^{-2}} \\ e^{\frac{\beta}{1-\alpha}} \end{bmatrix}, \quad D_4 f_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

27
\[
\mathcal{D}_5 f_{ss} = \begin{bmatrix}
0 & c_{ss}^{-1} \gamma \\
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0
\end{bmatrix},
\quad \mathcal{D}_6 f_{ss} = \begin{bmatrix}
0 & c_{ss}^{-1} \gamma \\
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0
\end{bmatrix}.
\]

\[
\mathcal{D}_7 f_{ss} = \begin{bmatrix}
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0 \\
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0
\end{bmatrix}, \quad \text{and} \quad \mathcal{D}_8 f_{ss} = \begin{bmatrix}
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0 \\
-\frac{c_{ss}^{-1} \gamma}{1 - \alpha} & 0
\end{bmatrix}.
\]

Given these derivatives, we can construct the necessary matrices for the solution.

The first sets of matrices are for the quadratic system, they are

\[
A(1, \mathcal{D}_1 g_{ss}(2)) = \begin{bmatrix}
\mathcal{D}_3 f_{ss} + p_{1,2} \mathcal{D}_1 f_{ss} \mathcal{D}_1 g_{ss}(2) & p_{1,1} \mathcal{D}_1 f_{ss}
\end{bmatrix},
\]

\[
A(2, \mathcal{D}_1 g_{ss}(1)) = \begin{bmatrix}
\mathcal{D}_3 f_{ss} + p_{2,1} \mathcal{D}_1 f_{ss} \mathcal{D}_1 g_{ss}(1) & p_{2,2} \mathcal{D}_1 f_{ss}
\end{bmatrix},
\]

and

\[
B = \begin{bmatrix}
-\mathcal{D}_5 f_{ss} & \mathcal{D}_2 f_{ss}
\end{bmatrix}.
\]

The second set of matrices are used for the derivative with respect to \( \varepsilon_t \), and they are

\[
\Theta_\varepsilon = \begin{bmatrix}
\mathcal{D}_2 f_{ss} & 0 \\
0 & \mathcal{D}_2 f_{ss}
\end{bmatrix},
\]

\[
\Phi_\varepsilon = \begin{bmatrix}
\mathcal{D}_1 f_{ss} (p_{1,1} \mathcal{D}_1 g_{ss}(1) + p_{1,2} \mathcal{D}_1 g_{ss}(2)) & 0 \\
0 & \mathcal{D}_1 f_{ss} (p_{2,1} \mathcal{D}_1 g_{ss}(1) + p_{2,2} \mathcal{D}_1 g_{ss}(2))
\end{bmatrix} + \begin{bmatrix}
\mathcal{D}_3 f_{ss} & 0 \\
0 & \mathcal{D}_3 f_{ss}
\end{bmatrix}
\]

and

\[
\Psi_\varepsilon = -\begin{bmatrix}
\mathcal{D}_6 f_{ss} \\
\mathcal{D}_6 f_{ss}
\end{bmatrix}.
\]

The third set of matrices are used for the derivative with respect to \( \chi \), and they are

\[
\Theta_\chi = \begin{bmatrix}
p_{1,1} \mathcal{D}_1 f_{ss} + \mathcal{D}_2 f_{ss} & p_{1,2} \mathcal{D}_1 f_{ss} \\
p_{2,1} \mathcal{D}_1 f_{ss} & p_{2,2} \mathcal{D}_1 f_{ss} \mathcal{D}_2 f_{ss}
\end{bmatrix},
\]
\[ \Phi_X = \begin{bmatrix}
D_{1fss} (p_{1,1} D_{1gss} (1) + p_{1,2} D_{1gss} (2)) & 0 \\
0 & D_{1fss} (p_{2,1} D_{1gss} (1) + p_{2,2} D_{1gss} (2)) 
\end{bmatrix} + \begin{bmatrix}
D_{3fss} & 0 \\
0 & D_{3fss} 
\end{bmatrix}, \]

and

\[ \Psi_X = -\begin{bmatrix}
D_{7fss} (p_{1,1} D\theta (0,1) + p_{1,2} D\theta (0,2)) + D_{8fss} D\theta (0,1) \\
D_{7fss} (p_{2,1} D\theta (0,1) + p_{2,2} D\theta (0,2)) + D_{8fss} D\theta (0,2)
\end{bmatrix}. \]

2.5.3 Solution

Now, to describe the solution, first consider the parameters. We consider the following standard parameters:

\[
\begin{array}{cccccc}
\alpha & \beta & \delta & \gamma & \sigma \\
0.33 & 0.99 & 0.025 & 2 & 0.002
\end{array}
\]

and for the Markov Process, we use

\[
\begin{array}{cccccc}
\mu (1) & \mu (2) & p_{1,1} & p_{2,2} \\
0.03 & 0.01 & 0.90 & 0.90
\end{array}
\]

The transition matrix implies that regimes 1 and 2 occur with equal frequency in the ergodic distribution, so our steady state depends upon \( \bar{\mu} = 0.02 \). The steady state values of capital and consumption are \( k_{ss} = 6.3825 \) and \( c_{ss} = 1.5278 \). Consequently the numerical values of the derivatives are

\[
\begin{align*}
D_{1fss} &= \begin{bmatrix}
0.5608 \\
0
\end{bmatrix}, & D_{2fss} &= \begin{bmatrix}
-0.5608 \\
1
\end{bmatrix}, & D_{3fss} &= \begin{bmatrix}
0.0041 \\
1.0303
\end{bmatrix} \\
D_{4fss} &= \begin{bmatrix}
0 \\
0
\end{bmatrix}, & D_{5fss} &= \begin{bmatrix}
0 \\
-1.0722
\end{bmatrix}, & D_{6fss} &= \begin{bmatrix}
0.0064 \\
0.0397
\end{bmatrix} \\
D_{7fss} &= \begin{bmatrix}
-0.0389 \\
0
\end{bmatrix}, & \text{and } D_{8fss} &= \begin{bmatrix}
1.2789 \\
7.9341
\end{bmatrix}.
\end{align*}
\]
Using these numerical values of the derivatives, we get a first-order solution of

\[ c_t = 1.5278 + 0.1073 (k_{t-1} - 6.3825) + 0.0025 \varepsilon_t + 0.0431 \]

\[ k_t = 6.3825 + 0.9365 (k_{t-1} - 6.3825) - 0.0178 \varepsilon_t - 0.1189 \]

in regime \( s_t = 1 \), and

\[ c_t = 1.5278 + 0.1073 (k_{t-1} - 6.3825) + 0.0025 \varepsilon_t - 0.0431 \]

\[ k_t = 6.3825 + 0.9365 (k_{t-1} - 6.3825) - 0.0178 \varepsilon_t + 0.1189 \]

in regime 2.

As an alternative parameterization, we consider the same parameters above, but with \( p_{1,1} = 0.5 \). In this case, the ergodic distribution across regimes, but instead regime 1 occurs with probability \( \frac{1}{6} \) and regime 2 occurs with probability \( \frac{5}{6} \). Then the steady state has \( \bar{\mu} = 0.0133 \), and we get a first order solution of

\[ c_t = 1.6901 + 0.0833 (k_{t-1} - 9.1087) + 0.0028 \varepsilon_t + 0.0361 \]

\[ k_t = 9.1087 + 0.9487 (k_{t-1} - 9.1087) - 0.0258 \varepsilon_t - 0.2276 \]

in regime \( s_t = 1 \), and

\[ c_t = 1.6901 + 0.0833 (k_{t-1} - 9.1087) + 0.0028 \varepsilon_t - 0.0072 \]

\[ k_t = 9.1087 + 0.9487 (k_{t-1} - 9.1087) - 0.0258 \varepsilon_t + 0.0455 \]

in regime 2.

There are two important properties of these first order solutions. First, for both the first case with a symmetric transition matrix and the second case with a non-symmetric transition matrix, the slope coefficients of the solutions are identical across regimes. Second, the additional constant term at the end of the solution is non-zero, which shows the non-certainty equivalence of the first-order solution,
and its magnitude depends upon the ergodic probabilities. Since the only regime-switching parameter is the level of growth, the only change in the decision rules is through the constant term, which represent deviations from the steady state due to Markov switching. In the symmetric parameterization, each regime occurs with equal probability in the ergodic distribution, so the steady state is exactly between each regime, and hence the deviations are equally above and below. In the non-symmetric transition matrix parameterization, since regime 2 occurs with a higher probability in the ergodic distribution, the additional constants are much smaller for regime 2, demonstrating that the steady state is closer to regime 2.

Figure 2.1 shows the policy functions for each regime when the transition matrix is symmetric if $\varepsilon_t = 0$, alongside the fixed regime case, which is no Markov switching but with TFP growth always at $\bar{\mu}$. The plot shows how the policy functions with Markov switching have identical slopes to those without switching, but the constant term associated with Markov switching scales the functions up and down. In the case with a symmetric transition matrix and hence equal ergodic probabilities, the fixed regime case lies exactly between the two lines when there is switching.

Figure 2.2 shows the policy functions for the non-symmetric transition matrix case. Again, this figure shows that the slopes are the same, but the Markov switching rules are scaled up and down by a constant. Since regime 2 occurs with higher probability in the ergodic distribution, the fixed regime policy function is very close to that for regime 2, while regime 1 is farther away.

2.5.4 Simulations

To illustrate how Markov switching can play a role in growth dynamics, especially through the non-certainty equivalence of the first-order approximation, we now turn to simulating the models discussed above and investigating their ergodic distributions. For both the symmetric and non-symmetric transition matrices, we simulate
the economy 1000 times for a length of 10000 periods, throwing out the first 1000 to eliminate the effects of initial conditions.

Figure 2.3 shows the simulated distributions of output and consumption growth for the symmetric transition matrix economy. Recall that in this specification, both regimes are equally highly persistent, so in the ergodic distribution, both occur with equal probability. While the fixed regime case has a single-peaked distribution, thereby exhibiting growth at close to a constant rate, the switching case has a twin-peaked distribution for both variables, one peak associated with each regime. The parameterization for the fixed regime case suggests that should be halfway between the growth rates of the two regimes, but simulations show that growth is higher on average in the switching case than the fixed regime case. This result follows from the non-certainty equivalence of the solution; when there is switching between high and low growth regimes, agents understand that they will experience both regimes, and, on average, this decision leads to higher consumption and output growth than if there was only a single regime.

Figure 2.4 shows the simulated distributions of output and consumption growth for the case of the non-symmetric transition matrix. In this case, regime 2 occurs much more often and is more persistent than regime 1, so the ergodic distribution has higher probability on regime 2. Again, the fixed regime case exhibits almost constant growth: the distribution is single-peaked. The Markov switching case, on the other hand, is no longer twin-peaked. In this case, there is one dominant peak of the distribution, which is associated with regime 2, but there are also several other smaller peaks to the distribution that correspond to different histories of the regimes. For example, the large peak is a result of regime 2 occurring approximately 5/6 of the time, but there will be long periods where only regime 2 occurs, and the small left-most peak is associated with these stretches. The other smaller peaks correspond to various lengths of regime 1 occurring, which happens with lower probability. As
in the symmetric case, the ergodic mean of growth in the fixed case is lower than when there is switching, again this is because of the lack of certainty equivalence in the two regimes.

2.6 Conclusion

This paper has developed a perturbation methodology for solving dynamic stochastic general equilibrium models with Markov switching. Our methodology has the advantage that it starts from first principles. Also, while the focus has been on first-order approximations, perturbation allows for higher-order approximations, which may be useful in capturing effects of uncertainty over the exogenous shocks. As in Schmitt-Grohe and Uribe (2004), finding second- and higher-order approximations is straightforward. Importantly, our method also shows non-certainty equivalence of the first-order approximation, which is a feature that other solution methods omit. Also, we allow for checking for determinacy, which an important feature of Markov switching models.

We illustrated our methodology using a simple real business cycle model, but there are many interesting applications, including switches in monetary or fiscal policy. One important avenue for future work is to evaluate the accuracy of perturbation in various contexts. Since perturbation is by definition an approximation to the solution of the economy, quantifying how accurate the approximation is in different applications is an important next step.

