Control of Vibratory Energy Harvesters in the Presence of Nonlinearities and Power-Flow Constraints

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Civil and Environmental Engineering in the Graduate School of Duke University
2012
Abstract
(Civil and Environmental Engineering)

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Abstract

Over the past decade, a significant amount of research activity has been devoted to developing electromechanical systems that can convert ambient mechanical vibrations into usable electric power. Such systems, referred to as vibratory energy harvesters, have a number of useful applications, ranging in scale from self-powered wireless sensors for structural health monitoring in bridges and buildings to energy harvesting from ocean waves. One of the most challenging aspects of this technology concerns the efficient extraction and transmission of power from transducer to storage. Maximizing the rate of power extraction from vibratory energy harvesters is further complicated by the stochastic nature of the disturbance. The primary purpose of this dissertation is to develop feedback control algorithms which optimize the average power generated from stochastically-excited vibratory energy harvesters.

This dissertation will illustrate the performance of various controllers using two vibratory energy harvesting systems: an electromagnetic transducer embedded within a flexible structure, and a piezoelectric bimorph cantilever beam. Compared with piezoelectric systems, large-scale electromagnetic systems have received much less attention in the literature despite their ability to generate power at the watt–kilowatt scale. Motivated by this observation, the first part of this dissertation focuses on developing an experimentally validated predictive model of an actively controlled electromagnetic transducer. Following this experimental analysis, linear-quadratic-Gaussian control theory is used to compute unconstrained state feedback controllers...
for two ideal vibratory energy harvesting systems. This theory is then augmented to account for competing objectives, nonlinearities in the harvester dynamics, and non-quadratic transmission loss models in the electronics.

In many vibratory energy harvesting applications, employing a bi-directional power electronic drive to actively control the harvester is infeasible due to the high levels of parasitic power required to operate the drive. For the case where a single-directional drive is used, a constraint on the directionality of power-flow is imposed on the system, which necessitates the use of nonlinear feedback. As such, a sub-optimal controller for power-flow-constrained vibratory energy harvesters is presented, which is analytically guaranteed to outperform the optimal static admittance controller. Finally, the last section of this dissertation explores a numerical approach to compute optimal discretized control manifolds for systems with power-flow constraints. Unlike the sub-optimal nonlinear controller, the numerical controller satisfies the necessary conditions for optimality by solving the stochastic Hamilton-Jacobi equation.
To my parents Karyn Lerner and Lawrence Cassidy
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Symbols

Bolded symbols denote vectors or matrices. Below is a non-exhaustive list of symbols used in this dissertation.

- **A**  Dynamics matrix.
- **A_0**  Acceleration amplitude.
- **B**  Control input matrix.
- **B**  Rotational viscous damping coefficient.
- **C**  Output matrix.
- **F**  Force matrix.
- **F_c**  Total Coulomb friction force.
- **G**  Exogenous disturbance input matrix.
- **H**  Hamiltonian matrix.
- **J**  Rotational inertia coefficient.
- **K**  State feedback gain matrix.
- **\tilde{K}**  Partial-state feedback gain matrix.
- **K_e**  Back-emf voltage magnitude.
- **K_t**  Back-emf motor constant.
- **L_c**  Stator coil inductance.
- **P_{gen}**  Average power generated.
$\bar{P}_L$ Average power generated across a resistive load.

$R$ Effective resistance of the electronics.

$R_c$ Stator coil resistance.

$R_L$ Load resistance.

$R_m$ MOSFET resistance.

$S$ Covariance matrix.

$V_d$ Diode conduction voltage.

$V_S$ Storage bus voltage.

$Y^{\text{max}}$ Maximum static admittance.

$Y_s$ Static admittance.

$a$ Disturbance acceleration.

$c_e$ Electromechanical damping coefficient.

$e_{an}, e_{bn}, e_{cn}$ Back-emf voltages for three stator phases.

$f$ Total transducer force.

$f_c$ Coulomb friction force amplitude.

$f_e$ Electromechanical force.

$i$ Control current.

$i_a, i_b, i_c$ Electrical currents in three stator phase windings.

$l$ Ballscrew lead.

$v$ Harvester voltage.

$v_{an}, v_{bn}, v_{cn}$ Line-to-neutral voltages for stator coils.

$w$ White noise process.

$x$ Harvester relative displacement.

$x$ State vector.

$\alpha$ Electrical and mechanical natural frequencies squared.

$\zeta_a$ Disturbance filter damping ratio.
\( \theta_m \) Angular rotor position.

\( \sigma_a \) Disturbance acceleration standard deviation.

\( \Phi_a \) Disturbance acceleration power spectral density.

\( \omega_a \) Disturbance filter bandpass frequency.

\( \nabla \) Gradient operator.

\( \mathcal{E} \) Expectation operator.

Abbreviations

Below is a non-exhaustive list of abbreviations used in this dissertation.

CCM Continuous conduction mode.

CO Clipped-optimal.

DCM Discontinuous conduction mode.

emf Electromotive force.

HJE Hamilton-Jacobi equation.

LMI Linear matrix inequality.

LQG Linear-quadratic-Gaussian.

MOSFET Metal-oxide-semiconductor field-effect transistor.

PG Performance-guaranteed.

PS Pseudospectral.

PSF Partial-state feedback.

PWM Pulse-width modulation.

PZT Lead zirconate titanate.

RTHT Real-time hybrid testing.

SA Static admittance.


TMD Tuned mass damper.
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Electromechanical systems to harvest energy from mechanical vibrations have been the subject of considerable research activity over the last decade. Most of this research has focused on powering wireless intelligence systems embedded within smart structures. However, energy harvesting from large vibrating structures (e.g., buildings, bridges, vehicles, and offshore structures) has recently been shown to be a viable source of renewable energy. An example of a generic single-transducer vibratory energy harvester can be seen in Figure 1.1. As shown, energy conversion is accomplished by a passive electromechanical system, which consists of a flexible

\[ 	ext{Figure 1.1: Generic single-transducer vibratory energy harvester.} \]
mechanical structure with an embedded transducer to generate power from an exoge-
nous disturbance. The transducer must be interfaced with some type of electronic
circuitry, which is used to convert the AC power extracted by the transducer to DC
power that can be delivered to a storage device, such as a rechargeable battery or a
supercapacitor. Furthermore, the use of feedback control can be used to operate the
electronic circuitry and enhance the efficiency of the energy harvester, which is the
focus of this dissertation.

1.1 Energy Harvesting Transducers

Vibratory energy harvesting research has primarily focused on low-power applica-
tions requiring energy-autonomy, such as wireless sensing and embedded computing
systems [5, 9, 61, 93, 96]. There are several modes of transduction available for
these milliwatt-scale applications. The diagrams in Figure 1.2 illustrate the three
most common types of vibratory energy harvesting transducers. Piezoelectric trans-
ducers, such as the one shown in Figure 1.2(a), have received the most attention.

![Diagram of energy harvesting transducers](image)

**Figure 1.2:** Common vibratory energy harvesting transducers: (a) piezoelectric,
(b) electromagnetic, and (c) electrostatic.
The type of piezoelectric material selected for an energy harvesting application can have a major impact on the functionality and performance of the device. To date, a number of different piezoelectric materials have been developed. The most common type of piezoelectric material used in energy harvesting applications is lead zirconate titanate (PZT), which is a piezoelectric ceramic. Another less common piezoelectric material, which was investigated in [68], is polyvinylidene fluoride.

Sodano et al. [111, 112] compared and experimentally tested the energy harvesting capability of three types piezoelectric devices. The materials used in those studies included a traditional PZT patch, a quick pack actuator, and a macro-fiber composite. Regardless of the type of piezoelectric material used, the most important feature is its ability to convert strain into electrical charge. A transducer’s ability to convert mechanical energy into electrical energy is called its electromechanical coupling. For piezoelectric transducers, two coupling modes have been extensively studied in the literature: the 31 mode and the 33 mode. The 31 mode refers to the case where a force is applied in the direction perpendicular to the poling direction (e.g., the bending of a beam that is poled in the direction perpendicular to its top and bottom surfaces). In the 33 mode, a force is applied in the same direction as the poling direction (e.g., the compression of a block that is poled in same direction as the loading). Conventionally, the 31 mode has been the most commonly used coupling mode. However, it has been shown that the 31 mode yields a lower electromechanical coupling coefficient than the 33 mode [8].

The 31 coupling mode has led to a number of studies involving the bonding configuration of piezoelectric materials to a cantilever beam. Many conventional systems consist of a single piezoelectric layer, which is referred to as a unimorph [56]. Another common configuration, referred to as the bimorph, consists of two bonded piezoelectric layers in bending. A number of studies have investigated the theoretical and experimental benefits of using the bimorph configuration to harvest
energy from base excitations [29, 36, 54, 86, 109, 113]. In particular, Ng and Liao [86] experimentally tested the energy harvesting capability of a bimorph beam with the piezoelectric patches connected in series and parallel and compared those results to a unimorph beam. The results of that study showed that the best configuration depends on the load resistance and excitation frequency.

The available power from a piezoelectric energy harvester depends on the frequency content of the excitation. For harvesters operating in their linear dynamic range, power availability is maximized near resonance by designing the harvester to be very lightly damped. However, this also makes power availability very sensitive to the excitation frequency, which results in much lower power levels when this frequency is uncertain. To overcome this sensitivity to excitation frequency, there has been growing research interest in exploiting the nonlinear behavior of systems with multiple equilibria to enhance the bandwidth of piezoelectric systems. The most common nonlinear vibratory energy harvesting configuration consists of a bistable cantilever beam with piezoelectric coupling. In the study by Stanton et al. [118], an experimental system involving a piezoelectric cantilever with a permanent magnet end mass interacts with the field of oppositely poled stationary magnets to create a bistable system. It is shown that adjusting the location of the stationary magnets can result in hardening and softening effects in the system. Similar nonlinear vibratory energy harvesting systems have been developed in [6, 75, 119]. However, it was recently pointed out in [27] that the theoretical expected power generation of Duffing-type (i.e., nonlinear stiffness) energy harvesters excited by broadband disturbances is independent of the nonlinearity. That study also demonstrated how damping and inertia nonlinearities may be used to enhance the theoretical expected power generation over the linear case. Recently, the study by Green et al. [46] verified these results using an experimental Duffing-type energy harvester with Coulomb damping.

4
Electromagnetic transducers have also been investigated for vibratory energy harvesting applications. Consider the transducer in Figure 1.2(b), which consists of a permanent magnet oscillating within a coil. This type of system generates electrical power through Faraday's law of induction. The amount of electricity generated from electromagnetic transducers depends upon the strength of the magnetic field, the velocity of the relative motion between the magnet and coil, and the number of turns of the coil. Several electromagnetic energy harvesting devices have been proposed in the literature for milliwatt-scale applications [34, 43, 128]. In particular, the study by Glynne–Jones et al. [43] investigated several experimental device prototypes consisting of different magnet and coil configurations.

Large-scale energy harvesting from vibrating structures (e.g. multi-story buildings and bridges) using electromagnetic transducers has recently been considered as a viable source of renewable energy. For example, it has been estimated that kilowatts of power are available from wind excitations on tall buildings [87]. In addition, the recent studies by Scruggs and Lattanzio [66, 107], investigated the use of feedback controllers to maximize power generation of a wave energy converter in a stochastic sea. In those studies, the wave energy converter consisted of a floating buoy that is tethered to electromagnetic generators at the ocean floor. Linear electromagnetic transducers also have application in vehicles, including automotive suspensions [135] and railway systems [83]. Choi and Wereley [24] present another potential application of energy harvesting from vibrations in large structures. In that study, an electromagnetic transducer provides parasitic power for other semi-active structural control devices, such as magnetorheological dampers, during seismic events.

The third mode of transduction for small-scale vibratory energy harvesting applications is electrostatic, which is shown in Figure 1.2(c). The basis of electrostatic energy conversion is the variable capacitor [4, 81, 82]. For example, the electrostatic transducer in Figure 1.2(c) is referred to as an in-plane overlap converter because the
change in capacitance arises from the changing overlap area of the many interdigitated fingers. As the center plate vibrates in the vertical direction, the capacitance of the fingers changes, which generates electric power. Two additional types of electrostatic transducers are the in-plane gap closing converter and the out-of-plane gap closing converter. The operation of these three types of transducers are discussed in more detail in [96]. Since the focus of the examples in this dissertation is limited to piezoelectric and electromagnetic transduction, interested readers should consult the review articles in [9, 96] for more information on electrostatic transducers.

1.2 Power Conditioning Circuits for Vibratory Energy Harvesting

Regardless of the application or scale, one of the most challenging aspects of vibratory energy harvesting technology concerns the efficient extraction and transmission of power from the transducer to a storage device. At the very least, such power

\[
\begin{align*}
&\text{(a)} \quad \text{transducer} \quad + \quad i \quad \text{storage} \quad - \quad V_s \\
&\text{(b)} \quad \text{transducer} \quad + \quad i \quad \text{storage} \quad - \quad V_s \\
&\text{(c)} \quad \text{transducer} \quad + \quad i \quad \text{storage} \quad - \quad V_s \\
&\text{(d)} \quad \text{transducer} \quad + \quad i \quad \text{storage} \quad - \quad V_s 
\end{align*}
\]

**Figure 1.3:** Power conditioning circuits interfaced with energy storage: (a) passive diode bridge rectifier, (b) PWM-switched controllable DC/DC (buck-boost) converter, (c) parallel SSHI circuit with passive rectifier, and (d) PWM-switched controllable bi-directional H-bridge.
conversion requires rectification of the transducer voltage through a standard diode bridge interfaced directly with storage [4, 96, 111]. This type of power conditioning circuit is shown in Figure 1.3(a). Utilizing a bridge rectifier is advantageous for applications in which the disturbance is sinusoidal because it creates an entirely passive system. However, there are two distinct limitations to this approach. First, the transducer voltage will only connect to the storage bus voltage when the diode bridge is forward biased (i.e., when $|v| > V_S$). Second, when power does flow from the transducer to storage there is no direct way to control the rate of this power-flow. In the absence of such control, the rectifier circuit may impose excessive damping on the harvester, which suppresses the transducer voltage and results in very low amounts of power generation. To reduce some of the losses associated with biasing, some researchers have proposed replacing the diodes in the bridge rectifier with MOSFETs in a diode-tied configuration (i.e., the source is shorted to the gate) [67].

Motivated by such observations, a number of researchers have proposed the use of high-frequency PWM-switched DC/DC converters to control the power extracted by vibratory energy harvesting systems [60, 122] The particular circuit shown in Figure 1.3(b), which was demonstrated in [69], is called a buck-boost converter because the output voltage magnitude $|V_S|$ can be less than or greater than the input voltage magnitude $|V_T|$. Power-flow in such converters is regulated via high-frequency pulse-width modulation (PWM) switching control of the MOSFETs in the circuitry. Switching converters are beneficial in small-scale vibratory energy harvesting applications because they only require that a single MOSFET be switched, thus reducing gating losses. However, one drawback of the such systems is that the directionality of power-flow is limited to extraction.

The study by Ottman et al. [89] demonstrated the operation of an experimental buck converter in discontinuous conduction mode (DCM) to extract power from a piezoelectric transducer for energy harvesting applications. Subsequent studies using
buck-boost converters in DCM have been examined in [64, 69, 90]. In the DCM regime, the inductor fully demagnetizes (i.e., $i_L$ drops to zero) before the end of the switching cycle, and remains so until the MOSFET gates on again at the leading edge of the next switching cycle. An advantage of operating a buck-boost converter in DCM is that it has an input admittance which is theoretically decoupled from the behavior of the storage voltage. Furthermore, if the transducer side capacitance, $C_T$, is sufficiently small, then the admittance is approximately linear and resistive, with the value of the effective resistance being proportional to the inverse-square of the switching duty cycle. Therefore, the design of these converters for optimal operation proceeds by first determining the effective shunt resistance which maximizes power generation and then synthesizing the duty cycle from this resistance. This technique is often referred to in the literature as “resistive impedance matching.”

For vibratory energy harvesting applications in which the disturbance is concentrated at a single frequency, such converters can be incorporated into a larger passive network and can be tuned to further optimize the power generated. Many researchers [99, 109, 120] have observed that the theoretical optimal power extraction from a linear energy harvester is attained by imposing a circuit with a linear input impedance (assuming the transmission losses in the circuit can be neglected). Specifically, this optimal impedance is matched to the complex conjugate transpose of the harvester’s driving point impedance as measured from the transducer terminals. For example, the optimal energy harvesting circuit for a single-transducer piezoelectric system is comprised of a linear resistor together with an inductor. Thus, for any single-transducer system with linear dynamics that is driven by a sinusoidal disturbance, the optimal energy harvesting circuit can be realized by implementing an appropriate DC/DC converter in DCM in combination with a passive linear reactance.

In many applications, the additional passive reactive components required to facilitate this matching condition are too large to be practical. For example, a
piezoelectric transducer responding at resonance exhibits a sizable internal capacitive reactance, resulting in a very low power factor (i.e., the cosine of the phase angle between the velocity-proportional internal current source for the transducer’s Norton equivalent circuit and its open-circuit voltage). As previously mentioned, the reactive component of the matched impedance in piezoelectric transducers is an inductance, the value of which can be hundreds of Henries. An inductance of this size is orders of magnitude too large for small-scale applications. Efforts to implement power factor correction in piezoelectric applications, while requiring smaller inductors, have given rise to various nonlinear circuits based on synchronized switch harvesting on inductor (SSHI) concepts. The circuit in Figure 3.3(c) illustrates the “parallel SSHI” configuration, which was investigated in [7, 47, 70]. Interested readers should see [74, 122, 127] for recent reviews of the rapidly evolving literature on SSHI circuits.

The parallel SSHI circuit in Figure 3.3(c) is operated by closing the switch $S_1$ every time a peak in transducer displacement is detected. Once the switch closes, the parallel $L-C$ circuit formed by the switched inductor and the internal piezoelectric capacitance quickly oscillates through one half-cycle. Typically, the value of $L$ is chosen to be small enough such that the half-cycle period is much shorter than the excitation period. At the end of the half-cycle, the inductor current drops to zero, which triggers the switch to open again and remain so until the next displacement peak. This switching operation forces a near instantaneous reversal of the magnitude of transducer voltage, thus causing $v$ to exhibit zero crossings at approximately the same times as the transducer velocity. As such, the fundamental harmonic of power generation (i.e. at the excitation frequency) exhibits an effective power factor close to unity. However, unlike the linear impedance-matched circuit, the inductance necessary to implement SSHI circuits is only limited by practical considerations, such as ratings on peak current and dissipation.

The study by Liu et al. [76] points out that SSHI circuits have a distinct dis-
advantage, in that the inductor current is uncontrollable during the time when the switch changes positions. As such, the only way to adjust the design to bound the peak value of the inductor current is to increase $L$. However, increasing $L$ slows the voltage reversal time during switching. Additionally, large inductor currents result in higher root mean square (RMS) losses for the circuit, and thus, there is an advantage to implementing a circuit that can regulate current during the voltage reversal, while still yielding fast reversal times. Motivated by this observation, Liu et al. [76] advocate the use of a fully active PWM-controlled H-bridge circuit, as illustrated in Figure 1.3(d), which allows for full control of the transducer current. This circuit is referred to a bi-directional converter because power can flow both ways through the converter (i.e., from $v$ to $V_S$ and vice versa).

As with the single-directional DC/DC converter in Figure 1.3(b), current control in an H-bridge is accomplished via high-frequency PWM switching. However, operation of the H-bridge requires control of four MOSFETs and thus requires more parasitic power than a one-directional converter, which is the price paid for two-way directionality of power-flow. Several previous studies have used H-bridge drives for various piezoelectric applications; [22, 79], including piezoelectric energy harvesting [76]. H-bridge control of currents for electromagnetic transducers is standard, and their use in electromagnetic energy harvesting applications has been investigated in [126].

Regardless of the application, the H-bridge must be capable of high-bandwidth current tracking. This technology is well understood and can be accomplished using hysteretic switching or PWM techniques [59]. By appropriate switching of its four MOSFETs, an H-bridge can be controlled to raise or lower the transducer current $i$ arbitrarily to track a desired value. The suitability of implementing an H-bridge circuit for vibratory energy harvesting depends on the power scale of the application, and how this compares with technological details relating to parasitic losses associ-
ated with electronic control. The focus of the first half of this dissertation will center around the use of H-bridges to optimally control the rate of power extraction from vibratory energy harvesters excited by stochastic disturbances.

1.3 Dissertation Overview

1.3.1 Research Contributions

The purpose of this dissertation is to develop feedback controllers for vibratory energy harvesters; i.e., the laws governing how much current the electronics should extract, based on sensor feedback signals. Specifically, this work addresses the challenge of augmenting standard linear-quadratic-Gaussian (LQG) control theory to optimize stochastic power output, as well as account for nonlinearities in the system dynamics, non-quadratic performance measures, and mixed constraints on the states and control input. These challenges arise as a result of the dynamics of the energy harvester itself as well as the power electronic circuitry used to regulate the rate of power extraction. Traditionally, vibratory energy harvesting research focuses on the design and modeling of flexible electromechanical systems and treats the maximization of electrical power generation as an afterthought. For example, a significant amount of theoretical and experimental work has investigated the response of piezoelectric patches bonded to base-excited cantilever beams. The load resistance that maximizes the average power generation from these systems is then computed using standard resistive impedance matching techniques. However, this type of analysis neglects the fact that the load resistance must be synthesized by a power electronic converter, which is not 100% efficient. Furthermore, it is often assumed that the harvester is excited at resonance by a sinusoidal disturbance when in general, it is more appropriate to model the disturbance as a stochastic process.

