Multi-Variable Period Polynomials Associated to Cusp Forms

by

Oliver Gjoneski

Department of Mathematics
Duke University

Date: ______________________

Approved:

____
Leslie Saper, Supervisor

____
Richard M. Hain

____
William L. Pardon

____
Mark Stern

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2011

Abstract

(Mathematics)

Multi-Variable Period Polynomials Associated to Cusp Forms

by

Oliver Gjoneski

Department of Mathematics
Duke University

Date: ________________________

Approved:

____
Leslie Saper, Supervisor

____
Richard M. Hain

____
William L. Pardon

____
Mark Stern

An abstract of a dissertation submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2011

Copyright © 2011 by Oliver Gjoneski
All rights reserved except the rights granted by the
Creative Commons Attribution-Noncommercial Licence

Abstract

My research centers on the cohomology of arithmetic varieties. More specifically, I am interested in applying analytical, as well as topological methods to gain better insight into the cohomology of certain locally symmetric spaces. An area of research where the intersection of these analytical and algebraic tools has historically been very effective, is the classical theory of modular symbols associated to cusp forms. In this context, my research can be seen as developing a framework in which to compute modular symbols in higher rank.

An important tool in my research is the well-rounded retract for GL_n. In particular, in order to study the cohomology of the locally symmetric space associated to GL_3 more effectively I designed an explicit, combinatorial contraction of the well-rounded retract. When combined with the suitable cell-generating procedure, this contraction yields new results pertinent to the notion of modular symbol I am researching in my thesis.

Contents

List of Tables

List of Figures

Acknowledgements

I would like to thank my advisor, Les Saper, for his limitless support during my tenure as a graduate student. His guidance and infinite patience have helped me tremendously while here at Duke. I would also like to thank my other committee members, Dick Hain, Bill Pardon, and Mark Stern, for their valuable contributions in the completion of my doctoral dissertation.

I would like to thank my parents for all of their sacrifices in giving me the opportunity to pursue education in the United States. Finally, I would like to thank my family and my fiancée Kelly for their love and constant encouragement.

1

Introduction

1.1 Background

Let $S_k\big(SL_2(\mathbb{Z})\big)$ denote the space of cusp forms of weight k with respect to $SL_2(\mathbb{Z})$, and $\mathcal{H}^+ \subset \mathbb{R}^2$ the upper half-plane. Furthermore let (ρ_2, V_2) be the standard representation of $GL_2(\mathbb{C})$, and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Given $f \in S_{n+2}\big(SL_2(\mathbb{Z})\big)$, and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V_2$, the two variable *period polynomial* $r_f\,(\mathbf{x}, \mathbf{y})$ is defined as

$$r_f(\mathbf{x}, \mathbf{y}) := \int_0^{i\infty} f(z)(z\,\mathbf{x} + \mathbf{y})^n dz.$$

As a result of this definition, we have a mapping:

$$S_{n+2}\big(SL_2(\mathbb{Z})\big) \to \mathrm{Sym}^n(V_2),$$

$$f \mapsto r_f^-(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(r_f(\mathbf{x}, \mathbf{y}) - r_f(\mathbf{x}, -\mathbf{y})),$$

where the range space of this map can be thought of as the space of homogeneous polynomials of degree n in 2 variables. The theory of two-variable period polynomials was originally established by Eichler [Eic57] and Shimura [Shi59], and later extended upon by Zagier [Zag91] and others. It describes the image of this map, and it presents a one-to-one correspondence between the space of cusp forms $S_{n+2}\big(SL_2(\mathbb{Z})\big)$ and a space of polynomials $\mathcal{Z}_n^- \subset \mathrm{Sym}^n(V_2)$ described in more detail in Chapter 5.

A key ingredient in the Eichler-Shimura correspondence is understanding the relationship between cusp forms, deRham differential forms, and cuspidal Eilenberg-MacLane groups co-cycles. In particular, in [Shi59], the author shows how starting with $p(\mathbf{x},\mathbf{y}) \in \mathcal{Z}_n^-$ one can construct unique closed, cuspidal group co-cycle ϕ vanishing at the cusps such that $\phi_i(S) = p(\mathbf{x},\mathbf{y})$. Furthermore, we can find a unique cusp form f, such that $r_f^-(\mathbf{x},\mathbf{y}) = p(\mathbf{x},\mathbf{y}) = \phi(S)$. If we use

$$H^1_{cusp}(SL_2(\mathbb{Z})\backslash\,\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^{(1,0)}$$

to denote the holomorphic (deRham) cuspidal cohomology of $SL_2(\mathbb{Z})$, we summarize the discussion above in the diagram below:

$$\begin{array}{ccccc} & \mathcal{Z}_n^- & \cong S_{n+2}\big(SL_2(\mathbb{Z})\big) \cong & H^1_{cusp}(SL_2(\mathbb{Z})\backslash\,\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^{(1,0)} \\ r_f^-(\mathbf{x},\mathbf{y}) = \phi(S) & \longleftarrow & f & \longmapsto \omega_f = f(z)(z\,\mathbf{x}+\mathbf{y})^n dz \end{array}$$

In this light, period polynomials can be appreciated for their ability to explain the Eilenberg-MacLane cohomology of the arithmetic group $SL_2(\mathbb{Z})$. We emphasize that starting with a polynomial $p(\mathbf{x},\mathbf{y}) \in \mathcal{Z}_n^-(\mathbf{x},\mathbf{y})$, there is a unique closed group co-cycle $\phi : SL_2(\mathbb{Z}) \to \mathrm{Sym}^n(V_2)$ restricting to zero at the cusps, such that $\phi(S) = p(\mathbf{x},\mathbf{y})$. (This sentence is best understood if revisited after reading the introductory exposition in Chapter 5.)

1.2 Goal: Period polynomial in 3-variables

We consider the natural question:

Can one define and compute an algebraic invariant analogous to the period polynomial in higher rank?

More specifically, let D_m denote the globally symmetric space of positive definite quadratic forms on $\mathbb{R}^m$ modulo homothety, and $\overline{D_m}$ the Borel-Serre bordification. Let V_m denote the standard m-dimensional representation for $GL_m(\mathbb{C})$. The underlying vector space for the induced representation $\mathrm{Sym}^n(V_m)$ on the symmetric product can be viewed as the space of homogeneous polynomials of degree n in m variables. We would like to invent a suitable notion of m-variable period polynomials and use them to explain the twisted cohomology of the locally symmetric space $GL_m(\mathbb{Z})\backslash D_m$ with coefficients in $\mathrm{Sym}^n(V_m)$.

In doing so, we will consider Eilenberg-MacLane cocycles for $GL_m(\mathbb{Z})$ created via the isomorphism presented in [Dup78],

$$H^\bullet\left(GL_m(\mathbb{Z})\backslash D_m, \mathbb{S}\mathrm{ym}^n(V_m)\right) \xrightarrow{\sim} H^\bullet\left(GL_m(\mathbb{Z}), \mathrm{Sym}^n(V_m)\right), \tag{1.1}$$
$$[\omega] \to [\phi_\omega]$$

where, $\phi_\omega : GL_m(\mathbb{Z})^k \to \mathrm{Sym}^n(V_m)$ is defined as

$$\phi_\omega(\gamma_1, \ldots, \gamma_k) = \int_{\sigma(\gamma_1,\ldots,\gamma_k)} \omega,$$

for σ a *filling* (defined in §5.3.2) of the symmetric space D_m.

The problem has two aspects. One is topological, namely the construction and choice of a suitable filling σ for the symmetric space, as a means of associating k-cells in D_m to each k-tuple $(\gamma_1, \ldots, \gamma_k) \in GL_m(\mathbb{Z})^k$. The other is analytic, the choice of differential forms representing cohomology classes. A goal of the project is to show that finitely many polynomials in n-variables determine each Eilenberg-MacLane co-cycle $[\phi_\omega]$ arising from (1.1).

1.2.1 *Analytical perspective: Eisenstein series*

The analytical dimension of this problem is captured by the $GL_m(\mathbb{Z})$-invariant, $\mathrm{Sym}^n(V_m)$-valued differential forms that need to be integrated to define the Eilenberg-MacLane group cocycles as in (1.1). While in rank one these are realized through cuspidal automorphic forms, in rank two the square-integrable cohomology groups vanish. Along these lines, Goncharov in [Gon01] proves a result relating the cohomology of $GL_3(\mathbb{Z})$ with that of $SL_2(\mathbb{Z})$. In Chapter 4, I offer an alternative proof of this result, and flesh out the relationship using the theory of Eisenstein series. In the statement of the theorem below, P is a maximal parabolic subgroup of GL_3, and $e'(P)$ is the associated face in $\partial\left(GL_3(\mathbb{Z})\backslash\overline{D_3}\right)$.

Theorem 1. *If* $[\nu] \in H^1_{cusp}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^- \cong H^3_{cusp}(e'(P), \mathbb{S}\mathrm{ym}^n(V_3))$, *then the corresponding Eisenstein series* $E^P(\nu,\Lambda)$ *is holomorphic at the point* $\Lambda_w = -w(\lambda+\rho) = \rho$, *for a suitable choice of* $w \in W^P$. *Furthermore,* $E^P(\nu,\Lambda_w)$ *is a harmonic form representing a non-trivial cohomology class in* $H^3(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$, *that restricts back to* ν *on* $e'(P)$. *Consequently,*

$$H^3\left(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3)\right) \cong H^1_{cusp}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^-.$$

1.2.2 *Topological perspective: combinatorics of the well-rounded retract*

As early as the case for $m = 3$, it quickly becomes clear that the traditional approach to constructing a filling for D_m, as described in [Dup78, Chapter 9], is not a suitable candidate. In fact, there is no hope that one may be able to catalog the three-cells produced by such a filling, even up to $GL_3(\mathbb{Z})$ equivalence. A key ingredient to my research is to instead consider a filling based on the contraction of a lower dimensional, $GL_3(\mathbb{Z})$-invariant subspace with a compact quotient.

The *well-rounded retract* for GL_3 was originally defined by Souĺe, and later generalized by Ash for GL_n. Roughly, we can describe the well-rounded retract as those

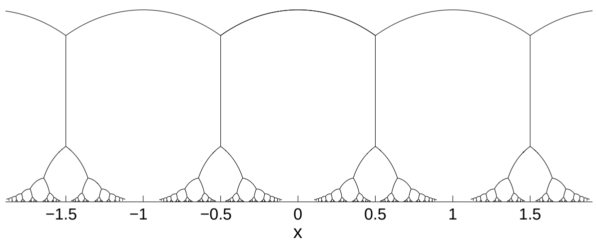


FIGURE 1.1: Well rounded retract in $\mathcal{H}^+$

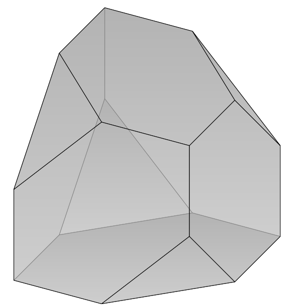


FIGURE 1.2: Soule Cube

equivalence classes of quadratic forms in D_m that have *enough* minimal vectors in the standard lattice $\mathbb{Z}^m \subset \mathbb{R}^m$. In Chapter 5 we present the definition in complete detail, and write down a deformation of the symmetric space onto the well-rounded retract $W_m \subset D_m$. One can then show that W_m is indeed $GL_3(\mathbb{Z})$-invariant, and has dimension equal to the virtual cohomological dimension of GL_3, or in other words, W_m is a *spine* for $GL_3(\mathbb{Z})$.

In the upper half plane we can visualize the well-rounded retract as the Serre tree, $\left\{\gamma(x+iy) \mid \gamma \in GL_2(\mathbb{Z}), x^2+y^2=1, |x| \leq \frac{1}{2}\right\}$, as seen in Figure 1.1.

In Chapter 5 we recast the classical rank-one theory in this new light. Using a contraction of the well rounded retract in the upper half plane we find that the Eilenberg-MacLane cocycle Φ_{ω_f} arising via (1.1) as an integral of a cohomology class associated to a cusp form f, can be recovered from a single polynomial.

Theorem 2. *If $f \in S_{n+2}\big(SL_2(\mathbb{Z})\big)$, $h : I \times W \to W$ is a contraction of the well-rounded retract in the upper half-plane to the point with (x, y)-coordinates $(-1/2, \sqrt{3}/2)$, and σ is the corresponding filling, then the Eilenberg-MacLane group cocycle*

$$\Phi_{\omega_f}(\gamma) = \int_{\sigma(\gamma)} f(z)(z\,\mathbf{x}+\mathbf{y})^n dz$$

can be completely recovered from the 2-variable polynomial $\Phi_{\omega_f}(S)$.

The key fact is that all one-cells in the filling created using the contraction of the well-rounded retract are unions of $GL_2(\mathbb{Z})$ translates of the arc:

$$\{\, x^2 + y^2 = 1 \mid |x| \leq 1/2 \,\}.$$

Carrying this methodology over to the case $m = 3$, we first demonstrate that each top dimensional cell in the well rounded retract, as seen in Figure 1.2, is endowed with local information identical to that of a spherical apartment in the building at infinity in $\overline{D_3}$. More precisely, to a 3-cell defined as those well-rounded quadratic forms whose minimal set of vectors include $\{v_1, v_2, v_3\}$, we associate the spherical apartment defined by the set of maximal parabolic subgroups each stabilizing a flag $\{0 \subset \mathbb{Q}\{v_i, v_j\} \subset \mathbb{Q}^3\}$, where $1 \leq i, j \leq 3$. Making use of the known contraction of the one-dimensional well rounded trees in the Borel-Serre faces associated to this spherical apartment, in Chapter 5 I am able to explicitly construct a contraction h with properties summarized in the following technical result:

Theorem 3. *There exists a piecewise smooth, finite time contraction of* W*, the well-rounded retract for* GL_3*, such that:*

- *h is defined independently of any contraction of the ambient five-dimensional globally symmetric space;*
- *h is a "local lift" of the lower rank contractions defined on the Serre trees inside the Levi components of a subset of the Bore-Serre faces at infinity;*
- *the three cells generated using a filling based on the contraction h, are unions of* $GL_3(\mathbb{Z})$ *translates of a finite collection of* fundamental *3-cells (different from the Soule Cube in Figure 1.2.)*

Consequently, pairing these *fundamental* cells with closed differential forms, we arrive at:

Corollary 1. *There is a finite list of polynomials that completely determine each Eilenberg-MacLane cocycle for $GL_3(\mathbb{Z})$, up to coboundaries.*

2

Algebraic Groups and the Geometry of Locally Symmetric Spaces

In general, we will use $\mathbf{G}$ to denote a connected, reductive, algebraic group defined over $\mathbb{Q}$. More often than not, we will focus on *linear algebraic groups*, namely those that embed in GL_n via a faithful rational representation. We note that the radical of $\mathbf{G}$, $\mathbf{Rad}(\mathbf{G})$ is also defined over $\mathbb{Q}$, and we require $\mathbf{G}$ modulo $\mathbf{Rad}(\mathbf{G})$ to have a strictly positive $\mathbb{Q}$-rank. By this we mean that the dimension of the maximal $\mathbb{Q}$-split algebraic torus of $\mathbf{G}\,/\,\mathbf{Rad}(\mathbf{G})$ is strictly greater than zero. Furthermore, we let $X(\mathbf{G})$ be the group of rational characters $\xi : \mathbf{G} \to GL_1$, and by extension $X_{\mathbb{Q}}(\mathbf{G})$ those charcters that are defined over $\mathbb{Q}$. Often we will consider the subgroup ${}^0\mathbf{G}$ defined as $\{g \in \mathbf{G} \mid |g^{\xi}| = 1 \text{ for all } \xi \in X_{\mathbb{Q}}(\mathbf{G})\}$. We let $G = \mathbf{G}(\mathbb{R})$, and throughout we will make use of the convention to use boldface font to refer to algebraic groups, and unless otherwise specified, we will use regular font for the groups of real points of the corresponding bold face algebraic group.

In later sections we will focus our attention on $\mathbf{G} = GL_m$. In such instances, we use (ρ_m, V_m) to denote the standard representation of $GL_m(\mathbb{C})$, and $(\rho_m^n, \mathrm{Sym}^n(V_m))$

for the n-th symmetric product of the standard representation.

We use $\mathcal{P}$ to denote the set of parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$, namely those subgroups of $\mathbf{G}$ defined over $\mathbb{Q}$ containing a Borel subgroup. Similarly, we use $\mathcal{P}_1$ to denote the set of maximal parabolic subgroups in $\mathcal{P}$.

Let $\mathbf{A_G}$ denote the maximal $\mathbb{Q}$-split torus in the center $\mathbf{Z_G}$ of $\mathbf{G}$, and $A_{\mathbf{G}}$ the identity component of $\mathbf{A_G}(\mathbb{R})$. We use $K \subset G$ to denote a maximal compact subgroup, and let

$$D_G = G/KA_{\mathbf{G}},$$

be the associated globally symmetric space. For Γ, an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$, we use X to denote the corresponding locally symmetric space, namely,

$$X = \Gamma\backslash D_G.$$

When it is clear from context that $\mathbf{G} = GL_m$, we will abbreviate notation and use D_m to denote D_G.

Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$, $\mathbf{N_P}$ its unipotent radical, $\mathbf{S_P}$ a maximal $\mathbb{Q}$-split torus in the Levi quotient $\mathbf{L_P} = \mathbf{P}\,/\,\mathbf{N_P}$, and $A_{\mathbf{P}}$ the identity component of S_P. We consider the map $\kappa : \mathbf{P} \to \mathbf{L_P}$. For a fixed maximal compact subgroup K and the base-point $o = eKA_{\mathbf{G}} \in D_G$, let $\boldsymbol{\theta}_K$ denote the isometry of $\mathbf{G}$ extending the Cartan involution θ_K of G associated to K (and therefore fixing the base-point o in the qotient D_G.) We consider $\mathbf{L}_{\mathbf{P},o} \subset \mathbf{P}$ characterized as the unique lift of $\mathbf{L_P}$ under κ stable under the involution θ_K. Along these lines, $\mathbf{A_P}$ lifts to a unique subgroup of $\mathbf{A}_{\mathbf{P},o} \subset \mathbf{L}_{\mathbf{P},o}$, and in fact $\mathbf{L}_{\mathbf{P},o}$ is the centralizer of $\mathbf{A}_{\mathbf{P},o}$ in $\mathbf{G}$. Furthermore we fix a unique complement $\mathbf{M}_{\mathbf{P},o} = {}^0\mathbf{L}_{\mathbf{P},o}$. This yields the Langlands decomposition,

$$\mathbf{P} = \mathbf{N_P}\,\mathbf{L}_{\mathbf{P},o} = \mathbf{N_P}\,\mathbf{A}_{\mathbf{P},o}\,\mathbf{M}_{\mathbf{P},o},$$

which in turn decends to the groups of real points:

$$P = N_{\mathbf{P}} L_{\mathbf{P},o}.$$

Finally, one can show that $L_{\mathbf{P},o} = (\mathbf{A}_{\mathbf{P},o}\mathbf{M}_{\mathbf{P},o})(\mathbb{R}) = A_{\mathbf{P},o}M_{\mathbf{P},o}$, where we break from tradition and use $A_{\mathbf{P},o}$ to specify the identity component of the real points on $\mathbf{A}_{\mathbf{P},o}$. Therefore we have,

$$P = N_{\mathbf{P}}L_{\mathbf{P},o} = N_{\mathbf{P}}A_{\mathbf{P},o}M_{\mathbf{P},o}.$$

We can extend this decomposition to D_G. Namely, since $G = PK$, and $K \cap P = K \cap M_{\mathbf{P},o}$,

$$D_G \equiv N_{\mathbf{P}} \times A_{\mathbf{P},o} \times D_{\mathbf{P},o}, \tag{2.1}$$

where $D_{\mathbf{P},o} = M_{\mathbf{P},o}/K \cap M_{\mathbf{P},o}$.

Hereafter, in order to simplify notation we adopt the convention that whenever we are considering the intersection of two groups $A, B \subseteq G$, A_B will be used to denote $A \cap B$. Furthermore, whenever the group B has a subcript, it will be omitted in the nomenclature for the intersection so as to avoid using double subscripts. However we will resort to using the double subscript whenever the groups in question are not clear from the context. Therefore, $K \cap M_{\mathbf{P},o} = K_M$.

Finally, we make the following point. Since $G = PK$, an element $g \in G$ may be written as $g = pk = namk$, $n \in N_{\mathbf{P}}, a \in A_{\mathbf{P},o}, m \in M_{\mathbf{P},o}, k \in K$. We note that this decomposition is not unique. However, since $K \cap P = K \cap {}^0P$, we see that there is no ambiguity in the element a belonging to $A_{\mathbf{P},o}$ in the above decomposition of g. Therefore, we introduce the function $a_{\mathbf{P}}(.) : G \to A_{\mathbf{P},o}$, such that for g as above, $a_{\mathbf{P}}(g) = a$. We note that while this function does not quite decend to D_G, a function $a(g)^\chi$, for χ a character on $A_{\mathbf{P},0}$ that is trivial on the center certainly does.

2.1 Lie algebras and root systems

We use $\mathfrak{g}_{\mathbb{C}}$ to denote be the Lie algebra of $\mathbf{G}$, the complexification of $\mathfrak{g}$ the Lie algebra of G. Moreover, let $\mathfrak{m}_{\mathbf{G},\mathbb{C}} = \bigcap \ker d\xi$ where the intersection is taken over all $\xi \in X(\mathbf{G})$. We maintain this notation throughout, the regular gothic scripts $\mathfrak{g}, \mathfrak{m}_G, \mathfrak{p}, \mathfrak{a}, \mathfrak{n}$ are Lie algebras of Lie groups (often the real points of an algebraic group) while we use the subscript $\mathbb{C}$ to denote the respective complexifications.

We fix $\mathbf{T}$ a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$, and we fix a minimal parabolic subgroup $\mathbf{P}_0 \in \mathcal{P}$ containing $\mathbf{T}$. We let $\Phi_{\mathbb{Q}} = \Phi(\mathbf{G}, \mathbf{T})$ denote the $\mathbb{Q}$-root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to the adjoint action of $\mathbf{T}$. Then, $\mathbf{P}_0$ corresponds to a choice of a positive root system $\Phi_{\mathbb{Q}}^+$ on the $\mathbb{Q}$-root system $\Phi(\mathbf{G}, \mathbf{T})$. We maintain this ordering throughout, and use Δ_0 to denote the simple positive $\mathbb{Q}$-roots in $\Phi_{\mathbb{Q}}^+$. In line with standard notation, we call a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ *standard* with respect to $\mathbf{T}$ and $\Phi_{\mathbb{Q}}^+$, if $\mathbf{P} \subset \mathbf{P}_0$. Furthermore, for such a parabolic subgroup $\Phi_{\mathbb{Q}}^+$ induces a choice of positive roots in $\Phi(\mathbf{L}_{\mathbf{P}}, \mathbf{T})$, and consequently a set of simple roots $\Delta(\mathbf{L}_{\mathbf{P}}, \mathbf{T})$.

Next we let $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan sub-algebra containing the Lie algebra of $\mathbf{T}$, and choose an ordering on the complete root system $\Phi_{\mathbb{C}} = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, that is compatible with the order on $\Phi_{\mathbb{Q}}$. For a standard parabolic subgroup $\mathbf{P}$, let $\mathbf{M}_{\mathbf{P},o}$ be as before with a Lie algebra $\mathfrak{m}_{\mathbf{P},\mathbb{C}}$. Then $\mathfrak{b}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{m}_{\mathbf{P},\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{m}_{\mathbf{P},\mathbb{C}}$, and consequently we define the root system $\Phi(\mathfrak{m}_{\mathbf{P},\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$. The choice of ordering on $\Phi_{\mathbb{C}}$, induces a choice of simple roots $\Delta(\mathfrak{m}_{\mathbf{P},\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ that is consistent with the set of simple roots $\Delta(\mathbf{L}_{\mathbf{P}}, \mathbf{T})$. Let $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the Weyl group of $\mathbf{G}$ associated to $\mathfrak{h}_{\mathbb{C}}$. For a standard parabolic subgroup $\mathbf{P}$ we define the set of *Kostant representatives*,

$$W^{\mathbf{P}} = \{w \in W \mid w^{-1}\left(\Delta(\mathfrak{m}_{\mathbf{P},\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})\right) \subset \Phi_{\mathbb{C}}^+\}.$$

Traditionally, we set $\rho = \frac{1}{2}\sum_{\alpha \in \Phi_{\mathbb{C}}^+} \alpha$. In addition for a standard parabolic subgoup $\mathbf{P}$, we define $\rho_{\mathbf{P}} \in \mathfrak{a}_{\mathbf{P}}^*$, as $\rho|_{\mathfrak{a}_{\mathbf{P}}}$. Alternatively, we can define $\rho_{\mathbf{P}}$ as one half the sum of the roots for the action of $A_{\mathbf{P}}$ on $\mathfrak{n}_{\mathbf{P}}$ counted with multiplicity.

There is a natural inclusion $\mathbf{A}_{\mathbf{G}} \subset \mathbf{A}_{\mathbf{P}}$ and a complement $\mathbf{A}_{\mathbf{P}}^{\mathbf{G}} \subset \mathbf{A}_{\mathbf{P}}$ given as the joint kernel of the characters in $X_{\mathbb{Q}}(\mathbf{G})$. There is an adjoint action of $z \in \mathbf{L}_{\mathbf{P}}$ on the Lie algebra $\mathfrak{n}_{\mathbf{P},\mathbb{C}}$ given by lifting z to $\mathbf{L}_{\mathbf{P},o}$ and then acting by the adjoint action. We note that the lift to $\mathbf{L}_{\mathbf{P},o}$ is arbitrary and we can in fact lift z to any of the isomorphic pre-images of $\mathbf{L_P}$ in $\mathbf{P}$. While this will affect the adjoint action of $\mathbf{L_P}$ as we have defined it, it will not affect the weights with respect to the action of $\mathbf{A_P^G}$. We let $\Delta_{\mathbf{P}} = \{\alpha_1, \ldots, \alpha_r\}$ be the indecomposable elements among the $\mathbf{A_P^G}$-weights of $\mathfrak{n}_{\mathbf{P},\mathbb{C}}$, and $\widehat{\Delta}_{\mathbf{P}} = \{\beta_1, \ldots, \beta_r\}$ be the dual basis with respect to a Weyl group invariant inner product. With this notation in place, and the fact that elements of $\Delta_{\mathbf{P}}$ are $\mathbb{R}$-valued on $A_{\mathbf{P}}^{\mathbf{G}}$ we take note of the following isomorphism:

$$A_{\mathbf{P}}^{\mathbf{G}} \xrightarrow{\sim} (\mathbb{R}_+)^r \tag{2.2}$$
$$a \mapsto (a^{\alpha_1}, \ldots, a^{\alpha_r}).$$

The parabolic subgroups $\mathbf{Q}$ defined over $\mathbb{Q}$ containing $\mathbf{P}$ are in one-to-one order preserving correspondence with subsets $\Delta_{\mathbf{P}}^{\mathbf{Q}} \subset \Delta_{\mathbf{P}}$. In particular,

$$\mathbf{A_Q^G} \equiv \bigcap_{\alpha \in \Delta_{\mathbf{P}}^{\mathbf{Q}}} \ker\alpha \subseteq \mathbf{A_P^G},$$

and the set of simple roots $\Delta_{\mathbf{Q}}$ is the set of restrictions $\alpha|_{\mathbf{A_Q^G}}$ for $\alpha \in \Delta_{\mathbf{P}} \backslash \Delta_{\mathbf{P}}^{\mathbf{Q}}$. In fact,

$$\mathbf{A_P^G} = \mathbf{A_Q^G} \times \mathbf{A_P^Q},$$

where $\mathbf{A_P^Q}$ is the complement of $\mathbf{A_Q^G}$ in $\mathbf{A_P^G}$ defined as

$$\mathbf{A_P^Q} \equiv \bigcap_{\alpha \in \Delta_{\mathbf{P}} \backslash \Delta_{\mathbf{P}}^{\mathbf{Q}}} \ker\beta_\alpha.$$

Here, $\beta_\alpha \in \widehat{\Delta}_{\mathbf{P}}$ the dual element to α.

2.2 The Borel-Serre compactification

It follows from classical theorems in geometry that D_G is diffeomoprhic to Euclidean space. While this space lacks topological structure, its arithmetic quotient X certainly does not. In fact, we will almost exclusively be interested in the case when X

is a non-compact orbifold. In [BS73], the authors construct a bordification of $\overline{D_G}$, leading to a compactification $\overline{X} = \Gamma\backslash\overline{D_G}$ of X. This construction is "topological" in the sense that the inclusion map $i : X \hookrightarrow \overline{X}$ is a homotopy equivalence.

In this section we relate the classical construction of the Borel-Serre compactification, following the exposition given in [BS73], [Sap03], and [Sap97].

As summarized in [Sap97], the construction of $\overline{D_G}$ can be broken down in three steps:

1. Form (partial) bordifications of $A_{\mathbf{P}}^{\mathbf{G}}$ for each $\mathbf{P} \in \mathcal{P}$;
2. Extend the partial bordification to D_G;
3. Use the functoriality of the partial bordification to define the full bordification $\overline{D_G}$.

We explain each of these steps in greater detail.

2.2.1 Partial bordification of $A_{\mathbf{P}}^{\mathbf{G}}$

The construction of $\overline{A}_{\mathbf{P}}^{\mathbf{G}}$, the partial bordification of $A_{\mathbf{P}}^{\mathbf{G}}$, follows directly from the isomorphism presented in 2.2. In particular, we extend the positive real line and include infinity,

$$\overline{A}_{\mathbf{P}}^{\mathbf{G}} \xrightarrow{\sim} (\mathbb{R}_+^* \cup \infty)^{|\Delta_{\mathbf{P}}|}, \quad a \mapsto (a^{\alpha_1}, \ldots, a^{\alpha_r}).$$

We note that for $\mathbf{Q} \supseteq \mathbf{P}$, the inclusion $A_{\mathbf{Q}}^{\mathbf{G}} \hookrightarrow A_{\mathbf{P}}^{\mathbf{G}}$ extends to the partial bordifications. Furthermore the inverses of the root coordinates yield a real analytic structure for $\overline{A}_{\mathbf{P}}^{\mathbf{G}}$.

In addition to the inclusion $\overline{A}_{\mathbf{Q}}^{\mathbf{G}} \hookrightarrow \overline{A}_{\mathbf{P}}^{\mathbf{G}}$, we would also like to say how $A_{\mathbf{P}}^{\mathbf{Q}}$ lies inside $\overline{A}_{\mathbf{P}}^{\mathbf{G}}$. To do so, we fix a system of base points corresponding to each subset of $\Delta_{\mathbf{P}}^{\mathbf{Q}} \subseteq \Delta_{\mathbf{P}}$, and therefore to each parabolic subgroup $\mathbf{Q} \supseteq \mathbf{P}$. Namely, let $o_{\mathbf{Q}}$ be the point with coordinates

$$\alpha(o_{\mathbf{Q}}) = \begin{cases} \infty & \alpha \in \Delta_{\mathbf{P}} \backslash \Delta_{\mathbf{P}}^{\mathbf{Q}} \\ 1 & \alpha \in \Delta_{\mathbf{P}}^{\mathbf{Q}} \end{cases} .$$

We note that the stabilizer of $o_{\mathbf{Q}} \in \overline{A}_{\mathbf{P}}^{\mathbf{G}}$ are precisely those $a \in A_{\mathbf{P}}^{\mathbf{G}}$ such that $\alpha(a) = 1, \alpha \in \Delta_{\mathbf{P}}^{\mathbf{Q}}$, or in other words the stabilizer is $A_{\mathbf{Q}}^{\mathbf{G}}$. Hence the orbit of $o_{\mathbf{Q}}$ is $A_{\mathbf{P}}^{\mathbf{Q}} \cdot o_{\mathbf{Q}}$, providing us with the following stratification:

$$\overline{A}_{\mathbf{P}}^{\mathbf{G}} = \coprod_{Q \supseteq P} A_{\mathbf{P}}^{\mathbf{Q}} \cdot o_{\mathbf{Q}}. \tag{2.3}$$

2.2.2 Geodesic action and partial bordification of D_G

We note that P acts transitively on D_G, so $z \in D_G$ may be expressed as $z = po$, where o is the basepoint in D_G associated with the maximal compact subgroup K. The geodesic action of $A_{\mathbf{P}}$ is defined by

$$a \circ z = pa_o o \quad (a \in A_{\mathbf{P}}, z \in D_G),$$

where $a_o \in A_{\mathbf{P},o}$ is the unique lift of $a \in A_{\mathbf{P}}$ stable under the Cartan involution θ_K. We note that the geodesic action is independent of the choice of a basepoint and computes with the usual action (multiplication on the left) of P. Furthermore, for $P \subseteq R$, we have $A_{\mathbf{R}} \subseteq A_{\mathbf{P}}$, and the geodesic action of $A_{\mathbf{R}}$ is the restriction of the geodesic action of $A_{\mathbf{P}}$.

Next to each $\mathbf{Q} \in \mathcal{P}$ we associate the *Borel-Serre* boundary face $e(\mathbf{Q}) \equiv N_{\mathbf{Q}} \times D_{\mathbf{Q},o}$. Note, from (2.1), it follows that $e(\mathbf{Q}) \equiv D_G/A_{\mathbf{Q}}^{\mathbf{G}}$, where the quotient is taken with respect to the geodesic action of $A_{\mathbf{Q}}$.

Borrel and Serre define the *corner* $D_G(P)$ *associated to* P as the total space of the associated bundle with typical fibre $\overline{A}_{\mathbf{P}}$:

$$D_G(P) = X \times^{A_{\mathbf{P}}} \overline{A}_{\mathbf{P}} = \overline{A}_{\mathbf{P}} \times e(\mathbf{P}).$$

Saper points out that one should view $e(P)$ as a stratum of $D_G(P)$ by identifying it with $o_{\mathbf{P}} \times e(\mathbf{P}) = \{\infty\}^{\Delta_{\mathbf{P}}} \times e(\mathbf{P})$. Also, we note that in view of (2.3), we may write

$$D_G(P) = \coprod_{\mathbf{Q} \supseteq \mathbf{P}} D_G(\mathbf{P}, \mathbf{Q}),$$

where,

$$D_G(\mathbf{P}, \mathbf{Q}) = D_G \times^{A_{\mathbf{P}}} A_{\mathbf{P}}^{\mathbf{Q}} \cdot o_{\mathbf{Q}}$$

2.2.3 The full bordification of D_G

In this step we make use of the functoriality of the construction above, namely for $Q \supseteq P$, one can naturally identify $e(\mathbf{Q})$ with the subset $D_G(\mathbf{P}, \mathbf{Q}) \subset D_G(P)$, yielding $D(P) = \coprod_{\mathbf{Q} \supseteq \mathbf{P}} e(\mathbf{Q})$. The full bordification of D is then defined as,

$$\overline{D}_G = \bigcup_{\mathbf{P}} D_G(P) = \coprod_{\mathbf{P}} e(\mathbf{P}).$$

Theorem 4. *1. The action of $\mathbf{G}(\mathbb{Q})$ on D_G extends continuously to an action on $\overline{D}_G$ by homeomorphisms, such that for $g \in \mathbf{G}(\mathbb{Q}), ge(\mathbf{P}) = e({}^{g}\mathbf{P})$.*

2. The quotient $\overline{X} = \Gamma \backslash \overline{D}_G$ is a compact space containing X as its interior. The boundary $\partial(\overline{X})$ is a manifold with finite quotient singularities and decomposes as a finite union over the Γ-conjugacy classes of parabolic subgroups in $\mathcal{P}$,

$$\partial(\overline{X}) = \coprod_{\mathbf{P} \in \Gamma \backslash \mathcal{P}} \Gamma \cap \mathbf{P} \backslash e(\mathbf{P}) = \coprod_{\mathbf{P} \in \Gamma \backslash \mathcal{P}} e'(\mathbf{P}).$$

3. The inclusion $i : X \hookrightarrow \overline{X}$ is a homotopy equivalence whenever $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is torsion-free.