2.7 Appendix A

This appendix explains how to solve the system given in (2.16) for each $s_t$. Again, the system is given by

$$ A \left( s_t, \{ \mathcal{D}_{1,n_x} g_{ss} (i) \}_{i=1,i \neq s_t}^{n_x} \right) \left[ \mathcal{D}_{1,n_x} h_{ss} (s_t) \right]_{s_t} = B \left[ \mathcal{D}_{1,n_x} g_{ss} (s_t) \right]_{s_t}. $$
As discussed previously, given values for \( \{D_{1,n_x gss (i)}\}_{i=1; i \neq s_t}^{n_s} \), this system for \( s_t \) is quadratic with \((n_x + n_y) n_x\) equations, with the \((n_x + n_y) n_x\) unknowns

\[
\{D_{1,n_x hss (s_t)}, D_{1,n_x gss (s_t)}\}
\]

. To simplify notation for now, we will drop the dependence on \( s_t \) \( \{D_{1,n_x gss (i)}\}_{i=1; i \neq s_t}^{n_s} \) and denote the system

\[
A \begin{bmatrix} I & \nabla \end{bmatrix} D_{1,n_x hss (s_t)} = B \begin{bmatrix} I & \nabla \end{bmatrix} D_{1,n_x gss (s_t)}.
\]

(2.18)

To solve this system, we will either use the Eigenvalue Decomposition of \((A, B)\).

### 2.7.1 Eigenvalue Decomposition

This subsection describes how to solve the quadratic equation (2.18) by using the Eigenvalue Decomposition of the matrix pair \((A, B)\).

The Jordan Decomposition says that the square matrix \( D_{1,n_x hss (s_t)} \) can be decomposed into it’s Jordan form with a block diagonal matrix of eigenvalues \( \Lambda \) with a matrix of corresponding eigenvectors \( \Gamma \) so that \( D_{1,n_x hss (s_t)} \Gamma = \Gamma \Lambda \). Define

\[
Z = \begin{bmatrix} I & \nabla \end{bmatrix} D_{1,n_x gss (s_t)} \Gamma.
\]

so the expression (2.18) becomes, after postmultiplying by \( \Gamma \):

\[
AZ\Lambda = BZ.
\]

The Eigenvalue Decomposition for matrices \((A, B)\) says that these matrices can be block diagonal matrix \( \Delta \) with the generalized eigenvalues of \((A, B)\) and a matrix \( V \) of corresponding eigenvectors such that

\[
A \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix} = B \begin{bmatrix} V_1 & V_2 \end{bmatrix}.
\]

34
Note here that $\Delta$ and $V$ are not unique in the sense that the eigenvalues can be reordered along the diagonal – and consequently between $\Delta_{11}$ and $\Delta_{22}$ – so long as the columns of $V$ are reordered in the exact same fashion. For our purposes, the only distinction that matters is the partition of eigenvalues into $\Delta_{11}$ or $\Delta_{22}$.

Suppose for now the eigenvalues are ordered in some given manner. Then we can set

$$\Lambda = \Delta_{11}$$

and by the definition of $Z$,

$$\begin{bmatrix} I \\ D_{1,n_x gss}(s_t) \end{bmatrix} \Gamma = V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$$

and so

$$D_{1,n_x h_{ss}}(s_t) = V_{11} \Delta_{11} V_{11}^{-1}$$

and

$$D_{1,n_x g_{ss}}(s_t) = V_{12} V_{11}^{-1}.$$  

### 2.7.2 Ordering of Eigenvalues

The previous subsection described how to solve the quadratic system (2.18) conditional on an ordering of the generalized eigenvalues of $(A, B)$. Now, we consider possible orderings of these eigenvalues.

Since $A$ and $B$ are dimension $n \times n$, there are $n$ eigenvalues, with any imaginary eigenvalues necessarily appearing as conjugate pairs. Also, as noted above, the exact ordering of the eigenvalues does not matter, but only the partition of eigenvalues into the matrices $\Delta_{11}$ and $\Delta_{22}$, so we can view the problem as one of allocating $n_x$ eigenvalues to $\Delta_{11}$.

Note that $\Delta_{11}$ is used to construct $D_{1,n_x h_{ss}}(s_t)$, which governs the evolution of the predetermined - both exogenous and endogenous - variables. Let $n_{exog}$ be the
number of exogenous predetermined variables, and \( n_{\text{endo}} \) be the number of endogenous variables (both predetermined and nonpredetermined). Then \( n_{\text{exog}} + n_{\text{endo}} = n \), and \( n_{\text{exog}} \leq n_x \). Define \( n'_x = n_x - n_{\text{exog}} \) to be the number of endogenous predetermined variables. Clearly, since the exogenous variables are associated with the predetermined variables, the eigenvalues associated with these exogenous variables must be in the set of \( \Delta_{11} \). Then we can further split \( \Delta_{11} \) so that \( \Delta \) is given by

\[
\Delta = \begin{bmatrix}
\Delta_{111} & 0 & 0 \\
0 & \Delta_{112} & 0 \\
0 & 0 & \Delta_{22}
\end{bmatrix}
\]

where \( \Delta_{111} \) is size \( n_{\text{exog}} \times n_{\text{exog}} \), \( \Delta_{112} \) is size \( n'_x \times n'_x \), and \( \Delta_{22} \) is size \( n_y \times n_y \). The focus is now on ordering the remaining (endogenous) eigenvalues between \( \Delta_{112} \) and \( \Delta_{22} \). This problem can be expressed as allocating \( n_{\text{endo}} \) eigenvalues to the \( n_x - n_{\text{exog}} \) slots of \( \Delta_{112} \), of which there are

\[
\binom{n_{\text{endo}}}{n'_x} = \frac{n_{\text{endo}}!}{n'_x! (n_{\text{endo}} - n'_x)!}
\]

possible combinations. We will facilitate the notation for choice of eigenvalues by defining a set of selection vectors

\[
V = \left\{ v : \dim(v) = n_{\text{endo}}, v_i \in \{0, 1\}, \sum_i v_i = n'_x \right\}
\]

so \( V \) is the set of vectors of length \( n_{\text{endo}} \), with individual elements being either 0 or 1, with elements that sum to \( n'_x \). Let \( n_v \) be the number of selection vectors, and suppose \( V \) is ordered in some manner with \( v^1 \) being the first element, \( v^2 \) the second, etc.

Consider first ordering the remaining eigenvalues by increasing absolute value, with eigenvalues \( \lambda_i, i \in \{1, \ldots, n_{\text{endo}}\} \). For some vector \( v \in V \), we will allocate the eigenvalue \( \lambda_i \) to \( \Delta_{112} \) if and only if \( v_i = 1 \). Consequently, to construct all
possible solutions, we will loop over the selection vectors \( v(s_t) \in \mathbb{V} \) for each \( s_t \). The purpose of the algorithm is to construct a solution for each possible selection vector for each possible regime, and then check mean square stability of that solution. So to construct all solutions, we will consider

\[
(v(1), ... v(n_s)) \in \left\{ \left(v^{k_1}, ..., v^{k_{n_s}}\right) \right\}_{k_1=1, ..., k_{n_s}=1}^{n_v}
\]

For each series of selection vectors, use Algorithm 1 to construct the solution, and then use (2.17) to check for MSS of the solution. An algorithm is as follows

**Algorithm 2** Take the following steps

1. Set \( (v(1), ..., v(n_s)) = (v^1, ..., v^1, v^1) \).

2. Solve the system using Algorithm 1. In particular, at every iteration, for the system of for regime \( s_t \), order the non-predetermined eigenvalues in increasing absolute value and then allocate the ones chosen by selection vector \( v(s_t) \) to \( \Lambda_{112} \).

3. Check the MSS of the solution from step 2 using the eigenvalues of (2.17).

4. Set \( (v(1), ..., v(n_s)) = (v^1, ..., v^1, v^2) \) and return to 2. Continue in this way until the last set of selection vectors \( (v^{n_v}, ..., v^{n_v}, v^{n_v}) \) is reached. Stop if two MSS solutions are found, which implies indeterminacy.

2.8 Appendix B: Figures
Figure 2.1: Decision Rules, Symmetric Case
Figure 2.2: Decision Rules, Non-Symmetric Case
Figure 2.3: Simulations, Symmetric Case
Figure 2.4: Simulations, Non-Symmetric Case
3

Financial Crises, Unconventional Monetary Policy
Exit Strategies, and Agents Expectations

3.1 Introduction

In the fall of 2008, the US economy experienced a financial crisis, which was marked by a rapid slowing of real economic activity, along with a deterioration in financial conditions. At the same time, the Federal Reserve expanded its purchases of financial assets in order to inject additional capital into the economy. The increase in demand for financial assets provided by the Federal Reserve helped bolster asset values and alleviate some of the pressure on financial institutions by lessening the drop in the value of assets on their respective balance sheets. The Federal Reserve accomplished this expansion in asset purchases by instituting a number of new lending facilities, such as expanding its purchase of mortgage backed securities and commercial paper. In total, the size of the Federal Reserve’s balance sheet grew by over $1 trillion. Figure 3.1 shows the sizeable increase in this balance sheet along with a measure of interest rate spreads that jumped during the crisis showing the increased level of uncertainty during that time.
An additional feature of the financial crisis and credit market intervention is that even after the crisis ended and interest rate spreads came down from their peak, the size of the Federal Reserve’s balance sheet remained elevated and currently remains high. In other words, the financial crisis triggered a start in unconventional monetary policy, but the end of the crisis did not trigger an end in the unconventional policy. Consequently, it remains to be seen how the Federal Reserve will unwind the size of its balance sheet, and what the effects of this unwind are for the macroeconomy. Current debate has focused on how long the Federal Reserve should hold its accumulated assets, when to start selling them off, and at what rate.

In addition to exit strategies, given that the Federal Reserve has intervened in credit markets during this latest crisis, one concern is how expectations about intervention policy during crises affects pre-crisis economic behavior. That is, if the central bank is expected to intervene during crises, does this expectation distort economic outcomes prior to a crisis occurring, and if so, what are the magnitudes of those distortions? Many of the concerns about intervention during the crisis revolved around the concern of setting a precedent, and that this precedent would have negative repercussions during non-crisis times by encouraging reckless risk-taking. Even if intervention is considered good policy during crises, if setting a precedent of intervention has negative effects during non-crisis times, it may be that this precedent is on whole a poor policy choice. On the other hand, if expectations of intervention allow agents in the economy to be less worried about small probability events and allow credit to flow more freely, then guaranteeing intervention may be an entirely positive policy choice.

This consideration of the effects of intervention expectations along with the effects during crises motivates an analysis of the welfare benefits of commitment to intervention during crises. The main concern with this welfare analysis is a form of time inconsistency of commitment: it may be the case that ex-ante – that is,
before a crisis occurs – committing to intervention is welfare decreasing relative to committing to no intervention, but ex-post – when a crisis occurs – committing to intervention is welfare increasing over a no intervention commitment.

This paper addresses these questions about exit strategies, effects of pre-crisis expectations, and welfare costs by building a dynamic stochastic general equilibrium (DSGE) model with a financial sector where financial crises occasionally occur, and then conditional on a crisis occurring, the central bank may or may not intervene in credit markets. If the central bank does intervene, it will not do so forever, but instead, at some point it will start exiting the credit market, selling off its accumulated assets at a specified rate. Using Markov Regime Switching, the model developed in this paper allows agents to have rational expectations about transitions between regimes where the central bank intervenes and does not. This Markov Switching framework then allows the study of exit strategies after intervention occurs, the effects of expectations on pre-crisis economic activity, and the welfare gain or loss from different policy commitments.

There has been a rapidly growing literature on the implications of financial frictions in the macroeconomy. Many DSGE models, such as Christiano and C. (2005) and Wouters (2007), do not incorporate a financial sector, and are therefore unable to explain movements associated with the banking system. A standard framework to incorporate a financial sector is to use a financial accelerator model, as developed in Bernanke and Gertler (1989), Kiyotaki and Moore (1997), and Bernanke et al. (1999)), which allows for frictions in the financial sector that slow the flow of funds from households to firms. Gertler and Karadi (2010) build upon the financial accelerator literature by incorporating a central bank equipped with a mechanism to intervene in credit markets during crises, and show that intervention can lessen the magnitude of downturns associated with financial crises. Other models that allow for financial frictions or government intervention during crises are Brunnermeier and
Sannikov (2009), Christiano et al. (2009), or Del Negro et al. (2010).