Motivated by the above observations, this dissertation presents four main research contributions to the field of vibratory energy harvesting:
1. An experimental demonstration of the energy harvesting capability of an actively controlled electromagnetic transducer. Unlike a majority of research that exists in the literature, which focuses on systems capable of generating power at the microwatt–milliwatt scale using piezoelectric or electrostatic transducers, we present an in-depth analysis of an electromagnetic energy harvester for large-scale (i.e., watt-kilowatt scale) structural vibration applications. In addition, we develop a predictive model that can be used to characterize the mechanical and electrical dynamics of the transducer over a wide range of operating conditions.

2. The derivation of state feedback controllers for ideal stochastically-excited vibratory energy harvesters. Specifically, it is shown that for stochastic disturbances characterized by second-order, bandpass-filtered white noise, energy harvesters can be passively “tuned” such that the optimal feedback controller only requires half of the system states. One can view these tuning techniques as a “stochastic counterpart” to the tuning techniques used in harvesting applications characterized by deterministic disturbances. Motivated by this result, fixed-structure feedback optimization techniques are utilized to determine feedback controllers for the case in which the availability of specific system states is fixed in the design.

3. A feedback control procedure to account for competing objectives and nonlinearities in the harvester dynamics. We employ a linear matrix inequality (LMI)-based approach to balance the energy harvesting objective against other structural response objectives. This problem can be formulated as a multi-objective optimization problem in which the controller seeks to minimize a structural response performance measure subject to a constraint on power generation. In addition, we account for mechanical nonlinearities in the harvester dynamics
using statistical linearization, whereby the nonlinearities are replaced by equivalent linear terms calculated from the stationary covariance of the closed-loop system. It is shown that the covariance matrix and optimal feedback gain matrix can be computed by implementing an iterative algorithm. This theory is then further extended to account for a non-quadratic transmission loss model in the power electronics.

4. The synthesis of controllers for power-flow-constrained vibratory energy harvesters. If a harvester is controlled using a converter capable of only single-directional power-flow (e.g., the buck-boost converter in Figure 1.3(b)), then a constraint is placed on the domain of feasible feedback laws. We address this constraint using two different techniques. The first technique utilizes a nonlinear feedback controller that is analytically guaranteed to outperform the optimal static admittance controller in stationary stochastic response. The second technique is based on Hamilton-Jacobi theory whereby a numerical control manifold is found as an approximate solution to the stochastic Hamilton-Jacobi equation.

1.3.2 Organization

This material in this dissertation is organized as follows. Chapter 2 focuses on the experimental characterization of an actively controlled vibratory energy harvesting transducer with electromagnetic coupling. Chapter 3 outlines the general vibratory energy harvesting problem for ideal systems that are excited by filtered white noise and derives state feedback controllers for such systems using LQG control theory. In Chapter 4, this theory is augmented to account for competing structural response objectives using an LMI-based approach. Chapter 5 further expands the theory in Chapter 3 to account for systems with nonlinearities and non-quadratic electronic transmission losses using a statistical linearization approach. An iterative algorithm
is proposed to simultaneously compute the optimal stationary covariance matrix and the optimal state feedback gain matrix. Chapter 6 introduces the concept of power-flow-constrained vibratory energy harvesting (i.e., the case where the directionality of power-flow is limited to extraction). Two sub-optimal nonlinear feedback controllers are derived for such constraints and the performances of the controllers are compared through simulation of the two ideal systems. Finally, Chapter 7 uses Hamilton-Jacobi theory to derive the optimal numerical controller for an energy harvesting system with both nonlinearities and power-flow constraints. The discretized control manifold for a given system is computed using a combined pseudospectral and successive approximation approach.
Modeling and Experimental Characterization of an Electromagnetic Transducer

2.1 Background

This chapter focuses on the modeling and experimental characterization of an electromagnetic transducer for large-scale vibratory energy harvesting applications. While this system is a scaled-down version of a system which would be appropriate for full-scale structures, it serves as a bench-scale demonstration of the technology. The basic conversion system consists of a back-driven precision ballscrew, which is coupled to the shaft of a three-phase, permanent-magnet synchronous machine. The machine terminals are interfaced with a three-phase active power electronic drive, which delivers harvested energy to a power bus. A similar device was investigated by Scruggs and Iwan [104, 105] for the suppression of earthquake induced vibrations in buildings and later by Cassidy et al. [21] for energy harvesting applications.

Zuo et al. [135] studied energy harvesting from vibrations in vehicle suspensions using a linear electromagnetic transducer. That study illustrates how the magnetic flux intensity of the device can be improved using finite element modeling of the
coil and magnet assemblies. Recently, a study by Zhu et al. [134] experimentally investigated the capability of a linear electromagnetic device for vibration damping and energy harvesting. Additional studies [85, 91] have examined the capability of electromagnetic actuators to be used as dampers by dissipating the absorbed energy in various electrical subsystems. In particular, the study by Palomera-Arias et al. [91] provides a detailed theoretical model for a linear electromagnetic actuator, which could be used to design similar transducers for structural control or energy harvesting applications.

Evaluating the usefulness of a ballscrew transducer to harvest energy from large-scale structures requires the development of a high-fidelity model for use in feedback design and analysis. This task is challenging because the device is nonlinear. Several studies have developed analogous approaches to the one taken here, for characterizing the nonlinearities in magnetorheological (MR) devices [116, 131] as well as in electrorheological (ER) devices [41, 42, 57] for semi-active control applications. The nonlinearities that arise in MR and ER dampers are mainly due to the yielding stress of the fluid. However, the nonlinearities present in the proposed device are a result of the sliding friction of the ballscrew. In addition, hysteresis in the force-velocity plane is observed, which can be attributed to the elasticity of the timing belt that connects the ballscrew to the shaft of the synchronous machine. Considering these effects, a new model is proposed that is numerically tractable and effectively portrays the behavior of the transducer.

2.2 Device Modeling

An illustration of the electromagnetic transducer modeled in this chapter is shown in Figure 2.1. Linear-to-rotational conversion is accomplished via a precision ballscrew. The particular ball screw mechanism used here is a Thomson EC3 unit [63]. It has a lead of 16mm/rev, and a diameter of 16mm, resulting in a helical angle of
Figure 2.1: Illustration of the electromagnetic transducer consisting of a back-driven ballscrew and a permanent magnetic synchronous machine.

approximately 17 degrees. The force rating on the screw is 7.2kN, well in excess of the force range for the experiments conducted in this chapter. The ballscrew is interfaced with the motor shaft via a timing belt with a 1:1 ratio. The configuration shown, in which the motor is placed in tandem with the screw, was chosen because it allowed for easier mounting of the device at either end via clevis brackets. However, a custom configuration in which the motor shaft is mounted directly to the rotating screw, and which also permits a clevis bracket to be mounted behind the motor housing, would reduce the need for the timing belt and also reduce Coulomb friction in the system by reducing side-loads on both rotating shafts.

The linear velocity $\dot{x}$ of the device is related to its mechanical angular velocity $\dot{\theta}_m$ via the lead conversion $l$; that is,

$$\dot{x} = l\dot{\theta}_m \tag{2.1}$$

The linear-to-rotational conversion of the ballscrew can be modeled as relating the linear force $f$ to the electromechanical force $f_e$ of the motor, via the equation

$$f = f_e + f_b - \frac{J}{I^2} \ddot{x} - \frac{B}{I^2} \dot{x} \tag{2.2}$$

where the sign convention is that $f$ and $\dot{x}$ have the same directional sense. In the
above expression, $B$ and $J$ are the rotational viscosity and inertia, respectively, of the combined motor shaft and screw, and $f_b$ is the nonlinear damping force associated with the bearing friction of the ballscrew. For the remainder of this chapter we will express the total device force as

$$f = f_e + f_d$$  

(2.3)

where $f_d$ includes the inertia, viscous damping, and bearing friction of the ballscrew as well as other characteristics of the device. A detailed model describing this term is further discussed in Section 2.3.

The motor itself is a Kollmorgen AKM42E [62] three-phase permanent-magnet synchronous motor with polarity $p = 5$ [65]. An illustration of the cross section of a typical three-phase permanent-magnet synchronous machine can be seen in Figure 2.2(a). The choice of a rotary three-phase AC motor (as opposed to a rotary DC motor or a linear motor) was made based on practical considerations. Generally, for motors with comparable power ratings, ones with higher velocity (and lower torque) capability tend to be less massive, as they require less iron to sustain the necessary magnetic field. This becomes even more pronounced in the comparison between rotary and linear motors with comparable power ratings. Additionally, rotary synchronous three-phase machines are almost always more efficient as power generators, because they can generate a larger back-emf for a given mechanical velocity, while also possessing a lower coil resistance.

The internal voltages (i.e., back-emfs) for phases $a$, $b$, and $c$, relative to the neutral node $n$, can be expressed as a function of the mechanical angle $\theta_m$, as

$$e_{an} = K_e \dot{\theta}_m \cos (p\theta_m)$$  

(2.4a)

$$e_{bn} = K_e \dot{\theta}_m \cos (p\theta_m + \frac{2\pi}{3})$$  

(2.4b)

$$e_{cn} = K_e \dot{\theta}_m \cos (p\theta_m - \frac{2\pi}{3})$$  

(2.4c)
where $K_e$ is the back-emf constant. (See Figure 2.2(b) for a schematic.) Let the line-to-neutral voltages at the terminals of the three phases be $v_{an}$, $v_{bn}$, and $v_{cn}$. Ultimately, these voltages will be imposed by the circuitry to be attached to the motor. Then the line current into phase $a$ evolves according to

$$\frac{d}{dt}i_a = \frac{1}{L_c} \left( -R_c i_a + v_{an} - e_{an} \right) \quad (2.5)$$

where $L_c$ and $R_c$ are the line-to-neutral coil inductance and coil resistance, respectively, of the motor. Analogous equations hold for the $b$ and $c$ phases. For the remainder of this chapter $L_c$ and $R_c$ are assumed to be the same for each of the phases (this was confirmed experimentally). Thus, in matrix form, the equation describing the electrical dynamics of the motor is

$$\frac{d}{dt}i_{abc} = \frac{1}{L_c} \left( -R_c i_{abc} + v_{abc} - e_{abc} \right), \quad (2.6)$$

where $i_{abc}$, $v_{abc}$, and $e_{abc}$ are vectors of the line-to-neutral currents, voltages, and back-emfs, respectively. The parameter values for the EC3 ballscrew and AKM42E motor from Danaher Motion are listed in Table 2.1.

It is convenient to express the dynamics of a three-phase machine in terms of
Table 2.1: Parameter values for the EC3 ballscrew and AKM42E motor.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_e$</td>
<td>0.77N-m/A</td>
</tr>
<tr>
<td>$R_c$</td>
<td>2.41Ω</td>
</tr>
<tr>
<td>$L_c$</td>
<td>$8.93 \times 10^{-3}$H</td>
</tr>
<tr>
<td>$l$</td>
<td>$2.55 \times 10^{-3}$m/rad</td>
</tr>
<tr>
<td>$J$</td>
<td>$1.5 \times 10^{-4}$N-m-s$^2$</td>
</tr>
<tr>
<td>$B$</td>
<td>$1.2 \times 10^{-4}$N-m-s</td>
</tr>
</tbody>
</table>

We start by defining the vector of quadrature ($q$), direct ($d$), zero (0) coordinates, via the Park transformation [92].

We start by defining the vector of $q$, $d$, and 0 currents as a linear transformation from the line currents; i.e.,

$$i_{qd0} = P(\theta_m)i_{abc} \quad (2.7)$$

where

$$P(\theta_m) = \begin{bmatrix}
cos(p\theta_m) & \cos(p\theta_m + \frac{2\pi}{3}) & \cos(p\theta_m - \frac{2\pi}{3}) \\
sin(p\theta_m) & \sin(p\theta_m + \frac{2\pi}{3}) & \sin(p\theta_m - \frac{2\pi}{3}) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}. \quad (2.8)$$

Similarly, for the $q$, $d$, and 0 voltages,

$$v_{qd0} = P(\theta_m)v_{abc}. \quad (2.9)$$

Substituting Equations (2.7) and (2.9) into Equation (2.6) gives

$$\frac{d}{dt}i_{qd0} - p\dot{\theta}_m Qi_{qd0} = \frac{1}{L_c}(-R_ci_{qd0} + v_{qd0} - P(\theta_m)e_{abc}) \quad (2.10)$$

where the matrix $Q$ can be found as

$$Q = \left(\frac{\partial}{\partial \theta_m}P(\theta_m)\right)P^{-1}(\theta_m) = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \quad (2.11)$$

We now have three nonlinear differential equations that describe the dynamics of the
motor; i.e.,

$$\frac{d}{dt}i_q = \frac{1}{L_c} \left( -R_c i_q + v_q - p\dot{\theta}_m L_c i_d - \frac{3K_e}{2} \dot{\theta}_m \right)$$  \hspace{1cm} (2.12a)

$$\frac{d}{dt}i_d = \frac{1}{L_c} \left( -R_c i_d + v_d + p\dot{\theta}_m L_c i_q \right)$$  \hspace{1cm} (2.12b)

$$\frac{d}{dt}i_0 = \frac{1}{L_c} (-R_c i_0 + v_0)$$  \hspace{1cm} (2.12c)

The neutral node $n$ is inaccessible, and because of this, external voltages can only be applied from line-to-line. This results in the constraint $v_0 = \frac{1}{2} (v_{an} + v_{bn} + v_{cn}) = 0$. In the equations above, this results in $i_0 = 0$, which may be alternatively confirmed by Kirchhoff’s current summation law at the neutral node. Thus, Equation (2.12c) can be eliminated from the dynamical description of the system, which is fully characterized by Equations (2.12a) and (2.12b). Therefore, the electromechanical force that can be applied by the device is proportional to the quadrature current and can be expressed as

$$f_e = \frac{3K_e}{2l} i_q = K_l i_q$$  \hspace{1cm} (2.13)

where $K_l$ is the back-emf constant of the motor.

We now consider the case in which the transducer will ultimately be used in an application as shown in Figure 2.3. As shown, the transducer is attached to a single-degree-of-freedom (SDOF) oscillator with mass $m$, damping $c$, and stiffness $k$. Specifically, we assume that the SDOF oscillator consists of a 3000kg mass with a 0.5Hz natural frequency and 5% damping. One of the simplest ways to harvest power from this application is to have the electronics simulate resistive loads $R_L$ across the terminals of the motor. This can be realized by simply attaching three resistive loads in a star connection to the motor, which is illustrated in Figure 2.3. With this configuration, we have that $v_{abc} = -R_L i_{abc}$, and consequently $v_{q0} = -R_L i_{q0}$. As
such, the equations describing the $i_q$ and $i_d$ can be expressed as

$$\frac{d}{dt} i_q = \frac{1}{L_c} \left( -(R_c + R_L) i_q - p\dot{\theta}_m L_c i_d - \frac{3K_e}{2} \dot{\theta}_m \right)$$  \hspace{1cm} (2.14a)

$$\frac{d}{dt} i_d = \frac{1}{L_c} \left( -(R_c + R_L) i_d + p\dot{\theta}_m L_c i_q \right).$$  \hspace{1cm} (2.14b)

We next make the approximation that $i_d \approx 0$ under normal operating conditions. This assumption can be justified as follows. Assume that $\dot{\theta}_m$ is slowly-varying, one can solve for the $\dot{\theta}_m$-dependent equilibrium of $i_q$ and $i_d$ as

$$i_q = -\frac{3K_e(R_c + R_L)\dot{\theta}_m}{2(R_c + R_L)^2 + 2(L_c p\dot{\theta}_m)^2}$$  \hspace{1cm} (2.15)

$$i_d = -\frac{3K_e L_c p\dot{\theta}_m^2}{2(R_c + R_L)^2 + 2(L_c p\dot{\theta}_m)^2}.$$  \hspace{1cm} (2.16)

From these expressions, we have that the dynamics of $i_d$ can be neglected as long as $p\dot{\theta}_m \ll (R_c + R_L)/L$. For the experiments presented in this paper, the load resistance $R_L$ will never be below the coil resistance $R_c$. Therefore, if we conservatively make $R_L = R_c$ then in terms of the linear velocity of the device, the dynamics of $i_d$ can be neglected as long as $\dot{x}$ is well below 27.2cm/s. If the device were run at higher velocities, then vector control [92] using velocity feedback should be used to explicitly control $i_d$ to be zero. Although this would complicate the conceptual approach...
illustrated in Figure 2.3 (because the harvested currents would be controlled in a manner which is more complicated than a simple star-connected resistive impedance), the derived model would not change.

Using the simplifications that $i_0 = 0$ and $i_d = 0$, we have that the differential equation describing $i_q$ is

$$\frac{d}{dt}i_q = \frac{1}{L_c} \left( -(R_c + R_L)i_q + \frac{3K_e}{2}\dot{\theta}_m \right).$$

(2.17)

This expression is analogous to the electrical dynamics of a single phase DC machine. Next, we substitute

$$i_q = -\frac{v_q}{R_c + R_L}$$

(2.18)

into Equation (2.13), and we have that at low frequencies, where $i_q$ may be assumed to be a slowly-varying function of $\dot{\theta}_m$, the electromechanical force can be expressed as a function of the linear velocity of the device; i.e.,

$$f_e = -\frac{K_e^2}{R_c + R_L}\ddot{x} = -c_e\dot{x}.$$  

(2.19)

We can think of $c_e$ as the equivalent electromechanical viscous damping term associated with connecting resistive loads across the terminals of the motor.

Harvesting power from the resistive load configuration in Figure 2.3 can be accomplished by connecting the three-phase motor to a servo drive, which interfaces the machine with a DC power bus (also called a “DC link”). The block diagram in Figure 2.4 demonstrates this concept. The particular three-phase servo drive that is used in this chapter is an S16A8 analog servo drive from Advanced Motion Controls [2]. The servo drive requires command signals for two of the phases (denoted by $i^*_a$ and $i^*_b$) and determines the command for the third phase by enforcing the fact that the sum of the line currents is equal to zero. Tracking of the desired command signals
is accomplished through high-frequency pulsewidth modulation (PWM) switching control of six MOSFETs. The switching frequency of the MOSFETs has been set at 33kHz. Line-to-neutral voltages of two of the phases are sensed with analog differential amplifiers and the output voltage measurements are then sent to a dSpace DS1103 data acquisition system. In dSpace, the line-to-neutral voltages are divided by the desired load resistance and the resulting current command signals are sent to the servo drive.

The main advantage of using a motor drive is that it is capable of four quadrant regenerative operation. However, there are two main drawbacks to using an off-the-shelf drive for structural vibration applications. First, the switching frequency of the MOSFETs is 33kHz and this setting cannot be adjusted. This switching frequency is more than an order of magnitude higher than what is necessary to track the desired currents that the drive will impose on the motor. For energy harvesting applications, it has been shown in [89] that there is an optimal switching frequency of power electronic drives at which harvested power is maximized. Second, is the
way in which the supply power is converted into logic power. An analog step-down converter is used to reduce the supply voltage $V_S$ (which can range from 60–80V) to 12V to power various drive intelligence components. During this process a significant amount of power is dissipated. Both of these drawbacks result in significant parasitic power losses, which can be reduced by using a drive that customized for the specific application.

To demonstrate the force capability of the device, the servo drive is used to simulate various resistive loads for a sinusoidal displacement input. The device is back-driven with 0.1Hz displacement with an amplitude of 7.5cm and the results are presented in Figure 2.5. Resistance values of $30\Omega$, $9\Omega$, and $3\Omega$ in addition to the open circuit case were chosen to illustrate the full force range of the device at this velocity. The hysteresis plots in Figure 2.5(b) appear to exhibit predominantly

![Figure 2.5: Experimentally measured transducer force for various load resistances simulated by the servo drive: (a) force versus time, (b) force versus displacement, and (c) force versus velocity.](image-url)
viscous damping behavior, with a Coulomb friction force threshold. The high frequency oscillations that are present in each test are due to the bearing friction in the ballscrew. A detailed model describing these effects is presented in Section 2.3. Resistance control could be used to compensate for the oscillations in the hysteresis plots, due to spatially-dependent Coulomb friction. However, this issue is beyond the scope of this chapter, which is limited to a basic characterization and demonstration of the device model used later in this dissertation.

2.3 Experimental Characterization

2.3.1 Mechanical Model Formulation

To illustrate the characteristics of the electromagnetic transducer described in the previous section, experiments were conducted in which the transducer displacement was controlled via a hydraulic actuator, and various response quantities were measured. As previously discussed, it is known from the manufacturing specifications that the ballscrew itself has some amount of inertia and viscous damping. In addition, sinusoidal oscillations are present in the force-displacement plane and are due to the sliding friction force of the ballscrew. In most applications of precision ballscrews (which are for the purpose of positioning) this force is compensated for via feedback, and often its characteristics are not extensively tested by manufacturers. Furthermore, hysteresis in the force-velocity plane can be seen at low velocities (i.e., as the motion of the ballscrew changes direction). To formulate a mechanical model that effectively captures these nonlinear effects, we compare the fit responses of two different models to experimental responses.

Consider the mechanical model of the electromagnetic transducer in Figure 2.6. Let the inertia present in the ballscrew be represented by a rack-and-pinion (i.e., inerter) element [110] with equivalent mass $m_d$. Similarly, let the viscous damping present in the ballscrew be represented by $c_d$. In addition, we include the linear spring
\( k_d \) to account for the stiffness that is present in the device. The bearing friction of the ballscrew is modeled by a summation of two sinusoids that are a function of displacement \( x \); i.e.,

\[
f_{\text{bearings}} = (\gamma_1 \sin (\lambda_1 x + \phi_1) + \gamma_1 \sin (\lambda_1 x + \phi_1)) \text{sgn}(\dot{x})
\]  

(2.20)

where \( \lambda_1 = 1/l \) and \( \lambda_2 = 2\pi/r \). The distance \( r \) is the distance in the \( x \)-direction that the transducer travels to cause a bearing to roll the length of its diameter in the track of the ballscrew (assuming a no-slip condition between the bearings and the track); i.e.,

\[
r = 2d \sin(\beta)
\]  

(2.21)

where \( d \) is equal to the diameter of one bearing (i.e., \( d = 0.29 \text{cm} \)) and \( \beta \) is equal to the helical angle of the ballscrew (i.e., \( \beta = 17 \) degrees).

