Categorification of Quantum $\mathfrak{sl}_3$ Projectors and the $\mathfrak{sl}_3$ Reshetikhin-Turaev Invariant of Framed Tangles

by

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Duke University

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2012

Abstract

(Mathematics)

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Abstract

Quantum $\mathfrak{sl}_3$ projectors are morphisms in Kuperberg's $\mathfrak{sl}_3$ spider, a diagrammatically defined category equivalent to the full pivotal subcategory of the category of (type 1) finite-dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_3)$ generated by the defining representation, which correspond to projection onto the highest weight irreducible summand. These morphisms are interesting from a topological viewpoint as they allow the combinatorial formulation of the $\mathfrak{sl}_3$ tangle invariant (in which tangle components are labelled by the defining representation) to be extended to a combinatorial formulation of the invariant in which components are labelled by arbitrary finite-dimensional irreducible representations. They also allow for a combinatorial description of the $SU(3)$ Witten-Reshetikhin-Turaev 3-manifold invariant.

There exists a categorification of the $\mathfrak{sl}_3$ spider, due to Morrison and Nieh, which is the natural setting for Khovanov's $\mathfrak{sl}_3$ link homology theory and its extension to tangles. An obvious question is whether there exist objects in this categorification which categorify the $\mathfrak{sl}_3$ projectors.

In this dissertation, we show that there indeed exist such "categorified projectors," constructing them as the stable limit of the complexes assigned to k-twist torus braids. These complexes satisfy categorified versions of the defining relations of the (decategorified) $\mathfrak{sl}_3$ projectors and map to them upon taking the Grothendieck group. We use these categorified projectors to extend $\mathfrak{sl}_3$ Khovanov homology to a homology theory for framed links with components labeled by arbitrary finite-dimensional

irreducible representations of $\mathfrak{sl}_3$.

Contents

List of Abbreviations and Symbols

Symbols

$\mathfrak{sl}_n$	The Lie algebra of traceless, complex $n \times n$ matrices.
$\mathcal{U}_q(\mathfrak{sl}_n)$	The quantized universal enveloping algebra of $\mathfrak{sl}_n$.
$fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_n))$	The category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_n)$.
$\mathrm{Kom}(\mathcal{A})$	The category of complexes in an additive category $\mathcal{A}$.
$\mathrm{K}(\mathcal{A})$	The homotopy category of complexes in an additive category $\mathcal{A}$.
$\mathcal{K}_{\oplus}(\mathcal{A})$	The split Grothendieck group of an additive category $\mathcal{A}$.
$\mathcal{K}_{\triangle}(\mathcal{T})$	The triangulated Grothendieck group of a triangulated category $\mathcal{T}$.
$\mathcal{S}$	The $\mathfrak{sl}_3$ spider.
$\langle T \rangle$	The $\mathfrak{sl}_3$ invariant of a tangle T.
$\mathrm{Hom}_{\mathcal{S}}(v, w)$	The vector space of webs, modulo isotopy and local relations, mapping from the word v to the word w.
$\mathcal{F}$	The foam 2-category.
$[\![T]\!]$	The categorified $\mathfrak{sl}_3$ invariant of a tangle T.
$[\![T]\!]_s$	The shifted categorified $\mathfrak{sl}_3$ invariant of a tangle T.
$\mathrm{Hom}_{\bullet}(v, w)$	The Hom-category in $\mathcal{F}$ between words v and w (or the corresponding Hom-category in $\mathrm{K}(\mathcal{F})$).
$\mathrm{Hom}_{\circ}(V, W)$	The vector space of degree-zero foams, modulo isotopy and local relations, between webs V and W (or the corresponding Hom-space in $\mathrm{K}(\mathcal{F})$).
$\widehat{\mathrm{Hom}}(V, W)$	The graded vector space of foams, modulo isotopy and local relations, between webs V and W.

$C^{\cdot}\{k\}$ A complex $C^{\cdot}$ in $\mathrm{K}(\mathcal{F})$ shifted up by k in q-degree.

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1

Introduction

Categorification is the process of lifting a mathematical invariant or structure to a higher-categorical version of that invariant or structure. We begin with an easy and instructive example. Let

$$\chi(-) : \mathcal{T}op \longrightarrow \mathbb{Z}$$

denote Euler characteristic and let

$$H_*(-;\mathbb{Q}) : \mathcal{T}op \longrightarrow gr\mathcal{V}ect$$

denote (singular) homology; here $\mathcal{T}op$ is the category of (nice enough) topological spaces and $gr\mathcal{V}ect$ is the category of graded vector spaces. Recall that these invariants are related via the formula

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \dim(H_i(X;\mathbb{Q})). \tag{1.0.1}$$

Homology is a nicer invariant than Euler characteristic for many reasons. First, equation (1.0.1) shows that $H_*(X;\mathbb{Q})$ contains as much information as does $\chi(X)$; in fact it contains more information since there are topological spaces which are distinguished by their singular homology but not by their Euler characteristic, e.g. S^3 and $S^1 \times S^2$. Secondly, homology is functorial, meaning that it not only assigns a (graded) vector space to each topological space, but also assigns to a continuous map $f : X \to Y$ a linear map $f_* : H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$. This structure can be used to deduce additional information about the properties of topological spaces, e.g. the standard proof of the Brouwer fixed-point theorem uses functoriality in a crucial way. Note that this second feature is only possible since homology takes values in a category rather than a set.

We say that homology categorifies Euler characteristic - it is a stronger invariant taking values in a higher-categorical structure from which Euler characteristic can be easily recovered. We also say that the category of (graded) vector spaces categorifies the integers since the former is a (1-)category from which the latter set (i.e. 0-category) can be obtained, in this case by taking the alternating sum of dimensions.

The program of categorification, first proposed by Crane and Frenkel [9], aims to apply the above framework in other mathematical contexts. That is, to find higher-categorical versions of known mathematical invariants and mathematical structures. Obviously, in order to categorify an invariant one must first have categorified the structure in which that invariant takes values, so these two flavors of categorification are intimately linked.

Khovanov's seminal work [12] is one of the earliest and most influential realizations of this program. In this paper, he constructs a categorification of the Jones polynomial invariant of links[1] as a bi-graded link homology theory. The Jones polynomial can be recovered from this invariant, now widely known as Khovanov homology, by taking the graded Euler characteristic. As one might expect, this invariant is formally stronger than the Jones polynomial, e.g. it distinguishes some knots which have identical Jones polynomials [1]. Moreover, this categorification possesses the

[1] In this dissertation, we allow links to have only one component, i.e. $\{knots\} \subset \{links\}$.

other property exhibited in the Euler characteristic/(singular) homology example in that Khovanov homology is functorial with respect to link cobordism [6], [2]. This structure has been used by Rasmussen to give a purely combinatorial proof of the Milnor conjecture concerning the slice genus of torus knots [20].

In [2, 3], Bar-Natan re-interpreted Khovanov homology in terms of a categorification of the Temperley-Lieb category. The latter, denoted $\mathcal{TL}$, is a diagrammatically defined category describing the finite-dimensional representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ which provides a natural setting for the extension of the Jones polynomial from an invariant of links to an invariant of tangles. Bar-Natan constructs a 2-category $\mathcal{BN}$ from which $\mathcal{TL}$ can be recovered by taking the Grothendieck group in each Hom-category. Moreover, he uses this structure to extend Khovanov homology to an invariant of tangles.

The Jones-Wenzl projectors are morphisms in $\mathcal{TL}$ which correspond to projection onto highest weight irreducible summands of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. These morphisms are important in low-dimensional topology as they can be used to give a combinatorial description of the colored Jones polynomial as well as the $SU(2)$ Witten-Reshetikhin-Turaev invariant of 3-manifolds. There has been recent interest in categorifying these projectors, that is, finding 1-morphisms in (the homotopy category of complexes in) $\mathcal{BN}$ which map to the Jones-Wenzl projectors under taking the Grothendieck group. Such categorified projectors have been constructed by Rozansky [22], Cooper and Krushkal [7], and Frenkel, Stroppel, and Sussan [10]. These projectors have been used to give a categorification of the colored Jones polynomial [7] and it is believed that these categorified projectors will be useful in constructing a categorification of the $SU(2)$ Witten-Reshetkhin-Turaev 3-manifold invariant [8].

In this thesis, we consider the $\mathfrak{sl}_3$ analog of the above results. The $\mathfrak{sl}_3$ analog of $\mathcal{TL}$ is Kuperberg's $\mathfrak{sl}_3$ spider [15], denoted $\mathcal{S}$ throughout, a diagrammatically defined category which describes the finite-dimensional representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_3)$. The $\mathfrak{sl}_3$ tangle invariant, the analog of the Jones polynomial, has a simple combinatorial description in terms of this category. There are highest weight $\mathfrak{sl}_3$ projectors which again are useful in low-dimensional topology since they allow for combinatorial constructions of the (colored) $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of framed tangles and the $SU(3)$ Witten-Reshetikhin-Turaev 3-manifold invariant [19].

There exists $\mathfrak{sl}_3$ link homology categorifying the $\mathfrak{sl}_3$ link invariant [13] and there exists a 2-category, denoted $\mathcal{F}$, which categorifies $\mathcal{S}$ and allows this invariant to be extended to a categorified tangle invariant [18]. We show that there exist 1-morphisms in (an extension of) this 2-category which categorify the $\mathfrak{sl}_3$ projectors. To do so, we follow Rozansky's approach to the $\mathfrak{sl}_2$ case, where he realizes the categorified Jones-Wenzl projectors as the stable limit of the categorified $\mathfrak{sl}_2$ invariant assigned to k-twist torus braids. We show that in the $\mathfrak{sl}_3$ case, categorified highest weight projectors can again be realized as the stable limit of the categorified invariant of k-twist torus braids. We then use these categorified projectors to give a categorification of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of framed tangles.

2

Background

In this chapter we review the necessary background on quantum $\mathfrak{sl}_n$, monoidal categories, quantum link invariants, and $\mathfrak{sl}_3$ link homology.

2.1 The quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$

We shall follow the exposition in [11], [17], and [5].

We first recall basic facts concerning $\mathfrak{sl}_n$ and its representation theory before turning our attention to the quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$. Our motivation for considering the latter is the following idea: the representation theory of $\mathfrak{sl}_n$ is interesting (it gives a braided, pivotal category), but not interesting enough (the braiding is trivial, hence so are the knot invariants derived from this structure). We thus would like to consider an object whose representation theory is a deformation of that of $\mathfrak{sl}_n$. Since semi-simple Lie algebras are 'rigid' (one can't deform a semi-simple Lie algebra to a non-isomorphic semi-simple Lie algebra), we instead consider a deformation of its universal enveloping algebra in the category of Hopf algebras.

2.1.1 Classical $\mathfrak{sl}_n$

Let $\mathfrak{sl}_n$ denote the Lie algebra of traceless, complex $n \times n$ matrices. If we let $E_{i,j}$ denote the matrix with 1 in the $(i,j)^{th}$ entry and all other entries zero then $\mathfrak{sl}_n$ is spanned by the elements $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ and $h_i = E_{i,i} - E_{i+1,i+1}$ for $i = 1, \ldots, n-1$. These elements generate $\mathfrak{sl}_n$ subject to the relations:

1. $[h_i, h_j] = 0$,
2. $[e_i, f_i] = \delta_{ij} h_i$,
3. $[h_i, e_j] = a_{ij} e_j$,
4. $[h_i, f_j] = -a_{ij} f_j$,
5. $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$,
6. $(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$,

where $A = (a_{ij})_{i,j=1,\ldots,n-1}$ is the Cartan matrix for $\mathfrak{sl}_n$ defined by $a_{ii} = 2$, $a_{ij} = -1$ if $|i-j| = 1$, and $a_{ij} = 0$ otherwise. Taking ϵ_i to be the linear functional on the space of $n \times n$ complex matrices given by $\epsilon_i(M) = m_{ii}$ where $M = (m_{ij})$ and setting $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$, we have that

$$a_{ij} = \alpha_i(h_j).$$

The (abelian) subalgebra $\mathfrak{h}$ spanned by the h_i's is called the Cartan subalgebra and it's dual space $\mathfrak{h}^*$ is spanned by the α_i's.

Recall that a representation of a Lie algebra $\mathfrak{g}$ on a (complex) vector space V is a Lie algebra homomorphism

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

We will concern ourselves only with finite-dimensional representations. In the case of $\mathfrak{sl}_n$ (or more generally any semi-simple Lie algebra) the finite-dimensional representation theory is semi-simple, meaning that any finite-dimensional representation is isomorphic to a direct sum of irreducible representations, those which have no nontrivial subrepresentations.

Let $\mathcal{U}(\mathfrak{sl}_n)$ be the universal enveloping algebra of $\mathfrak{sl}_n$. The Poincaré-Birkhoff-Witt theorem implies that the canonical map $\mathfrak{sl}_n \to \mathcal{U}(\mathfrak{sl}_n)$ is an inclusion and the universal property of $\mathcal{U}(\mathfrak{sl}_n)$ then implies that the representations of $\mathcal{U}(\mathfrak{sl}_n)$ are precisely the representations of $\mathfrak{sl}_n$.

The lattice

$$\Lambda = \{\lambda \in \mathfrak{h}^* | \lambda(h_i) \in \mathbb{Z} \ \forall i = 1, \ldots, n-1\}$$

is called the weight lattice for $\mathfrak{sl}_n$. Each finite-dimensional irreducible representation (and hence every finite-dimensional representation) V of $\mathfrak{sl}_n$ (and hence of $\mathcal{U}(\mathfrak{sl}_n)$) decomposes into a direct sum

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

where $V_\lambda = \{v \in V | hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}$. The set

$$\mathfrak{h}^*_+ = \{\lambda \in \mathfrak{h}^* | \lambda(h_i) \geqslant 0 \ \forall i = 1, \ldots, n-1\}$$

is the positive Weyl chamber and $\Lambda_+ = \Lambda \cap \mathfrak{h}^*$ is the set of dominant (integral) weights.

A representation V is called a highest weight representation with highest weight $\lambda \in \Lambda$ provided there exists a vector $v_\lambda \in V_\lambda \smallsetminus \{0\}$ so that $e_i v_\lambda = 0$ for all $i = 1, \ldots, n-1$ and $V = \mathcal{U}(\mathfrak{sl}_n)v_\lambda$. The following theorem is standard:

Theorem 2.1.1. *Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_n$, then V is a highest weight representation for some $\lambda \in \Lambda_+$ and is determined up to isomorphism by its highest weight.*

Recall now that if $\mathfrak{g}$ is a (complex) Lie algebra then $\mathcal{U}(\mathfrak{g})$ has a natural comultiplication

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{\Delta} \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) ,$$

counit

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{\varepsilon} \mathbb{C} ,$$

and antipode

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{S} \mathcal{U}(\mathfrak{g})$$

given on elements $x \in \mathfrak{g}$ by

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\varepsilon(x) = 0$$

$$S(x) = -x.$$

These maps give $\mathcal{U}(\mathfrak{g})$ the structure of a (noncommutative, cocommutative) Hopf algebra and give the category $fd\mathcal{R}ep(\mathcal{U}(\mathfrak{g}))$ of finite-dimensional representations of $\mathcal{U}(\mathfrak{g})$ the structure of a braided pivotal category (see section 2.2 below). We shall see that the quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$ is a noncommutative, noncocommutative deformation of $\mathcal{U}(\mathfrak{sl}_n)$ in the category of Hopf algebras and that its category of representations possesses a similar structure.

2.1.2 Quantum $\mathfrak{sl}_n$

Let A be the Cartan matrix for $\mathfrak{sl}_n$ defined above. Define the quantum group, or quantized enveloping algebra, $\mathcal{U}_q(\mathfrak{sl}_n)$ to be the unital associative algebra over $\mathbb{C}(q)$ generated by the elements e_i, f_i, K_i, and K_i^{-1} for $i = 1, \dots, n-1$ subject to the relations

1. $K_i K_i^{-1} = 1 = K_i^{-1} K_i$,

2. $K_i K_j = K_j K_i$,

3. $K_i e_j = q^{a_{ij}} e_j K_i$,

4. $K_i f_j = q^{-a_{ij}} f_j K_i$,

5. $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$,

6. $e_i^2 e_{i\pm 1} - [2] e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0$,

7. $e_i e_j = e_j e_i$ if $|i-j| \geqslant 2$,

8. $f_i^2 f_{i\pm 1} - [2] f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0$,

9. $f_i f_j = f_j f_i$ if $|i-j| \geqslant 2$,

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the quantum integer. $\mathcal{U}_q(\mathfrak{sl}_n)$ again has the structure of a Hopf algebra with comultiplication, counit, and antipode given by

$$\begin{aligned}
\Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1} \\
\Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i \\
\Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i \\
\varepsilon(K_i^{\pm 1}) &= 1 \\
\varepsilon(e_i) &= 0 = \varepsilon(f_i) \\
S(K_i^{\pm 1}) &= K_i^{\mp 1} \\
S(e_i) &= -e_i K_i \\
S(f_i) &= -K_i^{-1} f_i
\end{aligned}$$

and the category $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_n))$ of (type 1) finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_n)$ again has the structure of a braided pivotal category, although this is a much more difficult result than in the case of $\mathcal{U}(\mathfrak{sl}_n)$ (see [5, Corollary 10.1.20]).

The (type 1) finite-dimensional representation theory of $\mathcal{U}_q(\mathfrak{sl}_n)$ is again semi-simple and each such representation V^q has a weight space decomposition

$$V^q = \bigoplus_{\lambda \in \Lambda} V^q_\lambda$$

where $V^q_\lambda = \{v \in V^q | K_i^{\pm} v = q^{\pm\lambda(h_i)} v \ \forall i = 1, \ldots, n-1\}$. As in the classical case, a (type 1) representation V^q is called a highest weight representation with highest weight $\lambda \in \Lambda$ provided there exists a vector $v_\lambda \in V^q_\lambda$ so that $e_i v_\lambda = 0$ for all $i = 1, \ldots, n-1$ and $V^q = \mathcal{U}_q(\mathfrak{sl}_n) v_\lambda$. Again, we have a complete description of the (type 1) irreducible finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_n)$.

Theorem 2.1.2. *Let V^q be an irreducible (type 1) finite-dimensional representation of $\mathcal{U}_q(\mathfrak{sl}_n)$, then V^q is a highest weight representation for some $\lambda \in \Lambda_+$ and is determined up to isomorphism by its highest weight.*

Moreover, the representation rings of $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{U}_q(\mathfrak{sl}_n)$ are isomorphic, and if for $\lambda \in \Lambda_+$ we denote by $V(\lambda)$ and $V^q(\lambda)$ the highest weight representations of $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{U}_q(\mathfrak{sl}_n)$ with highest weight λ then

$$\dim_{\mathbb{C}} V(\lambda)_\mu = \dim_{\mathbb{C}(q)} V^q(\lambda)_\mu.$$

These results show that $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_n))$ and $fd\mathcal{R}ep(\mathcal{U}(\mathfrak{sl}_n))$ are similar categories. Their crucial difference is that the braiding on $fd\mathcal{R}ep(\mathcal{U}(\mathfrak{sl}_n))$ is trivial while the braiding on $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_n))$ is not.

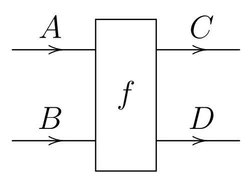
2.2 Monoidal categories and link invariants

We now recall the basics concerning monoidal categories, their graphical language, and their relation to link invariants.

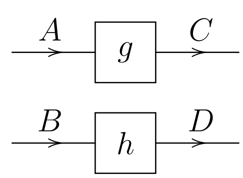
2.2.1 Graphical language of monoidal categories

Graphical language allows one to avoid writing the complicated (and mostly unenlightening) commutative diagrams which specify certain structures on monoidal categories. We follow the excellent source [23]. Let $(\mathcal{C}, \otimes, I)$ denote a monoidal category. We require that tensor product is associative and I is unital, both up to coherent isomorphism.

We will graphically denote objects in $\mathcal{C}$ (or equivalently their identity morphisms) with strands directed rightward labeled by the object and non-identity morphisms by (labeled) boxes with edges entering and leaving specifying the tensor factors of the domain and codomain. If a strand is to be labeled by I we simply omit it. Tensor product of both objects and morphisms will be denoted by stacking. For example, a morphism $f : A \otimes B \to C \otimes D$ is represented graphically by the diagram



and the tensor product of morphisms $g : A \to C$ and $h : B \to D$ is represented by



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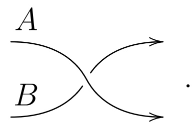
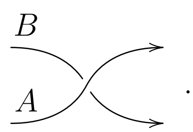
The coherence axioms for monoidal categories imply that two diagrams denote the same morphism provided they are related by a planar isotopy (relative to the boundary) keeping all strands directed rightward.

A monoidal category $\mathcal{C}$ is braided provided for every pair of objects A, B in $\mathcal{C}$ there is a natural isomorphism $c_{A,B} : A \otimes B \to B \otimes A$ satisfying the so-called "hexagon axioms." Rather than write down these axioms, we shall explain their implications in the graphical language for $\mathcal{C}$. We represent the braiding $c_{A,B}$ graphically by

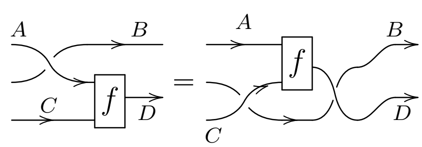


Proposition 2.2.1. *Two diagrams represent the same morphisms in a braided category provided they are related by braid isotopy, i.e.* 3*-dimensional isotopy (relative to the boundary) which preserves the property that all strands are oriented rightward.*

Note that $c_{A,B}^{-1} : B \otimes A \to A \otimes B$ is thus given graphically by



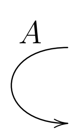
To illustrate the proposition, we have, for example,



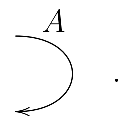
where $f : A \otimes C \to D$.

The final notions relevant to our discussion are those of pivotal and ribbon categories. A pivotal category is a monoidal category in which every object A has a dual object A^* together with distinguished morphisms $\eta_A : I \to A^* \otimes A$ and $\varepsilon_A : A \otimes A^* \to I$ satisfying certain properties and natural isomorphisms $A \cong (A^*)^*$. As in the above cases, we do not detail the properties these morphisms need to satisfy, but rather explain the implications for the graphical language.

We denote A^* by a strand oriented leftward labeled by A and denote the map η_A by



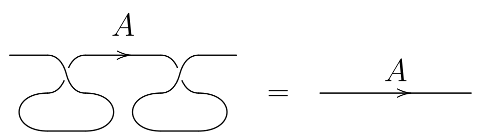
and ε_A by



Proposition 2.2.2. *Two diagrams represent the same morphism in a pivotal category provided they are related by planar isotopy (relative to the boundary). Moreover, diagrams represent the same morphism in a braided, pivotal category provided they are related by regular* 3*-dimensional isotopy.*

Note that such isotopies are allowed to 'turn around' strands, unlike in the non-pivotal case.

A braided, pivotal category is called ribbon provided the following graphical relation holds



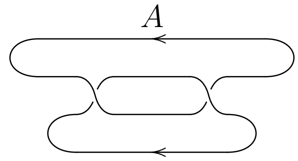
Morphisms in ribbon categories are depicted similarly to those in other tensor categories except that strands should be replaced by ribbons.

Proposition 2.2.3. *Two diagrams represent the same morphism in a ribbon category provided they are related by framed* 3*-dimensional isotopy (relative to the boundary).*

This observation is the crucial fact used to apply monoidal category theory in low-dimensional topology.

2.2.2 Link and tangle invariants

The graphical language of monoidal categories leads to link invariants. Fixing a ribbon category $\mathcal{C}$, a link diagram may be viewed as an endomorphism of the unit object in $\mathcal{C}$ by labeling each component of the link by an object in $\mathcal{C}$. For example the labeled diagram



denotes the morphism $I \to I$ given by

$$\varepsilon_{A*} \circ (\mathrm{id}_{A*\otimes A} \otimes \varepsilon_B) \circ (\mathrm{id}_{A*} \otimes c_{B,A} \otimes \mathrm{id}_{B*}) \circ (\mathrm{id}_{A*} \otimes c_{A,B} \otimes \mathrm{id}_{B*}) \circ (\mathrm{id}_{A*\otimes A} \otimes \eta_{B*}) \circ \eta_A$$

(this example should make evident why we prefer the graphical language!). By Proposition 2.2.3, this morphism is invariant under framed 3-dimensional isotopy and hence can be viewed as an invariant of the framed link determined by the diagram.

As mentioned above, the category $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_n))$ has the structure of a braided pivotal category; in fact, this category is ribbon. The invariant obtained by labeling each component by an irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_n)$ is the $\mathfrak{sl}_n$ Reshetikhin-Turaev invariant of framed links [21]. Since the unit object in this category is $\mathbb{C}(q)$, the invariant takes values in $\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{sl}_n)}(\mathbb{C}(q), \mathbb{C}(q)) = \mathbb{C}(q)$.

When each component is labeled by the defining representation of $\mathcal{U}_q(\mathfrak{sl}_n)$ this invariant is (up to a reparametrization) the $\mathfrak{sl}_n$ specialization of the HOMFLY polynomial [21]. In particular, for $n = 2$ this gives the Jones polynomial.

More generally, one can view a tangle with components labeled by objects in $\mathcal{C}$ as a morphism in $\mathcal{C}$. By proposition 2.2.3 this gives an invariant of framed tangles taking value in the appropriate Hom-set.