2.3 Automorphic cohomology

We begin by relating the cohomology of X to the relative Lie algebra cohomology of $\mathfrak{g}$ with respect to the Lie algebra $\mathfrak{k}$ of K. In particular, for (π, E) a $(\mathfrak{g}, K)$-module as in [BW00], consider the complex $D^\bullet(\mathfrak{g}, K, E)$ defined as:

$$D^\bullet(\mathfrak{g}, K, E) = \mathrm{Hom}_K\left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), E\right).$$

It follows from [BW00] that if we fix V, a representation of $\mathbf{G}$ over $\mathbb{C}$, there is a chain complex isomorphism $D^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G) \otimes V) \cong \Omega^\bullet(D_G, V)^\Gamma$, and consequently an isomorphism,

$$H^\bullet(X, \mathbb{V}) \cong H^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G) \otimes V).$$

Above, we use $\mathbb{V}$ for the local system on X associated to V. More specifically, if we let $\mu : D_G \to X$, then $\mathbb{V}$ is the locally constant sheaf defined on an open set $U \in X$ as

$$\mathbb{V}(U) = \left\{ s : \mu^{-1}(U) \to V \mid \begin{array}{c} s \text{ is locally constant} \\ s(\gamma x) = \gamma . s(x), \gamma \in \Gamma \end{array} \right\}.$$

We will use this description of the cohomology of X in order to describe the notions of *automorphic, square-integrable,* and *interior cohomology* later in this section.

In particular, we say a smooth function $f : \Gamma\backslash G \to \mathbb{C}$ is *automorphic* if

1. f is of moderate growth with respect to some norm $||\cdot||$ on G. More specifically, there exists constants C and d such that for all $g \in G, |f(g)| \leq C||g||^d$.

2. f is K-finite on the right.

3. f is $\mathcal{Z}(\mathfrak{g})$-finite on the left, where $\mathcal{Z}(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

The space of such functions is denoted $\mathcal{A}(\Gamma, G)$. If in addition, for all $g \in G$ and all proper parabolic subgroups $\mathbf{P} \in \mathcal{P}$,

$$\int_{\Gamma \cap N_{\mathbf{P}} \backslash N_{\mathbf{P}}} f(ng) = 0,$$

we say f is a *cuspidal* automorphic function. The inclusion $i : \mathcal{A}(\Gamma, G) \hookrightarrow C^\infty(\Gamma\backslash G)$ gives rise to a map

$$H^\bullet(\mathfrak{g}, K, \mathcal{A}(\Gamma, G) \otimes V) \to H^\bullet(\mathfrak{g}, K, C^\infty(\Gamma\backslash G) \otimes V) \cong H^\bullet(X, \mathbb{V}).$$

The image of this map is called *automorphic cohomology of* X.

In this section we also introduce the notion of an *automorphic form* as in [HC68], the vector-valued analogue of the automorphic function defined above. In particular, given any finite-dimensional representation (σ, W) of K, we denote by $\mathcal{A}(\Gamma\backslash G, \sigma)$ the space of all smooth functions $f : \Gamma\backslash G \to W$, such that

1. f is of moderate growth with respect to a choice of norms on G and W.

2. f is a σ-function:

$$f(kg) = \sigma(k)f(g) \quad \text{for all } k \in K, g \in G.$$

3. f is $\mathcal{Z}(\mathfrak{g})$-finite on the left.

We call functions $f \in \mathcal{A}(\Gamma\backslash G, \sigma)$, *automorphic forms of type* σ on G with respect to Γ.

2.4 Interior and cuspidal cohomology

The introducion of cuspidal, and square-integrable cohomology in this section is motivated by the exposition in [Sch90]. More specifically, consider $L^2(\Gamma\backslash G)$, the space of square-integrable, complex valued functions on the quotient $\Gamma\backslash G$. This Hilbert space is also a G-module under the right regular action of the group, and as such it admits a decomposition as a direct sum of the discrete and continuous spectrum,

$$L^2(\Gamma\backslash G) = L^2_d \oplus L^2_{\text{cont}}.$$

Furthermore, if we let L_0^2 denote the space of those square-integrable functions on $\Gamma\backslash G$ that are cuspidal automorphic forms, it follows from [HC68] that $L_0^2 \subset L_d^2$. In addition, this space decomposes into a Hilbert sum of irreducible subspaces H_π,

$$L_0^2 = \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) H_\pi, \tag{2.4}$$

where π ranges over the irreducible, unitary representations of G, and $\mu(\pi, \Gamma) < \infty$.

Much like in the case with automorphic cohomology, we map L_0^2, and $L_d^2 \cap C^\infty(\Gamma\backslash G)$ into $C^\infty(\Gamma\backslash G)$ with the goal of defining a map on (relative Lie algebra) cohomology. In particular the inclusion of the cuspidal square-integrable functions (resp. smooth, square-integrable functions) on $\Gamma\backslash G$ into $C^\infty(\Gamma\backslash G)$ gives rise to an maps j_0 (resp. j_d) on the level of cohomology [Bor81, Theorem 5.4, 5.5]. The image of j_0 is denoted $H^\bullet_{\mathrm{cusp}}(X, \mathbb{V})$ and is called *cuspidal* cohomology. On the other hand, the image of j_d is denoted $H^\bullet_{(2)}(X, \mathbb{V})$ and is called the *square-integrable* cohomology of X.

Let $\overline{\mathbb{V}}$ be the extension of the local system defined by $\mathbb{V}$ to $\overline{X}$. The pair $\left(\overline{X}, \partial\overline{X}\right)$ gives rise to the long exact sequence,

$$\ldots \longrightarrow H^\bullet_c(X, \mathbb{V}) \to H^\bullet(X, \mathbb{V}) = H^\bullet(\overline{X}, \overline{\mathbb{V}}) \xrightarrow{r^*} H^\bullet\big(\partial(\overline{X}), \overline{\mathbb{V}}\big) \longrightarrow \ldots \tag{2.5}$$

We call the kernel of r^* which coincides with the image of $H^\bullet_c(X, \mathbb{V})$ in $H^\bullet(X, \mathbb{V})$ *interior cohomology*, and we denote it by $H^\bullet_!(X, \mathbb{V})$. Since for each class $[w] \in H^\bullet_!(X, \mathbb{V})$ we can choose w to be compactly supported (and smooth), we have a natural map

$$\nu : H^\bullet_!(X, \mathbb{V}) \to H^\bullet_{(2)}(X, \mathbb{V}).$$

In fact, we conclude that:

$$H^\bullet_{\mathrm{cusp}}(X, \mathbb{V}) \subseteq H^\bullet_!(X, \mathbb{V}) \subseteq H^\bullet_{(2)}(X, \mathbb{V}).$$

3

Cuspidal Cohomology of GL_2

In the following chapter we relate the cohomology of $GL_2(\mathbb{Z})$ to that of $SL_2(\mathbb{Z})$. Results from this chapter will be particularly useful in Chapter 4 where we compute the cohomology of GL_3.

3.1 Cuspidal cohomology of $SL_2(\mathbb{Z})$

We begin by describing the cuspidal cohomology $H^1_{\text{cusp}}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\text{ym}^n(V_2))$.

We fix a basis of $V_2 = \text{Span}_{\mathbb{C}}\left\{\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{y} = \begin{pmatrix}0\\1\end{pmatrix}\right\}$, and analogously we fix a basis of $\text{Sym}^n(V_2)$, namely $\{\mathbf{x}^n, \mathbf{x}^{n-1}\,\mathbf{y}, \ldots, \mathbf{x}\,\mathbf{y}^{n-1}, \mathbf{y}^n\}$.

Let f be a cusp form of weight $n+2$ for the full modular group $SL_2(\mathbb{Z})$, and consider the $\text{Sym}^n(V_2)$-valued differential forms,

$$\omega_f = f(z)(z\,\mathbf{x}+\mathbf{y})^n dz,\;\; \omega_{\bar{f}} = \overline{f(z)}(\bar{z}\,\mathbf{x}+\mathbf{y})^n d\bar{z}.$$

Lemma 5. *The differential forms ω_f, and $\omega_{\bar{f}}$ are $SL_2(\mathbb{Z})$-equivariant,$(\rho_2^n, \text{Sym}^n(V_2))$-valued differential forms,* i.e.,

$$L_\gamma^*(\omega_f) = \rho_2^n(\gamma).\omega_f,\;\; L_\gamma^*(\omega_{\bar{f}}) = \rho_2^n(\gamma).\omega_{\bar{f}},\;\; \gamma \in SL_2(\mathbb{Z}).$$

Above the action on the left hand side is induced by the automorphism on the upper half-plane mapping z to $\gamma.z$, and on the right hand side, the action of γ is on the vector values of these differential forms.

Proof. We prove the lemma for ω_f. The proof for $\omega_{\bar{f}}$ follows in similar fashion. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Therefore,

$$\rho_2(\gamma)(\mathbf{x}) = \gamma\,\mathbf{x} = a\,\mathbf{x} + c\,\mathbf{y}; \;\; \rho_2(\gamma)(\mathbf{y}) = \gamma\,\mathbf{y} = b\,\mathbf{x} + d\,\mathbf{y}\,.$$

In particular,

$$\begin{aligned} L_\gamma^*(\omega_f) &= f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)\left(\frac{az+b}{cz+d}\,\mathbf{x} + \mathbf{y}\right)^n \\ &= (cz+d)^{n+2} f(z) \frac{dz}{(cz+d)^2}\left(\frac{az+b}{cz+d}\,\mathbf{x} + \mathbf{y}\right)^n \\ &= f(z)dz\,((az+b)\,\mathbf{x} + (cz+d)\,\mathbf{y})^n \\ &= f(z)dz\,(z(a\,\mathbf{x} + c\,\mathbf{y}) + (b\,\mathbf{x} + d\,\mathbf{y}))^n = \rho_2^n(\gamma)\omega_f. \end{aligned}$$

□

In [Eic57] and [Shi59], Eichler and Shimura established an important relationship between the space of holomorphic cusp of weight $n+2$ for the full modular group, denoted $S_{n+2}(SL_2(\mathbb{Z}))$ and the cuspidal cohomology of $SL_2(\mathbb{Z})\backslash\mathcal{H}^+$. In particular,

Theorem 6 (Eichler-Shimura)**.** *The cuspidal cohomology of $SL_2(\mathbb{Z})$ is entirely described by the space of cusp forms via the following isomorphism:*

$$S_{n+2}(SL_2(\mathbb{Z})) \oplus \overline{S_{n+2}(SL_2(\mathbb{Z}))} \xrightarrow{\sim} H^1_{cusp}\left(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2)\right)$$

$$f_1 + \bar{f}_2 \mapsto \omega_{f_1} + \omega_{\bar{f}_2}.$$

3.2 The action of $GL_2(\mathbb{Z})$ on the upper half-plane

Consider the map

$$\chi : GL_2(\mathbb{R})/O_2(\mathbb{R})(\lambda\cdot\mathrm{Id}_2)_{\lambda>0} \xrightarrow{\sim} SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

$$gO_2(\mathbb{R})(\lambda\cdot\mathrm{Id}_2)_{\lambda>0} \mapsto \tilde{g}SO_2(\mathbb{R})$$

where

$$\tilde{g} = \begin{cases} \frac{1}{\sqrt{\det(g)}} g & \det(g) > 0 \\ \frac{1}{\sqrt{|\det(g)|}} g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \det(g) < 0. \end{cases}$$

Since we are accustomed to identifying the right hand side with the upper half-plane, we see that this isomorphism allows us to define an action of $GL_2(\mathbb{Z})$ on $\mathcal{H}^+$. In particular, using the decomposition,

$$GL_2(\mathbb{Z}) = \left\{ \mathrm{Id}_2, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \ltimes SL_2(\mathbb{Z}),$$

and defining the action of $GL_2(\mathbb{Z})$ on $\mathcal{H}^+$ as,

$$\gamma.z = \chi(\gamma\chi^{-1}(z)), \;\; \gamma \in GL_2(\mathbb{Z})$$

we see that this action differs from the usual action of $SL_2(\mathbb{Z})$ on $\mathcal{H}^+$ only in the action of the element $\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Tracing through the mapping, we see that this element maps z with $-\bar{z}$. Therefore, we can identify the $GL_2(\mathbb{Z})$-equivariant cohomology of $GL_2(\mathbb{R})/O_2(\mathbb{R})(\lambda \cdot \mathrm{Id}_2)_{\lambda>0}$ with coefficients in the local system $\mathbb{V}$ associated to the $GL_2(\mathbb{C})$-module (ρ, V) with:

$$H^\bullet(SL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{V})^+ = \{[\omega] \in H^\bullet(SL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{V}) \mid L_\varepsilon^*(\omega) = \rho(\varepsilon)\omega\}.$$

3.3 Cuspidal cohomology of $GL_2(\mathbb{Z})$

We now turn our attention to identifying the cuspidal cohomology of $GL_2(\mathbb{Z})$, or in other words, the portion of the cuspidal cohomology of $SL_2(\mathbb{Z})$ that is also equivariant under the action of ε.

Theorem 7. *The cuspidal cohomology of $GL_2(\mathbb{Z})$ with coefficients in* $\mathrm{Sym}^n(V_2)$ *is* *isomorphic over* $\mathbb{R}$ *to the space of cusp forms* $S_{n+2}(SL_2(\mathbb{Z}))$ *via the isomorphism*

$$S_{n+2}(SL_2(\mathbb{Z})) \xrightarrow{\sim} H^1_{cusp}\left(GL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2)\right)$$

$$f \mapsto \omega_f - \omega_{\bar{f}}$$

Proof. Let $f \in S_{n+2}(SL_2(\mathbb{Z}))$. Using the change of variables $q = e^{2\pi i z}$, we may write $f(z) = \sum_{i=k}^{\infty} a_k q^k$. In order to understand how ω_f transforms under the mapping $z \mapsto -\bar{z}$, we need to gain a better a better understanding of the coefficients $\{a_k\}_{k=1}^{\infty}$ in the q-expansion of f. While there are many ways to approach this problem, I will show that these coefficients are rational numbers by analyzing the structure of the ring of all modular forms $M_\bullet(SL_2(\mathbb{Z})) = \oplus_{k=4}^{\infty} M_k(SL_2(\mathbb{Z}))$.

For $r \in \mathbb{Z}_{>0}$, let $\sigma_r : \mathbb{Z}_{>0} \to \mathbb{Z}$, be the divisor sum defined as,

$$\sigma_r(n) = \sum_{d|n} d^r.$$

The normalized Eisenstein series of weight 4 and 6 respectively are defined as:

$$G_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \quad G_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

These are holomorphic functions on $\mathcal{H}^+$, and one can show that in fact, G_4 and G_6 are modular forms of weight 4 and 6 respectively. In a similar fashion one may define G_{2k} for $k \in \mathbb{Z}_{\geq 2}$, a modular form of weight $2k$ with a unit constant coefficient. At this point we make the observation that $\dim M_k(SL_2(\mathbb{Z})) = \dim S_k(SL_2(\mathbb{Z})) + 1$. This is so because for every modular form of weight k not equal to (a multiple) of G_k, we may subtract a suitable multiple of G_k in order to obtain a cusp form of weight k. We can describe the ring $M_\bullet(SL_2(\mathbb{Z}))$ in even greater detail via the following

Theorem 8 (Proposition 1.3.4, [Bum97])**.** *Suppose that* k *is an even nonnegative integer. Let* $k = 12j + r$ *where* $0 \leq r \leq 10$*. Then*

$$\dim M_{12j+r}\left(SL_2(\mathbb{Z})\right) = \begin{cases} j+1 & \textit{if } r = 0, 4, 6, 8 \textit{ or } 10; \\ j & \textit{otherwise.} \end{cases}$$

The ring $\oplus_{k=0}^{\infty} M_k\left(SL_2(\mathbb{Z})\right)$ *of modular forms is generated by* G_4 *and* G_6*.*

Since the coefficients of G_4 and G_6 in their respective q-expansions are rational, this concludes the proof that for a cusp form f, there exists a (non-unique) $\tau \in \mathbb{C}$ such that $\tau f = \sum_{k=1}^{\infty} a_k q^k$, where the coefficients $\{a_k\}_{k=1}^{\infty}$ are rational as well. Hereafter we assume this is always the case for a cusp form f. Therefore for $z = \alpha + i\beta$,

$$f(-\bar{z}) = \sum_{k=1}^{\infty} a_k e^{2\pi i k(-\bar{z})} = \sum_{k=1}^{\infty} a_k e^{2\pi i k(-x)} e^{2\pi k(-y)} = \overline{f(z)}.$$

Consequently,

$$\begin{aligned} L_\varepsilon^*(\omega_f - \omega_{\bar{f}}) &= [f(-\bar{z})(-\bar{z}\,\mathbf{x} + \mathbf{y})^n d(-\bar{z})] - \left[\overline{f(-\bar{z})}(\overline{-\bar{z}}\,\mathbf{x} + \mathbf{y})^n d\overline{(-\bar{z})}\right] \\ &= \left[\overline{f(z)}(\bar{z}\,\mathbf{x} - \mathbf{y})^n d(\bar{z})\right] - [-f(z)(-z\,\mathbf{x} + \mathbf{y})^n dz] \\ &= f(z)(z(-\mathbf{x}) + \mathbf{y}) - \overline{f(z)}(\bar{z}(-\mathbf{x}) + \mathbf{y})^n d(\bar{z}) = \rho_2^n(\varepsilon)(\omega_f - \omega_{\bar{f}}). \end{aligned}$$

Similarly, one can show that,

$$L_\varepsilon^*(\omega_f + \omega_{\bar{f}}) = -\rho_2^n(\varepsilon)(\omega_f - \omega_{\bar{f}}).$$

If we define

$$H^\bullet(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{V})^- = \{[\omega] \in H^\bullet(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{V}) \mid L_\varepsilon^*(\omega) = -\rho(\varepsilon)\omega\},$$

then we conclude that

$$\begin{array}{ccccc} H^1_{\mathrm{cusp}}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^- & \xleftarrow{\sim} & S_{n+2}\left(SL_2(\mathbb{Z})\right) & \xrightarrow{\sim} & H^1_{\mathrm{cusp}}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^+ \\ \omega_f + \omega_{\bar{f}} & \leftarrow\!\shortmid & f & \mapsto & \omega_f - \omega_{\bar{f}}. \end{array}$$

□

We make a final note with regards to the cohomology of $GL_2(\mathbb{Z})$. More specifically, for $\det : GL_2 \to \mathbb{C}^*$ the standard character given by the determinant, we consider the cuspidal cohomology

$$H^\bullet_{\mathrm{cusp}}(GL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2) \otimes \det) = H^\bullet_{\mathrm{cusp}}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2) \otimes \det)^+.$$

Since the determinant character is trivial on $SL_2(\mathbb{Z})$, we focus on the invariance condition with regards to the element ε:

$$L_\varepsilon^*(\omega) = \rho_2^n(\varepsilon) \otimes \det(\varepsilon)\omega = -\rho_2^n(\varepsilon)\omega, \;\; [\omega] \in H^\bullet_{\text{cusp}}(SL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{S}\text{ym}^n(V_2) \otimes \det).$$

Therefore we discover that,

$$H^\bullet_{\text{cusp}}(GL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{S}\text{ym}^n(V_2) \otimes \det) = H^\bullet_{\text{cusp}}(SL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{S}\text{ym}^n(V_2))^-.$$

3.4 Eichler-Shimura via automorphic forms

For the time being let us assume we are in the framework of [MM63] and let Γ be torsion free, G connected and semi-simple, and $\Gamma\backslash G$ compact. We fix a maximal compact subgroup K in G and let (E, ρ) be a representation of $\mathbf{G}(\mathbb{C})$. Though the beginning of this discussion is presented in such generalities, we will always keep in mind the case when $G = SL_2(\mathbb{R})$, $K = SO_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$. Even though the torsion free and compactness assumption are violated, this general discussion will still apply.

Adhering to standard notation, we write the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, the Lie algebra of K, as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $A^p(\Gamma, D_G, \rho)$ denote the vector space of all E-valued differential forms ω on D_G such that $(L_\gamma)^*\omega = \rho(\gamma)\omega$ for all $\gamma \in \Gamma$. Matsushima and Murakami demonstrate an isomorphism between $A^p(\Gamma, D_G, \rho)$ and the complex $A_0^p(\Gamma, D_G, \rho)$ of E valued p-forms ω^0 on $\Gamma\backslash G$ satisfying

$$\xi(X)\omega^0 = -\rho(X)\omega^0, i(X)\omega^0 = 0, \tag{3.1}$$

for all $X \in \mathfrak{k}$, where $\xi(X)$ denotes the operator of Lie derivation by the vector field X. We choose a basis $\{X_1, \ldots, X_N\}$, and $\{X_{N+1}, \ldots, X_n\}$ of $\mathfrak{p}$ and $\mathfrak{k}$ respectively, such that,

$$B(X_i, X_j) = \delta_{ij}, B(X_a, X_b) = -\delta_{ab}, \qquad \text{for } 1 \leq i, j \leq N, \text{ and}, N+1 \leq a, b \leq N.$$

In the future, we maintain the above notation where we use indices i, j to denote basis elements of $\mathfrak{p}$ and letters a, b to denote basis elements of $\mathfrak{k}$. Since a form $\omega \in A_0^p(\Gamma, D_G, \rho)$ is entirely determined by the $E-$valued functions $\omega_{i_1,\ldots,i_p} = \omega_0(X_{i_1}, \ldots, X_{i_p})$ we can phrase the conditions in equation (3.1) as

$$X_a \omega_{i_1,\ldots,i_p} = -\rho(X_a)\omega_{i_1,\ldots,i_p} - \sum_{u=1}^{p}\sum_{j=1}^{N} c_{a i_u j}\omega_{i_1,\ldots,(j)_u,\ldots,i_p}, \tag{3.2}$$

where $(j)_u$ denotes that the index i_u is replaced by the j, and $[X_\lambda, X_\mu] = \sum_\nu c_{\nu\lambda\mu} X_\nu$. We seek to explain how the forms in $A_0^p(\Gamma, D_G, \rho)$ can be explained using the theory of Automorphic Forms.

Let $\pi : \Gamma\backslash G \to \Gamma\backslash D_G$ denote the canonical projection, T^X denote the tangent bundle of $\Gamma\backslash D_G$, and consider the pullback bundle $\pi^*(T^X) \cong \Gamma\backslash G \times \mathfrak{p}$. We see that vectorfields on $\Gamma\backslash D_G$ can be identified with smooth functions

$$f : \Gamma\backslash G \to \mathfrak{p}$$

which satisfy,

$$f(gk) = \mathrm{Ad}(k) f(g),$$

i.e. the tangent bundle of $\Gamma\backslash D_G$ is induced by the adjoint representation of K on $\mathfrak{p}$, or yet in another way $T^X \cong G \times \mathfrak{p}/\sim$, where $(g_1, v_1) \sim (g_2, v_2)$ if there exists a $k \in K$ such that $(g_1, v_1) = (g_2 k, \mathrm{Ad}(k^{-1})v_2)$. This in turn, immediately tells us that we can think of p forms on $\Gamma\backslash D_G$ as smooth functions

$$f : \Gamma\backslash G \to \wedge^p \mathfrak{p}^*,$$

which satisfy,

$$f(gk) = \left(\bigwedge^p \mathrm{Ad}^*\right)(k^{-1})(f(g)).$$

The transition to vector valued differential forms is painless. As in [MM63], we can identify the bundle $E(\rho)$ over $\Gamma\backslash D_G$ with the bundle $\tilde{E}(\rho)$ defined as

$$(\Gamma\backslash G \times E) / \sim$$

where $(g_1, v_1) \cong (g_2, v_2)$ if, and only if, there exists a $k \in K$ such that $(g_1, v_1) = (g_2 k, \rho(k^{-1})v_2)$. This immediately yields that, $A^p(\Gamma, D_G, \rho)$ can be identified with functions

$$f : \Gamma\backslash G \to \Lambda^p \mathfrak{p}^* \otimes E$$

satisfying

$$f(gk) = (\Lambda^p \mathrm{Ad}^*)(k^{-1}) \otimes \rho(k^{-1})(f(g)). \tag{3.3}$$

In fact, we can recover condition (3.2) from the above equation. Note, that an equivalent way of stating (3.3) is to say that, for $X_a \in \mathfrak{k}$,

$$(X_a f)(g) = \big(- \Lambda^p \mathrm{ad}^*(X_a) \otimes 1 - 1 \otimes \rho(X_a)\big)(f(g)).$$

Therefore,

$$X_a f(X_{i_1} \dots X_{i_p}) = -\rho(X_a) f(X_{i_1} \dots X_{i_p}) - \sum_{u=1}^{p} f(X_{i_1}, \dots, [-X_a, X_{i_u}], \dots, X_{i_p}) \tag{3.4}$$

$$= -\rho(X_a) f(X_{i_1} \dots X_{i_p}) + \sum_{u=1}^{p} f(X_{i_1}, \dots, \sum_{j=1}^{N} c_{jai_u} X_j, \dots, X_{i_p})$$

$$= -\rho(X_a) f(X_{i_1} \dots X_{i_p}) - \sum_{u=1}^{p} \sum_{j=1}^{N} c_{ai_uj} f(X_{i_1}, \dots, X_j, \dots, X_{i_p})$$

Above, we used the fact that $c_{jai_u} = -c_{ai_uj}$ [MM63, 4.7]. Note, equation (3.4) is precisely the same as the (3.2).

Remark 1. *We can also recover (3.3) from (3.1). Namely, let us begin with,*

$$R_k^*\eta = \rho(k^{-1})\eta.$$

The left hand side evaluated at a point $g \in G$ *and a vector* $L_{g_*}X$*, where* $X \in \mathfrak{g}$ *is,*

$$\begin{aligned} R_k^*\eta(g)(L_{g*}X) &= \eta(gk)(R_{k*}L_{g*}X) = \eta(gk)(L_{g*}L_{k*}L_{k^{-1}*}R_{k*}X) \\ &= \eta(gk)(L_{gk*}Ad(k^{-1})X). \end{aligned}$$

Suppressing some notation, and evaluating the right hand side as well we obtain,

$$\eta(gk)(Ad(k^{-1})X) = \rho(k^{-1})\eta(g)(X).$$

This is equivalent to,

$$\rho(k)Ad^*(k)R_k^*\eta = \eta.$$

This is precisely the same as equation (3.3).

In this framework it is quite clear how differential forms, subject to certain growth conditions, can be thought of as automorphic forms. Also, this gives us the opportunity to recast Eichler and Shimura's construction in the context of automorphic forms.

Namely, let as before $f \in S_n(\Gamma)$, and consider $\nu_f = f(z)(z\,\mathbf{x}+\mathbf{y})^n$, a 0-degree differential form on D_G that is Γ equivariant. As such we know that, to ν_f we may associate a 0-degree, $\mathrm{Sym}^n(V_2)$-valued differential form ν_f^0 on $\Gamma\backslash G$ satisfying condition (3.3). Following [MM63] this form is explicitely constructed as,

$$\nu_f^0(g) = \rho_2^n(g)^{-1}(\pi^*\nu_f)(g).$$

To explore the above expression, we make a change of basis, namely rather than $\{\mathbf{x},\mathbf{y}\}$, we take the following basis of V_2, $X = \begin{pmatrix} 1 \\ -i \end{pmatrix}, Y = \begin{pmatrix} 1 \\ i \end{pmatrix}$. This choice of basis elements is not arbitrary: note that X and Y are weight vectors for the restriction of the standard representation to K. We would like to write $\nu_f^0 = \sum_{j=0}^n f_j X^j \otimes Y^{n-j}$, for f_j suitable functions on $\Gamma\backslash G$. Note, the fact that ν_f^0 satisfies condition (3.3) immediately tells us that $f_j(gk) = \chi_j(k) f_i(g)$ for χ_j a character of K. An explicit computation yields even more information. Namely, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$:

$$\begin{aligned}
\nu_f^0(g) = \rho_2^n(g^{-1})\pi^*(\nu_f)(g) &= f(g.i)\left((g.i)\rho_2(g^{-1})\,\mathbf{x} + \rho_2(g^{-1})\,\mathbf{y}\right)^n \\
&= f(g.i)\left(\frac{ai+b}{ci+d}(d\,\mathbf{x} - c\,\mathbf{y}) + (-b\,\mathbf{x} + a\,\mathbf{y})\right)^n \\
&= \frac{1}{(ci+d)^n} f(g.i)\left((ai+b)(d\,\mathbf{x} - c\,\mathbf{y}) + (ci+d)(-b\,\mathbf{x} + a\,\mathbf{y})\right)^n \\
&= \frac{1}{(ci+d)^n} f(g.i)(i\,\mathbf{x} + \mathbf{y})^n = (i)^n h_f(g) X^n,
\end{aligned}$$

where $h_f(g) = \frac{1}{(ci+d)^n} f(g.i)$. Therefore, in terms of writing $\nu_f^0 = \sum_{j=0}^n f_j X^j \otimes Y^{n-j}$ we have,

$$f_j(g) = \begin{cases} (i)^n h_f(g) & j = n \\ 0 & \text{otherwise,} \end{cases} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We would like to replicate this construction for 1-forms. Rather than the Eichler-Shimura approach of starting with forms on the upper half-plane, we would rather like to arrive at ω_f^0 by means of automorphic forms, starting and ending with functions on $\Gamma\backslash G$. To this end, for $f \in S_k\left(SL_2(\mathbb{Z})\right)$ we set

$$h_f = \left(\frac{1}{(ci+d)}\right)^k f(g(i)).$$

The map $f \mapsto h_f$, from $S_k(\Gamma)$ to $C^\infty(\Gamma\backslash G, \mathbb{C})$ is a classical construction which yields automorphic functions on the group G with respect to Γ. We rehash condition (3.3): namely, Sym^n-valued differential forms of degree 1 are functions $\omega : \Gamma\backslash G \to$

$\mathfrak{p}^* \otimes \mathrm{Sym}^n(V_2)$ such that $\omega(gk) = \mathrm{Ad}^*(k^{-1}) \otimes \rho_{n+2}(k^{-1})\omega(g)$. Given, $f \in S_{n+2}(\Gamma)$ we construct ω_f^0 that satisfies this transformation law. As previously, a good starting point is $(i)^n h_f(g)$. To complete this function on $\Gamma\backslash G$ to a vector valued differential form we need to find a subspace of $\mathfrak{p}^* \otimes \mathrm{Sym}^n(V_2)$ of the same K-type as h_f. We already have a basis $\{X^i Y^{n-i}\}_{i=0}^n$ of $\mathrm{Sym}^n(V_2)$ consisting of weight vectors for K. We find a similar basis for $\mathfrak{p}^*$. Namely, let

$$k_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, P_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Then $\mathrm{Ad}(k_\theta)(P_1) = e^{-2i\theta} P_1$, and $\mathrm{Ad}(k_\theta)(P_2) = e^{2i\theta} P_2$. We let $\{P_i^*\}$ be the dual basis for $\mathfrak{p}^*$. Note, $\mathrm{Ad}^*(k_\theta)(P_1^*) = e^{2i\theta} P_1^*$, and $\mathrm{Ad}^*(k_\theta)(P_2^*) = e^{-2i\theta} P_2^*$. Therefore,

$$k_\theta(P_1^* \otimes X^n) = \mathrm{Ad}^*(k_\theta^{-1})(P_1^*) \otimes \rho_2^n(k_\theta^{-1}) X^n = e^{-i(n+2)\theta} P_1^* \otimes X^n.$$

Therefore, we set

$$\omega_f^0(g) = (i)^n h_f(g) P_1^* \otimes X^n.$$

By construction, (3.3) is satisfied. Furthemore, from the above discussion, it is clear that

$$\omega_f^0 = \rho_2^n(g)^{-1}(\pi^* \omega_f)(g).$$

On the other hand, for $f \in S_{n+2}(\Gamma)$ parallel to the construction of h_f, we consider $h_{\bar{f}} = \overline{h_f}$. Then, $h_{\bar{f}}(gk_\theta) = e^{i(n+2)\theta} h_{\bar{f}}(g)$. By the discussion above, it is clear that the differential form,

$$\omega_{\bar{f}}^0 = (i)^n h_{\bar{f}}(g) P_2^* \otimes Y^n$$

also satisfies (3.3).

Remark 2. *Note the differential in the upper half-plane* $\frac{\partial}{\partial x}|_i$ *is given by the left-invariant vector-field evaluating to* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ *at the identity. To see this in detail,* *note that,*

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . f\right](e) = \left\{\frac{d}{dt} f\left(\exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right)\right\}_{t=0}$$

$$= \frac{d}{dt} f(i+t)|_{t=0} = \frac{\partial}{\partial x} f(z)|_{z=i}.$$

We find that modulo $\mathfrak{k}$, $\frac{\partial}{\partial x}|_i$ *is represented by by* $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Similarly* $\frac{\partial}{\partial y}|_i$ *is given by* $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Therefore,*

$$\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = -\frac{i}{4}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = -\frac{i}{4} P_1.$$

Hence, dz *does indeed correspond to (a scalar multiple of)* P_1^*, *as desired.*

4

Cohomology of GL_3

4.1 Introduction

Several authors have studied the vector valued cohomology of GL_3. Concretely, in [Gon01, Theorem 6.2] Goncharov exhibits an isomorphism between a certain space of cusp forms of $SL_2(\mathbb{Z})$ and $H^\bullet(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$. Below we offer an alternative, constructive argument, involving the theory of Eisenstein series. The advantage of this approach is that it provides insight in the manner in which specific cohomology classes of differential forms restrict to different boundary components in the Borel - Serre compactification of X.

4.2 The A_2-root system

Throughout this chapter, unless otherwise specified, $\mathbf{G} = GL_3$, $\mathbf{T}$ is the maximal $\mathbb{Q}$-split torus consisting of all diagonal elements in $\mathbf{G}$, and $\mathbf{P}_0 \supset \mathbf{T}$ is the minimal parabolic subgroup of all upper triangular matrices in $\mathbf{G}$. We note that $\mathfrak{a}_{\mathbf{P}_0,\mathbb{C}}$ is a

Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and let

$$l_i : \mathfrak{a}_{\mathbf{P}_0,\mathbb{C}} \to \mathbb{C}$$

$$\begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \mapsto d_i.$$

In Figure 4.1, we depict the A_2 root system, together with our choice of simple roots $\alpha_1 = l_1 - l_2$, and $\alpha_2 = l_2 - l_3$.