Next, we note that the internal degree-of-freedom represented by \( y \) is related to the tandem mounting of the synchronous machine and the ballscrew via the timing belt. Because the timing belt has some elasticity, we can model this effect by a nonlinear spring; i.e.,

\[
f_{\text{belt}} = k_{b1} y + k_{b3} y^3
\]  

(2.22)
where $k_{b1}$ and $k_{b3}$ are spring constants. An additional viscous damper $c_b$ is added to this degree-of-freedom to account for viscous damping associated with the shaft of the motor.

To determine the analytical expression for the total device force $f$, we start by balancing the forces about the rigid bar associated with the $y$ degree-of-freedom; i.e.,

$$f_{belt} + c_b \dot{y} = m_d (\ddot{x} - \ddot{y}) + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn}(\dot{x} - \dot{y}) .$$  \hfill (2.23)

Next, solving Equation (2.23) for $\ddot{y}$ results in

$$\ddot{y} = \frac{1}{m_d} (m_d \ddot{x} + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn}(\dot{x} - \dot{y}) - f_{belt} - c_b \dot{y}) \hfill (2.24)$$

where $y$ and $\dot{y}$ can be found through numerical integration with the measurements $x$, $\dot{x}$, and $\ddot{x}$ treated as inputs to the differential equation. The total force generated by the system can be found by summing the forces in the upper and lower sections of the system in Figure 2.6. Thus, the total force can be expressed by

$$f = m_d (\ddot{x} - \ddot{y}) + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn}(\dot{x} - \dot{y}) + f_{bearings} + c_e \dot{x} \hfill (2.25a)$$

$$= f_{belt} + c_b \dot{y} + f_{bearings} + c_e \dot{x} . \hfill (2.25b)$$

### 2.3.2 Parameter Fit and Model Verification

To determine the fit for the parameters in Figure 2.6, we use a similar procedure to the one in Spencer et al. [116]. In that study, the force and displacement data used to determine the parameters for the mechanical model is measured for a single-frequency sinusoidal displacement. However, we fit the proposed model by backdriving the device over a range of frequencies. A sine-sweep displacement ranging from 0.2Hz to 2Hz with a velocity envelope ranging from 4.5cm/s to 7.5cm/s was generated over 44s and used for this test. A plot of this displacement input can be seen in Figure 2.7. It should be noted that the terminals of the motor are left open for this test (i.e., $c_e = 0$). During this test, force, displacement, velocity, and
acceleration data was collected using a dSpace DS1103 data acquisition system at a sample rate of 1kHz.

Furthermore, we quantify the error between the experimentally measured force and the predicted force using the error norms defined in [116]. In this study, error norms for time, displacement, and velocity can be calculated by the following three expressions

\[
E_t = \sqrt{\frac{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2}{\sum_{i=1}^{m} (f_i - \mu_f)^2}}, \\
E_x = \sqrt{\frac{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2 |\dot{x}_i|}{\sum_{i=1}^{m} (f_i - \mu_f)^2}}, \\
E_{\ddot{x}} = \sqrt{\frac{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2 |\ddot{x}_i|}{\sum_{i=1}^{m} (f_i - \mu_f)^2}}
\]

(2.26)

where \( f \in \mathbb{R}^m \) is a vector of measured force values, \( \mu_f \) is the mean value of the measured force vector, \( \hat{f} \in \mathbb{R}^m \) is a vector of predicted force values, \( \dot{x} \in \mathbb{R}^m \) is a vector of measured velocity values, and \( \ddot{x} \in \mathbb{R}^m \) is a vector of measured acceleration values.

The measured data was used as inputs for the proposed model device force \( f \) (i.e., Equation (2.25)) and a Levenberg-Marquardt method was implemented to solve for the optimal parameters in the model. Optimal fit values for the parameters are listed in Table 2.2. The fit responses using the optimal parameters are compared to measured responses in Figure 2.8. The error norms given in Equation (2.26) were
Table 2.2: Optimal parameter values for the proposed model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_d$</td>
<td>$2.26 \times 10^{-5} \text{kg}$</td>
</tr>
<tr>
<td>$c_d$</td>
<td>$575 \text{N-s/m}$</td>
</tr>
<tr>
<td>$k_d$</td>
<td>$627 \text{N/m}$</td>
</tr>
<tr>
<td>$f_c$</td>
<td>$143.1 \text{N}$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$6.53 \text{N}$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>$1.97 \text{rad/s}$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>$11.4 \text{N}$</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>$0.974 \text{rad/s}$</td>
</tr>
<tr>
<td>$k_{b1}$</td>
<td>$114 \text{N/m}$</td>
</tr>
<tr>
<td>$k_{b3}$</td>
<td>$1.54 \times 10^9 \text{N/m}^3$</td>
</tr>
<tr>
<td>$c_b$</td>
<td>$1.60 \times 10^4 \text{N-s/m}$</td>
</tr>
</tbody>
</table>

Figure 2.8: Comparison between experimentally measured responses and the proposed model fit to this data: (a) force versus time, (b) force versus displacement, and (c) force versus velocity.
calculated to be \( E_t = 0.209 \), \( E_x = 0.325 \), and \( E_{\dot{x}} = 0.919 \) for this test.

To verify the accuracy of the proposed model when the terminals of the motor are left open, the device was excited with a random displacement. The random displacement is characterized by bandpass filtered white noise. The particular filter used to generate the random displacement is a second-order filter where we have set the center of passband equal to 0.5Hz with an acceleration quality factor of 0.5. Figure 2.9 illustrates the random displacement used for this test. Using the fit parameters from the sine-sweep test (i.e., column 2 of Table 2.2), we compare the predicted response of the device using the proposed model to experimental data. The plots in Figure 2.10 show that the model accurately predicts the behavior of the device. For this test, the error norms given in Equation (2.26) were determined to be \( E_t = 0.177 \), \( E_x = 0.231 \), and \( E_{\dot{x}} = 0.618 \). Thus, we can conclude that the proposed model with the parameters fit using a sine-sweep displacement can be used to accurately predict the response of the device. We will use the proposed model for the remainder of the tests in this chapter.

2.3.3 Response to a Fluctuating Electromechanical Force

The response of the electromagnetic transducer to all of the previous tests have been for the case where the terminals of the motor have been left open. However, the performance of an energy harvesting system that utilizes this device will depend on the response of the device with electronics controlling the rate power extraction. Thus, we must show that the proposed model can accurately predict the behavior of the device when the electronics introduce an additional fluctuating electromechanical force. As previously mentioned, one of the simplest ways to harvest power from an electromagnetic transducer is to have the electronics simulate a resistive load across the terminals of the transducer. In this case, we have that the additional electromechanical force is the force from Equation (2.19). Three tests are presented to
Figure 2.9: Displacement input applied to the device during the random displacement test.

Figure 2.10: Comparison between predicted and experimentally obtained responses using the proposed model for the random displacement test: (a) force versus time, (b) force versus displacement, and (c) force versus velocity.
demonstrate the accuracy of the proposed model with a fluctuating electromechanical force, including (1) step response; (2) constant $R_L$, random displacement; and (3) random $R_L$, random displacement.

The step response test consisted of applying a triangular displacement with a constant velocity to the device with a step change in the load resistance as the device passes through the middle of the stroke. A triangle wave with an amplitude of 5.25cm at 0.25Hz was used for this test and can be seen in Figure 2.11. The plots presented in Figure 2.12 show the response of the device to this test. At the beginning of the test, the servo drive is used to simulate the device in open circuit mode. During the mid-way point of the second oscillation, a commanded load resistance of 10Ω is applied to the servo drive, which results in the device reaching its rated force in 6ms. This rise time is comparable to a MR damper of similar size and force rating [116]. The error norms given in Equation (2.26) were calculated to be $E_t = 0.132$, $E_x = 0.234$, and $E_{\dot{x}} = 0.497$ for this test.

During the step response test, we observe some amount of overshoot and ringing in the device force after the step change in load resistance is applied to the servo drive. The close-up view of the force response during the step change in resistance can be seen in Figure 2.13. The oscillations occurring before the force measurement reaches a steady state is a result of the dynamic interaction between servo drive and

![Figure 2.11: Displacement input applied to the device for the step response test.](image-url)
the finite coil inductance of the motor. Because internal current tracking of the servo drive is accomplished by proportional control, a step in the load resistance causes a ringing effect in the force measurement for a brief period of time until the coils become fully magnetized. We see that the predicted force response in Figure 2.13 does not capture the oscillations. This is due to the fact that the oscillations are
Figure 2.14: Comparison between predicted and experimentally obtained responses for the constant $R_L$, random displacement test: (a) force versus time, (b) force versus displacement, and (c) force versus velocity.

a result of the dynamics of the servo drive, which is not included in the proposed model. It would be possible to design a compensator using feedback to reduce these oscillations, but this is beyond the scope of this thesis.

In the second test conducted to verify the model, the device was excited with the same random displacement as in Figure 2.9, but the servo drive is used to simulate a constant load resistance across the terminals of the motor. In this case a 16Ω load resistance was used to ensure that the device would not exceed its rated force limit. The response of the device to this test can be seen in Figure 2.14. As seen here, the model accurately predicts the behavior of the device. The error norms given in Equation (2.26) were calculated to be $E_t = 0.115$, $E_x = 0.203$, and $E_{\dot{x}} = 0.378$.

For the final verification test, the load resistance simulated by the servo drive was chosen to randomly saturate between two values. The upper bound on the load
**Figure 2.15:** Applied $R_L$ command to the device in the random displacement, random resistance test.

**Figure 2.16:** Comparison between predicted and experimentally obtained responses for the random $R_L$, random displacement test: (a) force versus time, (b) force versus displacement, and (c) force versus velocity.
resistance is 800Ω (which is approximately open circuit) and the lower bound on the load resistance is 16Ω. These values were chosen so that the electromechanical force of the device would fluctuate between the force resulting from open circuit mode and the rated force. A plot of the random load resistance can be seen in Figure 2.15. Again, the random displacement that is applied to the device is the same as in the previous test. The results presented in Figure 2.16 confirm that we have excellent agreement between the experimental response and the model. For this test, the error norms given in Equation (2.26) were determined to be $E_t = 0.138$, $E_x = 0.237$, and $E_x = 0.484$.

2.4 Summary

In this chapter, we presented a model that accurately predicts the response of an experimental device for use in large-scale vibratory energy harvesting applications. The model includes an expression for the high-frequency force oscillations attributable to the mechanics of the ballscrew as well as a nonlinear stiffening spring which is responsible for the hysteresis in the force-velocity plane. We showed that back-driving the transducer with a sine-sweep can be used to determine the optimal parameters for the model through a Levenberg-Marquardt nonlinear least squares algorithm. Comparing the model with the optimal parameters to the experimental response of the device results in excellent agreement for a wide range of operating conditions.

In addition to accurately predicting the response of the actively-controlled electromagnetic transducer, the model we have developed adheres to many of the idealized assumptions made in later chapters of this dissertation. With the exception of the Coulomb friction force, many of the other nonlinearities (e.g., high-frequency force oscillations and hysteresis), while important to understand, would not significantly affect power generation, or the dynamics of a structure in which this type of device would be embedded.
3 State Feedback Controllers for Linear Vibratory Energy Harvesters

3.1 Background

This chapter explains the derivation of state feedback controllers for stochastically-excited vibratory energy harvesters with linear dynamics. In many vibratory energy harvesting applications, the disturbance cannot be characterized by a sinusoid. Instead, it is much more appropriate to model the disturbance as a stochastic process, possibly with a low quality factor. For such cases, optimization of the dynamic behavior of the electronics for maximal power generation is more challenging because the system must simultaneously harvest energy from a continuous band of frequencies.

In [48], Halvorsen investigated the broadband vibratory energy harvesting problem, in which the disturbance acceleration is modeled as white noise, and determined the optimal $R - L - C$ networks to extract power from electromagnetic and piezoelectric transducers. A follow-up study for determining admittances for this class of problems was proposed by Adhikari et al. [1]. That study used a Laplace-domain
analysis to determine the analytical expression for the resistance which maximizes the mean power harvested from a piezoelectric beam subjected to a stationary white noise disturbance. However, in [98], Scruggs showed that for harvesters excited by white noise, the optimal causal power generation must be achieved using active control. Furthermore, that study showed that the determination of the optimal power extraction can be framed as a feedback optimization problem, where the optimal feedback controller is the solution to a linear-quadratic-Gaussian (LQG) control problem.

Vibratory energy harvesting from colored disturbance noise of arbitrary quality factor was first presented in [99]. In that study, it is shown that the frequency content of the optimized power-flow is such that in frequency bands near resonance, average power flows from transducer to storage, but in other frequency bands the average power flows the other way. This implies that the optimal harvesting circuit cannot be made equivalent to any passive network, as is done for the deterministic energy harvesting problem. Hence, that study advocates the realization of a synthetic dynamic admittance using a bi-directional (i.e., four-quadrant) power electronic converter, such as an H-bridge.

For stochastic vibratory energy harvesting applications, the theoretically-optimal power generation current is a feedback function of the system response, including the transducer voltage, as well as other response states that may be available via sensor feedback. Allowable complexity of this algorithm depends on the hardware used to realize it, which in turn depends on the scale of the problem. More complex algorithms can be realized with a microprocessor or programmable controller, and for problems in which transduction power is low, such implementations may demand static power consumption levels comparable with transduction power levels. In such applications, it is therefore useful to simplify feedback algorithms to decision processes that can be realized using simple analog networks, containing as few active components as possible. This observation leads to an interesting connection between
stochastic vibratory energy harvesting problems, and fixed-structure and static control synthesis concepts.

Motivated by the above observation, the primary contributions of this chapter are twofold. First, building upon the techniques developed in [98, 99] and the results presented in [101], we first show that for stochastic disturbances characterized by second-order, bandpass-filtered white noise, harvesters can be passively “tuned” such that optimal stationary power generation only requires half of the system states for feedback in the active circuit. Interestingly, these states are the ones most easy to sense (e.g., voltage and acceleration for the SDOF electromagnetic energy harvester). One can view these tuning techniques as a “stochastic counterpart” to the tuning techniques used in harvesting applications characterized by monochromatic disturbances. In such cases, electrical tuning techniques can be used to maximize harvested power at the excitation frequency using only transducer voltage feedback, resulting in a harvesting circuit with a resistive input impedance. By contrast, even with the use of tuning techniques, optimal power generation in a stochastic context appears to almost always require knowledge of other quantities in addition to transducer voltage.

The fact that only half the states are required for feedback in tuned harvesters is a by-product of a special structure which arises in the matrix solution to the associated Riccati equation used to synthesize the optimal feedback gains. In effect, the passive tuning techniques have the effect of causing many of the entries in this matrix to be identically zero. However, in many instances such tuning techniques may be impractical, due, for example, to the large inductances they often require for piezoelectric applications. In other applications, even with the tuning techniques implemented as described, it may not be practical to feed back the reduced subset of system states required for optimality. Motivated by this observation, the second contribution of this chapter is the application of fixed-structure feedback optimization
techniques to maximize power generation for the case in which the availability of various system states for feedback is fixed in the design. Partial-state feedback gain optimization was first investigated in [16] and later in [20]. For the partial-state feedback controller, we assume that the transducer current is determined based only on concurrent measurements of a set of prescribed states, resulting in a decision process which can be realized with a very simple and efficient analog circuit.

3.2 Vibratory Energy Harvesting as a Feedback Problem

Independent of the application, one can view the vibratory energy harvesting problem by the block diagram shown in Figure 3.1. As shown, this problem will be limited to systems in which we have one controllable input. The block labeled “harvester” has inputs of disturbance acceleration $a(t)$ and transducer current $i(t)$. We make the assumption that $a(t)$ can be characterized by a stochastic process, which will be further explained in the following section. In addition, the output of the harvester block is transducer voltage $v(t)$.

![Block diagram of the general vibratory energy harvesting problem.](image)

**Figure 3.1:** Block diagram of the general vibratory energy harvesting problem.
The electronics that condition and regulate the power extracted from the harvester may be characterized by the following two quantities:

1. A feedback law used to map present and past values of \( v(t) \) into \( i^*(t) \). In general, the presence of feedback does not necessarily imply active control. It also arises as a consequence of the passive and/or switched dynamics of the power electronic circuits in Figure 1.3 and simply reflects the fact that the current, \( i(t) \), affects the harvester’s dynamics. For the remainder of this dissertation, we assume the feedback law is a quantity that can be designed to satisfy certain criteria. The feedback law may be a function of \( v(t) \) as well as additional feedback signals, which are denoted as “harvester states” in Figure 3.1. We assume that these states can either be measured directly from the harvester itself or estimated using an observer [78]. For the case where an observer is used to estimate system states, we assume that the Separation Principle [30] can be used to design the feedback law and observer separately, and then combine them into a single dynamic admittance filter.

2. A transmission loss model \( P_d(t) \) which determines the parasitic power losses in the energy harvesting circuit. Typically, the loss model depends on a combination of the transducer current and voltage, the internal “electronic states” of the circuit (e.g., capacitor voltage \( v_T(t) \) and inductor current \( i_L(t) \)), the bus voltage \( V_S \), and the switching frequency. As such, \( P_d(t) \) is characterized by the specific electronic hardware used to realize the circuit and feedback law.

The feedback law determines the desired command current \( i^*(t) \), and we assume that current tracking hardware within the power electronic circuit is used to track the command current such that \( i(t) \approx i^*(t), \forall t \). Specifically, a high-bandwidth proportional-integral-derivative (PID) controller may be used accomplish this task. In addition, we make the assumption that the tracking dynamics of the power elec-
tronics lie outside the frequency band of the disturbance. These assumptions are not unreasonable, as we have experimentally demonstrated in Chapter 2.

3.2.1 Quadratic Performance Objective

To determine the general energy harvesting performance objective, we first define the power delivered to storage as the power extracted by the harvester minus the transmission losses in the power electronic circuitry. Thus, the power delivered to storage can be defined as

\[ P_S(t) = -i(t)v(t) - P_d(t) \]  \hspace{1cm} (3.1)

For this chapter, we will make the simplifying assumption that the losses in the electronics are purely resistive; i.e., \( P_d(t) = R i^2(t) \) for some \( R > 0 \). This assumption is made because it yields the most straight-forward analysis which still accounts for transmission dissipation in some way. Given these assumptions, we have that the quadratic performance objective can be defined as the expectation of Equation (3.1); i.e.,

\[ P_{\text{gen}} = -\mathcal{E}\{iv + Ri^2\} \]  \hspace{1cm} (3.2)

This equation can be thought of as the average power generated by an energy harvesting system in stationary stochastic response. The focus of the remainder of this chapter will be on the optimization of Equation (3.2).

3.2.2 Harvester and Disturbance Model

To determine the controller that optimizes the performance objective in Equation (3.2), we first must outline the assumptions that are made for the harvester and disturbance dynamics. We assume the harvester is a dissipative system and that the harvester dynamics can be approximated by a finite-dimensional linear state space. Let \( G_{va}(s) \) and \( G_{vi}(s) \) be the transfer functions from the disturbance acceleration
\( a(t) \) and the control current \( i(t) \) to voltage \( v(t) \), respectively, where \( s \) denotes the Laplace variable. We will assume both \( G_{va}(s) \) and \( G_{va}(s) \) to be strictly proper. The condition that \( G_{va}(s) \) is strictly proper ensures a finite bandwidth of the response of the harvester. In addition, we assume that \( G_{vi}(s) \) is weakly strictly positive real (WSPR) [15]. Since \( G_{vi}(s) \) can be thought of as the driving point impedance of the harvester as seen at the terminals of the transducer, this implies that \( G_{vi}(s) \) can always be realized by an asymptotically-stable circuit with passive components (i.e., resistors, capacitors, and inductors).

With these assumptions, there always exists a self-dual state space realization [77] for the harvester, of the form

\[
\dot{x}_h(t) = A_h x_h(t) + B_h i(t) + G_h a(t) \tag{3.3a}
\]

\[
v(t) = B_h^T x_h(t) \tag{3.3b}
\]

The state-space representation of the harvester implies that the transfer functions from current \( i(t) \) and acceleration \( a(t) \) to voltage \( v(t) \) can be represented, respectively, as

\[
G_{vi}(s) = \frac{\hat{v}(s)}{\hat{i}(s)} = B_h^T (sI - A_h)^{-1} B_h \tag{3.4}
\]

\[
G_{va}(s) = \frac{\hat{v}(s)}{\hat{a}(s)} = B_h^T (sI - A_h)^{-1} G_h \tag{3.5}
\]

In the above realization, the total dissipation in the harvester at time \( t \) is \(-\frac{1}{2} x_h^T(t) [A_h + A_h^T] x_h(t) \geq 0\), and the total energy stored in the harvester is \( \frac{1}{2} x_h^T(t) x_h(t) \). The WSPR assumption allows us to assume that the pair \( (A_h, A_h + A_h^T) \) is observable, which implies that no free response of the harvester can exhibit zero internal dissipation over any finite interval.