2.3 Combinatorial description of $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$

We now restrict ourselves to the case $n = 3$. In this case, (a full subcategory of) the category $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$ can be completely described graphically.

2.3.1 The $\mathfrak{sl}_3$ spider

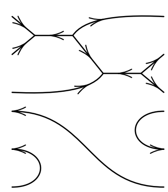
Kuperberg's $\mathfrak{sl}_3$ spider [15], denoted $\mathcal{S}$ for the duration, is the pivotal category defined as follows. Objects of $\mathcal{S}$ are words in the symbols $+$ and $-$, tensor product of objects is given by concatenation of words, and the dual of a word is obtained by reversing the word and interchanging $\pm$, e.g.

$$(+-+)\otimes(--+) = (+-+--+)$$

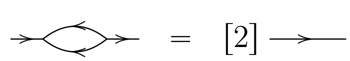
and

$$(+-+--+)^* = (-++-+-).$$

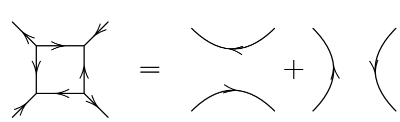
Morphisms in $\mathcal{S}$ are given by $\mathbb{C}(q)$-linear combinations of webs - oriented, trivalent planar graphs whose edges are all directed into or out from a trivalent vertex - with appropriate boundary, modulo isotopy (rel. boundary) and local relations. Edges should be directed into $+$ and out from $-$ in the codomain and vice-versa in the domain, e.g. the web



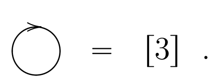
denotes a morphism $(+++--+) \to (+++-+-)$. Using these conventions, it becomes unnecessary to explicitly specify the domain and codomain of a web. The local relations are given as follows:



(2.3.1)



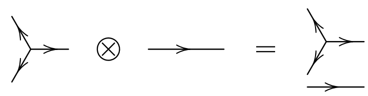
(2.3.2)



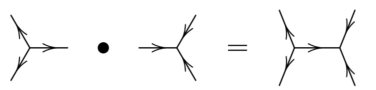
(2.3.3)

A web with no digon, square, or circular faces is called non-elliptic. Using the above relations, any web can be expressed as a $\mathbb{Z}[q^{-1}, q]$-linear sum of non-elliptic webs. We will refer to the power of q multiplying a web in such an expression as the quantum degree (or q-degree) of the web. An Euler characteristic argument shows that there cannot exist a non-empty, closed, non-elliptic web.

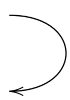
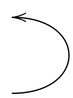
Tensor product is given by stacking webs vertically and composition, which we shall denote by $\bullet$, is given by gluing webs along their boundaries. For example,



and



Note that web composition is denoted in the non-traditional but diagrammatically more pleasing order; when considering morphisms in other categories we will use the traditional order for composition. The dual of a web is obtained by rotating the web 180° and the pairing giving the pivotal structure is given by the webs

 and  .

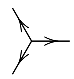
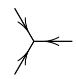
We will call these webs and their duals U-webs. The graphical description of this category immediately gives that it is a pivotal category.

The main result of [15] is that $\mathcal{S}$ completely encodes the category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_3)$.

Theorem 2.3.1 ([15])**.** *$\mathcal{S}$ is equivalent to the full pivotal subcategory of $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$ generated by the defining representation of $\mathcal{U}_q(\mathfrak{sl}_3)$.*

The defining representation V of $\mathcal{U}_q(\mathfrak{sl}_3)$ is the quantum analog of the obvious representation of $\mathfrak{sl}_3$ on $\mathbb{C}^3$. In down to earth terms, the pivotal subcategory generated by V has objects finite tensor products whose factors are all either V or V^*.

The correspondence between this category and $\mathcal{S}$ is easy to describe. The symbols $+$ and $-$ correspond to V and V^* and the morphisms

 and 

correspond to the unique (up to scalar multiple) maps $V^* \otimes V^* \to V$ and $V \otimes V \to V^*$. We will call these morphisms and their duals Y-webs. All other webs can be obtained from Y-webs and U-webs via composition and tensor product.

The reader may at this point object to the assertion that $\mathcal{S}$ completely describes the finite-dimensional representation theory of $\mathcal{U}_q(\mathfrak{sl}_3)$. Recall that in the classical case, every finite-dimensional irreducible representation of $\mathfrak{sl}_3$ appears as the highest weight irreducible summand of a tensor product composed of the defining representation and its dual. This behavior persists in the quantum case, hence by passing to the Karoubi envelope[1] of $\mathcal{S}$ we recover $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$.

We record here one final result from [15] which follows from (the proof of) Theorem 2.3.1.

Proposition 2.3.2. *For any words v and w, there are only finitely many non-elliptic webs in* $\mathrm{Hom}(v, w)$ *and these webs form a basis for this* Hom*-set.*

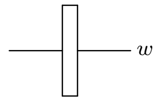
2.3.2 Highest weight projectors

Kuperberg introduced 'internal clasps' in his initial study of $\mathcal{S}$. These clasps are idempotent elements $P_w \in \mathrm{Hom}_{\mathcal{S}}(w, w)$ which via Theorem 2.3.1 correspond to projection onto the highest weight irreducible summand (and then inclusion). Here w is a word of $+$'s and $-$'s.

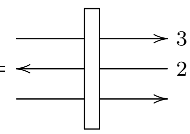
We shall denote these morphisms, referred to as (quantum $\mathfrak{sl}_3$) projectors for the

[1] This is also known as the idempotent completion; see [4] for a description of this construction.

duration, graphically by

$P_w =$ 

when we don't wish to specify the word w and by orienting the strands and labeling them with numbers corresponding to their multiplicities when we do. For instance,

$P_{(+++--+)} =$ 

where strands unlabeled on the right have multiplicity one.

These projectors can be described without reference to $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$ as follows. Let the weight of a word w be given by

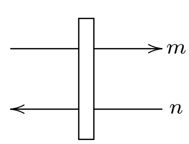
$$\mathrm{wt}(w) = (w_+, w_-) \in \mathbb{Z}^2_{\geqslant 0}$$

where $w_\pm$ denotes the number of $\pm$ signs appearing in the word. There is a partial order on words generated by the relations

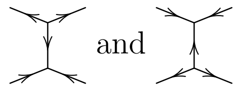
$$(w_+, w_-) > (w_+ + 1, w_- - 2)$$
$$(w_+, w_-) > (w_+ - 2, w_- + 1),$$

corresponding to the partial order on the weight lattice for $\mathfrak{sl}_3$. The projector P_w is the unique non-zero idempotent element in $\mathrm{Hom}_\mathcal{S}(w, w)$ satisfying the condition that if $\mathrm{wt}(v) < \mathrm{wt}(w)$ then $P_w \bullet W_1 = 0$ for any $W_1 \in \mathrm{Hom}_\mathcal{S}(w, v)$ and $W_2 \bullet P_w = 0$ for any $W_2 \in \mathrm{Hom}_\mathcal{S}(v, w)$ (recall our conventions for the order of tangle composition!). It follows that $(P_w)^* = P_{w^*}$.

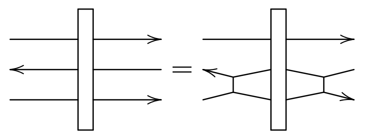
Of particular importance are the segregated projectors, those of the form



for $m, n \geqslant 0$; we will refer to the domain (= codomain) of a segregated projector as a segregated word. All other projectors can be obtained from these by inserting 'H-webs,' those of the form

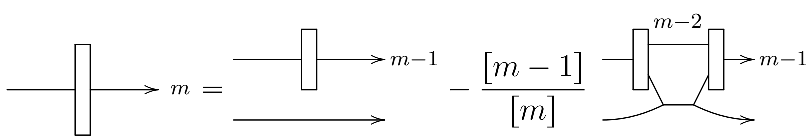


to permute the order of the +'s and −'s. For example $P_{(+-+)}$ can be obtained from $P_{(++-)}$ as follows:



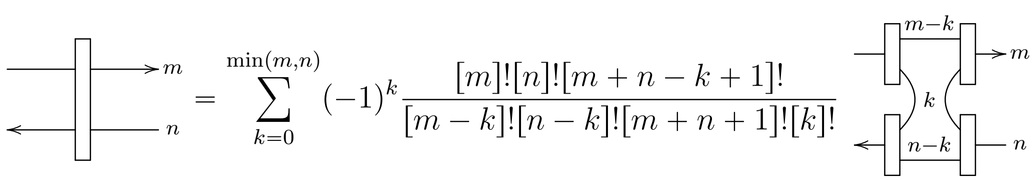
The following result, proved in [19], gives a recursive formula for the segregated projectors.

Proposition 2.3.3. *For $m > 0$,*



(2.3.4)

and for $m, n > 0$



(2.3.5)

From these formulas it is evident that a projector is the sum of a lone identity web, denoted id_w for the duration, in quantum degree zero with a $\mathbb{C}(q)$-linear combination of non-identity webs. The next proposition, which follows from the definition of the projectors, gives a characterization of P_w that we will eventually categorify.

Proposition 2.3.4. *The following properties characterize P_w.*

1. $P_w = \mathrm{id}_w + \sum_{i=1}^{r} f_i(q) \cdot W_i$ *with* $f_i \in \mathbb{C}(q)$ *and* $W_i \in \mathrm{Hom}_{\mathcal{S}}(w, w) \smallsetminus id_w$.

2. *If* $\mathrm{wt}(v) < \mathrm{wt}(w)$ *then* $P_w \bullet W_1 = 0$ *for any* $W_1 \in \mathrm{Hom}_{\mathcal{S}}(w, v)$ *and* $W_2 \bullet P_w = 0$ *for any* $W_2 \in \mathrm{Hom}_{\mathcal{S}}(v, w)$.

3. $P_w \bullet P_w = P_w$.

If the projector is segregated, Proposition 2.3.3 shows that the non-identity webs in the sum take the form $V_1 \bullet W \bullet V_2$ where V_1 and V_2 are (the tensor product of identity webs with) U-webs or Y-webs, and W is an arbitrary web. This observation, together with the semisimplicity of $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$, implies that the second defining property above can be replaced by the following in the case of a segregated projector:

2.' P_w annihilates Y-webs and U-webs (when two of the boundary points are attached to P_w).

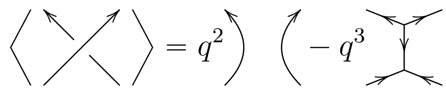
This can also be deduced from the following result of Kuperberg which shall be used in the sequel.

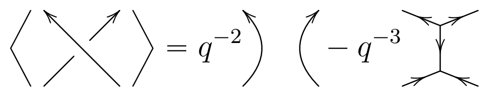
Proposition 2.3.5 ([15]). *If w is a segregated word and v is a word of lower or incomparable weight, then any non-elliptic web in* $\mathrm{Hom}_{\mathcal{S}}(w, v)$ *factors through a Y-web or a U-web, i.e. has a Y-web or U-web with two of its boundary points attached to w.*

A similar result holds for $\mathrm{Hom}_{\mathcal{S}}(v, w)$ by taking duals.

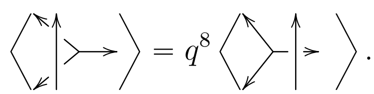
2.4 $\mathfrak{sl}_3$ tangle invariant

We now aim to describe the quantum $\mathfrak{sl}_3$ tangle invariant combinatorially. We begin with the invariant in which every component is labeled by V or V^*. It suffices to express the braiding morphisms from $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$ as morphisms in $\mathcal{S}$ via Theorem 2.3.1 and assign these values to the crossings. Such formulas are written down explicitly in [15]; however, it is more convenient from a categorification standpoint to instead use the skein relations


(2.4.1)


(2.4.2)

from [13] and [18]. The morphisms determined by these formulas are scalar multiples of the standard braiding on $fd\mathcal{R}ep(\mathcal{U}_q(\mathfrak{sl}_3))$, but do not themselves determine a braiding since, for example, we have the equation


(2.4.3)

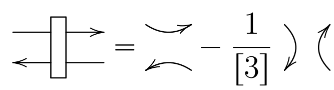
There are similar corrections of q^8 for other orientations of this diagram with the factor of q^8 always appearing on the side of the equation with smaller writhe.

Nevertheless, equations (2.4.1)and (2.4.2) still determine a well defined map from tangles to morphisms in $\mathcal{S}$ which turns out to be independent of framing. In particular, links can be viewed as morphisms in $\mathrm{Hom}_{\mathcal{S}}(\varnothing, \varnothing) \cong \mathbb{C}(q)$ so this gives a $\mathbb{C}(q)$-valued invariant of links. The local relations (2.3.1), (2.3.2), and (2.3.3) and skein relations (2.4.1) and (2.4.2) show that this link invariant is actually $\mathbb{Z}[q^{-1}, q]$-valued.

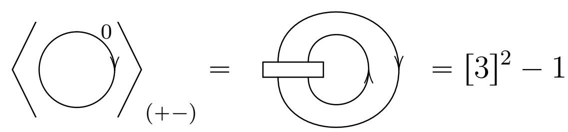
Using the $\mathfrak{sl}_3$ projectors we can extend this formulation to give a combinatorial description of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of framed tangles, i.e. the invariant of framed tangles in which each component is labeled by a finite-dimensional irreducible representation of $\mathfrak{sl}_3$. Recall that such a representation is determined by its highest weight, i.e. by $\lambda = \mathrm{wt}(w)$ for some object w in $\mathcal{S}$. To compute this invariant $\langle T \rangle_{\{\lambda_i\}_{i=1}^r}$ for an r-component tangle T, consider any word w_i corresponding to the highest weight λ_i of the irreducible representation labeling the i^{th} component.

Take the cable of the tangle corresponding to the tangle's framing with strands directed according to w_i and insert P_{w_i} anywhere along the cabling. Finally, use the skein relations to evaluate the (sum of) tangled webs.

Example 2.4.1. *Equation* (2.3.5) *gives*



so we compute



where the zero denotes the framing.

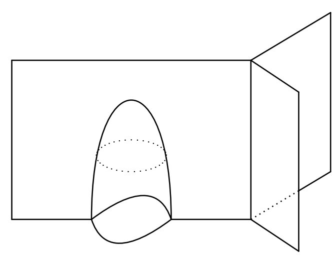
It is possible to show that for each labeling this gives an invariant of framed tangles; in particular it does not depend on where we insert the projector on each component.

2.5 The categorified $\mathfrak{sl}_3$ tangle invariant

Using the cohomology rings of projective space and flag varieties and certain singular surfaces called foams, Khovanov constructed a categorification of the quantum $\mathfrak{sl}_3$ link invariant in [13]. This construction gives a bigraded homology theory for links from which the quantum $\mathfrak{sl}_3$ link invariant can be recovered by taking the graded Euler characteristic. In [16], Mackaay and Vaz gave a geometric reformulation of this theory in the spirit of [2], which was later refined by Morrison and Nieh in [18] to an invariant of tangles. This latter theory is the setting for our categorification of the quantum $\mathfrak{sl}_3$ projectors.

We now outline Morrison and Nieh's construction, referring the reader to their work for complete details. The invariant of a tangle takes values in the Hom-categories of the homotopy category of complexes in a certain (weak) 2-category[2] $\mathcal{F}$ which we now describe.

The objects of $\mathcal{F}$ are words in the symbols $+$ and $-$, just as in $\mathcal{S}$. Given two words v and w, the category $\mathrm{Hom}(v, w)$ is the additive category whose objects (the 1-morphisms in $\mathcal{F}$) are q-graded formal direct sums of webs[3] with boundary determined by v and w. Morphisms between such objects (the 2-morphisms) are matrices of $\mathbb{C}$-linear combinations[4] of isotopy classes of degree-zero foams - oriented surfaces with singular arcs which locally look like the product of the letter Y and an interval - having the webs as boundary. The 'vertical' boundaries of a foam are required to be the product of the boundary of the webs it maps between with an interval. For example, we have the foam



which we read as mapping from the web at the bottom to the web at the top (we have omitted orientations from these webs).

A foam $F : q^{k_1} W_1 \to q^{k_2} W_2$ is graded via

$$\deg(F) = 2\chi(F) - |\partial| + \frac{|V|}{2} + k_2 - k_1$$

where χ is the Euler characteristic, ∂ is the boundary of W_1 (or W_2 - their boundaries agree!), and V is the set of trivalent vertices in $W_1 \coprod W_2$. Certain (degree homogeneous) local relations are imposed on these foams; we refer the reader to [18] for the

[2] Morrison and Nieh take the viewpoint that this structure is a canopolis, which is essentially a 2-category with some dualities. They denote what we call $\mathcal{F}$ by $\mathbf{Mat}(\mathcal{C}ob(\mathfrak{su}_3))$.

[3] Note that we don't quotient by local relations or isotopy. These relations will instead be encoded by certain 2-isomorphisms in $\mathcal{F}$.

[4] We could actually work over any ring in which 2 and 3 are invertible.

precise definition of a foam and for the local relations, rather than re-LaTeX-ing the complicated pictures here.

Vertical composition of 2-morphisms, given by gluing foams along webs, is denoted by $\circ$. Horizontal composition, the bifunctor

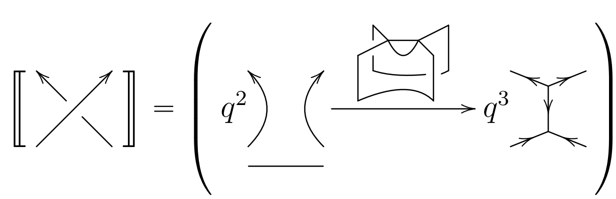
$$\mathrm{Hom}(v, w) \times \mathrm{Hom}(w, u) \to \mathrm{Hom}(v, u)$$

given by gluing 1-morphisms along their boundary and gluing 2-morphisms along the corresponding vertical segments, is denoted by $\bullet$. As in $\mathcal{S}$ this composition is written in the non-traditional, but diagrammatically pleasing, order. Tensor product is defined on 1-morphisms by stacking vertically, as in $\mathcal{S}$, and in the obvious manner for 2-morphisms.

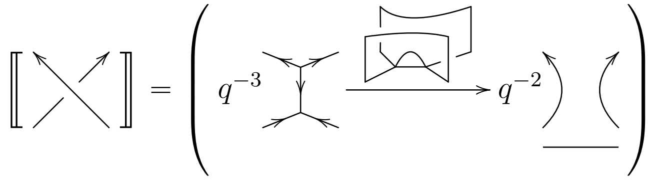
For the duration, we will denote the Hom-category between objects v and w by $\mathrm{Hom}_\bullet(v, w)$ and the vector space of morphisms between two 1-morphisms V and W by $\mathrm{Hom}_\circ(V, W)$. A useful perspective on the 2-category $\mathcal{F}$ is that we extend the planar diagrammatics for $\mathcal{S}$ into a third dimension via (singular) surfaces.

The categorified $\mathfrak{sl}_3$ tangle invariant takes values in $\mathrm{K}^b(\mathcal{F})$, the homotopy 2-category of bounded (cochain) complexes in $\mathcal{F}$. This 2-category is obtained from $\mathcal{F}$ by replacing each $\mathrm{Hom}_\bullet$-category by the homotopy category of bounded (cochain) complexes in this category. Vertical composition is then given by composition of (homotopy classes of) chain maps. Horizontal composition and tensor product are extended to $\mathrm{K}^b(\mathcal{F})$ in a manner analogous to the tensor product of complexes of R-modules.

The invariant for a tangle T, denoted $[\![T]\!]$, is given on crossings by



(2.5.1)



(2.5.2)

and extended to all tangles using horizontal composition and tensor product. Note that these foams, and hence the differentials in all complexes, have degree zero. We shall refer to the foam in equation (2.5.1) as a zip and the foam in (2.5.2) as an unzip, often denoting these morphisms by z and u respectively. Here and for the duration we will underline the term of a complex in homological degree zero; if no underline is present (and the homological degrees are not explicitly specified) then the leftmost term is assumed to be in homological degree zero.

Applying a Reidemeister move to a tangle changes the corresponding complex by a homotopy equivalence, so we obtain a $\mathrm{K}^b(\mathcal{F})$-valued invariant of tangles. Taking the Euler characteristic of the complex assigned to a tangle gives the quantum $\mathfrak{sl}_3$ invariant, as is evident by comparing equations (2.5.1) and (2.5.2) to equations (2.4.1) and (2.4.2).

This construction generalizes the $\mathfrak{sl}_3$ Khovanov homology of links as defined in [13]. Let $W \mapsto \widehat{\mathrm{Hom}}(\varnothing, W)$ be the functor which assigns to a closed web W the graded vector space of foams mapping from $\varnothing$ to W. Applying this functor to the complex in $\mathrm{K}^b(\mathcal{F})$ assigned to a link gives a complex of graded vector spaces and taking (co)homology recovers Khovanov's $\mathfrak{sl}_3$ link invariant.

2.6 Grothendieck groups and categorification

We now aim to make precise the statement that $\mathcal{F}$ categorifies $\mathcal{S}$. In order to do so, we must first discuss Grothendieck groups of additive and triangulated categories.

The main result of this section is Theorem 2.6.1 which equates the (split) Grothendieck group of an additive category with the (triangulated) Grothendieck group of the homotopy category of bounded complexes over that additive category. It has recently been pointed out to the author that this result is known to those working in the field of algebraic K-theory. The proof presented here was found prior to this and independently of any references.

2.6.1 Grothendieck groups

Let $\mathcal{C}$ be an abelian category. Recall that the Grothendieck group of $\mathcal{C}$, denoted $\mathcal{K}_0(\mathcal{C})$ is the quotient of the free abelian group of isomorphism classes $[C]$ of objects of $\mathcal{C}$ by the relation that $[C_2] = [C_1] + [C_3]$ for every short exact sequence

$$0 \to C_1 \to C_2 \to C_3 \to 0$$

in $\mathcal{C}$.

Since the categories discussed above are not abelian, we need the analog of this construction in additive (and triangulated) categories. Let $\mathcal{A}$ be an additive category. Recall that this means that $\mathcal{A}$ has a zero object, finite biproducts, and that $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ is an abelian group for any objects A_1, A_2 in $\mathcal{A}$ with addition distributing over composition. The split Grothendieck group of $\mathcal{A}$, denoted $\mathcal{K}_{\oplus}(\mathcal{A})$, is the abelian group generated by isomorphism classes $[A]$ of objects in $\mathcal{A}$ modulo the relations $[A_1 \oplus A_2] = [A_1] + [A_2]$ for all objects A_1, A_2 in $\mathcal{A}$. We can think of this as the analog of the Grothendieck group of an abelian category where we impose a relation for the only notion of exact sequence that makes sense in an additive category, that of a split exact sequence.

Now suppose that $\mathcal{T}$ is not only additive but in addition triangulated. In this case, we define the triangulated Grothendieck group, denoted $\mathcal{K}_{\triangle}(\mathcal{T})$, to be the abelian group generated by isomorphism classes $[T]$ of objects in $\mathcal{T}$ modulo the relations $[T_2] = [T_1] + [T_3]$ for all distinguished triangles $T_1 \to T_2 \to T_3$ in $\mathcal{T}$.

We'll note here for later use that if in either of the above constructions the category $\mathcal{A}$ or $\mathcal{T}$ is graded then we can keep track of this grading when we take the Grothendieck group. This endows the Grothendieck group of such a category with the structure of a $\mathbb{Z}[q^{-1}, q]$-module.

Now, fix an additive category $\mathcal{A}$ and recall that $\mathrm{K}^b(\mathcal{A})$ has the structure of a triangulated category. A natural (and relevant!) question is the relation between the abelian groups $\mathcal{K}_{\oplus}(\mathcal{A})$ and $\mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$.

Let $A^{\cdot} = \left(A^k \xrightarrow{d^k} \cdots \xrightarrow{d^{l-1}} A^l \right)$ be a (bounded) complex in $\mathrm{K}^b(\mathcal{A})$ and let $A[m]^{\cdot}$ denote the complex shifted up[5] by m in homological degree (i.e. $A[m]^i = A^{i-m}$). Recall for later use that the differential on $A[m]^{\cdot}$ is given by the differential on $A^{\cdot}$ multiplied by $(-1)^m$.

The distinguished triangle

$$A^{\cdot} \to 0 \to A[-1]^{\cdot}$$

gives that

$$\left[A[-1]^{\cdot}\right] = -[A^{\cdot}] \tag{2.6.1}$$

and the distinguished triangle

$$A^k \to \left(\underline{A}^{k+1} \xrightarrow{d^{k+1}} \cdots \xrightarrow{d^{l-1}} A^l \right) \to A[-k-1]^{\cdot}$$

shows (via induction) that

$$[A^{\cdot}] = \chi(A^{\cdot}) \tag{2.6.2}$$

in $\mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$. Here $\chi(A^{\cdot}) := \sum_{i=-\infty}^{\infty} (-1)^i [A^i]$ and A^i is shorthand for the complex with the object A^i in degree zero and all other terms zero. From this we see that $\mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$ and $\mathcal{K}_{\oplus}(\mathcal{A})$ are generated by the same elements.

[5] My apologies to any algebraic geometers in the audience; I learned homological algebra from [25].

Given complexes $A_1^{\cdot}$ and $A_2^{\cdot}$, the distinguished triangle

$$A_1^{\cdot} \to A_1^{\cdot} \oplus A_2^{\cdot} \to A_2^{\cdot}$$

shows that

$$[A_1^{\cdot} \oplus A_2^{\cdot}] = [A_1^{\cdot}] + [A_2^{\cdot}]. \tag{2.6.3}$$

Equations (2.6.2) and (2.6.3) then imply that there is a surjective map $\mathcal{K}_{\oplus}(\mathcal{A}) \to \mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$ which sends (the equivalence class of) an object A in $\mathcal{A}$ to the (equivalence class of the) complex with A in homological degree zero and all other terms zero.