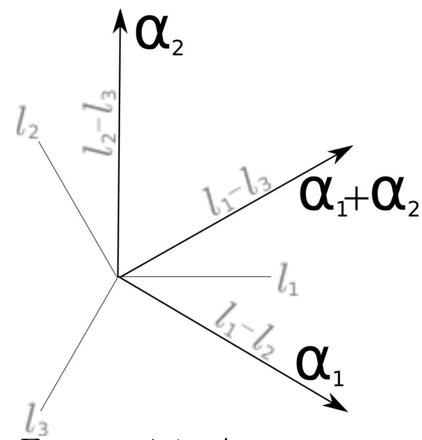


FIGURE 4.1: A_2 root system

With this choice of a fundamental Weyl chamber, the standard representation V_3 has highest weight $\mu = l_1$. In terms of the choice of simple roots as seen Figure 4.1, $\mu = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$. Hence, the highest weight of $\operatorname{Sym}^n V_3$ is $\lambda = n\mu = \frac{2}{3}n\alpha_1 + \frac{1}{3}n\alpha_2$. In terms of the basis l_1, l_2, l_3 of $\mathfrak{a}^*_{\mathbf{P}_0,\mathbb{C}}$, $\operatorname{Sym}^n V_3$ is represented by the coefficient vector $\lambda = (n, 0, 0)$.

Next we express the dual roots $\beta_{\alpha_1}, \beta_{\alpha_2}$ (also the fundamental weights) in terms of the basis $\{l_i\}$. Recall, for $i, j = 1, 2$, these are defined as $\langle \beta_{\alpha_i}, \alpha_j \rangle = \delta_{ij}$, where $\langle , \rangle$ is the inner product with respect to the Killing form on $\mathfrak{a}^*_{\mathbf{P}_0,\mathbb{C}}$, and δ_{ij} is the usual Kroneker δ. Here we make a note that,

$$\langle \sum_i a_i l_i, \sum_j b_j l_j \rangle = \sum_i a_i b_i - 2 \sum_{i,j} a_i b_j.$$

Therefore, accounting for the fact that β_{α_i} needs to vanish on the center, we have $\beta_{\alpha_1} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \beta_{\alpha_2} = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

We can write λ in terms of the basis $\{\beta_{\alpha_1}, \beta_{\alpha_1}\}$ as:

$$\lambda = \langle \lambda, \beta_{\alpha_1} \rangle \alpha_1 + \langle \lambda, \beta_{\alpha_2} \rangle \alpha_2.$$

Finally, we can enumerate the six elements in the Weyl group W as

$$W = \{\mathrm{Id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1} s_{\alpha_2}, s_{\alpha_2} s_{\alpha_1}, s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}\}.$$

Here we used s_{α_i} to denote the reflection in $\mathfrak{a}^*_{\mathbf{P}_0,\mathbb{C}}$ through the hyperplane perpendicular to α_i.

In §4.4.1 we will benefit from working with the following maximally compact Cartan subalgebra of $\mathfrak{g}$:

$$\mathfrak{b} = \left\{ \begin{pmatrix} -2t & 0 & 0 \\ 0 & t & -s \\ 0 & s & t \end{pmatrix} \oplus c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c, t, s \in \mathbb{R} \right\}.$$

The roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{b}_\mathbb{C}$ are, $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C}) = \{\pm 2is, \pm 3t \pm is\}$. We choose a positive root system, by defining the simple roots to be $\Delta = \{\tilde{\alpha}_1 = -3t - is, \tilde{\alpha}_2 = 2is\}$.

4.3 Vanishing of interior cohomology for $V = \mathrm{Sym}^n(V_3)$

We can say something about the parameter n, in $H^\bullet(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$, even before we begin our investigations. Namely, since $-Id \in GL_3(\mathbb{Z})$ acts non-trivially when n is odd, we conclude that in this instance, $H^\bullet(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3)) = 0$. Therefore, hereafter we assume that n is even.

Lemma 9. $\mathrm{Sym}^n(V_3)$ *as a representation of* GL_3 *is not conjugate self-contragradient.*

Proof. Since this representation is real, we need to show that this representation (when restricted to the derived subgroup) is not self-contragradient. Note, we can obtain the highest weight for the contragradient representation as $-w_G(\lambda)$, where w_G is the longest element of the Weyl group of GL_3. It is immediately apparent that $w_G = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ maps α_1 to $-\alpha_2$, and α_2 to $-\alpha_1$. Therefore,

$$-w_G(\lambda) = \langle \lambda, \beta_{\alpha_2} \rangle \alpha_1 + \langle \lambda, \beta_{\alpha_1} \rangle \alpha_2.$$

Therefore, for a general representation E of highest weight $\nu = (a_1, a_2, a_3)$, with $\sum_i a_i = 0$, we can restate the contragradient condition as

$$\langle \nu, \beta_{\alpha_1} \rangle = \langle \nu, \beta_{\alpha_2} \rangle, \text{ or } a_1 - 2a_2 + a_3 = 0.$$

As this is certainly not the case for $\mathrm{Sym}^n(V_3)$, this concludes the proof of the theorem. □

Corollary 10. *The square integral cohomology $H^\bullet_{(2)}(X, \mathbb{S}\mathrm{ym}^n(V_3))$ vanishes in all degrees.*

Proof. This result follows from considerations in [BC83, Section 2.2]. □

With this in mind, referencing the long exact sequence (2.5) we conclude that the restriction map,

$$r^* : H^\bullet\left(\overline{X}, \overline{\mathbb{S}\mathrm{ym}^n(V_3)}\right) \to H^\bullet\left(\partial\left(\overline{X}\right), \overline{\mathbb{S}\mathrm{ym}^n(V_3)}\right)$$

is injective.

4.4 Cohomology of the face $e'(\mathbf{P})$

For a fixed parabolic subgroup $\mathbf{P} \subset \mathcal{P}$, in this section we introduce the notation $\Gamma_{\mathbf{L_P}} = \kappa(\Gamma_P)$, and $K_{\mathbf{L_P}} = \kappa(K \cap P)$. This map induces a fibration of the face $e'(\mathbf{P})$,

$$\Gamma_N \backslash N_{\mathbf{P}} \to e'(\mathbf{P}) \to \Gamma_{\mathbf{L_P}} \backslash Z_{\mathbf{M_P}}, \tag{4.1}$$

over the locally symmetric space $\Gamma_{\mathbf{L_P}} \backslash Z_M$ with compact fibers, where $Z_M = {}^oL_{\mathbf{P}}/K_{\mathbf{L_P}}$. Since κ induces an isomorphism $\phi : L_{\mathbf{P},o} \xrightarrow{\sim} L_{\mathbf{P}}$, we can realize the base space for the fibration (4.1) as $\phi^{-1}(\Gamma_{\mathbf{L_P}}) \backslash M_{\mathbf{P},o}/K_M$. Finally, since in our case $\Gamma_P = \Gamma_M \Gamma_N$, we have $\phi^{-1}(\Gamma_{\mathbf{L_P}}) = \Gamma_M$. With this in mind, we abuse notation and write $\Gamma_{\mathbf{L_P}} \backslash Z_M$ when we mean $\Gamma_M \backslash M_{\mathbf{P},o}/K_M$.

The Hochschild-Serre spectral sequence associated with the above fibration degenerates at the E_2 level [Sch83, Theorem 2.7] yielding,

$$H^\bullet(e'(\mathbf{P}), \mathbb{S}\mathrm{ym}^n(V_3)) = H^\bullet(\Gamma_M \backslash Z_M, H^\bullet(\mathfrak{n}_{\mathbf{P}}, \mathrm{Sym}^n(V_3))) = H^\bullet(\Gamma_M, H^\bullet(\mathfrak{n}_{\mathbf{P}}, \mathrm{Sym}^n(V_3))).$$

The last expression should be interpreted as (one of the notions of) group cohomology of $\Gamma_{\mathbf{M_P}}$.

We take this one step further, and use Kostant's theorem to write down a decomposition of the cohomology groups above. In particular, if λ is the highest weight of $\mathrm{Sym}^n(V_3)$,then it follows from Kostant's work in [Kos61] that the cohomology groups of $\mathfrak{n}_{\mathbf{P}}$ with coefficients in $\mathrm{Sym}^n(V_3)$ have the following decomposition

$$H^q(\mathfrak{n}_{\mathbf{P}}, \mathrm{Sym}^n(V_3)) = \bigoplus_{\substack{w \in W^P \\ l(w)=q}} F_{\mu_w},$$

into irreducible $\mathbf{L_P}(\mathbb{C})$-modules F_{μ_w} of highest weight $\mu_w = w(\lambda + \rho) - \rho$. Consequently, in conjunction with this decomposition, we arrive at the following result,

$$H^n(e'(\mathbf{P}), \mathbb{S}\mathrm{ym}^n(V_3)) = \bigoplus_{p+q=n} \bigoplus_{\substack{w \in W^P \\ l(w)=q}} H^p(\Gamma_M, F_{\mu_w}). \tag{4.2}$$

Finally, we incorporate Kostant's decomposition (4.2) with the decomposition of the space of cuspidal square integrable functions into a sum of irreducible modules indexed by irreducible unitary representations (2.4). In particular, if we let

$L_0^2(\Gamma_M\backslash M_{\mathbf{P},o}) = \bigoplus_{\pi\in\widehat{M}_{\mathbf{P},o}} m(\pi,\Gamma_M)V_\pi$, then:

$$H^n_{\mathrm{cusp}}(e'(\mathbf{P}),\mathbb{S}\mathrm{ym}^n(V_3)) = \bigoplus_{p+q=n}\bigoplus_{\substack{w\in W^P\\ l(w)=q}} H^p(\Gamma_M\backslash Z_M, F_{\mu_w})$$

$$= \bigoplus_{p+q=n}\bigoplus_{\substack{w\in W^P\\ l(w)=q}}\bigoplus_{\pi\in\widehat{M}_{\mathbf{P},o}} m(\pi,\Gamma_M)H^p(\mathfrak{m}_{\mathbf{P}},K_M,V_\pi\otimes F_{\mu_w}).$$

Consequently, for a class $[\phi]\in H^\bullet_{\mathrm{cusp}}(e'(\mathbf{P}),\mathbb{S}\mathrm{ym}^n(V_3))$, we say $[\phi]$ is a *cuspidal class of type* (π,w), if $[\phi]$ is in $H^\bullet(\mathfrak{m}_{\mathbf{P}},K_M,V_\pi\otimes F_{\mu_w})$.

4.4.1 An explicit application of Kostant's Theorem

We focus on $\mathbf{P}=\mathbf{P}_{\{\alpha_2\}}$. For this parabolic subgroup, it is an easy exercise to check that $W^{P_{\{\alpha_2\}}} = \{\mathrm{Id}, s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}\}$. Therefore,

$$H^3(e'(\mathbf{P}_{\{\alpha_2\}}),\mathbb{S}\mathrm{ym}^n(V_3)) = H^1(\Gamma_M\backslash Z_M, F_{\mu_w}),$$

where $w=s_{\alpha_1}s_{\alpha_2}$, and $\mu_w = w(\lambda+\rho)-\rho$, for $\lambda=(n,0,0)$. A brief computation reveals $\mu_w=(-2,n+1,1)$, and consequently using Küneth's formula to decompose the cohomology of $\Gamma_M=\{\pm 1\}\times GL_2(\mathbb{Z})$ we discover that,

$$H^3(e'(\mathbf{P}_{\{\alpha_2\}}),\mathbb{S}\mathrm{ym}^n(V_3)) = H^1(GL_2(\mathbb{Z})\backslash\mathcal{H}^+,\mathbb{S}\mathrm{ym}^n(V_2)\otimes\det).$$

Recall from §3.4,

$$\omega_f^0(g) = h_f(g)P_1^*\otimes X^n, \omega_{\bar{f}}^0 = h_{\bar{f}}(g)P_2^*\otimes Y^n,$$

where $X^n = \begin{pmatrix}1\\-i\end{pmatrix}^n$, and $Y^n = \begin{pmatrix}1\\i\end{pmatrix}^n$ are respectively the highest and lowest weight vector of F_{μ_w} as an M-module. In this section we carefully work through Kostant's Theorem [Kos61, Lemma 5.12] to realize X and Y elements of $\mathrm{Hom}(\Lambda^2\mathfrak{n},\mathrm{Sym}^n(V_3))$.

Theorem 1 ([Kos61])**.** *If $s_{w\lambda}$ is the weight vector for the extremal weight $w(\lambda)$ of E, then the highest weight vector in F_{μ_w} is the cohomology class having*

$$e'_{-\Phi_w} \otimes s_{w\lambda},$$

as a representative (harmonic) cocycle.

To begin unraveling this theorem, first we briefly simplify the problem by considering the standard representation V_3 rather than $\mathrm{Sym}^n(V_3)$. The standard representation decomposes as

$$V_3 = V_{-2t} \oplus V_{t+is} \oplus V_{t-is},$$

where the weights are with respect to the Cartan subalgebra $\mathfrak{b}_{\mathbb{C}}$. We immediately note that $-2t = (t+is) + \tilde{\alpha}_1 = (t-is) + \tilde{\alpha}_1 + \tilde{\alpha}_2$, and therefore $\lambda = -2t = \frac{2}{3}(\tilde{\alpha}_1 + \frac{1}{2}\tilde{\alpha}_2)$ is the highest weight of V_3 with respect to the maximally compact Cartan subalgebra $\mathfrak{b}_{\mathbb{C}}$ and our choice of positive root system $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$. The corresponding highest weight vector is $A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Note that $\mathbf{P} = \mathbf{P}_{\{\alpha_2\}} = \mathbf{P}_{\{\tilde{\alpha}_2\}}$, and consequently we remain interested in $w = s_{\alpha_1}s_{\alpha_2} = s_{\tilde{\alpha}_1}s_{\tilde{\alpha}_2} \in W^P$. We compute the extremal weight

$$w(\lambda) = \frac{2}{3}s_{\tilde{\alpha}_1}s_{\tilde{\alpha}_2}(\tilde{\alpha}_1 + \frac{1}{2}\alpha_2) = t + is,$$

with weight vector $B = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$. We now return to the case of $\mathrm{Sym}^n(V_3)$ and discover that the extremal weight $w(\lambda)$ is equal to $n(t+is)$, and has a weight vector B^n. Next, we turn to deciphering $e'_{-\Phi_w}$. First, guided by the definition given in Kostant's paper, we find that

$$\Phi_w = w\Phi^-(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) \cap \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{-3t \pm is\} = \{\tilde{\alpha}_1, \tilde{\alpha}_1 + \tilde{\alpha}_2\}.$$

Then, $e'_{-\Phi_w}$ is defined as $e^*_{\tilde{\alpha}_1} \wedge e^*_{\tilde{\alpha}_1+\tilde{\alpha}_2}$. Here $e^*_{\tilde{\alpha}_1}$ (resp. $e^*_{\tilde{\alpha}_1+\tilde{\alpha}_2}$) is the dual vector to the root vector $e_{\tilde{\alpha}_1}$ (resp. $e^*_{\tilde{\alpha}_1+\tilde{\alpha}_2}$) for the positive root $\tilde{\alpha}_1$ (resp. $\tilde{\alpha}_1 + \tilde{\alpha}_2$.) Easily enough, we discover that $e_{\tilde{\alpha}_1} = \begin{pmatrix} 0 & 1 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $e_{\tilde{\alpha}_1+\tilde{\alpha}_2} = \begin{pmatrix} 0 & 1 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. To simplify matters later, if we allow s to be the coordinate on N representing the entry n_{12}, and t to be the coordinate representing the entry n_{13}, then

$$e'_{-\Phi_w} = e^*_{\tilde{\alpha}_1} \wedge e^*_{\tilde{\alpha}_1+\tilde{\alpha}_2} = \frac{i}{2} ds \wedge dt.$$

The above consideration lead to the conclusion that, according to Kostant's Theorem, X^n considered as an element of $\mathrm{Hom}(\Lambda^2\mathfrak{n}, \mathrm{Sym}^n(V_3))$ is equal to $\frac{i}{2}B^n \otimes ds \wedge dt$.

Kostant's Theorem does not immediately yield the expression for Y^n, the lowest weight vector of F_{μ_w}. However choosing $\{\tilde{\alpha}_1 + \tilde{\alpha}_2, -\tilde{\alpha}_2\}$ as simple roots of $\mathfrak{b}$, we find ourselves again in the framework of Kostant's paper, this time, with Y^n the highest weight vector. With respect to this choice of simple roots the highest weight vector of V_3 is again A, with weight $\lambda = -2t$. The extremal weight $w(\lambda)$ is equal to $t - is$, with weight vector $C = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$. The change of simple roots, has not altered the set $\Phi_w = \{-3t \pm is\}$. Therefore, we find that $Y^n = \frac{i}{2}C^n \otimes ds \wedge dt$ as an element of $\mathrm{Hom}(\Lambda^2\mathfrak{n}, \mathrm{Sym}^n(V_3))$.

4.5 Eisenstein cohomology

4.5.1 Eisenstein series

In this section we will show how, under special circumstances, one may extend a cuspidal cohomology classes in $H^\bullet_{\mathrm{cusp}}(e'(\mathbf{P}), \mathbb{S}\mathrm{ym}^n(V_3))$ to the full Borel-Serre compactification $\overline{X}$, following mostly the exposition in [Sch83], and [Sch94]. For notational convenience we will abbreviate $\mathrm{Sym}^n(V_3)$ simply with E.

First we begin by defining Eisenstein series for a subset of the space of functions in $C^\infty(\Gamma_P A_{\mathbf{P}} N_{\mathbf{P}} \backslash G)$ and then we will use this definition to define Eisenstein cohomology classes. Considering $f \in C^\infty(\Gamma_P A_{\mathbf{P}} N_{\mathbf{P}} \backslash G)$ a K-finite function such that for each $g \in G$, the function $m \mapsto f(mg)$ is a square-integrable automorphic form on M for Γ_M. For $\Lambda \in \mathfrak{a}^*_{\mathbf{P},\mathbb{C}}$, we define f_Λ as,

$$f_\Lambda(x) = f(x)a(x)^{\Lambda+\rho}, \;\; x \in G.$$

Furthermore we define

$$(\mathfrak{a}^*_{\mathbf{P}})^+ = \{\lambda \in \mathfrak{a}^*_{\mathbf{P}} \mid \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta_{\mathbf{P}}\},$$

$$(\mathfrak{a}^*_{\mathbf{P},\mathbb{C}})^+ = \{\Lambda \in \mathfrak{a}^*_{\mathbf{P},\mathbb{C}} \mid \mathrm{Re}\Lambda \in \rho_{\mathbf{P}} + (\mathfrak{a}^*_{\mathbf{P}})^+\}.$$

The series defined as,

$$E(f,\Lambda)(x) = \sum_{\gamma \in \Gamma_{\mathbf{P}} \backslash \Gamma} f_\Lambda(\gamma x), \;\; x \in G$$

converges absolutely on compact subsets of $G \times (\mathfrak{a}^*_{\mathbf{P},\mathbb{C}})^+$, and is called *Eisenstein series.* It is a smooth, automorphic function for Γ on $G \times (\mathfrak{a}^*_{\mathbf{P},\mathbb{C}})^+$ that is holomorphic in Λ. In addition, one can define an analytical continuation of $E(f,\Lambda)(x)$ to a meromorphic function on all of $\mathfrak{a}^*_{\mathbf{P},\mathbb{C}}$.

Now we let ϕ be a harmonic square-integrable differential form in $\Omega^\bullet(e'(\mathbf{P}), E)$. Making use of the decomposition $\Gamma_P \backslash D_G = \Gamma_P \backslash P/K_{\mathbf{P}} = e'(\mathbf{P}) \times A^G_{\mathbf{P}}$, for $\Lambda \in \mathfrak{a}^*_{\mathbf{P},\mathbb{C}}$, we define an extension of ϕ to all of D_G as,

$$\phi_\Lambda(namK) = \phi(nm)a^{\Lambda+\rho} \in \Omega^*(\Gamma_P \backslash X, E).$$

As explained in [MM63], to ϕ_Λ we can associate an E-valued differential form on $\Gamma_P \backslash G$. Abusing notation and using ϕ_Λ to denote this form as well, we have

$$\phi_\Lambda = \sum_J (f_J a^{\Lambda+\rho}) w^J.$$

Above, $f_J \in C^\infty(\Gamma_P A_{\mathbf{P}} N_{\mathbf{P}} \backslash G)$,$\{w^i\}_{i=1}^{\dim \mathfrak{g}}$ is a basis of left-invariant one forms G, and $J = \{i_1, \ldots, i_q\}$ is a subset of $I = \{1, \ldots, \dim \mathfrak{g}\}$. In addition for all x and J, the function $m \mapsto f_J(mx)$ is a square integrable automorphic form for Γ_M on $M_{\mathbf{P}}$.

Definition 1. *To* ϕ *we associate the* Eisenstein differential form *defined as,*

$$E(\phi, \Lambda) := \sum_J E(f_J, \Lambda) w^J.$$

As in the case of $E(f, \Lambda)$, these differential forms are first defined for $\Lambda \in (\mathfrak{a}^*_{\mathbf{P},\mathbb{C}})^+$. For such Λ, $E(f, \Lambda)$ is a Γ-invariant, E-valued differential form on D_G. One can then define $E(f, \Lambda)$ for all $\Lambda \in \mathfrak{a}^*_{\mathbf{P},\mathbb{C}}$ via analytic continuation.

4.5.2 Maximal parabolic subgroups

We represent the two $GL_3(\mathbb{Z})$-conjugacy classes of maximal parabolic subgroups of GL_3, by the two standard subgroups,

$$P_{\{\alpha_1\}} = P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, P_2 = P_{\{\alpha_2\}} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Furthermore, we use the fact that Weyl group W permutes the set $\{l_i\}_{i=1}^3$ to simplify our notation. In particular we will use $i_1 i_2 i_3$ to denote the element $w \in W$ that maps l_j to l_{i_j} for $1 \leq i, j \leq 3$. Thus the element $s_{\alpha_1} \in W$ which permutes l_1 and l_2 while fixing l_3, is written as 213.

Table 4.1 offers a summary of information regarding the sets W^P for the two conjugacy classes of parabolic subgroups:

In Table 4.2 we summarize the information regarding the weights $w(\lambda + \rho) - \rho$, for $w \in W$, $\lambda = (n, 0, 0), \rho = (1, 0, -1)$.

After analyzing the Meyer-Vietoris sequence coming from the faces associated to the maximal parabolic subgroups, as well as the face associated to the minimal

Table 4.1: Kostant representatives

	$W^{P_{\{\alpha_1\}}}$	$W^{P_{\{\alpha_2\}}}$
$l(w) = 0$	123	123
$l(w) = 1$	132	213
$l(w) = 2$	231	312

Table 4.2: Contributions to the cohomology of $e(\mathbf{P})$

w	$w(\lambda + \rho) - \rho$
123	$(n, 0, 0)$
132	$(n, -1, 1)$
213	$(-1, n+1, 0)$
312	$(-2, n+1, 1)$
231	$(-1, -1, n+2)$
321	$(-2, 0, n+2)$

parabolic subgroup, Goncharov in the aforementioned paper computes the cohomology of the Borel - Serre boundary as,

$$H^i\left(\partial\left(\overline{X}\right), \overline{\mathbb{S}\mathrm{ym}^n(V_3)}\right) = \begin{cases} H^1_{cusp}(GL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2)), & i = 1, \\ H^1_{cusp}(GL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2)\otimes\det), & i = 3. \end{cases}$$

Therefore $H^3(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$ injects into $H^1_{cusp}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^-$, via the restriction map. The following argument shows that the image of this injection is in fact, the whole of $H^1_{cusp}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^-$.

In [Sch94] the author identifies the Eisenstein cohomology classes originating from compactly supported cohomology on the Borel - Serre faces associated to maximal parabolic subgroups. Though we are working with a family of representations that have a non-generic heighest weight, we may adopt some of the arguments presented in [Sch94, Theorem 6.3] to arrive at a similar conclusion. We make this clear in the following

Theorem 2. *Let* $[\phi] \in H^1_{cusp}(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))^- = H^3_{cusp}(e'(P_{\{\alpha_2\}}), \mathbb{S}\mathrm{ym}^n(V_3))$. *Then the corresponding Eisenstein series* $E^{P_{\{\alpha_2\}}}(\phi, \Lambda)$, *are holomorphic at the point*

$\Lambda_w = -w(\lambda+\rho) = \rho$, for $w = 312$. Furthermore, $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)$ is a closed, harmonic form representing a non-trivial cohomology class in $H^3(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$. Finally, $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)$ restricts back to ϕ on $e'(P_{\{\alpha_2\}})$.

Proof. Rather than showing explicitly that $\Lambda_w \in (\mathfrak{a}^*_{\mathbf{P},\mathbb{C}})^+$ and as such gives rise to a harmonic differential form, we make use of the following neat argument that takes into account the vanishing of square integrable cohomology and still yields the same result. Namely, assuming that the Eisenstein series $E^{P_{\{\alpha_2\}}}(\phi, \Lambda)$ do have a pole at Λ_w, then by taking the residue one gets a non-trivial cohomology class in the square integrable cohomoogy $H^\bullet_{(2)}(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_2))$. By our considerations above, the square integrable cohomology is zero, therefore the Eisenstein series are indeed holomorphic at Λ_w. The fact that $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)$ is a closed, harmonic form giving rise to a non-trivial cohomology class follows from [Sch83, Theorem 4.11].

It remains to consider the restriction of $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)$ to the face $e'(P_{\{\alpha_2\}})$. As proven in [Sch83, Theorem 1.10], the restriction $r^*_S(E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w))$ of the Eisenstein series to $e'(S)$ for S a parabolic subgroup, is equal to the restriction of the constant term in the Fourier expansion of $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)$ with respect to S. If we use $E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)_S$ to denote this constant term, then,

$$r^*_S(E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)) = r^*_S(E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)_S).$$

On the other hand, the constant term is expressed as the following sum:

$$E^{P_{\{\alpha_2\}}}(\phi, \Lambda_w)_S = \sum_{s \in \Omega(\mathfrak{a}_{P_{\{\alpha_2\}}}, \mathfrak{a}_S)} c(s, \Lambda_w)(\phi),$$

where the sum is taken over the set $\Omega(\mathfrak{a}_{P_{\{\alpha_2\}}}, \mathfrak{a}_S)$ of all linear isomorphisms from $\mathfrak{a}_{P_{\{\alpha_2\}}}$ to $\mathfrak{a}_S$) obtained by restricting elements of the weyl group W. In the case we are interested, $S = P_{\{\alpha_2\}}$, and $\Omega(\mathfrak{a}_{P_{\{\alpha_2\}}}, \mathfrak{a}_{P_{\{\alpha_2\}}}) = \{1\}$, since $P_{\{\alpha_2\}}$ is not conjugate to to its opposite $\bar{P}_{\{\alpha_2\}}$, see [Sch86, Section 3.1]. This concludes the proof, since $c(1, \Lambda_w)(\phi) = \phi$. □

5

Modular Symbols, and Period Polynomials of Modular Forms

5.1 Group cohomology

In this chapter we will make extensive references to *Eilenberg-MacLane* group cohomology. In this section we define in more detail this notion of group cohomology for a general group G with coefficients in a G-module (ρ, V). In particular, we consider the complex $C^{\bullet}(G,V)$ of all functions $f : G^{\bullet} \to V$. To this complex we associate the co-boundary operator $d^{\bullet} : C^{\bullet}(G,V) \to C^{\bullet+1}(G,V)$ defined as:

$$(d^i f)(g_1, \ldots, g_{i+1}) = \rho(g_1) f(g_2, \ldots, g_{i+1}) \\ + \sum_{k=1}^{k=n} (-1)^k f(\ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

We state without proof that this is a co-chain map, namely $(d^{i+1} \circ d^i) f = 0$. Therefore, in line with standard notation, we say

$$Z^i(G,V) = \left\{ f \in \mathrm{Ker}\left(d^i\right) \right\} \subset C^i(G,V)$$

are the i-co-chains. Again, following standard conventions, we define co-boundaries as,

$$B^i(G,V) = \left\{ f \in \operatorname{Im}\left(d^{i-1}\right) \right\} \subseteq C^i(G,V), i \geq 1,$$

and $B^0(G,V) = 0$. With this notation in place, we can compute Eilenberg-MacLane group cohomology as,

$$H^\bullet(G,V) = Z^\bullet(G,V)/B^\bullet(G,V)$$

5.1.1 Cuspidal co-cycles for $\Gamma = SL_2(\mathbb{Z})$

In this section we define a subspace of the Eilnberg-MacLane cohomology groups that should roughly correspond to the notion interior deRham cohomology. More specifically we say a co-cycle $f \in Z^1(SL_2(\mathbb{Z}), \operatorname{Sym}^n(V_2))$ *cuspidal* if for all $\gamma \in \Gamma_{\mathbf{P}}$, where $\mathbf{P}$ is a maximal rational parabolic subgroup of $SL_2(\mathbb{R})$, there exists a $v_{\mathbf{P}}^f \in \operatorname{Sym}^n(V_m)$ such that,

$$f(\gamma) = (1-\gamma)v_{\mathbf{P}}^f. \tag{5.1}$$

In other words, we require that $f \in B^1(\Gamma_{\mathbf{P}}, \operatorname{Sym}^n(V_2))$. We use $Z_c^1(SL_2(\mathbb{Z}), \operatorname{Sym}^n(V_2))$ to denote the space of cuspidal co-cycles. We note that by this definition, all co-boundaries satisfy the cuspidal condition. Therefore, we define the *cuspidal group cohomology* in degree one for $\Gamma = SL_2(\mathbb{Z})$ with coefficients in $\operatorname{Sym}^n(V_2)$ as

$$H^1_{\text{cusp}}(SL_2(\mathbb{Z}), \operatorname{Sym}^n(V_2)) = Z_c^1(SL_2(\mathbb{Z}), \operatorname{Sym}^n(V_2))/B^1(SL_2(\mathbb{Z}), \operatorname{Sym}^n(V_2)).$$

We note that $\Gamma_{\mathbf{P}}$ is cyclic for all maximal rational parabolics $\mathbf{P}$. Furthermore, as all maximal parabolic subgroups are conjugate to each other, for a co-cycle to be cuspidal it is sufficient to satisfy the cuspidal condition in (5.1) for $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the generator of $\Gamma_{\mathbf{P}}$ when $\mathbf{P} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

Furthermore, let γ_0 be a non-trivial element in a parabolic subgroup $\Gamma_{\mathbf{Q}}$. We define the cohomology groups,

$$H^1_{\text{cusp}}\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right) = Z^1_c\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right) / B^1\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right),$$

where in defining $Z^1_c\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right)$ and $B^1\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right)$ we require that the co-cycles and co-boundaries vanish on γ_0.

For a co-cycle $f \in Z^1_c\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2)\right)$, consider the co-boundary defined as,

$$h(\gamma) = (1-\gamma)v^f_{\mathbf{Q}}.$$

Then it follows that $f - h \in Z^1_c\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2), \gamma_0\right)$. Consequently,

$$Z^1_c\left(\Gamma, \text{Sym}^n(V_2)\right) = B^1\left(\Gamma, \text{Sym}^n(V_2)\right) \oplus Z^1_c(\Gamma, \text{Sym}^n(V_2), \gamma_0).$$

Therefore we have an isomorphism,

$$H^1_{\text{cusp}}\left(\Gamma, \text{Sym}^n(V_2), \gamma_0\right) \cong H^1_{\text{cusp}}\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2)\right).$$

Now we let $S, T \in \Gamma$ be as before, and $U = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Any two of these three elements generate $SL_2(\mathbb{Z})$. Consequently for $f \in Z^1_c(\Gamma, \text{Sym}^n(V_2), T)\}$, f is completely determined by the value $f(S)$. This follows from the fact that $f(T) = 0$, and an arbitrary $\gamma \in \Gamma$ can be written as a product of S's and T's. Finally, a repeated application of the co-cycle condition yields an expression for $f(\gamma)$ in terms of $f(S)$. Consider the following spaces of polynomials

$$\mathcal{Z}_n = \{f(S) \mid f \in Z^1_c(\Gamma, \text{Sym}^n(V_2), T)\},$$

$$\mathcal{B}_n = \{f(S) \mid f \in B^1(\Gamma, \text{Sym}^n(V_2), T)\},$$

and let $\mathcal{R}_n = \mathcal{Z}_n / \mathcal{B}_n$. The sum of these considerations yields an isomorphism

$$H^1_{\text{cusp}}\left(SL_2(\mathbb{Z}), \text{Sym}^n(V_2)\right) \cong \mathcal{R}_n.$$

We would like to accurately describe the space of polynomials $\mathcal{R}_n$. First we focus on polynomials in $\mathcal{B}_n$. These are of the form $(1-\rho_2^n(S))v$ for $v \in \mathrm{Sym}^n(V_2)$ such that $v = \rho_2^n(T)v$. It is an easy exercise to show that this condition on v is very restrictive, and in particular $v = \mathbb{C}\,\mathbf{x}^n$. Therefore $\mathcal{B}_n = \mathbb{C}(\mathbf{x}^n - \mathbf{y}^n)$. On the other hand, consider $p(\mathbf{x}, \mathbf{y}) = f(S) \in \mathcal{Z}_n$. Considering that,

$$0 = df(S, S) = \rho_2^n(S)f(S) - f(-\mathrm{Id}_2) + f(S),$$

and the fact that $f(-\mathrm{Id}_2) = 0$, we discover that $(1 + \rho_2^n(S))p(\mathbf{x}, \mathbf{y}) = 0$. Note also that making use of the co-cycle condition we discover that,

$$0 = df(T, S) = \rho_2^n(T)f(S) - f(U) + 0,$$

and therefore $f(U) = \rho_2^n(T)p(\mathbf{x}, \mathbf{y})$. Furthermore, it follows from

$$0 = df(U^2, U) = U^2 f(U) - f(-\mathrm{Id}_2) + f(U^2),$$

$$0 = df(U, U) = Uf(U) - f(U^2) + f(U),$$

that $(1 + \rho_2^n(U) + \rho_2^n(U^2))Tp(\mathbf{x}, \mathbf{y}) = 0$. Making use of the fact that $\rho_2^n(S)p(\mathbf{x}, \mathbf{y}) = -p(\mathbf{x}, \mathbf{y})$, and that $\rho_2^n(-\mathrm{Id}_2)p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, \mathbf{y})$, we discover that,

$$\begin{aligned}(1 + \rho_2^n(U) + \rho_2^n(U^2))Tp(\mathbf{x}, \mathbf{y}) &= (1 + \rho_2^n(U) + \rho_2^n(U^2))TS(-p(\mathbf{x}, \mathbf{y}))\\ &= (1 + \rho_2^n(U) + \rho_2^n(U^2))U(-p(\mathbf{x}, \mathbf{y}))\\ &= -(\rho_2^n(U) + \rho_2^n(U^2) + 1)p(\mathbf{x}, \mathbf{y}) = 0.\end{aligned}$$

Therefore, it is certainly true that,

$$\mathcal{Z}_n \subseteq \left\{ p(\mathbf{x}, \mathbf{y}) \in \mathrm{Sym}^n(V_2) \;\middle|\; \begin{array}{c} (1 + \rho_2^n(S))p(\mathbf{x}, \mathbf{y}) = 0 \\ (\rho_2^n(U) + \rho_2^n(U^2) + 1)p(\mathbf{x}, \mathbf{y}) = 0 \end{array} \right\}.$$

In fact, making use of the finite presentation of $SL_2(\mathbb{Z})$ one can show that the above inclusion is an equality.