Next, we describe the disturbance filter. Let \( a(t) \) be the acceleration of the disturbance that excites the energy harvesting system. We assume that \( a(t) \) is characterized
as filtered noise with a power spectral density equal to

$$\Phi_a(\omega) = \left| \frac{qj\omega}{-\omega^2 + 2\omega_a\zeta_a j\omega + \omega_a^2} \right|^2$$  \hspace{1cm} (3.6)

where $\omega_a$ is the center of the passband of $a(t)$, and $\zeta_a$ determines the spread of its frequency content. This particular power spectral density is equivalent to a second-order band-pass filter. For such a process, it is straightforward to represent the disturbance dynamics by a two-dimensional state space of the form

$$\dot{x}_a(t) = A_a x_a(t) + B_a w(t)$$ \hspace{1cm} (3.7a)

$$a(t) = C_a x_a(t)$$ \hspace{1cm} (3.7b)

where $w(t)$ is a white noise process with spectral intensity equal to unity. The parameter $q$ in Equation (3.6) is adjusted such that irrespective of $\omega_a$ and $\zeta_a$, $a(t)$ has a consistent standard deviation of $\sigma_a$; i.e.,

$$\sigma_a = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_a(\omega) d\omega}.$$  \hspace{1cm} (3.8)

This allows us to compare disturbances of varying spectral content and equal intensity. We refer to the “narrowband limit” for the disturbance model as the case in which $\zeta_a \rightarrow 0$. Similarly, we refer to “broadband limit” as the case in which $\zeta_a \rightarrow \infty$.

The value of $q$ that satisfies Equation (3.8) can be found by solving a Lyapunov equation for the stationary covariance of the disturbance states. If we define the disturbance state space matrices as

$$A_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta_a\omega_a \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ q \end{bmatrix}, \quad C_a = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and the stationary covariance matrix of the disturbance states as $S_a = \mathcal{E}\{x_a x_a^T\}$, then the Lyapunov equation that is satisfied in stationarity is

$$A_a S_a + S_a A_a^T + B_a B_a^T = 0.$$  \hspace{1cm} (3.9)
It turns out that the stationary covariance matrix \( S_a \) can be defined as

\[
S_a = \begin{bmatrix}
(\sigma_a/\omega_a)^2 & 0 \\
0 & \sigma_a^2
\end{bmatrix}.
\] (3.10)

Since we know that \( \sigma_a^2 = C_a S_a C_a^T \), we can solve Equation (3.9) for \( q \) in terms of the known parameters. In this case we have

\[
q = 2\sigma_a \sqrt{\zeta_a \omega_a}.
\] (3.11)

With state space defined for the disturbance filter, the transfer function from white noise \( w(t) \) to disturbance acceleration \( a(t) \) is

\[
G_{aw}(s) = \frac{\hat{a}(s)}{\hat{w}(s)} = C_a(sI - A_a)^{-1}B_a.
\] (3.12)

From this relationship, we have that if we set \( s = j\omega \), then \( \Phi_a(\omega) = |G_{aw}(j\omega)|^2 \). Next, we illustrate the influence of varying \( \zeta_a \) on magnitude and phase of the disturbance filter \( G_{aw}(s) \) in Figure 3.2. For this plot, we set \( \sigma_a = 9.81 \text{m/s}^2 \) and \( \omega_a = 1 \text{rad/s} \), and plot the magnitude and phase of \( G_{aw}(j\omega) \) for three values of \( \zeta_a \). As \( \zeta_a \) increases, we see that the passband of the filter becomes wider and there is less roll-off for frequencies away from \( \omega_a \).

Finally, if we combine the harvester and disturbance dynamics such that \( x(t) = [x_h^T(t) \ x_a^T(t)]^T \), then the augmented system obeys

\[
\dot{x}(t) = Ax(t) + B_i(t) + Gw(t)
\]  

\[
v(t) = B^T x(t)
\] (3.13a)

(3.13b)

where the matrices \( A, B, \) and \( G \) are defined as

\[
A = \begin{bmatrix}
A_h & G_h C_a \\
0 & A_a
\end{bmatrix}, \quad B = \begin{bmatrix}
B_h \\
0
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
B_a
\end{bmatrix}.
\]

We make the assumption that \( (A, B^T) \) is observable and \( (A, [B \ G]) \) is controllable. If this is not the case for the original system model, we assume that the dimension of the augmented state space has been reduced to a minimal realization.
3.2.3 Examples

Consider the single-degree-of-freedom (SDOF) resonant oscillator with mass $m$, damping $b$, and stiffness $k$ in Figure 3.3(a). An actively-controlled transducer is attached between the base and the moving mass such that $f_e(t) = K_t i(t)$ where $K_t$ is the back-emf motor constant. This system is similar to experimental energy harvester presented in Chapter 2. However, for the electromagnetic example considered in this

\[ \begin{align*}
|G_{aw}(j\omega)| (\text{dB}) \\
\angle G_{aw}(j\omega) (\text{deg})
\end{align*} \]

\[ \begin{align*}
\omega (\text{rad/s})
\end{align*} \]

Figure 3.2: Magnitude and phase of the disturbance bandpass filter $G_{aw}(j\omega)$ for $\sigma_a = 9.81 \text{m/s}^2$ and $\omega_a = 1 \text{rad/s}$.

Figure 3.3: Ideal actively-controlled energy harvesters: (a) SDOF oscillator with electromagnetic coupling, and (b) piezoelectric bimorph cantilever beam with an inductor connected in parallel.
chapter, we neglect the nonlinearities that are present in the transducer. Similar electromagnetic energy harvesters have been demonstrated in [43, 100, 126]. The governing equation for this system is

\[ m \ddot{x}(t) + c \dot{x}(t) + kx(t) = ma(t) + Kt i(t) \]  

(3.14)

where \( x(t) \) is the relative displacement of the mass \( m \).

In order to generalize the analysis, we relate time \( t \) to a nondimensional time \( \tau \) as \( t = \sqrt{\frac{m}{k}} \tau \) and relate the physical variables of the problem to nondimensionalized versions (denoted by overbars) as

\[ x(t) = \left( \frac{m \sigma_a}{k} \right) \bar{x}(\tau), \quad i(t) = \left( \frac{m \sigma_a}{K_t} \right) \bar{i}(\tau), \quad v(t) = \left( K_t \sigma_a \sqrt{\frac{m}{k}} \right) \bar{v}(\tau). \]

Thus, we have that the nondimensional harvester state space matrices are

\[
A_h = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

where \( d = c/\sqrt{mk} \).

The power spectrum of \( \bar{a}(\tau) \) is the same as in Equation (3.6), but in terms of normalized frequency \( \bar{\omega} = \omega \sqrt{m/k} \) and associated normalized parameters \( \{ \bar{\omega}_a, \bar{q} \} \).

We assume that the harvester has been tuned such that its natural frequency is in the center of the disturbance passband, resulting in the normalized matching condition \( \bar{\omega}_a = 1 \). The quantity \( \bar{q} \) chosen such that \( \int_{-\infty}^{\infty} \Phi_{\bar{a}}(\bar{\omega}) \, d\bar{\omega} = 1 \). It turns out that the value of \( \bar{q} \) which brings this about is \( \bar{q} = 2\sqrt{\zeta_a} \). Thus, we have that the nondimensional disturbance state space matrices are

\[
A_a = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta_a \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ 2\sqrt{\zeta_a} \end{bmatrix}, \quad C_a = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

The nondimensional harvester and disturbance dynamics can be combined into an
equivalent augmented state space where
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -d & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -2\zeta_a
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
0 \\
2\sqrt{\zeta_a}
\end{bmatrix}.
\]

Next, consider the ideal actively-controlled piezoelectric bimorph cantilever beam in Figure 3.3(b). We assume that the piezoelectric patches are made of lead zirconate titanate (PZT) and have an equivalent capacitance \( C_p \) and dielectric leakage resistance \( R_p \). In addition, we assume that the deflection at the end of the cantilever beam can be modeled by a finite summation of Galerkin mode shapes. As such, the differential equations describing the dynamics of the beam can be approximated through a standard Rayleigh-Ritz projection. Keeping only the fundamental vibratory mode, we approximate the beam by a mass \( m \), a damping \( c \), a stiffness \( k \), and a coupling coefficient \( \theta \). As seen in Figure 3.3(b) the terminals of the piezoelectric patches are connected in parallel to an inductor with inductance \( L \). Given the assumptions about the dynamics of the beam, the governing equations for this system are
\[
\begin{align*}
mx''(t) + cx'(t) + kx(t) &= \Gamma a(t) - \theta \lambda'(t) \\
C_p \lambda''(t) + \frac{1}{R_p} \lambda'(t) + \frac{1}{L} \lambda(t) &= i(t) + \theta \dot{x}(t)
\end{align*}
\] (3.15a)

where \( \lambda(t) \) is the flux linkage across the piezoelectric patch and \( \Gamma \) is an equivalent mass term.

We nondimensionalize \( t \) and \( x(t) \) in the same way as the electromagnetic example and further nondimensionalize the remaining dynamic quantities as
\[
\lambda(t) = \left( \frac{m\sigma_a}{k} \sqrt{\frac{m}{C_p}} \right) \tilde{\lambda}(\tau), \quad a(t) = \left( \frac{m\sigma_a}{\Gamma} \right) \tilde{a}(\tau), \quad i(t) = \left( \theta \sigma_a \sqrt{\frac{m}{k}} \right) \tilde{i}(\tau).
\]

The nondimensionalized harvester model then has the second-order differential equa-
\[ \ddot{q}(\tau) + D \dot{q}(\tau) + Nq(\tau) = B_1 \ddot{i}(\tau) + B_2 \dddot{a}(\tau) \]  

(3.16)

where \( q(\tau) = [\bar{x}(\tau) \quad \bar{\lambda}(\tau)]^T \) and the coefficient matrices are

\[
D = \begin{bmatrix} d & 1 \\ -1 & \beta \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \end{bmatrix}.
\]

In the above, the mechanical damping \( d \) remains unchanged from the previous example, while the electrical damping is \( \beta = \frac{1}{C_p R_p} \sqrt{\frac{m}{k}} \). The additional nondimensional variable \( \alpha \) is the ratio of the squared natural frequencies of the electrical and mechanical subsystems; i.e.,

\[
\alpha = \frac{m}{k C_p L}.
\]

(3.17)

We assume that the state space describing the disturbance dynamics is the same as the state space defined in the previous example. As such, the augmented state space can be expressed as

\[
A = \begin{bmatrix} 0 & I & 0 & 0 \\ -N & -D & 0 & B_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2\zeta_a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_1 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\sqrt{\zeta_a} \end{bmatrix}.
\]

We will henceforth uniformly assume that the energy harvesters have been nondimensionalized as described above. To ease the notation, we will do away with all overbars on the parameters and refer to the nondimensional time as \( t \).

### 3.3 State Feedback Controllers

#### 3.3.1 Optimal State Feedback Controller

Given the augmented energy harvesting system in Equation (3.13), we can express the quadratic performance measure in Equation (3.2) as

\[
\bar{P}_{gen} = -\mathcal{E} \left\{ \begin{bmatrix} x \\ i \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2B^T} & \frac{1}{2B} \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix} \right\}.
\]

(3.18)
Maximization of Equation (3.18) is equivalent to a nonstandard LQG optimal control problem where \( i(t) \) is treated as the control input. In general, this problem would not be well-posed (i.e., \( \bar{P}_{\text{gen}} \) would be unbounded from above). However, it turns out that since the harvester is WSPR, then \( \bar{P}_{\text{gen}} \) has a maximum value that is finite. The following theorem introduces the concept of optimal energy harvesting as an LQG control problem. A proof of this theorem can be found in Appendix A.1.

**Theorem 1.** Let the energy harvesting system in Equation (3.13) be WSPR. Then for any causal, stabilizing mapping \( v(t) \mapsto i(t) \), we have that the energy harvesting performance is

\[
\bar{P}_{\text{gen}} = -G^T P G - R E \{ (K x - i)^2 \} 
\]

where \( P = P^T < 0 \) is the solution to the nonstandard Riccati equation

\[
A^T P + PA - \frac{1}{R} \left( P + \frac{1}{2} I \right) B B^T \left( P + \frac{1}{2} I \right) = 0
\]

and \( i(t) = K x(t) \) is the optimal feedback control law where

\[
K = -\frac{1}{R} B^T \left( P + \frac{1}{2} I \right)
\]

If the full state \( x(t) \) is available for feedback, then the causal limit on power generation is that attained by the LQG controller; i.e.,

\[
\bar{P}_{\text{gen}}^{\text{LQG}} = -G^T P G
\]

The expression for \( \bar{P}_{\text{gen}}^{\text{LQG}} \) in Equation (3.22) is the causal limit on average power generated due to the combined dissipation in the harvester and the electronics. In [99], it was shown that the power electronic hardware necessary to implement the control law \( i(t) = K x(t) \) must be capable of extracting as well as injecting power into the system, such as with an H-bridge drive.
3.3.2 Optimal Partial-State Feedback Controller

Implementing the optimal state feedback controller \( i(t) = Kx(t) \) generally requires knowledge of every state in the system. In other words, every component in the feedback gain matrix \( K \) will in general be nonzero. However, it turns out that if the augmented system can be expressed in the realization presented in Equation (3.13), then the stabilizing solution \( P \) to the Riccati equation in Equation (3.20) has a special structure. Specifically, \( P \) has several block entries that are equal to zero and several non-zero block entries that are repeated. As a result of this, the solution for \( K \) in Equation (3.21) has many entries which are also zero. We now present a theorem which formally introduces these concepts. A proof of this theorem can be found in Appendix A.2.

**Theorem 2.** If the harvester dynamics can be expressed as

\[
\begin{align*}
A_h &= \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix}, & B_h &= \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, & G_h &= \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\
\end{align*}
\quad (3.23)
\]

and the disturbance dynamics can be expressed as

\[
\begin{align*}
A_a &= \begin{bmatrix} 0 & I \\ -I & -Z \end{bmatrix}, & B_a &= \begin{bmatrix} 0 \\ Q \end{bmatrix}, & C_a &= \begin{bmatrix} 0 & I \end{bmatrix} \\
\end{align*}
\quad (3.24)
\]

then the unique, stabilizing solution to the Riccati equation in Equation (3.20) is

\[
P = \begin{bmatrix}
P_{22} & 0 & P_{24} & 0 \\
0 & P_{22} & 0 & P_{24} \\
P_{24}^T & 0 & P_{44} & 0 \\
0 & P_{24}^T & 0 & P_{44} \\
\end{bmatrix}
\quad (3.25)
\]

where \( P_{22}, P_{24}, \) and \( P_{44} \) can be solved sequentially as

\[
P_{22}D + D^TP_{22} + \frac{1}{R} (P_{22} + \frac{1}{2}I) B_1B_1^T (P_{22} + \frac{1}{2}I) = 0 ,
\quad (3.26a)
\]

\[
P_{22}B_2 - P_{24}Z - D^TP_{24} = \frac{1}{R} (P_{22} + \frac{1}{2}I) B_1B_1^TP_{24} = 0 ,
\quad (3.26b)
\]

\[
P_{24}^T B_2 + B_2^TP_{24} - P_{44}Z - Z^TP_{44} = \frac{1}{R} P_{24}^T B_1B_1^TP_{24} = 0 .
\quad (3.26c)
\]
For the state vector partitioned analogously to $P$, the corresponding optimal gain matrix is

$$K = \begin{bmatrix} 0 & -\frac{1}{R}B_1^T (P_{22} + \frac{1}{2}I) & 0 & -\frac{1}{R}B_1^T P_{24} \end{bmatrix}$$

(3.27)

and the optimal power generation is

$$\dot{P}_{gen}^{LQG} = -Q^T P_{44} Q.$$  

(3.28)

From this theorem we obtain the interesting result that only half of the states are required for the optimal energy harvesting current.

### 3.3.3 Examples

Returning to the electromagnetic energy harvesting example, it is clear that the state space matrices for this system satisfy the conditions in Theorem 2. As such, the corresponding decoupled solution to the Riccati equation in Equation (3.25), with scalar quantities $P_{22}$, $P_{24}$, and $P_{44}$, is found by solving Equations (3.26a) – (3.26c). Each of these equations is scalar, and their solutions may be solved symbolically in terms of the system parameters. However, for brevity, we merely state that the solution for the optimal energy harvesting current (which depends on $P$) is

$$i(t) = \left( \frac{dR - \sqrt{d^2 R^2 + dR}}{R} \right) v(t) - \left( -\left( dR + \frac{1}{2} \right) + \sqrt{dR^2 + dR} \right) a(t)$$

(3.29)

where $v(t)$ and $a(t)$ are the second and fourth states in $x(t)$, respectively. Furthermore, we have a symbolic solution for the corresponding optimal harvested power, as

$$\dot{P}_{gen}^{LQG} = -2R - \left( dR + \frac{1}{2} \right) + \sqrt{d^2 R^2 + dR} + \left( -\left( dR + \frac{1}{2} \right) + \sqrt{dR^2 + dR} \right)^2.$$  

(3.30)

It is interesting to examine the symbolic dependency of the optimal feedback law on the parameters $d$, $\zeta_a$, and $R$. Referring to the gains $K_v$ and $K_a$ in Equation
(3.29) (i.e., \( i(t) = K_v v(t) + K_a a(t) \)), we notice that as \( R \rightarrow 0 \), both of these gains go to infinity. It is also interesting that \( K_v \) is independent of the bandwidth of \( a(t) \). Meanwhile, \( K_a \) reduces in magnitude from its value for harmonic excitation (with \( \zeta_a = 0 \)) to the infinite-broadband case, for which \( K_a \rightarrow 0 \). For the narrowband case, \( K_a \) is significantly nonzero, implying that even when \( a(t) \) is nearly harmonic, explicit knowledge of \( a(t) \) may still be leveraged to improve harvesting performance. However, in the limit as \( \zeta_a \rightarrow 0 \), both \( a(t) \) and \( v(t) \) become purely sinusoidal, and exactly in phase. (The phase condition is a consequence of the fact that the harvester is assumed to be tuned to the center of the passband for \( a(t) \).) Thus, in this limiting case, knowledge of both \( v(t) \) and \( a(t) \) is redundant, as one is known to be a scaled version of the other. We can therefore conclude that in this case, the optimal \( i(t) \) is in fact attained by imposing a static admittance; i.e., \( i(t) = K_v v(t) \). This observation is harmonious with what we expect from approaching the harmonic energy harvesting problem for tuned harvesters from an impedance matching perspective [99].

The optimal average power generated, as expressed in Equation (3.30), is always positive. Furthermore, we see that it increases monotonically as \( \zeta_a \) decreases. This result is expected because it is easier to harvest energy from disturbances with more signal strength concentrated near resonance. Although it is less obvious from Equation (3.30), the power generation decreases monotonically as \( R \) increases. This is also to be expected, as it stands to reason that as the electronics become less efficient, the harvesting potential decreases.

Although the contributions of this chapter are primarily theoretical, we pause now to consider the implementation of feedback law Equation 3.29. A diagram depicting one possible implementation is shown in Figure 3.4 (where the supply voltage is denoted by \( V_S \) and the logic voltage is denoted by \( V_L \)). There are several components to this diagram. The first component is the H-bridge gate drive circuit, which is used to control the four MOSFETs (\( Q_1, Q_2, Q_3, \) and \( Q_4 \)) in the bridge.
typical H-bridge gate drive circuit requires two inputs: a PWM signal and a triangle wave signal, which is not pictured. The circuit tracks a desired current command signal by switching the MOSFETs on and off at a high frequency using PWM such that the average value of the current imposed across the terminals of the transducer is approximately equal to the command signal.

The next components in this circuit are the voltage, current, and acceleration sensing. The differential voltage across the terminals of the transducer is measured by a simple differential amplifier. Similarly, the current flowing into or out of the transducer can be sensed by measuring the differential voltage across a low resistance sensing resistor, $R_{\text{sense}}$. An accelerometer is attached to the base of the structure and is used to output a voltage signal proportional to the disturbance acceleration. Next, the voltage and acceleration signals are sent to two inverting amplifiers that multiply the signals by their respective optimal partial-state gains. These gains can be tuned using two potentiometers (represented by Pot 1 and Pot 2 in Figure 3.4).

The final component in this circuit is a summing amplifier with proportional and integral control in feedback. The voltage signal, acceleration signal, and current signal are summed together to produce the error signal. The error signal then passes through a proportional-integral (PI) controller, which outputs the PWM signal and

**Figure 3.4:** PSF circuit diagram consisting of voltage and acceleration measurements with gain adjustment.
sends it to the H-bridge gate drive circuit. If the chosen resistor values are made such that $R_1 = R_2 = R_3$, then the proportional gain is just $-R_4/R_1$. The integral gain can be adjusted by changing the capacitor value $C_1$ that is in series with $R_4$. The values of the proportional and integral gains should be tailored for the particular components used in the circuit as well as the bandwidth constraints on the current command signal.

Next, we return to the piezoelectric energy harvesting example. The augmented harvester and disturbance model for this system satisfies the conditions in Theorem 2 if $N = I$, which can be accomplished by setting $\alpha = 1$. Recall that $\alpha$ is the ratio of squared natural frequencies of the mechanical and electrical systems. As such, for a typical piezoelectric energy harvesting system, an extremely large inductor would typically be required for $\alpha$ to equal unity. However, for the purpose of this example, we assume that the value of $\alpha$ is unconstrained and can be tuned via the inductance value $L$. It turns out that tuning a passive network containing an inductor and a resistor in parallel is exactly what one would do to impedance-match a piezoelectric energy harvester for resonant performance [1].

With $\alpha = 1$, the system model meets the requirements of Theorem 2, and the corresponding solution to the Riccati equation thus decouples as in Equation (3.25) (with $P_{44}$ a scalar in this case). We can thus solve for $P_{22}$, $P_{24}$, and $P_{44}$ by sequentially solving Equations (3.26a) – (3.26c). For this example it is not as easy to find a symbolic solution for $P$, because Equation (3.26a) is a $2 \times 2$ Riccati equation and $D$ is a full $2 \times 2$ matrix. Nonetheless, in terms of $P_{22}$ and $P_{24}$, we find the optimal harvested power is $P_{gen}^{LQG} = -4\zeta_a P_{44}$, and the optimal energy harvesting current is the feedback law

$$i(t) = -\frac{1}{R} \left[ B_1^T \left( P_{22} + \frac{1}{2} I \right) \dot{q}(t) + B_1^T P_{24} a(t) \right] + K\ddot{x}(t) + K_v v(t) + K_a a(t). \quad (3.31a)$$

$$= K\ddot{x}(t) + K_v v(t) + K_a a(t). \quad (3.31b)$$
with appropriate definitions for $K_x$, $K_v$, and $K_a$.