Theorem 2.6.1. *The map* $\mathcal{K}_{\oplus}(\mathcal{A}) \to \mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$ *is an isomorphism.*

Before proving this result, we recall the following fact concerning cones of (co)chain maps and homotopy equivalence. Our convention is that for a map $f : A_1^{\cdot} \to A_2^{\cdot}$, the complex $\mathrm{cone}(f)$ is given by

$$\mathrm{cone}(f)^i = A_1^{i+1} \oplus A_2^i$$

with differential $\begin{pmatrix} -d_{A_1} & 0 \\ -f & d_{A_2} \end{pmatrix}$.

Proposition 2.6.2. *A chain map* $\varphi : A_1^{\cdot} \to A_2^{\cdot}$ *is a homotopy equivalence iff* $\mathrm{cone}(\varphi)^{\cdot}$ *is null-homotopic.*

For an elementary proof of this fact for complexes of abelian groups (which carries over to the case of arbitrary additive categories) see [24, pg. 167]. We now prove Theorem 2.6.1.

Proof. It suffices to show that there are no additional relations imposed on $\mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{A}))$ other than those given in equations (2.6.1), (2.6.2), and (2.6.3). Given a map

$f : A_1 \to A_2$, these equations show that

$$[\mathrm{cone}(f)^{\cdot}] = \sum_{j=\infty}^{\infty} \Big((-1)^j [A_2^j] + (-1)^{j+1} [A_1^j] \Big) \tag{2.6.4}$$

$$= [A_2^{\cdot}] - [A_1^{\cdot}]$$

so distinguished triangles of the form

$$A_1^{\cdot} \xrightarrow{f} A_2^{\cdot} \longrightarrow \mathrm{cone}(f)^{\cdot} \tag{2.6.5}$$

contribute no new relations. Since all distinguished triangles are isomorphic to those of the form (2.6.5) and isomorphism in $\mathrm{K}^b(\mathcal{A})$ is homotopy equivalence, it suffices to show that $A_1^{\cdot} \simeq A_2^{\cdot}$ implies that $\chi(A_1^{\cdot}) = \chi(A_2^{\cdot})$ when the latter are viewed as elements in the group $\mathcal{K}_{\oplus}(\mathcal{A})$. Here and for the duration, we let the symbol $\simeq$ denote homotopy equivalence of complexes, i.e. isomorphism in $\mathrm{K}^b(\mathcal{A})$.

To this end, suppose that $\varphi : A_1^{\cdot} \to A_2^{\cdot}$ is a homotopy equivalence and consider the distinguished triangle

$$A_1^{\cdot} \xrightarrow{\varphi} A_2^{\cdot} \longrightarrow \mathrm{cone}(\varphi)^{\cdot} \ .$$

Proposition 2.6.2 gives that $\mathrm{cone}(\varphi) \simeq 0$ so equation (2.6.4) implies that it suffices to show that if $A^{\cdot}$ is a null-homotopic complex then $\chi(A^{\cdot}) = 0$ when viewed as an element of $\mathcal{K}_{\oplus}(\mathcal{A})$.

Let $A^{\cdot}$ be such a complex; we may assume that

$$A^{\cdot} = \left(A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{2k}} A^{2k+1} \right)$$

contains all of the non-zero terms of $A^{\cdot}$. It suffices to show that

$$\bigoplus_{i=0}^{k} A^{2i} \cong \bigoplus_{i=0}^{k} A^{2i+1}$$

which we shall do by explicitly writing down matrices (which determine morphisms between finite direct sums in additive categories) giving the isomorphism.

Since $A^{\cdot}$ is null-homotopic there exist maps $h^j : A^j \to A^{j-1}$ so that

$$\mathrm{id}_j = d^{j-1}h^j + h^{j+1}d^j.$$

Using these equations, we can deduce the relations

$$h^j h^{j+1} \cdots h^{j+2l+1} = d^{j-2}h^{j-1}h^j \cdots h^{j+2l+1} + h^j \cdots h^{j+2l+1}h^{j+2l+2}d^{j+2l+1}.$$

For instance, we can compute

$$\begin{aligned} h^j h^{j+1} &= h^j \, \mathrm{id}_j \, h^{j+1} \\ &= h^j d^{j-1} h^j h^{j+1} + h^j h^{j+1} d^j h^{j+1} \\ &= h^j h^{j+1} - d^{j-2}h^{j-1}h^j h^{j+1} + h^j h^{j+1} - h^j h^{j+1} h^{j+2} d^{j+1} \end{aligned}$$

and

$$\begin{aligned} h^j h^{j+1} h^{j+2} h^{j+3} &= h^j h^{j+1} \, \mathrm{id}_{j+1} \, h^{j+2} h^{j+3} \\ &= h^j h^{j+1} d^j h^{j+1} h^{j+2} h^{j+3} + h^j h^{j+1} h^{j+2} d^{j+1} h^{j+2} h^{j+3} \\ &= h^j h^{j+1} h^{j+2} h^{j+3} - h^j d^{j-1} h^j h^{j+1} h^{j+2} h^{j+3} \\ &\quad + h^j h^{j+1} h^{j+2} h^{j+3} - h^j h^{j+1} h^{j+2} h^{j+3} d^{j+2} h^{j+3} \\ &= d^{j-2} h^{j-1} h^j h^{j+1} h^{j+2} h^{j+3} + h^j h^{j+1} h^{j+2} h^{j+3} h^{j+4} d^{j+3}. \end{aligned}$$

and similar computations (or induction on l) show the result in general.

Consider now the maps

$$R : \bigoplus_{i=0}^{k} A^{2i} \to \bigoplus_{i=0}^{k} A^{2i+1}$$

and

$$L : \bigoplus_{i=0}^{k} A^{2i+1} \to \bigoplus_{i=0}^{k} A^{2i}$$

given by

$$R = \begin{pmatrix} d^0 & \alpha_0 h^2 & \alpha_1 h^2 h^3 h^4 & \alpha_2 h^2 \cdots h^6 & \cdots & \alpha_{k-1} h^2 \cdots h^{2k} \\ 0 & d^2 & \alpha_0 h^4 & \alpha_1 h^4 h^5 h^6 & \cdots & \alpha_{k-2} h^4 \cdots h^{2k} \\ 0 & 0 & d^4 & \alpha_0 h^6 & \cdots & \alpha_{k-3} h^6 \cdots h^{2k} \\ 0 & 0 & 0 & d^6 & \cdots & \alpha_{k-4} h^8 \cdots h^{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d^{2k} \end{pmatrix} \tag{2.6.6}$$

and

$$L = \begin{pmatrix} \alpha_0 h^1 & \alpha_1 h^1 h^2 h^3 & \alpha_2 h^1 \cdots h^5 & \alpha_3 h^1 \cdots h^7 & \cdots & \alpha_k h^1 \cdots h^{2k+1} \\ d^1 & \alpha_0 h^3 & \alpha_1 h^3 h^4 h^5 & \alpha_2 h^3 \cdots h^7 & \cdots & \alpha_{k-1} h^3 \cdots h^{2k+1} \\ 0 & d^3 & \alpha_0 h^5 & \alpha_1 h^5 h^6 h^7 & \cdots & \alpha_{k-2} h^5 \cdots h^{2k+1} \\ 0 & 0 & d^5 & \alpha_0 h^7 & \cdots & \alpha_{k-3} h^7 \cdots h^{2k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_0 h^{2k+1} \end{pmatrix} \tag{2.6.7}$$

where $\{\alpha_k\}$ are integers defined by the recursion $\alpha_0 = 1$, $\alpha_1 = -1$, and

$$\alpha_k = -\sum_{j=0}^{k-1} \alpha_j \alpha_{k-1-j}.$$

It is easy to see that in fact $\alpha_k = (-1)^k c_k$ where c_k is the k^{th} Catalan number.

We now compute the entries of the matrices RL and LR. For $i < j$ we have

$$\begin{aligned} (RL)_{ij} &= \alpha_{j-i} d^{2i-2} h^{2i-1} \cdots h^{2j-1} + \alpha_0 \alpha_{j-i-1} h^{2i} \cdots h^{2j-1} + \cdots \\ &\quad + \alpha_{j-i-1} \alpha_0 h^{2i} \cdots h^{2j-1} + \alpha_{j-i} h^{2i} \cdots h^{2j} d^{2j-1} \\ &= \alpha_{j-i} (d^{2i-2} h^{2i-1} \cdots h^{2j-1} + h^{2i} \cdots h^{2j} d^{2j-1} - h^{2i} \cdots h^{2j-1}) \\ &= 0 \end{aligned}$$

and $(RL)_{ij} = 0$ for $i > j$. We also compute

$$(RL)_{jj} = \alpha_0 (d^{2j-2} h^{2j-1} + h^{2j} d^{2j-1}) = \mathrm{id}_{2j-1}$$

which shows that $RL = \text{id}$. Similarly, for $i < j$ we have

$$\begin{aligned}(LR)_{ij} &= \alpha_{j-i} d^{2i-3} h^{2i-2} \cdots h^{2j-2} + \alpha_0 \alpha_{j-i-1} h^{2i-1} \cdots h^{2j-2} + \cdots \\ &\quad + \alpha_{j-i-1}\alpha_0 h^{2i-1} \cdots h^{2j-2} + \alpha_{j-i} h^{2i-1} \cdots h^{2j-1} d^{2j-2} \\ &= \alpha_{j-i}(d^{2i-3} h^{2i-2} \cdots h^{2j-2} + h^{2i-1} \cdots h^{2j-1} d^{2j-2} + h^{2i-1} \cdots h^{2j-2}) \\ &= 0\end{aligned}$$

and $(LR)_{ij} = 0$ for $i > j$. We also see that

$$(LR)_{jj} = \alpha_0 (d^{2j-3} h^{2j-2} + h^{2j-1} d^{2j-2}) = id_{2j-2}$$

so $LR = \text{id}$. □

We record a corollary of the above proof for later use.

Corollary 2.6.3. *Let $A^{\cdot}$ be a null-homotopic, bounded below (cochain) complex in $\mathcal{A}$, then*

$$\bigoplus_{i=-\infty}^{\infty} A^{2i} \cong \bigoplus_{i=-\infty}^{\infty} A^{2i+1}.$$

Proof. We can assume that $A^{\cdot}$ is supported in non-negative homological degree. The infinite matrices determined by taking $k \to \infty$ in equations (2.6.6) and (2.6.7) then give the desired isomorphism. □

2.6.2 Categorification

Given an abelian group A, we say that an abelian, additive, or triangulated category $\mathcal{C}$ categorifies[6] A if the (appropriate) Grothendieck group of $\mathcal{C}$ is isomorphic to A. If A additionally has the structure of a $\mathbb{Z}[q^{-1}, q]$-module then we require that $\mathcal{C}$ is graded and that this grading gives the module structure when we 'decategorify' - i.e. take the Grothendieck group.

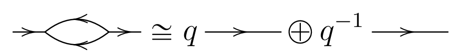
[6] See [14] for a more refined notion of categorification.

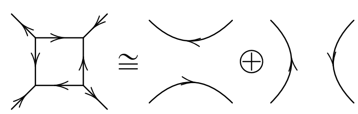
Similarly, if we have an additive category $\mathcal{C}$, we say that a 2-category $2\mathcal{C}$ categorifies $\mathcal{C}$ if we can recover $\mathcal{C}$ by taking the Grothendieck group in each Hom-category of $2\mathcal{C}$. Again, if the Hom-groups of $\mathcal{C}$ are $\mathbb{Z}[q^{-1}, q]$-modules then we require the Hom-categories of $2\mathcal{C}$ to be graded and for the grading to give the module structure on the Grothendieck group.

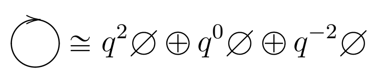
Now, consider the subcategory $\mathcal{S}_{int} \subset \mathcal{S}$ whose morphisms are given by $\mathbb{Z}[q^{-1}, q]$-linear sums of webs. Equations (2.3.1), (2.3.2), and (2.3.3) show that this is indeed a subcategory. The following result relating $\mathcal{F}$ and $\mathcal{S}_{int}$ is proved in [18].

Theorem 2.6.4. *The* 2*-category* $\mathcal{F}$ *categorifies* $\mathcal{S}_{int}$*.*

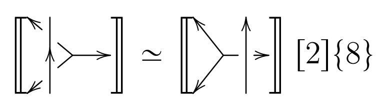
In particular, categorified versions of equations (2.3.1), (2.3.2), and (2.3.3) hold:


(2.6.8)


(2.6.9)


(2.6.10)

where $\cong$ denotes isomorphism in $\mathcal{F}$. See [18, Theorem 3.11] for the foams giving these isomorphisms. The categorified version of equation (2.4.3) also holds:


(2.6.11)

in $\mathrm{K}^b(\mathcal{F})$, where $\{k\}$ denotes a shift up in q-degree by k.

Theorems 2.6.1 and 2.6.4 pair to give the following result.

Corollary 2.6.5. *The* 2*-category* $\mathrm{K}^b(\mathcal{F})$ *categorifies* $\mathcal{S}_{int}$*.*

Observe now that equations (2.4.1) and (2.4.2) imply that the $\mathfrak{sl}_3$ invariant of a tangle actually takes values in the appropriate Hom-set of $\mathcal{S}_{int}$, i.e. is defined over the ring $\mathbb{Z}[q^{-1}, q]$. We have the commutative diagram

$$\begin{array}{ccc} & & \mathrm{K}^b(\mathcal{F}) \\ & \overset{[\![-]\!]}{\nearrow} & \downarrow \chi \\ \mathcal{T}ang & \xrightarrow{\langle - \rangle} & \mathcal{S}_{int} \end{array} \tag{2.6.12}$$

which sums up the relation between the $\mathfrak{sl}_3$ tangle invariant and its categorified counterpart. In this diagram $\mathcal{T}ang$ is the collection of tangles and the vertical map χ assigns to each complex the corresponding morphism in $\mathcal{K}_{\triangle}(\mathrm{K}^b(\mathcal{F}))$. Concretely, this is the alternating sum of the (equivalence classes of the) terms in the complex, i.e. its (graded) Euler characteristic. We shall denote by χ the operation of taking the alternating sum of the terms of a complex for the duration.

3

Summary of results

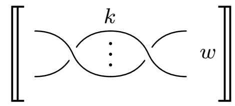
Having digested the relevant background information, we now summarize the results of this thesis. Our main result is the following.

Theorem 3.0.1. *For each word w (in the symbols $+$ and $-$) there exists a complex $\tilde{P}_w$ with terms in $\mathrm{Hom}_\bullet(w,w)$ supported in non-negative homological degree so that*

1. id_w *appears only once in* $\tilde{P}_w$ *and does so in quantum and homological degree zero,*
2. *all other webs in the complex factor through words of lower weight,*
3. *if* $\mathrm{wt}(v) < \mathrm{wt}(w)$ *then* $\tilde{P}_w \bullet W_1 \simeq 0$ *for any* $W_1 \in \mathrm{Hom}_\bullet(w,v)$ *and* $W_2 \bullet \tilde{P}_w \simeq 0$ *for any* $W_2 \in \mathrm{Hom}_\bullet(v,w)$*, and*
4. $\tilde{P}_w \bullet \tilde{P}_w \simeq \tilde{P}_w$.

It follows from this description that such a complex is unique up to homotopy. One should compare this theorem to Proposition 2.3.4, observing that the properties characterizing the categorified projector $\tilde{P}_w$ are categorical analogs of those characterizing P_w.

We construct such complexes as the stable limit (up to homotopy) of the complexes

 $[k \cdot c_-]\{k\,(3c_- - 2c_+)\}$ (3.0.1)

as $k \to \infty$. The notation indicates that there are k full twists on strands directed according to the word w and $c_\pm$ is the number of $\pm$ crossings in one twist.

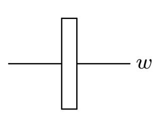
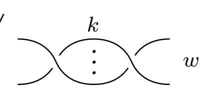
Our inspiration for this construction is Rozansky's work [22], where he proves the analogous result in the $\mathfrak{sl}_2$ case, constructing categorified Jones-Wenzl projectors as the stable limit of the categorified $\mathfrak{sl}_2$ invariant assigned to such 'torus braids.'

Although the complexes $\tilde{P}_w$ will be semi-infinite (i.e. only bounded below), we will show that it is possible to take their graded Euler characteristic and that the next result holds.

Theorem 3.0.2. $\chi(\tilde{P}_w) = P_w$.

Theorems 3.0.1 and 3.0.2 will be proved in Chapter 5. As a consequence of the above we obtain an alternate characterization of the $\mathfrak{sl}_3$ projectors, known to experts in the field. The author is unaware of a proof appearing in the literature.

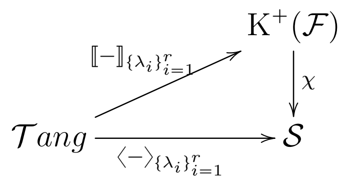
Corollary 3.0.3. *Let w be a word, then*

 $= \lim_{k\to\infty} q^{k(3c_- - 2c_+)} \Big\langle$  $\Big\rangle$.

This limit is a finite sum of webs with coefficients in $\mathbb{Z}[q^{-1}, q]]$ and corresponds to the left hand side using the inclusion of coefficients $\mathbb{C}(q) \hookrightarrow \mathbb{C}[q^{-1}, q]]$.

Finally, we shall use the categorified projectors $\tilde{P}_w$ to give a categorification of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariants of framed tangles. Let $\mathrm{K}^+(\mathcal{F})$ denote the homotopy category of bounded below complexes in $\mathcal{F}$.

Theorem 3.0.4. *Let T be an r-component framed tangle with components labeled by finite-dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_3)$ with highest weights $\{\lambda_i\}_{i=1}^r$. There exists a complex $[\![T]\!]_{\{\lambda_i\}_{i=1}^r}$ in $\mathrm{K}^+(\mathcal{F})$, invariant up to homotopy under regular isotopy, which gives a categorification of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant; that is, the diagram*



commutes.

This result is proved in Chapter 6. As in the unlabeled categorified $\mathfrak{sl}_3$ invariant, we can apply the functor $\widehat{\mathrm{Hom}}(\varnothing, -)$ to the complex assigned to a (labeled, framed) link and compute (co)homology to obtain an honest-to-goodness link homology theory.

4

Some homological algebra

In this chapter we present the requisite homological algebra for our results. The first section contains elementary results. The following two sections present the 'calculus' of chain complexes. The aim of these results is to establish a theoretical framework to define the limit of the complexes in equation (3.0.1). Almost all of the results in the latter two sections (or rather the dual statements) are taken from [22], but we repeat them in the interest of giving a self contained treatment and in order to provide some proofs omitted there.

4.1 Basic technical tools

Let $\mathcal{A}$ be an additive category. We will consider both the category $\mathrm{Kom}(\mathcal{A})$ of (cochain) complexes of objects in $\mathcal{A}$ and the category $\mathrm{K}(\mathcal{A})$, the homotopy category of $\mathrm{Kom}(\mathcal{A})$. As above, we use the superscripts b and $+$ to denote the full subcategories of these categories consisting of bounded and bounded below complexes. We will use $\cong$ to indicate isomorphism in $\mathrm{Kom}(\mathcal{A})$ and $\simeq$ to denote isomorphism in $\mathrm{K}(\mathcal{A})$, that is, homotopy equivalence.

The first result, a technical tool from [3], concerns Gaussian elimination homotopy equivalences. Such homotopy equivalences are ubiquitous in the study of Khovanov homology and its generalizations.

Proposition 4.1.1 (Gaussian elimination)**.** *Let*

$$\cdots \longrightarrow A \xrightarrow{\begin{pmatrix} * \\ \alpha \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \psi & \beta \\ \gamma & \delta \end{pmatrix}} D \oplus E \xrightarrow{\begin{pmatrix} * & \epsilon \end{pmatrix}} F \longrightarrow \cdots$$

be a complex in $\mathrm{Kom}(\mathcal{A})$ *where* $\psi : B \to D$ *is an isomorphism, then this complex is homotopy equivalent to the following complex:*

$$\cdots \longrightarrow A \xrightarrow{\alpha} C \xrightarrow{\delta - \gamma \circ \psi^{-1} \circ \beta} E \xrightarrow{\epsilon} F \longrightarrow \cdots \quad .$$

Moreover, if $\mathcal{A}$ *is graded and the differentials in the complex are degree* 0 *then so are the maps giving the homotopy equivalence.*

Our next result relates the cone of a chain map with the cone of that chain map composed with a homotopy equivalence.

Proposition 4.1.2. *Let* φ *and* ψ *be homotopy equivalences and* α *be a chain map, then* $\mathrm{cone}(\varphi \circ \alpha \circ \psi) \simeq \mathrm{cone}(\alpha)$.

Proof. Given chain maps f and g, there is a homotopy equivalence

$$\mathrm{cone}\left(\mathrm{cone}(f) \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & g \end{pmatrix}} \mathrm{cone}(g \circ f) \right) \simeq \mathrm{cone}(g)$$

so there is a distinguished triangle

$$\mathrm{cone}(f) \longrightarrow \mathrm{cone}(g \circ f) \longrightarrow \mathrm{cone}(g) \longrightarrow \mathrm{cone}(f)[-1] \, .$$

Considering the rotations of this triangle, the result follows from Proposition 2.6.2 assuming in turn that f or g is a homotopy equivalence. □

Let $\{A^{i,j}\}$ be a double complex. By convention, the horizontal d_h and vertical d_v differentials anti-commute. Given a double complex we can obtain an element in $\mathrm{Kom}(\mathrm{Kom}(\mathcal{A}))$ by negating the differentials in every other row, and vice-versa. We will use this trick to show the following.

Proposition 4.1.3 (Replacement)**.** *Let $\{A^{i,j}\}$ be a double complex with $0 \leqslant i \leqslant \infty$ and $0 \leqslant j \leqslant m$ (a triply-bounded double complex). Suppose that for each j there exist complexes $D^{\cdot,j}$ and homotopy equivalences $\varphi_j : A^{\cdot,j} \simeq D^{\cdot,j}$, then $\mathrm{Tot}(\{A^{i,j}\})$ is homotopy equivalent to a complex D_m which has $\oplus_{i+j=k} D^{i,j}$ in homological degree k.*

Proof. We proceed via induction on m. The case $m = 0$ is obvious. We will show the $m = 1$ case as this informs the proof of the general case. We consider a double complex $\{A^{i,j}\}$ of the form

$$\begin{array}{ccccccc} A^{0,1} & \xrightarrow{d_h} & A^{1,1} & \xrightarrow{d_h} & A^{2,1} & \xrightarrow{d_h} & \cdots \\ \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \\ A^{0,0} & \xrightarrow{d_h} & A^{1,0} & \xrightarrow{d_h} & A^{2,0} & \xrightarrow{d_h} & \cdots \end{array} \tag{4.1.1}$$

where each square anti-commutes. We find that

$$\mathrm{Tot}(\{A^{i,j}\}) = \underline{A^{0,0}} \xrightarrow{\begin{pmatrix} d_h \\ d_v \end{pmatrix}} A^{1,0} \oplus A^{0,1} \xrightarrow{\begin{pmatrix} d_h & 0 \\ d_v & d_h \end{pmatrix}} A^{2,0} \oplus A^{1,1} \xrightarrow{\begin{pmatrix} d_h & 0 \\ d_v & d_h \end{pmatrix}} \cdots . \tag{4.1.2}$$

Negating the top row of equation (4.1.1) to view d_v as a chain map between the complexes $(A^{\cdot,0}, d_h)$ and $(A^{\cdot,1}, -d_h)$ we find that $\mathrm{Tot}(\{A^{i,j}\}) = \mathrm{cone}(d_v)[1]$. Consider the composition $\varphi_1 \circ d_v \circ \varphi_0^{-1} : D^{\cdot,0} \to D^{\cdot,1}$. Proposition 4.1.2 shows that

$$\mathrm{cone}(\varphi_1 \circ d_v \circ \varphi_0^{-1}) \simeq \mathrm{cone}(d_v)$$

which gives the result since the degree k term of $\mathrm{cone}(\varphi_1 \circ d_v \circ \varphi_0^{-1})[1]$ is $D^{k,0} \oplus D^{k-1,1}$.

We now prove the general case. Let $\{A^{i,j}\}_{j \leqslant m+1}$ be a double complex with $0 \leqslant i \leqslant \infty$ and $0 \leqslant j \leqslant m+1$ and let $\{A^{i,j}\}_{j \leqslant m}$ be the double complex obtained by truncating the last row of $\{A^{i,j}\}_{j\leqslant m+1}$. We find that

$$\mathrm{Tot}(\{A^{i,j}\}_{j\leqslant m+1}) \cong \mathrm{cone}\left(\mathrm{Tot}(\{A^{i,j}\}_{j\leqslant m}) \xrightarrow{d_v} A^{\cdot,m+1}\,[m]\right)[1]$$

where we have negated the differential on $A^{\cdot,m+1}[m]$ in the case that m is even to view d_v as a chain map (recall how $[-]$ acts on differentials). By induction, there exists a homotopy equivalence $\psi : D_m \to \mathrm{Tot}(\{A^{i,j}\}_{j\leqslant m})$ and the result follows from

$$\mathrm{cone}(\varphi_{m+1}[m] \circ d_v \circ \psi) \simeq \mathrm{cone}(d_v)$$

as above. □

We will typically apply this result to double complexes of the form $\{A^i \bullet B^j\}$ where $A^\cdot$ and $B^\cdot$ are complexes in $\mathrm{Kom}(\mathcal{F})$ and the complexes $A^i \bullet B^\cdot$ can be simplified using Gaussian elimination.