Many of the papers that consider government intervention during financial crises lack the expectations and transitions between the intervention and no intervention regimes that are included in this paper. When expectations and transitions are ignored, any change in policy is entirely unexpected and considered permanent. Therefore, without the regime switching introduced in this paper, the effects of exit strategies and pre-crisis expectations have to be ignored as well. Following the rare event literature (Rietz (1988), Barro (2006), and Barro (2009)) this paper allows financial crises to occur with a small probability. Agents form expectations over the central bank’s decision to intervene conditional upon that rare even occurring. However, as in Barro et al. (2010), the model also allows crises to be persistent – that is, to last several periods before ending – and studies the implications of uncertain crisis duration. This uncertainty over crisis duration may have implications for the magnitude of the drop in real activity: if agents are uncertain about how long asset prices will remain suppressed, the economy may not rebound as quickly as if agents know that the crisis will be brief.

Markov Switching in government policies has become a popular way to model discrete changes in government policy that are expected with some probability. Perhaps the most widely used application is changing monetary policy rules, such as Davig and Leeper (2007), Farmer et al. (2008), and Bianchi (2009). With Markov Switching, since policy changes are expected with some probability, expectations over future policy rules affect current dynamics of the economy. For example, in the context of switches in monetary policy, expected changes in the inflation target or sensitivity to inflation can affect current inflation. In this paper, the probability of changing to a regime where the central bank intervenes in credit markets can affect pre-crisis dynamics, and expectations on exit strategies can affect effectiveness in the initial portion of the crisis. Foerster et al. (2011) develop perturbation methods
for Markov Switching models, which allows a high degree of flexibility in the modelling of the regime switching in order to capture the various specifications of regime switching to be considered here.

The paper proceeds as follows. Section 3.2 discusses the model, with special emphasis on the financial sector. Section 3.3 details how the parameters of the economy change according to a Markov Process, and the transitions between regimes. The response of the economy to crises with and without intervention is discussed in Section 3.4, as are the effects of different exit strategies. Section 3.5 analyzes the effects of expectations of crisis policies on the pre-crisis economy. Section 3.6 discusses the welfare implications of policy announcements, and Section 3.7 concludes. All tables and plots are included in the Appendices.

3.2 Model

This section describes the basic model, which is based upon that used in Gertler and Karadi (2010). It is a standard DSGE model, similar to Christiano and C. (2005), or Wouters (2007), with the addition of a financial sector. The purpose of the financial sector is to serve as an intermediary between households and nonfinancial firms, channeling funds from households to the firms.

At this point in time, the parameters that will change with regime switching are described simply as time-varying parameters. The next section will describe regime-switching in more detail.

3.2.1 Households

The economy is populated by a continuum of households of unit measure. These households consume, supply labor, and save by lending money to financial intermediaries or potentially to the government.

Each household is comprised of a fraction \((1 - f)\) of workers, and a fraction \(f\) of
bankers. Workers supply labor to nonfinancial firms, earning wages for the household in the process. Each banker is the owner of a financial intermediary that returns its earnings to the household. Bankers become workers with probability \((1 - \theta)\), so a total fraction of \((1 - \theta) f\) transition to become workers; the same fraction transition from being workers to being bankers to keep the measure of each type constant. In addition, the probability of transitioning occupations is independent of duration. Upon exit, bankers transfer their net worth to the household, and new bankers receive some initial funds from the household. Within the household, there is perfect consumption insurance.

The households maximize their lifetime utility function

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log (C_t - hC_{t-1}) - \frac{\kappa}{1 + \varphi} L_{t}^{1+\varphi} \right\}
\]  

(3.1)

where \(E_0\) is the conditional expectations operator, \(\beta \in (0, 1)\) is the discount factor, \(C_t\) is household consumption at time \(t\), \(h\) controls the degree of habit formation in consumption, \(L_t\) is household labor supply, \(\kappa\) controls the disutility of labor, and \(\varphi\) is the inverse of the Frisch labor supply elasticity.

As previously noted, workers earn a wage \(W_t\) on their labor supplied \(L_t\), there is an amount \(\Delta_t\) of net profits from financial and nonfinancial firms, which is profits and banker earnings returned to the household from exiting bankers less some start-up funds for new bankers, and lump sum transfers \(T_t\). Households save by purchasing bonds \(B_t\) either from financial intermediaries or the government, these bonds pay a gross real return of \(R_t\) in period \(t + 1\). In equilibrium, both sources of bonds are riskless and hence identical from the household’s perspective, so \(R_t\) is the risk-free rate of return. Households then have income \(R_{t-1}B_{t-1}\) from bonds. Consequently, the household’s budget constraint is given by

\[
C_t + B_t = W_t L_t + \Delta_t + T_t + R_{t-1}B_{t-1}.
\]  

(3.2)
Using a multiplier \( \varrho_t \) on the budget constraint, the household’s optimality conditions are given by an equation for the marginal utility of consumption

\[
(C_t - hC_{t-1})^{-1} - \beta hE_t (C_{t+1} - hC_t)^{-1} = \varrho_t,
\]

(3.3)

one for the marginal utility of savings

\[
\beta R_t E_t \frac{\varrho_{t+1}}{\varrho_t} = 1,
\]

(3.4)

and equation for the marginal disutility of labor

\[
\kappa L_t^p = \varrho_t W_t.
\]

(3.5)

### 3.2.2 Financial Intermediaries

The purpose of financial intermediaries is to serve as a channel for funds between the households and nonfinancial firms. Financial intermediaries, which are indexed by \( j \), accumulate net worth \( N_{j,t} \) and collect deposits from households \( B_{j,t} \). Using these two sources of funding, they purchase claims on non-financial firms \( S_{j,t} \) which have relative price \( Q_t \). The intermediaries’ balance sheet dictates that the overall value of claims on non-financial firms must equal the value of the intermediaries net worth plus deposits, so

\[
Q_t S_{j,t} = N_{j,t} + B_{j,t}.
\]

(3.6)

As discussed in the previous subsection, in period \( t + 1 \) deposits made by the household at time \( t \) pay a risk-free rate \( R_t \). The claims on non-financial firms purchased at time \( t \), on the other hand, pay out at \( t + 1 \) a stochastic return of \( R_{t+1}^k \). The evolution of net worth is the difference in interest received from non-financial firms and interest paid out to depositors:

\[
N_{j,t+1} = R_{t+1}^k Q_t S_{j,t} - R_t B_{j,t}
\]

(3.7)

\[
= (R_{t+1}^k - R_t) Q_t S_{j,t} + R_t N_{j,t}.
\]
Hence, the intermediary’s net worth will grow at the risk-free rate, with any growth above that level being the excess return on assets \((R_{t+1}^k - R_t) Q_{t+j,t}\). Faster growth in net worth therefore must come from higher realized interest rate spreads \(R_{t+1}^k - R_t\) or an expansion of assets \(Q_{t+j,t}\).

Given that the evolution in net worth depends upon the interest rate spread, a banker will not fund assets if the discounted expected return is less than the discounted cost of borrowing. So, the banker’s participation constraint is

\[
\mathbb{E}_t \beta^{i+1} \frac{\theta_{t+1+i}}{\theta_t} (R_{t+1+i}^k - R_{t+i}) \geq 0, \text{ for } i \geq 0, \tag{3.8}
\]

where \(\beta^{i+1} \frac{\theta_{t+1+i}}{\theta_t}\) is the stochastic discount factor applied to returns in period \(t + 1 + i\). Note here the inequality is a key differential with financial frictions. In a standard economy without constrained financial intermediaries this participation constraint exactly binds by no arbitrage. In a model with financial frictions, financial intermediaries may be unable to take advantage of positive expected interest rate spreads due to constraints on their borrowing.

As noted previously, each period bankers exit the financial intermediary sector and become standard workers with probability \((1 - \theta)\). This probability limits the lifespan of bankers, and hence eliminates their ability to build up net worth to a point where they would forego deposits and fund their purchase of claims entirely out of their own net worth. If the participation constraint (3.8) holds, a banker has incentive to accumulate as much net worth as possible upon exit. The banker’s objective function is to maximize the present value of their net worth at exit. The expected terminal net worth is then

\[
V_{j,t} = \mathbb{E}_t (1 - \theta) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\theta_{t+1+i}}{\theta_t} N_{j,t+1+i} \tag{3.9}
\]

\[
= \mathbb{E}_t (1 - \theta) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\theta_{t+1+i}}{\theta_t} \left( (R_{t+1+i}^k - R_{t+i}) Q_{t+i+j,t+i} + R_{t+i} N_{j,t+i} \right)
\]
This expression shows that, following from the expression (3.7) describing growth in net worth, the value of being a financial intermediary is increasing in expected future interest rate spreads, \( (R_{t+1}^k - R_t) \), future asset levels \( Q_{t+i}S_{j,t+i} \), plus the risk-free return on net worth.

Now, to formulate the expected terminal net worth (3.9) in terms of a banker’s current position, define the growth of assets from \( t-1 \) to \( t \) as

\[
x_t = \frac{Q_t S_{j,t}}{Q_{t-1} S_{j,t-1}}
\]  

and the growth of net worth from \( t-1 \) to \( t \) as

\[
z_t = \frac{N_{j,t}}{N_{j,t-1}}.
\]

Then (3.9) can be expressed as a function of current assets \( Q_t S_{j,t} \) and net worth \( N_{j,t} \) as

\[
V_{j,t} = v_t Q_t S_{j,t} + \eta_t N_{j,t}
\]

where the discounted marginal gain from expanding assets \( Q_t S_{j,t} \) is given by \( v_t \):

\[
v_t = \mathbb{E}_t \left[ (1 - \theta) \beta \frac{\partial x_t}{\partial t} \left( R_{t+1}^k - R_t \right) + \beta \theta \frac{\partial x_t}{\partial t} x_{t+1} v_{t+1} \right]
\]

and the discounted marginal gain from expanding net worth \( N_{j,t} \) is given by \( \eta_t \):

\[
\eta_t = \mathbb{E}_t \left[ (1 - \theta) \beta \frac{\partial z_t}{\partial t} R_t + \theta \beta \frac{\partial z_t}{\partial t} z_{t+1} \eta_{t+1} \right]
\]

In a frictionless environment, if the expected interest rate spread \( (R_{t+1}^k - R_t) \) is positive, financial intermediaries will want to expand their assets infinitely by borrowing additional funds from the household. However, consider the following constraint that will limit a banker’s ability to expand without limit. In each period, the banker can divert a fraction \( \lambda \) of its assets \( Q_t S_{j,t} \) back to the household. If the
does choose to divert assets, depositors are able to recover the remaining fraction \((1 - \lambda)\) of assets. Consequently, the incentive constraint for the banker requires that the expected value of not diverting and remaining until exit exceeds the value of diverted funds in each period:

\[
V_{j,t} \geq \lambda Q_t S_{j,t}
\] (3.15)

The constraint (3.15) binds so long as \(\lambda > v_t\), which implies that marginal increases in assets have more benefit to the banker being diverted than as an increase in expected terminal wealth. For the purposes of this paper, this constraint will always bind, which implies, using (3.12) with (3.15) that assets can be expressed as a function of net worth by

\[
Q_t S_{j,t} = \phi_t N_{j,t}
\] (3.16)

where

\[
\phi_t = \frac{\eta_t}{\lambda - v_t}
\] (3.17)

is the leverage ratio of the financial intermediary. Consequently, the incentive constraint puts a limit on the amount of assets, and hence deposits, that an intermediary can have relative to its net worth.

To continue the characterization of the financial sector, consider again the evolution of net worth in (3.7), which can be expressed using the leverage ratio (3.17) as

\[
N_{j,t} = \left[ (R_t^k - R_{t-1}) \phi_{t-1} + R_{t-1} \right] N_{j,t-1}.
\] (3.18)

Then the growth in net worth of the an intermediary, defined in (3.11) is written as

\[
z_t = (R_t^k - R_{t-1}) \phi_{t-1} + R_{t-1}
\] (3.19)

and, similarly, the growth in assets defined in (3.10) is expressed as

\[
x_t = \frac{\phi_t}{\phi_{t-1}} z_t.
\] (3.20)
Since the price $Q_t$ and the leverage ratio $\phi_t$ then are independent of banker-specific characteristics, total intermediary demand is a result of summing over all independent intermediaries $j$:

$$Q_t S_{I,t} = \phi_t N_t. \quad (3.21)$$

So the total value of intermediated assets $Q_t S_{I,t}$ is equal to the economy’s leverage ratio $\phi_t$ times aggregate intermediary net worth $N_t$. The key feature of this expression is that the total amount of assets supplied by the financial intermediaries is in part determined by their net worth. During financial crises, sharp declines in financial intermediary net worth therefore limits the amount of assets the sector can provide for the economy.