Keeping in mind that n is even, the space $\mathrm{Sym}^n(V_2)$ decomposes as,

$$\mathrm{Sym}^n(V_2) = \mathrm{Sym}^n(V_2)^+ \oplus \mathrm{Sym}^n(V_2)^-,$$

where the "+" (resp. "−") in the superscript is used to denote the subspace of polynomials where the monomials have an even (resp. odd) exponent in both $\mathbf{x}$ and $\mathbf{y}$. Carrying over this decomposition to $\mathcal{R}_n$ we discover that,

$$\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^- = \left(\mathcal{Z}_n^+/\langle \mathbf{x}^n - \mathbf{y}^n\rangle\right) \oplus \mathcal{Z}_n^-$$

5.2 Period polynomials and group co-cycles

Consider the map

$$H_c^1(SL_2(\mathbb{Z})\backslash \mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2)) \hookrightarrow H^1(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2)),$$

$$\omega \mapsto \phi_\omega,$$

where the right hand side denotes Eilenberg-MacLane group cohomology. The function $\phi_\omega : SL_2(\mathbb{Z}) \to \mathrm{Sym}^n(V_2)$ is defined by,

$$\phi_\omega(\gamma) = \int_0^{\gamma(0)} \omega,$$

where the integral is taken along the geodesic with respect to the $SL_2(\mathbb{R})$ invariant metric on $\overline{\mathcal{H}^+}$ connecting 0 and $\gamma(0)$. We note that ϕ_ω is closed by the virtue of Stokes' Theorem:

$$\begin{aligned} d\phi_{\omega_f}(\gamma_1, \gamma_2) &= \rho_2^n(\gamma_1)\phi_{\omega_f}(\gamma_2) - \phi_{\omega_f}(\gamma_1\gamma_2) + \phi_{\omega_f}(\gamma_1) \\ &= \rho_2^n(\gamma_1)\int_0^{\gamma_2(0)} - \int_0^{\gamma_1\gamma_2(0)} + \int_0^{\gamma_1(0)} \\ &= \int_{\gamma_1(0)}^{\gamma_1\gamma_2(0)} + \int_{\gamma_1\gamma_2(0)}^{0} + \int_0^{\gamma_1(0)} = 0. \end{aligned}$$

First we observe that ϕ_ω is a parabiolic co-cycle. We show that the cuspidal condition is satisfied for $\mathbf{P} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, generated by T. As all $\mathbb{Q}$-parabolic subgroups of $SL_2(\mathbb{R})$ are conjugate, the general case follows immediately from,

$$\phi_\omega(T) = \int_0^1 \omega = \int_0^{i\infty} \omega - \int_0^{\infty} \omega = \phi_\omega(S) - \rho_2^n(T)\phi_\omega(S).$$

Consequently, consider the co-boundary $h_\omega(\gamma) = (1 - \rho_2^n(\gamma))\phi_\omega(S)$, and the co-cycle

$$\nu_\omega = \phi_\omega - h_\omega.$$

The above considerations show that $\nu_\omega \in H^1(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2), T)$. We also note that,

$$\nu_\omega(S) = \phi_\omega(S) - h(S) = \phi_\omega(S) - (1 - \rho_2^n(S))\phi_\omega(S) = \rho_2^n(S)\phi_\omega(S) = -\phi_\omega(S).$$

Therefore, as in §5.1.1, we have $\phi_\omega(S) = -\nu_\omega(S) \in \mathcal{Z}_n$.

Finally we focus on $[\omega_f], [\omega_{\bar{f}}] \in H_c^1(SL_2(\mathbb{Z})\backslash\mathcal{H}^+, \mathbb{S}\mathrm{ym}^n(V_2))$ as introduced in Chapter 3. The above exposition yields a map

$$\alpha : \mathcal{S}_{n+2} = S_{n+2}(SL_2(\mathbb{Z})) \oplus \overline{S_{n+2}(SL_2(\mathbb{Z}))} \to \mathcal{R}_n$$
$$f_1 \oplus \overline{f_2} \mapsto \phi_{\omega_{f_1}}(S) + \phi_{\omega_{\overline{f_2}}}(S),$$

Where it is understood that the expression for the image on the right hand side is in fact the equivalence class of the polynomial $\phi_{\omega_{f_1}}(S) + \phi_{\omega_{\overline{f_2}}}(S)$. We can do even better by making use of the decomposition of $\mathcal{R}_n$ into polynomials with all even and all odd exponents. In particular, following our earlier convention we write $f(z) = \sum_{k=1}^{\infty} a_k q^k$, with $q = e^{2\pi i z}, a_k \in \mathbb{Q}$. It follows that for $z \in \mathcal{H}^+$ such that $\mathrm{Re}(z) = 0$, $f(z) \in \mathbb{R}$, and therefore $\omega_{\bar{f}} = \overline{\omega_f}$. Therefore,

$$\phi_{\omega_f - \omega_{\bar{f}}}(S)(\mathbf{x}, \mathbf{y}) = \int_0^{i\infty} \omega_f - \overline{\int_0^{i\infty} \omega_f} = 2\mathrm{Im}\phi_{\omega_f}(S).$$

On the other hand, we note that coefficients of monomials in $\phi_{\omega_f}(S) = \int_0^{i\infty} \omega_f$ are either imaginary or real valued monomials, with the former corresponding to monomials with even exponents in $\mathbf{x}$, and therefore an even exponent in $\mathbf{y}$. Consequently, we arrive at

$$\phi_{\omega_f - \omega_{\bar{f}}}(S)(\mathbf{x}, \mathbf{y}) = 2\phi_{\omega_f}(S)^+.$$

Along the same lines we make the observation that,

$$\phi_{\omega_f + \omega_{\bar{f}}}(S)(\mathbf{x}, \mathbf{y}) = 2\phi_{\omega_f}(S)^-.$$

Therefore, we identify the following two subspaces,

$$\mathcal{S}^{\mp} = \{(f, \pm f) \subset \mathcal{S}\} \cong S_{n+2}(SL_2(\mathbb{Z})).$$

The map α above induces maps,

$$\alpha^{\pm} : \mathcal{S}^{\pm} \to \mathcal{R}^{\pm}.$$

The essence of Eichler-Shimura-Manin theory is that the above maps $\alpha^{\pm}$ are in fact isomorphisms of the space of cusp forms onto the spaces of polynomials $\mathcal{R}^{\pm}$.

Finally, we are in a position to expand on some of the concepts touched upon in Chapter 1. There we defined the two variable period polynomial associated to a cusp form $f \in S_{n+2}(\mathbb{Z})$ as

$$r_f(\mathbf{x}, \mathbf{y}) = \int_0^{i\infty} w_f = \int_0^{i\infty} f(z)(z\,\mathbf{x} + \mathbf{y})^n dz.$$

In fact, we now know that this is the special value $\phi_{\omega_f}(S)$. From our prior considerations, this value completely determines the co-cycle $\nu_{\omega_f} \in \mathcal{Z}^1(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2), T)$. Next, we show that starting with $r_f(\mathbf{x}, \mathbf{y})$, we can compute the value of the co-cycle ϕ_{ω_f} at all elements of $SL_2(\mathbb{Z})$.

Consider the case that $\gamma(0) = \frac{b}{a} \in \mathbb{Q}$. Without loss of generality we may assume that $\frac{b}{a}$ is in lowest terms. Therefore there exists an $a' \in \mathbb{Z}_{>0}$ such that $ba' \equiv 1\,($

mod a). Then, $b' = \frac{ba'-1}{a} \in \mathbb{Z}$, and therefore $A = \begin{pmatrix} b & b' \\ a & a' \end{pmatrix} \in SL_2(\mathbb{Z})$. We use the closed condition on ϕ_{ω_f} and conclude that

$$0 = d\phi_{\omega_f}(A, S) = \rho_2^n(A)\phi_{\omega_f}(S) - \phi_{\omega_f}(AS) + \phi_{\omega_f}(A)$$

Considering that $AS(0) = \gamma(0) = \frac{b}{a}, A(0) = \frac{b'}{a'}$ we have, $\phi_{\omega_f}(\gamma) = \rho_2^n(A)\phi_{\omega_f}(S) + \phi_{\omega_f}(A)$. The proof is concluded by inducting on a.

The reader should note that we have made a choice in choosing the base point zero. Next we show that the group co-chain obtained by choosing a different base point differs from ϕ_{ω_f} by a co-boundary. Specifically, let us fix $z_0 = i$ as a new base point. Let $\tilde{\phi}_f$ be the function from $SL_2(\mathbb{Z})$ to $\mathrm{Sym}^n(V_2)$ defined similarly as above, only with base point z_0. We show that ϕ_{ω_f} and $\tilde{\phi}_f$ differ by a co-boundary. Namely consider the section of the upper half plane encircled by the geodesics between 0 and i, i and $\gamma(i)$, $\gamma(i)$ and $\gamma(0)$, and $\gamma(0)$ and 0. Using Stoke's theorem we get,

$$\tilde{\phi}_f(\gamma) = \int_i^{\gamma(i)} \omega_f = \int_i^0 \omega_f + \int_0^{\gamma(0)} \omega_f + \int_{\gamma(0)}^{\gamma(i)} \omega_f = \phi_{\omega_f}(\gamma) + (1 - \rho(\gamma))\, v,$$

where $v = \int_0^i \omega_f \in \mathrm{Sym}^n(V_2)$. Note, $(1 - \rho(\gamma))\, v = d\chi(\gamma)$, where we think of $\chi = v$ as a function from the set containing a single point to $\mathrm{Sym}^n(V_2)$. Therefore, indeed $\tilde{\phi}_f$ is cohomologous to ϕ_{ω_f}, which is determined entirely by its value on S.

In the next section we aim to repeat this construction in the case of Γ_P, for $\Gamma = GL_3(\mathbb{Z})$ and $\mathbf{P}$ a maximal parabolic subgroup of GL_3. More specifically, as in the case of $SL_2(\mathbb{Z})$, we associate group 3-co-cycles to closed differential 3-forms on the symmetric space. We show that these group co-cycles are determined by their values on a single element of Γ_P^3, and explore how these values relate to the classical two variable period polynomials.

5.3 Constructing 3-variable period polynomials: A first attempt

5.3.1 The setting

Throughout the rest of this chapter $\mathbf{G} = GL_3$, and $\Gamma = GL_3(\mathbb{Z}$. Recall that in §4.4.1 we explicitly computed the cohomology of of the Borel-Serre boundary face associated to the parabolic subgroup $\mathbf{P} = \mathbf{P}_{\{\alpha_2\}}$. Topologically, the face $e'(\mathbf{P})$ is a quotient by $\Gamma_{\mathbf{P}} = \Gamma \cap P$ of the trivial bundle over the upper half-plane with a fiber isomorphic to $\mathbb{R}^2 \cong \mathbb{C}$. Let us write the Cartan decomposition of $\mathfrak{m}_{\mathbf{P}}$, the Lie algebra of $M_{\mathbf{P},o}$ as $\mathfrak{m}_{\mathbf{P}} = \mathfrak{k}_M \oplus \mathfrak{p}_1$. Note ${}^0\mathfrak{p} = \mathfrak{k}_M \oplus \mathfrak{r}$, where $\mathfrak{r} = \mathfrak{p}_1 \oplus \mathfrak{n}$, is the Cartan decomposition of the Lie algebra of ${}^0P$. In §4.4.1 to each $f \in S_{n+2}(\mathbb{Z})$ we associated a cohomology class $[\tilde{\omega}_f] \in H^3(e'(P), \mathbb{S}\mathrm{ym}^n(V_3))$. From the same section we may transform $\tilde{\omega}_f$ to a function on the quotient $\Gamma_M \backslash M$:

$$\tilde{\omega}_f^0 = \omega_f^0 + \omega_{\bar{f}}^0 : \Gamma_M \backslash M \to \mathrm{Hom}(\mathfrak{p}_1, H^2(\mathfrak{n}, \mathrm{Sym}^n(V_3))),$$

satisfying an analogue of (3.3). In particular,

$$\tilde{\omega}_f^0 = (i)^n \left(h_f P_1^* \otimes (ds \wedge dt) \otimes \frac{i}{2} B^n + h_{\bar{f}} P_2^* \otimes (ds \wedge dt) \otimes \frac{i}{2} C^n \right).$$

Using the natural embedding $\mathfrak{p}_1 \oplus \Lambda^2 \mathfrak{n}_{\mathbf{P}} \hookrightarrow \Lambda^3(\mathfrak{p}_1 \oplus \mathfrak{n}_{\mathbf{P}}) = \Lambda^3 \mathfrak{r}$, and extending $\tilde{\omega}_f^0$ trivially to the $N_{\mathbf{P}}$ factor, we write it as a function:

$$\tilde{\omega}_f^0 : \Gamma_P \backslash {}^0P \to \mathrm{Hom}(\Lambda^3 r, \mathrm{Sym}^n(V_3)),$$

satisfying

$$\tilde{\omega}_f^0(pk) = (\Lambda^3 \mathrm{Ad}^*)(k^{-1}) \otimes \rho_3^n(k^{-1})(\tilde{\omega}_f^0(p)), \quad p \in {}^0P, k \in K_P = K_M.$$

Finally, following [MM63], and [Har75], we relate $\tilde{\omega}_f^0$ to $\tilde{\omega}$, the Γ_P-invariant differential form on ${}^0P/K_P$ by a twist of the representation ρ_3^n:

$$\tilde{\omega}_f(gK) = \rho_3^n(g) \tilde{\omega}_f^0(g).$$

5.3.2 Fillings

As in the case of the 2-variable period polynomial, to each cohomology class $[\tilde{\omega}_f] \in H^\bullet(e'(P), \mathbb{S}\mathrm{ym}^n(V_3))$, we would like to associate an Eilenberg-MacLane cohomology class $[\phi_{\tilde{\omega}_f}] \in H^\bullet(\Gamma_P, \mathrm{Sym}^n(V_3))$, via the map outlined in §1.2. We can accomplish this by making use of Dupont's *fillings*:

Definition 2 ([Dup78], Ch.9)**.** *A* filling *of $D_m = GL_m(\mathbb{R})/O_m(\mathbb{R})(\lambda Id_m)_{\lambda>0}$ is a family of C^∞ singular simplices*
$\sigma(\gamma_1, \ldots, \gamma_k)\colon \Delta^k \to D_m$*,* $\gamma_1, \ldots, \gamma_k \in GL_m(\mathbb{Z})$*,* $k = 0, 1, 2, \ldots,$ *such that for* $k = 1, 2, \ldots,$

$$\sigma(\gamma_1, \ldots, \gamma_k) \circ \xi^i = \begin{cases} \gamma_1 \sigma(\gamma_2, \ldots, \gamma_k) & i = 0, \\ \sigma(\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_k) & 0 < i < k, \\ \sigma(\gamma_1, \ldots, \gamma_{k-1}) & i = k. \end{cases}$$

Above, we used ξ^i to denote the standard face operators associated with the regular k-simplex in $\mathbb{R}^{k+1}$. A standard way of constructing a filling is to start with a contraction $h : [0,1] \times D_m \to D_m$ to a point $x_0 \in D_m$, and define σ recursively as

$$\sigma(\emptyset) = x_0; \quad \sigma(\gamma_1, \ldots, \gamma_k) = h\left(t, \gamma_1 \sigma(\gamma_2, \ldots, \gamma_k)\right), 0 \le t \le 1.$$

It is in fact this construction that we will use in order to write down a filling of ${}^0P/K_P$. Namely, consider the (piecewise smooth) contraction $\tilde{h}_\bullet$ described in words as: first we collapse $\mathbb{C} \times \overline{\mathcal{H}^+}$ to the fibre $\mathbb{C}$ over the point 0 in $\overline{\mathcal{H}^+}$ (following geodesics in $\overline{\mathcal{H}^+}$), and then contract this fibre to the origin, using a straight-line contraction. More specifically,

$$\tilde{h}_s = \begin{cases} Id \times \phi_{2s} 0 \le s \le \frac{1}{2} & \\ l_{2s-1} \times Id & \frac{1}{2} \le s \le 1 \end{cases} \qquad s \in [0,1].$$

Above, $l_\bullet$ is the linear contraction of the complex fiber to the origin. The component of the contraction taking place in $\overline{\mathcal{H}^+}$ traces geodesics in (the closure of) the upper half-plane with respect to the hyperbolic metric. In particular, consider a point $z = x + iy$ in $\overline{\mathcal{H}^+}$. If $x \neq 0$, the geodesic connecting z to zero with respect to the hyperbolic metric is a circular arc parametrized in constant speed by $\alpha(t) : [0,1] \to \mathbb{C}, \alpha(0) = z, \alpha(1) = 0$. Hence we specify that $\phi_t(z) = \alpha(t), t \in [0,1]$. Similarly, if $y = 0$, then $\alpha(t)$ parametrizes a portion of the $y = 0$ axis in the upper half-plane, and the specification of $\phi_\bullet$ remains the same.

The advantage of using the contraction of $\overline{\mathcal{H}^+} \times \mathbb{C}$ is that the image of $\sigma(\gamma_1, \gamma_2, \gamma_3)$ can be expressed as a union of subsets of the form $N_i \times M_i$, $N_i \subset N_{\mathbf{P}}, M_i \subset M_{\mathbf{P},0}/K_M$ making integration a somewhat simpler affair. More specifically,

$$\phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3) = \int_{\sigma(\gamma_1,\gamma_2,\gamma_3)} \rho\tilde{\omega}_f = \sum_i \int_{N_i \times M_i} \rho\tilde{\omega}_f.$$

Depending on the elements $\gamma_1, \gamma_2, \gamma_3 \in \Gamma_{\mathbf{P}}$, M_i is represents either one- or two-dimensional subset of $\bar{\mathcal{H}^+}$. In the latter case, it looks like a region in the upper half plane bounded by geodesics with respect to the hyperbolic metric.

Theorem 3. *The value of $\int_{N_i \times M_i} \rho\tilde{\omega}_f$ is zero in the event that M_i is two dimensional.*

Proof. First note that Fubini-Tonelli allows us to transform the integral as,

$$\int_{N_i \times M_i} \rho(nm)\tilde{\omega}_f^0 = \int_{N_i} \rho(n) \int_{M_i} \rho(m)\tilde{\omega}_f^0.$$

Comparing with the first few lines of the computation presented in §5.3.4, we find that $\int_{M_i} \rho(m)\tilde{\omega}_f^0$ amounts to the integral,

$$\int_{M_i} \omega_f + \int_{M_i} \omega_{\bar{f}},$$

where $\omega_f, \omega_{\bar{f}}$ are as in §3.1. Considering the shape of the region M_i, this integral is zero by Stokes's theorem, and standard arguments in complex analysis. □

Therefore, the value of $\phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3)$ is determined by integrals over three-dimensional regions of the form $N_i \times M_i$, where M_i is a one-dimensional geodesic path in the upper half plane connecting two points in $P^1(\mathbb{Q})$.

For $\gamma_1 \in \Gamma_{\mathbf{P}}$, let $\gamma_1 = \gamma_{1,N}\gamma_{1,M}$, where $\gamma_{1,M} \in M_{\mathbf{P},o}, \gamma_{1,N} \in N_{\mathbf{P}}$. We make use of the closed co-cycle condition to reduce computing $\phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3)$ to the case when $\gamma_1 = \gamma_{1,M}$. In particular we note that since,

$$\begin{aligned}
0 &= d\phi_{\tilde{\omega}_f}(\gamma_{1,n}, \gamma_{1,M}, \gamma_2, \gamma_3) \\
&= \rho_3^n(\gamma_{1,N})\phi_{\tilde{\omega}_f}(\gamma_{1,M}, \gamma_2, \gamma_3) - \phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3) + \phi_{\tilde{\omega}_f}(\gamma_{1,N}, \gamma_{1,M}\gamma_2, \gamma_3) \\
&+ \phi_{\tilde{\omega}_f}(\gamma_{1,N}, \gamma_{1,M}, \gamma_2\gamma_3) - \phi_{\tilde{\omega}_f}(\gamma_{1,N}, \gamma_{1,M}, \gamma_2).
\end{aligned}$$

It is an easy exercise to check that $\sigma(\gamma_{1,N}, \gamma_{1,M}\gamma_2, \gamma_3), \sigma(\gamma_{1,N}, \gamma_{1,M}, \gamma_2\gamma_3, \sigma(\gamma_{1,N}, \gamma_{1,M}, \gamma_2)$ are unions of three cells of the form $N_i \times M_i$, none of which have the correct dimension. Therefore,

$$\phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3) = \rho_3^n(\gamma_{1,N})\phi_{\tilde{\omega}_f}(\gamma_{1,M}, \gamma_2, \gamma_3).$$

We invite the reader to verify that all cells in $\sigma(\gamma_{1,M}, \gamma_2, \gamma_3)$ that are one-dimensional in the upper half-plane, and two-dimensional in the $\mathbb{C}$-fiber, have a vertical cross section over the relevant portion of the upper half-plane in the form of a triangle with a vertex set $\{(0,0), (s_1, t_1), (s_2, t_2)\}$, where (s, t)-is a coordinate description of $N_{\mathbf{P}}$ as explained in §4.4.1.

Using our knowledge from group co-cycles on the upper half-plane presented in a previous section, we find that $\int_{M_i} \rho(m)\tilde{\omega}_f^0$ can be expressed in terms of the same integral taken from zero to ∞. Consequently, if we let $\int_{M_i} \rho(m)\tilde{\omega}_f = \sum_k \rho_3^n(\gamma_k) \int_0^{i\infty} \left(\omega_f + \omega_{\bar{f}}\right)$, for $\gamma_k \in GL_2(\mathbb{Z}) \subset \Gamma_M \subset \Gamma_P$,

$$\begin{aligned}\int_{N_j} \rho(n) \int_{M_j} \rho(m)\tilde{\omega}_f^0 &= \sum_k \int_{N_j} \rho(n\gamma_k) \int_0^{i\infty} \omega_f + \omega_{\bar{f}} \\ &= \sum_k \int_{n' \in \gamma_k \gamma_k^{-1} N_j \gamma_k} \rho(n') \int_0^{i\infty} \omega_f + \omega_{\bar{f}} \\ &= \sum_k \rho(\gamma_k) \int_{N_j'} \rho(n') \int_0^{i\infty} \omega_f + \omega_{\bar{f}}\end{aligned}$$

where in the last step we used the Γ_P-equivariance. We note that since $\gamma_k \in M_{\mathbf{P},o}$, they stabilize $N_{\mathbf{P}}$, and therefore $N_j' \subset N_{\mathbf{P}}$. Furthermore since conjugation by an element of $M_{\mathbf{P},o}$ maps straight lines in $\mathbb{C} \cong N_{\mathbf{P}}$ to straight lines, N_j' still describes a triangle anchored at the origin in the $\mathbb{C}$-fiber over the upper half-plane.

Therefore, the value of $\phi_{\tilde{\omega}_f}(\gamma_1, \gamma_2, \gamma_3)$ is determined by integrals over regions of the form $\{(s_1, t_1), (s_2, t_2)\} \times (0, \infty)$ where $\{(s_1, t_1), (s_2, t_2)\}$ describes a triangle with a vertex set $\{(0, 0), (s_1, t_1), (s_2, t_2)\}$ in the $\mathbb{C}$-fiber over the upper half-plane.

5.3.3 Three-variable period polynomial: Definition

We fix the following (standard) basis of V_3:

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We define the three-variable period polynomial associated to a cusp form $f \in S_{n+2}(SL_2(\mathbb{Z}))$:

$$r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = \phi_{\tilde{\omega}_f}(S_3, V^{-1}, H),$$

where

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is an easy exercise to verify that $\sigma(S_3, V^{-1}, H)$ is a 3-cell in $\overline{\mathcal{H}^+} \times \mathbb{C}$ that lies over the geodesic from zero to $i\infty$ in the upper half-plane. Furthermore the intersection of $\sigma(S_3, V^{-1}, H)$ with each $\mathbb{C}$-fiber over a point on the y-axis in the upper half plane is the triangle Δ_1 supported on the vertex set $\{0, (0, 1), (1, 1)\}$, as seen in Figure 5.1.

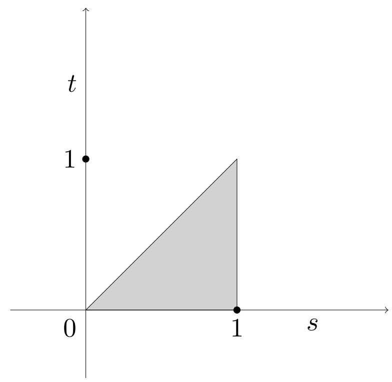


FIGURE 5.1: Triangle in $\mathbb{C}$-fibre over $(0, i\infty) \subset \mathcal{H}^+$ in $\sigma(S_3, V^{-1}, H)$

Recall, with this definition in place, we would like to be able to compute all values of $\phi_{\tilde{\omega}_f}$ simply by knowing the value of $r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$. We will do this in §5.3.6 after first presenting how one would go about explicitly calculating this value, as well as exploring certain relations that are satisfied by $r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$.

5.3.4 A computation

Below, we present a computation of the 3-variable period polynomial. By doing so we would like to relate this value to a period polynomial calculation for the discrete group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ communicated by Richard Hain in [Hai09]. There, Hain's polynomials are made out of monomials in $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$, with coefficients expressed as:

$$\int_0^{i\infty} \frac{z^k f(z)}{z - \bar{z}} \left(\int_{F_z} \zeta^j d\zeta d\bar{\zeta} \right) dz, \tag{5.2}$$

where $F_z = \{t + sz \mid 0 \leq s \leq 1, 0 \leq t \leq s\}$.

As the period polynomials we defined earlier are for the group $\Gamma_P = GL_2(\mathbb{Z} \ltimes \mathbb{Z}^2$, we expect these two notions to differ in the same way the 2-variable period polynomial for $SL_2(\mathbb{Z})$, $r_f(\mathbf{x},\mathbf{y}) = \int_0^{i\infty} \omega_f$, differs from the analogous value for $GL_2(\mathbb{Z})$,

$$\int_0^{i\infty} \omega_f - \omega_{\bar{f}} = 2r_f^+(\mathbf{x},\mathbf{y}).$$

In this section we parametrize the y-axis in the upper half-plane using the subgroup

$$A = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \in M_{\mathbf{P},o} \right\}.$$

We will oft abuse notation and denote by a the element $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \in A$, as well as the coordinate on the group A.

$$\int_{n\in\Delta_1}\int_A \rho_n(na)\tilde{\omega}_f^0$$

$$= (i)^n \int_A \int_{\Delta_1} \rho_n(n) \left(h_f P_1^* \otimes \frac{i}{2}\rho_n(a)B^n + h_{\bar{f}} P_2^* \otimes \frac{i}{2}\rho_n(a)C^n \right) dsdt$$

$$= (i)^n \frac{i}{2} \int_A \int_{\Delta_1} \rho_n(n) \left(h_f P_1^* \otimes (a\,\mathbf{x_2} - \frac{i}{a}\,\mathbf{x_3})^n + h_{\bar{f}} P_2^* \otimes (a\,\mathbf{x_2} + \frac{i}{a}\,\mathbf{x_3})^n \right) dsdt$$

$$= (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_f P_1^* \otimes \left(a(s\,\mathbf{x_1} + \mathbf{x_2}) - \frac{i}{a}(t\,\mathbf{x_1} + \mathbf{x_3}) \right)^n \right) dsdt$$

$$+ (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_{\bar{f}} P_2^* \otimes \left(a(s\,\mathbf{x_1} + \mathbf{x_2}) + \frac{i}{a}(t\,\mathbf{x_1} + \mathbf{x_3}) \right)^n \right) dsdt$$

$$= (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_f P_1^* \otimes \left(\mathbf{x_1}(as - \frac{it}{a}) + a\,\mathbf{x_2} - \frac{i}{a}\,\mathbf{x_3} \right)^n \right) dsdt$$

$$+ (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_{\bar{f}} P_2^* \otimes \left(\mathbf{x_1}(as + \frac{it}{a}) + a\,\mathbf{x_2} + \frac{i}{a}\,\mathbf{x_3} \right)^n \right) dsdt$$

$$= (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_f(a) P_1^* \otimes \left(\mathbf{x_1}(as - \frac{it}{a}) + a\,\mathbf{x_2} - \frac{i}{a}\,\mathbf{x_3} \right)^n \right) dsdt$$

$$+ (i)^n \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(h_{\bar{f}}(a) P_2^* \otimes \left(\mathbf{x_1}(as + \frac{it}{a}) + a\,\mathbf{x_2} + \frac{i}{a}\,\mathbf{x_3} \right)^n \right) dsdt.$$

Next, recall that $h_f(a) = (a)^{n+2} f(a.(i)), h_{\bar{f}}(a) = (a)^{n+2}\bar{f}(a.(i))$. Thus,

$$\int_{\Delta_1}\int_A \rho_n(n)(a)\tilde{\omega}_f^0$$

$$= \frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=s} \left(f(a.(i))(a^2 P_1^*) \otimes (\mathbf{x_1}(t + ia^2 s) + ia^2\,\mathbf{x_2} + \mathbf{x_3})^n \right) dsdt+$$

$$\frac{i}{2} \int_A \int_{s=0}^{s=1} \int_{t=0}^{t=1} \left(\bar{f}(a.(i))(a^2 P_2^*) \otimes (\mathbf{x_1}(-t + ia^2 s) + ia^2\,\mathbf{x_2} - \mathbf{x_3})^n \right) dsdt.$$

We introduce the parameter $z = g.i$ on $\mathcal{H}^+$, and the parameter $\zeta = t + sz$ on $\mathbb{C}_z$, the fiber over z. We also use F_z to denote $\{t + sz \mid 0 \leq s \leq 1, 0 \leq t \leq s\} \subset \mathbb{C}_z$. Also note, $d\zeta d\bar{\zeta} = (z-\bar{z})dsdt$. Therefore, the above expression is equal to,

$$= \frac{i}{2}\int_{z=0}^{z=i\infty}\int_{F_z}\left(\frac{f(z)}{z-\bar{z}}(\mathbf{x_1}\,\zeta + z\,\mathbf{x_2} + \mathbf{x_3})^n\right) d\zeta d\bar{\zeta} dz$$

$$+ \frac{i}{2}\int_{z=0}^{z=i\infty}\int_{F_z}\left(\frac{\bar{f}(z)}{z-\bar{z}}(-\,\mathbf{x_1}\,\bar{\zeta} + z\,\mathbf{x_2} - \mathbf{x_3})^n\right) d\zeta d\bar{\zeta} d\bar{z}$$

Above, I have used the fact that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\frac{1}{ci+d}\left(P_2^*\right)_g = dz|_{gK}$. This is certainly true at $g = \mathrm{Id}_2$. To verify this everywhere on the upper half-plane we note that $d(g.z) = \frac{1}{(cz+d)^2}dz$, and check that $\frac{1}{ci+d}\left(P_2^*\right)_g$ obeys the same transformation law. In particular for $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 1, 2$:

$$\frac{1}{\left((a_1c_2 + c_1d_2)i + b_1c_2 + d_1d_2\right)^2}\left(P_2^*\right)_{g_2g_1} = \frac{1}{\frac{\left((a_1c_2+c_1d_2)i+b_1c_2+d_1d_2\right)^2}{(ci+d)^2}}\frac{\left(P_2^*\right)_{g_2g_1}}{(ci+d)^2}$$

$$= \frac{1}{\left(c_2(g_1.i) + d_2\right)^2}\frac{\left(P_2^*\right)}{(ci+d)^2},$$

as desired. Therefore $a^2P_1^* = dz$, and similarly, $a^2\left(P_1^*\right) = d\bar{z}$. Finally, we note that $-z = \bar{z}$, and therefore,

$$\left(\frac{\bar{f}(z)}{z-\bar{z}}(-\,\mathbf{x_1}\,\bar{\zeta} + z\,\mathbf{x_2} - \mathbf{x_3})^n\right) = -\overline{\left(\frac{f(z)}{z-\bar{z}}(\mathbf{x_1}\,\zeta + z\,\mathbf{x_2} + \mathbf{x_3})^n\right)},$$

$$d\zeta d\bar{\zeta} = -d\bar{\zeta} d\zeta = -\overline{d\zeta d\bar{\zeta}}.$$

In conclusion we arrive at

$$r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = \int_{\Delta_1}\int_A \rho_n(n)(a)\tilde{\omega}_f^0$$

$$= i\mathrm{Re}\left(\int_{z=0}^{z=i\infty}\int_{F_z}\left(\frac{f(z)}{z-\bar{z}}(\zeta\,\mathbf{x_1} + z\,\mathbf{x_2} + \mathbf{x_3})^{2n}\right) d\zeta d\bar{\zeta} dz\right). \qquad (5.3)$$

This expression yields monomials in $\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ that differ from the ones presented in [Hai09] precisely as expected.

5.3.5 A word on relations

In deriving relations in this section we will make use of the Γ_P-equivariance of $\tilde{\omega}_f$. More specifically, we note that for $\gamma \in \Gamma_P$:

$$\int L_\gamma^*(\tilde{\omega}_f) = \rho_3^n(\gamma) \int \tilde{\omega}_f.$$

We also note that these relations were also obtained by Hain in [Hai09]. In particular, our derivation of the Dilation relation is closely motivated by his exposition.

Symmetry relation

Therefore,

$$\begin{aligned}\rho_3^n(S_3) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) &= \int_{\Delta_1} \int_0^{i\infty} L_{S_3}^*(\tilde{\omega}_f) = \int_{S_3\Delta_1} \int_0^{i\infty} \tilde{\omega}_f \\ &= \int_{S_3\Delta_1 S_3^{-1}} \int_{S_3(0)}^{S_3(i\infty)} \tilde{\omega}_f = \int_{S_3\Delta_1 S_3^{-1}} \int_{i\infty}^{0} \tilde{\omega}_f\end{aligned}$$

Since $S_3 \in \Gamma_M$ it normalizes $N_{\mathbf{P}}$, and therefore $S_3\Delta_1 S_3^{-1}$ is also in $N_{\mathbf{P}}$. Even better since all operations are linear, $S_3\Delta_1 S_3^{-1}$ again describes a triangle in the $\mathbb{C}$-fibre over the upper half-plane. In fact, it is not difficult to check that conjugation by S_3 maps Δ_1 to the triangle see in Figure 5.2.