The final result we shall need describes how tensor products interact with cones and homotopy equivalence.

Proposition 4.1.4. *Suppose $\mathcal{A}$ is a monoidal category, then*

$$\mathrm{cone}\left(A^\cdot \xrightarrow{f} B^\cdot \right) \otimes C^\cdot = \mathrm{cone}\left(A^\cdot \otimes C^\cdot \xrightarrow{f \otimes id_C} B^\cdot \otimes C^\cdot \right)$$

and if $A^\cdot \simeq B^\cdot$ then $A^\cdot \otimes C^\cdot \simeq B^\cdot \otimes C^\cdot$.

Similar results holds for other operations that behave like the tensor product of complexes. In particular, we will use the analogous result for horizontal composition in $\mathrm{Kom}(\mathcal{F})$.

4.2 Homological calculus in the category of chain complexes

In this section, we study the calculus of chain complexes in $\mathrm{Kom}(\mathcal{A})$. Let $A^\cdot$ be such a complex.

Definition 4.2.1. *The k^{th} truncation of $A^\cdot$ is the complex $t_{\leqslant k}A^\cdot$ with*

$$t_{\leqslant k}A^i = \begin{cases} A^i & \textit{if} \quad i \leqslant k \\ 0 & \textit{if} \quad i > k \end{cases}$$

and the obvious differentials.

Extend $t_{\leqslant k}$ to a functor $\mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ by defining $t_{\leqslant k}f$ for a chain map $f : A^\cdot \to B^\cdot$ to be the obvious map $t_{\leqslant k}A^\cdot \to t_{\leqslant k}B^\cdot$. We will say that a chain complex $A^\cdot$ is $\mathrm{O}^h(k)$ if $A^i = 0$ for all $i < k$. Equivalently, $A^\cdot$ is $\mathrm{O}^h(k)$ if and only if $t_{\leqslant (k-1)}A^\cdot = 0$.

Definition 4.2.2. *The isomorphism order of a chain map $f : A^\cdot \to B^\cdot$ is given by*

$$|f|_{\cong} = \sup_k \{k \mid t_{\leqslant k}f \textit{ is an isomorphism}\}$$

Clearly, a chain map f is a chain isomorphism if and only if $|f|_{\cong} = \infty$.

Proposition 4.2.3. *Let $A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{\iota} \mathrm{cone}(f) \xrightarrow{\delta} A[-1]^\cdot$ be a distinguished triangle and suppose $B^\cdot$ is $\mathrm{O}^h(k)$, then $|\delta|_{\cong} \geqslant k-1$.*

Proof. The result follows since δ is given by the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{i+1} \oplus B^i & \longrightarrow & \cdots & \longrightarrow & A^k \oplus B^{k-1} & \longrightarrow & A^{k+1} \oplus B^k & \longrightarrow & \cdots \\ & & \big\downarrow {\scriptstyle =} & & & & \big\downarrow {\scriptstyle =} & & \big\downarrow {\scriptstyle ?} & & \\ \cdots & \longrightarrow & A^{i+1} & \longrightarrow & \cdots & \longrightarrow & A^k & \longrightarrow & A^{k+1} & \longrightarrow & \cdots \end{array}$$

noticing that the degree i term of $\mathrm{cone}(f)$ is $A^{i+1} \oplus B^i$. □

Define an inverse system in $\mathrm{Kom}(\mathcal{A})$ to be a sequence of chain complexes linked by chain maps:

$$\mathbf{A} = \left(A_0^\cdot \xleftarrow{f_0} A_1^\cdot \xleftarrow{f_1} \cdots \right).$$

Definition 4.2.4. *An inverse system $\mathbf{A}$ is stabilizing if* $\lim_{l\to\infty} |f_l|_{\cong} = \infty$.

Definition 4.2.5. *An inverse system has a* Kom*-limit, denoted* $\lim_{\text{Kom}} \mathbf{A}$*, if there exist chain maps* $\lim_{\text{Kom}} \mathbf{A} \xrightarrow{\tilde{f}_l} A^{\cdot}_l$ *so that*

$$\begin{array}{ccc} & \lim_{\text{Kom}} \mathbf{A} & \\ {\scriptstyle \tilde{f}_{l-1}} \swarrow & & \searrow {\scriptstyle \tilde{f}_l} \\ A^{\cdot}_{l-1} & \xleftarrow{f_{l-1}} & A^{\cdot}_l \end{array} \tag{4.2.1}$$

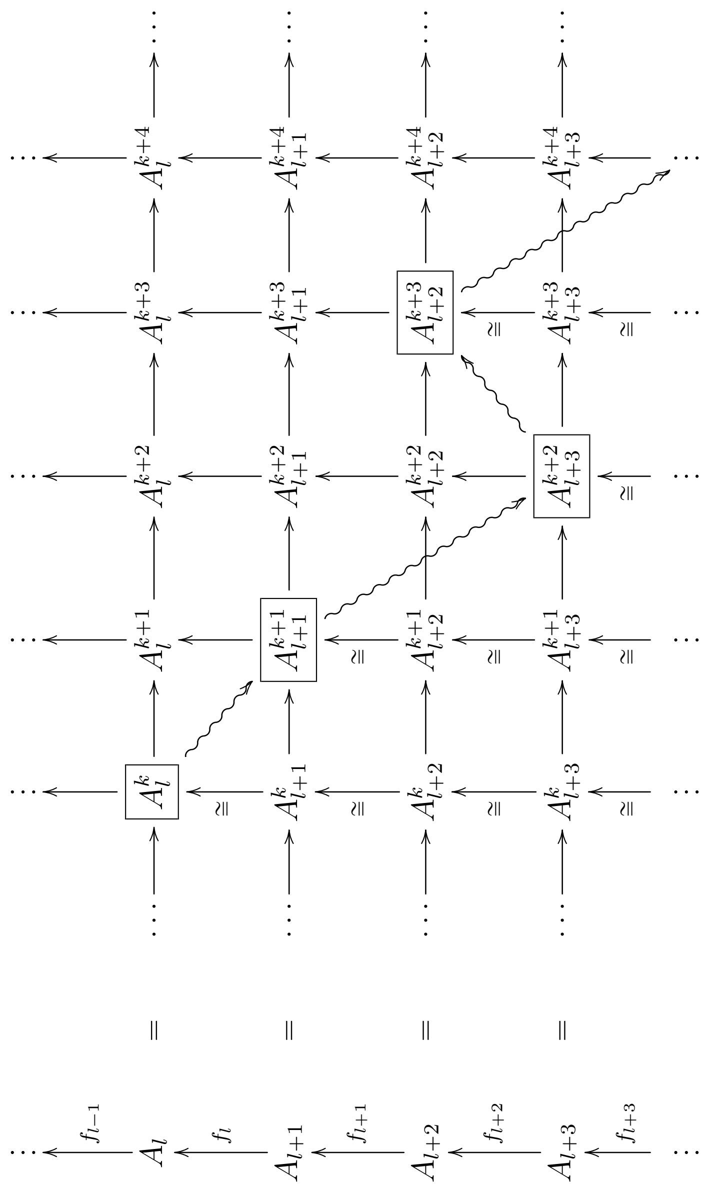
commutes and $\lim_{l\to\infty} \left|\tilde{f}_l\right|_{\cong} = \infty$.

In fact, these preceding two notions coincide.

Theorem 4.2.6. *An inverse system* $\mathbf{A}$ *has a* Kom*-limit if and only if it is stabilizing. Such a limit is unique up to isomorphism.*

Proof. First, assume that $\mathbf{A} = \left(A^{\cdot}_0 \xleftarrow{f_0} A^{\cdot}_1 \xleftarrow{f_1} \cdots\right)$ is a stabilizing inverse system. We shall give a direct construction of $A^{\cdot} = \lim_{\text{Kom}} \mathbf{A}$.

For each homological degree k, there exists a minimal $l(k)$ so that $f^k_l : A^k_{l+1} \to A^k_l$ is an isomorphism for all $l \geqslant l(k)$; let $A^k = A^k_{l(k)}$. Since all of the f_l are chain maps, there is an obvious choice of boundary map $A^k \to A^{k+1}$. Indeed, the construction is demonstrated in the following diagram:



where the boundary maps are uniquely determined as any path between the selected 'nodes' since the grid commutes. Define the maps $\tilde{f}_l : A^{\cdot} \to A_l$ using the obvious maps $A^k = A^k_{l(k)} \longrightarrow A^k_l$.

We now show that $A^{\cdot}$ is a Kom-limit for $\mathbf{A}$. Since $\mathbf{A}$ is stabilizing, for every N there exists l_N so that $l > l_N$ implies $|f_l|_{\cong} \geqslant N$. This in turn implies that if $l > l_N$ then $\left|\tilde{f}_l\right|_{\cong} \geqslant N$, showing that $\lim_{l\to\infty} \left|\tilde{f}_l\right|_{\cong} = \infty$. The result now follows since the diagram (4.2.1) commutes by construction.

Next, assume that $\mathbf{A}$ has a Kom-limit. Commutativity of (4.2.1) shows that

$$|f_l|_{\cong} \geqslant \min\left(\left|\tilde{f}_l\right|_{\cong}, \left|\tilde{f}_{l+1}\right|_{\cong}\right)$$

so $\lim_{l\to\infty} |f_l|_{\cong} = \infty$, i.e. $\mathbf{A}$ is stabilizing.

Finally, we show that Kom-limits are unique up to isomorphism in $\mathrm{Kom}(\mathcal{A})$. Suppose that both $A^{\cdot}$ and $(A')^{\cdot}$ are Kom-limits. For each homological degree k_0, there exists $m(k_0)$ so that for all $m \geqslant m(k_0)$ the maps $\tilde{f}^k_m$ and $\tilde{f'}^k_m$ are isomorphisms for all $k \leqslant k_0$. Define the isomorphism $(A')^{\cdot} \xrightarrow{\cong} A^{\cdot}$ in the k_0^{th} homological degree by

$$(\tilde{f}^k_{m(k_0)})^{-1} \circ \tilde{f'}^k_{m(k_0)} : (A')^k \to A^k.$$

Commutativity of (4.2.1) implies that this gives a chain isomorphism, well defined independent of the choice of $m(k_0)$. □

4.3 Homological calculus in the homotopy category of chain complexes

We now describe the extension of the definitions and results from the previous section to the homotopy category $\mathrm{K}(\mathcal{A})$.

Let $A^{\cdot}$ be a complex in $\mathrm{K}(\mathcal{A})$. Define the homological order of $A^{\cdot}$ via

$$|A^{\cdot}|_h = \sup\{k \mid A^{\cdot} \simeq B^{\cdot} \text{ where } B^{\cdot} \text{ is } \mathrm{O}^h(k)\}.$$

We think of a complex as homologically negligible if $|A^\cdot|_h$ is large. In the same vein, we view complexes $A^\cdot$ and $B^\cdot$ as homologically close if there is a chain map $A^\cdot \xrightarrow{f} B^\cdot$ so that $\operatorname{cone}(f)$ is homologically negligible.

The next two definitions generalize the notions of stabilizing inverse system and Kom-limit.

Definition 4.3.1. *An inverse system* $\mathbf{A} = \left(A_0^\cdot \xleftarrow{f_0} A_1^\cdot \xleftarrow{f_1} \cdots\right)$ *is Cauchy if* $\lim_{l\to\infty} |\operatorname{cone}(f_l)|_h = \infty$.

Definition 4.3.2. *An inverse system* $\mathbf{A} = \left(A_0^\cdot \xleftarrow{f_0} A_1^\cdot \xleftarrow{f_1} \cdots\right)$ *has a* K*-limit, denoted* $\lim_{\mathrm{K}} \mathbf{A}$*, if there exist chain maps* $\lim_{\mathrm{K}} \mathbf{A} \xrightarrow{\tilde{f}_l} A_l^\cdot$ *so that*

$$\begin{array}{ccc} & \lim_{\mathrm{K}} \mathbf{A} & \\ {\scriptstyle \tilde{f}_{l-1}} \swarrow & & \searrow {\scriptstyle \tilde{f}_l} \\ A_{l-1}^\cdot & \xleftarrow{\quad f_{l-1} \quad} & A_l^\cdot \end{array} \tag{4.3.1}$$

commutes in $\mathrm{K}(\mathcal{A})$ *and* $\lim_{l\to\infty} \left|\operatorname{cone}(\tilde{f}_l)\right|_h = \infty$.

We now aim to state and prove the analog of Theorem 4.2.6 in the homotopy category. Before doing so, we need two preparatory lemmata.

Lemma 4.3.3. *Suppose we have the commutative diagram*

$$\begin{array}{ccc} A^\cdot & & \\ {\scriptstyle f}\downarrow & \searrow {\scriptstyle k} & \\ B^\cdot & \xrightarrow{\ g\ } & C^\cdot \end{array}$$

in $\mathrm{K}(\mathcal{A})$ *(i.e.* $k \simeq g \circ f$*), then*

$$|\operatorname{cone}(g)|_h \geqslant \min\{|\operatorname{cone}(k)|_h\,, |\operatorname{cone}(f)|_h - 1\},$$

$$|\operatorname{cone}(f)|_h \geqslant \min\{|\operatorname{cone}(g)|_h + 1, |\operatorname{cone}(k)|_h\},$$

and

$$|\text{cone}(k)|_h \geqslant \min\{|\text{cone}(f)|_h, |\text{cone}(g)|_h\}.$$

Proof. The result follows from the various rotations of the distinguished triangle

$$\text{cone}(f) \to \text{cone}(g \circ f) \to \text{cone}(g) \to \text{cone}(f)[-1],$$

noting that for any chain map $X^\cdot \xrightarrow{\alpha} Y^\cdot$ we have the inequality $|\text{cone}(\alpha)|_h \geqslant \min\{|Y^\cdot|_h, |X^\cdot|_h - 1\}$. □

Theorem 4.3.4. *An inverse system* **A** *has a* K*-limit if and only if it is Cauchy.*

The following proof is essentially taken from [22], but we reproduce the argument in our context as a construction contained therein will be used later.

Proof. First, assume that $\mathbf{A} = \left(A_0^\cdot \xleftarrow{f_0} A_1^\cdot \xleftarrow{f_1} \cdots\right)$ is Cauchy. This implies that there exist complexes $C_l^\cdot$ so that

1. $C_l[-1]^\cdot \simeq \text{cone}(f_l)$,
2. $C_l^\cdot$ is $\text{O}^h(m_l)$,
3. and $\lim_{l\to\infty} m_l = \infty$.

We now construct a new inverse system $\mathbf{B} = \left(B_0^\cdot \xleftarrow{\delta_0} B_1^\cdot \xleftarrow{\delta_1} \cdots\right)$ with $B_l^\cdot \simeq A_l^\cdot$ via the following procedure. Let $B_0^\cdot = A_0^\cdot$; to define $B_l^\cdot$ for $l \geqslant 1$, consider the distinguished triangle

$$A_l^\cdot \xrightarrow{f_{l-1}} A_{l-1}^\cdot \xrightarrow{\iota_{l-1}} \text{cone}(f_{l-1}). \tag{4.3.2}$$

Using the diagram

$$
\begin{array}{ccccccc}
A_{l-1}[1]^\cdot & \longrightarrow & \mathrm{cone}(f_{l-1})[1] & \longrightarrow & A^\cdot_l & \xrightarrow{f_{l-1}} & A^\cdot_{l-1} \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\
A_{l-1}[1]^\cdot & \longrightarrow & C^\cdot_{l-1} & \longrightarrow & A^\cdot_l & \longrightarrow & A^\cdot_{l-1} \\
\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \simeq} \\
B_{l-1}[1]^\cdot & \xrightarrow{j_{l-1}} & C^\cdot_{l-1} & \longrightarrow & A^\cdot_l & \longrightarrow & B^\cdot_{l-1} \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle =} \\
B_{l-1}[1]^\cdot & \xrightarrow{j_{l-1}} & C^\cdot_{l-1} & \longrightarrow & \mathrm{cone}(j_{l-1}) & \xrightarrow{\delta_{l-1}} & B^\cdot_{l-1}
\end{array}
$$

whose first row is obtained from rotating (4.3.2), we define $B^\cdot_l = \mathrm{cone}(j_{l-1})$. Note that indeed $B^\cdot_l \simeq A^\cdot_l$ and under these homotopies we have the commutative diagram

$$
\begin{array}{ccc}
A^\cdot_l & \xrightarrow{f_{l-1}} & A^\cdot_{l-1} \\
\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\
B^\cdot_l & \xrightarrow{\delta_{l-1}} & B^\cdot_{l-1}
\end{array}
\quad . \tag{4.3.3}
$$

Consider now the inverse system $\mathbf{B} = \left(B^\cdot_0 \xleftarrow{\delta_0} B^\cdot_1 \xleftarrow{\delta_1} \cdots\right)$ whose objects fit into distinguished triangles

$$B_l[1]^\cdot \xrightarrow{j_l} C^\cdot_l \longrightarrow B^\cdot_{l+1} \xrightarrow{\delta_l} B^\cdot_l.$$

Since $C^\cdot_l$ is $\mathrm{O}^h(m_l)$, Proposition 4.2.3 gives that $|\delta_l|_{\cong} \geqslant m_l - 1$; this in turn implies that $\mathbf{B}$ is stabilizing so it has a Kom-limit $B^\cdot = \lim_{\mathrm{Kom}} \mathbf{B}$.

We now see that $B^\cdot$ gives a K-limit for $\mathbf{A}$. The commutative square 4.3.3 together with the fact that $B^\cdot$ is a Kom-limit gives the diagram

$$
\begin{array}{ccccccc}
 & & & B^\cdot & & & \\
 & \swarrow{\scriptstyle \tilde{\delta}_0} & \swarrow{\scriptstyle \tilde{\delta}_1} & \big\downarrow{\scriptstyle \tilde{\delta}_2} & \searrow & & \\
B^\cdot_0 & \xleftarrow[\delta_0]{} & B^\cdot_1 & \xleftarrow[\delta_1]{} & B^\cdot_2 & \longleftarrow & \cdots \\
{\scriptstyle \simeq}\big\downarrow{\scriptstyle \alpha_0} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \alpha_1} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \alpha_2} & & \\
A^\cdot_0 & \xleftarrow[f_0]{} & A^\cdot_1 & \xleftarrow[f_1]{} & A^\cdot_2 & \longleftarrow & \cdots
\end{array}
\tag{4.3.4}
$$

where $\lim_{l\to\infty} \left|\tilde{\delta}_l\right|_{\cong} = \infty$ (and where the squares commute up to homotopy). Since α_l is a homotopy equivalence, $\operatorname{cone}(\alpha_l \circ \tilde{\delta}_l) \simeq \operatorname{cone}(\tilde{\delta}_l)$ which implies $\left|\operatorname{cone}(\alpha_l \circ \tilde{\delta}_l)\right|_h = \left|\operatorname{cone}(\tilde{\delta}_l)\right|_h$. Define $\tilde{f}_l : B^{\cdot} \to A_l^{\cdot}$ via $\tilde{f}_l = \alpha_l \circ \tilde{\delta}_l$; Gaussian elimination of complexes implies that $\left|\operatorname{cone}(\tilde{f}_l)\right|_h \geqslant \left|\tilde{f}_l\right|_{\cong}$ so we have

$$\begin{aligned}\lim_{l\to\infty} \left|\operatorname{cone}(\tilde{f}_l)\right|_h &\geqslant \lim_{l\to\infty} \left|\tilde{\delta}_l\right|_{\cong} \\ &= \infty \ .\end{aligned}$$

Since $f_l \circ \tilde{f}_{l+1} \simeq \tilde{f}_l$ this shows that $B^{\cdot}$ is a K-limit for **A**.

Now, suppose that **A** has a K-limit. Equation (4.3.1) and Lemma 4.3.3 give that

$$|\operatorname{cone}(f_l)|_h \geqslant \min\left\{\left|\operatorname{cone}(\tilde{f}_l)\right|_h, \left|\operatorname{cone}(\tilde{f}_{l+1})\right|_h - 1\right\}$$

so

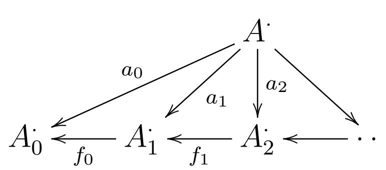
$$\lim_{l\to\infty} |\operatorname{cone}(f_l)|_h = \infty$$

showing that **A** is Cauchy. □

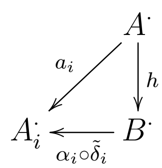
Similar to the case of limits in $\operatorname{Kom}(\mathcal{A})$, there is also a uniqueness statement; however, we need a few preparatory results before giving its proof.

The next result (which will be used in the proof of uniqueness) will later pair with the uniqueness result to show that K-limits actually give categorical limits.

Lemma 4.3.5. *Let* **A** *be a Cauchy inverse system and let* $B^{\cdot}$ *be the* K*-limit constructed in the proof of Theorem 4.3.4, then if* $A^{\cdot}$ *is a complex and there are maps* $A^{\cdot} \xrightarrow{a_i} A_i^{\cdot}$ *so that the diagram*

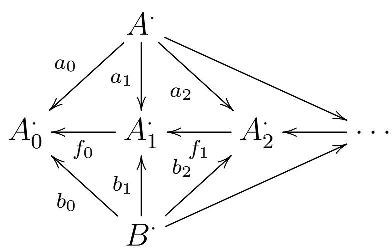


commutes in $\mathrm{K}(\mathcal{A})$*, then there exists a map* $A^{\cdot} \xrightarrow{h} B^{\cdot}$ *so that the triangles*

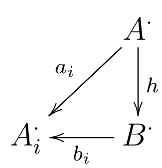


commute in $\mathrm{K}(\mathcal{A})$.

Proof. Consider the diagram

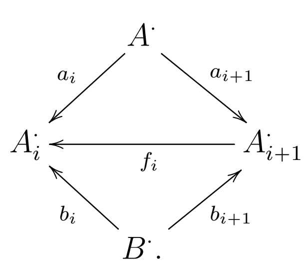


where $b_i = \alpha_i \circ \tilde{\delta}_i$ (in the notation from the proof of Theorem 4.3.4). Using (4.3.4) we can assume that $b_i = f_i \circ b_{i+1}$ and by throwing out terms and re-indexing we can suppose that $|b_i|_{\cong} \geqslant i$. It suffices to show that such an $h : A \to B$ exists with the (remaining) triangles

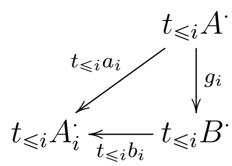


commuting up to homotopy.

To this end, consider the sub-diagrams given by



We begin by constructing maps $g_i : t_{\leqslant i}A^{\cdot} \to t_{\leqslant i}B^{\cdot}$ so that g_{i+1} is homotopic to a map $\hat{g}_{i+1}$ with $t_{\leqslant j}\hat{g}_{i+1} = t_{\leqslant j}g_i$ for $j < i$ and so that the triangles



commute. Define $g_i = (t_{\leqslant i}b_i)^{-1} \circ t_{\leqslant i}a_i$. Since $a_i \simeq f_i \circ a_{i+1}$ there exist maps H_i^k so that

$$a_i^k = f_i^k \circ a_{i+1}^k + d_{A_i} \circ H_i^{k-1} + H_i^k \circ d_A. \tag{4.3.5}$$

Consider the map $\hat{g}_{i+1} : t_{\leqslant i+1}A^{\cdot} \to t_{\leqslant i+1}B^{\cdot}$ defined in homological degree k by

$$\hat{g}_{i+1}^k = \begin{cases} g_{i+1}^{i+1} & \text{if} \quad k = i+1 \\ g_{i+1}^i + d_B \circ (b_i^{i-1})^{-1} \circ H_i^{i-1} & \text{if} \quad k = i \\ g_{i+1}^k + d_B \circ (b_i^{k-1})^{-1} \circ H_i^{k-1} + (b_i^k)^{-1} \circ H_i^k \circ d_A & \text{if} \quad k < i \end{cases}$$

which is a chain map homotopic to g_{i+1}. Note that for $k < i$

$$\begin{aligned} \hat{g}_{i+1}^k &= g_{i+1}^k + d_B \circ (b_i^{k-1})^{-1} \circ H_i^{k-1} + (b_i^k)^{-1} \circ H_i^k \circ d_A \\ &= (b_i^k)^{-1} \circ f_i^k \circ a_{i+1}^k + d_B \circ (b_i^{k-1})^{-1} \circ H_i^{k-1} + (b_i^k)^{-1} \circ H_i^k \circ d_A \\ &= (b_i^k)^{-1} \circ \left(f_i^k \circ a_{i+1}^k + d_{A_i} \circ H_i^{k-1} + H_i^k \circ d_A\right) \\ &= (b_i^k)^{-1} \circ a_i^k \\ &= g_i^k \end{aligned}$$

so the g_i have the desired properties.