Now, to determine the law of motion for aggregate intermediary net worth $N_t$, note that total assets are equal to those of existing bankers $N_{e,t}$ plus new bankers $N_{n,t}$:

$$N_t = N_{e,t} + N_{n,t}. \quad (3.22)$$

Since bankers exit with probability $(1 - \theta)$, existing banker net worth makes up a fraction $\theta$ of the growth in net worth from the previous period,

$$N_{e,t} = \theta \left[ (R^k_t - R_{t-1}) \phi_{t-1} + R_{t-1} \right] N_{t-1} \quad (3.23)$$

In every period, a fraction $(1 - \theta)$ of bankers exit and become workers, transferring their accumulated net worth to the household. At the same time, an identical measure of workers become bankers, and receive an initial level of net worth from the household. Specifically, new bankers receive start-up funds equal to a fraction $\omega_{1-\theta}$ of the assets of exiting bankers $(1 - \theta) Q_t S_{t-1}$:

$$N_{n,t} = \frac{\omega}{1-\theta} (1 - \theta) Q_t S_{t-1} = \omega Q_t S_{t-1} \quad (3.24)$$

Combining the decomposition of net worth (3.22), the expression for existing net worth (3.23), and the expression for new net worth (3.24), the law of motion for net
worth is given by
\[ N_t = \theta \left[ (R^k_t - R_{t-1}) \phi_{t-1} + R_{t-1} \right] N_{t-1} + \omega Q_t S_{t-1}. \] (3.25)

3.2.3 Credit Policy

The previous section discussed the financial intermediary sector, and how bankers use their net worth and borrowing from households to purchase claims on nonfinancial firms. Now consider that sometimes the central bank may enter the credit market to assist in the flow of funds from households to nonfinancial firms. In particular, the government owns an amount of claims \( S_{g,t} \) on nonfinancial firms, these have relative price \( Q_t \), and so the total value of central bank assets is \( Q_t S_{g,t} \). Since \( Q_t S_{I,t} \) is the total value of privately intermediated assets, the total value of all assets in the economy is \( Q_t S_t \), where

\[ Q_t S_t = Q_t S_{I,t} + Q_t S_{g,t} \] (3.26)

The central bank purchases these assets in a manner similar to private financial intermediaries: by issuing debt to households \( B_{g,t} \) at time \( t \) that pays the risk free rate \( R_t \) in period \( t+1 \). In addition, the central bank’s claims on nonfinancial firms earn the stochastic rate \( R^k_{t+1} \) in period \( t+1 \). The government then will earn returns equal to \( (R^k_{t+1} - R_t) \) \( B_{g,t} \).

Unlike private financial intermediaries, which are balance sheet constrained because of the constant opportunity to divert a fraction \( \lambda \) of their assets, the government does not face a similar moral hazard problem – it always repays its debts. Consequently, the central bank faces no constraints on its balance sheet, it can borrow and lend without limit. However, for every unit of assets that the central bank owns, they pay a resource cost of \( \tau \). This resource cost captures any possible inefficiencies from government intervention, such as costs of raising government debt or that the central bank finds it more difficult to screen for quality investments.
For the moment, assume that the government has a rule – which will be discussed in Section 3.2.7 – that determines the fraction $\psi_t$ of total intermediated assets it will supply. That is, it sets its purchases such that

$$Q_t S_{g,t} = \psi_t Q_t S_t.$$  \hfill (3.27)

To characterize the full leverage ratio of the economy, that which includes private plus public assets, first note that using the government share (3.27) and the private intermediaries’ total demand (3.21) in the decomposition of total assets (3.26) yields

$$Q_t S_t = \phi_t N_t + \psi_t Q_t S_t.$$  

So the total funds are equal to the total leverage ratio $\phi_t^c$ times intermediary net worth

$$Q_t S_t = \phi_t^c N_t.$$  \hfill (3.28)

The total leverage ratio, which is that for private and public funds, is given by

$$\phi_t^c = \frac{\phi_t}{1 - \psi_t}.$$  \hfill (3.29)

Note here that by setting $\psi_t$, the central bank manipulates the private leverage ratio $\phi_t$. If the central bank increases its fraction of supplied assets given a fixed private leverage ratio, the total leverage ratio increases at an increasing rate.

### 3.2.4 Intermediate Goods Firms

Intermediate goods firms operate in a competitive environment, producing using capital and labor. Firms purchase capital by issuing claims $S_t$ to financial intermediaries, and then use the funds from issuing those claims to purchase capital for next period. After production, the firm then pays to repair its depreciated capital and sells its entire capital on the open market. The price of a unit of capital and the price of a claim are $Q_t$, so
\[ Q_t K_t = Q_t S_t \]

Given a level of capital \( K_{t-1} \), the firm decides on labor demand, which pays wage \( W_t \), and a capital utilization rate \( U_t \), and produces the intermediate good \( Y_t^m \) using a Cobb-Douglas production function

\[
Y_t^m = A_t (U_t \xi_t K_{t-1})^\alpha L_t^{1-\alpha}
\]  
(3.30)

and sells this output at price \( P_t^m \). Firms are also subject to changes in total factor productivity \( A_t \), where

\[
\log A_t = \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t}
\]  
(3.31)

and changes in a capital quality measure \( \xi_t \) which evolves according to the process

\[
\log \xi_t = (1 - \rho_\xi (s_t)) \log \xi_{ss} (s_t) + \rho_\xi (s_t) \log \xi_{t-1} + \sigma_\xi \varepsilon_{\xi,t}
\]  
(3.32)

where \( s_t \) indicates a hidden Markov state at time \( t \). This Markov process affects the mean of the process \( \log \xi_{ss} (s_t) \), as well as persistence around the mean \( \rho_\xi (s_t) \). The role of the capital quality shock \( \xi_t \) is to alter the effective capital stock of the economy \( \xi_t K_{t-1} \) and thereby exogenously change the value of capital in the economy.

A more detailed description of the Markov Process is in Section 3.3.

Since the firm faces no adjustment costs, its problem is static, and so period-by-period the firm chooses its labor demand and capital utilization such that

\[
W_t = P_t^m (1 - \alpha) \frac{Y_t^m}{L_t}
\]  
(3.33)

and

\[
P_t^m \alpha \frac{Y_t^m}{U_t} = \delta' (U_t) \xi_t K_{t-1}.
\]  
(3.34)
The depreciation rate varies with utilization and is assumed to follow the functional form

$$\delta (U_t) = \tilde{\delta} - \frac{\tilde{\delta}}{1 + \zeta} + \frac{\tilde{\delta}}{1 + \zeta} U_t^{1+\zeta}$$

where $\tilde{\delta}$ is determined by the steady state.

After producing in time $t$, the firm sells its non-depreciated capital at price $Q_t$ less a replacement price of depreciated capital of 1. The firm earns zero profits state-by-state, it pays the realized return on capital to the intermediary, which is given by

$$R^k_t = \left[ \frac{P_{tm} \alpha \gamma_{km} + Q_t - \delta (U_t)}{Q_{t-1}} \right] \xi_t.$$ (3.35)

This expression highlights how changes in the capital quality measure $\xi_t$ produce exogenous changes in the return on capital.

### 3.2.5 Capital Producing Firms

Capital producers are competitive firms that buy used capital from intermediate goods firms, repair depreciated capital, build new capital, and sell it to the intermediate goods firms. So gross investment is defined as

$$I_t = K_t - (1 - \delta (U_t)) \xi_t K_{t-1}$$ (3.36)

which is simply the total change in capital. Net investment is gross investment less depreciation, so

$$I^n_t = I_t - \delta (U_t) \xi_t K_{t-1}. \quad (3.37)$$

Firms face quadratic adjustment costs on construction of new capital but not depreciated capital. In particular, the costs are quadratic in deviations of the growth in net investment plus steady-state gross investment from a value of unity:

$$f \left( \frac{I^n_t + \bar{I}}{I^n_{t+1} + \bar{I}} \right) = \frac{\eta_i}{2} \left( \frac{I^n_t + \bar{I}}{I^n_{t+1} + \bar{I}} - 1 \right)^2$$ (3.38)
The present value of profits for a capital producer are

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{Q_t}{q_0} \left\{ (Q_t - 1) I^r_t - f \left( \frac{I^I_t + \bar{I}}{I^I_{t-1} + \bar{I}} \right) (I^r_t + \bar{I}) \right\}$$

The optimal choice of net investment implies the price of capital is given by

$$Q_t = 1 + f \left( \frac{I^r_t + \bar{I}}{I^I_{t-1} + \bar{I}} \right) + f' \left( \frac{I^r_t + \bar{I}}{I^I_{t-1} + \bar{I}} \right) \left( \frac{I^r_t + \bar{I}}{I^I_t + \bar{I}} \right)^2 \right)$$  \hspace{1cm} (3.39)

3.2.6 Retail Firms

Retail firms repackage intermediate output $Y^m_t$ into differentiated products $Y^f_t$ which it sells at price $P^f_t$, where $f \in [0, 1]$ denotes differentiated products. Final output is a CES aggregate of retail firm goods, so

$$Y_t = \left( \int_0^1 Y_{f,t}^{\frac{\varepsilon+1}{\varepsilon}} df \right)^{\frac{\varepsilon}{\varepsilon+1}}$$  \hspace{1cm} (3.40)

Consumers of the final good use cost minimization, so standard optimality conditions imply that demand for good $f$ is a function of the relative price of the good times aggregate demand:

$$Y_{f,t} = \left( \frac{P}{P_{f,t}} \right)^{-\varepsilon} Y_t.$$  \hspace{1cm} (3.41)

and that the aggregate price level is related to the individual prices by

$$P^{1-\varepsilon}_t = \int_0^1 P^{1-\varepsilon}_{f,t} df$$

Since retail firms repackage intermediate output, their marginal cost is $P^m_t$. Firms set their price according to Calvo pricing, meaning a firm can re-optimize their price each period with probability $(1 - \gamma)$. Those firms that do not re-optimize
prices re-index prices with respect to inflation and the parameter $\mu$. A firm that can choose its price at time $t$ maximizes the present value of profits according to

$$
\max_{P_{f,t}} \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{Q_{t+i}}{q_t} \left( \prod_{k=1}^{i} \Pi_{t+k-1}^{\mu} \frac{P_{f,t}}{P_{t+i}} - P_{t+i}^m \right) Y_{f,t+i} \quad (3.42)
$$

subject to

$$
Y_{f,t+i} = \left( \prod_{k=1}^{i} \Pi_{t+k-1}^{\mu} \frac{P_{f,t}}{P_{t+i}} \right)^{-\varepsilon} Y_{t+i} \quad (3.43)
$$

The first-order condition produces the optimal price setting level $P_{f,t}^*$ and using the symmetric equilibrium $P_{f,t}^* = P_t^*$:

$$
\sum_{i=0}^{\infty} \gamma^i \beta^i \frac{Q_{t+i}}{q_t} \left( (\varepsilon - 1) \left( \prod_{k=1}^{i} \Pi_{t+k-1}^{\mu} \frac{P_{f,t}^*}{P_{t+i}} \right)^{1-\varepsilon} - \varepsilon \left( \prod_{k=1}^{i} \Pi_{t+k-1}^{\mu} \frac{P_{t+i}^m}{P_{t+i}} \right)^{-\varepsilon} \right) Y_{t+i} = 0.
$$

Defining the auxiliary variable $a_t$, this term is written in recursive form as two expressions, the first being

$$
a_t = \tilde{P}_t^* Y_t + \gamma \beta \frac{\tilde{P}_t^*}{P_{t+1}} \left( \frac{\Pi_{t}^{\mu}}{\Pi_{t+1}} \right)^{1-\varepsilon} \frac{q_{t+1}}{q_t} a_{t+1} \quad (3.44)
$$

and the second

$$
a_t = \frac{\varepsilon}{(\varepsilon - 1)} P_t^m Y_t + \gamma \beta \left( \frac{\Pi_{t}^{\mu}}{\Pi_{t+1}} \right)^{-\varepsilon} \frac{q_{t+1}}{q_t} a_{t+1}. \quad (3.45)
$$

where the relative price is defined as $\tilde{P}_t^* = P_t^*/P_t$.