We note however, that there are other transformations that one can apply to Δ_1 to arrive at the same triangle. In particular, we also consider Γ_N acting on $N_{\mathbf{P}}$ via multiplication on the left - no need for conjugation as Γ_N is already in $N_{\mathbf{P}}$. Furthermore we introduce the element

$$X_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \Gamma_M \cap K,$$

and note that conjugation by X_r acts as reflection through the s-axis in the $\mathbb{C}$-fibre over the upper half-plane. We observe in Figure 5.3, that

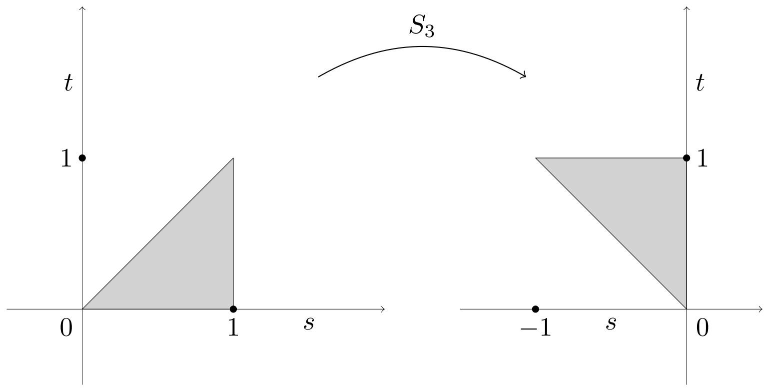


FIGURE 5.2: S_3 acts on Δ_1 via conjugation

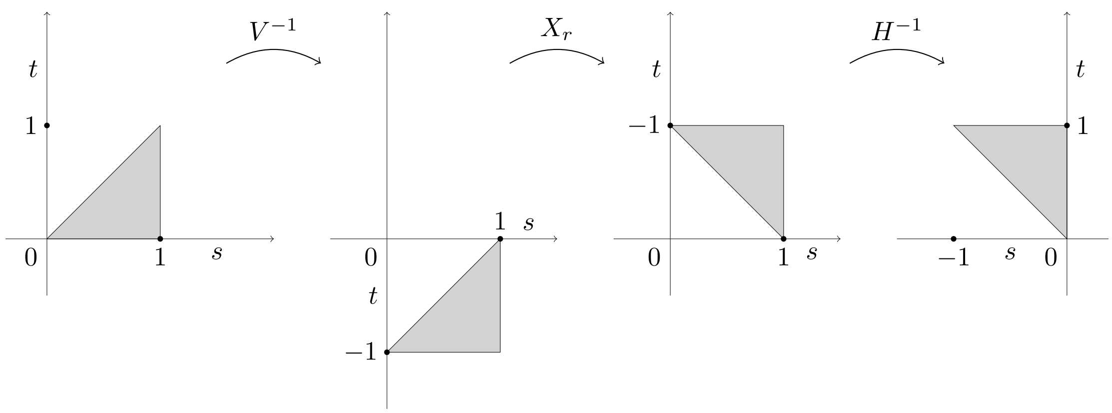


FIGURE 5.3: Transforming Δ_1.

Therefore,

$$\begin{aligned}\rho_3^n(S_3)r_f(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) &= \int_{S_3\Delta_1 S_3^{-1}}\int_{i\infty}^{0}\tilde{\omega}_f = -\int_{\Delta_1}\int_{0}^{i\infty} L^*_{H^{-1}X_rV^{-1}}(\tilde{\omega}_f)\\ &= -\rho_3^n\left(H^{-1}X_rV^{-1}\right)r_f(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}).\end{aligned}$$

Unraveling the above equality we discover that:

$$\begin{aligned} r_f(\mathbf{x_1}, \mathbf{x_3}, -\mathbf{x_2}) = \rho_3^n(S_3) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) &= -\rho_3^n \left(H^{-1} X_r V^{-1}\right) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) \\ &= -\rho_3^n \left(H^{-1} X_r\right) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} - \mathbf{x_1}) \\ &= +\rho_3^n \left(H^{-1}\right) r_f(\mathbf{x_1}, \mathbf{x_2}, -\mathbf{x_3} - \mathbf{x_1}) \\ &= r_f(\mathbf{x_1}, \mathbf{x_2} - \mathbf{x_1}, -\mathbf{x_3} - \mathbf{x_1}), \end{aligned}$$

where in the second to last equality we accounted for the full action of X_r on the coefficients, including a -1 coming from the determinant character as it pertains to the action of X_r on $\mathrm{Sym}^n(V_2) \otimes \det \hookrightarrow \mathrm{Sym}^n(V_3)$.

Dilation relation

From our discussion in the previous section it should be easy to see that the value $(\mathrm{Id}_3 + \rho_3^n(HS_3)) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is calculated by integrating $\tilde{\omega_f}$ over the region in $\overline{\mathcal{H}^+} \times \mathbb{C}$ described as the imaginary axis in the upper half-plane, and the square with vertex set $\{0, (1,0), (1,1)(0,1)\}$ in the $\mathbb{C}$-fiber. We also note that $\rho_3^n(H^p V^q)(\mathrm{Id}_3 + \rho_3^n(HS_3)) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is in fact the integral over the imaginary axis, and the square in the fiber described by the vertex set $\{(p,q), (p+1,q), (p+1,q+1), (p,q+1)\}$. Therefore, the value of

$$\sum_{p,q=0}^{k-1} \rho_3^n(H^p V^q)(\mathrm{Id}_3 + \rho_3^n(HS_3)) r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$$

is in fact calculated as the integral over

$$\left\{\mathcal{H}^+ \supset \{\mathrm{Re}(z) = 0\}\right\} \times \{\{(s,t) \mid 0 \le s, t \le k\} \subset \mathbb{C}\}.$$

Therefore, referring back to (5.3), and letting $\tilde{F}_z^k = \{t + sz \mid 0 \leq s, t \leq k\} \subset \mathbb{C}_z$, we find that

$$\sum_{p,q=0}^{k-1} \rho_3^n(H^p V^q)(\mathrm{Id}_3 + \rho_3^n(HS_3))r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$$

$$= i\mathrm{Re}\left(\int_{z=0}^{z=i\infty} \int_{\tilde{F}_z^k} \left(\frac{f(z)}{z-\bar{z}}(\zeta\,\mathbf{x_1} + z\,\mathbf{x_2} + \mathbf{x_3})^{2n}\right) d\zeta d\bar{\zeta} dz\right)$$

$$= k^2 i\mathrm{Re}\left(\int_{z=0}^{z=i\infty} \int_{\tilde{F}_z^1} \left(\frac{f(z)}{z-\bar{z}}(k\zeta\,\mathbf{x_1} + z\,\mathbf{x_2} + \mathbf{x_3})^{2n}\right) d\zeta d\bar{\zeta} dz\right)$$

$$= k^2(\mathrm{Id}_3 + \rho_3^n(HS_3))r_f(k\,\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}).$$

5.3.6 As it relates to the 2-variable period polynomial

We note that, if the odd part of the 2-variable period polynomial associated to f, is denoted as $r_f^-(\mathbf{x}, \mathbf{y}) = \sum_{ij} a_{ij}\,\mathbf{x}^i\,\mathbf{y}^j$, then

$$r_f(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = \int_{\Delta_1} \rho(n) \int_0^{i\infty} \rho(m)\tilde{\omega}_f^0 = \int_{\Delta_1} \rho(n) r_f^-(\mathbf{x_2}, \mathbf{x_3})$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=s} r_f^-(\mathbf{x_2} + s\,\mathbf{x_1}, \mathbf{x_3} + t\,\mathbf{x_1}) ds dt$$

$$= \sum_{\substack{ij \\ r_1+r_2=r}} a_{ij}\,\mathbf{x_1}^r\,\mathbf{x_2}^{i-r_1}\,\mathbf{x_3}^{j-r_2} \int_{s=0}^{s=1} \int_{t=0}^{t=s} s^{r_1} t^{r_2} ds dt \qquad (5.4)$$

The reader should note that (up to a constant), the odd part of the two variable period polynomial can be recovered from the three variable polynomial by setting $\mathbf{x_1} = 0$.

In terms of the relations we derived from the previous section, we note that when we set $\mathbf{x_1} = 0$ they reduce to known relations of the two-variable period polynomial. More specifically, in terms of the symmetry relation, since all monomial exponents in $r_f^-(\mathbf{x_2}, \mathbf{x_3})$ are odd for both $\mathbf{x_2}$ and $\mathbf{x_3}$, we find that the relation reduces to,

$$r_f^-(\mathbf{x_3}, -\mathbf{x_2}) = -r_f^-(\mathbf{x_2}, \mathbf{x_3}).$$

We note that this is indeed a relation satisfied by the odd part of the two-variable period polynomial. We can see this by first realizing that following our discussion in §5.2

$$r_f^-(\mathbf{x},\mathbf{y}) = \phi_{\omega_f+\omega_{\bar{f}}}(S).$$

Making use of the closed condition, and the fact that $S^2 = -\mathrm{Id}_2$, we discover:

$$0 = d\phi_{\omega_f+\omega_{\bar{f}}}(S,S) = \rho_2^n(S)\phi_{\omega_f+\omega_{\bar{f}}}(S) - \phi_{\omega_f+\omega_{\bar{f}}}(S^2) + \phi_{\omega_f+\omega_{\bar{f}}}(S).$$

Therefore,

$$r_f^-(\mathbf{x},\mathbf{y}) = -\rho_2^n(S)r_f^-(\mathbf{x},\mathbf{y}) \iff r_f^-(\mathbf{x},\mathbf{y}) = -r_f^-(\mathbf{y},-\mathbf{x}).$$

We state without proof that the same is true for the Dilation relation, namely setting $\mathbf{x_1} = 0$, it reduces to a known relation of the even part of the two variable period polynomial.

Next we exploit this relationship between the 3- and 2-variable period polynomials and demonstrate that indeed, knowing $r_f(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ allows us to compute $\phi_{\tilde{\omega}_f}(\gamma_1,\gamma_2,\gamma_3)$ for all $(\gamma_1,\gamma_2,\gamma_3) \in \Gamma_P^3$.

For any triangle Δ_2 supported on the origin in the $\mathbb{C}$-fiber over the upper half-plane:

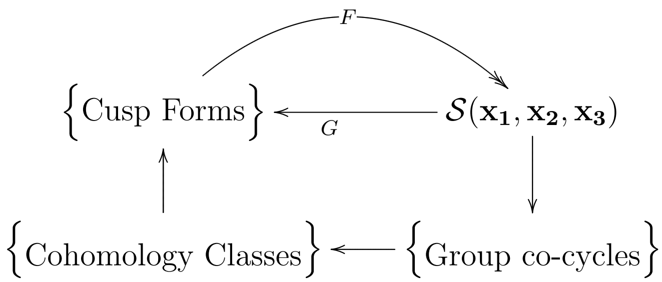
$$\int_{\Delta_2} \rho(n) \int_0^{i\infty} \rho(m)\tilde{\omega}_f^0 = \sum_{\substack{ij \\ r_1+r_2=r}} a_{ij}\, \mathbf{x_1}^r\, \mathbf{x_2}^{i-r_1}\, \mathbf{x_3}^{j-r_2} \int_{\Delta_2} s^{r_1}t^{r_2}dsdt$$

It is evident from (5.4) that knowing $r_f(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$, we can easily compute $\int_{\Delta_2}\int_0^{\infty} \rho\tilde{\omega}_f^0$. By this we mean that once we know $r_f(0,\mathbf{x_2},\mathbf{x_3})$, computing the coefficients of monomials with a non-trivial $\mathbf{x_1}$ term reduces to computing a double integrals of the form $\int_{\Delta_2} s^{r_1}t^{r_2}dsdt$. Since Δ_2 is an arbitrary triangle in the $\mathbb{C}$-fiber, our considerations in §5.3.2 conclude the argument that knowing $r_f(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = \phi_{\tilde{\omega}_f}(S_3,V^{-1},H)$ we can proceed to compute $\phi_{\tilde{\omega}_f}(\gamma_1,\gamma_2,\gamma_3)$ for all $(\gamma_1,\gamma_2,\gamma_3) \in \Gamma_P^3$.

With these considerations in mind, we are finally ready to flesh out the space of 3-variable period polynomials as defined in this section. In particular, we define

$$\mathcal{S}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = \left\{ \begin{array}{c} p(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = c_{ijk}\,\mathbf{x_1}^i\,\mathbf{x_2}^j\,\mathbf{x_3}^k \in \mathrm{Sym}^n(V_3) \mid p(0,\mathbf{x_2},\mathbf{x_3}) \in \mathcal{Z}_n^-; \\ c_{pqr} = \sum_{\substack{p_1+p_2=p \\ p_1<n-q \\ p_2<n-r}} c_{0(q+p_1)(r+p_2)} \binom{q+p_1}{p_1}\binom{r+p_2}{p_2} \int_{s=0}^{1}\int_{t=0}^{s} s^{p_1}t^{p_2} \end{array} \right\}.$$

The sum of our work in this section yields the following commutative diagram,



Above, the map F, is given by $F(f) = \phi_{\tilde{\omega}_f}(S_3, V^{-1}, H)$, and the map G is the composition of the maps leading from $\mathcal{S}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) \subset \mathrm{Sym}^n V_3$ to the space of cusp forms. From our discussion thus far, we know F to be one-to-one. Similarly, one can easily show that $G = F^{-1}$. Therefore these maps yield isomorphisms between the space of cusp forms and $\mathcal{S}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$, space consisting of certain special 3-variable polynomials.

This in fact concludes our first attempt at defining a 3-variable period polynomial. In summary, the end result is a space of polynomials that completely describes the cohomology of $\Gamma_{\mathbf{P}}$. While, new and original, and certainly useful as it pertains to $\Gamma_{\mathbf{P}}$, the space $\mathcal{S}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ is in a sense a simple 3-variable analogue of $\mathcal{Z}_n^-$. In the next chapter, we hope that we can use this type of modular symbol investigation to explore the Eilenberg-MacLane cohomology of $GL_3(\mathbb{Z})$, and in doing so define a space of polynomials that differs from $\mathcal{Z}_n^-$ more substantially.

6

A fresh look at Modular Symbols

As described in the introduction, a guiding principle in defining a new, and improved notion of a 3-variable period polynomial is to make use of the combinatorial information encoded in the well-rounded retract for GL_3. More specifically, we will use this combinatorial structure in order to define a filling giving rise to 3-cells that are entirely contained within the well-rounded retract itself. Furthermore, in constructing this filling we succeed in maintaining a degree of control over the shape and form of these cells, much like in the case of $e'(\mathbf{P})$ where we had an explicit description of the geometry of all relevant cells we were integrating over.

In this chapter we present a framework in which the computation is done via an integration pairing where the cells are compactly supported inside the well-rounded retract for GL_2 and GL_3. Most of the introductory exposition on the well-rounded retract in this chapter is motivated by the discussions in [Ash84], [AM97], and [Gun].

6.1 The well-rounded retract

Let $G = GL_m(\mathbb{R})$, $K = O_m(\mathbb{R}), \Gamma = GL_m(\mathbb{Z})$ and A_+ the group of scalar matrices corresponding to the positive real homotheties. Let

$$L_0 := \{a_1e_1 + a_2e_2 + \cdots + a_ne_n \mid a_i \in \mathbb{Z}, i = 0, 1, \ldots, n\} \cong \mathbb{Z}^m,$$

where $\{e_i\}_{i=1}^m$ is the standard basis in $\mathbb{R}^m$.

Throughout this exposition, it will be useful to think of D_m as both the space of positive definite quadratic forms on $\mathbb{R}^m$ modulo homothety, and as the space of marked lattices in $\mathbb{R}^m$ modulo rotation and homothety. We make this relationship explicit below.

Namely, for each $g \in G$, we define the *associated quadratic form* as,

$$Q_g = ({}^tg)^{-1}g^{-1}.$$

We note that $Q_g = Q_{gk}$. Keeping in mind that a positive definite symmetric matrix can be diagonalized using orthogonal matrices yielding a $({}^tg)^{-1}g^{-1}$ decomposition, it is obvious that the space of positive definite quadratic forms is equivalent to G/K. Going one step further, we see that D_m can be identified with the G/KA_+.

Similarly, for $g \in G$, we can define the *corresponding marked lattice* as,

$$\begin{aligned} f_g : L_0 &\to \mathbb{R}^n \\ x &\mapsto g^{-1}x. \end{aligned}$$

In addition, we identify two marked lattices that differ by a homothety, namely, $\tilde{f} \sim f$ whenever $\tilde{f} = a \cdot f$, where $a \in A_+$. Therefore we have established the following relationship,

$$\left\{\begin{matrix}\text{Positive definite quadratic forms}\\ \text{modulo homothety}\end{matrix}\right\} \cong G/KA_+ \cong \left\{\begin{matrix}\text{Marked lattices}\\ \text{modulo rotation}\end{matrix}\right\}$$

$$Q_g = ({}^tg)^{-1}g^{-1} \longleftarrow gKA_+ \longmapsto f_g.$$

For a positive definite quadratic form Q_g fixed within its homothety class, we define the *arithmetic minimum* of Q_g as,

$$m_g = m(Q_g) = \min\left\{\sqrt{Q_g(x)} \mid x \in L_0, x \neq 0\right\}.$$

The set of *minimal vectors* of Q_g is defined to be

$$M_g = M(Q_g) = \{x \in L_0 \mid Q_g(x) = m_g\}/\sim,$$

where $v_1 \sim v_2 \iff v_1 = \pm v_2$. Hereafter we abuse notation and write $v \in M_g$, when we mean $[v] \in M_g$. Equivalently, following the exposition in [AM97], we can define these as,

$$m_g = m(f_g) = \min\{|f_g(x)| \mid x \in L_0, x \neq 0\},$$

$$M_g = M(f_g) = \{x \in L_0 \mid |f_g(x)| = m_g\}/\sim,$$

where $|\cdot|$ is the norm with respect to the standard Euclidean inner product.

Lemma 11. *We make the following observations*:

1. *The quantity m_g is finite and positive for all $g \in G$.*

2. *The set M_g is finite and nonempty.*

3. *For $v = (v_1, \ldots, v_m)^t \in M_g, \gcd(v_1, \ldots, v_m) = 1$.*

Proof. To prove the first observation, assume by contradiction that $f_g(x_i) \to 0$ for some sequence $\{x_i\}_{i=0}^{\infty}, x_i \in L_0$, and $x_i \neq 0$. If the sequence $\{x_i\}_i$ is bounded, then it contains a convergent subsequence. Therefore there exists an N such that for $i, j > N$, $x_i = x_j \neq 0$. This contradicts the hypothesis that $f_g(x_i) \to 0$. If, on the other hand, the sequence $\{x_i\}_i$ is unbounded, then consider the map given as multiplication by g. This is a continuous function mapping zero to zero, however it also maps the sequence $\{f_g(x_i) = g^{-1}x_i\}$ which converges to zero, to the sequence of x_i which is unbounded. This is a clear contradiction.

As far as the second observation is concerned, let $\{x_i\}_i$ be a sequence in $L_0 \setminus \{0\}$ such that $f_g(x_i)$ approaches m_g in norm. The set $\{x_i\}_i$ is bounded. For if it is not, the compact set $g \cdot \left(\overline{B}(m_g + \varepsilon)\right)$, where $\overline{B}(m_g + \varepsilon)$ is the closed ball with radius $m_g + \varepsilon$ in $\mathbb{R}^n$, has elements with arbitrarily large norm. Therefore the set $\{x_i\}_i$ is bounded, and as such it contains a subsequence converging to some $x \in L_0 \setminus \{0\}$. From the continuity of f_g it follows that $|f_g(x)| = m_g$, and therefore M_g is nonempty. Since any $x \in M_g$ is a lattice point in the compact set $g \cdot \overline{B}(m_g)$, M_g is finite.

The last observation follows immediately from the minimality of $Q_g(v)$ for $v \in M_g$. Integral vectors in $\mathbb{R}^m$ having coefficients that are mutually prime are called *primitive.* □

Definition 3. *A marked lattice f_g is* well-rounded *if M_g spans L_0 as a $\mathbb{Z}$-module.*

Definition 4. *We call this The* well-rounded retract *in X_m is the set W of well-rounded elements.*

6.1.1 The well-rounded retraction

The exposition in this section follows the one in [AM97, Section 4.1] very closely. Let $Y = G/A_+$ be thought of as the space of marked lattices, and consider the following filtration for $i = 1, \dots, m$,

$$Y_i = \{f_g \in Y \mid \operatorname{rank}_{\mathbb{Z}}(\mathbb{Z} \cdot M_g) \geq i\}.$$

We note that, $Y = Y_1 \supset Y_2 \supset \cdots \supset Y_m = W$. The well-rounded retraction is defined as a collapsing composition of intermediate deformation retractions $r_t^{(i)} : Y_i \to Y_i, t \in [0, 1]$, with $r_1^i(Y_i) = Y_{i+1}$. Start with a marked lattice f_g, and let $L = f_g(L_0)$. Since we always identify marked lattices up to homothety, we may specify that $m_g = 1$.

We let $f_1 = f_g$, and inductively define $f_{i+1} = r_1^i(f_i)$. If $f_i \in Y_{i+1}$ then let $r_t^{(i)}(f_i) = f_i$. Otherwise, let $M^{(i)}$ be the $\mathbb{Z}$-span of $M(f_i)$. Furthermore, let $S = \mathbb{R} \otimes f_i(M^{(i)})$, and $T = S^\perp$. Consider the family of $\mathbb{R}$-linear maps

$$\phi_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$s + t \longmapsto s + \mu t,$$

where $\mu > 0$, $s \in S$, $t \in T$. According to [Ash84] there is a unique smallest μ_i such that $m_{\phi_{\mu_i} \cdot f_i} = 1$, and $\phi_{\mu_i} \cdot f_i \in Y_{i+1}$. We set $r_t^{(i)} = \phi_{1+(\mu_i - 1)t} \cdot f_i$, and define the full contraction as,

$$r_t(f) = r_{t(m-1)-(i-1)}^{(i)} \circ r_1^{(i-1)} \circ \cdots \circ r_1^{(1)}(f),$$

where i is determined by the rule $t \in \left[\frac{i-1}{m-1}, \frac{i}{m-1}\right]$ for $i \in \{1, \ldots, m-1\}$.

6.1.2 The action of Γ

Let $g \in G$ be such that f_g is well-rounded with $\{v_i\}_{i=1}^m \in M(f_g)$ spanning $\mathbb{Z}^m$ as a $\mathbb{Z}$-module. For $\gamma \in \Gamma$, γg gives rise to a well rounded marked lattice as $\{\gamma v_i\}_{i=1}^m$ form a set of minimal vectors for $f_{\gamma g}$ of maximal rank.

6.1.3 Visualizing the well-rounded retract for GL_2 *and* GL_3

In [Sou78] and [Ash84], Soulé and later Ash, present a cellular decomposition of the well-rounded retract that we can use to visualize the spine in low rank. In particular, each cell is determined by a set of minimal integral vectors (modulo multiplication by ± 1), shared by all quadratic forms in the cell. Therefore, for D_m, the top-dimensional cells in the well-rounded retract are decorated by m-integral, primitive vectors that span $\mathbb{Z}^m$ as a $\mathbb{Z}$-module. The cells of dimension one less, are decorated by $(m+1)$ such vectors, and the list continues on. Each cell is a convex linear set in the globally symmetric space. To see this, note that, if the quadratic forms Q_1 and Q_2 share $v \in \mathbb{Z}^n$ as a minimal vector, then for $0 \leq \lambda \leq 1$, it is certainly the case that,

$$Q_\lambda = \lambda Q_1 + (1 - \lambda) Q_2,$$

also has v as a minimal vector.

The well-rounded retract for GL_2

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$. Using standard manipulation, we may factor

$$g = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} k,$$

where $k \in O_2(\mathbb{R}),\ y, \lambda > 0$. Making use of the action of $SL_2(\mathbb{R})$ on the upper half-plane by fractional linear transformations, we can identify gKA_+ with the point $x + iy$ in the upper half plane $\mathcal{H}^+$.

On the other hand, g gives rise to a marked lattice $f_g : L_0 \to \mathbb{R}^2$, where $x \mapsto g^{-1}x$. We note that since, $g^{-1} = k^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}$, modulo homothety and rotation the marked lattice associated to gKA_+ is given by $v \mapsto \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix} v$. In order to be consistent with the coordinates on G/KA_+ introduced previously, we again identify this coset with the point $x + iy$. We note that, we can read off all relevant information about the lattice f_g in this manner, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is fixed, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is mapped to $-x+iy$. This is a convenient manner for working with coordinates in the upper half-plane. Namely, the point $z = \alpha + i\beta$ with $\beta > 0$, represents the marked lattice:

$$f : L_0 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}.$$

This identification between points in the upper half-plane and equivalence classes of marked lattices allows us to visualize the well-rounded retract. In Figure 6.1, each one-dimensional cell in the well-rounded retract is decorated with the minimal vectors for the marked lattices corresponding to points in that cell.

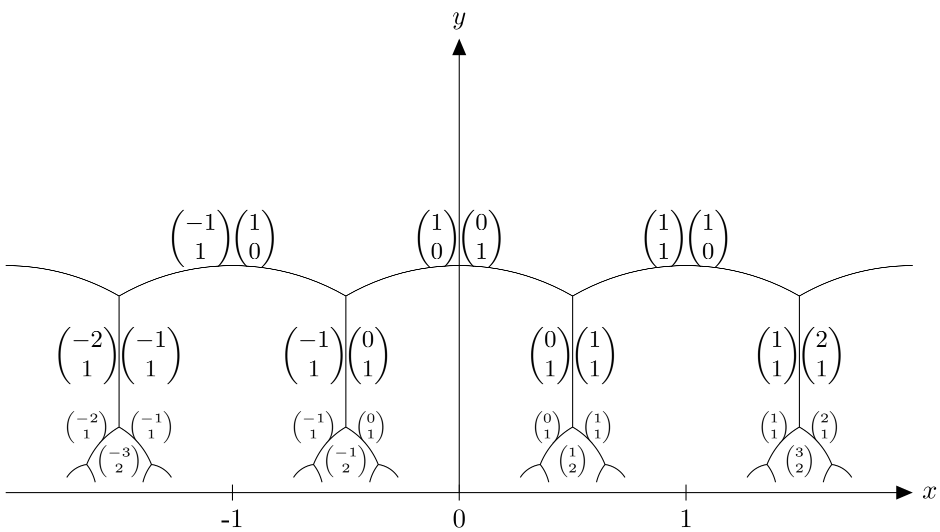


FIGURE 6.1: Well-rounded retract in the upper half-plane.

This illustration also contains all information necessary to decorate each zero-cell in the well-rounded retract with a set of minimal vectors. In particular, the 0-cell given as the intersection between the three one-cells decorated with,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

represents a marked lattice whose set of minimal vectors is given as the union of the three sets above,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Furthermore, once we have decorated the well-rounded retract, we can use the above illustration to visualize the action of Γ on the individual cells, even without knowing the action of Γ on the ambient space. Namely, we make use of the fact that the action on marked lattices can be thought of as an action on minimal vectors, as explained in 6.1.2. Therefore, it follows that the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$, sends the one-cell decorated with $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, to the one decorated with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

This is easily verified using what we know about the action of $SL_2(\mathbb{R})$ on the upper half-plane. As a final note, we define the *fundamental arc* of the well-rounded retract in the upper half-plane to be the set of points,

$$\{x + iy \mid x^2 + y^2 = 1, -1/2 \leq x \leq 1/2\}.$$

The well-rounded retract for GL_3

Offering a picture of the well-rounded retract in rank two, a three dimensional spine embedded in five dimensional space, is a more challenging task. Here however, we can lean on the work of Soulé in [Sou78], where using coordinates on the space of quadratic forms, he was able to illustrate a single three-dimensional cell in the well-rounded retract for GL_3. Namely, we identify the quadratic form

$$Q(u, v, w) = \begin{pmatrix} 2 & w & v \\ w & 2 & u \\ v & u & 2 \end{pmatrix},$$

with the triplet (u, v, w). As described before, each cell is a convex linear set, and using the coordinates (u, v, w) in Figure 6.2 we can see the unique three cell in the well-rounded retract containing the equivalence class of the quadratic form given by the identity matrix.

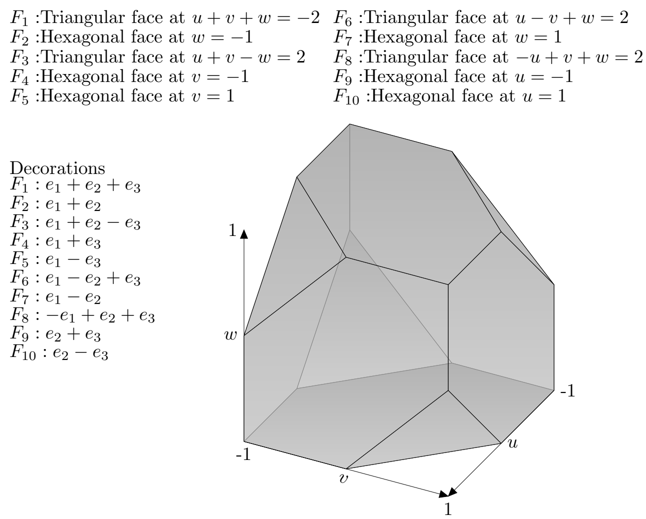


FIGURE 6.2: Top-dimensional cell in the Well-rounded retract for GL_3.

Each quadratic form Q in the interior of the cell in Figure 6.2 has a set of minimal vectors, $M(Q) = \{e_1, e_2, e_3\}$, where $\{e_i\}_{i=1}^3$ is the standard basis in $\mathbb{Z}^3$. In the figure we have also named and decorated the two cells in the boundary of the cube - the cell F_i is decorated by the set $\{e_1, e_2, e_3, v_i\}$, where $F_i : v_i$ is listed in Figure 6.2. For each Q_{F_i}, a quadratic form in a 2-dimensional boundary face F_i decorated with a vector v_i in Figure 6.2, we have $M(Q_B) = \{e_1, e_2, e_3, v_i\}$. Similarly as before, for quadratic forms in the interior of the one-cell given as the intersection of F_i and F_j, the set of minimal vectors is $\{e_1, e_2, e_3, v_i, v_j\}$. We note that, up to Γ-equivalence there is exactly one three-cell, one one-cell, and one zero-cell. On the other hand, there are two Γ-equivalence classes of two-cells, each visible in Figure 6.2 as a triangle, or a hexagon.

Each Soulé cube has ten 2-dimensional faces, namely six hexagons and four triangles. Each hexagon is shared between three cubes, and each triangle between four cubes. To make this more specific, consider the hexagon F_{10} in Figure 6.2 determined by $u = 1$. As explained, the quadratic forms in the interior of F_{10} are characterized by the set of minimal vectors,

$$M(F_{10}) = \left\{ e_1, e_2, e_3, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Apart from the cube portrayed above, the two other cubes sharing this hexagonal face are decorated with the different rank three subsets of $M(F_{10})$, namely,

$$\left\{ e_1, e_2, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}, \text{ and } \left\{ e_1, e_3, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Similarly, the four cubes tangent to the triangular face F_3 decorated with the vector $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in addition to the vectors e_1, e_2, and e_3, are the cubes corresponding to the four different rank three subsets of $\{e_1, e_2, e_3, v\}$. In summary, each cube is tangent to twenty four other cubes in codimension 1 faces. Table 6.1 offers a summary if the incidence analysis taking into accound lower dimensional cells. For $1 \leq i, j \leq 5, i > j$, a_{ij} is the number of column type j cells contained in the boundary of a cell of row type i. On the other hand, a_{ji} is the number of cells of column type i containing a cell of row type j in its boundary.

Table 6.1: Table of Incidences in W_3.

	vertex	edge	triangle	hexagon	Soulé cube
vertex	–	6	3	12	16
edge	2	–	1	4	8
triangle	3	3	–	–	4
hexagon	6	6	–	–	3
Soulé cube	16	24	4	6	–

We note that the information in this table departs from similar tables found in standard references, such as those in [MM89], and [Gun], in the value in the top right corner. Specifically, this value is the number of top-dimensional cells (cubes) incident in a given vertex of the well-rounded retract. Since all 0-cells in the well-rounded retract are $SL_3(\mathbb{Z})$-equivalent, it suffices to write down sixteen different cubes that contain the lower left corner of the cube seen in Figure 6.2. This particular vertex is decorated by the set $\{e_1, e_2, e_3, e_1+e_3, e_1+e_2, e_2-e_3\}$. We provide a list of elements $\{\gamma = \langle v_1 \mid v_2 \mid v_3\rangle \in SL_3(\mathbb{Z})\}$ such that that list of γ-translates of the cube in Figure 6.2 enumerates all the cubes containing the vertex in question:

$$\langle e_1 \mid e_2 \mid e_3\rangle, \langle e_1 \mid e_2 \mid -e_2+e_3\rangle, \langle e_1 \mid e_2 \mid e_1+e_3\rangle, \langle e_1 \mid e_3 \mid -e_2+e_3\rangle,$$

$$\langle e_1 \mid e_3 \mid -e_1-e_3\rangle, \langle e_2 \mid e_3 \mid e_1+e_2\rangle, \langle e_2 \mid e_3 \mid e_1+e_3\rangle, \langle e_1 \mid e_2-e_3 \mid e_1+e_3\rangle,$$

$$\langle e_1 \mid e_2-e_3 \mid e_1+e_2\rangle, \langle e_1 \mid e_1+e_3 \mid -e_1-e_2\rangle, \langle e_2 \mid e_2-e_3 \mid -e_1-e_2\rangle,$$

$$\langle e_2 \mid e_2-e_3 \mid -e_1-e_3\rangle, \langle e_2 \mid e_1+e_2 \mid -e_1-e_3\rangle, \langle e_3 \mid e_2-e_3 \mid -e_1-e_3\rangle,$$

$$\langle e_3 \mid e_2-e_3 \mid -e_1-e_2\rangle, \langle e_3 \mid e_1+e_2 \mid -e_1-e_3\rangle.$$

Finally, when studying the action of Γ on the well-rounded retract, it is again very useful to think of Γ as acting on the set of decorations. Therefore, $\gamma \in \Gamma$ maps the cube decorated with $\{v_1, v_2, v_3\}$, to the one determined by the set $\{\gamma(v_1), \gamma(v_2), \gamma(v_3)\}$. Note that $\gamma \in GL_3(\mathbb{Z})$ guarantees that this set has $\mathbb{R}$-rank equal to three. This immediately points to the fact that the cube above has a non-trivial stabilizer under the action of Γ. In particular, the cube is stabilized by all monomial elements in Γ. In Figure 6.3, we can see the fundamental domain for this action, triangulated as four neighboring tetrahedra supported on the center of the cube (the reason for this triangulation will be made apparent later).

6.1.4 The well-rounded retract as the Voronoi dual

Before we delve deeper into the properties of the well-rounded retract, it would also be helpful if we made more explicit the relationship between it and the Voronoi complex.

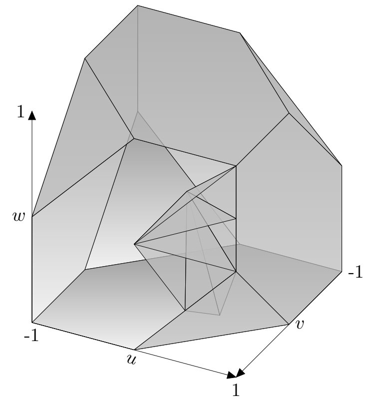


FIGURE 6.3: Fundamental domain for the action of Γ within the Soulé cube.