We now define maps $h_i : t_{\leqslant i}A^{\cdot} \to t_{\leqslant i}B^{\cdot}$ with $h_i \simeq g_i$ and so that $t_{\leqslant j}h_{i+1}$ agrees with $t_{\leqslant j}h_i$ for all $j < i$ as follows. Let $h_0 = g_0$ and $h_1 = \hat{g}_1 \simeq g_1$; we will construct h_{i+1} assuming that we have constructed $h_0, \dots, h_i$. Since $h_i \simeq g_i$ there exist maps G_i^k so that

$$\begin{aligned} h_i^i &= g_i^i + d_B \circ G_i^{i-1} \\ h_i^k &= g_i^k + d_B \circ G_i^{k-1} + G_i^k \circ d_A \quad \text{if} \quad k < i. \end{aligned}$$

Define h_{i+1} via

$$h_{i+1}^k = \begin{cases} g_{i+1}^{i+1} & \text{if} \quad k = i+1 \\ g_{i+1}^{i} + d_B \circ \big((b_i^{i-1})^{-1} \circ H_i^{i-1} + G_i^{i-1}\big) & \text{if} \quad k = i \\ g_{i+1}^{k} + d_B \circ \big((b_i^{k-1})^{-1} \circ H_i^{k-1} + G_i^{k-1}\big) & \\ \qquad + \big((b_i^k)^{-1} \circ H_i^k + G_i^k\big) \circ d_A & \text{if} \quad k < i \end{cases}$$

and observe that $h_{i+1} \simeq g_{i+1}$ (and that in fact we take $G_i^{i-1} = 0$). We also compute

$$\begin{aligned} h_{i+1}^k &= g_{i+1}^k + d_B \circ \big((b_i^{k-1})^{-1} \circ H_i^{k-1} + G_i^{k-1}\big) + \big((b_i^k)^{-1} \circ H_i^k + G_i^k\big) \circ d_A \\ &= \hat{g}_{i+1}^k + d_B \circ G_i^{k-1} + G_i^k \circ d_A \\ &= g_i^k + d_B \circ G_i^{k-1} + G_i^k \circ d_A \\ &= h_i^k \end{aligned}$$

for $k < i$, so the h_i have the desired properties.

Finally, let $h : A^{\cdot} \to B^{\cdot}$ be defined as the stable limit of the maps h_i, ie $h^k = h_i^k$ for any $i > k$. It remains to check that $b_i \circ h \simeq a_i$ for all i. Observe first that we have the equalities

$$\begin{aligned} h^k &= \begin{cases} h_{k+1}^k & \text{if} \quad k \geqslant i \\ h_i^k & \text{if} \quad k < i \end{cases} \\ &= \begin{cases} g_{k+1}^k + d_B \circ (b_k^{k-1})^{-1} \circ H_k^{k-1} & \text{if} \quad k \geqslant i \\ g_i^k + d_B \circ G_i^{k-1} + G_i^k \circ d_A & \text{if} \quad k < i \end{cases} \end{aligned}$$

and so

$$\begin{aligned} (b_i \circ h)^k &= \begin{cases} b_i^k \circ g_{k+1}^k + b_i^k \circ d_B \circ (b_k^{k-1})^{-1} \circ H_k^{k-1} & \text{if} \quad k \geqslant i \\ b_i^k \circ g_i^k + b_i^k \circ d_B \circ G_i^{k-1} + b_i^k \circ G_i^k \circ d_A & \text{if} \quad k < i \end{cases} \\ &= \begin{cases} b_i^k \circ (b_{k+1}^k)^{-1} \circ a_{k+1}^k & \\ \qquad + d_{A_i} \circ b_i^{k-1} \circ (b_k^{k-1})^{-1} \circ H_k^{k-1} & \text{if} \quad k \geqslant i \\ b_i^k \circ (b_i^k)^{-1} \circ a_i^k & \\ \qquad + d_{A_i} \circ b_i^{k-1} \circ G_i^{k-1} + b_i^k \circ G_i^k \circ d_A & \text{if} \quad k < i \end{cases} \\ &= \begin{cases} f_i^k \circ \cdots \circ f_k^k \circ a_{k+1}^k & \\ \qquad + d_{A_i} \circ f_i^{k-1} \circ \cdots \circ f_{k-1}^{k-1} \circ H_k^{k-1} & \text{if} \quad k \geqslant i \\ a_i^k + d_{A_i} \circ b_i^{k-1} \circ G_i^{k-1} + b_i^k \circ G_i^k \circ d_A & \text{if} \quad k < i. \end{cases} \end{aligned}$$

Using (4.3.5) we compute

$$f_i^k \circ \cdots \circ f_k^k \circ a_{k+1}^k = a_i^k - d_{A_i} \circ \big(H_i^{k-1} + f_i^{k-1} \circ H_{i+1}^{k-1} + \cdots$$
$$+ f_i^{k-1} \circ \cdots \circ f_{k-1}^{k-1} \circ H_k^{k-1}\big) - \big(H_i^k + f_i^k \circ H_{i+1}^k + \cdots$$
$$+ f_i^k \circ \cdots \circ f_{k-1}^k \circ H_k^k\big) \circ d_A$$

and

$$(b_i \circ h)^k = \begin{cases} a_i^k - d_{A_i} \circ \big(H_i^{k-1} + f_i^{k-1} \circ H_{i+1}^{k-1} + \cdots \\ \quad + f_i^{k-1} \circ \cdots \circ f_{k-2}^{k-1} \circ H_{k-1}^{k-1}\big) \\ \quad - \big(H_i^k + f_i^k \circ H_{i+1}^k + \cdots \\ \quad + f_i^k \circ \cdots \circ f_{k-1}^k \circ H_k^k\big) \circ d_A & \text{if} \quad k \geqslant i+2 \\ a_i^{i+1} - d_{A_i} \circ H_i^i - \big(H_i^{i+1} + f_i^{i+1} \circ H_{i+1}^{i+1}\big) \circ d_A & \text{if} \quad k = i+1 \\ a_i^i - H_i^i \circ d_A & \text{if} \quad k = i \\ a_i^{i-1} + d_{A_i} \circ b_i^{i-2} \circ G_i^{i-2} & \text{if} \quad k = i-1 \\ a_i^k + d_{A_i} \circ b_i^{k-1} \circ G_i^{k-1} + b_i^k \circ G_i^k \circ d_A & \text{if} \quad k \leqslant i-2 \end{cases}$$

from which it is evident that $b_i \circ h \simeq a_i$. □

The next result shows that if a complex is 'infinitely' homologically negligible, then it is contractible.

Lemma 4.3.6. *If $|A^{\cdot}|_h = \infty$ then $A^{\cdot}$ is contractible.*

The following proof is taken from [22].

Proof. Since $|A^{\cdot}|_h = \infty$ we have that $A^{\cdot} \simeq A_i^{\cdot}$ for complexes $A_i^{\cdot}$ which are $\mathrm{O}^h(m_i)$ with $\lim_{i\to\infty} m_i = \infty$. We have the following diagram in which each map is a homotopy equivalence

$$A^{\cdot} \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1^{\cdot} \rightleftarrows \cdots \rightleftarrows A_i^{\cdot} \underset{g_i}{\overset{f_i}{\rightleftarrows}} A_{i+1}^{\cdot} \rightleftarrows \cdots \quad ;$$

in particular $\mathrm{id}_{A_i} - g_i \circ f_i = d_{A_i} \circ H_i + H_i \circ d_{A_i}$ where the H_i are chain homotopies (and $A_0^{\cdot} = A^{\cdot}$).

Now, define maps $\tilde{f}_i = f_i \circ \cdots \circ f_0$, $\tilde{g}_i = g_0 \circ \cdots \circ g_i$, and $\tilde{H}_i = H_0 + \tilde{g}_0 \circ H_1 \circ \tilde{f}_0 + \cdots + \tilde{g}_{i-1} \circ H_i \circ \tilde{f}_{i-1}$ which are related by

$$\mathrm{id}_A - \tilde{g}_i \circ \tilde{f}_i = d_A \circ \tilde{H}_i + \tilde{H}_i \circ d_A. \tag{4.3.6}$$

The equality $\lim_{i\to\infty} m_i = \infty$ implies that the maps $\tilde{H}_i$ stabilize in each homological degree as $i \to \infty$, so we can define their stable limit $\tilde{H}$. Equation (4.3.6) stabilizes as well (for the same reason) to give

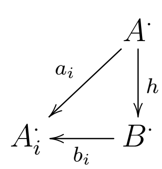
$$\mathrm{id}_A = d_A \circ \tilde{H} + \tilde{H} \circ d_A$$

which shows that $A^{\cdot}$ is contractible. □

We can now prove a uniqueness result concerning limits in $\mathrm{K}(\mathcal{A})$.

Theorem 4.3.7. *The limit of a Cauchy sequence* $\mathbf{A}$ *is unique up to homotopy equivalence.*

Proof. Since $\mathbf{A}$ is Cauchy we can construct the limit $B^{\cdot}$ as in the proof of Theorem 4.3.4. If $A^{\cdot}$ is another K-limit then Lemma 4.3.5 gives a map $A^{\cdot} \xrightarrow{h} B^{\cdot}$ so that the triangles



commute (up to homotopy) for all i.

Lemma 4.3.3 gives that

$$|\mathrm{cone}(h)|_h \geqslant \min\{|\mathrm{cone}(b_i)|_h + 1, |\mathrm{cone}(a_i)|_h\}$$

for all i. Taking the limit of the right hand side as $i \to \infty$ and using the fact that both $A^{\cdot}$ and $B^{\cdot}$ are K-limits, we find $|\mathrm{cone}(h)|_h = \infty$. Lemma 4.3.6 then shows that $\mathrm{cone}(h)$ is contractible, which is equivalent to h being a homotopy equivalence. □

We conclude this section with two easy results concerning K-limits.

Proposition 4.3.8. *If* $\mathbf{A} = \left(A^{\cdot}_0 \xleftarrow{f_0} A^{\cdot}_1 \xleftarrow{f_1} \cdots \right)$ *is a Cauchy system and* $\lim_{l\to\infty} |A^{\cdot}_l|_h = \infty$ *then* $\lim_{\mathrm{K}} \mathbf{A} \simeq 0$.

Proof. Let $\tilde{f}_l = \left(0 \longrightarrow A^{\cdot}_l \right)$. □

Proposition 4.3.9. *If* $\mathbf{A} = \left(A^{\cdot}_0 \xleftarrow{f_0} A^{\cdot}_1 \xleftarrow{f_1} \cdots \right)$ *is a Cauchy system in* $\mathrm{K}(\mathcal{A})$ *and* $\mathcal{A}$ *is a monoidal category then*

$$B^{\cdot} \otimes \mathbf{A} = \left(B^{\cdot} \otimes A^{\cdot}_0 \xleftarrow{id\otimes f_0} B^{\cdot} \otimes A^{\cdot}_1 \xleftarrow{id\otimes f_1} \cdots \right)$$

is a Cauchy system and $\lim_{\mathrm{K}} (B^{\cdot} \otimes \mathbf{A}) \simeq B^{\cdot} \otimes \lim_{\mathrm{K}} \mathbf{A}$.

Proof. Consider the maps $\lim_{\mathrm{K}} \mathbf{A} \xrightarrow{\tilde{f}_l} A^{\cdot}_l$ which satisfy

$$\lim_{l\to\infty} \left|\mathrm{cone}(\tilde{f}_l)\right|_h = \infty.$$

These give maps $B^{\cdot} \otimes \lim_{\mathrm{K}} \mathbf{A} \xrightarrow{id\otimes\tilde{f}_l} B^{\cdot} \otimes A^{\cdot}_l$ which also satisfy

$$\lim_{l\to\infty} \left|\mathrm{cone}(id \otimes \tilde{f}_l)\right|_h = \infty$$

since $\mathrm{cone}(id \otimes \tilde{f}_l) = B^{\cdot} \otimes \mathrm{cone}(\tilde{f}_l)$. The result follows since K-limits are unique up to homotopy. □

5

Categorified $\mathfrak{sl}_3$ projectors

In this chapter we construct the categorified projectors and prove Theorems 3.0.1 and 3.0.2. Section 5.1 contains the construction of $\tilde{P}_w$ for $w = (+\cdots+)$; in Section 5.2 we show that in this case $\chi(\tilde{P}_w) = P_w$, where χ is the map sending a complex to the alternating sum of (the equivalence classes of) its terms. The case $w = (+\cdots+-\cdots-)$ is treated in Section 5.3 and the results for general w are given in Section 5.4.

5.1 $\tilde{P}_w$ for $w = (+\cdots+)$

We begin by constructing the categorified projectors $\tilde{P}_w$ and giving a proof of Theorem 3.0.1 when $w = (+\cdots+)$. The general case differs from this one only in the technical details.

We will refer to the process of applying equations (2.6.8), (2.6.9), and (2.6.10) to express a web in terms of the direct sum of webs with fewer digon, square, and circular faces as reduction. Recall that a web which has no digon, square, or circular faces is called non-elliptic and that any web can be reduced to a direct sum of non-elliptic webs. When we write the complex $[\![D]\!]$ for a tangle diagram D we will assume that we have reduced all webs appearing to direct sums of non-elliptic webs. If we would like to consider the complex with terms unreduced we will denote it by $[\![D]\!]^{un}$.

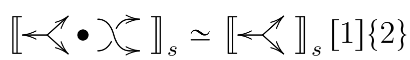
A shifted version of $\mathfrak{sl}_3$ knot homology will be useful for our considerations. Given a tangle diagram D, define the shifted complex by

$$[\![D]\!]_s = [\![D]\!]\,[c_-]\{3c_- - 2c_+\}$$

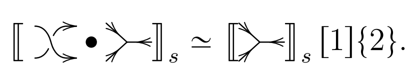
where $c_\pm$ is the number of $\pm$ crossings in D; the complex $[\![D]\!]_s^{un}$ is defined similarly. This complex is not an invariant of the tangle corresponding to D as it acquires shifts in both homological and quantum degree under $R1$ and $R2$ Reidemeister moves (but is invariant under $R3$). Nevertheless, this shifting convention will prove useful. In particular, for any diagram D the shifted complex $[\![D]\!]_s$ is supported in non-negative homological degree.

We begin with a basic result describing the complex assigned to a Y-web attached to a positive crossing.

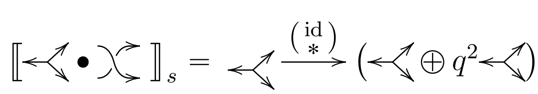
Lemma 5.1.1. *There are homotopy equivalences*



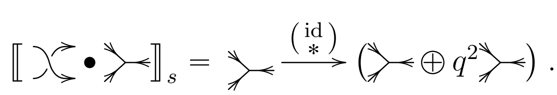
and



Proof. We have



and



The result then follows from Proposition 4.1.1. $\square$

Now consider the complex $\left\llbracket \rangle\!\vdots\!\langle\!\rightarrow_{m} \right\rrbracket_s^{un}$ assigned to the diagram of a full twist on m strands and note that every web W appearing in the complex except the lone identity web in homological degree zero takes the form

$$W = \succ\!\!\!- \bullet W' \bullet -\!\!\!\prec \quad , \tag{5.1.1}$$

where we have omitted multiplicities from the identity strands (we will often do this to simplify notation). Since reduction cannot affect such a decomposition, we find that

$$\begin{aligned} \left\llbracket \rangle\!\vdots\!\langle_{m} \right\rrbracket_s &= (\rightarrow m) \xrightarrow{z} C^1 \longrightarrow C^2 \longrightarrow \cdots \\ &= \operatorname{cone}\Big(\llbracket \rightarrow m \rrbracket_s \xrightarrow{z} C[-1]^{\cdot} \Big)[1] \end{aligned}$$

where every web appearing in each C^i is of the form (5.1.1) and non-elliptic (by definition we take $C^i = 0$ for $i \leqslant 0$). We thus have the distinguished triangle

$$\llbracket \rightarrow m \rrbracket_s \xrightarrow{z} C[-1]^{\cdot} \longrightarrow \left\llbracket \rangle\!\vdots\!\langle_{m} \right\rrbracket_s [-1] \xrightarrow{g[-1]} \llbracket \rightarrow m \rrbracket_s [-1]$$

where the map g is the identity in homological degree zero and zero in all other homological degrees. This implies that there is a homotopy equivalence

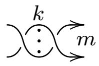
$$C[-1]^{\cdot} \simeq \operatorname{cone}\Big(\left\llbracket \rangle\!\vdots\!\langle_{m} \right\rrbracket_s \xrightarrow{g} \llbracket \rightarrow m \rrbracket_s \Big).$$

We now consider the inverse system

$$\mathbf{T}_w = \llbracket \rightarrow m \rrbracket_s \xleftarrow{g_0} \left\llbracket \rangle\!\vdots\!\langle_{m} \right\rrbracket_s \xleftarrow{g_1} \left\llbracket \rangle\!\overset{2}{\vdots}\!\langle_{m} \right\rrbracket_s \longleftarrow \cdots \tag{5.1.2}$$

for $w = \underbrace{(+\cdots+)}_{m}$ where g_k is defined as

$$\left\llbracket \rangle\!\vdots\!\langle_{m} \right\rrbracket_s \bullet \left\llbracket \rangle\!\overset{k}{\vdots}\!\langle_{m} \right\rrbracket_s \xrightarrow{\quad g \bullet \mathrm{id} \quad} \llbracket \rightarrow m \rrbracket_s \bullet \left\llbracket \rangle\!\overset{k}{\vdots}\!\langle_{m} \right\rrbracket_s .$$

Here  denotes k full twists on m strands.

Proposition 5.1.2. *The inverse system* $\mathbf{T}_w$ *is Cauchy.*

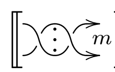
Proof. We inductively construct complexes $C_k^{\cdot}$ satisfying the following conditions.

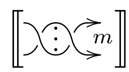
1. $C_k[-1]^{\cdot} \simeq \operatorname{cone}(g_k)$.

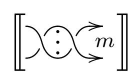
2. $C_k^{\cdot}$ is $\mathrm{O}^h(2k+1)$.

3. Every web appearing in $C_k^{\cdot}$ takes the form (5.1.1).

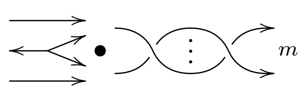
Let $C_0^{\cdot} = C^{\cdot}$ and suppose we have constructed $C_0^{\cdot}, \ldots, C_{k-1}^{\cdot}$ as above. Since Proposition 4.1.4 gives that

$$\operatorname{cone}(g_k) = \operatorname{cone}(g_{k-1}) \bullet \left[\!\left[\quad \right]\!\right]_s$$


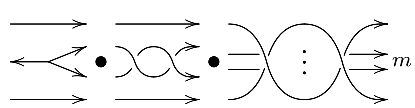
$$\simeq C_{k-1}[-1]^{\cdot} \bullet \left[\!\left[\quad \right]\!\right]_s$$


so we must show that $C_{k-1}^{\cdot} \bullet \left[\!\left[\quad \right]\!\right]_s$  is homotopy equivalent to a complex satisfying the second and third of the above conditions.

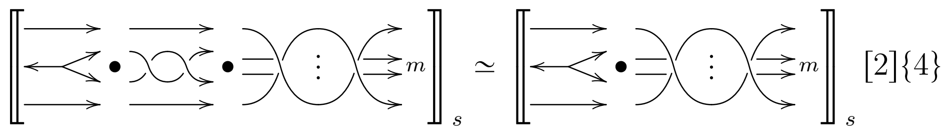
To this end, consider the tangled web



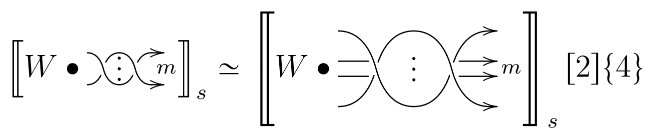
Using Reidemeister 3 moves, we can pull the crossings on the two strands aligning with the Y-web through so that they take place before any other crossings, giving the tangled web



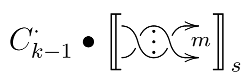
The rightmost tangle above is the result after pulling the crossings on the two strands through and out of the twist; we shall denote such a tangle in this way for the duration. Lemma 5.1.1 gives the homotopy equivalence



so we have homotopy equivalences

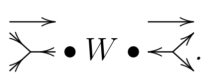

(5.1.3)

for each web W appearing in $C^{\cdot}_{k-1}$. Proposition 4.1.3 now gives that



is homotopy equivalent to a complex which is $\mathrm{O}^h(2k+1)$ and whose terms come from complexes taking the same form as the right side of equation (5.1.3); define $C^{\cdot}_k$ to be this complex. Since all such webs take the form (5.1.1), the result follows. □

Since $\mathbf{T}_w$ is Cauchy, Theorem 4.3.4 implies that $\lim_{\mathrm{K}} \mathbf{T}_w$ exists. Let $\tilde{P}_w$ denote the limiting complex for $\mathbf{T}_w$ explicitly constructed using the proof of Theorem 4.3.4. The next result follows from analysis of the details of that proof.

Proposition 5.1.3. *Let $w = (+ \cdots +)$. The web $id_w = (\rightarrow m)$ appears only once in $\tilde{P}_w$ and does so in quantum and homological degree zero. All other webs appearing in $\tilde{P}_w$ take the form* .

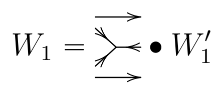
Proof. The limit of the Cauchy sequence is given as the Kom-limit of the stabilizing system **B** constructed in the proof of Theorem 4.3.4. In the current case, we see that

$$B_0^{\cdot} = [\![\rightarrow m]\!]_s$$
$$B_1^{\cdot} = \mathrm{cone}\,([\![\rightarrow m]\!]_s[1] \to C_0^{\cdot})$$
$$\vdots$$
$$B_k^{\cdot} = \mathrm{cone}\left(B_{k-1}^{\cdot}[1] \to C_{k-1}^{\cdot}\right).$$

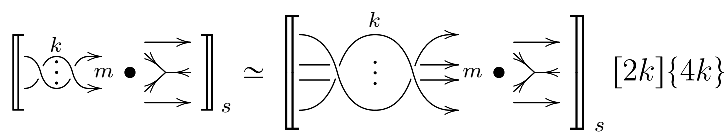
The result now follows from our description of the complexes $C_k^{\cdot}$ above. $\square$

Proposition 5.1.4. *Let* $w = (+\cdots+)$*. If* $\mathrm{wt}(v) < \mathrm{wt}(w)$ *then* $\tilde{P}_w \bullet W_1 \simeq 0$ *for any* $W_1 \in \mathrm{Hom}_\bullet(w, v)$ *and* $W_2 \bullet \tilde{P}_w \simeq 0$ *for any* $W_2 \in \mathrm{Hom}_\bullet(v, w)$*.*

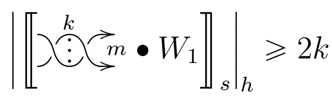
Proof. Let $W_1 \in \mathrm{Hom}_\bullet(w, v)$ and note that it suffices to consider the case when W_1 is non-elliptic. Proposition 2.3.5 then implies that



for some W_1'. Lemma 5.1.1 gives the homotopy equivalence



so we have



Propositions 4.3.8 and 4.3.9 now give

$$\tilde{P}_w \bullet W_1 = \left(\lim_{\mathrm{K}} \mathbf{T}_w\right) \bullet W_1$$
$$\simeq \lim_{\mathrm{K}} (\mathbf{T}_w \bullet W_1)$$
$$\simeq 0.$$

The proof concerning W_2 is completely analogous. $\square$

Proposition 5.1.5. *Let $w = (+\cdots+)$, then*

$$\tilde{P}_w \bullet \tilde{P}_w \simeq \tilde{P}_w.$$

Proof. Proposition 5.1.3 gives that

$$\tilde{P}_w = \Big((\to m) \xrightarrow{z} D^1 \longrightarrow D^2 \longrightarrow \cdots \Big)$$

where each web appearing in D^i takes the form (5.1.1). Setting $D^i = 0$ for $i \leqslant 0$ we have

$$\tilde{P}_w = \mathrm{cone}\,(\llbracket \to m \rrbracket_s \to D[-1]^{\cdot})\,[1]$$

which gives the distinguished triangle

$$\tilde{P}_w \longrightarrow \llbracket \to m \rrbracket_s \longrightarrow D[-1]^{\cdot}\ .$$

This in turn gives the distinguished triangle

$$\tilde{P}_w \bullet \tilde{P}_w \longrightarrow \tilde{P}_w \longrightarrow D[-1]^{\cdot} \bullet \tilde{P}_w$$

so by Proposition 2.6.2 it suffices to show that $D^{\cdot} \bullet \tilde{P}_w \simeq 0$.

We can write

$$D^{\cdot} = \mathrm{cone}\left(D^1[2] \to t_{\geqslant 2} D^{\cdot}\right)$$

where D^1 stands for the complex with all terms zero except D^1 sitting in homological degree zero and $t_{\geqslant k} D^{\cdot}$ denotes the truncation of $D^{\cdot}$ from below. As above, this gives the distinguished triangle

$$t_{\geqslant 2} D^{\cdot} \bullet \tilde{P}_w \longrightarrow D^{\cdot} \bullet \tilde{P}_w \longrightarrow D^1[1] \bullet \tilde{P}_w\ .$$

Proposition 5.1.4 implies that $D^1[1] \bullet \tilde{P}_w \simeq 0$ so $t_{\geqslant 2} D^{\cdot} \bullet \tilde{P}_w \simeq D^{\cdot} \bullet \tilde{P}_w$. Repeating this procedure gives

$$t_{\geqslant k} D^{\cdot} \bullet \tilde{P}_w \simeq D^{\cdot} \bullet \tilde{P}_w$$

for all $k > 0$. Lemma 4.3.6 then implies that $D^{\cdot} \bullet \tilde{P}_w \simeq 0$. □

Propositions 5.1.3, 5.1.4, and 5.1.5 give the proof of Theorem 3.0.1 when $w = (+\cdots+)$.