Given indexation and the fact that a fraction $(1 - \gamma)$ of firms change their prices, the evolution of the price level satisfies

$$
P_{t}^{1-\varepsilon} = (1 - \gamma) (P_{t}^*)^{1-\varepsilon} + \gamma \left( \Pi_{t-1}^{\mu} P_{t-1} \right)^{1-\varepsilon}
$$
and putting the expression in terms of relative price

\[ 1 = (1 - \gamma) \tilde{P}_t^{1+\varepsilon} + \gamma \left( \frac{\Pi_t^{\mu}}{\Pi_t} \right)^{1-\varepsilon} \]  

(3.46)

Finally, the domestic rate of absorption \( \varsigma_t \) is defined by

\[ \varsigma_t = \int_0^1 \left( \frac{P_{f,t}}{P_t} \right)^{-\varepsilon} \, df \]

\[ = (1 - \gamma) \left( \tilde{P}_t^{\varepsilon} \right)^{-\varepsilon} + \gamma \left( \frac{\Pi_t^{\mu}}{\Pi_t} \right)^{-\varepsilon} \varsigma_{t-1} \]  

(3.47)

and since \( Y_{f,t} = Y_t^m \), then intermediate output and final output are related by

\[ Y_t^m = \varsigma_t Y_t. \]  

(3.48)

3.2.7 Government Policy

There are two aspects to government policy: standard monetary policy and the credit market rule. Monetary policy sets the nominal interest rate \( r_t \) according to the Taylor rule

\[ \left( \frac{r_t}{\bar{r}} \right) = \left( \frac{r_{t-1}}{\bar{r}} \right)^{\rho_r(s_t)} \left( \left( \frac{\Pi_t^{\mu}}{\Pi_t} \right)^{\kappa_\pi} \left( \frac{Y_t}{Y_t^\pi} \right)^{\kappa_y} \right)^{1-\rho_r(s_t)} \exp(\sigma_r \varepsilon_{r,t}) \]  

(3.49)

where the smoothing parameter \( \rho_r(s_t) \) follows a Markov Process to be discussed in Section 3.3, \( \bar{\Pi} \) is the target inflation rate, \( \bar{r} \) is the steady state nominal rate, and \( \kappa_\pi \) and \( \kappa_y \) control responses to the inflation and output gap, respectively. The nominal and risk-free interest rates satisfy the Fisher equation

\[ r_t = R_t \mathbb{E}_t \Pi_{t+1} \]  

(3.50)

Second, as discussed in Section 3.2.3, the government sets its level of credit market intervention \( \psi_t \) according to the rule

\[ \psi_t = (1 - \rho_\psi(s_t)) \psi_{ss}(s_t) + \rho_\psi(s_t) \psi_{t-1} \]  

(3.51)
where the mean of the process $\psi_{ss} (s_t)$ and its persistence $\rho_\psi (s_t)$ change according to a Markov Process to be discussed in Section 3.3.

Finally, the government has a fixed amount of spending $G$ every period, plus it must pay a resource cost $\tau$ on any assets it purchases. It finances these via lump-sum taxes and the return from its previously held assets, which, as discussed previously, is the realized interest rate differential. Consequently, the government’s budget constraint is given by

$$G + \tau \psi_t Q_t K_t = T_t + (R^k_t - R_{t-1}) B_{g,t-1}$$  \hspace{1cm} (3.52)

The economy wide resource requires that output be used for consumption, investment plus capital adjustment costs, and government spending including the resource cost of intervention. Assume that the level of government spending $G$ equals a fraction $g$ of steady state output $\bar{Y}$. Therefore, the economy’s resource constraint is

$$Y_t = C_t + I_t + f \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} \right) (I^n_t + \bar{I}) + G + \tau \psi_t Q_t K_t$$  \hspace{1cm} (3.53)

and the economy wide evolution of capital is

$$K_t = \xi_t K_{t-1} + I^n_t - f \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} \right) (I^n_t + \bar{I})$$  \hspace{1cm} (3.54)

### 3.3 Regime Switching and Equilibrium

This section builds upon the previous one by discussing the regime changes in the economy. There are three equations that have parameters that switch according to a Markov process.

#### 3.3.1 Markov Switching in the Capital Quality Process

The first is the exogenous process for capital quality

$$\log \xi_t = (1 - \rho_\xi (s_t)) \log \xi_{ss} (s_t) + \rho_\xi (s_t) \log \xi_{t-1} + \sigma_\xi \varepsilon_{\xi,t}.$$
This process allows for changes in the mean of the process through the term $\xi_{ss} (s_t)$, and changes in the persistence $\rho_\xi (s_t)$, where $s_t$ denotes the state of the Markov Process. By allowing for changes in the mean and the persistence, this equation can capture a wide variety of possible switching dynamics. As mentioned in Section 3.2.4, this capital quality measure will drive exogenous fluctuations in the value of capital, and significant declines will create a financial crisis.

For the purposes of this paper, the two switching parameters $\xi_{ss} (s_t)$ and $\rho_\xi (s_t)$ will each take on two values, depending on whether the economy is in a financial crisis or not. Specifically, if the economy is not in a financial crisis, then the mean of the process is assumed to be $\xi_{ss}^n = 1$, and the persistence is assumed to be $0 < \rho_{\xi}^n < 1$, where the superscript $n$ denotes "normal times." During a financial crisis, the mean of the process is a crisis level $\xi_{ss}^c < 1$, where the superscript $c$ indicates "crisis" and the persistence is $\rho_{\xi}^c = 0$.

This dual change in parameters between normal and crisis times has two effects. First, when the economy enters a crisis, the fact that the crisis mean is low implies that the capital quality measure moves to a lower level than in normal times. The fact that the crisis persistence $\rho_{\xi}^c = 0$ implies that the capital quality jumps downward to the new, lower mean. Second, when the economy leaves a crisis, the mean of the process $\xi_{ss}^n = 1$ implies that the capital quality measure is at its higher normal level, but the persistence $\rho_{\xi}^n$ implies that the reversion to this high mean is slow. These two features are meant to capture the fact that entry into financial crises tends to be rapid, with a quick transition to a low capital quality, while after the crisis ends the economy takes time to return back to its pre-crisis levels.
3.3.2 Markov Switching in the Taylor Rule

The second equation that contained parameters that switch along with the Markov Process is the Taylor Rule

\[
\frac{r_t}{\bar{r}} = \left( \frac{r_{t-1}}{\bar{r}} \right)^{\rho_r(s_t)} \left( \frac{\Pi_t}{\Pi} \right)^{\kappa_x} \left( \frac{Y_t}{Y_{t-1}} \right)^{\kappa_y} \exp(\sigma_r \varepsilon_{r,t})
\]

where the switching affects the interest rate smoothing \(\rho_r(s_t)\). Similar to the assumptions on switching for the process of capital quality, the smoothing parameter takes two different values depending upon whether the economy is in a financial crisis or not. If the economy is in normal times, then \(\rho_r(s_t)\) is equal to \(0 < \rho_r^a < 1\), which indicates that the central bank is smoothing when setting the nominal interest rate. When the economy enters a crisis, the central bank ceases to smooth interest rates however, making their policy decision completely dependent upon current conditions, and sets the smoothing parameter \(\rho_r(s_t) = \rho_r^c = 0\).

3.3.3 Markov Switching in Credit Policy

The third equation that has Markov Switching in the parameters is the equation governing the level of credit intervention by the central bank:

\[
\psi_t = (1 - \rho_\psi(s_t)) \psi_{ss}(s_t) + \rho_\psi(s_t) \psi_{t-1}
\]

where again the Markov Switching affects the mean of the process \(\psi_{ss}(s_t)\) and its persistence. In contrast to the previous two Markov Switching equations, which changed depending upon whether the economy was in a crisis or not, the level of credit intermediation will need to allow for the fact that the economy may not intervene during crises. Consequently, the distinction will be between ”intervene” and ”not intervene.”

When the central bank intervenes, it sets the mean of the process to be \(0 < \psi_{ss}^b < 1\) and persistence to be \(\rho_\psi^b = 0\), where the superscript \(b\) denotes ”bailout.”
This rule implies that when the central bank intervenes in credit markets, it always does so by purchasing a fraction $\psi_{ss}^b$ of total assets. When the central bank does not intervene, it sets the mean to be $\psi_{ss}^{nb} = 0$ and $0 \leq \rho_{\psi}^{nb} < 1$. These values imply two features about the no intervention case. First, if $\psi_{t-1} = 0$, so the central bank previously had no assets, then it will continue to have no assets. Second, even if the central bank is not intervening, if it does have assets, so $\psi_{t-1} > 0$, then it will continue to hold assets, but will be unwinding them according to an AR process. This unwinding has important implications for when the central bank is exiting the credit market after a crisis. If $0 < \rho_{\psi}^{nb} < 1$, when the intervention rule switches to the no intervention parameters, there will be a gradual unwind of the accumulated assets. On the other hand, if $\rho_{\psi}^{nb} = 0$ then when the intervention rule switches to the no intervention parameters, then instantly $\psi_t = 0$, meaning the central bank exits the credit market with an immediate sell-off of its assets.

### 3.3.4 Transitions and Timing

Having discussed the nature of the switching in parameters governed by a Markov Switching process, this subsection discusses exactly how the switches in the three equations coincide, and transitions between the various regimes. There are five regimes in the economy. Figure 3.2 depicts these transitions as a flow chart.

The first regime ($s_t = 1$) is defined as "Normal Times" and can be thought of as the regime the economy is in a high percentage of the time. In this regime, the capital quality measure has a high mean ($\xi_{ss}^n (1) = \xi_{ss}^n = 1$), and positive persistence ($\rho^\xi (1) = \rho^n_\xi \in (0, 1)$). The monetary authority practices smoothing of nominal interest rates ($\rho^r(s_t) = \rho^r (1) > 0$) and the central bank either is not intermediating or decreasing its level of intermediation ($\psi_{ss} (1) = \psi_{ss}^n = 0$ and $\rho^\psi (1) = \rho^n_\psi \in [0, 1]$).

With some exogenous small probability, $p_c$, which can be thought of as a rare event in the form of Barro (2007), the economy experiences a financial crisis each
period. Conditional on this crisis occurring, there is a simultaneous decision by the central bank to intervene or not, and it intervenes with probability $p_b$.

If the central bank does not intervene, which occurs with probability $1 - p_b$, the economy moves to the second regime ($s_t = 2$), which is "Crisis Without Intervention." In this regime, there is a financial crisis, so the capital quality measure drops immediately ($\rho_\xi (2) = \rho_\xi = 0$) to a lower steady state ($\xi_{ss} (2) = \xi_{ss}^n < 1$). The monetary authority ceases its policy of smoothing interest rates in order to react rapidly to current conditions ($\rho_r (2) = \rho_r^c = 0$), but the central bank does not intervene in credit markets, leaving the credit policy rule as during normal times ($\psi_{ss} (2) = \psi_{ss}^n = 0$ and $\rho_\psi (2) = \rho_\psi^n \in [0, 1]$).

After a crisis occurs, the economy exits the crisis with probability $p_e$ each period. If the economy is in regime 2, then upon exiting the crisis, it moves to the third regime ($s_t = 3$), called "Policy Continuation Without Intervention." In this regime, the crisis has ended, but the policy rules are still in their crisis modes. That is, the capital quality measure has high mean ($\xi_{ss} (3) = \xi_{ss}^n = 1$) and positive persistence ($\rho_\xi (3) = \rho_\xi^c \in (0, 1)$), which implies that after the drop in capital quality from the crisis, quality is gradually moving to its original pre-crisis level. However, since policy is still in crisis mode, the monetary authority does not smooth interest rates ($\rho_r (3) = \rho_r^c = 0$), and the central bank still does not intervene, leaving the credit policy rule as during normal times ($\psi_{ss} (2) = \psi_{ss}^n = 0$ and $\rho_\psi (2) = \rho_\psi^n \in [0, 1]$).

Now consider that, instead of not intervening when a crisis occurs, the central bank does choose to intervene in credit markets, which occurs conditional on a crisis with probability $p_b$. Then, instead of moving to regime 2, the economy moves to the fourth regime ($s_t = 4$), called "Crisis with Intervention." In this regime, the financial crisis has the capital quality measure dropping ($\rho_\xi (4) = \rho_\xi^c = 0$) to a lower steady state ($\xi_{ss} (4) = \xi_{ss}^n < 1$). The monetary authority ceases its smoothing of interest rates ($\rho_r (4) = \rho_r^c = 0$) and the central bank intervenes in credit markets,
purchasing a fixed fraction of assets \( (\psi_{ss} (4) = \psi_{ss}^c > 0 \text{ and } \rho_{\psi} (4) = \rho_{\psi}^c = 0) \).