3.4 Partial-State Feedback Gain Optimization

We now present a procedure for optimizing the feedback gains corresponding to states that are most important in terms of the average power generated by the harvester. Since most energy harvesting systems don’t satisfy the conditions in Theorem 2, they require knowledge of all of the states for optimal power generation. For practical purposes, measuring all of the states is often not feasible. Instead, transducer voltage measurements can be passed through a standard Luenberger observer, which is used to estimate the remaining system states. Observer gains can be chosen to achieve close tracking of states, but the construction of a dynamic observer also complicates the feedback circuitry.

Alternatively, it is possible to achieve performance almost as good as the optimal full state upper bound by imposing static feedback using only the states that have influence on the performance. We have seen in the previous section that when the Riccati equation decouples, only the derivatives of the harvester electromechanical coordinates and the disturbance acceleration (but not its integral) are required for feedback. From this insight we presume that these states have the most influence on performance even when the Riccati equation does not decouple. Furthermore, these states are often the easiest to measure. Implementing circuitry to sum static gains of measured states is rather simple.

In order to solve for the partial-state feedback (PSF) gains, we define the output vector $y(t) = Cx(t)$. The matrix $C$ is defined such that $y(t)$ contains the states that we choose to include for feedback. Without loss of generality, $C$ will be normalized such that $CC^T = I$. We wish to impose the PSF control law

$$i(t) = \tilde{K}Cx(t)$$

(3.32)
where $\tilde{K}$ is a vector of the PSF gains. Next, we substitute Equation (3.32) into Equation (3.18), which gives

$$\bar{P}_{gen} = \mathcal{E}\left\{ x^T \tilde{Q}(\tilde{K}) x \right\}$$

(3.33)

where

$$\tilde{Q}(\tilde{K}) = \frac{1}{2} C^T \tilde{K} B^T + \frac{1}{2} B \tilde{K} C + R C^T \tilde{K} B C.$$

(3.34)

In this case we have that the performance resulting from the PSF controller is

$$\bar{P}^{PSF}_{gen} = -G^T \tilde{P}$$

(3.35)

where $\tilde{P}$ is the solution to the Lyapunov equation

$$\begin{bmatrix} A + B \tilde{K} C \end{bmatrix}^T \tilde{P} + \tilde{P} \begin{bmatrix} A + B \tilde{K} C \end{bmatrix} + \tilde{Q}(\tilde{K}) = 0.$$

(3.36)

To optimize $\tilde{K}$, we use a gradient-descent method. Noting that $\bar{P}^{PSF}_{gen}$ is non-convex in $\tilde{K}$, in general there may be multiple local minima. One of the challenges associated with this problem is the choice of the initial condition for the algorithm. A study by Cai and Lim [16] suggest that the initial guess should be

$$\tilde{K}_0 = KC^T$$

(3.37)

where $K$ is the solution to the full state Riccati equation in Equation (3.20). For the examples presented in this chapter, this particular initial guess always resulted in stable closed loop dynamics. We now present a theorem which defines the gradient of the performance with respect to the gain matrix $\tilde{K}$. The proof of this theorem can be found in Appendix A.3.

**Theorem 3.** The performance resulting from the PSF controller is minimized by

$$\frac{\partial \bar{P}^{PSF}_{gen}}{\partial \tilde{K}} = -2 \left( B^T \tilde{P} + \frac{1}{2} B^T + R \tilde{K} C \right) SC^T = 0$$

(3.38)
where $\tilde{P}$ is the solution to the Lyapunov equation in Equation (3.36) and where $S = E\{xx^T\}$ is the solution to another Lyapunov equation; i.e.,

$$\begin{bmatrix} A + B\tilde{K}C \end{bmatrix} S + S \begin{bmatrix} A + B\tilde{K}C \end{bmatrix}^T + GG^T = 0 .$$  

(3.39)

This theorem leads to a simple first-order gradient ascent method for optimizing $\tilde{K}$, consisting of the following steps.

**Step 0**: Start with $\tilde{K}_0 = KC^T$ as the initial guess.

**Step 1**: For a matrix $\tilde{K}_0$, evaluate $\partial \bar{P}_{gen}^{PSF} / \partial \tilde{K}$ as in Equation (3.38).

**Step 2**: Compute $\tilde{K}$ by updating $\tilde{K}_0$ in the direction of the steepest-ascent, with a user-specified step size $\epsilon$, as

$$\tilde{K} = \tilde{K}_0 + \epsilon \left. \frac{\partial \bar{P}_{gen}^{PSF}}{\partial \tilde{K}} \right|_{\tilde{K}_0} .$$  

(3.40)

**Step 3**: Return to Step 1 with $\tilde{K}_0 \leftarrow \tilde{K}$ until convergence is reached.

Since the focus of this study is restricted to single-transducer energy harvesting systems, the efficiency of the optimization algorithm is of little concern. The gradient descent algorithm converges to a local solution in $\bar{P}_{gen}^{PSF}$ in 10–20 iterations for the examples discussed later in this chapter. This is likely due to the fact that the initial guess $\tilde{K}_0$ is close to a local minimum. A more robust method for systems with an arbitrary number of transducers is presented in [53]. The algorithm used in that study involves solving linear matrix inequalities within a scaled min/max optimization routine.

A special case of PSF arises when only the transducer voltage is used to determine the current command $i(t)$. In other words, electronics implement a static admittance
(SA) controller where \( i(t) = -Y_s v(t) \) by convention. The performance resulting from the SA controller can be written as

\[
\bar{P}_{gen}^{SA} = (Y_s - Y_s^2 R) B^T S B
\]  

(3.41)

where the covariance matrix \( S \) is the solution to the Lyapunov equation

\[
\begin{bmatrix}
A - Y_s B B^T
\end{bmatrix} S + S \begin{bmatrix}
A - Y_s B B^T
\end{bmatrix}^T + GG^T = 0.
\]  

(3.42)

Because the system only has one design parameter (i.e., \( Y_s \)) in this case, the most straight-forward way to optimize \( \bar{P}_{gen}^{SA} \) is via a one-dimensional line search. One can, for example, employ the bisection algorithm instead of the gradient descent method to converge rapidly to the optimal \( Y_s \), given \( \{A, B, G, R\} \). However, for more complicated systems involving multiple transducers, the optimal SA matrix can be found through a first-order gradient descent algorithm. This algorithm was first proposed in [106] for applications involving controllable dampers to suppress vibrations in structures excited by earthquakes.

3.5 Simulation Examples

The ratio of the performance when the electronics implement the SA or PSF controllers divided by the performance when the electronics implement the LQG controller gives an idea of the potential for improvement in energy harvesting performance. In this section, we present several simulation examples for both the electromagnetic energy harvester as well as the piezoelectric energy harvester to demonstrate these concepts.

3.5.1 Performance Ratios when the Riccati Equation Decouples

We first present results illustrating the performance ratios for the case when the Riccati equation decouples for both the electromagnetic and piezoelectric energy
Figure 3.5: Performance ratios for the electromagnetic harvester with $R$ values of 0, 0.05, 0.11, 0.22, 0.47, 1 (from bottom to top): (a) $d = 0.01$, and (b) $d = 0.1$.

Figure 3.6: Performance ratios for the piezoelectric harvester with $R$ values of 0, 0.05, 0.11, 0.22, 0.47, 1 (from bottom to top): (a) $d = 0.01$ and $\beta = 0.01$, (b) $d = 0.01$ and $\beta = 0.1$, (c) $d = 0.1$ and $\beta = 0.01$, and (d) $d = 0.1$ and $\beta = 0.1$. 
harvesting examples. Figure 3.5 shows the performance ratios for the electromagnetic energy harvester for various values of $d$ and $R$, and for ranges of $\zeta_a \in [0, 1]$. Similarly, Figure 3.6 shows the performance ratios for the piezoelectric energy harvester for various values of $d$, $\beta$, and $R$, and for ranges of $\zeta_a \in [0, 1]$. From these plots we see that there is a finite bandwidth for $a(t)$ at which knowledge of the derivative of the harvester states together with the disturbance acceleration is most beneficial. For the electromagnetic harvester, we see that knowledge of these states greatly improves performance over the entire range of $\zeta_a$ values for the asymptotic case where $R \to 0$. However, this is not true for the piezoelectric harvester, as we see that the performance ratio appears to bend upwards after it reaches a minimal value.

For the piezoelectric example we observe that qualitatively, if only the voltage and acceleration states are measured (but not the velocity) and their respective feedback gains are optimized, very little performance is sacrificed. This observation is useful because in many applications it may be advantageous to make power generation decisions without explicit knowledge of beam velocity.

3.5.2 Improved Performance via Electrical Tuning

For the piezoelectric energy harvester, we have shown several benefits in terms of the performance for setting $\alpha$ is equal to unity. However, it is possible to tune $\alpha$ such that the LQG controller gains result in additional improvement in $\bar{P}_{gen}^{LQG}$. By tuning $\alpha$ to maximize $\bar{P}_{gen}^{LQG}$, the solution to the Riccati equation may not always decouple. The plots in Figure 3.7 are surfaces showing the value of $\alpha$ that optimizes the power generation performance over the $\{\zeta_a, R\}$ domain and for increasing values of mechanical damping. This plot illustrates the dependency of the optimal $\alpha$ surface on $d$, and shows that $\alpha$ is much more sensitive to changes in $d$ than $\beta$.

It is interesting to note that there are three regions for values of $\alpha$ in these plots. The regions have distinct boundaries because, for given values of $R$ and $\zeta_a$,
\( \bar{P}_{LQG} \) has two local maxima over the range of possible \( \alpha \) values. For low levels of mechanical damping there is a region where high \( \alpha \) values, corresponding to lower inductance values, result in the optimal performance. This region decreases in size as \( d \) increases, and eventually becomes a region where the optimal performance is obtained by setting \( \alpha = 0 \); i.e., the case with no inductor. In addition, for low levels of mechanical damping there is a region where \( \alpha = 1 \) results in the optimal performance. This region occurs in an area corresponding to low levels of \( R \), which means that the decoupling of the Riccati equation results in the optimal performance for systems with efficient electronics.
3.5.3 Piezoelectric Energy Harvester Without an Inductor

Finally, we consider an example where the piezoelectric energy harvester in Figure 3.3(b) does not have an inductor connected in parallel to the terminals of its patches; i.e., $\alpha = 0$. The nondimensionalized augmented harvester and disturbance dynamics can be expressed as

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & -d & -1 & 0 & 1 \\
0 & 1 & -\beta & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -2\zeta_a
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
2\sqrt{\zeta_a}
\end{bmatrix}.
$$

In this case, the solution to the Riccati equation does not decouple and all of the states are required for feedback to obtain the upper bound on the performance. However, it is possible to improve upon the performance obtained from just feeding back voltage by feeding back both voltage and disturbance acceleration. This can be accomplished by optimizing the PSF gains. For this case we define $C$ as

$$
C = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (3.43)
$$

and implement the gradient descent method to determine $\tilde{K}$.

The plots in Figure 3.8 illustrate the improvement in performance through the use of PSF controller. Figure 3.8(a) shows $\bar{P}_{gen}^{SA}/\bar{P}_{gen}^{LQG}$, while Figure 3.8(b) shows $\bar{P}_{gen}^{PSF}/\bar{P}_{gen}^{LQG}$. We see that including the disturbance acceleration state and optimizing the PSF gains greatly improves the power generation performance for all values of $R$ over the range of $\zeta_a$.

3.6 Summary

This chapter investigated the potential for enhanced energy harvesting performance from stochastic disturbances through the use of a partial-state feedback control law.
It turns out that if the state space system describing the augmented harvester and disturbance dynamics satisfy the special tuning conditions in Theorem 2, with the harvester dynamics being WSPR, then the Riccati equation decouples. This leads to an energy harvesting current relationship that only requires half of the states for feedback. We showed that for both electromagnetic and piezoelectric harvesters with appropriate electrical networks, the states required for feedback are the easiest to measure. The main result of this chapter is that more average power can be harvested when the electronics implement the optimal partial-state feedback control law than by implementing the optimal static admittance. This result is illustrated for both the electromagnetic and piezoelectric energy harvesters via ratios of the performances obtained under different feedback assumptions.

We considered two additional scenarios to illustrate the improvement in performance for the piezoelectric energy harvester. First, we showed that the performance can be further enhanced by tuning the size of the inductor. It was found that for various levels of mechanical damping, and over the \( \{\zeta, R\} \) domain, there are three distinct regions where various inductor values optimize the LQG performance. Two
of the regions correspond to cases where the Riccati does and does not decouple while the third region corresponds to the case where there is no inductor. Next, the improvement in the performance using a gradient ascent routine was investigated when the piezoelectric energy harvester is not connected to an inductor. Despite the Riccati equation not decoupling, we illustrated the benefits in terms of the performance ratio for including the disturbance acceleration in addition to voltage measurements in the energy harvesting current relationship. We showed that for various values of $R$, and over the range of $\zeta_a$, optimizing the voltage and disturbance acceleration gains improves the performance ratio.
Simultaneous Structural Response Suppression and Vibratory Energy Harvesting

4.1 Background

In this chapter, we build upon the theory presented in Chapter 3 to develop controllers for simultaneous structural response suppression and vibratory energy harvesting. The main contributions of this chapter are twofold. First, we outline a procedure for designing multi-objective controllers which balance structural response objectives against the energy harvesting objective. The multi-objective control problem is formulated in terms of linear matrix inequalities (LMIs), and we show that the problem can be made convex using a transformation of variables. Second, we discuss a technique for designing dynamic voltage feedback controllers for energy harvesting systems. This technique further extends the theory in Chapter 3 by showing how the estimated system states can be accounted for in the LMI optimization problem.

Several previous studies [71, 73] have examined the tradeoff between energy harvesting and structural damping objectives. In [71], Lesieutre et al. determined the supplemental damping imposed on a piezoelectric energy harvester by varying
the equivalent shunt resistance of a DC/DC (buck) converter. These results were extended in the study by Liang and Liao [73], which compared the damping of a piezoelectric energy harvester with a shunt circuit to the damping imposed by a nonlinear switching circuit. More recent studies by Harne [49, 50] have experimentally investigated the ability of piezoelectric transducers to simultaneously harvest energy and suppress surface vibrations in plates. However, unlike those studies, the technique proposed in this chapter is applicable to additional competing structural response objectives besides supplemental damping.

Johnson and Erkus [55] proposed a similar multi-objective control formulation to the one discussed in this chapter for seismically-excited structures. In that study, a standard linear-quadratic-regulator (LQR) controller was modified to include dissipativity constraints for active and semi-active dampers, and computed using LMIs. Another study by Crews et al. [26] presented a different method to compute multi-objective controllers for semi-active vehicle suspensions. Specifically, they implemented a genetic algorithm and investigated the limits of controller performance through Pareto and other performance frontiers. Multi-objective optimal control of vibratory energy harvesters was investigated in [102] and the performance was illustrated using a simulated piezoelectric system. The main difference between the multi-objective optimization problem in that study and the one proposed in this chapter relates to the treatment of the structural control and energy harvesting objectives. In this chapter, we seek to minimize the structural response objective subject to a constraint on average power generation, whereas the opposite approach was taken in [102].

4.2 Formulation of the Multi-Objective Control Problem

Consider the block diagram of in Figure 4.1. The block labeled “harvester and disturbance” consists of the self-dual state space system outlined in Chapter 3, but
with an additional output term; i.e.,
\[
\begin{align*}
\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}i(t) + \mathbf{G}w(t) \quad (4.1a) \\
v(t) &= \mathbf{B}^T x(t) \quad (4.1b) \\
z(t) &= \mathbf{C}x(t) + \mathbf{D}i(t). \quad (4.1c)
\end{align*}
\]

where the output vector \(z(t) \in \mathbb{R}^{n_z}\) contains structural response quantities and where the augmented system state \(x(t) \in \mathbb{R}^n\) consists of both the harvester states and the disturbance states. In addition, we note that \(i(t)\) and \(v(t)\) are the transducer current and voltage, respectively, and \(w(t)\) is white noise with spectral intensity equal to unity.

We assume the electronics are actively controlled via voltage feedback. Specifically, we assume a linear dynamic admittance controller \(Y(s): \hat{v}(s) \rightarrow -\hat{i}(s)\), where \(Y(s)\) is of order \(n\) (i.e., the same as that of the system model), and is strictly proper. This implies a controller state space \(x_Y(t) \in \mathbb{R}^n\) characterized by
\[
\begin{align*}
\dot{x}_Y(t) &= \mathbf{A}_Y x_Y(t) + \mathbf{B}_Y v(t) \quad (4.2a) \\
i(t) &= \mathbf{C}_Y x_Y(t). \quad (4.2b)
\end{align*}
\]

With the above formulation, we note the matrix \(\mathbf{B}_Y\) is similar to an observer gain and the matrix \(\mathbf{C}_Y\) is similar to a state feedback control gain. Equivalently, the
admittance can be expressed in the frequency domain as the transfer function

\[ Y(s) = -\frac{\dot{i}(s)}{\dot{v}(s)} = -C_K (sI - A_K)^{-1} B_K. \]  (4.3)

Augmenting the system states with the control states results in the closed-loop state vector \( \chi(t) = [x^T(t) \ x_y^T(t)]^T \) and corresponding state space

\[
\begin{align*}
\dot{\chi}(t) &= A\chi(t) + Bw(t) \quad (4.4a) \\
v(t) &= C_v\chi(t) \quad (4.4b) \\
i(t) &= C_i\chi(t) \quad (4.4c) \\
z(t) &= C_z\chi(t) \quad (4.4d)
\end{align*}
\]

where

\[
A = \begin{bmatrix} A & BC_y \\ B_yB^T & A_y \end{bmatrix}, \quad B = \begin{bmatrix} G \\ 0 \end{bmatrix}, \\
C_v = \begin{bmatrix} B^T \\ 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & C_y \end{bmatrix}, \\
C_z = \begin{bmatrix} C & DC_y \end{bmatrix}.
\]

The goal is to optimize \( \{A_Y, B_Y, C_Y\} \) to minimize the response of the structure, subject to a constraint on the amount of power generated by the system. To formulate this problem in terms of multi-objective optimization, we first define the stationary structural response performance measure as

\[
J = \mathcal{E}\{z^Tz\} = \mathcal{E}\left\{ \begin{bmatrix} x \\ i \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix} \right\} \quad (4.5)
\]

where \( D^T D > 0 \). Next, we define the stationary power generation performance, for the case where we assume purely resistive transmission losses in the electronics, as

\[
P_{gen} = -\mathcal{E}\left\{ \begin{bmatrix} x \\ i \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2}B^T \\ \frac{1}{2}B & R \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix} \right\} \quad (4.6)
\]

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where \( R > 0 \). As such, we can state the power generation constraint as \( \tilde{P}_{\text{gen}} > \tilde{P}_{\text{gen}}^0 \) where \( \tilde{P}_{\text{gen}}^0 \) is a specified level of power generation. In addition to the power generation constraint, we also require that the poles of the closed-loop system (i.e., the eigenvalues of \( A \)) are in some region in the \( s \)-plane. Specifically, we require that \( \max_i |\lambda_i(A)| < \omega_0 \) where \( \lambda_i(A) \) is the \( i \)-th eigenvalue of \( A \) and \( \omega_0 > 0 \) is a specified threshold.

To summarize, we state the following multi-objective optimization problem

\[
\begin{align*}
\text{Minimize} & : & J \\
\text{Over} & : & \{A_Y, B_Y, C_Y\} \\
\text{Subject to} & : & \tilde{P}_{\text{gen}} > \tilde{P}_{\text{gen}}^0 \\
& & \max_i |\lambda_i(A)| < \omega_0
\end{align*}
\]

(4.7)

Including the additional constraint on the closed-loop poles of \( A \) allows us to explicitly account for the tracking bandwidth of the electronics in the design of the controller.