5.2 Decategorification for $w = (+\cdots+)$

We now aim to show that $\chi(\tilde{P}_w) = P_w$ when $w = (+\cdots+)$. Doing so requires several steps. We must show that it is possible to define and compute $\chi(\tilde{P}_w)$ since the graded Euler characteristic of an unbounded complex is not generally well-defined. Since P_w is an element in $\mathcal{S}$ (which is defined over $\mathbb{C}(q)$) and Euler characteristics of complexes are 'integral' objects, we must also explain the context for the above equality. Finally, we must prove this equality.

To resolve the first issue, we consider a full subcategory of $\mathrm{K}^+(\mathcal{F})$ in which we restrict the support of complexes $A^\cdot$. By definition, $\mathrm{supp}(A^\cdot)$ is the set of pairs $(h, l) \in \mathbb{Z}^2$ for which A^h has a non-zero summand in q-degree q^l. We shall identify $\mathrm{supp}(A^\cdot)$ with the corresponding discrete subset in $\mathbb{R}^2$; by a slight abuse of notation we shall call this latter set the (h, q)-plane.

If we wish to translate a subset of the (h, q)-plane, we will use the same notation which we use to shift complexes, viewing homological degree as the first (horizontal) coordinate and q-degree as the second (vertical) coordinate. For instance,

$$\{(h, q)|h \geqslant 2 \text{ and } q \geqslant 1\}[2]\{1\} = \{(h, q)|h \geqslant 4 \text{ and } q \geqslant 2\}.$$

Now consider the subset of $\mathbb{R}^2$ given by

$$R_t = \{(h, q) \in \mathbb{R}^2 | h \geqslant 0 \text{ and } q \geqslant t \cdot h\}$$

and let $\hat{\mathcal{S}}_{int}$ denote the $\mathfrak{sl}_3$ spider considered over the ring $\mathbb{Z}[q^{-1}, q]] := \mathbb{Z}[[q]][\frac{1}{q}]$, i.e. $\hat{\mathcal{S}}_{int} = \mathbb{Z}[q^{-1}, q]] \otimes_{\mathbb{Z}[q^{-1},q]} \mathcal{S}_{int}$. By Proposition 2.3.2, the following conditions are sufficient to guarantee that the Euler characteristic $\chi(A^\cdot)$ of a complex $A^\cdot$ in $\mathrm{K}^+(\mathcal{F})$ is a well-defined element in $\hat{\mathcal{S}}_{int}$:

1. $\operatorname{supp}(A^{\cdot}) \subset R_t[a]\{b\}$ for some $t > 0$ and $a, b \in \mathbb{Z}$.

2. All webs appearing in $A^{\cdot}$ are non-elliptic.

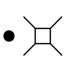
Denote by $\mathrm{K}^{\angle}(\mathcal{F})$ the full subcategory of $\mathrm{K}^{+}(\mathcal{F})$ whose objects satisfy the above conditions. This subcategory is obviously closed under taking direct sums, cones, and tensor products. The horizontal composition of two complexes in $\mathrm{K}^{\angle}(\mathcal{F})$ is isomorphic (in $\mathrm{K}^{+}(\mathcal{F})$) to a complex in $\mathrm{K}^{\angle}(\mathcal{F})$ by reducing each term. In this sense, we can view $\mathrm{K}^{\angle}(\mathcal{F})$ as closed under horizontal composition.

Our interest in $\mathrm{K}^{\angle}(\mathcal{F})$ stems from the fact that the $\mathfrak{sl}_3$ projectors can be viewed as elements of $\hat{\mathcal{S}}_{int}$. Indeed, we have an inclusion[1] $\mathcal{S} \hookrightarrow \hat{\mathcal{S}} = \mathbb{C}[q^{-1}, q]] \otimes_{\mathbb{C}(q)} \mathcal{S}$ induced by the inclusion $\mathbb{C}(q) \hookrightarrow \mathbb{C}[q^{-1}, q]]$. Proposition 2.3.3 shows that under this map P_w actually lies in $\hat{\mathcal{S}}_{int}$, so $\mathrm{K}^{\angle}(\mathcal{F})$ is a reasonable setting for our categorified projectors.

We now aim to show that $\tilde{P}_w$ is an object in $\mathrm{K}^{\angle}(\mathcal{F})$. Before doing so, we need some preparatory lemmata. Our first result enables us to bound the q-degree of the webs appearing when we express a web as a direct sum of non-elliptic webs.

Lemma 5.2.1. *Let W be a web with no closed components, then when reducing W to a direct sum of non-elliptic webs we can assume that no closed component forms. If W has r faces, $W \cong \oplus_{i=1}^{s} q^{k_i} \cdot W_i$ is the direct sum decomposition into non-elliptic webs resulting from reduction, and r_i is the number of faces in W_i, then $k_i \geqslant r_i - r$.*

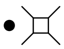
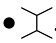
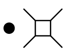
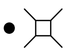
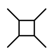
Proof. Suppose that a reduction does produce a closed component. We can assume that no reductions are possible which do not split off a closed component; otherwise, perform these reductions. Since a closed component can only form upon application of equation (2.6.9), we have that

$W = \big(U \bullet$  $\bullet V\big)$

[1] That this is indeed an inclusion can be proved using the pairing from Proposition 5.2.4 below.

and hence the isomorphism

$$W \cong \Big(U \bullet$$  $$\bullet V\Big) \oplus \Big(U \bullet)\, (\bullet V\Big).$$

We can assume that $(\bullet V$ has no internal digon faces. Indeed, if a digon face is formed then we must have $V =$  $\bullet V'$ so we can instead consider the reduction corresponding to $W = U' \bullet$  $\bullet V'$ where $U' = U \bullet$ . Similarly, we can assume that $(\bullet V$ has at most one internal square face since otherwise we have $V = V'' \bullet$  $\bullet V'$ and again we can consider the reduction corresponding to $W = U'' \bullet$  $\bullet V'$ where $U'' = U \bullet$  $\bullet V''$.

We analyze the closed web

$$C = (\bullet V$$

which has at most one internal square face and no internal digon faces. The orientations of edges around vertices shows that all faces must have an even number of edges. Considering the web on the surface of the 2-sphere creates an external face which may have any (even) number of edges. Since every edge borders two regions we compute

$$e_C \geqslant \frac{1}{2}(6(f_C - 2) + 4 + 2) = 3(f_C - 1)$$

where e_C is the number of edges in C and f_C is the number of faces bounded on the 2-sphere by C. Since C is trivalent we have $v_C = \frac{2}{3}e_C$ where v_C is the number of vertices in C. We thus find

$$\begin{aligned} 2 &= f_C - e_C + v_C \\ &= f_C - \frac{1}{3}e_C \\ &\leqslant f_C + (1 - f_C) = 1, \end{aligned}$$

a contradiction.

The second statement follows from the first by noticing that each reduction lowers the number of internal faces and that (2.6.10) need never be used. □

Given a diagram D, define the 0-resolution as the unique web appearing in $[\![D]\!]_s^{un}$ in homological degree zero. Concretely, this is the web obtained by taking the smooth resolution $)\,($ of each positive crossing and the singular resolution $\asymp$ of each negative crossing. Define the smooth resolution of D to be the web obtained by taking the smooth resolution of both positive and negative crossings.

Lemma 5.2.2. *Let D be a tangle diagram and let the smooth resolution of D have no closed components. Let r be the number of internal faces in the 0-resolution of D and c_+ be the number of positive crossings in D, then the complex $[\![D]\!]_s$ satisfies*

$$\operatorname{supp}\left([\![D]\!]_s\right) \subset R_{1/c_+}\{-r-1\}.$$

If a web W appearing in $[\![D]\!]_s$ has r_W internal faces then that term is supported in $R_{1/c_+}\{r_W - r - 1\}$. Moreover, if D has no negative crossings then the -1 shifts can be omitted from both statements.

Proof. Since any web in $[\![D]\!]_s^{un}$ is obtained from the smooth resolution by switching smoothings of crossings to singular resolutions, no web appearing in this complex has closed components. Also, note that changing the resolution of a crossing from the smooth resolution to the singular resolution produces at most one new internal face while changing from the singular resolution to the smooth resolution cannot produce new internal faces.

It follows that a web V appearing in $[\![D]\!]_s^{un}$ in homological degree h has at most $r + \min(h, c_+)$ internal faces. Next, note that the complex $[\![D]\!]_s^{un}$ is supported along the line $h = q$ in the (h, q)-plane. Hence by Lemma 5.2.1, if $q^l W$ appears in $[\![D]\!]_s$ in homological degree h then

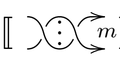
$$\begin{aligned} l &\geqslant h + r_W - r - \min(h, c_+) \\ &\geqslant r_W - r - 1 + \frac{h}{c_+} \end{aligned}$$

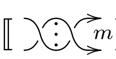
which gives the result.

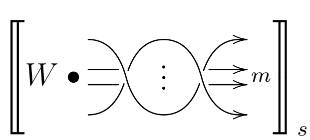
For the final statement, note that if D has no negative crossing then the 0-resolution and the smooth resolution agree. All webs appearing in $[\![D]\!]_s^{un}$ are thus obtained from the smooth resolution by changing a crossing to the singular resolution. Since no internal faces are formed when the first resolution is changed, we find that a web V appearing in $[\![D]\!]_s^{un}$ in homological degree $h > 0$ has at most $r + \min(h, c_+) - 1$ internal faces. The result then follows as above. □

Proposition 5.2.3. *Let $w = (+ \cdots +)$. The categorified projector $\tilde{P}_w$ lies in $\mathrm{K}^{\angle}(\mathcal{F})$.*

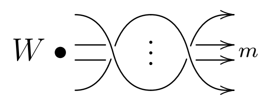
Proof. Since the terms of $\tilde{P}_w$ are the stable limit of the terms from the complexes B_k given in the proof of Proposition 5.1.3 and all webs appearing there are non-elliptic, it suffices to show that the complexes $C_k^{\cdot}$ lie in $A_{1/t}$ for some fixed $t > 0$.

Lemma 5.2.2 gives that each web W appearing in $[\![$  $]\!]_s$ is supported in $A_{1/M}\{r_W\}$ where $M = m(m-1)$ and r_W is the number of internal faces in W. It follows that the same is true for webs appearing in $C_0^{\cdot}$. We now show, via induction, that the same result holds for the complexes $C_k^{\cdot}$.

Recalling how $C_k^{\cdot}$ is constructed from $C_{k-1}^{\cdot} \bullet [\![$  $]\!]_s$, it suffices to show that if W is a web appearing in $C_{k-1}^{\cdot}$ then any web V appearing in

 $[2]\{4\}$

is supported in $A_{1/M}\{r_V - r_W\}$. Since the zero (= smooth) resolution of



is simply W there are r_W internal faces and no closed components. The result then follows from Lemma 5.2.2. □

We hence can consider the $\mathbb{Z}[q^{-1}, q]]$-linear combination of webs $\chi(\tilde{P}_w)$. The equality $\chi(\tilde{P}_w) = P_w$ (in $\hat{S}_{int}$) will follow easily from the following result.

Proposition 5.2.4. *Let* $A_1^{\cdot} \simeq A_2^{\cdot}$ *in* $\mathrm{K}^{\angle}(\mathcal{F})$, *then* $\chi(A_1^{\cdot}) = \chi(A_2^{\cdot})$.

Proof. As in the proof of Theorem 2.6.1, it suffices to show that if $A^{\cdot} \simeq 0$ then $\chi(A^{\cdot}) = 0$. Let $A^{\cdot}$ be such a complex; we wish to employ Corollary 2.6.3, but first must introduce the appropriate category in which the desired isomorphism holds.

Let $\mathcal{E}$ be an extension of (the skeleton of) the 2-category $\mathcal{F}$ where the objects in the category $\mathrm{Hom}_{\bullet}(v, w)$ are (countably) infinite formal q-graded direct sums of non-elliptic webs where the q-degrees are bounded below and there exists only finitely many direct summands in each q-degree. Morphisms between these direct sums are infinite matrices of $\mathbb{C}$-linear combinations of isotopy classes of degree-zero foams having only finitely many non-zero entries in each column. Note that (the skeleton of) $\mathcal{F} \subset \mathcal{E}$ (consisting of non-elliptic webs) is a full subcategory.

Corollary 2.6.3 now implies that

$$\bigoplus_{i=-\infty}^{\infty} A^{2i} \cong \bigoplus_{i=-\infty}^{\infty} A^{2i+1}.$$

in $\mathcal{E}$. It hence suffices to show that if $\oplus_{i=1}^{\infty} q^{l_i} W_{\alpha_i} \cong \oplus_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}$ as objects in $\mathcal{E}$, then the corresponding sums $\sum_{i=1}^{\infty} q^{l_i} W_{\alpha_i}$ and $\sum_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}$ are equal in $\hat{\mathcal{S}}_{int}$.

To do so, we will make use of the tools from [18, Section 5.4.2]. Fix words v and w and consider the pairing on webs in $\mathrm{Hom}_{\mathcal{S}}(v, w)$, denoted $\langle W, W' \rangle_{\mathfrak{sl}_3}$ for webs

$W, W' \in \mathrm{Hom}_{\mathcal{S}}(v, w)$, given by reversing the orientation of edges in W, gluing W and W' along their boundary, and evaluating the closed web.

If $\{W_1, \ldots, W_l\}$ is the basis for $\mathrm{Hom}_{\mathcal{S}}(v, w)$ consisting of all non-elliptic webs then this pairing satisfies the relation

$$\langle W_i, W_j \rangle_{\mathfrak{sl}_3} = q^{|v|+|w|}\, \mathrm{qdim}\, \widehat{\mathrm{Hom}}(W_i, W_j) \tag{5.2.1}$$

where $|v|$ and $|w|$ denote the lengths of the words v and w, qdim is the Laurent polynomial in q giving the graded dimension of a graded vector space, and $\widehat{\mathrm{Hom}}$ denotes the graded vector space of morphisms between webs (so $\mathrm{Hom}_{\circ}(W, W')$ is the degree-zero component of $\widehat{\mathrm{Hom}}(W, W')$). In addition, this pairing is non-degenerate, i.e. the matrix $A_{ij} = \langle W_i, W_j \rangle_{\mathfrak{sl}_3}$ is non-singular.

We can extend the pairing $\langle -, - \rangle_{\mathfrak{sl}_3}$ to a non-degenerate semi-linear[2] form on $\mathrm{Hom}_{\mathcal{S}_{int}}(v, w)$ by taking

$$\left\langle \sum_{i=1}^{l} p_i(q) W_i, \sum_{j=1}^{l} h_j(q) W_j \right\rangle_{\mathfrak{sl}_3} = \sum_{i,j=1}^{l} p_i(q^{-1}) h_j(q) \langle W_i, W_j \rangle_{\mathfrak{sl}_3}$$

for any $p_i(q), h_j(q) \in \mathbb{Z}[q^{-1}, q]$. Equation (5.2.1) then gives

$$\langle \chi(X), \chi(Y) \rangle_{\mathfrak{sl}_3} = q^{|v|+|w|}\, \mathrm{qdim}\, \widehat{\mathrm{Hom}}(X, Y)$$

for any 1-morphisms X and Y in $\mathrm{Hom}_{\bullet}(v, w)$ (in $\mathcal{F}$).

We would like to extend this fact to a relation between $\mathcal{E}$ and $\hat{\mathcal{S}}_{int}$, but cannot for a variety of reasons[3]. Nevertheless, suppose that $\oplus_{i=1}^{\infty} q^{l_i} W_{\alpha_i}$ is a 1-morphism in $\mathcal{E}$ and that W is a fixed web (with all webs mapping between words v and w), then

[2] This means that $\langle qV, W \rangle_{\mathfrak{sl}_3} = q^{-1} \langle V, W \rangle_{\mathfrak{sl}_3}$ and $\langle V, qW \rangle_{\mathfrak{sl}_3} = q \langle V, W \rangle_{\mathfrak{sl}_3}$ for webs V and W.

[3] Two such reasons are the following: the pairing cannot be extended to $\hat{\mathcal{S}}_{int}$ due to semi-linearity and the fact that $\mathbb{Z}[q^{-1}, q]]$ is not closed under the map $q \mapsto q^{-1}$; even if the pairing could be extended, the desired equality would not hold since the infinite formal direct sums in $\mathcal{E}$ as defined are coproducts but not products.

we have

$$\begin{aligned} q^{|v|+|w|}\operatorname{qdim}\widehat{\operatorname{Hom}}(\oplus_{i=1}^{\infty} q^{l_i} W_{\alpha_i}, W) &= q^{|v|+|w|}\operatorname{qdim}\prod_{i=1}^{\infty}\widehat{\operatorname{Hom}}(q^{l_i} W_{\alpha_i}, W) \\ &= q^{|v|+|w|}\sum_{i=1}^{\infty}\operatorname{qdim}\widehat{\operatorname{Hom}}(q^{l_i} W_{\alpha_i}, W) \\ &= \sum_{i=1}^{\infty} q^{-l_i}\langle W_{\alpha_i}, W\rangle_{\mathfrak{sl}_3} \end{aligned}$$

in the ring $\mathbb{Z}[[q^{-1},q] = \mathbb{Z}[[q^{-1}]][q]$.

Now, suppose that $\oplus_{i=1}^{\infty} q^{l_i} W_{\alpha_i}$ and $\oplus_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}$ are isomorphic 1-morphisms in $\mathcal{E}$. This implies that

$$\widehat{\operatorname{Hom}}(\oplus_{i=1}^{\infty} q^{l_i} W_{\alpha_i}, W) \cong \widehat{\operatorname{Hom}}(\oplus_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}, W)$$

for any fixed web W. This then implies that

$$\sum_{i=1}^{\infty} q^{-l_i}\langle W_{\alpha_i}, W\rangle_{\mathfrak{sl}_3} = \sum_{i=1}^{\infty} q^{-l'_i}\langle W_{\alpha'_i}, W\rangle_{\mathfrak{sl}_3}$$

i.e. that

$$\left\langle \sum_{i=1}^{\infty} q^{l_i} W_{\alpha_i} - \sum_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}, W \right\rangle_{\mathfrak{sl}_3} = 0 \tag{5.2.2}$$

for any fixed web W; in particular this holds when W is a (non-elliptic) basis web.

We now claim this implies that $\sum_{i=1}^{\infty} q^{l_i} W_{\alpha_i} = \sum_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i}$ in $\hat{\mathcal{S}}_{int}$. Indeed, write

$$\sum_{i=1}^{\infty} q^{l_i} W_{\alpha_i} - \sum_{i=1}^{\infty} q^{l'_i} W_{\alpha'_i} = \sum_{i=1}^{l} f_i(q) W_i$$

where $f_i(q) \in \mathbb{Z}[q^{-1},q]]$ and as before $\{W_1,\ldots,W_l\}$ are the non-elliptic basis. Equation (5.2.2) then gives that the row vector

$$\big(f_1(q^{-1}),\ldots,f_l(q^{-1})\big)$$

in $\mathbb{Z}[[q^{-1}, q]^l$ lies in the kernel of the map determined by left multiplication by the matrix A_{ij}. This in turn implies that $f_i(q) = 0$ for all i since A_{ij} is non-singular. $\square$

Propositions 5.1.5 and 5.2.4 now give that

$$\chi(\tilde{P}_w) \bullet \chi(\tilde{P}_w) = \chi(\tilde{P}_w \bullet \tilde{P}_w) = \chi(\tilde{P}_w)$$

and Proposition 5.1.4 gives that if $\mathrm{wt}(v) < \mathrm{wt}(w)$ then

$$\chi(\tilde{P}_w) \bullet W = \chi(\tilde{P}_w \bullet W) = 0$$

for any $W \in \mathrm{Hom}_\bullet(w, v)$ and

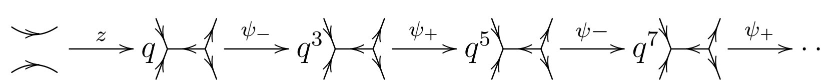
$$W' \bullet \chi(\tilde{P}_w) = \chi(W' \bullet \tilde{P}_w) = 0$$

for any $W' \in \mathrm{Hom}_\bullet(v, w)$. Propositions 5.1.3 and 5.2.3 show that

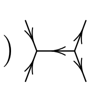
$$\chi(\tilde{P}_w) = \mathrm{id}_w + \sum_{i=1}^{r} f_i(q) \cdot W_i$$

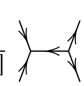
with $f_i \in \mathbb{Z}[q^{-1}, q]]$ and $W_i \in \mathrm{Hom}_\bullet(w, w) \smallsetminus \mathrm{id}_w$. These facts, together with Proposition 2.3.4, give a proof of Theorem 3.0.2 in the case that $w = (+ \cdots +)$ where we view the equality $\chi(\tilde{P}_w) = P_w$ as holding in $\hat{\mathcal{S}}_{int}$.

Example 5.2.5. *The computations given in [18, Section 6.1] show that*

$\tilde{P}_{(++)} =$ 

where $\psi_\pm$ are the morphisms defined in that section. Note that

$\chi(\tilde{P}_{(++)}) =$  $- (q - q^3 + q^5 - q^7 + \cdots)$ 

$=$  $- \frac{1}{[2]}$ 

$= P_{(++)}$

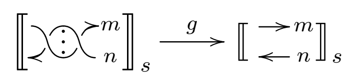
by Proposition 2.3.3.

5.3 $\tilde{P}_w$ and decategorification for $w = (+\cdots+-\cdots-)$

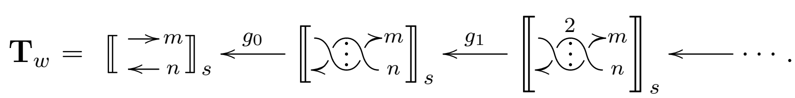
In this section we construct $\tilde{P}_w$ and prove Theorems 3.0.1 and 3.0.2 for

$$w = (\underbrace{+\cdots+}_{m}\underbrace{-\cdots-}_{n}).$$

Since we have already considered the case $n = 0$ (and $m = 0$ by taking duals everywhere in sight) we assume $m, n > 0$. We proceed by mimicking the proof for $w = (+\cdots+)$. Let  denote a full twist on $m+n$ strands directed as indicated. The first step is to construct a map



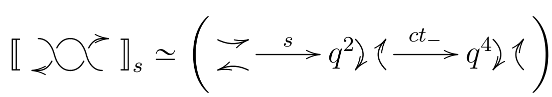
and use it to build the inverse system



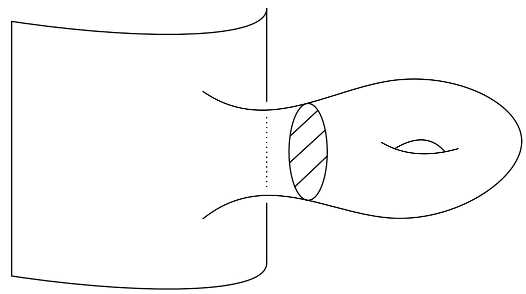
This is trivial when $w = (+\cdots+)$ since the degree zero term of the complex assigned to a single twist is the identity tangle. This fails for $w = (+\cdots+-\cdots-)$, but we shall see that it holds up to homotopy; this is sufficient to define the map g.

We begin with a lemma showing this for the case $m = 1 = n$, which will also be of use later.

Lemma 5.3.1.



Here s is a saddle cobordism and $ct_- = ct_L - ct_R$ where ct_L denotes the foam which is the identity foam on the right arc and has a 'choking torus:'



on the left arc. The foam ct_R is defined similarly.

Proof. We have

$$[\![\quad]\!]_s \cong \quad \oplus \quad \xrightarrow{A} q^2 \quad \oplus \quad \oplus q^2 \quad \oplus \quad \xrightarrow{B} q^4 \quad \oplus q^2 \quad \oplus \quad$$

where

$$A = \begin{pmatrix} s & * \\ 0 & \mathrm{id} \\ s & * \\ 0 & \mathrm{id} \end{pmatrix}$$

and

$$B = \begin{pmatrix} * & * & * & * \\ * & * & \mathrm{id} & * \\ 0 & * & 0 & -\mathrm{id} \end{pmatrix}.$$

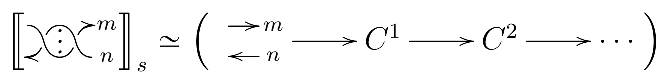
Using Gaussian elimination, we find that

$$[\![\quad]\!]_s \simeq \quad \xrightarrow{s} q^2 \quad \longrightarrow q^4 \quad .$$

We deduce the second map since, up to scalar multiple, it is the only degree zero map which makes the diagram a complex. A computation shows that it is indeed non-zero. □

We now prove the general case.

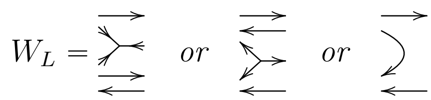
Proposition 5.3.2.



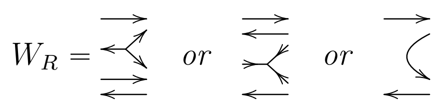
where every web appearing in $C^{\cdot}$ is non-elliptic and takes the form

$$W_L \bullet W \bullet W_R \tag{5.3.1}$$

for

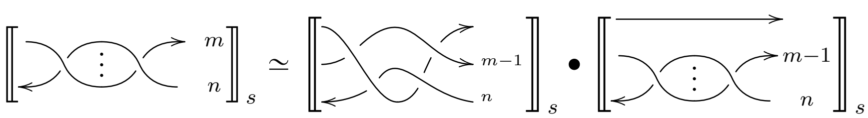


and

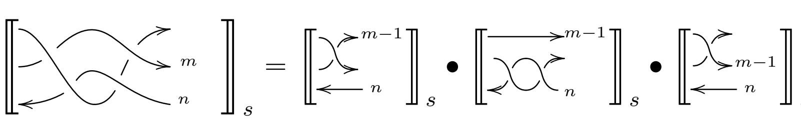


where the strands involved in the Y-webs and U-webs have multiplicity one (and the other strands may have higher multiplicities).