Again, the crisis ends with probability \( p_e \), and if the economy is in regime 4 it moves to the fifth regime \( (s_t = 5) \), "Policy Continuation with Intervention." Similar to the third regime, this regime is what happens after the crisis ends, and capital quality is slowly moving to the high steady state again \( (\xi_{ss} (5) = \xi_{ss}^n = 1, \rho_{\xi} (5) = \rho_{\xi}^c \in (0,1)) \), but policy is still in crisis mode. That is, the monetary policy still does not smooth interest rates \( (\rho_r (5) = \rho_r^c = 0) \) and the central bank continues to hold a fixed fraction of assets \( (\psi_{ss} (5) = \psi_{ss}^c > 0 \text{ and } \rho_{\psi} (5) = \rho_{\psi}^c = 0) \).

Finally, when the economy is in the policy continuation regimes, either without intervention \( (s_t = 3) \) or with intervention \( (s_t = 5) \), the policy continuation stops with probability \( p_s \) each period. Then the economy returns to the "Normal" regime \( (s_t = 1) \), and interest rate smoothing resumes and the central bank sets its credit policy parameters to the no intervention levels. Importantly, when the economy leaves policy continuation with intervention, the parameter \( \rho_n \) determines the rate at which the credit market intervention ends.

Table 3.1 summarizes this discussion of how the parameters change according to the state of the Markov Process. Table 3.2 summaries the transition probabilities.

### 3.3.5 Equilibrium and Model Solution

The described DSGE model with Markov Switching is solved using the perturbation method of Foerster et al. (2011). The advantage of this method is that it allows for the equilibrium conditions to be written with Markov Switching in a transparent manner. Appendix A lists the full set of equilibrium conditions for the model, and Table 3.3 lists the calibrated parameters. In addition, since the processes for credit policy and the exogenous capital quality include shifts in steady state, perturbation has the advantage that it considers the effects of switches in steady state parameters. In particular, Markov Switching models are not certainty equivalent,
so constructing approximations to the solution of the economy require an additional term that captures the effects of the parameters that switch.

3.4 Crisis Responses

Having discussed the basic model and the nature of regime switching, this section now turns to considering what happens during and immediately after financial crises. These impulse responses are responses to a ”typical” crisis. In this experiment, agents know that financial crises occur with probability $p_c$, and that when a crisis occurs, the credit market intervention occurs with probability $p_b$. Agents also know that the crisis ends with probability $p_e$ each period; in a typical crisis considered, the responses depict the case when the case where the crisis lasts the expected duration.

More specifically, two different assumptions on crisis duration will be considered: in the first, crises last one period only, so $p_e = 1$; in the second, crises end with probability $p_e = \frac{1}{4}$ and the typical crisis response is for a crisis that lasts exactly four periods. Similarly, policy continuation is such that the expected duration of intervention totals 20 periods, meaning the probability $p_s$ is chosen so that $\frac{1}{p_s} = 20$. In the typical crisis, after 20 periods of crisis policies in place, policies return to the ”normal” regime.

For both assumptions on crisis duration, the results below will consider to different specifications. First, they will consider the differences for the central bank in committing to not intervening ($p_b = 0$) and intervening ($p_b = 1$) with a slow unwind of assets ($\rho_n^\psi = 0.99$) and how these change the effects of crises. Second, the results will suppose that the central bank has committed to intervening, and then look at the effects of different types of exit strategies: one in which the central bank slowly unwinds assets after deciding to exit the credit market ($\rho_n^\psi = .99$) versus one in which they immediately exit the credit market ($\rho_n^\psi = 0$).
3.4.1 Single-Period Crisis

This section describes the response to a typical crisis when crises last one period. Agents know that the probability of exiting a crisis is $p_e = 1$, and so they realize that when a crisis occurs, the following period capital quality will move back towards its pre-crisis level.

*Intervention Versus No Intervention*

Figure 3.3 displays the impulse responses to a crisis when the government has committed to intervening versus when it has committed to not intervening. When the crisis lasts a single period, upon entering the crisis, there is an immediate five-percent decline in capital quality. In subsequent periods, however, the capital quality moves back up to its original level.

When the central bank commits to not intervening, the effect of the crisis is to diminish the net worth of financial intermediaries, which also causes a decline in the price of capital. Since bankers have a major decline in net worth, they are unable to take deposits from households and purchase claims on capital, and so the capital stock falls dramatically, and the interest rate spread increases. Output decreases about 5 percent at its trough, and takes a significant amount of time to recover.

The central bank committing to intervening helps alleviate the crisis by injecting credit into the economy by immediately purchasing six percent of assets. The resulting increase in demand for capital over the no intervention case implies that the price of capital does not fall as dramatically, and as a result the net worth of financial intermediaries does not fall as far, and the widening of the interest rate spread is diminished. The resulting loss of output is approximately 4 percent, and the rebound is slightly faster than in the case without intervention.

In the typical crisis, 20 periods after the crisis, the central bank ends its policy continuation. In the case with no credit market intervention, this means only that
interest rate smoothing begins again. In the case with credit intervention, the ending of crisis policies means that the central bank needs to unwind its assets, and in the current setup, does so according to an autoregressive process with coefficient $\rho_\psi^n = .99$, which implies they hold assets for a significant amount of time beyond the crisis. Because they unwind so gradually, when policy continuation ends, there is little response from the economy – all the plotted responses exhibit no noticeable jumps at $t = 20$.

**Exit Strategies**

Having determined that committing to intervention helps during crisis times because the injection of central bank funds lowers the strain on financial intermediaries and reduces the loss in output, now consider the effects of the exit strategy. In the previous crisis response, when the central bank ended its crisis policies, it unwound its accumulated assets very slowly; the persistence of the credit policy rule during normal times was $\rho_\psi^n = .99$. Suppose, on the other hand, that instead of unwinding its asset position gradually, the central bank completely exited the credit market when policy continuation ended. In terms of parameterizations, this policy is characterized by $\rho_\psi^n = 0$. Figure 3.4 depicts the differences between these two cases.

In both cases, agents’ expectations about the duration of policy continuation are the same, the only difference is the that agents know the difference in exit strategies once the central bank decides to stop intervention. It is important to note that during policy continuation both policies are identical. When agents expect a rapid sell-off of assets ($\rho_\psi^n = 0$) the initial downturn is slightly less significant when compared to when agents expect a gradual unwind ($\rho_\psi^n = .99$). The drop in the price of capital is slightly less, so the loss in financial intermediary net worth is not quite as dramatic, and the resulting loss of output is lower under the sell-off policy.

The big difference between the two policies occurs when the central bank decides
to unwind. In the typical crisis, 20 periods after the initial crisis, the central bank ends its crisis policies. As seen previously, when the central bank unwinds slowly, the transition back to the "normal" regime is relatively seamless: there is a resumption of interest rate smoothing, and because the central bank unwinds its positions very gradually, there is no abrupt change. In contrast, when the central bank ends policy intervention in the sell-off case, there is a rapid decrease in the demand for capital, the price of capital drops again, and output falls. The quick sell-off creates a "double-dip" recession approximately with a trough in output roughly half of that for the initial crisis.

3.4.2 Multi-Period Crisis

Having considered the effects of policy commitment and different unwind strategies in an environment where the initial crisis lasts only a single period, this subsection considers the possibility that crises last several periods. The motivation for this specification is seen in Figure 3.1 and comparing it to the crisis response functions for one-period crises. The crisis responses show a sharp spike in interest rate spreads that declines fairly rapidly, while the data on interest rate spreads show that interest rate spreads remained high after the initial spike for nearly a year. Consequently, the responses in this section will consider the possibility that crises last longer than one period. In the parameterization considered, the probability of exit is $p_e = \frac{1}{4}$, and so the typical crisis responses considered below have the crisis lasting four periods.

*Intervention Versus No Intervention*

Figure 3.5 displays the responses of variables to a typical crisis when the central bank has committed to intervention versus committed to no intervention. The initial impulse is again a five percent decline in capital quality, although in this case it remains at this level for four periods before gradually returning to its pre-crisis
levels.

The effects of a crisis and intervention are similar in this case to the one-period crisis, only magnified because of the persistence of the crisis. The initial drop in capital quality causes a massive decline in this case, almost eliminating the entire net worth of the financial intermediaries. The price of capital declines significantly in the process, and the interest rate spread jumps and stays elevated for several periods before declining. In all, output falls by over 10 percent, although the rebound is relatively fast.

The big difference between the multi-period crisis case and the single-period case is that multi-period crises have a second discontinuity in the responses that occurs when the crisis ends. Banker net worth declines significantly during the crisis, but then upon exiting the crisis rebounds to levels similar to in the one-period crisis almost immediately. Similarly, the interest rate spread has a rapid decline when the crisis ends.

In this environment, committing to credit intervention serves the same purpose as it did in the one-period crisis environment. The commitment to intervention implies that when the crisis occurs, the central bank purchases six percent of assets, this helps ease the downturn by mitigating the decline in the price of capital, which reduces the downturn of in banker net worth and ends with a smaller loss of output.

Exit Strategies

Considering again the effects of exit strategies in an environment of multi-period crises, the effects are similar. Figure 3.6 depicts the differences in crisis responses when the credit policy rule during normal times is slow unwind ($\rho^\psi_n = .99$) versus fast sell-off ($\rho^\psi_n = 0$).

Again, the only difference between the two cases is that agents know that when policy continuation ends there will be differing exit strategies pursued by the central
When agents expect the rapid sell-off, the magnitude of the initial drop is slightly less than when agents expect the gradual unwind. The big difference is again when policy continuation ends. If the central bank exits completely upon transition back to the normal regime, there is a "double dip" recession. Unlike the single period crisis, where the double dip was approximately half the size of the original drop, the second dip in this case is not half as large as the initial crisis. This difference is because of the large magnitude of the original crisis, the size of the drop in this case is actually comparable to the single-period crisis case.

3.5 Pre-Crisis: Effects of Expectations

The crisis responses in the previous section considered the effects of commitment on the economy during financial crises, as well as the effects of different exit strategies. Having considered what happens with these different policy specifications, the analysis now turns to how expectations about these policy responses to crises affect the economy in the "normal times" regime. In other words, this section asks the question: are there distortionary effects from announcing a policy of guaranteed intervention during crises *ex-ante*?

Since the Markov Switching processes discussed in Section 3.3 include switches in the steady state of some of the processes, there is a difference between the regime-specific steady state of the economy and the long-run steady state of the economy. The long-run steady state is the ergodic mean of the economy, which allows for the consideration of the long-run distribution of the economy across regimes. In contrast, the regime-specific steady state is the average level the economy would operate around if it remained in one regime indefinitely, although agents perceive that regime switches may occur. Put another way, the long-run steady state of the economy would be the unconditional mean, whereas the regime-specific steady state would be the mean conditional on a given regime.
Consequently, this section considers the regime-specific steady state for the "normal times" regime \((s_t = 1)\). Tables 3.5 and Normal Regime Conditional Mean, Multi-Period Crisis displays the normal regime’s conditional mean of several variables for different types of policy announcements. Table 3.5 shows the conditional mean if crises are assumed to last only one period, Table 3.6 has the values if crises occur for multiple periods. For each crisis assumption, the baseline is a policy announcement of guaranteed no intervention when crises occur. Using this benchmark, the table displays the percent change in the "normal times" steady state if a different policy is announced. One policy announcement is uncertain intervention where intervention occurs in half of crises \((p_b = .5)\), the second is where intervention is guaranteed \((p_b = 1)\). For each intervention probability, the two different unwind strategies, unwind and sell-off, are considered.

The results for both panels are similar in that the effects of the different policy announcements and strategies are small. When agents believe crises occur with probability \(p_c\), there are two competing effects. First, since output and consumption drop significantly during crises, households wish to boost their savings as a precautionary motive for smoothing consumption. This precautionary savings motive expands the supply of funds available in the economy, which increases the steady state level of capital, and hence the total output. On the other hand, if agents expect crises to occur with nonzero probability, and when crises happen there is a very low return on equity which implies a lower leverage ratio and a lower amount of capital in the economy unless bankers hold a higher level of net worth.

The extent to which expectations about intervention policy during crises are beneficial or not depends upon whether policy announcements negate the precautionary motive of households or the leverage ratio constraint on the banks. In both crisis scenarios, the effects on capital are mixed dependent upon the exit strategy. For both uncertain and certain intervention, expectations of a slow unwind lower the capital
stock, while expectations of a fast sell-off increase the capital stock. Recall the fast sell-off case produced a slightly less significant initial drop when a crisis occurred, but then upon the end of policy continuation, the rapid exit of the central bank from credit markets created a "double dip" recession. This second drop dominates the slight initial difference, and the economy builds only slightly more precautionary capital in this case.