We say that a quadratic form Q is *perfect* if it can be completely recovered from the information $\{m(Q), M(Q)\}$, or in other words, it is the unique positive definite quadratic form with an arithmetic minimum $m(Q)$, and a set of minimal vectors $M(Q)$. The standard example of a perfect quadratic form on $\mathbb{R}^n$ is Q_n defined by,

$$Q_n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} x_i x_j,$$

with $m(Q_n) = 1$, and $M(Q_n) = \{e_1, \dots, e_n, e_i - e_j \mid 1 \le i < j \le n\}$. It follows from [Vor07] that there are only finitely many classes of perfect forms modulo Γ. One can use information about the perfect forms and their sets of minimal vectors to construct a cellulation of the symmetric space. Namely, each perfect form Q gives rise to a top-dimensional cell described as the convex hull of the quadratic forms $\{v({}^t v) \mid v \in M(Q)\}$, where the hull is taken in the linear space of all quadratic forms, modulo homothety. Smaller dimensional cells in turn are obtained as convex hulls of subsets of the set $\{v({}^t v) \mid v \in M(Q)\}$, as Q ranges over all perfect quadratic forms. We make this explicit in the case of GL_2. Here, there is only one perfect form up to Γ-equivalency, namely, Q_2 with $M(Q_2) = \{e_1, e_1 + e_2, e_2\}$. The 2-cell

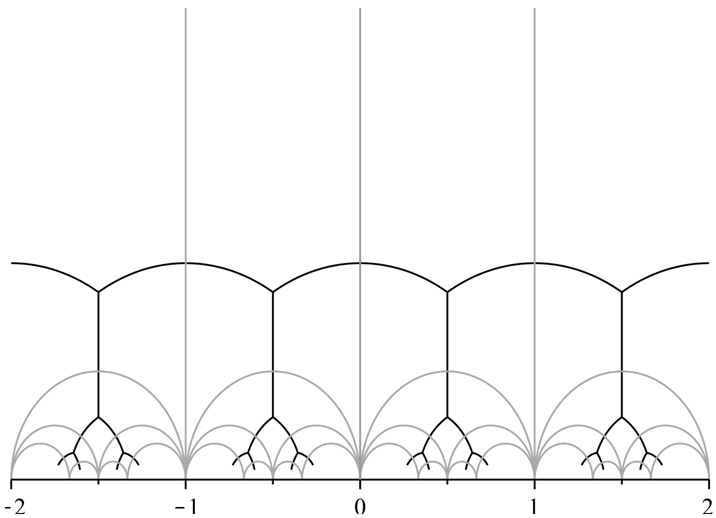


FIGURE 6.4: The well-rounded retract and the Voronoi complex in the upper half-plane.

corresponding to this quadratic form is the convex hull of the quadratic forms

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In $\overline{\mathcal{H}^+}$, we can visualize these points as $0, i\infty, -1$. In Figure 6.4, we have superimposed this cell and the neighboring translates, over the picture of the well-rounded retract in the upper half-plane. From the figure it is obvious that each 0-cell in the well-rounded retract is contained in a unique Voronoi 2-cell. Similarly, each 1-cell in the well-rounded retract intersects transversely a unique 1-cell in the Voronoi complex. This relationship goes beyond a one-to-one correspondence between the relevant cells, and extends to an isomorphism between the Voronoi complex and the dual of the well-rounded retract respecting the boundary operators.

6.1.5 Apartments and the well-rounded retract

In this section we make the connection between cells in the well-rounded retract and spherical apartments in the building associated to the boundary of the Borel-

Serre stratification of D_m. Recall, in the Borel-Serre stratification of D_m, for each $\mathbb{Q}$-rational parabolic subgroup P of G one adjoins a face $e(P)$ to the globally symmetric space.

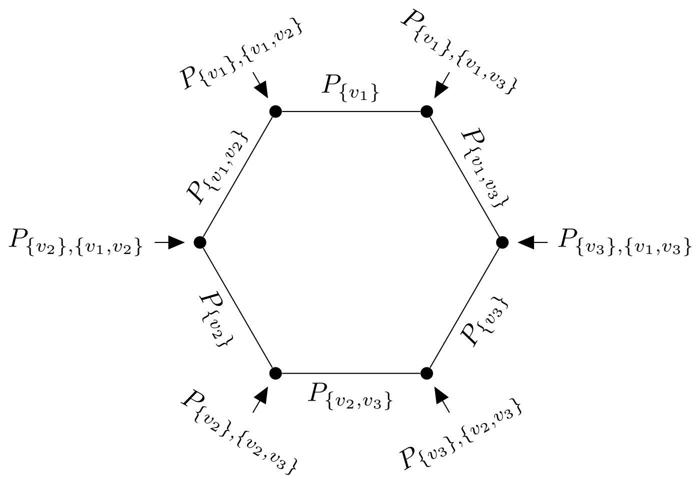


Figure 6.5: (Dual of a) rank two spherical apartment.

We note that in rank one, each nontrivial rational parabolic subgroup $P_{\{v\}}$ is the stabilizer of a flag $\{0 \subset \mathbb{Q}\{v\} \subset \mathbb{Q}^2\}$ and a spherical apartment can be represented by a pair of nodes, each node corresponding to a (maximal) parabolic subgroup. On the other hand, as seen in Figure 6.1, each one cell is decorated by a pair of vectors $\{v_1, v_2\}$. Therefore, to each one-cell in the well-rounded retract we can associate a unique spherical apartment, namely $\{P_{\{v_1\}}, P_{\{v_2\}}\}$. This relationship is more than superficial: one can show that if we were to extend the well-rounded contraction to the Borel-Serre boundary, say via a procedure making use of the tillings introduced in [Sap97] as in [AM97], then points in $r_1\left(e\left(P_{\{v_1\}}\right)\right) \cap r_1\left(e\left(P_{\{v_2\}}\right)\right)$ is precisely the one-cell in W decorated by $\{v_1, v_2\}$.

One can make a similar argument in the rank two case. Here, there are two conjugacy classes of maximal parabolic subgroups, as well as one conjugacy class of minimal parabolics. Recycling the notation from the previous paragraph, we use

P_U to denote the maximal parabolic subgroup stabilizing $U \subset \mathbb{R}^2$, and P_{U_1,U_2} for the minimal parabolic subgroup stabilizing the flag $\{0 \subset U_1 \subset U_2 \subset \mathbb{Q}^2\}$. For the $\mathbb{R}$-linearly independent set $\{v_1, v_2, v_3\}$, we can visualize a spherical apartment as the hexagon in Figure 6.5, with the lines corresponding to maximal, and the points to minimal parabolic subgroups. In fact, the spherical apartment is classically illustrated as the dimensional dual to Figure 6.5, but for the purposes of this exposition we do not make this distinction.

As discussed before, a 3-cell S in the well-rounded retract is decorated by a triple of vectors $\{v_1, v_2, v_3\}$, making up a $\mathbb{Z}$-basis of $\mathbb{Z}^3$. To this cell we associate the spherical apartment in Figure 6.5. As in rank one, we again observe that the intersection of the retracts of the boundary components is the relevant 3-cell. The connection between the spherical building at infinity and the well-rounded retract goes beyond this identification. Namely, consider a point $p \in S$ corresponding to the well-rounded marked lattice f_g. Let $g = p_{ij}k$, with $p_{ij} \in P_{\{v_i,v_j\}}$. Note that $p_{ij}|_{\mathbb{R}\{v_i,v_j\}} \in \mathrm{Aut}(\mathbb{R}\{v_i, v_j\})$. It is not difficult to see that $p_{ij}|_{\mathbb{R}\{v_i,v_j\}}$ gives rise to a well-rounded marked lattice $f_{p_{ij}|_{\mathbb{R}\{v_i,v_h\}}} : \mathbb{Z}\{v_i, v_j\} \to \mathbb{R}\{v_i, v_j\}$, with $M(f_{p_{ij}|_{\mathbb{Q}\{v_i,v_j\}}}) = \{v_i, v_j\} \subset \mathbb{Z}\{v_i, v_j\}$. We will refer to $p_{ij}|_{\mathbb{R}\{v_i,v_j\}}$ as the *projection* of g onto $\mathbb{R}\{v_i, v_j\}$. The projection is unique up to $K \cap P_{\{v_i,v_j\}}$, an ambiguity that has no effect on the equivalence class of the marked lattice $f_{p_{ij}|_{\mathbb{Q}\{v_i,v_j\}}}$ modulo rotations.

The case when $\{v_1, v_2, v_3\} = \{e_1, e_2, e_3\}$ as seen in Figure 6.2 is informative. The associated spherical apartment has six maximal parabolic subgroups, three of which stabilize a two-dimensional subspace: $P_{\{e_1,e_2\}}, P_{\{e_1,e_3\}}$, and $P_{\{e_2,e_3\}}$. For a point p in the cube, we consider the three projections $p_{ij}|_{\mathbb{R}\{e_i,e_j\}}, 1 \leq i < j \leq 3$. A simple calculation shows that all three projections are in the corresponding fundamental arc for the relevant two dimensional subspace. Furthermore, the u coordinate of the point parametrizes the position of the projection to the fundamental arc in the

$\mathbb{R}\{e_2, e_3\}$ subspace. Similarly, the v and w coordinates parametrize the projections to the fundamental arcs in the $\mathbb{R}\{e_1, e_3\}$, and $\mathbb{R}\{e_1, e_2\}$ subspaces respectively.

We can generalize this to a method for associating spherical apartments to top dimensional cells in the well-rounded retract in higher rank as well.

6.2 Contracting W_2

The rank one well-rounded retract is homeomorphic to a trivalent tree, and as such it has a contraction to any node we choose as a root. We aim to make this contraction explicit by using its combinatorial structure. In doing so we hope that we may then generalize the argument to the well-rounded retract in higher rank, where similar such combinatorial structure exists, however the topology is not nearly as nice.

Before we explain the contraction algorithm, we need to prove the following lemma.

Lemma 12. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v_1 \mid v_2) \in SL_2(\mathbb{Z})$ *give rise to a point on the well-rounded retract. If* g *neither stabilizes the fundamental arc in the well-rounded retract in* $\mathcal{H}^+$*, nor maps it to one of its neighboring arcs, then,*

1. *The sets* $\{v_1, v_2, v_1 + v_2\}$*, and* $\{v_1, v_2, v_1 - v_2\}$ *are totally ordered with respect to the Euclidean norm;*

2. $\min\{|v_1 \pm v_2|\} < \max\{|v_1|, |v_2|\}$.

Proof. Observe that if w_1, w_2 are the rows of $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, then $|v_1|, |v_2|, |v_1 \pm v_2|$ equals $|w_2|, |w_1|, |w_1 \pm w_2|$ respectively. Consequently, it suffices to prove the lemma assuming v_1, and v_2 are the rows of g^{-1}. Furthermore, since acting by $k \in SO_2(\mathbb{R})$ on the right rotates the rows, it is sufficient to prove the lemma for the rows of $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = g^{-1}k \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Let us assume, that in contradiction that one (or both) of the condition in the Lemma are false. We claim that this is equivalent to saying that g gives rise to a point inside the region in the upper half-plane shown in Figure 6.6. Consider the marked lattice:

$$f_{g^{-1}} : L_0 \longrightarrow L_0$$
$$v \mapsto g(v)$$

Moding out by rotation and homothety, we can assume $f_{g^{-1}}$ maps e_1 to e_1, and e_2 to a point $x + iy$.

First assume that condition (1) is violated by having $|v_1| = |v_2|$. This equality translates to $x^2 + y^2 = 1$. The only part of the unit circle intersecting the well-rounded retract is the fundamental arc, meaning that $f_{g^{-1}}$ corresponds to a point in the upper half-plane that is simultaneously in the $SL_2(\mathbb{Z})$-orbit of $z = i$, and on the fundamental arc. Clearly this point is $z = i$ itself, and g^{-1} (and therefore g) stabilizes the fundamental arc. Now assume that condition (1) is violated by $|v_1 + v_2|$ or $|v_1 - v_2|$ is equal to $|v_i|, i = 1$ or 2. Since, $\tilde{g} = (v_1 \pm v_2 \mid v_i)$ is again an element $SL_2^{\pm 1}(\mathbb{Z})$ the above argument applies, and as a result we can deduce that $\tilde{g}$ stabilizes the fundamental arc. Since $g = \tilde{g} \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$ we conclude that g maps the fundamental arc to a neighboring arc.

To prove the second assertion, note that the norm inequality $|v_1 \pm v_2| > \max\{|v_1\, |v_2|\}$ translates to the following condition on the basis $\{f_{g^{-1}}(e_1), f_{g^{-1}}(e_2)\}$ of the lattice $f_{g^{-1}}(L_0)$,

$$(x \pm 1)^2 + y^2 > |v_1| = \max\{1, x^2 + y^2\}.$$

An immediate consequence is that $|x| < \frac{1}{2}$. In addition, it follows from $(x \pm 1)^2 + y^2 > 1$ that $x + iy$ is a point above the the circular arcs in Figure 6.6.

Therefore, $f_{g^{-1}}$ corresponds to a point in the region outlined in Figure 6.6. Since the well-rounded retract intersects this region only in the fundamental arc, and taking

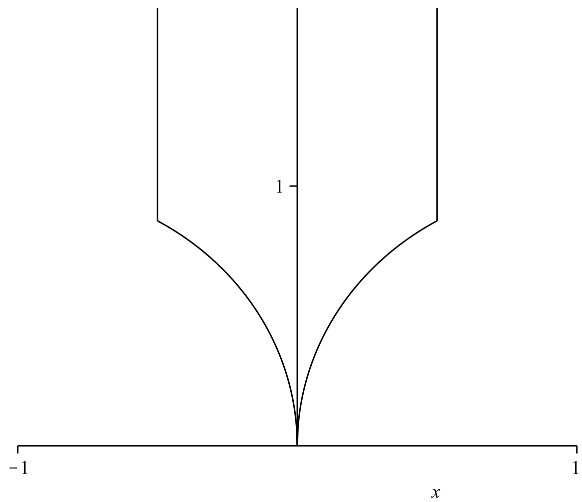


FIGURE 6.6: Region in the upper half-plane

into account that $g^{-1} \in SL_2(\mathbb{Z})$, it follows that g^{-1} stabilizes the point i when acting by fractional linear transformations, and therefore the fundamental arc. This concludes the proof the Lemma.

□

The main idea behind the contraction of the well-rounded retract in rank one is to use the decorations adorning each cell as directionals, tracing a route for each point in the well-rounded-retract.

First we begin with a 1-cell marked by two vectors $v_1, v_2 \in \mathbb{Z}^2$. The boundary of this cell consists of two vertices, which in addition to v_1 and v_2, are decorated with $v_1 + v_2$, and $v_1 - v_2$, respectively. This is easy to see, since each one cell is an $SL_2(\mathbb{Z})$-translate of the fundamental cell which is decorated by $\{e_1, e_2\}$ with one vertex decorated by $\{e_1, e_2, e_1 + e_2\}$ and the other by $\{e_1, e_2, e_1 - e_2\}$. We will need the following geometric lemma,

Lemma 13. *For* $v_1 \neq 0 \neq v_2 \in \mathbb{R}^m$*, if* $|v_1 + v_2| \leq \max\{|v_1|, |v_2|\}$*, then*

$$|v_1 - v_2| > \max\{|v_1|, |v_2|\}.$$

Proof. Without loss of generality, assume $|v_1| = \max\{|v_1|, |v_2|\}$. Then the condition $|v_1 + v_2| \leq \max\{|v_1|, |v_2|\}$ is equivalent to $\frac{(v_1,v_2)}{|v_2|^2} \leq \frac{-1}{2}$. On the other hand, the conclusion $|v_1 - v_2| > \max\{|v_1|, |v_2|\}$ is equivalent to $\frac{(v_1,v_2)}{|v_2|^2} < \frac{1}{2}$. The proof of the Lemma is immediate. □

6.2.1 *Specifying the contraction*

With this information in mind, we are ready to define the method of contraction. We will do so recursively, by first specifying how 1-cells in the well-rounded retract contract. For each vertex in the intersection of multiple 1-cells, we will define a consistent way of choosing a 1-cell that will determine how the vertex contracts.

An arbitrary 1-cell defined by $\{v_1, v_2\}$ can be oriented according to a direction pointing towards the fundamental arc in the well rounded retract. More specifically, we choose the direction to point towards the boundary point decorated by $\{v_1, v_2, \min\{v_1 \pm v_2\}\}$, where $\min\{v_1 \pm v_2\}$ denotes the vector of smaller euclidean norm. Here we have used Lemmas 12 and 13 to deduce that $\min\{v_1 \pm v_2\} < \max\{v_1 \pm v_2\}$, barring the special case of the fundamental arc.

On the other hand, an arbitrary 0-cell as described above, is in the boundary of three 1-cells (apartments). According to the first observation in Lemma 12, provided that the 0-cell is not in the boundary of the fundamental cell, we may order the three 1-cells by the euclidean norm of the vectors that make up their decorating set. Namely, we think of two apartments $\{v_1, v_2\}$, and $\{w_1, w_2\}$ as ordered lexicographically by the norm of the individual vectors. To determine which apartment is smaller, first we compare $\min\{v_1, v_2\}$ with $\min\{w_1, w_2\}$. If these are equal we compare the remaining two vectors. From our prior work, it is clear that the three apartments incident at a point that is not in the boundary of the fundamental arc, make a well ordered set. Therefore, we specify that the vertex should follow the contraction in the direction as specified by the smallest 1-cell (apartment.)

Finally for each point p in the well-rounded retract but outside of the fundamental arc, this recursive algorithm determines a unique path to a vertex in the boundary of the fundamental arc. We complete the path by specifying that points in the fundamental arc contract to the vertex $z_0 = -\frac{1}{2} + i\frac{\sqrt{2}}{2}$. We may parametrize this path in constant speed by,

$$\phi_p : [0,1] \to W, \quad \phi(0) = p, \quad \phi_p(1) = z_0.$$

The contraction of the well-rounded retract in rank one is defined as,

$$\begin{aligned} h : [0,1] \times W &\longrightarrow W \\ (t,p) &\longmapsto \phi_p(t). \end{aligned}$$

6.3 Contracting The Soulé Complex

In this section we generalize the algorithm we employed for the Serre tree to the higher dimensional Soulé complex. As before, our method for contracting the well-rounded retract will hinge on being able to continuously assign paths (trajectories) to the fundamental cube. More specifically, we would like to assign trajectories such that two points that are “close together” in the well-rounded retract, travel along paths that ”remain close together.” Once these are in place, the actual contraction mechanics will be similar to those at the end of the previous section.

Hereafter, S will denote an arbitrary 3-cell decorated by the triplet $\{v_1, v_2, v_3\}$, with $(v_1 \mid v_2 \mid v_3) \in SL_3(\mathbb{Z})$. At times, we will use Figure 6.2 to visualize S. We will abuse notation and refer to the coordinate system as (u, v, w), with the caveat that the coordinates in reality are g translates of the (u, v, w)-system used to describe the *fundamental cube* decorated by $\{e_1, e_2, e_3\}$. Similarly, the face F_{10} as seen in Figure 6.2, when used to depict S, is really the hexagonal face decorated by $g\{e_1, e_2, e_3, e_2 - e_3\} = \{v_1, v_2, v_3, v_2 - v_3\}$.

We begin by introducing a preordering on vectors in $\mathbb{Z}^3$, where the vectors are ordered first by euclidean norm, and then lexicographically by the size of each entry. More specifically,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \prec w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

whenever,

$$|v| < |w|, \text{ or } \{(|v| = |w|) \wedge (\exists\ 1 \leq i \leq 3 \mid |v_i| < |w_i|, |v_j| = |w_j|\ \forall\ 1 \leq j < i)\}.$$

We say that $v \approx w$, whenever $|v_i| = |w_i|, 1 \leq i \leq 3$. Finally, we say that $v \preceq w$ whenever $v \prec w$ or $v \approx w$. This in turn induces an preordering on finite collections of vectors. Namely, consider $\{v_1, \ldots, v_k\} \neq \{w_1, \ldots, w_k\}$. In determining which collection is smaller with respect to the induced ordering $\preceq$, we choose $v_M \in \{v_1, \ldots, v_k\}$ and $w_M \in \{w_1, \ldots, w_k\}$ such that $v_M \preceq v_i, w_M \preceq w_j$, for $1 \leq i, j \leq k$. If $v_M \prec w_M$ we say that $\{v_1, \ldots, v_k\} \prec \{w_1, \ldots, w_k\}$, and similarly, if $w_M \prec v_M$ we say that $\{w_1, \ldots, w_k\} \prec \{v_1, \ldots, v_k\}$. If $v \approx w$, then we proceed by comparing the sets $\{v_1, \ldots, v_k\} \backslash \{v_M\}$ and $\{w_1, \ldots, w_k\} \backslash \{w_M\}$. If this process terminates without being able to conclude which collection is smaller, we say that $\{v_1, \ldots, v_k\} \approx \{w_1, \ldots, w_k\}$.

We noticed in rank one that there is a unique $\mathbb{Z}$-basis $\{v_1, v_2\}$ of $\mathbb{Z}^2$ such that, $|v_1 \pm v_2| > \max\{|v_1|, |v_2|\}$, namely the one associated with the fundamental arc in the upper half-plane. In rank two, we make the following

Definition 5. *For a $\{v_1, v_2\}$ primitive vectors in $\mathbb{Z}^3$, we say that $\{v_1, v_2\}$ forms a* fundamental pair *whenever $v_1 \pm v_2 \succ v_1, v_2$.*

Lemma 14. *If a pair $\{v_1, v_2\}$ of primitive vectors in $\mathbb{Z}^3$ is not fundamental, then*

$$\max\{|v_1 \pm v_2|\} > \max\{|v_1|, |v_2|\} \geq \min\{|v_1 \pm v_2|\}.$$

Proof. This proof is a small adjustment to the one in Lemma 13. ☐

Theorem 4. *Consider a $\mathbb{Z}$-basis of $\mathbb{Z}^3$, $\{v_1, v_2, v_3\}$. If all pairs of vectors $\{v_i, v_j\}$, $1 \leq i \neq j, \leq 3$ are fundamental pairs, then $\{v_1, v_2, v_3\} \subset \{\pm e_1, \pm e_2, \pm e_3\}$.*

Proof. From the hypothesis, it follows that:

$$|v_i \pm v_j| \geq \{|v_i|, |v_j|\}. \tag{6.1}$$

We note that an equivalent system of inequalities is

$$|\mathrm{proj}_{v_1}(v_2)| \leq \frac{1}{2}|v_1|, \quad |\mathrm{proj}_{v_2}(v_1)| \leq \frac{1}{2}|v_2|.$$

Making use of the fact that $|\mathrm{proj}_{v_i}(v_j)| = \frac{|(v_i,v_j)|}{|v_i|}$, we can restate the above conditions in a unified form

$$|(v_1, v_2)| \leq \frac{1}{2}\min\{|v_1|^2, |v_2|^2\}. \tag{6.2}$$

If we focus on the plane spanned by v_1 and v_2, rescale and rotate so that v_1 is the vector $\begin{pmatrix}1\\0\end{pmatrix}$, we have seen in the proof of Lemma 12 that (6.1) implies that v_2 lies in the region of Figure 6.6. We note that, among other things, the angle θ between v_1 and v_2 has to always be between 60 and 120 degrees, and equal to 60 or 120 degrees only when the norm of v_1 is equal to that of v_2.

We will break the proof down in three cases with respect to how the norms of the three vectors relate to each other.

Case I: Let $|v_1| > |v_2|, |v_3|$

Let u_1 be the projection of v_1 onto the plane spanned by v_2 and v_3. Since v_1, v_2, and v_3 form an integral basis it follows that $|u_1|^2 > |v_1|^2 - 1 \geq |v_2|^2, |v_3|^2$. Otherwise, any integral linear combination of $\{v_1, v_2, v_3\}$ not in $\mathrm{Span}\{v_2, v_3\}$ will have norm greater than one, as the component of v_1 perpendicular to the plane spanned by $\{v_2, v_3\}$ will have norm greater than one itself. Note that

$$|(u_1, v_2)| = |(v_1, v_2)| \leq \frac{1}{2}|v_2|^2 < \frac{1}{2}|u_1|^2,$$

and similarly,

$$|(u_1, v_3)| = |(v_1, v_3)| \leq \frac{1}{2}|v_3|^2 < \frac{1}{2}|u_1|^2.$$

In addition, we also have $|(v_2, v_3)| \leq \frac{1}{2}\min\{|v_2|^2, |v_3|^2\}$. Therefore, the pairwise angles between the planar vectors u_1, v_2 and v_3 are all between 60 and 120 degrees. The only way to arrange three vectors in such a planar arrangement is so that each is 120 degrees from the other two. Furthermore, from an earlier observation following equation (6.2), the three vectors must then have the same norm. This however this violates the inequality $|u_1|^2 > |v_1|^2 - 1 \geq |v_2|^2, |v_3|^2$.

Case II: Let $|v_1| = |v_2| = |v_3|$.

Consider $g^{-1} = (v_1|v_2|v_3) \in SL_3(\mathbb{Z})$ and the associated quadratic form

$$q_g = {}^t\left(g^{-1}\right) g^{-1} = \begin{pmatrix} |v_1|^2 & (v_1, v_2) & (v_1, v_3) \\ (v_1, v_2) & |v_2|^2 & (v_2, v_3) \\ (v_1, v_3) & (v_2, v_3) & |v_3|^2 \end{pmatrix}.$$

After re-scaling so that $|v_i| = 2$, and making use of the inequalities $|(v_i, v_j)| \leq \frac{1}{2}|v_i|^2 = \frac{1}{2}|v_j|^2$, we find that,

$$q_g = \begin{pmatrix} 2 & w & v \\ w & 2 & u \\ v & u & 2 \end{pmatrix},$$

where $|w|, |v|, |u| \leq 1$. Note, these coordinates place q_g in the non-truncated cube decorated by $\{e_1, e_2, e_3\}$. We proceed by showing that quadratic forms in a corner of the non-truncated cube are not well-rounded.

In particular, we will show this to be the case for one of the missing corners in the Soulé Cube, namely the one characterized by the inequalities $-1 \leq w < 0, 0 < u, v \leq 1, 2 < u + v - w \leq 3$ (see Figure 6.2). The case when the quadratic form is in one of the three other corners of the non-truncated cube follows in a similar fashion.

For a quadratic form $q = \begin{pmatrix} 2 & w & v \\ w & 2 & u \\ v & u & 2 \end{pmatrix}$ representing a point in this corner we have

$$0 \leq q\left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right] = 2(w - v - u) + 6 < 2. \tag{6.3}$$

On the other hand, for an arbitrary primitive vector in $\mathbb{Z}^3$,

$$\begin{aligned} q\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] &= 2a^2 + 2b^2 + 2c^2 + 2wab + 2vac + 2ubc \\ &= (a-b)^2 + (b+c)^2 + (a+c)^2 + (2w+2)ab + (2v-2)ac + (2u-2)bc. \end{aligned}$$

It is easy to see that

$$\begin{aligned} 2w + 2 = 0 &\implies (a-b)^2 + (2w+2)ab \geq \min\{a^2 + b^2, (a-b)^2\} \\ 2v - 2 = 0 &\implies (a+c)^2 + (2v-2)ac \geq \min\{a^2 + c^2, (a+c)^2\} \\ 2u - 2 = 0 &\implies (b+c)^2 + (2u-2)bc \geq \min\{b^2 + c^2, (b+c)^2\}. \end{aligned}$$

Similarly, provided that the quantities on the left side of the implication signs above are not equal to zero, we note the following:

$$\begin{aligned} \operatorname{sign}((2w+2)ab) = \operatorname{sign}(a) \cdot \operatorname{sign}(b) &\implies (a-b)^2 + (2w+2)ab \geq \min\{a^2 + b^2, (a-b)^2\} \\ \operatorname{sign}((2v-2)ac) = -\operatorname{sign}(a) \cdot \operatorname{sign}(c) &\implies (a+c)^2 + (2v-2)ac \geq \min\{a^2 + c^2, (a+c)^2\} \\ \operatorname{sign}((2u-2)bc) = -\operatorname{sign}(b) \cdot \operatorname{sign}(c) &\implies (b+c)^2 + (2u-2)bc \geq \min\{b^2 + c^2, (b+c)^2\} \end{aligned}$$

As a consequence of the above set of inequalities and (6.3), $q(v) > 2$ whenever v is a primitive vector in $\mathbb{Z}^3$ not equal to $\pm(1, 1, -1)^t, \pm e_1, \pm e_2$, or $\pm e_3$. This concludes the proof that quadratic forms in a missing corner of the non-truncated cube decorated by $\{e_1, e_2, e_3\}$ are not well-rounded. Therefore q_g is in the closure of the truncated cube and consequently $\{v_1, v_2, v_3\} \subset \{\pm e_1, \pm e_2, \pm e_3\}$.

Case III: Let $|v_3| = |v_2| > |v_1|$.

Consider $g^{-1} = (v_1|v_2|v_3) \in SL_3(\mathbb{Z})$ and the associated quadratic form

$$q_g = {}^t(g^{-1})g^{-1} = \begin{pmatrix} |v_1|^2 & (v_1, v_2) & (v_1, v_3) \\ (v_1, v_2) & |v_2|^2 & (v_2, v_3) \\ (v_1, v_3) & (v_2, v_3) & |v_3|^2 \end{pmatrix}.$$

We may rescale this quadratic form using the scalar matrix $\frac{1}{|v_2|^2} I_3 = \frac{1}{|v_3|^2} I_3$, and obtain

$$q_g \sim \begin{pmatrix} \frac{|v_1|^2}{|v_2|^2} & \frac{(v_1,v_2)}{|v_2|^2} & \frac{(v_1,v_3)}{|v_3|^2} \\ \frac{(v_1,v_2)}{|v_2|^2} & 1 & u \\ \frac{(v_1,v_3)}{|v_3|^2} & u & 1 \end{pmatrix},$$

where (6.2) yields $|u| \leq \frac{1}{2}$.

It is apparent that if we multiply g on the left by the element $a = \begin{pmatrix} \frac{|v_1|}{|v_2|} & & \\ & 1 & \\ & & 1 \end{pmatrix}$,

we get an element of GL_3 which in turn yields a quadratic form belonging to the non-truncated cube decorated by $\{e_1, e_2, e_3\}$. Namely, (after re-scaling)

$$q_{ag} \sim \begin{pmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{pmatrix},$$

where $|u| \leq \frac{1}{2}$. Also $|w| = \frac{|(v_1,v_2)|}{|v_1||v_2|} = \frac{|(v_1,v_2)|}{|v_1|^2} \cdot \frac{|v_1|}{|v_2|} < \frac{1}{2}$. Similarly, $|v| < \frac{1}{2}$ as well.

In other words,

$$g = \begin{pmatrix} \frac{|v_2|}{|v_1|} & & \\ & 1 & \\ & & 1 \end{pmatrix} \tilde{g},$$

where $\frac{|v_1|}{|v_2|} < 1$ and $\tilde{g}$ gives rise to a quadratic form $q_{\tilde{g}}$ in the non-truncated cube decorated by $\{e_1, e_2, e_3\}$.

Note however that as shown in Case II, $\tilde{g}$ being in the non-truncated Soulé Cube means that the minimal vectors are either $\pm e_1, \pm e_2, \pm e_3$, ($\tilde{g}$ is well rounded), or there is a single minimal vector of the form $e_1 \pm e_2 \pm e_3$ ($\tilde{g}$ is not well rounded.) I contend that multiplying by the element a^{-1} produces a lattice that is not well rounded in both of these cases. This is apparent in the case that $f_{\tilde{g}}$ is well rounded with minimal vectors $\{e_1, e_2, e_3\}$. In this case, the vectors $\{\tilde{v_1} = f_{\tilde{g}}(e_1), \tilde{v_2} = f_{\tilde{g}}(e_2), \tilde{v_3} = f_{\tilde{g}}(e_3\}$ are of smallest norm in the lattice

$$f_{\tilde{g}}(L_0) = \{c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3 \mid c_1, c_2, c_3 \in \mathbb{Z}\}.$$

In the lattice

$$f_g(L_0) = f_{\tilde{g}} \begin{pmatrix} \frac{|v_1|}{|v_2|} & & \\ & 1 & \\ & & 1 \end{pmatrix} (L_0) = \left\{ c_1 \frac{|v_1|}{|v_2|}\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3 \mid c_1, c_2, c_3 \in \mathbb{Z} \right\},$$

there is a unique minimal vector, namely $\frac{|v_1|}{|v_2|}\tilde{v}_1$ as a corollary to the following

Lemma 15. *For $k < 1$,*

$$|\tilde{v}_1| < |c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3| \implies |k\tilde{v}_1| < |c_1k\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3|.$$

Proof. Let $c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3 = c_1\tilde{v}_1 + \tilde{c}_2 w_2 + \tilde{c}_3 w_3$, where the w_1, and w_2 are two linearly independent unit vectors perpendicular to v_1. Then,

$$|c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3|^2 = c_1^2|\tilde{v}_1|^2 + \tilde{c}_2^2 + \tilde{c}_3^2.$$

Therefore,

$$\begin{aligned} |c_1k\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3|^2 &= k^2c_1^2|\tilde{v}_1|^2 + \tilde{c}_2^2 + \tilde{c}_3^2 \\ &> k^2c_1^2|\tilde{v}_1|^2 + k^2\tilde{c}_2^2 + k^2\tilde{c}_3^2 \\ &= k^2|c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3|^2 \\ &> k^2|v_1|^2 = |kv_1|^2. \end{aligned}$$

□

Therefore, under these assumptions $M(f_g)$ will consist of only one vector, and f_g as such will not be well-rounded providing for a contradiction. In the case that $\tilde{g}$ gives rise to a quadratic form in one of the truncated corners of this cube (and therefore not well-rounded), the computation in Case II demonstrates that up to sign, the the lattice $f_{\tilde{g}}(L_0)$ has one vector in the interior of the sphere of radius 2, three vectors on its boundary (the images of e_1, e_2, and e_3), with all other vectors having norm greater than 2. Again we see that with this type of arrangement, minimizing the image of e_1 while keeping the image of e_2 and e_3 constant, will not produce three vectors in the lattice of same, minimal length. This concludes the argument for Case III, and completes the theorem. □

Keeping with the notation from the rank one case, we call the 3-cell decorated by $\{e_1, e_2, e_3\}$, the fundamental cell.

Lemma 16. *At an arbitrary* 0*-cell, the 6 integral vectors that make up its decoration are well ordered with respect to* $\prec$.