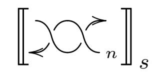
Proof. We proceed via induction on m, noting that the result holds trivially in the cases $m = 0$ or $n = 0$. We have



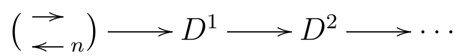
where the first tangle diagram on the right hand side indicates one strand wrapping around the others and the second denotes the tensor product of a strand with a full twist on $m + n - 1$ strands, directed as indicated. By induction, the second complex has the desired form. Since the composition of two webs of the form (5.3.1) is isomorphic to a direct sum of non-elliptic webs of this form, it suffices to show that the first complex has the desired form. We have



where the middle term on the right side is the tensor product of $m-1$ strands with a single strand wrapping around n strands (which do not twist themselves - note the subtle difference in notation!). Since the two outside terms on the right side have the desired form, it now suffices to show that

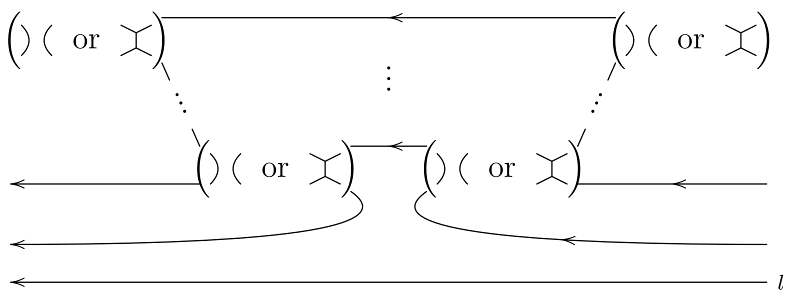


has this form. We claim that this complex is homotopy equivalent to a complex



(5.3.2)

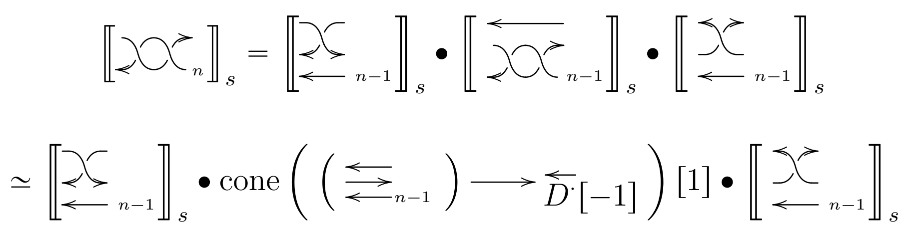
where each web in D^i takes the form



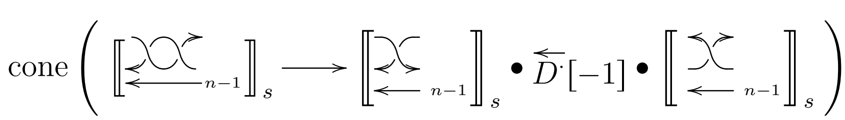
(5.3.3)

for $l \geqslant 0$.

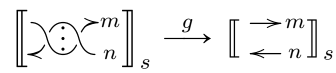
We proceed via induction on n. The $n = 1$ case follows from Lemma 5.3.1. We now compute

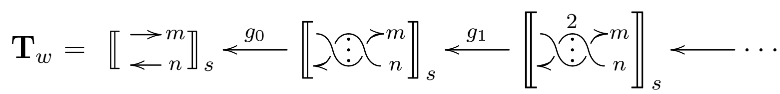


Here $\overleftarrow{D^{\cdot}}$ denotes the tensor product of the complex $D^{\cdot}$ with a single strand. By Proposition 4.1.4, the above complex is homotopy equivalent to



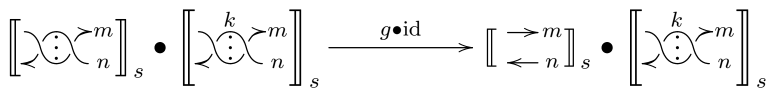
which has the form (5.3.2). Since reducing a web of the form (5.3.3) gives webs of the form (5.3.1), the result follows. ∎

There thus exists a map  which we use to construct the inverse system



(5.3.4)

for $w = (+\cdots+-\cdots-)$; g_k is defined as



In the case $w = (+\cdots+)$, both the proof that the limit of the system exists (i.e. that the system is Cauchy) and that the limit lies in $\mathrm{K}^{\angle}(\mathcal{F})$ involved analysis of the complexes $\mathrm{cone}(g_k)$. In this case, we combine these facts into one result.

Proposition 5.3.3. *Let $w = (+\cdots+-\cdots-)$. The inverse system $\mathbf{T}_w$ is Cauchy and its limit lies in $\mathrm{K}^{\angle}(\mathcal{F})$.*

Of course, since limits are unique only up to homotopy, we mean that there is a representative of the homotopy class of the limit which lies in $\mathrm{K}^{\angle}(\mathcal{F})$.

Proof. As in the proofs of Propositions 5.2.3 and 5.1.2, it suffices to construct complexes $C_k^{\cdot}$ satisfying the following conditions:

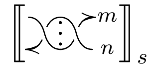
1. $C_k[-1]^{\cdot} \simeq \mathrm{cone}(g_k)$,

2. $C_k^{\cdot}$ is $\mathrm{O}(2k+1)$,

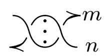
3. Every web appearing in $C_k^{\cdot}$ is non-elliptic and takes the form (5.3.1),

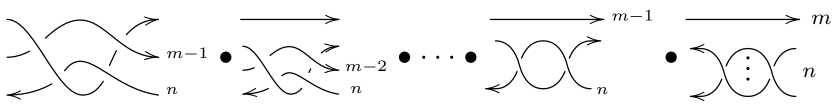
4. and $C_k^{\cdot}$ is supported in $A_{1/M}$ for some fixed M. Moreover, if a web W in $C_k^{\cdot}$ has r_W internal faces then that web is supported in $A_{1/M}\{r_W\}$.

We shall see that it suffices to take $M = 2(m+n)^2$, so fix this value.

We begin by reconsidering the complex  and showing that it is homotopic to a complex of the form

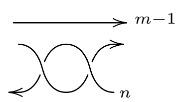
$$\left(\substack{\rightarrow m \\ \leftarrow n}\right) \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

which is supported in $A_{1/M}$ and where each web in C^i takes the form (5.3.1). Using only $R3$ moves we find that the tangle  is isotopic to the tangle

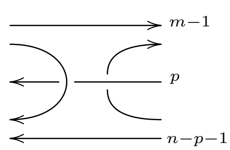


(5.3.5)

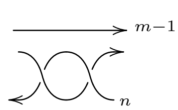
where all the strands directed to the right wrap (one by one, starting with the top strand) around the strands directed to the left at the beginning. We consider the complex assigned to this tangle. We next use Lemma 5.3.1 and Proposition 4.1.4 to express this complex in terms of complexes assigned to tangles obtained from (5.3.5) by replacing the tangle



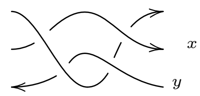
with tangles of the form



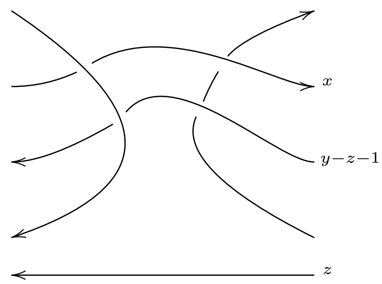
We can then use $R2$ moves to slide the left strand which 'turns back' through the entire tangle. We now repeat the procedure for all terms in (5.3.5), moving leftward from



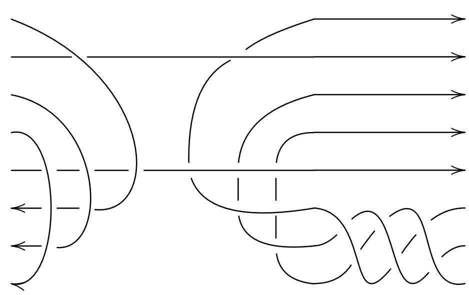
and one by one expressing complexes assigned to tangles of the form



in terms of complexes assigned to



then sliding the left strand which turns around through the tangle. In the end we find that the terms in the complex assigned to (5.3.5) come from the complexes assigned to tangles τ which, for example, take the form



. (5.3.6)

It hence suffices to show that $[\![\tau]\!]_s$, suitably shifted to take into account the shifts in quantum and homological degree that arise from the Gaussian elimination homotopies and $R2$ moves, is supported in $A_{1/M}$.

We first establish some notation. Noting that

$$[\![\ \asymp\ ]\!]_s \simeq [\![\ )\,(\ ]\!]_s\,[1]\{1\}$$

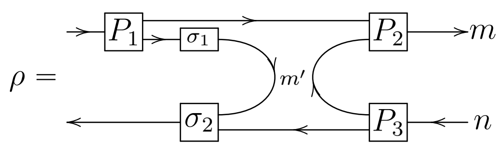
for all possible orientations of the strands, we call an $R2$ move which reduces the number of crossings in a tangle a 'good' $R2$ move. These moves are good in the sense that the corresponding Gaussian elimination homotopy equivalence yields a complex whose support is (properly) contained in that of the original complex.

Now, if m' is the number of strands which turn back on the left side of τ then, assuming $m' > 1$, we make at least

$$2\sum_{i=1}^{m'-1} i = (m')^2 - m'$$

good $R2$ moves to arrive at such a presentation. The right hand side of this formula also works in the cases when $m' = 0, 1$.

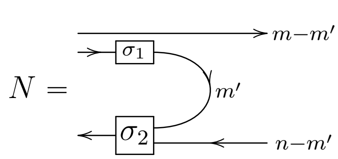
Next, let l denote half the total number of negative crossings in τ. We can apply l good $R2$ moves to eliminate all negative crossings involving the strands which turn back on the right to produce a tangle of the form



(5.3.7)

where the P_i are positive braids. The σ_i are negative braids, but of a particular sort - every braid determines an element of the symmetric group and these braids are the simplest ones corresponding to their particular element (i.e. there is no twisting). The braid σ_2 has the further property that all crossings consist of a strand leaving P_3 crossing under a strand leaving σ_1.

All of the internal faces in the 0-resolution of (5.3.7) come from the tangle



(5.3.8)

and it follows from inspection that if r is the number of such faces then

$$r \leqslant l - 1$$

unless $l = 0$ in which case there are no negative crossings in (5.3.7) and no internal faces in the zero resolution. We assume for now that we are not in the exceptional case $l = 0$.

For later use, we'll bound the values for l. The number of crossings in σ_1 is bounded by

$$\binom{m'}{2} = \frac{1}{2}m'(m'-1),$$

the length of the longest element in the symmetric group. The remaining crossings come from σ_2 where some of the $n - m'$ strands pass underneath some of the m' strands leaving σ_1, producing at most $m'(n - m')$ crossings. We thus have

$$l \leqslant \frac{1}{2}m'(m'-1) + m'(n-m') = \frac{1}{2}m'(2n - m' - 1) < mn.$$

Lemma 5.2.2 gives that any web W appearing in $[\![\rho]\!]_s$ is supported in

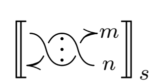
$$A_{1/c_+}\{-r - 1 + r_W\} \subseteq A_{1/c_+}\{-l + 1 - 1 + r_W\}$$

where r_W is the number of internal faces in W and c_+ is the number of positive crossings in ρ. Since

$$c_+ \leqslant (m+n)(m+n-1) < M$$

we see such a web is supported in

$$A_{1/M}\{-l + r_W\}.$$

Considering all the shifts due to Gaussian elimination homotopies and good $R2$ moves, we see that the contribution to  is given by

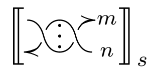
$$[\![\rho]\!]_s\,[(m')^2 - m' + a]\{(m')^2 - m' + a\}[m' + b]\{2m' + 2b\}[l]\{l\}$$

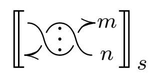
where $a \leqslant m^2 - (m')^2 - m + m' \leqslant m^2$ and $b \leqslant m' \leqslant m$. Indeed, the shifts of $[1]\{1\}$ come from the good $R2$ moves and the shifts of $[1]\{2\}$ come via Lemma 5.3.1 from the strands which turn back. The web W is hence supported in

$$A_{1/M}[(m')^2 + a + b + l]\{(m')^2 + m' + a + 2b + r_W\}.$$

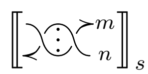
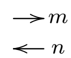
We have

$$(m')^2 + a + b + l \leqslant 2m^2 + m + mn \leqslant M$$

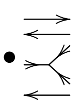
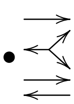
so we see that each web W in the contribution to  coming from $[\![\rho]\!]_s$ with $l \neq 0$ is supported in $A_{1/M}\{r_W\}$ (we have used the fact that $l \neq 0$ implies $m' \geqslant 1$ so $(m')^2 + m' + a + 2b \geqslant 1$).

In the case that $l = 0$, i.e. there are no negative crossings in τ, it follows from Lemma 5.2.2 that any web W contributing to  is supported in $A_{1/M}\{r_W\}$.

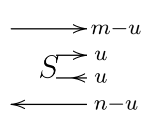
We have thus shown that

 $\simeq \Big($  $\longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots \Big)$

where each web W in C^i is supported in $A_{1/M}\{r_W\}$ and comes from the tangles (5.3.7). It is easy to see that all such tangles take the form (5.3.1). It will be useful for our further considerations to note that every web of the form (5.3.1) is of the form

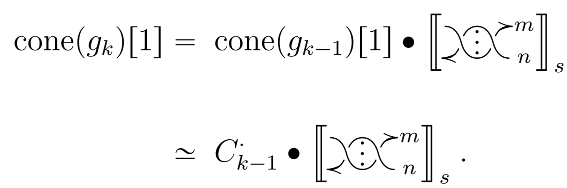
$W' \bullet R$, $W' \bullet$  , or $W' \bullet$  (5.3.9)

where R is a non-elliptic web of the form

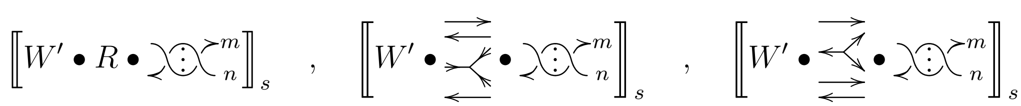


and S is a non-elliptic web with no left boundary. While the latter two webs in (5.3.9) do not preclude the first, we employ the convention that if we claim a web has either of these two forms it is implicit that it does not have the first. We will also assume that u is chosen maximal for the first type of web.

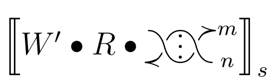
We now proceed with our construction of the complexes $C_k^{\cdot}$. Let $C_0^{\cdot}$ be the complex constructed above with the degree zero term truncated off. We now construct $C_k^{\cdot}$ assuming we have constructed $C_{k-1}^{\cdot}$; we have



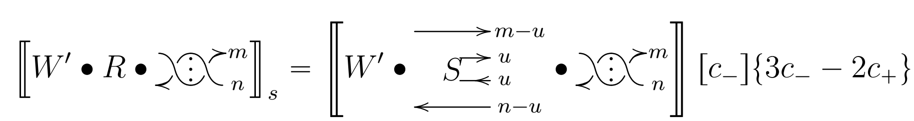
Since each web W in $C_{k-1}^{\cdot}$ has the form (5.3.9) (and also (5.3.1)), is supported in $A_{1/M}\{r_W\}$, and has homological degree at least $1+2(k-1)$, it suffices to show that each of the complexes



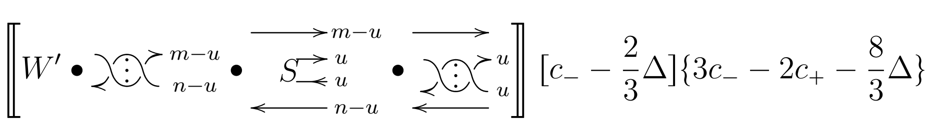
is homotopic to a complex with terms of the form (5.3.1), minimal homological degree 2, and with webs V supported in $A_{1/M}\{r_V - r_W\}$ with $W = W' \bullet R$ or W'. We shall analyze each case separately.

Case 1: 

We compute

 (5.3.10)

which is homotopy equivalent to the complex

 (5.3.11)

where $c_\pm$ is the number of $\pm$-crossings in $\bowtie^{m}_{n}$ and Δ is the change in writhe between the right side of (5.3.10) and (5.3.11). The shifts involving Δ can be deduced from (2.6.11).

We have the formulae

$$\begin{aligned} c_+ &= m^2 - m + n^2 - n \\ c_- &= 2mn \end{aligned}$$

and $\Delta = 0$. Taking into account the shifts due to the changes in the number of crossings, this gives that $\left\llbracket W' \bullet R \bullet \bowtie^{m}_{n} \right\rrbracket_s$ is homotopy equivalent to

$$\left\llbracket W' \bullet \bowtie^{m-u}_{n-u} \bullet \overset{\xrightarrow{\quad} m-u}{\underset{\xleftarrow{\quad} n-u}{S \rightleftarrows^{u}_{u}}} \bullet \overset{\xrightarrow{\quad}}{\underset{\xleftarrow{\quad}}{\bowtie^{u}_{u}}} \right\rrbracket_s [2u(m+n-2u)]\{2u(m+n-2u)\}.$$

Similarly, taking into account the change in writhe, we compute

$$\left\llbracket S \rightleftarrows^{u}_{u} \bullet \bowtie^{u}_{u} \right\rrbracket_s \simeq \llbracket S \rightleftarrows^{u}_{u} \rrbracket_s [2u^2]\{2u(u+2)\}$$

and so $\left\llbracket W' \bullet R \bullet \bowtie^{m}_{n} \right\rrbracket_s$ is homotopy equivalent to

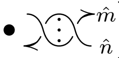
$$\left\llbracket W' \bullet \bowtie^{m-u}_{n-u} \bullet \overset{\xrightarrow{\quad} m-u}{\underset{\xleftarrow{\quad} n-u}{S \rightleftarrows^{u}_{u}}} \right\rrbracket_s [2u(m+n-u)]\{2u(m+n-u+2)\}. \qquad (5.3.12)$$

Since $u \geqslant 1$, $m + n - u \geqslant 1$, and each web appearing in (5.3.12) takes the form $W' \bullet R$, it suffices to show that every web V appearing in this complex is supported in $A_{1/M}\{r_V - r_{(W' \bullet R)}\}$.

We have

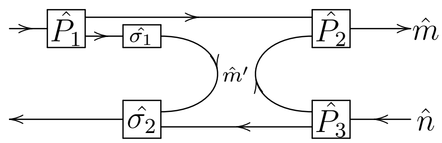
$$\mathrm{supp}\left(\left\llbracket W' \bullet \bowtie^{m-u}_{n-u} \bullet \overset{\xrightarrow{\quad} m-u}{\underset{\xleftarrow{\quad} n-u}{S \rightleftarrows^{u}_{u}}} \right\rrbracket_s \right) \subseteq \mathrm{supp}\left(\left\llbracket W' \bullet \bowtie^{m-u}_{n-u} \right\rrbracket_s \right)$$

so we will consider complexes

$$\left[\!\left[W' \bullet \right.\right.$$  $$\left.\left.\right]\!\right]_s$$

for $\hat{m} < m$ and $\hat{n} < n$. As before, we can express the complex assigned to the twist in terms of the complexes assigned to tangles

$\hat{\rho} =$



. (5.3.13)

We must show that the complexes $[\![W' \bullet \hat{\rho}]\!]_s$, with appropriate shifts, are supported in $A_{1/M}$. Let $\hat{n}'$ be the number of strands which leave $\hat{P}_3$ and actually cross strands in $\hat{\sigma}_2$. Refining our earlier estimate, if r is the number of internal faces in the zero resolution of ρ then

$$r \leqslant \max(0, \hat{l} - \hat{n})$$

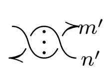
where $\hat{l}$ is the number of negative crossings in (5.3.13). The 0-resolution of $W' \bullet \hat{\rho}$ hence has at most

$$r_{W'} + \hat{l} - \hat{n}' + 2\hat{m}' + \hat{n}' - 1 = r_{W'} + \hat{l} + 2\hat{m}' - 1$$

internal faces where $r_{W'}$ is the number of internal faces in W' (the additional $2\hat{m}' + \hat{n}' - 1$ possible faces come from gluing W' to $\hat{\rho}$).

Since we chose u maximal, the smooth resolution of $W' \bullet \hat{\rho}$ has no closed components; Lemma 5.2.2 gives that $[\![W' \bullet \hat{\rho}]\!]_s$ is supported in

$$A_{1/M}\{-r_{W'} - \hat{l} - 2\hat{m}'\}.$$

The contribution to $\left[\!\left[W' \bullet \right.\right.$  $\left.\left.\right]\!\right]_s$ is obtained considering the shifts. As before we find it is given by

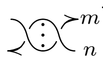
$$[\![W' \bullet \hat{\rho}]\!]_s\,[(\hat{m}')^2 - \hat{m}' + a]\{(\hat{m}')^2 - \hat{m}' + a\}[\hat{m}' + b]\{2(\hat{m}' + b)\}[\hat{l}]\{\hat{l}\}$$

with $a \leqslant \hat{m}^2$ and $b \leqslant \hat{m}$ so this complex is supported in

$$A_{1/M}[(\hat{m}')^2 + a + b + \hat{l}]\{(\hat{m}')^2 - \hat{m}' + a + 2b - r_{W'}\}.$$

Since $(\hat{m}')^2 + a + b + \hat{l} \leqslant M$ we see that this support is contained in

$$A_{1/M}\{-r_{W'} - 1\}.$$

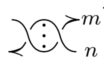
The contribution to $\left[\!\left[W' \bullet R \bullet \right.\right.$  $\left.\left.\right]\!\right]_s$ is supported in

$$A_{1/M}[2u(m + n - u)]\{2u(m + n - u + 2) - r_{W'} - 1\}$$

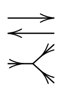
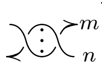
which is contained in $A_{1/M}\{-r_{W'}\}$ since $1 \leqslant u \leqslant \min(m, n)$ and

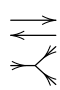
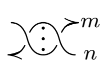
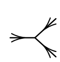
$$2u(m + n - u) \leqslant M.$$

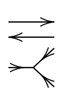
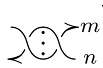
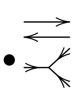
Moreover, applying the second statement from Lemma 5.2.2 throughout the preceding argument, we see that any web V in the contribution to

$\left[\!\left[W' \bullet R \bullet \right.\right.$  $\left.\left.\right]\!\right]_s$

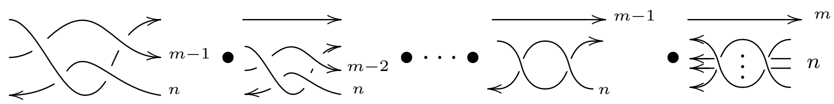
is supported in $A_{1/M}\{r_V - r_{(W' \bullet R)}\}$.

<u>Case 2:</u> $\left[\!\left[W' \bullet \right.\right.$  $\bullet$  $\left.\left.\right]\!\right]_s$

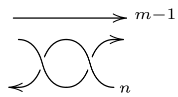
We assume that $W' \bullet$  does not take the form $W' \bullet R$ above. We begin by using $R3$ moves to express the tangle  as in (5.3.5). We then use $R3$ moves to pull the two strand twist which 'lines up' with  to the left through the tangle. Lemma 5.1.1 gives

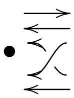
$\left[\!\left[W' \bullet \right.\right.$  $\bullet$  $\left.\left.\right]\!\right]_s \simeq \left[\!\left[W' \bullet \right.\right.$  $\left.\left. \bullet\, T \right]\!\right]_s [2]\{4\}$

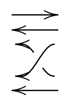
where

$T =$ 

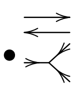
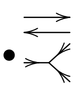
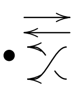
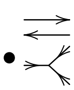
As in our analysis of the complex assigned to a single twist, we use Lemma 5.3.1 starting with

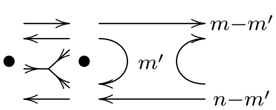


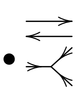
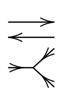
and moving left to express $[\![T]\!]_s$ in terms of complexes $[\![\rho]\!]_s$ or $\left[\!\!\left[\rho \bullet\right.\right.$  $\left.\left.\right]\!\!\right]_s$ with ρ as in (5.3.7). In the latter case the top strand of the crossing in



turns back to the right in ρ; this case arises due to the pair of non-twisting strands in T which prevent the use of a good $R2$ move which in our previous analysis removed the crossing.

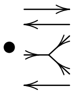
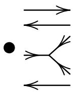
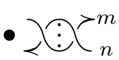
We hence analyze the complexes $\left[\!\!\left[W' \bullet\right.\right.$  $\left.\left.\bullet\, \rho\right]\!\!\right]_s$ and $\left[\!\!\left[W' \bullet\right.\right.$  $\bullet\, \rho \,\bullet$  $\left.\left.\right]\!\!\right]_s$ noting that every web appearing takes the form (5.3.1). It suffices to consider the support of such complexes and we begin with the former (and slightly easier) case. The smooth resolution of $W' \bullet$  $\bullet\, \rho$ takes the form

$W' \bullet$ 

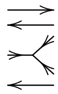
which has no closed components (since $W' \bullet$  $\neq W' \bullet R$). The 0-resolution of $W' \bullet$  $\bullet\, \rho$ has at most

$$r_{W'} + l - n' + 2m' + n' - 1 = r_{W'} + l + 2m' - 1$$

internal faces, where n' is the number of strands in ρ leaving P_3 and crossing under strands in σ_2.