On the other hand, Tables 3.5 and 3.6 also shows a difference in net worth and leverage in the economy when intervention is either uncertain or guaranteed. In all cases, the percent change in leverage is negative and the percent change in banker net worth is positive and of a slightly lower magnitude. This difference implies that when intervention is uncertain or guaranteed, bankers hold more net worth and are less levered than when the central bank guarantees no intervention.

Finally, the effect of expectations on output is always negative but small. In both the unwind and sell-off cases for the two different crisis assumptions expecting intervention with uncertainty or certainty lessens output, but the magnitudes are small. Figures 3.7 and 3.8 both show how the loss in output increases with the probability of intervention. Consumption, however, has an ambiguous effect, depending upon the exit strategy. The change in consumption is positive if the exit strategy is to sell-off assets immediately, but negative if the central bank unwinds gradually.

3.6 Welfare Calculations

Having considered the effects of policy announcements and expectations during and before crises, this section turns to evaluating the overall welfare gains or losses from different policy announcements. In particular, Section 3.4 discussed the fact that guaranteed intervention had benefits relative to no intervention during crises, since intervention helps bolster the economy and alleviate the crisis. However, there was a slight trade-off depending upon the exit strategy: the immediate sell-off case pro-
duced a slightly lower drop in output and consumption, but upon exit, the economy experienced a double-dip recession. In addition, as shown in Section 3.5, the effects of policy announcements prior to crises was mixed depending upon the exit strategy pursued. Consequently, this section will ask: what are the welfare gains or losses from the various policy specifications?

Importantly, in addition to the probability of intervention and the exit strategy considered, welfare costs will be affected by two factors. First, the resource cost $\tau$ of central bank intermediation will matter for welfare in that, if the cost is high, then the resource cost of intermediation will lower output and lower welfare. Second, the timing of the calculation matters for welfare costs. Specifically, the household’s gain or loss in welfare from different policies will depend upon whether they are in the midst of a crisis or not. In other words the ex-ante welfare costs measure the willingness to pay for intervention before a crisis, while the ex-post welfare costs measure willingness to pay when a crisis occurs.

For both ex-ante and ex-post welfare calculations, the welfare cost measure used is, for a given probability of intervention and exit strategy, the percent increase in lifetime consumption that would make households indifferent between the increase in consumption and a change to an environment of guaranteed intervention with the same exit strategy. Positive welfare measures indicate that guaranteed intervention is welfare-increasing, since households need additional consumption under the given specification to mimic guaranteed intervention. Negative welfare measures then imply guaranteed intervention is welfare-decreasing, as households are willing to give up consumption rather than have guaranteed intervention.

Returning to the households’s preferences (3.1), the value function for a given policy $A$ can be expressed as

$$V_t^A = E_0 \sum_{t=0}^{\infty} \beta^t U \left( C_t^A, L_t^A \right) = E_0 \sum_{t=0}^{\infty} \beta^t \log \left( C_t^A - hC_{t-1}^A \right) - \frac{\kappa}{1 + \varphi} (L_t^A)^{1+\varphi}.$$
and the welfare measure is $\Upsilon$ so that, for some other policy $B$,

$$V_t^A = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U \left(C_t^B (1 + \Upsilon), L_t^B \right)$$

The value function can be expressed recursively as

$$V_t = \log \left(C_t^A - h C_{t-1}^A \right) - \frac{\kappa}{1 + \varphi} \left(L_t^A \right)^{1+\varphi} + \beta \mathbb{E}_t V_{t+1}. \quad (3.55)$$

and this can be added to the equilibrium conditions to solve for $\Upsilon$.

Figures 3.9 and 3.10 show the welfare costs for both single and multi-period crises, respectively. The top panel shows the case when $\tau = 0$, which corresponds to no efficiency loss from intermediation, the second and third panels show $\tau = .001$ and $\tau = .005$, respectively. Each panel shows the welfare gain in consumption units from committing to intervention, both with the slow unwind ($\rho_{\psi}^n = .95$) and the fast sell-off ($\rho_{\psi}^n = 0$) cases. In addition, the solid lines show the welfare gain when the economy is in the normal regime, and the dotted line shows the welfare gain conditional on a crisis occurring and the intervention decision being unrealized.

The first panel of Figure 3.9 shows that both types of intervention are welfare improving in both crisis and pre-crisis scenarios. Since there is no resource cost of intervention, this fact is not surprising – there is effectively no downside to intervention. However, it is interesting to note that the slow unwind exit strategy dominates the fast sell-off strategy in both circumstances. Since the slow unwind tended to smooth output after crises, this fact implies that agents place emphasis on avoiding the double-dip recession that the sell-off can produce. In addition, conditional on an exit strategy, the benefit from commitment is higher once a crisis occurs. Again, this result is not surprising, as there is no cost to intervention, and so when a crisis occurs, the benefit of intervening is highest. Finally, the benefit of committing to intervention is decreasing as the intervention probability of intervention increases, simply because as $p_b$ approaches one the two alternatives become indistinguishable.
Considering the second and third panels of Figure 3.9 changes the implications of intervention however. In the second panel, when \( \tau = .001 \), the results about the type of intervention and timing change. For both types of exit strategy, the welfare benefits of commitment are negative when the economy is in the "normal times" regime, and positive when the economy enters a crisis. This case then shows a type of time-inconsistency in the commitment by the central bank. Before a crisis, it is welfare-improving to not commit to intervention because of the distortions caused by committing and the resource cost of intervention, but when a crisis occurs, it is welfare improving for commitment to intervention to occur. Further, conditional upon intervention, the welfare-preferred exit strategy changes from preferring slow unravel before a crisis to fast unwind when a crisis occurs. Because of the resource cost of intervention, welfare improves if the central bank exits as soon as possible, and the slow unwind strategy, while helping credit markets during crises, creates a drag on the economy that lowers output and welfare in the longer term.

Finally, in the third panel of Figure 3.9 which has \( \tau = .005 \), the welfare benefits are negative for all the cases, meaning committing to intervention is welfare decreasing. This fact holds true regardless of the exit strategy used, and regardless of whether the economy is in the normal times regime or has just entered a crisis. Further, the welfare losses are higher when the economy enters a crisis, and the losses are higher when the exit strategy is to unwind assets slowly. Both of these results stem from the relatively high resource cost of intervention: when intervention is costly in terms of output, there is a welfare loss from committing to intervention, especially when a crisis occurs – because intervention is immediately guaranteed – and welfare losses are higher when the intervention takes longer to unwind.

The three panels of Figure 3.10 have similar results to those in Figure 3.9. When the resource cost is zero, committing to intervention is welfare improving, and the slow unwind strategy is better from a welfare perspective. Increasing the resource
cost slightly implies a difference in welfare benefits based upon the regime the economy is in, and which exit strategy is preferred depends upon the timing. When the resource cost is high, then committing to intervention is always welfare decreasing and the welfare-dominant exit strategy is to unwind assets as fast as possible.

3.7 Conclusion

This paper has used a model of unconventional monetary policy along with regime switching to study the effects of exit strategies and agents pre-crisis expectations. It has shown that after intervention, if the central bank exits the credit market too quickly, the economy can experience a double-dip recession. In addition, it has shown that the effects of commitment may cause slight distortions in pre-crisis activity by altering agents’ expectations about intervention during crises. Finally, it has analyzed the welfare implications of committing to intervention, and argued that commitment can raise or lower welfare, and that the timing of the welfare calculation matters as well as the type of exit strategy used.

One interesting avenue for future research is to endogenize the probability and magnitude of intervention depending on the extent of the crisis. This paper has assumed that the magnitude of crises are fixed, although they may be persistent, the level of intervention is fixed, and all the probabilities are fixed. One might expect that crises of greater magnitude are accompanied by higher levels and probabilities of intervention. Similarly, the decision by the central bank to exit credit markets is governed by an exogenous probability in this paper, it is reasonable to think that this transition probability depends upon how quickly the economy rebounds from the initial drop from a crisis. Finally, this paper has focused on a given class of policy, optimal policy within this class is left for future work.
3.8 Markov Switching Equilibrium Conditions

With regime switching, the equilibrium conditions are

\[(C_{i,t} - hC_{i,t-1})^{-1} - \beta h \sum_{j=1}^{n_s} E_{j,t} (C_{j,t+1} - hC_{i,t})^{-1} = \varrho_{i,t} \]

\[\beta R_{i,t} \sum_{j=1}^{n_s} p_{i,j} E_{j,t} \frac{\varrho_{j,t+1}}{\varrho_{i,t}} = 1 \]

\[xL_{i,t}^e = \varrho_{i,t} W_{i,t} \]

\[v_{i,t} = \sum_{j=1}^{n_s} p_{i,j} E_{j,t} [(1 - \theta) \beta A_{j,t+1} (R_{j,t+1}^k - R_{i,t}) + \beta \theta A_{j,t+1} x_{j,t+1} v_{j,t+1}] \]

\[\eta_{i,t} = \sum_{j=1}^{n_s} p_{i,j} E_{j,t} [(1 - \theta) \beta A_{j,t+1} R_{i,t} + \theta \beta A_{j,t+1} z_{j,t+1} \eta_{j,t+1}] \]

\[\phi_{i,t} = \eta_{i,t} \frac{1}{\lambda - v_{i,t}} \]

\[z_{i,t} = (R_{i,t}^k - R_{i,t-1}) \phi_{i-1} + R_{i,t-1} \]

\[x_{i,t} = \frac{\phi_{i,t}}{\phi_{i-1}} z_{i,t} \]

\[N_{i,t} = \theta [(R_{i,t}^k - R_{i,t-1}) \phi_{i-1} + R_{i,t-1}] N_{i,t-1} + \omega Q_{i,t} \xi_{i,t} K_{i,t-1} \]

\[Q_{i,t} K_{i,t} = \phi_{i,t}^c N_{i,t} \]

\[\phi_{i,t}^c = \frac{\phi_{i,t}}{1 - \psi_{i,t}} \]

\[Y_{i,t}^m = A_{i,t} (U_{i,t} \xi_{i,t} K_{t-1})^\alpha L_{i,t}^{1-\alpha} \]

\[P_{i,t}^m (1 - \alpha) \frac{Y_{i,t}^m}{L_{i,t}} = W_{i,t} \]

\[P_{i,t}^m \alpha \frac{Y_{i,t}^m}{U_{i,t}} = \tilde{\delta} U_{i,t} \xi_{i,t} K_{t-1} \]

78
\[ R^k_{i,t} = \left[ P_{i,t} \beta_{i,t} K_{i,t-1} + \left( Q_{i,t} - \left( \tilde{\delta} - \frac{\delta}{1+\zeta} + \frac{\delta}{1+U_{i,t}^{1+\zeta}} \right) \right) \right] \xi_{i,t} \]

\[ I^n_t = I_t - \left( \tilde{\delta} - \frac{\tilde{\delta}}{1+\zeta} + \frac{\tilde{\delta}}{1+U_{i,t}^{1+\zeta}} \right) \xi_t K_{t-1} \]

\[ Q_{i,t} = 1 + \frac{\eta}{2} \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} - 1 \right)^2 + \frac{\eta}{2} \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} - 1 \right) \left( \frac{I^n_t + \bar{I}}{I^n_{t+1} + \bar{I}} \right)^2 \]

\[ a^1_{i,t} = \tilde{P}_{i,t} Y_{i,t} + \sum_{j=1}^{n_s} \hat{P}_{i,j} \hat{E}_{j,t} \gamma \beta \left( \frac{\Pi_{i,t}^{\gamma_p}}{\Pi_{i,t}^{\gamma_p}} \right)^{1-\epsilon} \frac{\theta_{j,t} + \bar{I}}{\theta_{i,t}} a^1_{j,t+1} \]

\[ a^1_{i,t} = \frac{\epsilon}{(\epsilon - 1)} P_{i,t} m Y_{i,t} + \sum_{j=1}^{n_s} \hat{P}_{i,j} \hat{E}_{j,t} \gamma \beta \left( \frac{\Pi_{i,t}^{\gamma_p}}{\Pi_{i,t}^{\gamma_p}} \right)^{-\epsilon} \frac{\theta_{j,t} + \bar{I}}{\theta_{i,t}} a^1_{j,t+1} \]

\[ 1 = (1 - \gamma) \tilde{P}_{i,t}^{1-\epsilon} + \gamma \left( \frac{\Pi_{i,t}^{\gamma_p}}{\Pi_{i,t}} \right)^{1-\epsilon} \]

\[ \varsigma_{i,t} = (1 - \gamma) \left( \tilde{P}_{i,t}^{1-\epsilon} \right)^{-\epsilon} + \gamma \left( \frac{\Pi_{i,t}^{\gamma_p}}{\Pi_{i,t}} \right)^{-\epsilon} \varsigma_{t-1} \]

\[ Y_{i,t}^m = \varsigma_{i,t} Y_{i,t} \]

\[ Y_{i,t} = C_{i,t} + I_{i,t} + \frac{\eta}{2} \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} - 1 \right)^2 \left( I^n_{i,t} + \bar{I} \right) + \bar{Y} + \tau \psi_{i,t} Q_{i,t} K_{i,t} \]

\[ K_{i,t} = \xi_{i,t} K_{i,t-1} + I^n_{i,t} - \frac{\eta}{2} \left( \frac{I^n_t + \bar{I}}{I^n_{t-1} + \bar{I}} - 1 \right)^2 \left( I^n_{i,t} + \bar{I} \right) \]

\[ \left( \frac{r_{i,t}}{\bar{r}} \right) = \left( \frac{r_{i,t} - \bar{r}}{\bar{r}} \right)^{\rho_{i,r}} \left( \frac{\Pi_{i,t}}{\Pi} \right)^{\kappa_r} \left( \frac{Y_{i,t}}{Y_{i,t}^{\kappa_r}} \right)^{1-\rho_{i,r}} \exp \left( \sigma_r \epsilon_{r,t} \right) \]