Proof. Since all 0-cells in the well rounded-retract are equivalent modulo $SL_3(\mathbb{Z})$, we may assume that an arbitrary 0-cell is decorated by the following collection of 6 vectors $\Sigma = \{v_1, v_2, v_3, v_1 + v_2.v_1 + v_3, v_1 + v_2 + v_3\}$, where $\{v_1, v_2, v_3\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$, and all 6 vectors are primitive. It is evident that for any pair of these vectors $\{w_1, w_2\} \subset \Sigma$ there exists a third vector w_3 within this sextet such that $w_1 + w_3 = \pm w_2$, or $w_1 - w_3 = \pm w_2$. Now assume $w_1 \approx w_2$, and let w_3 be as above. From the properties of the ordering, it is evident that the coefficients of w_3 are either 0, or twice the corresponding coefficient in w_1 in absolute value. However, this immediately tells us that w_3 is not a primitive integral vector, since 2 divides all of its entries. This is a contradiction and concludes the argument. □

This lemma immediately yields the following,

Corollary 17. *All cubes (apartments) incident at a point, are well ordered with respect to the ordering on collections of vectors induced by* $\prec$ *.*

Proof. A cube incident at a 0-cell decorated by $\Sigma = \{v_1, v_2, v_3, v_1 + v_2.v_1 + v_3, v_1 + v_2 + v_3\}$ is in turn decorated by a triplet which is a subset of Σ. The well ordering Σ tells us that if we compare two distinct triplets from Σ, one will always be smaller than the other. It is an easy exercise to show that the above is true for any point in the well-rounded retract, as the set decorating a p-cell containing an arbitrary point is always a subset of the set decorating a 0-cell in the complex. □

Definition 6. *For an arbitrary point p in the well-rounded retract, the* minimal cube for p *is the smallest* 3*-cell in its star.*

Definition 7. *For a* 3*-cell* S*, the* minimal set for S *denoted* $\Xi(S)$ *is the collection of lower dimensional cells* $C \in \partial(S)$ *such that the minimal cube for points in* C *is different than* S*. Cells in* $\Xi(S)$ *are called* minimal cells for S.

We caution the reader not to confuse $\Xi(S)$ with the decoration for S, $\{v_1, v_2, v_3\}$, which is in fact the set of integral vectors on which quadratic forms in S are minimal.

Armed with these lemmas and definitions, we are ready to qualitatively describe the contraction algorithm. As in rank one, points in the interior of the 3-cell S will contract to a cell in $\Xi(S)$ along trajectories within the 3-cell. Furthermore, we will also specify trajectories within S for points in its boundary for which S is minimal. The union of these trajectories will yield a continuous contraction of the closure $\overline{S}$ of S to $\Xi(S)$, which in turn remains fixed with respect to the contraction of S. This definition is recursive, for cells in $\Xi(S)$ are not minimal when considered as boundary components with respect to their minimal cubes. Consequently points in $\Xi(S)$ follow trajectories outside of S. The next two lemmas helps us visualize the geometry of $\Xi(S)$. This is important, since according to the description above, $\overline{S}$

contracts continuously to this minimal set. In particular we aim to show that the number of minimal faces of S is less than or equal to five, and that $\Xi(S)$ forms a connected set.

Lemma 18. *At most five of its ten 2-dimensional faces are minimal cells for S.*

Proof. In justifying this claim, first let us focus on hexagonal faces. Without loss of generality, consider the face defined by $v_1, v_2, v_3, v_1 + v_2$. The cubes incident to this face other than S all contain $v_1 + v_2$ in their decorating set. However, if for a moment we assume that $\{v_1, v_2\}$ do not form a fundamental pair, then according to Lemma 14, $\max\{|v_1 \pm v_2|\} > \max\{|v_1|, |v_2|\} \geq \min\{|v_1 \pm v_2|\}$. This in turn tells us that at most one of the two hexagonal faces defined by $\{v_1, v_2, v_3, v_1 \pm v_2\}$, is potentially in the minimal set for S, whereas for the other (opposite) hexagonal face, S is the minimal cube in its star. If instead we are in the case where $\{v_1, v_2\}$ form a fundamental pair, then both of the hexagonal faces $\{v_1, v_2, v_3, v_1 \pm v_2\}$ are not in $\Xi(S)$. Therefore, we can pair up the 6 hexagonal faces to determine that at most 3 of them will be in $\Xi(S)$. A similar argument leads to pairing of the four triangular faces. Namely, let the triangular face decorated by $\{v_1, v_2, v_3, v_1 + v_2 + v_3\}$ be in $\Xi(S)$. Therefore, $|v_1 + v_2 + v_3| \leq \max\{|v_1|, |v_2|, |v_3|\}$. Without loss of generality let $|v_1| = \max\{|v_1|, |v_2|, |v_3|\}$. Then, following an argument similar to the one in Lemma 13, it follows immediately that $|v_1 - (v_2 + v_3)| > |v_1|$. Hence, the triangular face decorated by $\{v_1, v_2, v_3, v_1 - v_2 - v_3\}$ is not in $\Xi(S)$. In this fashion one can organize the four triangular faces of S in two disjoint pairs such that if one face is in $\Xi(S)$, the other face it is paired up with is not. □

We also address the question of connectedness of the minimal set for each apartment.

Theorem 5. *The cells in $\Xi(S)$ form a connected set.*

Proof. Since two opposite hexagons can not both be minimal as seen in the proof of Lemma 18, the set of minimal hexagons always form a connected set. Therefore, we only need to concern ourselves with the triangles in any given 3-cell. First we prove the following

Lemma 19. *If* $|v_i + v_j| \geq \max\{|v_i|, |v_j|\}, (v_k, v_i + v_j) \geq 0$ *for some* $1 \leq i, j, k \leq 3$ *distinct, then* $|v_1 + v_2 + v_3| > \max\{|v_1|, |v_2|, |v_3|\}$.

Proof.

$$\begin{aligned}|v_1 + v_2 + v_3|^2 &= (v_1 + v_2 + v_3, v_1 + v_2 + v_3) \\ &= |v_k|^2 + 2(v_k, v_i + v_j) + |v_i + v_j|^2 > |v_1|^2, |v_2|^2, |v_3|^2.\end{aligned}$$

□

Now we pick up the proof of the theorem and assume that the triangular face decorated by $\{v_1, v_2, v_3, v_1 + v_2 + v_3\}$ is minimal in the 3-cell defined by $\{v_1, v_2, v_3\}$, i.e., $v_1 + v_2 + v_3 \prec \max\{v_1, v_2, v_3\}$. To conclude the proof it would suffice to show that one of the three neighboring hexagons, decorated by $\{v_1, v_2, v_3, v_i + v_j\}$ $1 \leq i < j \leq 3$, is also minimal. From the minimality of the triangle it follows that $|v_1 + v_2 + v_3| \leq \max\{|v_1|, |v_2|, |v_3|\}$. By the contrapositive of Lemma 19, it follows that for all distinct triples $1 \leq i, j, k \leq 3$, either $|v_i + v_j| < \max\{|v_i|, |v_j|\}$ or $(v_k, v_i + v_j) < 0$. If for at least one such triple, we have $v_i + v_j \prec \max\{v_i, v_j\}$, then the proof is complete. Therefore we may assume that for all distinct triples $1 \leq i, j, k \leq 3$,

1. $v_i + v_j \succeq v_i, v_j$;

2. $(v_k, v_i + v_j) < 0$.

However note that from (1), it follows that for all pairs of vectors we have $(v_i, v_j) \geq -\frac{1}{2}\min\{|v_i|^2, |v_j|^2\}$. On the other hand, since by (2), $(v_k, v_i + v_j) = (v_k, v_i) + (v_k, v_j) < 0$, it follows that for all pairs of vectors, $(v_i, v_j) < \frac{1}{2}\{|v_i|, |v_j|\}$, and consequently $|v_i - v_j| > \max\{|v_i|, |v_j|\}$. Thus for each pair of vectors, $v_i \pm v_j \succ |v_i|, |v_j|$. By Theorem

4, the only 3-cell for which this is true, is the fundamental cell with decorating set $\{e_1, e_2, e_3\}$, and in this case $\Xi(S) = \emptyset$.

$\square$

Example 1. *Consider the apartment S defined by the columns v_1, v_2, and v_3 of the following element of $SL_3(\mathbb{Z})$,* $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. *Here, clearly the minimal set of hexagonal faces are the three faces each decorated by the union of $\{v_1, v_2, v_3\}$ and one of the set $\{v_1 - v_2, v_1 - v_3, v_2 - v_3\}$. Also minimal are the two triangles decorated by $\{v_1, v_2, v_3, v_1 - v_2 + v_3\}$, and $\{v_1, v_2, v_3, v_1 + v_2 - v_3\}$. These are all of the minimal faces in $\Xi(S)$.*

6.3.1 Trajectories within each cube: Part I

In this section we describe one approach to contracting points within our generic cube S. Since points in $\Xi(S)$ are fixed with respect to the contraction of S itself, in order to preserve continuity our specification needs to be such that points inside S close to $\Xi(S)$ contract to $\Xi(S)$. This procedure described in this section will be a two step procedure, where in the first step, with the help of a function arising as a solution to a system of coupled ODEs, we contract the cube to a subset of $\partial(S)$ containing $\Xi(S)$, and in the second step we finish the contraction to $\Xi(S)$.

Without loss of generality, we will embed a 3-cell decorated by v_1, v_2, and v_3, in standard Euclidean 3-space, with the coordinate axes aligned with the coordinates u, v and w much like in Figure 6.2. We caution the reader that these are not the same coordinates as in Figure 6.2, rather, the translate of this coordinate system by $g = (v_1 \mid v_2 \mid v_3)$. Using an analysis as the one described above, yields a preferred directional flow within each 3-cell which we would like to exploit in our contraction.

More specifically, unless a pair $\{v_i, v_j\}$ is fundamental, it gives rise to a preferred direction in the corresponding coordinate axis. In particular, consider the two opposite hexagons decorated by $\{v_1, v_2, v_3, v_1 \pm v_3\}$. The pair $\{v_i, v_j\}$ not being fundamental is equivalent to the statement that exactly one of these hexagons is in $\Xi(S)$. Let $\bar{v} = \begin{pmatrix} v_{12} \\ v_{13} \\ v_{23} \end{pmatrix}$ with

$$v_{12} = \begin{cases} 1 & F_7 \in \Xi(S) \\ -1 & F_2 \in \Xi(S) \\ 0 & \{v_1, v_2\} \text{ is fundamental} \end{cases}, v_{13} = \begin{cases} 1 & F_4 \in \Xi(S) \\ -1 & F_5 \in \Xi(S) \\ 0 & \{v_1, v_3\} \text{ is fundamental} \end{cases},$$

$$v_{23} = \begin{cases} 1 & F_{10} \in \Xi(S) \\ -1 & F_9 \in \Xi(S) \\ 0 & \{v_2, v_3\} \text{ is fundamental} \end{cases}.$$

A good starting point in organizing the contraction flow within each cube, is the following coupled system of autonomous differential equations:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{pmatrix} v_{23}|v_{13} - v||v_{12} - w| \\ v_{13}|v_{23} - u||v_{12} - w| \\ v_{12}|v_{13} - v||v_{23} - u| \end{pmatrix}.$$

In Figure 6.7 we have computed the trajectories derived from solving the above system numerically, with $v = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}^t$. This is an example where the hexagons in the minimal set are F_5, F_7, and F_9. While these are all of the minimal hexagons, there may be other minimal 2-cells (triangles) in the minimal set for this cube.

Assume $f : I \times S \to S$ is a solution to the above system of differential equations (existence and uniqueness to be shown in following section), and define T_M as the set of triangular faces in $\partial(S)$ having a non-trivial intersection with at least one cell in $\Xi(S)$. Note, this set includes the minimal triangles, as well as triangles that share one or two edge with minimal hexagons. We describe a two step process of contracting S to $\Xi(S)$:

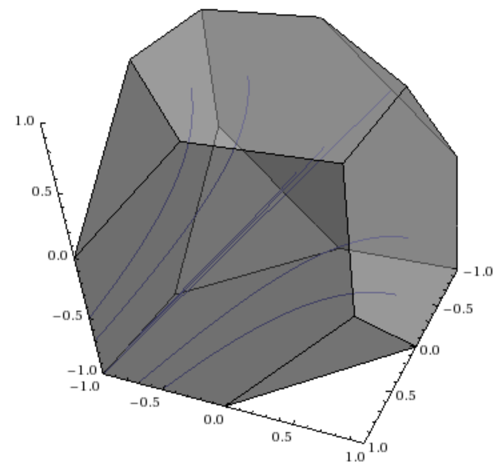


FIGURE 6.7: Trajectories Example

(1) Contract elements of S to $\Xi(S) \cup T_M$ by following trajectories specified by f for points in $S \setminus (\Xi(S) \cup T_M)$, while fixing points in $\Xi(S) \cup T_M$.

(2) Contract non-minimal triangles in T_M to $\Xi(S)$.

To show the contraction described in Step (1) is continuous, it suffices to show that the velocity vector specified by the ODE for points in $\Xi(S) \cup T_M$ points outside of the cube, or at worst, is planar with a cell in $\Xi(S)$. This is immediate for points in $\Xi(S) \setminus T_M$, as these are all points on minimal hexagons. Next we consider a triangle There are three cases to consider:

1. The triangle is incident to one minimal hexagon in the cube;

2. The triangle is incident to two minimal hexagons in the cube;

3. The triangle is incident to three minimal hexagons in the cube;

We will show that points on a triangle as in the first case have velocity vectors that are tangent to the cube, or that point to the outside of the cube. The other two cases follow in a similar fashion.

Therefore, consider the triangle $T : u + v + w = -2$ depicted as F_1 in Figure 6.2, with the directional vector as in the example $v = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$. In this case, the triangle is incident to only one minimal hexagon - $F9$, otherwise decorated by $\{v_1, v_2, v_3, v_2 + v_3\}$. In order for the velocity vector at points on the T (ignoring zero condition) to point outside of the cube, we need to verify that,

$$\frac{du}{dt} + \frac{dv}{dt} + \frac{dw}{dt} < 0.$$

Substituting from the system of equations above, we find that,

$$\begin{aligned} \frac{du}{dt} + \frac{dv}{dt} + \frac{dw}{dt} &= 1 + 2u - u(v + w) - vw = 1 + 4u + u^2 - vw \\ &= (u + 2)^2 - 3 - vw = v^2 + vw + w^2 - 3. \end{aligned}$$

Above, in the second as well as the last equality we used the defining equation for the triangle T. Therefore, proving that $\frac{du}{dt} + \frac{dv}{dt} + \frac{du}{dw} < 0$ amounts to proving that $v^2 + vw + w^2 < 3$. However we see that this is immediate for points on T, with the point $(0, -1, -1)$ being the exception where strictness is violated. Note this is not a problem as the velocity vector in this case is in the same plane as the triangle.

With regards to Step (2) of the contraction, we consider the incidence between a triangle in $T_M \setminus \Xi(S)$ and cells in $\Xi(S)$. First we prove that for such a triangle, at most two of its edges are in $\Xi(S)$. This follow from the following Lemma where we prove that in the case that the hexagonal faces F_2, F_4, and F_9 are minimal, it is also the case that the triangle represented by the face F_1 is also in $\Xi(S)$.

Lemma 20. *If* $|v_i + v_j| \leq \max\{|v_i|, |v_j|\}$ *then* $|v_1 + v_2 + v_3| < \max\{|v_1|, |v_2|, |v_3|\}$.

Proof. Without loss of generality let $|v_3| = \max\{|v_1|, |v_2|, |v_3|\}$. Note that,

$$|v_i + v_3| \leq |v_3| \implies (v_i, v_3) \leq -\frac{1}{2}|v_i|^2, \;\; i = 1, 2.$$

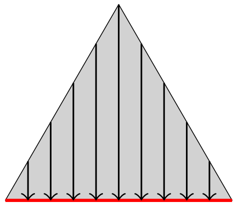


FIGURE 6.8: Single minimal edge.

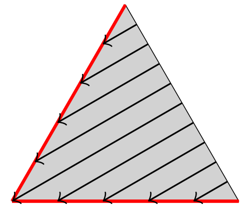


FIGURE 6.9: Two minimal edges.

Therefore,

$$(v_1 + v_2, v_3) \leq -\frac{1}{2}|v_1|^2 - \frac{1}{2}|v_2|^2.$$

Consequently,

$$\begin{aligned} |v_1 + v_2 + v_3|^2 = (v_1 + v_2 + v_3, v_1 + v_2 + v_3) &= |v_3|^2 + 2(v_1 + v_2, v_3) + |v_1 + v_2|^2 \\ &\leq |v_3|^2 - |v_1|^2 - |v_2|^2 + |v_1 + v_2|^2 < |v_3|^2. \end{aligned}$$

This concludes the proof of the Lemma. □

Therefore for a triangle in $T_M \setminus \Xi(S)$, its incidence with $\Xi(S)$ is covered by the following two cases:

- Case I: Triangle has two edges contained in $\Xi(S)$.
- case II: Triangle has one edge contained in $\Xi(S)$.

In Figures 6.8 and 6.9 we illustrate Step (2) of the contraction for each of these cases.

6.3.2 Existence and uniqueness to the system of ODEs

Having defined the system of differential equations governing the contraction within each cube we would like to say a few words regarding the existence and uniqueness of their solution.

Note, that we can unify the system of differential equations as,

$$\frac{dX}{dt} = F(X, t),$$

with X denoting the 3-dimensional "space" variable, and $F : \mathbb{R}^{3+1} \to \mathbb{R}^3$. In fact, we note that as our system of differential equations is autonomous, F is in fact a function of only the space variables, and not time.

Definition 8. *A vector valued function $F(X, t)$ is said to satisfy a Lipschitz condition in X for constant k if,*

$$|F(X_1, t) - F(X_2, t)| \leq k|X_1 - X_2|.$$

In our particular case, we may omit all references to t from the left hand side. Taking a closer look the coordinate description of our vector valued function F, we see that proving that F satisfies the Lipschitz condition reduces to showing that the absolute value function $|\cdot|$ satisfies the Lipschitz condition, which is a tautology. We offer the following well known theorems regarding existence and uniqueness of systems of first order differential equations.

Theorem 6 (Uniqueness; [Hur58], Section 5)**.** *Let $F(X, t)$ be continuous and satisfy a Lipschitz condition in X in some neighborhood of the fixed point (X_0, t_0). Then, there can exists at most one solution of*

$$\frac{dX}{dt} = F(X, t),$$

such that $X(t_0) = X_0$.

Theorem 7 (Existence; [Hur58], Section 5)**.** *Let $F(X, t)$ be continuous and satisfy a Lipschitz condition on X in the $(n + 1)-$dimensional region R,*

$$|X - X_0| \leq b, |t - t_0| \leq a.$$

Let M be the upper bound of F in R. Then there exists a solution $X(t)$ of this differential equation defined over the interval

$$|t - t_0| \leq h = \min(a, \frac{b}{M}).$$

The combination of these two theorems provides for a unique solution of our system of differential equations as long as our starting point X_0 is not on the boundary of the cube. If X_0 is in fact on the boundary, we can not apply these theorems immediately since F is not defined outside of the cube. However, assuming we can extend F to an arbitrary neighborhood outside of the cube in such a way as to maintain the Lipschitz property, we can solve the system for initial values on the boundary as well. We do this extension, by projecting the values of F attained on the boundary of the cube to the exterior, using a linear projection originating at the center of the Soulé cube.

6.3.3 Trajectories within each cube: Part II

There is something unsatisfactory about the contraction of each cube described in §6.3.1 - it is governed by a system of ODEs, granting little control over the 3-cell generated using a filling based on this contraction. However, we can improve this contraction by specifying a more explicit method of contracting each cube to its minimal set. We aim to do this in such a way, so that for $k = 0, 1, 2$, the contraction of a k-cell produces a union of $k+1$-cells, each a translate of one found in the tetrahedra making up the fundamental domain in Figure 6.3; see Theorem 8 later in §6.3.4.

As before, S is a cube in the Soulé complex decorated by $\{v_1, v_2, v_3\}$. We single out the case of the fundamental cube, where a point $p \in S$ is contracted to the center of the cube o along the line segment connecting p and o. In specifying trajectories for points in/on a generic cube S different from the fundamental cube, we will use the triangulation by translates of the tetrahedra seen in Figure 6.3. First we determine where in $\Xi(S)$ the center of the cube o contracts to. There are several cases to consider:

1. If there are three minimal hexagons:

(a) If there is an element $g \in SL_3(\mathbb{Z})$ stabilizing S, and which maps the three minimal hexagons to the ones decorated by $\{v_1, v_2, v_3, v_1+v_2\}, \{v_1, v_2, v_3, v_2+v_3\}, \{v_1, v_2, v_3, v_1+v_3\}$, then contract o linearly to the center of the triangular face decorated by $\{v_1, v_2, v_3, v_1+v_2+v_3\}$, which is minimal by Lemma 20.

(b) If there is an element $g \in SL_3(\mathbb{Z})$ stabilizing S, which maps the three minimal hexagons to the ones decorated by $H_1 = \{v_1, v_2, v_3, v_1+v_2\}, H_2 = \{v_1, v_2, v_3, v_2-v_3\}, H_3 = \{v_1, v_2, v_3, v_1-v_3\}$, then contract o linearly to the vertex of S that is the intersection of H_1, H_2, and H_3.

2. If there are two minimal hexagons, then contract o linearly to the vertex at the intersection of the two minimal hexagons and a triangular face in the cube.

3. If there is a single minimal hexagon, then contract o linearly to the center of the minimal hexagon.

We use $\tilde{o}$ to denote the point in $\Xi(S)$ that is the image of o under this contraction mapping. Next we classify the tetrahedra in S based on their relation to $\Xi(S)$, and $\tilde{o}$.

- Tier I: These are tetrahedra whose exterior face is on a 2-cell not in $\Xi(S)$.
- Tier II: Tetrahedra in S not in tier I or III. These can also be classified as tetrahedra having a 2-dimensional intersection with a cell in $\Xi(S)$, sharing 1-cells with no more than one minimal hexagon, and not intersecting any minimal triangle containing $\tilde{o}$ in its interior.
- Tier III: Tetrahedra in S sharing a 1-cell with two minimal hexagons, or with a minimal hexagon and a (minimal) triangle containing $\tilde{o}$ in its interior.

The reason for this classification has to do with the order of the contraction. In the first stage, we contract tier I tetrahedra to tier II. In the second and most involved stage, we contract tier II tetrahedra to the union of $\Xi(S)$ and tier III tetrahedra, and in the final stage three we contract tier III tetrahedra to $\Xi(S)$, by mapping o to $\tilde{o}$.

- Stage I: Consider a tier I tetrahedron described as a convex hull (in Euclidean space) of the points (t_1, t_2, t_3, t_4). From the description of tier I tetrahedra, we know that at least one of these points is not in $\Xi(S)$. Contract each of these non-minimal points linearly to the center of the cube, and using the convex hull description, extend this linearly to a contraction of all tier I tetrahedra to (the boundaries of) tier II and III tetrahedra.

- Stage II: Begin by contracting tier II tetrahedra having a 2-dimensional intersection with minimal triangles (not containing $\tilde{o}$ in their interior by the definition of tier II tetrahedra).

  - Case I: Triangle is flanked by only one minimal hexagon. We can visualize this case in Figure 6.10 where as an example we have used the case where the hexagon at $v = 1$ is minimal.

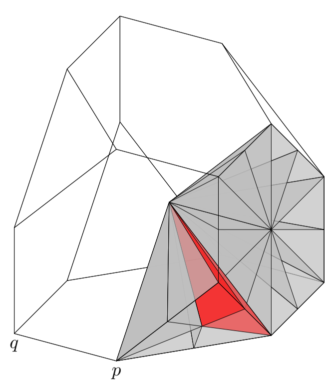


FIGURE 6.10: Minimal triangle flanked by a single minimal hexagon

We pair the gray tetrahedra supported on the minimal triangle into two pairs based on the non-minimal hexagons they share in addition to the minimal triangle. We contract these four gray colored tetrahedra in pairs by projecting linearly from the centers of the two non-minimal hexagonal faces they each share a 1-cell with respectively. The contraction stops upon entering the union of the red tetrahedra, the minimal triangle itself, and the 2-cell described by the points o, the center of the triangle, and the point p at the intersection of the triangle and the two non-minimal hexagons flanking it. Next we contract the red tetrahedra and this 2-cell by linearly projecting from the cube corner q shared by the two non-minimal hexagonal faces flanking the minimal triangle. The contraction stops upon entering the union of the minimal triangle and the tetrahedra having a 2-dimensional intersection with a minimal hexagon.

– Case II: Triangle is flanked by two minimal hexagons. We can visualize this case in Figure 6.11 where the tetrahedra supported on the minimal hexagons are colored light gray. We can again use the (u, v, w)-coordinates

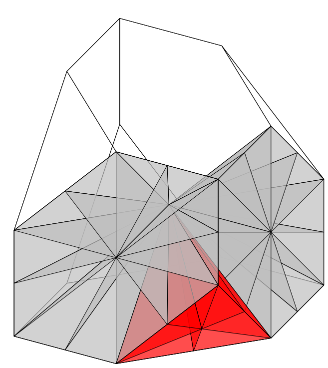


FIGURE 6.11: Minimal triangle flanked by two minimal hexagons

from Figure 6.2, and think of the minimal hexagons as sitting at $v = 1$, and $u = 1$. In this case, we contract the red tetrahedra using a linear projection from the point $(u, v, w) = (-1, -1, -1)$, as usual stopping the contraction when we reach the union of the minimal triangle itself and the tetrahedra having a 2-dimensional intersection with the two minimal hexagons.

We fix a small $\delta > 0$, and continue the Stage II contraction by focusing on the remaining tier II tetrahedra, namely those with a 2-dimensional intersection with a minimal hexagon. These too come in two flavors:

- Case I: Tetrahedron T shares a 2-dimensional face with a tier III tetrahedron: In this case consider C, the center of the 1-cell connecting o to the vertex of T on the minimal hexagon not on the adjacent tier III tetrahedron. Contract the points in this tetrahedron by linearly projecting from $\tilde{C}$ a point on the line segment connecting the center of the minimal hexagon to C, a distance δ outside of T. The projection stops upon entering the union of the minimal hexagon, or the adjacent tier III tetrahedron.
- Case II: Tetrahedron does not share a 2-dimensional face with a tier III tetrahedron. This case is more involved and to gain better understanding we label the vertex set $V_1, \ldots, V_4$, with V_1 being the center of the associated minimal hexagon, and $V_4 = o$. We refer the reader to Figure 6.12. Let C_1 and C_2 be the centers of the segments from V_4 to V_2 and V_3 respectively. As before let $\tilde{C}_1$ and $\tilde{C}_2$ be the points on the lines $\overline{V_1C_1}$ and $\overline{V_1C_2}$ a distance δ "behind" C_1 and C_2 respectively. In this case we contract based on a linear projection from a line segment L just outside of the tetrahedron, connecting $\tilde{C}_1$ and $\tilde{C}_2$. Specifically, for a point M inside the tetrahedron, we express it as a linear combination of the vectors $v_1 = V_1 - V_4, v_2 = V_2 - V_4$, and $v_3 = V_3 - V_4$: $M = \sum \lambda_i v_i$, with $\sum \lambda_i \leq 1$. Set $\tilde{M} = 0v_1 + \lambda_2 v_2 + \lambda_3 v_3$, with $\lambda_2 + \lambda_3 = c < 1$. Note also, that L can be described

as $L = \theta_1\tilde{v}_1 + \theta_2\tilde{v}_2$, where $\tilde{v}_1 = \tilde{C}_1 - V_4, \tilde{v}_2 = \tilde{C}_2 - V_4, \theta_1 + \theta_2 = 1$. We now pick, $p_M \in L$, based on the proportion of λ_2 to λ_3. More specifically, we let $p_M = \frac{\lambda_2}{c}\tilde{v}_1 + \frac{\lambda_3}{c}\tilde{v}_2$. We contract M along the line segment from p_M to M and stop when we reach either the face with vertices $\{V_1, V_2, V_3\}$ (which is in $\Xi(S)$), or the edge $\overline{V_1V_4}$ (which is in a tier III tetrahedron).

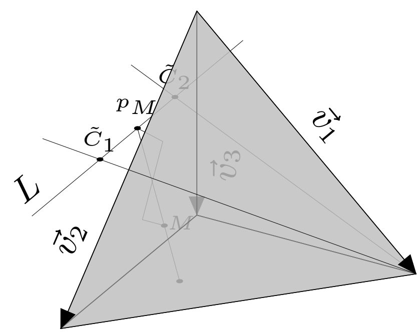


FIGURE 6.12: Contraction of a model tetrahedron.

- Stage III: Contract all tier III tetrahedra to $\Xi(S)$ by mapping o to $\tilde{o}$ linearly and extending linearly to the tetrahedra (viewed as convex hulls of their vertex sets).

This concludes the description of the contraction in each cube.

6.3.4 Specifying the contraction

Now that trajectories are in place within each cube, for $p \in S$ this recursive algorithm determines a unique path to the point o at the center of the fundamental cube. More specifically, making use of the cube contraction in §6.3.3, we can contract p to a point $\tilde{p} \in \Xi(S)$ in a continuous piecewise-linear fashion. In the next stage of the contraction, we make use of the fact $\tilde{p}$ is incident to a minimal cube $\tilde{S} \prec S$. Consequently, $\tilde{p} \notin \Xi(\tilde{S})$, and we may apply the contraction from §6.3.3 again. By repeatedly applying this argument until we reach the center o of the fundamental Soulé cube decorated by $\{e_1, e_2, e_3\}$, we can associate to p a path in W_3. Using a length function on paths, such as the one described in §6.3.5, we may parametrize this path with constant speed as,

$$\phi_p : [0,1] \to W_3, \quad \phi_p(0) = p, \quad \phi_p(1) = o.$$

The contraction of the well rounded retract in rank two is defined as,

$$\begin{aligned} h : [0,1] \times W_3 &\longrightarrow W_3 \\ (t,p) &\longmapsto \phi_p(t). \end{aligned}$$

We note that h is defined independently of any contraction of the ambient five-dimensional globally symmetric space.

Theorem 8. *There exists a piecewise smooth, finite time contraction $h : [0,1] \times W_3 \to W_3$ of the well-rounded retract for GL_3 such that:*

(i) *h is a "local lift" of the lower rank contractions defined on the Serre trees inside the Levi components of a subset of the Bore-Serre faces at infinity;*

(ii) *the 3-cells generated using a filling based on this contraction, are unions of $GL_3(\mathbb{Z})$-translates of the four tetrahedra making up the fundamental domain inside the Soulé cube, as seen in Figure 6.3.*

Proof. The proof of the continuity (and piecewise smoothness) of h is relegated to §6.3.5.

Proving part (i) of the theorem hinges on the relationship between the boundary of the Borel-Serre bordification of D_3 and the well-rounded retract as explained in §6.1.5. In particular consider a top dimensional cell in the well-rounded retract decorated by $\{v_1, v_2, v_3\}$. Consider the two opposing hexagons decorated by $\{v_1, v_2, v_3, v_2 - v_3\}$, and $\{v_1, v_2, v_3, v_2 + v_3\}$ respectively. Under the projection $D_3 \to D_2$ obtained from (2.1) with $P = P_{\{v_2,v_3\}}$, the points in this 3-cell remain well-rounded, and furthermore these two hexagons project to two distinct 0-cells incident to a single 1-cell in this lower-dimensional well-rounded retract. The manner in which we decide which one of these hexagons belongs to the minimal set for this top-dimensional cell reduces to comparing the Euclidean norm of $v_2 \pm v_3$ to that of v_2 and v_3. Comparing to the contraction algorithm for $W_2 \subset D_2$, we see that this assignment to the minimal set is consistent with the manner in which we choose which vertex is minimal for the 1-cell in question, resulting in a preferred direction for the lower-dimensional contraction. The same holds for the two other maximal parabolic subgroups conjugate to $P_{\{v_2,v_3\}}$ in the spherical apartment determined by the set $\{v_1, v_2, v_3\}$.

Finally for part (ii) of the theorem, let $\mathcal{C}$ be the set of 0-, 1-, 2-, and 3-cells in the union of all $SL_3(\mathbb{Z})$-translates of the four fundamental tetrahedra. Tracing through the manner in which we contract points in W_3 we see that for a 0-cell $p \in W_3$, $h\left([0,1],p\right)$ is a union of 1-cells in $\mathcal{C}$. Similarly, if we let $l \subset W_3$ denote a 1-cell in $\mathcal{C}$, $\cup_{p\in l} h\left([0,1],p\right)$ is a union of 1 - and 2-cells in $\mathcal{C}$. Finally for a 2-cell $s \in \mathcal{C}$, $\cup_{p\in s} h\left([0,1],p\right)$ is a union of 1-, 2-, and 3-cells in $\mathcal{C}$. This concludes the proof of part (iii). □

6.3.5 Continuity

In this section we prove the continuity of h described in §6.3.4. We note that an argument similar to the one that follows would prove the continuity of a contraction extrapolated recursively from the solution of the system of differential equations presented in §6.3.1.

The continuity in the time component of $h : [0,1] \times W_3 \to W_3$ follows from the continuity of our trajectories within each 3-cell. The challenging part is to prove overall continuity. We begin by endowing Soulé cube S_0, decorated by the set $\{e_1, e_2, e_3\}$, with a distance function:

$$d_W : S_0 \times S_0 \to \mathbb{R},$$

where $d_W(w_1, w_2)$ is equal to the Euclidean length of the shortest path connecting the points w_1 and w_2 as computed using the coordinates u, v, and w (see Figure 6.2). We note that these paths of shortest distance are invariably straight lines in S_0. Next consider an arbitrary 3-cell $S = \gamma S_0, \gamma \in SL_3(\mathbb{Z})$, and define d_W on points in S via,

$$d_W(\gamma w_1, \gamma w_2) = d_W(w_1, w_2).$$

Similarly, for a piecewise smooth path $c : [0, 1] \to S, c(0) = q_1, c(1) = q_2$, we define its length $|c|$ to be the length of the path $L_{\gamma^{-1}} \circ c$ contained in S_0. Note that there is no ambiguity in the above definitions since the subgroup of $SL_3(\mathbb{Z})$ that stabilizes S_0 is generated by reflections of the cube which act as isometries with respect to the metric associated to the distance function d_W. Finally, extend d_W to all of W_3, by defining $d_W(w_1, w_2)$ to mean the length of a shortest distance piecewise smooth path $l : [0, 1] \to W_3$ such that $l(0) = w_1, l(1) = w_2$. To calculate the length of a path l contained in the well-rounded retract we partition the unit interval $0 = t_0 < t_1 \leq \cdots \leq t_n < 1$ in such a way that

$$l(t) = l_k(t), \quad t \in [t_k, t_{k+1}], k = 0, \ldots, n, \tag{6.4}$$

where $l_k : [0, 1] \to S_k$, and $S_k \neq S_{k+1}$. In this setting, we define

$$|l| = \sum_{k=0}^{n} |l_k|.$$

We note that if we re-parametrize l to have constant speed, the partition decomposing l into a paths of disjoint 3-cells is given as $\left\{ \frac{\sum_{k=0}^{i} |l_k|}{|l|} \right\}_{i=0}^{n}$. Finally, for an interval $[t_0, t_1] \subset I$, by $|l([t_0, t_1])|$ we mean the length of the path l when restricted to $[t_0, t_1]$. We note that a path of shortest distance between two points in the well-rounded retract is invariably a union of straight line segments.