4.3 Matrix Inequality Approach

In order to solve the multi-objective optimization problem, we must first re-express the objective and constraint from Equation (4.7) as an associated linear matrix inequality (LMI). As such, we have that the stationary covariance matrix \( S > \mathcal{E}\{\chi\chi^T\} \) if and only if \( S > 0 \) and

\[
AS + SA^T + BB^T < 0.
\]

(4.8)

It follows that the response performance objective \( J \) is less than some threshold \( J_0 \) if and only if \( \exists S = S^T > 0 \), satisfying Equation (4.8) as well as

\[
J_0 > \text{tr}\{C_zS\}.
\]

(4.9)

Equivalently, by defining \( P = S^{-1} \) and through the use of Schur complements, we have that \( J < J_0 \) if and only if \( \exists P = P^T > 0 \) and \( W = W^T > 0 \) satisfying
\( J_0 = \text{tr}\{W\} \) and

\[
\begin{bmatrix}
A^T P + PA & PB \\
BP & -I
\end{bmatrix} < 0 \tag{4.10}
\]

\[
\begin{bmatrix}
W & C_z \\
C_z^T & P
\end{bmatrix} > 0 . \tag{4.11}
\]

To formulate the power generation constraint in terms of matrix inequalities, we invoke a result from Theorem 1. This theorem states that for any stabilizing feedback law \( K \), \( \bar{P}_{\text{gen}} \) may be expressed as

\[
\bar{P}_{\text{gen}} = \bar{P}_{\text{gen}}^{\text{max}} - RE\left\{ (Kx - i)^2 \right\} \tag{4.12}
\]

where \( \bar{P}_{\text{gen}}^{\text{max}} \) and \( K \) are defined in Equations (3.22) and (3.21), respectively. As such, for some \( \bar{P}_{\text{gen}}^0 > 0 \), we have that \( \bar{P}_{\text{gen}} > \bar{P}_{\text{gen}}^0 \) if and only if \( \exists S_P > 0 \) satisfying equation (4.8) together with

\[
R (C_i - K) S_P (C_i^T - K^T) - \bar{P}_{\text{gen}}^{\text{max}} + \bar{P}_{\text{gen}}^0 < 0 \tag{4.13}
\]

where \( K = [K \; 0] \). An equivalent condition is that \( \exists P_P > 0 \) and \( \theta > 0 \) such that \( P_P = S_P^{-1} \) along with

\[
\begin{bmatrix}
A^T P_P + P_P A & P_P B \\
P_P B^T & -I
\end{bmatrix} < 0 \tag{4.14}
\]

\[
\begin{bmatrix}
\theta & C_i - K \\
C_i^T - K^T & P_P
\end{bmatrix} > 0 \tag{4.15}
\]

\[
R\theta - \bar{P}_{\text{gen}}^{\text{max}} + \bar{P}_{\text{gen}}^0 < 0 . \tag{4.16}
\]

With no introduction of conservatism, we may set the equivalency \( P_P = P \). To see this, consider that for any \( \{J, \bar{P}_{\text{gen}}^0\} \) resulting in a feasible solution to the LMIs in Equations (4.10), (4.11), and (4.14)–(4.16), the region of feasibility is maximized by setting \( P_K = P = S_\epsilon^{-1} \) where \( S_\epsilon \) is the solution to

\[
AS_\epsilon + S_\epsilon A^T + GG^T + \epsilon I = 0 \tag{4.17}
\]
for $\epsilon \to 0$. Thus, we have that $\tilde{P}_{gen} > \tilde{P}_{gen}^0$ and $J < \text{tr}\{W\}$ if and only if $\exists \mathbf{P} > 0$ and $\mathbf{W} = \mathbf{W}^T > 0$ such that equation (4.16) holds along with
\[
\begin{bmatrix}
\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} & \mathbf{PB} \\
\mathbf{B}^T \mathbf{P} & -\mathbf{I}
\end{bmatrix} < 0
\]  
(4.18)
\[
\begin{bmatrix}
\mathbf{W} & \mathbf{C}_z \\
\mathbf{C}_z^T & \mathbf{P}
\end{bmatrix} > 0
\]  
(4.19)
\[
\begin{bmatrix}
\mathbf{C}_i^T - \mathbf{K}^T & \mathbf{C}_i - \mathbf{K} \\
\mathbf{P}
\end{bmatrix} > 0.
\]  
(4.20)

The pole placement constraint on $\mathbf{A}$ can also be formulated as an equivalent LMI using an approach outlined by Chilali and Gahinet [23]. In that study, they proved that if $\mathbf{A}$ is asymptotically stable then $\max_i |\lambda_i(\mathbf{A})| < \omega_0$ if and only if $\exists \mathbf{P}_\omega > 0$ such that
\[
\begin{bmatrix}
-\omega_0 \mathbf{P}_\omega & \mathbf{A}^T \mathbf{P}_\omega \\
\mathbf{P}_\omega \mathbf{A} & -\omega_0 \mathbf{P}_\omega
\end{bmatrix} < 0.
\]  
(4.21)

Again, we impose the conservative restriction that $\mathbf{P}_\omega = \mathbf{P}$ to the above equation. The practice of equating the various Lyapunov matrices (i.e., $\mathbf{P} = \mathbf{P}_P = \mathbf{P}_\omega$) is called “Lyapunov shaping” and is a conservative assumption imposed in order to arrive at a convex optimization problem.

4.3.1 Convex Design Problem

With the matrix inequalities from the previous section defined, we now use standard LMI techniques, as outlined in [97], to arrive at a convex semidefinite program. To summarize the standard results, we partition $\mathbf{P}$ and its inverse as
\[
\mathbf{P} = \begin{bmatrix}
\mathbf{Y} & \mathbf{N} \\
\mathbf{N}^T & \mathbf{\bullet}
\end{bmatrix},
\]  
(4.22a)
\[
\mathbf{P}^{-1} = \begin{bmatrix}
\mathbf{X} & \mathbf{M} \\
\mathbf{M}^T & \mathbf{\bullet}
\end{bmatrix}.
\]  
(4.22b)
where \bullet \text{ implies a matrix sub-block that does not need to be known, and where } X = X^T \text{ and } Y = Y^T. \text{ Next, we define the transformation matrix }

\[ \Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} \quad (4.23) \]

and note that the following transformations occur:

\[ \Pi_1^T P A \Pi_1 = \begin{bmatrix} A X + B \hat{C} & A \\ \hat{A} & Y A + B B^T \end{bmatrix} \]

\[ \Pi_1^T P B = \begin{bmatrix} G \\ Y G \end{bmatrix} \]

\[ C_v \Pi_1 = \begin{bmatrix} B^T X & B^T \end{bmatrix} \]

\[ C_i \Pi_1 = \begin{bmatrix} \hat{C} & 0 \end{bmatrix} \]

\[ C_z \Pi_1 = \begin{bmatrix} C X & C \end{bmatrix} \]

\[ K \Pi_1 = \begin{bmatrix} K X & K \end{bmatrix} \]

where the variables

\[ \hat{A} = N A_Y M^T + N B_Y B^T X + Y B C_Y M^T + Y A X \quad (4.24a) \]

\[ \hat{B} = N B_Y \quad (4.24b) \]

\[ \hat{C} = C_Y M^T \quad (4.24c) \]

become the transformed control design variables. The reason for performing these transformations is that they permit the LMIs in Equations (4.18)–(4.21) to become linear in the variables \{X, Y, \hat{A}, \hat{B}, \hat{C}, W, \theta\}. Specifically, they respectively become

\[ \begin{bmatrix} \Delta_1 + \Delta_1^T & A + \hat{A}^T \\ \Delta_2 + \Delta_2^T & Y G \end{bmatrix} < 0 \quad (4.25) \]

\[ \begin{bmatrix} -W & C X & C \\ -X & -I \end{bmatrix} < 0 \quad (4.26) \]

\[ \begin{bmatrix} \theta & \hat{C} - K X & -K \\ \hat{A} & X & I \end{bmatrix} > 0 \quad (4.27) \]

\[ \begin{bmatrix} -\omega_0 X & -\omega_0 I \\ -\omega_0 Y & -\omega_0 X \end{bmatrix} \]

\[ \begin{bmatrix} \Delta_1^T & \hat{A}^T \\ \Delta_2^T & \hat{A}^T \end{bmatrix} < 0 \quad (4.28) \]
where
\[
\Delta_1 = AX + B\hat{C}
\]
\[
\Delta_2 = YA + \hat{BB}^T.
\]

With the problem transformed as such, the following optimization problem is a standard convex LMI eigenvalue problem

\[
\begin{align*}
\text{Minimize :} & \quad \text{tr}\{W\} \\
\text{Over :} & \quad \{X, Y, \hat{A}, \hat{B}, \hat{C}, S, \theta\} \\
\text{Subject to :} & \quad \text{Equations (4.16) and (4.25)–(4.28)}
\end{align*}
\]

Let the optimal solution to Equation (4.31) be \(\{X^*, Y^*, \hat{A}^*, \hat{B}^*, \hat{C}^*, W^*, \theta^*\}\). Then because Equation (4.31) is always more conservative than Equation (4.7), we have that the resultant power generation \(\bar{P}_{gen}^*\) at the optimum is under-bounded by the maximum objective; i.e.,
\[
\bar{P}_{gen}^* \geq \bar{P}_{gen}^{\text{max}} - R\theta^*.
\]

To obtain a set of controller state space parameters \(\{A_Y, B_Y, C_Y\}\) that achieve the optimized performance bound in Equation (4.32), we first find \(M\) and \(N\). An infinite number of equivalent realizations exist, and consequently there will be an infinite number of \(\{M, N\}\) combinations. It is only important that they bring about the inverse relations in Equation (4.22), which is ensured by
\[
X^*Y^* + MN^T = I.
\]

Thus, with \(\{X^*, Y^*\}\) solved, we may find a valid pair \(\{M, N\}\) by performing the singular value decomposition
\[
U\Sigma V^T = I - X^*Y^*
\]
where \(\Sigma \geq 0\) is diagonal and \(\{U, V\}\) are unitary. It follows that
\[
M = U\Sigma^{1/2},
\]
\[
N = V\Sigma^{1/2}.
\]

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With these solved, we can find the inverse mapping of equation (4.24) as
\[
C_K = \hat{C}^* M^{-T} \quad (4.36a)
\]
\[
B_K = N^{-1} \hat{B}^* \quad (4.36b)
\]
\[
A_K = N^{-1} \left( \hat{A}^* - Y^*AX^* - Y^*B\hat{C}^* - \hat{B}^*B^TX^* \right) M^{-T}. \quad (4.36c)
\]

With the control parameters known, we may find the actual (i.e., non-conservative) value of $\bar{P}^*_{gen}$ achieved by the optimized controller by solving for the closed-loop covariance matrix via the Lyapunov equation
\[
AS^* + S^*A^T + BB^T = 0 \quad (4.37)
\]
with the optimal $\{A_Y, B_Y, C_Y\}$ parameters inserted into $A$ and $B$, and then evaluating
\[
\bar{P}^*_{gen} = \bar{P}^{max}_{gen} - R(C_i - K)S^*(C_i^T - K^T). \quad (4.38)
\]
Similarly, the actual (non-conservative) value of the structural response performance measure is $J^* = \text{tr}\{C_zS^*C_z^T\}$.

4.4 Simulation Example

Consider the actively controlled three-story structure in Figure 4.2. We assume that an ideal electromagnetic transducer with motor constant $K_t$ is embedded between the ground and the first story. For the structural model, we use the structure from the experimental study by Chung et al. [25]. Let $q_i(t)$ be the relative displacement of the $i$-th story with respect to the ground. As such, the dynamics of the structure-transducer system can be described by
\[
M_s\ddot{q}(t) + D_s\dot{q}(t) + S_sq(t) = \Gamma_s M_s a(t) + \Gamma_t K_t i(t) \quad (4.39)
\]
where
\[
M_s = \begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{bmatrix}, \quad
D_s = \begin{bmatrix}
c_1 + c_2 & -c_2 & 0 \\
-c_2 & c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3
\end{bmatrix},
\]
\[
S_s = \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3
\end{bmatrix}, \quad
\Gamma_a = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\Gamma_i = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

Now, if we define the harvester state vector as
\[
x_h(t) = \begin{bmatrix}
S_s^{1/2} & 0 \\
0 & M_s^{1/2}
\end{bmatrix}
\begin{bmatrix}
q(t) \\
\dot{q}(t)
\end{bmatrix}
\]
(4.40)
then the harvester state space matrices can be expressed as
\[
A_h = \begin{bmatrix}
0 & \frac{S_s^{1/2}M_s^{-1/2}}{M_s^{-1/2} - M_s^{-1/2}D_sM_s^{-1/2}} \\
-M_s^{-1/2}S_s^{1/2} & \frac{S_s^{1/2}M_s^{-1/2}}{M_s^{-1/2} - M_s^{-1/2}D_sM_s^{-1/2}}
\end{bmatrix}, \quad
B_h = \begin{bmatrix}
0 \\
K_1M_s^{-1/2}\Gamma_i
\end{bmatrix}, \quad
G_h = \begin{bmatrix}
0 \\
M_s^{1/2}\Gamma_a
\end{bmatrix}.
\]

Let the disturbance acceleration \(a(t)\) be characterized by bandpass filtered white noise as in [103], which has the state vector \(x_a(t)\) and state space matrices
\[
A_a = \begin{bmatrix}
0 & 1 \\
-\omega_a^2 & -2\zeta_\omega \omega_a
\end{bmatrix}, \quad
B_a = \begin{bmatrix}
0 \\
\alpha
\end{bmatrix}, \quad
C_a = \begin{bmatrix}
0 & 2\zeta_\omega \omega_a
\end{bmatrix}.
\]
Augmenting the harvester and disturbance state vectors such that \( \mathbf{x}(t) = [\mathbf{x}_h^T(t) \quad \mathbf{x}_a^T(t)]^T \) results in state space matrices in Equations (4.1a) and (4.1b) with appropriate definitions for \( \{A, B, G\} \). The structural response quantities \( \mathbf{z}(t) \) were chosen as normalized drift and acceleration quantities for the structure. As such, \( \mathbf{z}(t) \) can be expressed as

\[
\mathbf{z}(t) = \left[ \frac{\ddot{q}_1(t)}{\bar{d}} \quad \frac{\ddot{\bar{q}}_1(t)}{\bar{a}} \quad \frac{\ddot{q}_3(t)}{\bar{a}} \right]
\]

(4.41)

where \( \bar{d} \) and \( \bar{a} \) are the drift and acceleration weights, respectively. The structural response matrices \( \{C, D\} \) in Equation (4.1c) can be obtained from the formulation of \( \mathbf{z}(t) \).

Relevant values for the structure, transducer, disturbance, and performance parameters are listed in Table 4.1. The value of the motor constant \( K_t \) corresponds to the experimental transducer in Chapter 2.1. The disturbance parameters \( \omega_a \) and \( \alpha \) have been scaled by factors of 4 and 0.25, respectively, to reflect the scale of the three-story structure. Originally, the disturbance parameters were selected as the “near-field without pulse” ground motion case from [103]. For this example, we consider two structural response scenarios, which we will refer to as “Case 1” and “Case 2.” The main difference between these two scenarios is that we place a higher weight on suppressing accelerations in Case 2 (which is reflected in the lower value of \( \bar{a} \) for Case 2). Furthermore, we note that the pole placement parameter \( \omega_0 \) was chosen to

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{1,2,3} )</td>
<td>1000kg</td>
<td>( K_t )</td>
<td>453N/A</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>332N-s/m</td>
<td>( \omega_a )</td>
<td>5.2Hz</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>58.5N-s/m</td>
<td>( \zeta_a )</td>
<td>1.1</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>408N/m</td>
<td>( \alpha )</td>
<td>0.219m/s²</td>
</tr>
<tr>
<td>( k_1 )</td>
<td>( 1.09 \times 10^6 )N/m</td>
<td>( \bar{d} )</td>
<td>1cm</td>
</tr>
<tr>
<td>( k_2 )</td>
<td>( 1.68 \times 10^6 )N/m</td>
<td>( \bar{a} ) (Case 1)</td>
<td>1g</td>
</tr>
<tr>
<td>( k_3 )</td>
<td>( 1.41 \times 10^6 )N/m</td>
<td>( \bar{a} ) (Case 2)</td>
<td>0.3g</td>
</tr>
</tbody>
</table>
be a decade below the switching frequency of the motor drive used to control the transducer in Chapter 2.1 (i.e., $\omega_0 = 2\pi \times f_s/10 = 2.07 \times 10^4 \text{rad/s}$). We will not specify $R$ or $P_{gen}^0$, but instead will vary these two quantities to examine how $J$ and $P_{gen}$ depend on them.

We proceed by analyzing the closed-loop system for an example scenario. Let the transfer function of the closed-loop system in Equation (4.4) be defined by

$$G_{vw}(s) = \frac{\hat{v}(s)}{\hat{w}(s)} = C_v (sI - A)^{-1} B . \quad (4.42)$$

As such, the plot in Figure 4.3 illustrates the poles and zeros of $G_{vw}(s)$ for Case 1 where we have set $R = 5\Omega$ and $P_{gen}^0 = 0$. From Figure 4.3(a) we see that there is a set of complex conjugate poles that are much further away from the rest of the poles and zeros in the system. These poles are equal to $-2.81 \times 10^3 \pm 1.06 \times 10^4j$, and have magnitude less than $\omega_0$. Therefore, we have verified that the pole placement LMI has restricted the poles of $A$ to be less than the specified value of $\omega_0$. In addition, we plot the magnitude and phase of $G_{vw}(j\omega)$ in Figure 4.4.

Next, we examine how varying $R$ affects the energy harvesting and structural response performance measures for the two weighting scenarios. In addition, we set
Figure 4.4: Magnitude and phase of the closed-loop system $G_{vw}(j\omega)$ for Case 1 with $R = 5\Omega$ and $\bar{P}_{gen}^0 = 0$.

$\bar{P}_{gen}^0 = 0$, which we will refer to as the regenerative constraint on the average power generation. The plots in Figures 4.5 and 4.6 illustrate $J$ versus $R$ and $\bar{P}_{gen}$ versus $R$ for Case 1 and Case 2, respectively. As shown, the performance of the multi-objective (MO) controller is compared to the performances of the LQG and static admittance (SA) controllers. We note that the LQG and SA controllers are designed to minimize $J$ with no restrictions on $\bar{P}_{gen}$.

From these two plots, there are three values of $R$ (denoted by $R_1$, $R_2$, and $R_3$) that we will discuss further. The first resistance $R_1$ corresponds to the electronic efficiency that results in $\bar{P}_{gen}^{LQG} = 0$. Values of $R$ below $R_1$ result in $J^{LQG} = J^{MO}$ as well as $\bar{P}_{gen}^{LQG} = \bar{P}_{gen}^{MO}$, which means that the performance of the MO controller is identical to the LQG solution. The second resistance $R_2$ corresponds to the electronic efficiency that results in $\bar{P}_{gen}^{SA} = 0$. However, $J^{MO} < J^{SA}$ at $R = R_2$, which means that the MO controller outperforms the SA controller when $\bar{P}_{gen}^{SA} = \bar{P}_{gen}^{MO} = 0$. Finally, the third resistance $R_3$ corresponds to the electronic efficiency that results in the
Figure 4.5: Plot of the two performance measures versus $R$ for Case 1 and $\bar{P}_{gen}^0 = 0$: (a) $J$ versus $R$, and (b) $\bar{P}_{gen}$ versus $R$. Note: $\{R_1, R_2, R_3\} = \{3.62\Omega, 4.31\Omega, 7.48\Omega\}$

Figure 4.6: Plot of the two performance measures versus $R$ for Case 2 and $\bar{P}_{gen}^0 = 0$: (a) $J$ versus $R$, and (b) $\bar{P}_{gen}$ versus $R$. Note: $\{R_1, R_2, R_3\} = \{1.60\Omega, 7.74\Omega, 15.5\Omega\}$

$J^{MO} = J^{SA}$. The range of $R$ values between $R_1$ and $R_3$ is important because it represents electronic efficiency values where the MO controller outperforms the SA controller (in terms of $J$) while still enforcing the regenerative constraint. The main difference between Case 1 in Figure 4.5 and Case 2 in Figure 4.6 is there is wider range of $R$ values in $\{R_1, R_3\}$ for Case 2. In other words, when the suppression of accelerations becomes more important than displacements, the MO controller is able to outperform the SA controller over a wider range of electronic efficiency values while still maintaining the regenerative constraint on power generation.
Now, we modify our example by varying $\bar{P}_\text{gen}$ in addition to $R$. This is motivated by the fact that controllable transduction systems may require some additional parasitic power to run the electronics and control intelligence. As such, we examine how increasing $\bar{P}_\text{gen}$ affects the structural response performance measure. The plots in Figure 4.7 illustrate the structural response performance ratio $J_{LQG}^{\text{MO}}$ over a range of $\{R, \bar{P}_\text{gen}\}$ values. In both plots there is an area labeled as the “infeasible region,” which corresponds to the case where $\bar{P}_\text{gen} > \bar{P}_\text{max}$. When $\bar{P}_\text{gen} > \bar{P}_\text{max}$, the optimization problem in Equation (4.31) is infeasible because $\bar{P}_\text{max}$ is the theoretically-optimal bound on generation for a given value of $R$. From these plots, we see that there is a region of $\{R, \bar{P}_\text{gen}\}$ values that corresponds to $J_{LQG}^{\text{MO}} \approx 1$. As expected, the value of $J^{\text{MO}}$ increases as $\bar{P}_\text{gen}$ and $R$ are increased away from this region, until the problem becomes infeasible.

4.5 Summary

This chapter has illustrated that active stochastic feedback controllers can be designed to minimize the response of a structure subject to a constraint on average power generation. These requirements may arise due to the dual functionality of a
transduction system for both energy harvesting and structural control. As such, we outlined a procedure for explicitly balancing structural control objectives against the energy harvesting objective, in the context of broadband stochastic vibrations. In our analysis, the energy harvesting objective manifests itself as a constraint on the minimization of the structural response performance measure. We showed that the multi-objective optimization problem can be formulated as a standard convex LMI eigenvalue problem. However, as illustrated in Scherer et al. [97], many other types of control objectives, including $H_\infty$ performance objectives, peak-to-peak gain bounds, overshoot bounds, tracking objectives, and robustness objectives can be addressed using the LMI techniques discussed in this chapter.
Control of Nonlinear Vibratory Energy Harvesters

5.1 Background

In order to fully maximize the potential power generation from an actively controlled vibratory energy harvester, the nonlinearities in the system must be accounted for in the control design. In this chapter, we augment the energy harvesting control theory derived in Chapter 3 by incorporating the mechanical and electrical nonlinearities observed in the electromagnetic transducer that was studied in Chapter 2.

In the energy harvesting literature, several researchers have investigated the effects of nonlinearities on power generation. For example, a recent study by Stanton et al. [117] fit a nonlinear model of a piezoelectric energy harvester to experimental data. The study showed that nonlinear damping in the cantilever beam as well as nonlinear electromechanical coupling in the piezoelectric patch must be accounted for in the model in order to accurately predict the response of the beam. In addition, Chapter 2 developed a predictive model to account for the nonlinearities present in an electromagnetic transducer consisting of a ballscrew actuator coupled to a permanent-magnetic synchronous machine. The nonlinearities in that device
are caused by the sliding friction interaction between the ballscrew and ball bearings as well as the elasticity of the belt that connects the ballscrew to the shaft of the motor.