Lemma 5.2.2 now gives that any web V appearing in $\left[\!\left[W' \bullet\right.\right.$  $\left.\left.\bullet\, \rho \right]\!\right]_s$ is supported in $A_{1/M}\{r_V - r_{W'} - l - 2m'\}$. The contribution to $\left[\!\left[W' \bullet\right.\right.$  $\bullet$ 

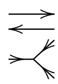
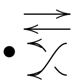
$\left.\right]\!\right]_s$ is computed considering the shifts and as before we see it is given by

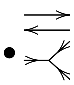
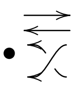
$\left[\!\left[W' \bullet\right.\right.$  $\left.\left.\bullet\, \rho \right]\!\right]_s$ $[(m')^2 - m' + a]\{(m')^2 - m' + a\}[m' + b]\{2(m' + b)\}[l]\{l\}[2]\{4\}$

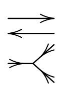
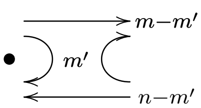
with $a \leqslant m^2$ and $b \leqslant m$. Any web V in this complex is supported in

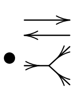
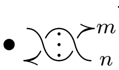
$$A_{1/M}[(m')^2 + a + b + l + 2]\{(m')^2 - m' + a + 2b + r_V - r_{W'} + 4\}$$

and since $(m')^2 + a + b + l + 2 \leqslant M$ this is contained in $A_{1/M}\{r_V - r_{W'}\}$.

We now consider $\left[\!\left[W' \bullet\right.\right.$  $\bullet\, \rho \,\bullet$  $\left.\right]\!\right]_s$. The smooth resolution of

$W' \bullet$  $\bullet\, \rho \,\bullet$ 

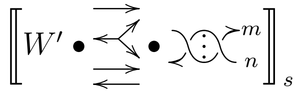
takes the form $W' \bullet$  $\bullet$ 

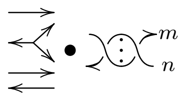
which again has no closed components. Due to the alignment of the negative crossing, the 0-resolution has the same number of internal faces as the case where it is not present. It follows that the only difference from the previous case is that we make one less good $R2$ move. It follows that any web V in the contribution to $\left[\!\left[W' \bullet\right.\right.$  $\bullet$ 

$\left.\right]\!\right]_s$ is supported in

$$A_{1/M}[(m')^2 + a + b + l + 1]\{(m')^2 - m' + a + 2b + r_V - r_{W'} + 3\}$$

which is contained in $A_{1/M}\{r_V - r_{W'}\}$ and has minimal homological degree 2 (since we necessarily have $l \geqslant 1$).

Case 3: 

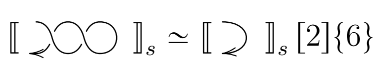
This case can be handled completely analogously to case 2 above. In some detail, begin by rotating the diagram  180° about a horizontal axis and reversing the direction of all strands. We are then in case 2 (with m and n switched). Apply the above analysis (noting that M is symmetric in m and n) then rotate every web appearing in the complexes 180° about a horizontal axis and reverse the direction of all strands. $\square$

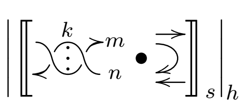
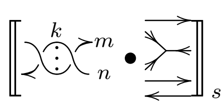
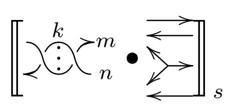
Now, let $\tilde{P}_w$ be the limit of $\mathbf{T}_w$ lying in $\mathrm{K}^{\angle}(\mathcal{F})$ constructed using the complexes $C_k^{\cdot}$ from the above proof.

Proposition 5.3.4. *Let* $w = (+\cdots+-\cdots-)$. *The web* id_w *appears only once in* $\tilde{P}_w$ *and does so in quantum and homological degree zero; all other webs in the complex take the form* (5.3.1). *If* $\mathrm{wt}(v) < \mathrm{wt}(w)$ *then* $\tilde{P}_w \bullet W_1 \simeq 0$ *for any* $W_1 \in \mathrm{Hom}_\bullet(w, v)$ *and* $W_2 \bullet \tilde{P}_w \simeq 0$ *for any* $W_2 \in \mathrm{Hom}_\bullet(v, w)$. *Finally,* $\tilde{P}_w \bullet \tilde{P}_w \simeq \tilde{P}_w$.

Proof. The first statement follows as in the proof of Proposition 5.1.3 from the description of the complexes $C_k^{\cdot}$ in the proof of Proposition 5.3.3. The third statement follows from the first and second as in the proof of Proposition 5.1.5.

It hence suffices to prove the second statement. Pulling the strands lining up with a U-web through the twist and using the homotopy equivalence

 (5.3.14)

we see that  $\geqslant 2k$. A similar analysis using Lemma 5.1.1 shows the same for the complexes  and . Using Proposition

2.3.5 and following the proof of Proposition 5.1.4 we have that $\tilde{P}_w \bullet W_1 \simeq 0$. The result for W_2 follows similarly. □

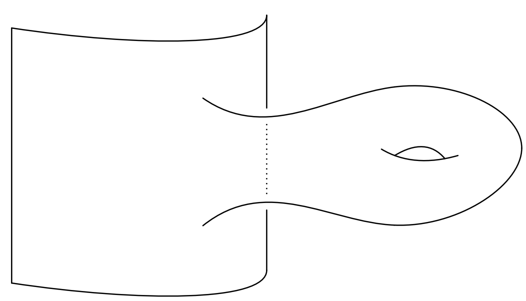
This gives Theorem 3.0.1 for the case $w = (+\cdots+-\cdots-)$. Since we have already seen that $\tilde{P}_w$ lies in $\mathrm{K}^{\angle}(\mathcal{F})$, Theorem 3.0.2 follows from Proposition 5.2.4 as in the case $w = (+\cdots+)$.

The methods used to show that $\tilde{P}_w$ is supported in $\mathrm{K}^{\angle}(\mathcal{F})$ can be employed to simplify their computation. We exhibit this in the following computation of $\tilde{P}_{(+-)}$.

Proposition 5.3.5.

$$\tilde{P}_{(+-)} = \asymp \xrightarrow{s} q^2 \,)(\, \xrightarrow{ct_-} q^4 \,)(\, \xrightarrow{\frac{1}{3}ct_B - T_+} q^8 \,)(\, \xrightarrow{ct_-} \cdots \tag{5.3.15}$$

where $ct_B = ct_L \circ ct_R$ with ct_-, ct_L and ct_R as in Lemma 5.3.1. $T_+ = T_L + T_R$ where T_L is the identity foam on the right arc and is the foam:



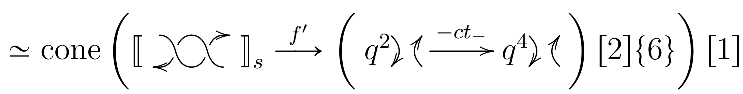
on the left arc; T_R is defined similarly.

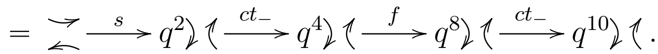
Proof. Lemma 5.3.1 gives that

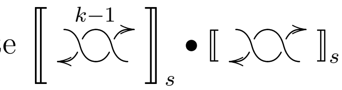
$$\llbracket \bowtie \rrbracket_s \simeq \mathrm{cone}\left(\llbracket \asymp \rrbracket \xrightarrow{s} \left(q^2)(\xrightarrow{-ct_-} q^4)(\right) \right)[1]$$

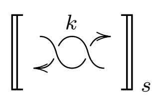
from which we compute that $\llbracket \bowtie \rrbracket_s \bullet \llbracket \bowtie \rrbracket_s$ is homotopy equivalent to

$$\mathrm{cone}\left(\llbracket \asymp \rrbracket \xrightarrow{s} \left(q^2)(\xrightarrow{-ct_-} q^4)(\right) \right)[1] \bullet \llbracket \bowtie \rrbracket_s$$





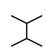
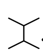
A direct (and tedious!) computation shows that $f = \frac{1}{3}ct_B - T_+$, although it can be argued based on degree that f must be a multiple of this map. Repeating this procedure inductively to compute  shows that



is given by the complex (5.3.15) truncated at homological degree $2k$. □

5.4 $\tilde{P}_w$ and decategorification for non-segregated words

In the decategorified case, we construct the projector P_w for a non-segregated word w by considering the segregated projector of the same weight and (horizontally) composing with (a composition of) H-webs on both sides. This procedure works in the categorified setting as well.

Indeed, suppose that w' is a word for which we have constructed $\tilde{P}_{w'}$ satisfying the conditions of Theorem 3.0.1 and w is a word of the same weight obtained by transposing one pair of adjacent $+$ and $-$ signs in w'. Let h be the web in $\mathrm{Hom}_\bullet(w', w)$ given by the tensor product of  (oriented appropriately) and identity webs. Consider $\bar{h} \bullet \tilde{P}_{w'} \bullet h$ where $\bar{h}$ is the web in $\mathrm{Hom}_\bullet(w, w')$ obtained from h by reversing the orientation of the strands in . If V is a web in $\mathrm{Hom}_\bullet(v, w)$ with $\mathrm{wt}(v) < \mathrm{wt}(w)$ then

$$V \bullet \bar{h} \cong \bigoplus_\alpha q^{l_\alpha} W_\alpha$$

for W_α in $\mathrm{Hom}_\bullet(v, w')$. Since $\mathrm{wt}(v) < \mathrm{wt}(w')$ we have that

$$\begin{aligned} V \bullet \bar{h} \bullet \tilde{P}_{w'} \bullet h &\cong \left(\bigoplus_\alpha q^{l_\alpha} W_\alpha \right) \bullet \tilde{P}_{w'} \bullet h \\ &\cong \bigoplus_\alpha \left(q^{l_\alpha} W_\alpha \bullet \tilde{P}_{w'} \right) \bullet h \\ &\simeq 0. \end{aligned}$$

A similar computation shows that $\bar{h} \bullet \tilde{P}_{w'} \bullet h \bullet V \simeq 0$ for V in $\mathrm{Hom}_\bullet(w, v)$ with $\mathrm{wt}(v) < \mathrm{wt}(w)$.

If h is as above, $h \bullet \bar{h} \cong \mathrm{id}_w \oplus W$ where $W \bullet \tilde{P}_{w'} \simeq 0$ so we have

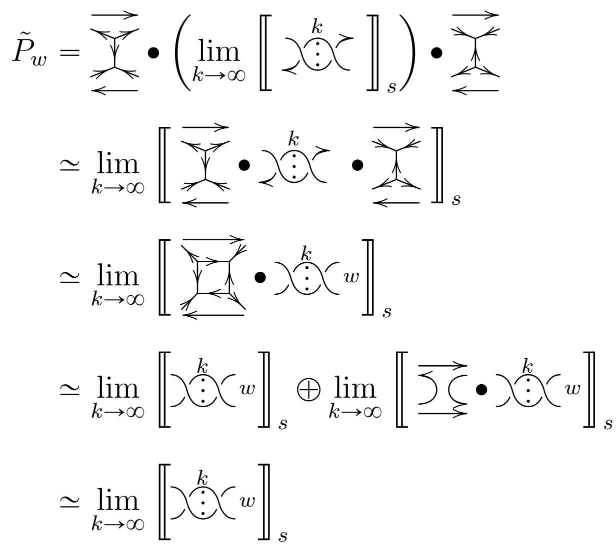
$$\begin{aligned} \left(\bar{h} \bullet \tilde{P}_{w'} \bullet h\right) \bullet \left(\bar{h} \bullet \tilde{P}_{w'} \bullet h\right) &\cong \bar{h} \bullet \tilde{P}_{w'} \bullet (\mathrm{id}_w \oplus W) \bullet \tilde{P}_{w'} \bullet h \\ &\simeq \bar{h} \bullet \tilde{P}_{w'} \bullet \tilde{P}_{w'} \bullet h \\ &\simeq \bar{h} \bullet \tilde{P}_{w'} \bullet h. \end{aligned}$$

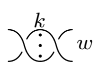
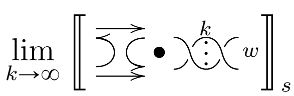
Since all non-identity webs appearing in $\tilde{P}_{w'}$ factor through a word of lower weight, these webs will not contribute an identity web to $\bar{h} \bullet \tilde{P}_{w'} \bullet h$. Noting that $\bar{h} \bullet h$ is the direct sum of an identity web and a web factoring through a word of lower weight, this shows that $\bar{h} \bullet \tilde{P}_{w'} \bullet h$ gives the categorified projector $\tilde{P}_w$.

Since any word can be obtained from the segregated word of the same weight via a sequence of permutations of the symbols $+$ and $-$, this proves Theorem 3.0.1 for arbitrary w. Theorem 3.0.2 also follows since $\mathrm{K}^\angle(\mathcal{F})$ is (essentially) closed under horizontal composition and

$$\begin{aligned} \chi(\tilde{P}_w) &= \chi(\bar{h} \bullet \tilde{P}_{w'} \bullet h) \\ &= \bar{h} \bullet \chi(\tilde{P}_{w'}) \bullet h \\ &= \bar{h} \bullet P_{w'} \bullet h \\ &= P_w. \end{aligned}$$

Since the above construction of $\tilde{P}_w$ for non-segregated w is somewhat indirect, a natural question to ask is whether this projector can also be realized as the stable limit of torus braids. In fact, the answer is yes. To illustrate this, let w' be a segregated word and suppose that w is the word that results from switching the last $+$ in w' with the first $-$. We then have



where  denotes k full twists on strands oriented according to w. The fact that  is null-homotopic follows from equation (5.3.14) and Lemma 4.3.6. Repeating this argument to switch all desired $+$'s and $-$'s gives the general result.

6

Categorification of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of framed tangles

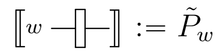
We now use the categorified projectors $\tilde{P}_w$ to give a categorification of the $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of framed tangles. Recall that this invariant assigns an element of the appropriate Hom-set in $\mathcal{S}$ for each labeling of the components of the framed tangle by finite-dimensional irreducible representations of $\mathfrak{sl}_3$. In particular, this assigns an element of $\mathbb{C}(q)$ to each labeled, framed link.

Our definition of the categorified $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant is given by replacing every structure in the combinatorial formulation of this invariant, described in Section 2.4, by its categorified counterpart.

Definition 6.0.1. *The categorified $\mathfrak{sl}_3$ Reshetikhin-Turaev invariant of an r-component framed tangle T with i^{th} component labeled by the finite-dimensional irreducible representation of highest weight λ_i, denoted $[\![T]\!]_{\{\lambda_i\}_{i=1}^r}$, is computed as follows: choose words w_i with weights λ_i, cable each component according to the framing with strands directed according to the corresponding words, insert $\tilde{P}_{w_i}$ anywhere along the relevant components, and evaluate using the categorified $\mathfrak{sl}_3$ skein relations (equations* (2.5.1) *and* (2.5.2)*) to obtain a complex in* $\mathrm{K}^{\angle}(\mathcal{F})$.

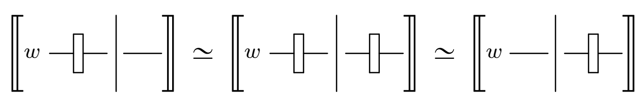
To prove that this defines an invariant of framed tangles it suffices to show that the resulting complex is invariant up to homotopy under $R2$ and $R3$ Reidemeister moves and does not depend on where $\tilde{P}_{w_i}$ is inserted along the i^{th} component.

We first establish some diagrammatic notation. Let



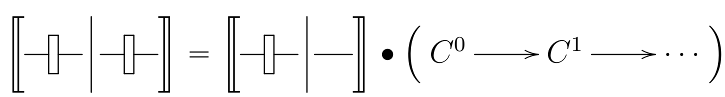
and given a tangle T denote the composition ⟦w ─▯─⟧ $\bullet$ ⟦T⟧ by ⟦w ─▯─ $\bullet$ T⟧ (and similarly for composition on the left).

Lemma 6.0.2. *Let* w *be a word. We have*

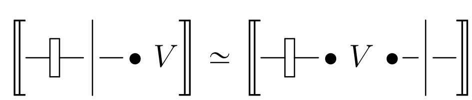


where the vertical strand can be oriented in either direction. A similar result holds for sliding a categorified projector over a strand.

Proof. It suffices to show the first homotopy equivalence. We have (dropping the word specifying the projector)

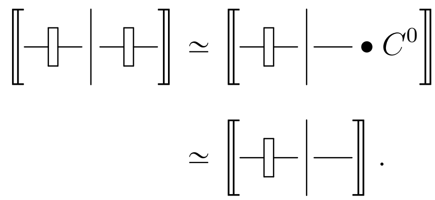


where C^i is a direct sum of webs annihilated by $\tilde{P}_w$ for $i > 0$ and C^0 is the direct sum of an identity web and webs annihilated by $\tilde{P}_w$. If V is one such (non-identity) web, equation (2.6.11) gives



$$\simeq 0.$$

It then follows, using Propositions 4.1.3 and 4.3.6, that



□

Note that a form of the above argument previously appeared in [7].

Proposition 6.0.3. *The complex assigned to a labeled, framed tangle according to Definition 6.0.1 is invariant up to homotopy equivalence under* $R2$ *and* $R3$ *Reidemeister moves and choice of where along each component the categorified projector is inserted.*

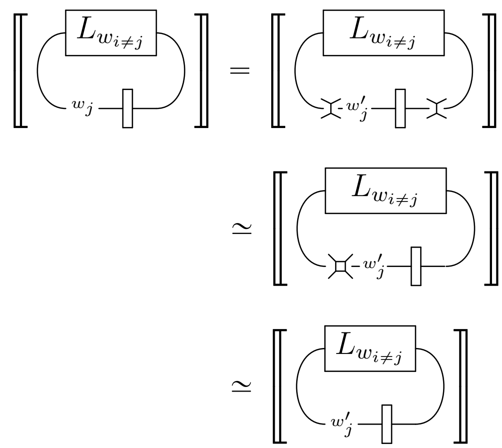
Proof. The preceding lemma shows that the categorified projector can be slid along a component to any desired location without changing the complex up to homotopy. Invariance under $R2$ and $R3$ Reidemeister moves follows since we can assume that the projector is not located in the region of the tangle diagram where the moves take place. □

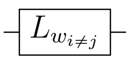
Theorem 3.0.4 now follows from Proposition 6.0.3, Theorem 3.0.2, and the analogous definitions of the categorified and decategorified invariants.

Note that $[\![T]\!]_{\{\lambda_i\}_{i=1}^r}$ appears to depend on the choice of words w_i with weight λ_i; in particular, if the tangle has boundary they determine the Hom-category in which this invariant takes values. All such Hom-categories are isomorphic, so this does not present an issue.

More troubling, perhaps, is the apparent dependence on the words labeling the closed components. We now show that the invariant does not depend on this choice. For simplicity, suppose that the tangle is in fact a link L. Let w_j be a word of weight

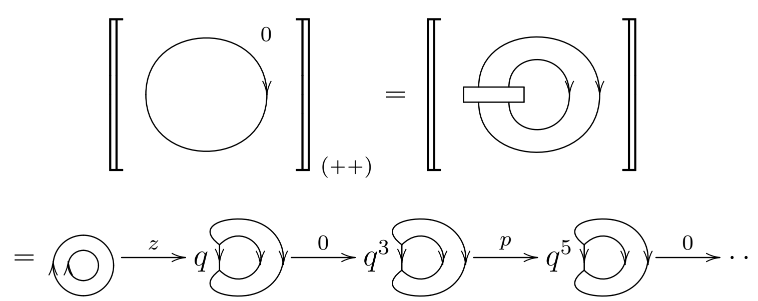
λ_j and w'_j be a word obtained from w_j by transposing the symbols $+$ and $-$ in w_j. We then compute



where  denotes the tangle obtained from removing an arc from the j^{th} component of L (cabled appropriately with $\tilde{P}_{w_i}$ inserted along the i^{th} component for $i \neq j$) and  and  denote the tensor product of these webs with appropriate identity webs. The first homotopy equivalence follows via sliding the H-web through the cabled link L. A similar argument applies to closed components in arbitrary tangles.

We conclude with some explicit computations of this invariant.

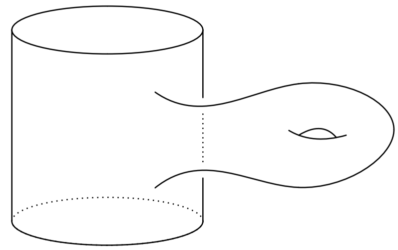
Example 6.0.4. *Using Example 5.2.5 we find that*



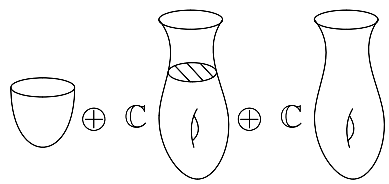
where p is the foam which zips and then unzips along the two downward arcs. Using Gaussian elimination (and a somewhat involved foam calculation), we find that this complex is homotopy equivalent to the complex

$$q^{-2}\bigcirc \longrightarrow 0 \longrightarrow q^{2}\bigcirc \xrightarrow{T} q^{6}\bigcirc \xrightarrow{0} q^{6}\bigcirc \xrightarrow{T} q^{10}\bigcirc \xrightarrow{0} \cdots$$

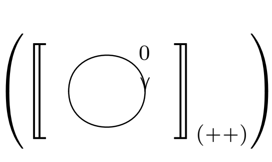
where

$T =$ 

and we have omitted the orientation of the circles. Applying the functor $\widehat{\mathrm{Hom}}(\varnothing, -)$ *to obtain a complex of graded vector spaces, we have that*

$\widehat{\mathrm{Hom}}(\varnothing, \bigcirc) = \mathbb{C}$ 

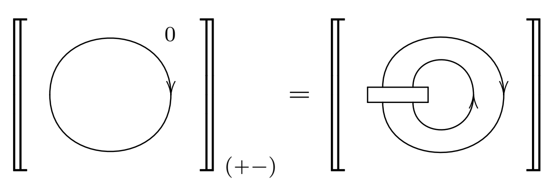
in gradings 2, 0, *and* -2 *respectively. We find that the map* $\widehat{\mathrm{Hom}}(\varnothing, T)$ *has rank* 1, *giving the cohomology as*

$\mathcal{H}^{i,j}$ 

$$= \begin{cases} \mathbb{C} & i = 0 \text{ and } j = -4, -2, 0 \\ \mathbb{C} & i = 2k \text{ and } j = 4k-4, 4k-2 \text{ for } k > 0 \\ \mathbb{C} & i = 2k+1 \text{ and } j = 4k+2, 4k+4 \text{ for } k > 0 \\ 0 & \text{else .} \end{cases}$$

In the above formula, i *denotes homological degree while* j *denotes the vector space grading.*

Example 6.0.5. *Using Proposition 5.3.5 we compute*



$$= \circledcirc \xrightarrow{s} q^2 \bigcirc \xrightarrow{0} q^4 \bigcirc \xrightarrow{-3T} q^8 \bigcirc \xrightarrow{0} q^{10} \bigcirc \xrightarrow{-3T} \cdots$$

$$\simeq q^{-2}\bigcirc \oplus \bigcirc \longrightarrow 0 \longrightarrow q^4\bigcirc \xrightarrow{T} q^8\bigcirc \xrightarrow{0} q^{10}\bigcirc \xrightarrow{T} q^{14}\bigcirc \xrightarrow{0} \cdots$$

where again we have omitted the orientation of the circles. Applying $\widehat{\mathrm{Hom}}(\varnothing, -)$ *and taking cohomology gives*

$$\mathcal{H}^{i,j}\left(\left[\!\left[\ \bigcirc^{0}\ \right]\!\right]_{(+-)}\right) = \begin{cases} \mathbb{C}^2 & i=0 \textit{ and } j=-2,0 \\ \mathbb{C} & i=0 \textit{ and } j=-4,2 \\ \mathbb{C} & i=2k \textit{ and } j=6k-4, 6k-2 \textit{ for } k>0 \\ \mathbb{C} & i=2k+1 \textit{ and } j=6k+2, 6k+4 \textit{ for } k>0 \\ 0 & \textit{else} \ . \end{cases}$$

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Biography

David Emile Vatcher Rose was born on December 31, 1983 in Virginia Beach, Virginia. He attended the Hebrew Academy of Tidewater, Providence Elementary School, Kemps Landing Magnet School, Kempsville High School, and The College of William and Mary, obtaining a B.S. in Mathematics and Physics summa cum laude with highest (research) honors from the latter in May 2006. He also was awarded the William and Mary Prize in Mathematics. While at William and Mary, David captained the club ice hockey team, played soprano and alto sax in the William and Mary saxophone quartet, and was a member of the Sigma Alpha Epsilon frat.

David spent the 2006-2007 academic year completing Part III of the Mathematical Tripos at Christ's College, University of Cambridge. He obtained the Certificate of Advanced Studies with merit in 2007 and was also a member of the varsity ice hockey team, earning a half-blue.

In Fall 2007, David entered the Duke Mathematics Department. He obtained an M.A. in 2009 and a Ph.D. in 2012, under the supervision of Lenny Ng. During his time at Duke, he won both the L.P. and Barbara Smith award for beginning teachers and the Bass fellowship. David will continue his academic career as a Busemann Assistant Professor (NTT) at the University of Southern California, beginning Fall 2012.