Consider the map,

$$\varphi : W_3 \to C\left([0,1], W_3\right)$$
$$w \mapsto h(\cdot, w).$$

It follows from [Bou98, Chapter X, §3.4], that the map h is continuous if and only if φ is continuous when $C\left([0,1], W_3\right)$ is endowed with the compact-open topology. We state without proof that the L^∞ distance function on $C\left([0,1], W_3\right)$ defined as:

$$d_{L^\infty}(f_1, f_2) = \sup_{t\in[0,1]} \{d_W(f_1(t), f_2(t))\},$$

gives rise to a topology that is finer than the compact-open topology. Therefore, to show that h is continuous, it suffices to show that φ is continuous when we consider $C\left([0,1], W_3\right)$ together with the L^∞ distance function.

Let $w_1 \in W_3$, and choose $\epsilon > 0$. We would like to show that we can find a $\delta > 0$ such that for $d_W(w_1, w_2) < \delta$, $d_{L^\infty}(\varphi(w_1), \varphi(w_2)) < \epsilon$.

Observation 1. *Before we proceed in full with the proof we focus on a particular cube S and consider the line segments obtained using the contraction method outlined in* §6.3.3. *In particular, we consider the map*

$$\varphi_S : S \to C\left([0,1], S\right),$$

where $\varphi_S(w)$ is defined to be the constant speed parametrization of the trajectory connecting the point w with a point in $\Xi(S)$, as explained in §6.3.3. *In this sense, the notation is conveying that φ_S is somehow both a restriction and a projection to the top dimensional cell S. Defined in this fashion, it follows from the definition of the trajectories within each cube as well as the compactness of the domain for φ_S that this map is continuous in the sense discussed above.*

More specifically, let T_S^i, T_S^{ii}, and T_S^{iii} each denote the unions of tier I, II, and III tetrahedra within S, respectively. We then have $S = T_S^i \cup T_S^{ii} \cup T_S^{iii}$, with the *pairwise intersections containing at most 2-dimensional cells. From the manner in which we assign trajectories within each cube as described in* §6.3.3 *it is clear that in fact the trajectories taking S to $\Xi(S)$ can be decomposed as $T_S^i \to T_S^{ii} \to T_S^{iii} \to \Xi(S)$. Therefore, consider the maps*

$$\varphi_S^k : T_S^k \to C\left([0,1], T_S^k\right), \quad k = i, ii, iii,$$

where we have parametrized the path within each tier to have constant speed.

Lemma 21. *The maps $\varphi_S^k, k = i, ii, iii$ are continuous.*

Proof. The restrictions of these maps to individual tetrahedra are certainly continuous. This follows from the fact that tetrahedra are contracted either via projections, or using a convex hull description coupled with a specified contraction for one or more of the 0-cells in each hull. We leave it to the reader to verify that these contraction methods give rise to continuous maps.

Since the description of the trajectories agrees on intersections of tetrahedra within each tier, we conclude that the maps φ_S^k are indeed continuous. □

Corollary 1. *Let $w_1 \in S$. For all $\epsilon > 0$, we can find a $\delta > 0$ such that $d_W(w, w_1) < \delta \implies ||\varphi_S(w)| - |\varphi_S(w_1)|| < \epsilon$.*

Proof. Without loss of generality, let $w_1 \in T_S^i, w_{1,ii} = \varphi_S^i(w_1)(1), w_{1,iii} = \varphi_S^{ii}(w_{1,ii})(1)$. This statement in the corollary then follows from Lemma 21 together with the equality

$$|\varphi_S(w_1)| = |\varphi_S^i(w_1)| + |\varphi_S^{ii}(w_{1,ii})| + |\varphi_S^{iii}(w_{1,iii})|.$$

□

Finally we offer a proof that φ_S is itself continuous. In particular, without loss of generality, let $w_1 \in T_S^i \subset S$, and let $\epsilon > 0$. From Lemma 21 and Corollary 1 it *follows that we can find* $\delta_\epsilon > 0$ *such that for* $d_W(w, w_1) < \delta_\epsilon$ *we have:*

$$\sup_{t\in[0,1]} \{d_W(\varphi^i_S(w_1)(t), \varphi^i_S(w)(t))\} < \frac{\epsilon}{2},$$

$$\sup_{t\in[0,1]} \{d_W(\varphi^k_S(w_{1,k})(t), \varphi^k_S(w_k)(t))\} < \frac{\epsilon}{2}, \ i < k \leq iii;$$

$$||\varphi^i_S(w_1)| - |\varphi^i_S(w)|| < \frac{\epsilon}{3},$$

$$||\varphi^{ii}_S(w_{1,ii})| - |\varphi^{ii}_S(w_{ii})|| < \frac{\epsilon}{3},$$

$$||\varphi^{iii}_S(w_{1,iii})| - |\varphi^{iii}_S(w_{iii})|| < \frac{\epsilon}{3},$$

$$||\varphi_S(w_1)| - |\varphi_S(w)|| < \epsilon.$$

Let $t_0 \in [0, 1]$*, and* $\varphi_S(w_1)(t_0) \in T^k_S$*, where* $k = i, ii,$ *or* iii*. Hereafter we assume that* $\varphi_S(w_1)(t_0)$ *is in the interior of* T^k_S*, and we leave it to the reader to verify the special case when* $\varphi_S(w_1)(t_0)$ *is in a 2-dimensional intersection with* $T^j_S, j \neq k$*. Since the path* $\varphi_S(w_1)$ *is parametrized in constant speed,* $|\varphi_S(w_1)([0, t_0])| = t_0 \cdot |\varphi_S(w_1)|$*. Similarly, for* w *as above,*

$$t_0 \cdot (|\varphi_S(w_1)| - \epsilon) < |\varphi_S(w)([0, t_0])| < t_0 \cdot (|\varphi_S(w_1)| + \epsilon).$$

For the choice of δ *as above, if the portion of* $\varphi_S(w_1)([0, t_0])$ *inside* T^k_S *has length* $L^k_{w_1}$*, then the portion of* $\varphi_S(w)([0, t_0])$ *inside* T^k_S *has length* $L^k_{w_1} - \epsilon < L^k_w < L^k_{w_1} + \epsilon$*. Therefore,*

$$\varphi^k_S(w_{1,k})\left(\frac{L^k_{w_1}}{|\varphi^k_S(w_{1,k})|}\right) = \varphi_S(w_1)(t_0),$$

$$\varphi^k_S(w_k)\left(\frac{L^k_w}{|\varphi^k_S(w_k)|}\right) = \varphi_S(w)(t_0).$$

Let $\tilde{t}^1 = \frac{L^k_{w_1}}{|\varphi^k_S(w_{1,k})|}, \tilde{t} = \frac{L^k_w}{|\varphi^k_S(w_k)|}$. *One can show that the quantity* $|\tilde{t}^1 - \tilde{t}|$ *is bounded from above by* $2\epsilon M$, *where* M *is* $\frac{1}{|\varphi^k_S(w_{1,k})|}$. *Hence,*

$$\begin{aligned} d_W(\varphi_S(w_1)(t_0), \varphi_S(w)(t_0)) &= d_W\left(\varphi^k_S(w_{1,k})\left(\tilde{t}^1\right), \varphi^k_S(w_k)(\tilde{t})\right) \\ &\leq d_W\left(\varphi^k_S(w_{1,k})\left(\tilde{t}\right), \varphi^k_S(w_k)(\tilde{t})\right) \\ &+ d_W\left(\varphi^k_S(w_{1,k})\left(\tilde{t}^1\right), \varphi^k_S(w_{1,k})(\tilde{t})\right). \qquad (6.5) \end{aligned}$$

In the last step, we use the uniform continuity of $\varphi^k_S(w_{1,k})$ *to find a value of* $\tilde{\delta} > 0$ *such that* $|t_1 - t_2| < \tilde{\delta} \implies d_W(\varphi^k_S(w_{1,k})(t_1), \varphi^k_S(w_{1,k})(t_2)) < \frac{\epsilon}{2}$. *To complete the proof, let* $\delta = \min\{\tilde{\delta}, \delta_{\frac{\delta_2}{2M}}\}$. *It follows from the considerations above that for* $t_0 \in [0, 1]$,

$$d_W(w, w_1) < \delta \implies d_W(\varphi_S(w_1)(t_0), \varphi_S(w)(t_0)) < \epsilon.$$

This completes the proof that φ_S *is continuous.*

We now turn our attention to the complete map φ. We notice that the argument needed to prove continuity of φ is in many ways similar to the one needed to prove the continuity for φ_S. In particular, both maps give rise to paths from the unit interval that are re-parametrizations of concatenated trajectories inside smaller spaces, φ_S, and φ^k_S respectively. In both cases, we know the (restricted) maps are continuous (Observation 1 and Lemma 21 respectively). Let $\varphi(w_1) = \coprod_{k=0}^n l^1_k$ as in (6.4). Furthermore if a path component l_k is contained in more than one 3-cell, we chose to reference the minimal such cell. We enumerate the target 3-cells referenced in this decomposition as $S^1_1, \ldots S^1_n$. We consider a point w_2 *close* to w_1, and let $\varphi(w_2) = \coprod_{k=0}^m l^2_k$ with target 3-cells $S^2_1, \ldots S^2_m$. If the two sequences of top dimensional cells match, then it follows from Observation 1 that for w_2 sufficiently close to w_1, the L^∞ distance between $\varphi(w_1)$, and $\varphi(w_2)$ will indeed be less than ϵ. For the general case we prove a Lemma that should be viewed as an analog to Corollary 1.

Lemma 22. *Fix $w_1 \in W, \epsilon > 0$. There exists $\delta > 0$ such that for $d_W(w_1, w) < \delta$ we have,*

$$||\phi(w_1)| - |\phi(w)|| < \epsilon.$$

Proof. We begin the proof with the following observation describing some of the finer points of the recursive algorithm that assigns trajectories to points in the well-rounded retract.

Observation 2. *Let us now focus on the first instance where the two lists $\{S_j^1\}$ and $\{S_j^2\}$ differ, i.e., i, such that $S_k^1 = S_k^2, k < i$, $S_i^1 \neq S_i^2$. This occurs when w_1 and w_2 contract to different boundary components within $S_{i-1}^2 = S_{i-1}^1$. Let $\tilde{w}_1$ and $\tilde{w}_2$ denote the "last" points in l_{i-1}^1 and l_{i-1}^2, respectively.*

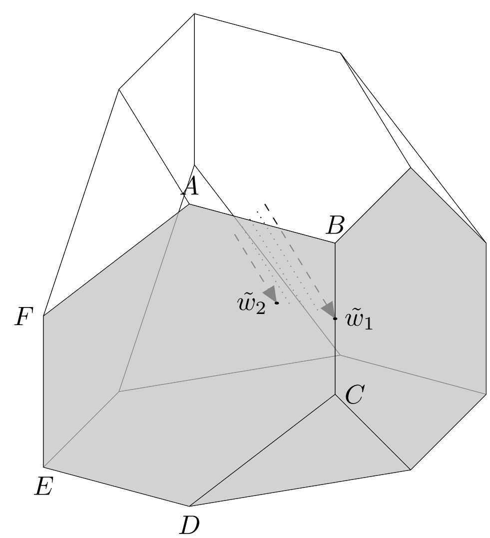


FIGURE 6.13: Illustration of $S_{i-1}^1 = S_{i-1}^2$. The dashed lines represent l_{i-1}^1 and l_{i-1}^2.

In Figure 6.13 we have illustrated an example of how this situation might occur. The hexagon $F_{10} = ABCDEF$ is in $\Xi(S_{i-1}^1) = \Xi(S_{i-1}^2)$. We note that the set of cubes incident to the edge $F_{10,5} = BC$ strictly dominates the set of cubes incident to F_{10}, and it is therefore possible for S_i^1, the minimal cube in the star of $\tilde{w}_1$, to be *smaller than S_i^2, the minimal cube in the star of $\tilde{w}_2$. At worst, S_i^1 and S_i^2 are two cubes incident in a 0-, 1-, or a 2-cell (in the configuration shown in Figure 6.13, they would be incident in the ednge $F_{10,5}$, or possibly a 2-dimensional face containing $F_{10,5}$). As such these are ordered, and without loss of generality let $S_i^1 \prec S_i^2$. In this case $S_i^1 \cap S_i^2 \subset \Xi(S_i^2)$, and consequently points in S_i^2 that are sufficiently close to $S_i^1 \cap S_i^2$, will contract to a cell in the closure of S_i^2 whose boundary contains $S_i^1 \cap S_i^2$, or to $S_i^1 \cap S_i^2$ itself. One such point is $\tilde{w}_2 \in S_i^2$, and as a consequence of Observation 1, by controlling the distance between w_1 and w_2, we can control the distance between $\tilde{w}_1$ and $\tilde{w}_2$. In Figure 6.13 this is portrayed with the help of the dotted lines that help us visualize how moving w_2 closer to w_1 will also enable us to move $\tilde{w}_2$ closer to $\tilde{w}_1$. Therefore, we can ensure that the length of the path l_i^2 from $\tilde{w}_2$ to the face in $\Xi(S_i^2)$ containing $S_i^1 \cap S_i^2$ is arbitrarily small. If $S_{i+1}^2 \neq S_i^1$ we repeat the argument until we arrive at the first instance when $S_{i+q}^2 = S_i^1$. We know from Table 6.1 that $1 \leq q < 16$, as this is the number of cubes incident at a vertex, and as such it is the largest number of cubes having a non-trivial intersection. Furthermore, controlling the distance $\delta > d_W(w_2, w_1)$, the above arguments demonstrate that we can make $\max\{|l_{i+r}^2| \mid 1 \leq r < q\}$ arbitrarily small.*

A repeated application of the above argument completes the proof of the Lemma.

□

With this result in place, as explained previously when we noted the similarity between the manner in which φ_S and φ are constructed, one can follow an arugment like the one in Observation 1 to show that φ is continuous. We leave the details to the reader.

6.4 Two-variable modular symbols

We fix a base point $z_0 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \in \mathcal{H}^+$, and let $z_1 = T(z_0) = S(z_0) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Furthermore, we let $h : I \times W_2 \to W_2$ be the contraction of the well-rounded retract presented in Chapter 5, and σ a filling based on this contraction. To each $f \in S_{n+2}(SL_2(\mathbb{Z}))$ we associate the Eilenberg-MacLane group co-cycle Φ_{ω_f} defined as,

$$\Phi_{\omega_f}(\gamma) = \int_{\sigma(\gamma)} \omega_f.$$

We make the following observations, consistent with our prior exposition:

- For $\gamma \in SL_2(\mathbb{Z})$, $\sigma(\gamma)$ is a union of $SL_2(\mathbb{Z})$ translates of the fundamental arc in the upper half-plane.
- For $\gamma \in SL_2(\mathbb{Z})$, the value $\Phi_{\omega_f}(\gamma)$ is completely determined by the integral of ω_f on the fundamental arc, otherwise written as $\Phi_{\omega_f}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \Phi_{\omega_f}(S)$.

The relationship between Φ_{ω_f} and ϕ_{ω_f} can be abstracted from Dupont's results where he shows that co-cycles differing by a choice of a contraction are cohomologous. For the sake of completeness we show this in more detail.

We note that Φ_{ω_f} is cuspidal as $\Phi_{\omega_f}(T) = \Phi_{\omega_f}(S) = \left(\int_{z_0}^{i\infty} - \int_{z_1}^{i\infty}\right)\omega_f = (\mathrm{Id}_2 - \rho_2^n(T)) \int_{z_0}^{i\infty} \omega_f$. Consider the co-cycle ν_{ω_f} cohomologous to Φ_{ω_f} and vanishing at T:

$$\nu_{\omega_f} = \Phi_{\omega_f} - h_{\omega_f},$$

where $h_{\omega_f}(\gamma) = (\mathrm{Id}_2 - \rho_2^n(\gamma)) \int_{z_0}^{i\infty} \omega_f$. Below we compute the value of ν_{ω_f} at S:

$$\nu_{\omega_f}(S) = \Phi_{\omega_f}(S) - (\mathrm{Id}_2 - \rho_2^n(S)) \int_{z_0}^{i\infty} \omega_f = \left(\int_{z_0}^{z_1} - \int_{z_0}^{i\infty} + \int_{z_1}^{0}\right)\omega_f$$

$$= \left(\int_{z_0}^{0} - \int_{z_0}^{i\infty}\right)\omega_f = -\int_{0}^{i\infty} \omega_f = -\phi_{\omega_f}(S).$$

Considerations in Section 5.2 yield that ν_{ω_f} is cohomologous to ϕ_{ω_f}, which in turn immediately tells us that ϕ_{ω_f} is cohomologous to Φ_{ω_f}.

However stopping here would leave us with a somewhat incomplete picture of these new, revised 2-variable period polynomials. In particular, we note that Eichler's and Shimura's approach to defining the 2-variable period polynomial does not allow for associating polynomials to non-cuspidal differential forms (Eisenstein differential forms). This was remedied recently, primarily in the works of Zagier, who defines a notion of an extended period polynomial obtained by integrating the cuspidal part of the Eisenstein forms over the same non-compact cells in the upper half-plane. We immediately note an advantage of the method outlined in this section, namely, as all cells we integrate over are compact, there is no inherent obstacle to incorporating non-cuspidal differential forms in our analysis.

To make use of this property, we would like to extend Eichler's and Shimura's original analysis, and find an equivalence between Eilenberg-MacLane group cohomology and a space of polynomials that goes beyond classifying only cuspidal cohomology.

We define the element of order 6, $\check{U} = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, and recall that earlier we defined another element of order 6, $U = TS$. We begin by proving the following

Lemma 23. *A co-cycle $f : SL_2(\mathbb{Z}) \to \mathrm{Sym}^n(V_2)$ is cohomologous to $\tilde{f}$ where $\tilde{f}(\check{U}) = 0$.*

Proof. It suffices to show that there exists a $v \in \mathrm{Sym}^n(V_2)$ such that $(\rho_2^n(\mathrm{Id}_2) - \rho_2^n(\check{U}))v = f(\check{U})$, or equivalently, it suffices to show that $f(\check{U}) \in \mathrm{Im}(\rho_2^n(\mathrm{Id}_2) - \rho_2^n(\check{U}))$. On the other hand, repeated applications of the closed condition yield:

$$0 = f(-\mathrm{Id}_2) = f(\check{U}^3) = \rho_2^n(\check{U})f(\check{\mathrm{U}}^2) + f(\check{\mathrm{U}}) = \rho_2^n(\check{U})(\rho_2^n(\check{U})f(\check{\mathrm{U}}) + f(\check{\mathrm{U}})) + f(\check{\mathrm{U}})$$
$$= (1 + \rho_2^n(\check{U}) + \rho_2^n(\check{U}^2))f(\check{\mathrm{U}}).$$

Therefore we know $f(\check{U}) \in \mathrm{Ker}\{1 + \rho_2^n(\check{U}) + \rho_2^n(\check{U}^2)\}$. Consequently we are left to show that $\mathrm{Ker}\{\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^n(\check{U}) + \rho_2^n(\check{U}^2)\} \subseteq \mathrm{Im}(\rho_2^n(\mathrm{Id}_2) - \rho_2^n(\check{U}))$. In fact, it is true that these two spaces are equal. To show this, first we note that $\rho_2^{\mathrm{n}}(\check{U})$ satisfies the polynomial

$$m_{\check{U}}(x) = (x-1)(x^2+x+1) = (x-1)(x-z_0)(x-z_0^2),$$

where z_0 is as before. Therefore, $\rho_2^{\mathrm{n}}(\check{U})$ is diagonizable and after picking an appropriate basis we have,

$$\rho_2^{\mathrm{n}}(\check{U}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 0 & \dots & z_0 & 0 & 0 & 0 & \dots & 0 \\ & & & & \ddots & & & & \\ 0 & \dots & & & & z_0^2 & 0 & \dots & 0 \\ 0 & \dots & & & & & \ddots & & \end{pmatrix}.$$

Therefore,

$$\begin{aligned} &\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^n(\check{U}) + \rho_2^n(\check{U}^2) = \\ &= \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 0 & \dots & 1+z_0+z_0^2 & 0 & 0 & 0 & \dots & 0 \\ & & & & \ddots & & & & \\ 0 & \dots & & & & 1+z_0^2+z_0 & 0 & \dots & 0 \\ 0 & \dots & & & & & \ddots & & \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ & & & & \ddots & & & & \\ 0 & \dots & & & & 0 & 0 & \dots & 0 \\ 0 & \dots & & & & & \ddots & & \end{pmatrix} \end{aligned}$$

At the same time,

$$\rho_2^n(\mathrm{Id}_2) - \rho_2^n(\check{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 0 & \dots & 1-z_0 & 0 & 0 & 0 & \dots & 0 \\ & & & & \ddots & & & & \\ 0 & \dots & & & & 1-z_0^2 & 0 & \dots & 0 \\ 0 & \dots & & & & & \ddots & & \end{pmatrix}$$

These above expressions immediately yield the equality:

$$\mathrm{Ker}\{\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^n(\check{U}) + \rho_2^n(\check{U}^2)\} = \mathrm{Im}(\rho_2^n(\mathrm{Id}_2) - \rho_2^n(\check{U})),$$

which in turn proves the lemma. □

Finally, a co-cycle for $SL_2(\mathbb{Z})$ is zero on both the identity and $-\mathrm{Id}_2$. On the other hand $PSL_2(\mathbb{Z})$ is finitely presented as

$$\{S, \check{U} \mid S^2 = \check{U}^3 = 1\}.$$

Therefore, in order to define f, a closed co-cycle for $SL_2(\mathbb{Z})$ it suffices to define values $f(\check{U})$ and $f(S)$ such that

$$0 = f(\check{U}^3) = (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^{\mathrm{n}}(\check{U}) + \rho_2^{\mathrm{n}}(\check{U}^2))f(\check{U}), \tag{6.6}$$

and

$$0 = f(S^2) = (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^{\mathrm{n}}(\check{U}))f(S), \tag{6.7}$$

where the equlities follow from the group co-cycle condition. Notice, $f(\check{U}) = 0$ automatically satisfies 6.6. Therefore to define a co-cycle for $SL_2(\mathbb{Z})$ it is sufficient to define $f(S) = p(x,y) \in \mathrm{Sym}^n(V_2)$, where $(\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^{\mathrm{n}}(S))p(x,y) = 0$. The sum of these considerations yield,

$$H^1(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2)) \cong \frac{\{p(x,y) \in \mathrm{Sym}^n(V_2) \mid (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) + \rho_2^{\mathrm{n}}(S))p(x,y) = 0\}}{\left\{ (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) - \rho_2^{\mathrm{n}}(S))g(x,y) \;\middle|\; \begin{array}{c} g(x,y) \in \mathrm{Sym}^n(V_2) \\ (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) - \rho_2^{\mathrm{n}}(\check{U}))g(x,y) = 0 \end{array} \right\}}$$

$$[f] \mapsto [\tilde{f}(S)], \tag{6.8}$$

where $\tilde{f}$ is cohomologous to f, and $\tilde{f}(\check{U}) = 0$.

We quickly examine what restrictions the cuspidality condition places on $\tilde{f}(S)$. In particular $\tilde{f}(S) = \tilde{f}(T) = (\rho_2^{\mathrm{n}}(\mathrm{Id}_2) - \rho_2^{\mathrm{n}}(T))v$, for some $v \in \mathrm{Sym}^n(V_2)$, if and only if $\tilde{f}(S)$ is lacking a $\mathbf{y}^n$ term. Since $\rho_2^{\mathrm{n}}(S)\tilde{f}(S) = -\tilde{f}(S)$, we see that this means that $\tilde{f}(S)$ has no $\mathbf{x}^n - \mathbf{y}^n$ term. We also note that the co-boundary values appearing in the quotient on the right hand side on 6.8 have no $\mathbf{x}^n - \mathbf{y}^n$ term. In particular, if we let $g(x,y) = \sum_i a_i x^i y^{n-i}$, we note that,

$$(\rho_2^{\mathrm{n}}(\mathrm{Id}_2) - \rho_2^{\mathrm{n}}(\check{U}))g(x,y) = 0 \iff g(x,y) = g(y, y-x) \implies a_n = a_0.$$

Taking into account that $\rho_2^{\mathrm{n}}(S)g(x,y) = g(y,-x)$, we conclude that $(\rho_2^{\mathrm{n}}(\mathrm{Id}_2) - \rho_2^{\mathrm{n}}(S))g(x,y)$ has no $\mathbf{x}^n - \mathbf{y}^n$ term.

Therefore if $\mathcal{C}_n$ is the image of $H^1_{\mathrm{cusp}}(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2))$ under the isomophism 6.8, we discover that

$$H^1(SL_2(\mathbb{Z}), \mathrm{Sym}^n(V_2)) \cong \langle \mathbf{x}^n - \mathbf{y}^n \rangle \oplus \mathcal{C}_n. \tag{6.9}$$

Finally, we note that with this definition of a modular symbol, the fact that all values of Φ_{ω_f} can be recovered from $\Phi_{\omega_f}(S)$ comes not as a result of cuspidality, or as an application of Stokes' Theorem to a non-compact region. In fact, algebraically, we can see that this follows as a consequence of the fact that Φ_{ω_f} is a representative of $[\phi_f]$ that vanishes on the stabilizer of the elliptic point z_0. Geometrically, the fact that $\Phi_{\omega_f}(S)$ determines Φ_{ω_f} is encoded in the contraction of the well-rounded retract and the subsequent Dupont filling. We appreciate this as the cohomology of GL_3, where we hope to define a notion of modular symbol, is non-cuspidal. It is precisely the geometry of the cells generated via a filling based on the contraction of the well-rounded retract that allows us to find a finite list of polynomials that determine each group co-cycle.

6.5 Three-variable modular symbols

Let $f \in S_{n+2}(SL_2(\mathbb{Z}))$. Following the discussion in Section 4.5.2, to f we associate the Eisenstein differential form $E(\omega_f, \Lambda_w)$, a closed, harmonic differential form in $\Omega^3(GL_3(\mathbb{Z})\backslash D_3, \mathbb{S}\mathrm{ym}^n(V_3))$. This allows us to define an Eilenberg-MacLane co-cycle for the group $GL_3(\mathbb{Z})$ associated to f. In particular, if σ is the filling of D_3 based on the contraction of the well-rounded retract, let

$$\Phi_{E(\omega_f,\Lambda_w)}(\gamma_1, \gamma_2, \gamma_3) = \int_{\sigma(\gamma_1,\gamma_2,\gamma_3)} E(\omega_f, \Lambda_w).$$

As the cells $\sigma(\gamma_1, \gamma_2, \gamma_3)$ are compact, the above expression makes sense for any triplet of elements $(\gamma_1, \gamma_2, \gamma_3) \in GL_3(\mathbb{Z})^3$. Based on our results from §6.3 we observe that,

Theorem 9. *For $(\gamma_1, \gamma_2, \gamma_3) \in GL_3(\mathbb{Z})^3$, the value $\Phi_{E(\omega_f,\Lambda_w)}(\gamma_1, \gamma_2, \gamma_3)$ is completely determined by the integral of $E(\omega_f, \Lambda_w)$ over the four tetrahedra triangulating a fundamental domain in the Soule cube decorated by the set $\{e_1, e_2, e_3\}$.*

7

A Look Forward

In the future, I would like to focus on the following questions emerging from my current work:

- *Can one create a dictionary between the well rounded-retract in higher rank and the spherical building at infinity, much like the one I exploited in rank one and two?*

  I suspect this question has an affirmative answer. In higher rank, the building at infinity is more complicated, with even more conjugacy classes of maximal parabolics. In addition, the cells of the well-rounded retract are exponentially more difficult to pin down, even up to $GL_m(\mathbb{Z})$-equivalence. However for top dimensional cells at least, one could fairly easily write down a spherical apartment, using the same framework as in the case of GL_3.

- *Using this correspondence, does the projection of a top dimensional cell of the well-rounded retract in higher rank to the Levi components of a special subset of boundary faces in the associated spherical apartment, remain well rounded?*

  In odd rank at least, there are maximal parabolic subgroups that are conjugate to their opposites. I am curious if there is any additional information to be gleaned from considering the projections to these faces.

- *Based on the projection to the boundary, can one design a contraction consistent with the contractions of certain lower-rank copies of the well-rounded retract at infinity?*

  I believe that work on answering this question can begin immediately for all cases where the cells of the well rounded retract have been classified up to the action of the full arithmetic group. A logical progression would be to extend this work to the cases when the cells of the well rounded retract have not been cataloged. While there are certain aspects of the machinery I intend to use that are abstract and do not require knowledge of the full structure of the well rounded retract, there are others that would need to be modified. More specifically, a key ingredient to my current work is the ability to write down a total ordering for the set of vectors defining any vertex in well rounded retract. This relies heavily on the fact that for GL_3 there is only one equivalence class of 0-cells in the well rounded retract.

- *Can one make explicit the relations satisfied by polynomials from Corollary 9, obtained by integrating $GL_m(\mathbb{Z})$-invariant differential forms over (a finite number of) cells in the well rounded retract?*

  There are multiple ways of attacking this problem that are not mutually exclusive. In all cases one would have to choose a finite presentation of the arithmetic group, and make use of both the Eilenberg-MacLane co-cycle condition, as well as the geometry of the cells one integrates over to generate relations.

- *Do these relations completely describe the image of the map sending a differential form to the corresponding 3-variable period polynomial? If not, is the image unique up to coboundaries?*

In rank one one can prove that there are no coboundaries in the space of 2-variable period polynomials. While the proof would not generalize easily to the case of 3-variable polynomials, it is curious to see what, if any, coboundaries do appear in the image of this map.

- *Is there a connection between the well rounded retract and Goncharov's* Modular Complex *[Gon98]?*

  In studying multiple polylogarithms and multi ζ-values, Goncharov introduces the *modular complex* $(M_\bullet, d)$ as a bridge between the Voronoi complex $(V_\bullet, d)$ for X_3 on one hand, and the dihedral Lie coalgebra on the other. More specifically, in [Gon98] Goncharov proves that $(M_\bullet, d)$ and $(V_\bullet/dV_5, d)$ are quasi-isomorphic. There may be a more explicit relationship between the modular complex and the well-rounded retract, namely an isomorphism, after one deforms the well-rounded retract in such a way so as to contract all triangular 2-dimensional cells.

- *Can one describe explicitly the family of retracts arising from* tiling *degeneration?*

  In [Sap97] Saper hints that through a process of tile degeneration, one should be able to define a deformation onto a lower-dimensional subspace of the symmetric space. In particular, he goes on to say that "*...it is reasonable to speculate that such a process would yield retractions generalizing those in the work of Ash.*" We have shown this to be a fact, and discovered tiling parameters yielding the deformation onto the well rounded retract. Furthermore, one may choose parameters yielding a deformation onto $\theta(W)$, where θ is the automorphism of the symmetric space arising from the Cartan involution associated to O_3. Making use of Saper's tiling framework which allows for a two-dimensional choice of parameters in rank two, a similar process may yield a possibly infinite family of deformation retracts of the symmetric space. I would like to describe these in greater detail, including any that arise for a choice of θ-stable parameters. In addition, I am curious if this family of retracts contains a "triangle-free" version, like the one I described in connection with Goncharov's modular complex.

Bibliography

[AM97] Avner Ash and Mark McConnell. Cohomology at infinity and the well-rounded retract for general linear groups. *Duke Math. J.*, 90(3):549–576, 1997.

[Ash84] Avner Ash. Small-dimensional classifying spaces for arithmetic subgroups of general linear groups. *Duke Math. J.*, 51(2):459–468, 1984.

[BC83] A. Borel and W. Casselman. L^2-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.*, 50(3):625–647, 1983.

[Bor81] Armand Borel. Stable real cohomology of arithmetic groups. II. In *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, volume 14 of *Progr. Math.*, pages 21–55. Birkhäuser Boston, Mass., 1981.

[Bou98] Nicolas Bourbaki. *General topology. Chapters 5–10.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2000.

[Dup78] Johan L. Dupont. *Curvature and characteristic classes.* Lecture Notes in Mathematics, Vol. 640. Springer-Verlag, Berlin, 1978.

[Eic57] M. Eichler. Eine Verallgemeinerung der Abelschen Integrale. *Math. Z.*, 67:267–298, 1957.

[Gon98] A. B. Goncharov. Multiple polylogarithms, cyclotomy and modular complexes. *Math. Res. Lett.*, 5(4):497–516, 1998.

[Gon01] A. B. Goncharov. The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$. *Duke Math. J.*, 110(3):397–487, 2001.

[Gun] Paul Gunnells. Appendix: Computing in higher rank. In *Explicitly Computing Modular Forms*, pages 205–254. unpublished manuscript.

[Hai09] Richard Hain. Brief notes on 3-variable modular symbols. *Personal Communication*, 2009.

[Har75] G. Harder. On the cohomology of dis8crete arithmetically defined groups. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 129–160. Oxford Univ. Press, Bombay, 1975.

[HC68] Harish-Chandra. *Automorphic forms on semisimple Lie groups*. Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62. Springer-Verlag, Berlin, 1968.

[Hur58] Witold Hurewicz. *Lectures on ordinary differential equations*. The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass., 1958.

[Kos61] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

[MM63] Yozô Matsushima and Shingo Murakami. On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds. *Ann. of Math. (2)*, 78:365–416, 1963.

[MM89] Robert MacPherson and Mark McConnell. Classical projective geometry and modular varieties. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 237–290. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Sap97] Leslie Saper. Tilings and finite energy retractions of locally symmetric spaces. *Comment. Math. Helv.*, 72(2):167–202, 1997.

[Sap03] Leslie Saper. On the cohomology of locally symmetric spaces and of their compactifications. In *Current developments in mathematics, 2002*, pages 219–289. Int. Press, Somerville, MA, 2003.

[Sch83] Joachim Schwermer. *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, volume 988 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[Sch86] Joachim Schwermer. Holomorphy of Eisenstein series at special points and cohomology of arithmetic subgroups of $\mathrm{SL}_n(\mathbf{Q})$. *J. Reine Angew. Math.*, 364:193–220, 1986.

[Sch90] Joachim Schwermer. Cohomology of arithmetic groups, automorphic forms and L-functions. In *Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989)*, volume 1447 of *Lecture Notes in Math.*, pages 1–29. Springer, Berlin, 1990.

[Sch94] Joachim Schwermer. Eisenstein series and cohomology of arithmetic groups: the generic case. *Invent. Math.*, 116(1-3):481–511, 1994.

[Shi59] Goro Shimura. Sur les intégrales attachées aux formes automorphes. *J. Math. Soc. Japan*, 11:291–311, 1959.

[Sou78] Christophe Soulé. The cohomology of $\mathrm{SL}_3(\mathbf{Z})$. *Topology*, 17(1):1–22, 1978.

[Vor07] G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *Journal für die Reine und Angewandte Mathematik*, 133:97–178, 1907.

[Zag91] Don Zagier. Periods of modular forms and Jacobi theta functions. *Invent. Math.*, 104(3):449–465, 1991.

Biography

Oliver Gjoneski was born on March 6, 1983 in Skopje, Macedonia. Oliver graduated with a B.A. in Mathematics and Economics from Bates College in 2005, and earned an M.A. and Ph.D. in Mathematics from Duke University in 2006 and 2011, respectively.