Nonlinearities also occur in the electronics that interface the transducer with energy storage. For the simplest passive energy harvesting circuit, which consists of a standard diode bridge, a small amount of parasitic power is dissipated in a nonlinear manner as a result of the voltage threshold required for the diodes to conduct. More elaborate active energy harvesting circuits, such as buck-boost converters [69] or H-bridges [76, 126], are operated via high frequency PWM switching control of MOSFETs. The parasitic power losses associated with these switching converters are highly nonlinear and result from the way in which the transducer current is controlled to track a desired current. The study by Scruggs et al. [102] derived an approximate loss model for the behavior of an H-bridge operated in discontinuous conduction for a piezoelectric energy harvester excited by a broadband disturbance. The nonlinearities in the model derived in that paper can be attributed to the conduction losses in the MOSFETs and diodes as well as gating and transition losses.

The main objective of this chapter is to develop a way to account for dynamic nonlinearities in the harvester, while optimizing the controller for maximum power generation. To this end, we use statistical linearization to account for the influence of the nonlinearities on the stochastic response. This concept has been applied in piezoelectric energy harvesting applications by Ali et al. [3], but not in the context of optimal control. However, problems involving simultaneous statistical linearization and optimal control have been investigated in other applications. These techniques were first developed to account for saturation constraints on control inputs in stochastic systems [130]. In another study by Gökçek et al. [44], saturating linear-quadratic-regulator (SLQR) and saturating linear-quadratic-Gaussian (SLQG) feedback gains were developed for linear systems with saturating actuators. Several additional stud-
ies [84, 88, 132] have developed sub-optimal control designs to account for nonlinear systems subjected to stochastic disturbances. In [84], an iterative algorithm was proposed in which the optimal controller for the statistically linearized system is updated until specified response statistics converge. However, a standard Riccati equation is solved for the statistically linearized system at each iteration, which results in the performance being sub-optimal. Controllers developed for the systems studied in [88, 132] were computed by first statistically linearizing the nonlinear system, and then solving a standard Riccati equation as if the linearized coefficients were independent of the feedback law. This also leads to a sub-optimal solution.

The first section of this chapter deals with the experimental system from Chapter 2, in which the electromagnetic transducer is embedded within a SDOF oscillator and controlled with an active power electronic drive. For this system, we derive an analytical expression for the average power generated by the device when the terminals of the motor are connected to resistive shunts. This expression can be used to determine the impedance matched resistive load that maximizes the average power generated. We compare the analytical expression for the average power generated to the experimental system when the disturbance acceleration is sinusoidal with known amplitude and frequency. The experimental analysis is performed using real-time hybrid testing.

The remainder of this chapter focuses on deriving the statistically linearized feedback controller for the experimental system when it is excited by a bandpass-filtered white noise process, which was first presented in [19]. The control objective in this chapter is different from the ones presented in [44, 84, 88, 130, 132]. The performance objective in those studies minimizes the variance of the system’s output, while the performance objective in the present study maximizes the average power generated by the transducer. We show that the optimal feedback gains for the nonlinear system can be computed by solving two nonlinear, coupled algebraic equations. Solving
these two equations can be accomplished through an iterative algorithm, which solves the Riccati equation using standard linear matrix inequality (LMI) [14] techniques. In addition to the nonlinearities in the transducer, we develop a non-quadratic loss model for the H-bridge operating in continuous conduction mode (CCM). The final section of this chapter augments the iterative algorithm to account for this non-quadratic loss model in the optimization of the controller.

5.2 Impedance Matching of a Nonlinear Energy Harvester

Consider the diagram of the experimental system in Figure 5.1, which consists of the electromagnetic transducer and servo drive configuration from Chapter 2. Recall that the linear force \( f(t) \) of the device can be related to the electromechanical force \( f_e(t) \) of the motor, via the equation

\[
f(t) = f_e(t) - F_c \text{sgn}(\dot{x}(t)) - m_d \ddot{x}(t) - c_d \dot{x}(t) - k_d x(t) \tag{5.1}
\]

where \( m_d \) and \( c_d \) are the equivalent linear mass and viscous damping resulting from the rotational inertia and viscous damping of the ballscrew and shaft of the motor, respectively. It was experimentally determined in Chapter 2 that Coulomb friction and stiffness forces, which are represented by \( F_c \) and \( k_d \), are also present in the device. We note that the total Coulomb friction force \( F_c \) is a conservative summation of the

\[ a \]

\[ x \]

\[ m_x \]

\[ k_x \]

\[ c_x \]

\[ f \]

\[ V_s \]

\[ i^* \]

\[ R_f \]

\[ dSpace DS1103 \]

Figure 5.1: Experimental setup consisting of the electromagnetic transducer and SDOF oscillator from Chapter 2.
identified friction parameters in Chapter 2 (i.e., $F_c = f_c + \gamma_1 + \gamma_2$).

As shown in Figure 5.1, the back-emf voltage $v(t)$ at the terminals of the motor is measured using a differential amplifier. Next, a command current $i^*(t)$ is obtained by dividing $v(t)$ by a specified load resistance $R_L$, which is accomplished in dSpace. Finally, a S16A8 servo drive from Advanced Motion Controls imposes the desired current $i(t)$ onto the transducer. When the terminals of the motor are connected to $R_L$, the electromechanical force can be expressed as

$$f_e(t) = -\frac{K_t^2}{R_c + R_L} \dot{x}(t) = -c_e \dot{x}(t)$$ (5.2)

where $K_t$ is the back-emf motor constant and $R_c$ is the coil resistance. We can think of $c_e$ as the equivalent electromechanical viscous damping term associated with connecting the motor to resistive loads.

The electromagnetic transducer is embedded between the ground and moving mass of the single-degree-of-freedom (SDOF) resonant oscillator. The SDOF oscillator is characterized by a mass $m_s$, a damping $c_s$, and a stiffness $k_s$, and is excited at its base by the stochastic disturbance acceleration $a(t)$. Thus, the coupled dynamics of the SDOF oscillator and electromagnetic transducer is

$$m_s \ddot{x}(t) + c_s \dot{x}(t) + k_s x(t) = m_s a(t) + f(t)$$ (5.3)

where $r(t)$ is the relative displacement of the mass of the structure. Next, inserting the expression for $f(t)$ into Equation (5.3) results in the following nonlinear differential equation

$$m \ddot{x}(t) + c \dot{x}(t) + k x(t) + F_s \operatorname{sgn}(\dot{x}(t)) = m_s a(t)$$ (5.4)

where $m = m_d + m_s$, $c = c_d + c_s + c_e$, and $k = k_d + k_s$. Furthermore, for this section, we assume that the SDOF oscillator is excited at its base by a sinusoidal acceleration with amplitude $A_0$ and frequency $\omega_0$, where $\omega_0 = \sqrt{k/m}$. We summarize the values
Table 5.1: Parameter values for the electromagnetic energy harvester.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_t$</td>
<td>453N/A</td>
<td>$F_c$</td>
<td>160N</td>
</tr>
<tr>
<td>$R_c$</td>
<td>2.41Ω</td>
<td>$m_a$</td>
<td>3000kg</td>
</tr>
<tr>
<td>$m_d$</td>
<td>20kg</td>
<td>$c_s$</td>
<td>395N-s/m</td>
</tr>
<tr>
<td>$c_d$</td>
<td>575N-s/m</td>
<td>$k_s$</td>
<td>$3\times10^4$N/s</td>
</tr>
<tr>
<td>$k_d$</td>
<td>630N/m</td>
<td>$A_0$</td>
<td>0.18m/s^2</td>
</tr>
</tbody>
</table>

for the experimental system in Table 5.1. The energy harvester’s mass, damping, and stiffness correspond to values for a scaled tuned mass damper within a multi-story building.

The experimental electromagnetic transducer is back-driven by a hydraulic actuator, which simulates the dynamics of the SDOF oscillator using real-time hybrid-testing (RTHT). RTHT is accomplished in dSpace by feeding back force measurements from the electromagnetic transducer in real-time to a simulated disturbance and structure model. The force measurement and simulated disturbance force are summed together and used to excite the simulated structure. The output of the simulated structure is a displacement command signal, which is tracked by the digital controller used to operate the hydraulic actuator. Our experimental RTHT design follows the procedure in the study by Darby et al. [28] and Figure 5.2 shows a block diagram describing this procedure.

![Hybrid testing block diagram.](image)

Figure 5.2: Hybrid testing block diagram.
We begin by deriving an expression for the equivalent viscous damping from the Coulomb friction. This can be accomplished through an energy balance approach, where we set the amount of energy dissipated from the Coulomb friction during one oscillation equal to the amount of energy dissipated by an equivalent viscous damper. This quantity can be expressed as

\[ c_{eq} = \frac{4F_c}{\pi \dot{X}} \]  

(5.5)

where \( \dot{X} \) is the relative velocity amplitude. The expression for the velocity amplitude can be written as

\[ \dot{X} = \frac{m_s A_0 \omega_0}{\sqrt{(-m\omega_0^2 + k)^2 + (c + c_{eq})^2 \omega_0^2}} \]  

(5.6)

Since \( c_{eq} \) depends on \( \dot{X} \), the expression in Equation (5.6) is quadratic in \( \dot{X} \). Substituting the expression for \( c_{eq} \) into Equation (5.6) and solving for \( \dot{X} \) results in

\[ \dot{X} = \frac{-4cF_c \omega_0^2 + \sqrt{\omega_0^2 \left(-16F_c^2 (-m\omega_0^2 + k)^2 + A_0^2 m_s^2 \pi^2 \left(c^2 \omega_0^2 + (-m\omega_0^2 + k)^2\right)\right)} \pi \left(c^2 \omega_0^2 + (-m\omega_0^2 + k)^2\right)}{\pi \left(c^2 \omega_0^2 + (-m\omega_0^2 + k)^2\right)} \]  

(5.7)

where we have ignored the negative root. Since we know that \( \omega_0 = \sqrt{k/m} \), the expression for the velocity amplitude simplifies to

\[ \dot{X} = \frac{m_s A_0 - 4F_c/\pi}{c} \]  

(5.8)

We can now express the average total power generated during one oscillation as a function of the back-emf of the motor; i.e.,

\[ \bar{P}_T = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{v^2(t)}{R_c + R_L} \, dt \]  

(5.9)
where \( v(t) = K_t \dot{x}(t) \) is the back-emf of the motor. Therefore, solving for \( \bar{P}_T \) results in

\[
\bar{P}_T = \frac{K_t^2 \dot{X}^2}{2(R_c + R_L)}. \tag{5.10}
\]

Finally, we have that the average power generated across the load resistance \( R_L \) is

\[
\bar{P}_L = \bar{P}_T \frac{R_L}{R_c + R_L} = \frac{K_t^2 \dot{X}^2 R_L}{2(R_c + R_L)^2}. \tag{5.11}
\]

To determine the resistance that maximizes average power, we take the partial derivative of \( \bar{P}_L \) with respect to \( R_L \) and set this expression equal to zero. After some manipulation, the optimal resistive load can be written as

\[
R_L^* = R_c + \frac{K_t^2}{c_s + c_d}. \tag{5.12}
\]

Substituting the expression for \( R_L^* \) into Equation (5.11) results in an analytical expression for the optimal power generated; i.e.,

\[
\bar{P}_L^* = \frac{K_t^2 (\pi m_s A_0 - 4F_c)^2}{8\pi^2(c_s + c_d)(K_t^2 + R_c(c_s + c_d))}. \tag{5.13}
\]

It is interesting to note that the amount of Coulomb friction in the device does not influence the optimal resistive load. However, neglecting the Coulomb friction in the device will significantly overestimate the amount of power that can be harvested given any resistive load. In other words, including Coulomb friction effects is crucial to accurately predicting the amount of power that can be harvested from a SDOF oscillator by an electromagnetic transducer.

As previously mentioned, we compare the analytical expression for the average power generated across the load resistance \( R_L \) with the experimental system using RTHT. Three levels of passive viscous damping \( \zeta_0 \) are used in the simulated SDOF
Figure 5.3: Comparison of predicted and experimental average power generated over a range of load resistance values: (a) $\zeta_0 = 0.15$, (b) $\zeta_0 = 0.2$, and (c) $\zeta_0 = 0.3$.

Figure 5.4: Plot of the optimal load resistance and the corresponding optimal average power generated over a range of $\zeta_0$ values: (a) $R_L^*$ versus $\zeta_0$, and (b) $P_L^*$ versus $\zeta_0$. 

92
oscillator, where we define $\zeta_0$ as

$$\zeta_0 = \frac{c_s + c_d}{2\sqrt{mk}}. \quad (5.14)$$

The three levels of damping that we have chosen to investigate are $\zeta_0 = 0.15$, $\zeta_0 = 0.2$, and $\zeta_0 = 0.3$. Although these levels of damping are much higher than typical levels for a TMD in a tall building, they illustrate three $\bar{P}_L$ versus $R_L$ curves with three distinct optimal values. For each test at a given $R_L$ value, we collect data for 20 seconds at a sample rate of 1000Hz.

The analytical expression for the predicted average power generated across $R_L$ is determined by using optimal parameter values for the proposed model in Table 2.2. We compare the predicted power to the experimental RTHT results in Figure 5.3 for three levels of mechanical damping in the SDOF oscillator. This plot shows that we have excellent agreement between the predicted and experimental average power values over the range of load resistances and for the three levels of mechanical damping. The optimal load resistances given in Equation (5.12) were calculated to be 38Ω, 55Ω, and 73Ω while the optimal power generated values given in Equation (5.13) were calculated to be 2.62W, 4.02W, and 5.42W for $\zeta_0 = 0.3$, $\zeta_0 = 0.2$, and $\zeta_0 = 0.15$, respectively. In addition, we plot the variation in the optimal load resistance $R_L^*$ and the optimal average power generated $\bar{P}_L^*$ for a range of $\zeta_0 \in [0.05, 0.5]$ in Figure 5.4. As expected, both values decrease monotonically as $\zeta_0$ increases.

5.3 Statistically Linearized Energy Harvesting

In this section, we extend the theory presented in Chapter 3 to account for the Coulomb friction force present in the electromagnetic transducer. Again we assume that the losses in the electronics are purely resistive.

Recall that the coupled dynamics of the SDOF oscillator and electromagnetic
transducer in Figure 5.1 can be expressed by the nonlinear differential equation

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) + F_c \text{sgn}(\dot{x}(t)) = m_s a(t) + f_e(t) \]  

(5.15)

where \( x(t) \) is the relative displacement of the mass of the structure, \( m = m_d + m_s \), \( c = c_d + c_s \), and \( k = k_d + k_s \). If we define the harvester state vector as

\[ \mathbf{x}_h(t) = \begin{bmatrix} \sqrt{k} \\ 0 \end{bmatrix} x(t) \begin{bmatrix} 0 \\ \sqrt{m} \end{bmatrix} \dot{x}(t) \]  

(5.16)

then the harvester dynamics can be expressed by the self-dual state space

\[ \dot{\mathbf{x}}_h(t) = \mathbf{A}_h \mathbf{x}_h(t) + \mathbf{F}_h \text{sgn}(\dot{x}(t)) + \mathbf{B}_h \mathbf{i}(t) + \mathbf{G}_h a(t) \]  

(5.17a)

\[ v(t) = \mathbf{B}_h^T \mathbf{x}_h(t) \]  

(5.17b)

\[ \dot{x}(t) = \mathbf{C}_h \mathbf{x}_h(t) \]  

(5.17c)

where

\[ \mathbf{A}_h = \begin{bmatrix} 0 & \sqrt{k/m} \\ -\sqrt{k/m} & -c/m \end{bmatrix}, \quad \mathbf{B}_h = \begin{bmatrix} 0 \\ K_t/\sqrt{m} \end{bmatrix}, \quad \mathbf{G}_h = \begin{bmatrix} 0 \\ m_s/\sqrt{m} \end{bmatrix}, \]

\[ \mathbf{F}_h = \begin{bmatrix} 0 \\ -F_c/\sqrt{m} \end{bmatrix}, \quad \mathbf{C}_h = \begin{bmatrix} 0 & 1/\sqrt{m} \end{bmatrix}. \]

We characterize the disturbance acceleration by the second-order bandpass filter

\[ \dot{x}_a(t) = \mathbf{A}_a x_a(t) + \mathbf{B}_a w(t) \]  

(5.18a)

\[ a(t) = \mathbf{C}_a x_a(t) \]  

(5.18b)

where

\[ \mathbf{A}_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta_a\omega_a \end{bmatrix}, \quad \mathbf{B}_a = \begin{bmatrix} 0 \\ 2\sigma_a\sqrt{\zeta_a\omega_a} \end{bmatrix}, \quad \mathbf{C}_a = \begin{bmatrix} 0 & 1 \end{bmatrix}. \]

We assume the input \( w(t) \) is a white noise process with spectral intensity equal to unity. In addition, we have that \( \sigma_a \) is the standard deviation of the disturbance
acceleration, \( \omega_a = \sqrt{k/m} \) is the passband of the disturbance filter, and \( \zeta_a \) determines the quality factor of the disturbance filter. We combine the harvester states with the disturbance states such that the augmented state space \( \mathbf{x}(t) = [\mathbf{x}_h^T(t) \, \mathbf{x}_a^T(t)]^T \) obeys

\[
\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + F \text{sgn}(\dot{\mathbf{x}}(t)) + B\mathbf{i}(t) + G\mathbf{w}(t) \\
v(t) = B^T\mathbf{x}(t) \\
\dot{\mathbf{x}}(t) = C\mathbf{x}(t)
\]

(5.19a) (5.19b) (5.19c)

where

\[
A = \begin{bmatrix} A_h & G_h C_a \\ 0 & A_a \end{bmatrix}, \quad B = \begin{bmatrix} B_h \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ B_a \end{bmatrix}, \quad F = \begin{bmatrix} F_h \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_h & 0 \end{bmatrix}.
\]

5.3.1 Stationary Covariance

The general state space model for an energy harvesting system with nonlinearities is

\[
\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \phi(\mathbf{x}(t), t) + B\mathbf{i}(t) + G\mathbf{w}(t) \\
v(t) = B^T\mathbf{x}(t) \\
y(t) = C\mathbf{x}(t)
\]

(5.20a) (5.20b) (5.20c)

where we assume the function \( \phi(\mathbf{x}(t), t) \) is nonlinear. We assume \( \phi(0, t) = 0 \), and that it is anti-symmetric; i.e., \( \phi(-\mathbf{x}(t), t) = -\phi(\mathbf{x}(t), t) \). In addition, we assume that \( \mathbf{x}(t) \) has a probability distribution which can be approximated as Gaussian with zero mean (because \( \phi(\mathbf{x}(t), t) \) is assumed to be anti-symmetric) and covariance \( \Sigma(t) \). The corresponding probability density function (pdf) of \( \mathbf{x}(t) \) is

\[
p(\mathbf{x}(t), t) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma(t)}} \exp\left\{-\frac{1}{2} \mathbf{x}^T(t)\Sigma^{-1}(t)\mathbf{x}(t)\right\}.
\]

(5.21)

If we implement any stabilizing full-state feedback control law \( \mathbf{i}(t) = K\mathbf{x}(t) \), then the solution to the covariance \( \Sigma(t) \) can be found via statistical linearization [95].
Specifically, the dynamic evolution of $\Sigma(t)$ is governed by the differential equation

$$
\dot{\Sigma}(t) = \mathcal{E}\{\nabla^T_x \phi^T_{cl}(x(t), t)\}^T \Sigma(t) + \Sigma(t) \mathcal{E}\{\nabla^T_x \phi^T_{cl}(x(t), t)\} + G G^T \tag{5.22}
$$

where the closed-loop nonlinear function $\phi_{cl}(x(t), t)$ is

$$
\phi_{cl}(x(t), t) = A x(t) + B K x(t) + \phi(x(t), t) \tag{5.23}
$$

and where $\nabla_x$ is the gradient operator with respect to the variable $x$. The covariance, $S$, in stationary response is then found by finding the equilibrium of the above, i.e., the solution to the Lyapunov-like equation

$$
\mathcal{E}\{\nabla^T_x \phi^T_{cl}(x)\}^T S + S \mathcal{E}\{\nabla^T_x \phi^T_{cl}(x)\} + G G^T = 0 \tag{5.24}
$$

For the case where the nonlinearity is Coulomb friction, we replace $\phi(x(t), t)$ with $F \text{sgn}\{\dot{x}(t)\}$ in Equation (5.23) where $\dot{x}(t) = y(t)$. Taking the gradient of $\phi_{cl}(x(t), t)$ with respect to $x(t)$, results in

$$
\nabla^T_x \phi^T_{cl}(x(t), t) = A^T + K^T B^T + 2 C^T F^T \delta(y(t)) \tag{5.25}
$$

where $\delta(\cdot)$ is the Dirac delta function. Next, taking the expectation of both sides of Equation (5.25) results in the following expression

$$
\mathcal{E}\{\nabla^T_x \phi^T_{cl}(x(t), t)\} = A^T + K^T B^T + 2 C^T F^T \int \delta(y(t)) p(y(t), t) \, dy \tag{5.26}
$$

By assumption, the pdf for $y(t)$ is a zero-mean Gaussian function, with scalar variance $s_y(t) = C \Sigma(t) C^T$; i.e.,

$$
p(y(t), t) = \frac{1}{\sqrt{2\pi s_y(t)}} \exp\{-y^2(t)/2s_y(t)\} \tag{5.27}
$$

Thus, we have that Equation (5.26) is

$$
\mathcal{E}\{\nabla^T_x \phi^T_{cl}(x(t), t)\} = A^T + K^T B^T + V(t)^T \tag{5.28}
$$
where

\[ V(t) = \sqrt{\frac{2}{\pi}} \frac{FC}{\sqrt{Cz(t)C^T}}. \]  

(5.29)

Substituting Equation (5.28) into Equation (5.22) results in an equation for \( \Sigma(t) \) as

\[ \dot{\Sigma}(t) = A_{cl}(t)\Sigma(t) + \Sigma(t)A_{cl}(t)^T + GG^T \]  

(5.30)

where \( A_{cl}(t) = A + BK + V(t) \). It is important to note that the matrix \( V(t) \) augments the dynamics matrix \( A \) by adding an additional term which supplements the viscous damping in the system. This additional term is the statistically-equivalent linear viscous damping due to the Coulomb friction force.