79
\[ r_{i,t} = R_{i,t} \sum_{j=1}^{n_s} E_{j,t} \Pi_{j,t+1} \]

\[ \psi_{i,t} = (1 - \bar{\rho} + \rho_{i,\psi}) \left( \hat{\psi} + \chi_{\psi_{i,ss}} \right) + \rho_{i,\psi} \psi_{t-1} \]

\[ \log A_{i,t} = \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t} \]

\[ \log \xi_{i,t} = (1 - \rho_{i,\xi}) \left( \hat{\xi} + \chi_{\xi_{i,ss}} \right) + \rho_{i,\xi} \log \xi_{t-1} + \sigma_\xi \varepsilon_{\xi,t} \]

\[ V_{i,t} = \log (C_{i,t} - hC_{t-1}) - \frac{\kappa}{1 + \varphi} L_{i,t}^{1+\varphi} + \beta \sum_{j=1}^{n_s} p_{i,j} E_{j,t} V_{j,t+1} \]
### 3.9 Tables and Figures

**Table 3.1: Markov Switching Parameters**

<table>
<thead>
<tr>
<th>$s_t$</th>
<th>$\rho_r (s_t)$</th>
<th>$\xi_{ss} (s_t)$</th>
<th>$\rho_s (s_t)$</th>
<th>$\psi_{ss} (s_t)$</th>
<th>$\rho_\psi (s_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) “Normal”</td>
<td>$\rho_r^n &gt; 0$</td>
<td>$\xi_{ss}^n = 1$</td>
<td>$\rho_s^n &gt; 0$</td>
<td>$\psi_{ss}^n = 0$</td>
<td>$\rho_\psi^n \geq 0$</td>
</tr>
<tr>
<td>2) ”Crisis w/o Interv”</td>
<td>$\rho_r^c = 0$</td>
<td>$\xi_{ss}^c &lt; 1$</td>
<td>$\rho_s^c = 0$</td>
<td>$\psi_{ss}^c = 0$</td>
<td>$\rho_\psi^c \geq 0$</td>
</tr>
<tr>
<td>3) ”Policy Cont w/o Interv”</td>
<td>$\rho_r^c = 0$</td>
<td>$\xi_{ss}^n = 1$</td>
<td>$\rho_s^c &gt; 0$</td>
<td>$\psi_{ss}^c = 0$</td>
<td>$\rho_\psi^c \geq 0$</td>
</tr>
<tr>
<td>4) ”Crisis w/ Interv”</td>
<td>$\rho_r^b = 0$</td>
<td>$\xi_{ss}^c &lt; 1$</td>
<td>$\rho_s^c = 0$</td>
<td>$\psi_{ss}^b &gt; 0$</td>
<td>$\rho_\psi^b = 0$</td>
</tr>
<tr>
<td>5) ”Policy Cont w/ Interv”</td>
<td>$\rho_r^b = 0$</td>
<td>$\xi_{ss}^n = 1$</td>
<td>$\rho_s^c = 0$</td>
<td>$\psi_{ss}^b &gt; 0$</td>
<td>$\rho_\psi^b = 0$</td>
</tr>
</tbody>
</table>

**Table 3.2: Markov Switching Probabilities**

<table>
<thead>
<tr>
<th>$s_t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 - p_c$</td>
<td>$p_c (1 - p_b)$</td>
<td>0</td>
<td>$p_c p_b$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$p_e p_s$</td>
<td>$1 - p_e$</td>
<td>$p_c (1 - p_s)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$(1 - p_c) p_s$</td>
<td>$p_c$</td>
<td>$(1 - p_c) (1 - p_s)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$p_c p_s$</td>
<td>0</td>
<td>0</td>
<td>$1 - p_e$</td>
<td>$p_c (1 - p_s)$</td>
</tr>
<tr>
<td>5</td>
<td>$(1 - p_c) p_s$</td>
<td>0</td>
<td>0</td>
<td>$p_c$</td>
<td>$(1 - p_c) (1 - p_s)$</td>
</tr>
<tr>
<td>Parameter</td>
<td>Description</td>
<td>Value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>--------------------------------------------------</td>
<td>----------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount Factor</td>
<td>0.99</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h$</td>
<td>Degree of Habit Persistence</td>
<td>0.815</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Disutility of Labor</td>
<td>3.409</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Inverse Frisch Elasticity of Labor</td>
<td>0.276</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Divertable Fraction of Banker Assets</td>
<td>0.381</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>Transfer to New Bankers</td>
<td>0.002</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta$</td>
<td>Survival Rate of Bankers</td>
<td>0.972</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Capital Share</td>
<td>0.33</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{U}$</td>
<td>Steady State Capital Utilization</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{\delta}$</td>
<td>Steady State Depreciation</td>
<td>0.025</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Elasticity of Depreciation to Utilization</td>
<td>7.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta_i$</td>
<td>Inverse Elasticity of Net Invest. to Capital Price</td>
<td>1.728</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Elasticity of Substitution Between Final Goods</td>
<td>4.167</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Probability of No Optimization of Prices</td>
<td>0.779</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>Degree of Price Indexation</td>
<td>0.241</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{g}$</td>
<td>Fraction of Steady State Output for Government</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_\pi$</td>
<td>Response of Interest Rate to Inflation</td>
<td>2.043</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_y$</td>
<td>Response of Interest Rate to Output Gap</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>Persistence of TFP</td>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>Std Dev of TFP</td>
<td>0.005</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_\xi$</td>
<td>Std Dev of Capital Quality Process</td>
<td>0.005</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_R$</td>
<td>Std Dev of Monetary Policy Shock</td>
<td>0.0025</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3.4: Regime Switching Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_c$</td>
<td>Probability of Crisis Occurring</td>
<td>0.005</td>
</tr>
<tr>
<td>$p_b$</td>
<td>Probability of Intervention</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>$p_e$</td>
<td>Probability of Exiting Crisis</td>
<td>${\frac{1}{4}, 1}$</td>
</tr>
<tr>
<td>$p_s$</td>
<td>Probability of Stopping Crisis Policy</td>
<td>${\frac{1}{15}, \frac{1}{19}}$</td>
</tr>
<tr>
<td>$\rho_{rn}$</td>
<td>Interest Rate Smoothing, Normal</td>
<td>0.80</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>Interest Rate Smoothing, Crisis</td>
<td>0.00</td>
</tr>
<tr>
<td>$\xi_{n}$</td>
<td>Capital Quality Mean, Normal</td>
<td>1.00</td>
</tr>
<tr>
<td>$\xi_c$</td>
<td>Capital Quality Mean, Crisis</td>
<td>0.95</td>
</tr>
<tr>
<td>$\rho_{n\xi}$</td>
<td>Capital Quality Persistence, Normal</td>
<td>0.66</td>
</tr>
<tr>
<td>$\rho_{c\xi}$</td>
<td>Capital Quality Persistence, Crisis</td>
<td>0.00</td>
</tr>
<tr>
<td>$\phi_{ns}$</td>
<td>Intermediation Share Mean, No Intervention</td>
<td>0.00</td>
</tr>
<tr>
<td>$\phi_{bs}$</td>
<td>Intermediation Share Mean, Intervention</td>
<td>0.06</td>
</tr>
<tr>
<td>$\rho_{n\psi}$</td>
<td>Intermediation Share Persistence, No Intervention</td>
<td>[0, 1)</td>
</tr>
<tr>
<td>$\rho_{b\psi}$</td>
<td>Intermediation Share Persistence, Intervention</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3.5: Normal Regime Conditional Mean, Single-Period Crisis

<table>
<thead>
<tr>
<th>Variable</th>
<th>No Intervention</th>
<th>Uncertain Unwind</th>
<th>Sell-Off</th>
<th>Intervention Unwind</th>
<th>Sell-Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output ($y$)</td>
<td>0.0</td>
<td>-0.0091</td>
<td>-0.0134</td>
<td>-0.0148</td>
<td>-0.0269</td>
</tr>
<tr>
<td>Consumption ($c$)</td>
<td>0.0</td>
<td>-0.0104</td>
<td>0.0049</td>
<td>-0.0198</td>
<td>0.0099</td>
</tr>
<tr>
<td>Capital ($k$)</td>
<td>0.0</td>
<td>-0.0338</td>
<td>0.0149</td>
<td>-0.0183</td>
<td>0.0300</td>
</tr>
<tr>
<td>Leverage ($\phi$)</td>
<td>0.0</td>
<td>-0.3539</td>
<td>-0.2046</td>
<td>-0.6978</td>
<td>-0.4094</td>
</tr>
<tr>
<td>Net Worth ($N$)</td>
<td>0.0</td>
<td>0.3108</td>
<td>0.2049</td>
<td>0.6161</td>
<td>0.4107</td>
</tr>
</tbody>
</table>
Table 3.6: Normal Regime Conditional Mean, Multi-Period Crisis

<table>
<thead>
<tr>
<th>Variable</th>
<th>No Intervention</th>
<th>Uncertain Unwind</th>
<th>Sell-Off</th>
<th>Intervention Unwind</th>
<th>Sell-Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output ((y))</td>
<td>0.0</td>
<td>-0.0088</td>
<td>-0.0175</td>
<td>-0.0140</td>
<td>-0.0351</td>
</tr>
<tr>
<td>Consumption ((c))</td>
<td>0.0</td>
<td>-0.0102</td>
<td>0.0042</td>
<td>-0.0196</td>
<td>0.0084</td>
</tr>
<tr>
<td>Capital ((k))</td>
<td>0.0</td>
<td>-0.0341</td>
<td>0.0118</td>
<td>-0.0645</td>
<td>0.0237</td>
</tr>
<tr>
<td>Leverage ((\phi))</td>
<td>0.0</td>
<td>-0.3572</td>
<td>-0.2323</td>
<td>-0.7060</td>
<td>-0.4649</td>
</tr>
<tr>
<td>Net Worth ((N))</td>
<td>0.0</td>
<td>0.3143</td>
<td>0.2300</td>
<td>0.6244</td>
<td>0.4613</td>
</tr>
</tbody>
</table>

Figure 3.1: Federal Reserve Balance Sheet
Figure 3.2: Flow Chart of the Economy
Figure 3.3: Single Period Crisis - Responses Under Commitment
Figure 3.4: Single Period Crisis - Exit Strategies
Figure 3.5: Multi-Period Crisis - Responses Under Commitment
Figure 3.6: Multi-Period Crisis - Exit Strategies
Figure 3.7: Single Period Crisis - Effects of Expectations on Pre-Crisis Output
Figure 3.8: Multi-Period Crisis - Effects of Expectations on Pre-Crisis Output
Figure 3.9: Single Period Crisis - Welfare Costs
Figure 3.10: Multi-Period Crisis - Welfare Costs
Bibliography


94


Biography

Andrew Thomas Foerster was born in Rochester, NY on May 19, 1982.

He holds a B.A. in Economics from Davidson College, earned in 2004, a M.S. in Mathematical Sciences from Virginia Commonwealth University, earned in 2006. From Duke University he earned a M.A. in Economics in 2007 and a Ph.D. in Economics from Duke University in 2011.

After graduation, he will work as an Economist at the Federal Reserve Bank of Kansas City.