It follows that the stationary covariance is the equilibrium solution of Equation (5.30); i.e., the solution with \( \dot{\Sigma}(t) = 0 \). Although the resultant equation is reminiscent of an algebraic Lyapunov equation, it is in fact nonlinear, because \( V(t) \) depends on \( \Sigma(t) \). In general, the equilibrium solution to Equation (5.30) can only be found iteratively.

It is also important to recognize that equilibrium solutions of Equation (5.30) may not necessarily be stable, and mean-square stability of any equilibrium solution must be checked. From classical linear system theory, we know that if \( A_{cl}(t) \) were constant (i.e., if it did not depend on \( \Sigma(t) \)) then Equation (5.30) would have a unique equilibrium, provided that \( A_{cl}(t) \) does not have any two eigenvalues that sum to zero. Asymptotic stability of this unique equilibrium (and, therefore, achievement of stationarity of the covariance \( \Sigma(t) \)) then follows if and only if \( A_{cl}(t) \) is asymptotically stable. However, because \( A_{cl}(t) \) varies with \( \Sigma(t) \), the situation at hand is somewhat more complicated than this.

Let \( S \) be an equilibrium solution to Equation (5.30), and then consider that for small \( \| \Sigma(t) - S \| \), the stability of \( \Sigma(t) \) can be ascertained by examining the linearized version of Equation (5.30), with \( \Sigma(t) = S \) used as the linearization point.
The resultant linearized covariance equation for the deviation \( \Xi(t) = \Sigma(t) - S \) is

\[
\dot{\Xi}(t) = A_{cl} \Xi(t) + \Xi(t)A_{cl}^T - \sqrt{\frac{1}{2\pi}} \frac{C\Xi(t)C^T}{(CSC^T)^{3/2}} \left[ FCS + SC^TFT \right]
\]

(5.31)

where the time-invariant closed-loop dynamics matrix is \( A_{cl} = A_{cl}(t)|_{\Sigma(t)=S} \). In order for equilibrium solution \( S \) to be a valid stationary solution, the above linearized differential equation must be asymptotically stable about the origin.

Note that Equation (5.31) is not a standard Lyapunov differential equation, due to the last term on the right. Similar equations arise in the literature on stochastic systems with multiplicative noise inputs [129], where it is well-known that asymptotic stability is not in general guaranteed by asymptotic stability of \( A_{cl} \). Even when this condition is satisfied, the last term on the right-hand side can have a destabilizing effect. For a given set of parameters \( \{A, B, C, F, K\} \) and a known equilibrium solution \( S \), stability of Equation (5.31) can be inferred exactly by converting this equation into vectorized form, using Kronecker algebra, and then examining the asymptotic stability of associated \( n^2 \times n^2 \) dynamics matrix. However, here we instead introduce the following theorem that is sufficient to guarantee stationarity. The proof of this theorem can be found in Appendix A.4.

**Theorem 4.** Assume that the time-invariant closed-loop dynamics matrix \( A_{cl} \) is asymptotically stable. Then the equilibrium solution of Equation (5.31) is also asymptotically stable if

\[
\left( CSTSC^T \right)^{1/2} \left( F^TTF \right)^{1/2} \left( CSC^T \right)^{3/2} < \sqrt{\frac{\pi}{2}}
\]

(5.32)

where \( T > 0 \) is the solution to the Lyapunov equation

\[
A_{cl}^T T + TA_{cl} + C^TC = 0 .
\]

(5.33)

Finally, we note that, because statistical linearization is merely an approximation of the true system dynamics, it is important that certain precautions be taken to
ensure that a stable covariance matrix is indeed a justifiable approximation of the true mean-square system behavior. At the bare minimum, it should be ensured that the true system is bounded-input bounded-state stable, in order for the approximate ensemble averages to be meaningful. For the type of nonlinear system we consider here (i.e., where the nonlinearities arise due to Coulomb friction), this may be done by checking that the matrix $A + BK$ is asymptotically stable. It is a straight-forward Lyapunov analysis to show that if this condition holds, then the ratio $\|x_h\|_\infty/\|a\|_\infty$ is always bounded, independently of the amount of Coulomb friction present in the system.

5.3.2 Stationary Optimal Energy Harvesting

Recall that the energy harvesting objective is to maximize the average power generated in stationarity; i.e.,

$$P_{gen} = -\text{tr}\left\{ \frac{1}{2} K^T B^T + \frac{1}{2} BK + R K^T K \right\} S$$

(5.34)

over the feedback gain matrix $K$. Since this optimization is subject to the constraint in to Equation (5.30), we define the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} = -P_{gen} + \text{tr}\left\{ P \left( A_{cl} S + S A_{cl}^T + G G^T \right) \right\}$$

(5.35)

where $P = P^T$ is a Lagrange multiplier matrix which enforces the stationary solution to Equation (5.30) as a constraint in the optimization. Thus, we have the following minimax problem

$$K = \arg \min_K \left[ \min_{S=S^T} \max_{P=P^T} \mathcal{H} \right].$$

(5.36)

To find the optimal solution to the problem in Equation (5.36), we take the partial derivative of the Hamiltonian with respect to each of the decision variables and set these quantities equal to zero. This procedure constitutes a standard approach to solving an optimal control problem [72]. For brevity, we suppress the intermediate
steps required to compute the partial derivatives and merely highlight their final analytical expressions. We start by taking the partial derivative of $\mathcal{H}$ with respect to $S$; i.e.,

$$\frac{\partial \mathcal{H}}{\partial S} = \frac{1}{2} K^T B^T + \frac{1}{2} B K + R K^T K + P A_{cl} + A_{cl}^T P - U V - V^T P U^T = 0 \quad (5.37)$$

where $V = V(t)|_{\Sigma(t)=S}$ and

$$U = \frac{1}{2} C^T C S \quad (5.38)$$

Next, we take the partial derivative of $\mathcal{H}$ with respect to $K$; i.e.,

$$\frac{\partial \mathcal{H}}{\partial K} = S B^T + 2 R S K^T + 2 S P B = 0 \quad (5.39)$$

Pre-multiplying Equation (5.39) by $S^{-1}$ and solving for $K$ results in Equation (3.21), but with the new $P$ found via Equation (5.37) rather than the Riccati equation in Equation (3.20). It is not necessary to take the partial derivative of $\mathcal{H}$ with respect to the Lagrange multiplier $P$ as this will just result in the equilibrium condition for $S$ in Equation (5.30). Finally, we can substitute Equation (3.21) into Equations (5.30) and (5.37) to arrive at two coupled, nonlinear algebraic equations for $S$ and $P$ that must hold at the optimum; i.e.,

$$\left[ A + V - \frac{1}{R} B B^T (P + \frac{1}{2} I) \right] S + S \left[ A + V - \frac{1}{R} B B^T (P + \frac{1}{2} I) \right]^T + G G^T = 0 \quad (5.40)$$

$$[A + V]^T P + P [A + V] - \frac{1}{R} (P + \frac{1}{2} I) B B^T (P + \frac{1}{2} I) - U V - V^T P U^T = 0 . \quad (5.41)$$

### 5.3.3 Iterative Algorithm

Because $U$ and $V$ depend on $S$, Equations (5.40) and (5.41) are coupled nonlinear algebraic equations. As such, solutions for the stationary covariance matrix $S$ and
the Lagrange multiplier $P$ must be computed iteratively. To do this, we begin by linearizing Equation (5.35) about $S = S_0$; i.e.,

$$\tilde{H} = -\tilde{P}_{\text{gen}} + \text{tr}\left\{ P \left[ [A + BK + V_0]S + S [A + BK + V_0] + GG^T 
+ \frac{1}{2} (V_0S_0 + S_0V_0^T) \left( 1 - \frac{\text{csc}^T}{\text{csc}_0} \right) \right] \right\}$$

where $V_0 = V|_{S = S_0}$. Regrouping terms, we have that

$$\tilde{H} = \text{tr}\left\{ P \left[ GG^T + \frac{1}{2} (V_0S_0 + S_0V_0^T) \right] + S \left[ \frac{1}{2} KTB^T + \frac{1}{2} BK + RK^TK 
+ P [A + BK + V_0] + [A + BK + V_0]^T P - U_0PV_0 - V_0^TPU_0^T \right] \right\}$$

where $U_0 = U|_{S = S_0}$.

If $\{ K, P, S \}$ are the optimal parameters for the original problem, they will also be optimal parameters for the linearized problem with $S_0$ equal to its optimal value.

Furthermore, if we assume that the optimal $K$ is such that $A_{cl}$ is asymptotically stable, then $S_0 > 0$ can be assumed to hold at the optimal solution as well. If this is the case, then using the linearized Hamiltonian and substituting the optimal relationship between $K$ and $P$ in Equation (3.21), gives a bound on the value of $\tilde{H}$; i.e.,

$$\tilde{H} > \text{tr}\left\{ P \left[ GG^T + \frac{1}{2} (V_0S_0 + S_0V_0^T) \right] \right\}$$

where $P$ is subject to the constraint

$$[A + V_0]^T P + P [A + V_0] - \frac{1}{R} (P + \frac{1}{2}I) BB^T (P + \frac{1}{2}I) - U_0PV_0 - V_0^TPU_0^T > 0.$$
the optimization

$$P = \arg \max_P \left\{ \text{tr}\left\{ P \left[ GG^T + \frac{1}{2} (V_0 S_0 + S_0 V_0^T) \right] \right\} \right\} \quad (5.46)$$

subject to Equation (5.45), or equivalently, the LMI

$$\begin{bmatrix} (A + V_0)^T P + P (A + V_0) & -U_0 PV_0 - V_0^T PU_0^T \cr B^T (P + \frac{1}{2} I) & \end{bmatrix} > 0 \quad (5.47)$$

will give the same optimal solution for $P$.

Motivated by the above, we define the matrix function $P = \Theta(S_0)$, over the domain $S_0 > 0$, as

$$\Theta(S_0) = \text{sol}_P \begin{cases} \text{Given :} & S_0 > 0 \\ \text{Maximize :} & \text{tr}\left\{ P \left[ GG^T + \frac{1}{2} (V_0 S_0 + S_0 V_0^T) \right] \right\} \\ \text{Over :} & P \\ \text{Subject to :} & \text{LMI constraint in Equation (5.47)} \end{cases}$$

(5.48)

The optimization in Equation (5.48) is convex and is feasible for any $S_0 > 0$, and may therefore be viewed as a unique function over the domain $S_0 > 0$. Using this function, the following iterative algorithm can be used to solve for the values of $S$ and $P$ at the optimum:

**Step 0:** Initialize $S_0$ by solving the linear energy harvesting problem (i.e., with $U = V = 0$).

**Step 1:** Compute new values for $V_0$ and $U_0$ using $S_0$.

**Step 2:** Compute $P = \Theta(S_0)$.

**Step 3:** In Equation (5.40), fix $V \leftarrow V_0$ and solve the resultant Lyapunov equation for $S$.

**Step 4:** Set $S_0 \leftarrow S$ and return to Step 1.
The algorithm has converged when the absolute value of the change in $\bar{P}_{gen}$ between the current and previous iteration is below a certain tolerance. The value of $\bar{P}_{gen}$ at the first iteration can be calculated using Equation (3.22) while the value of $\bar{P}_{gen}$ at any subsequent iteration can be calculated using Equation (5.34).

Here, we make no claim that this algorithm always converges, although it did converge for all examples considered in this chapter. Once convergence is reached, asymptotic stability of $A_{cl}$ at the optimum should be verified. Furthermore, it should also be verified that the matrix $A + BK$ is asymptotically stable, and that the inequality in Equation (5.32) holds. All conditions were found to hold uniformly in the solutions for the examples considered here.

For the energy harvesting example considered in this chapter, the above algorithm was found to converge within 10–20 iterations. Using the system defined in Equation (5.19), we illustrate the convergence of the proposed algorithm in figure 5.5. For this example, we fix $R = 5\Omega$ and $\zeta_a = 0.5$ and run the algorithm with a convergence tolerance of $1e^{-6}$. As shown, the algorithm converges to $\bar{P}_{gen} = 10.1W$ in 14 iterations.
5.3.4 Simulation Example

We can gain some valuable insight into the statistically linearized energy harvesting problem by comparing the performance of the statistically linearized SA controller with the performance of the statistically linearized LQG controller. The plots in Figure 5.6 illustrate this comparison for the SDOF energy harvester characterized by Equation (5.19). We see that the curves in both plots monotonically decrease as $\zeta_a$ increases and that the $\bar{P}_{gen}^{SA}$ curves in Figure 5.6(a) are lower than the $\bar{P}_{gen}^{LQG}$ curves in Figure 5.6(b), for all values of $\zeta_a$ and $R$.

To illustrate the improvement in performance that can be achieved by implementing the statistically linearized LQG controller, we define plot $\bar{P}_{gen}^{SA}/\bar{P}_{gen}^{LQG}$ in Figure 5.7. From this plot, we obtain the interesting result that there is a finite bandwidth for $a(t)$ at which the statistically linearized LQG controller is most beneficial. Another important result of this analysis can be seen in the narrowband limit as $\zeta_a \to 0$. At this limit, we see that the performance ratio is equal to unity. As pointed out in Chapter 3 (for the case without Coulomb friction), this is due to the fact that the velocity and acceleration gains are the only gains in $K$ required for the statistically

![Figure 5.6: Comparison of the average power generated by: (a) the statistically linearized SA controller, and (b) the statistically linearized LQG controller; for $R$ values of 2Ω, 5Ω, 10Ω, 20Ω, and 50Ω (from top to bottom).](image-url)
linearized LQG controller. In the narrowband limit, $\dot{x}(t)$ and $a(t)$ become purely sinusoidal and exactly in phase, which means that knowledge of both is redundant. Thus, we can conclude that for the case where Coulomb friction is included in the dynamics of the harvester, the optimal $i(t)$ is attained by imposing a static admittance in the narrowband limit.

Finally, we illustrate the effect Coulomb friction has on the average power generated in Figure 5.8. The “friction ratio” curves in this plot are defined as the ratio of performance resulting from the statistically linearized LQG controller over per-
formance resulting from the LQG controller for the case where Coulomb friction is neglected. As shown, the ratios decrease as $\zeta_a$ increases for all values of resistive losses. In addition, we see that in the limit as $\zeta_a \to 0$, the ratios approach the same value, which is independent of $R$. From this analysis, we can conclude that systems with higher resistive losses are more sensitive (in terms of performance) to the effects of Coulomb friction when the system is excited by broadband disturbances.

5.4 Non-Quadratic Electronic Loss Model

5.4.1 Loss Model of an H-bridge in CCM

Consider again the H-bridge in Figure 1.3(b). In this circuit, each MOSFET/diode pair is operated like a power-electronic switch. For this chapter, we consider the operation of this system in bi-directional continuous conduction mode (CCM). In this operating regime, the transducer’s current is controlled to take on the shape shown in Figure 5.9. As shown, current from the transducer is controlled to be a triangle wave, which is triggered by a switching clock with period $t_s$. The switching frequency $f_s = 1/t_s$ is presumed to be at least a decade above the predominant dynamics of the harvester (i.e., $f_s > 10\omega_a/2\pi$), and as such, the high-frequency component of the current is filtered out by the inductance of the transducer. Consequently, only the low-frequency switch-averaged current significantly influences the overall system response. For clarity, we refer to $i(t)$ and $v(t)$ as the switch-averaged current and

![Figure 5.9: Transducer current with the electronics operating in CCM.](image-url)
Figure 5.10: Equivalent circuit made by the H-bridge and transducer for (a) $t \in [0, Dt_s]$, and (b) $t \in [Dt_s, t_s]$.

voltage, respectively, and $\tilde{i}(t)$ and $\tilde{v}(t)$ as the actual current and voltage, respectively, with the high-frequency content included.

For the analysis of the H-bridge presented here, we assume that $\tilde{i}(t) > 0$ for $t \in [0, t_s]$. Making the assumption that $\tilde{i}(t) < 0$ for $t \in [0, t_s]$ would result in the same expression for the losses. As such, the H-bridge operated in CCM works as follows. At the leading edge of each switching cycle, MOSFETs $Q_1$ and $Q_4$ are gated on for the first $Dt_s$ seconds, which increases $\tilde{i}(t)$. The equivalent circuit made by this current path can be seen in Figure 5.10(a). Then, at time $t = Dt_s$, the conducting MOSFETs $Q_1$ and $Q_4$ are gated off, causing the free-wheeling diodes $Q_2$ and $Q_3$ to conduct. This causes $\tilde{i}(t)$ to decrease until the switching period $t_s$ is reached. The equivalent circuit made by this current path can be seen in Figure 5.10(b).

The power dissipated during this process is what is known as conductive dissipation. We assume this dissipation to be a consequence of forward conduction losses in the MOSFETs and diodes. These two devices exhibit very different dissipative characteristics, which are nonlinear. To simplify the analysis, we make the conservative assumption that energy is dissipated like a diode in series with a resistor over the entire switching period. We define the total forward conduction voltage of the diodes as $V_d$ and the total resistance of the conducting MOSFETs in series with the equivalent resistance of transducer’s coil as $R_m$. Additional losses, such as transition and gating losses in the MOSFETs, are neglected in this analysis as they will have a
negligible effect on the loss model for the levels of power considered in this chapter. However, for piezoelectric energy harvesting applications, where the levels of power are on the order of $\mu$W–mW, transition and gating losses will have a much greater effect on the efficiency of the electronics and they should be included in the loss model. See [102] for an analysis which includes these losses.

Our goal is to determine an expression for the losses in the H-bridge in terms of $i(t)$ and the non-dynamic parameters of the drive. We begin by following Kirchoff’s voltage law in the direction of positive current for the circuits in Figure 5.10. Thus, we have that the voltage across the inductor is

$$v_L(t) = \begin{cases} V_S - \tilde{v}(t) - V_d - R_m \tilde{i}(t) : & t \in [0, D t_s] \\ -V_S - v(t) - V_d - R_m i(t) : & t \in [D t_s, t_s] \end{cases} .$$

(5.49)

Next, we make the assumption that the current ripple on the inductor is small (i.e., $|i(t)| \gg \tilde{I}$ in Figure 5.9) during steady-state operation of the drive. As such, we temporarily approximate $\tilde{i}(t)$ and $\tilde{v}(t)$ as being constant over one switching period $t_s$ (i.e., $\tilde{i}(t) \approx i$ and $\tilde{v}(t) \approx v$). Then we have that the average value of the voltage across the inductor, $\bar{v}_L$, during each switching cycle is equal to 0; i.e.,

$$\bar{v}_L = \frac{1}{t_s} \int_0^{t_s} v_L(t) \, dt \approx D t_s (V_S - v - V_d - R_m i)$$

$$+ (1 - D) t_s (-V_S - v - V_d - R_m i) = 0 .$$

(5.50)

The approximation in Equation (5.50), referred to “inductor volt-second balance,” constitutes a standard approach for analyzing power electronic converters that are operating in CCM [35]. From Equation (5.50), we can then solve for the value of the steady-state duty cycle $D$ as

$$D = \frac{V_S + v + V_d + R_m i}{2V_S} .$$

(5.51)

The power dissipated in the electronics over the course of a switching cycle, $P_d(i)$, can be computed by integrating the total resistive losses and the total diode losses
over the switching period $t_s$; i.e.,

$$P_d(i) = \frac{1}{t_s} \left\{ \int_0^{t_s} R_m \tilde{i}^2(t) dt + \int_0^{t_s} V_d |\tilde{i}(t)| dt \right\}$$

(5.52)

$$= R_m \left( \bar{i}^2(t) + \frac{1}{3} \bar{I}^2 \right) + V_d|i(t)|.$$  

(5.53)

where Equation (5.53) assumes the triangular current waveform in Figure 5.9. We can derive an expression for the magnitude of the current ripple using the expression for the inductor voltage during the interval $t \in [0, Dt_s]$; i.e.,

$$\tilde{I} = \frac{D t_s}{2L} v_L(t)$$

(5.54)

$$= \frac{t_s}{2L} \frac{V_S^2}{2V_S} - \frac{(v + V_d + R_m i)^2}{2V_S}.$$  

(5.55)

We can conservatively approximate the magnitude of the current ripple by assuming that $V_S \gg v + V_d + R_m i$. Thus, we have that the upper bound for $\tilde{I}$ is

$$\tilde{I} = \frac{V_S}{4Lf_s}$$

(5.56)

and the expression for power dissipation in the electronics is conservatively approximated by

$$P_d(i) = \frac{R_m V_S^2}{48L^2 f_s^2} + R_m \bar{i}^2(t) + V_d|i(t)|.$$  

(5.57)

5.4.2 Energy Harvesting with Non-Quadratic Loss Models

We now propose a technique for expanding the theory presented in Section 5.3 to accommodate non-quadratic loss models. If the dynamics of the closed-loop system are statistically linearized, then the response distribution for the augmented system state $x(t)$ is assumed to be Gaussian. Consequently, the distribution of the current $i(t)$ is also Gaussian, with zero mean and variance $s_i = \mathcal{E}\{i^2\}$. We may evaluate the
average power dissipation by taking the expectation of Equation (5.57); i.e.,

\[
\bar{P}_d = \mathcal{E} P_d(i) = \frac{1}{\sqrt{2\pi s_i}} \int_{-\infty}^{\infty} \exp\{-i^2(t)/2s_i\} P_d(i) \, di(t) \tag{5.58}
\]

\[
= \frac{R_m V_s^2}{48L^2 f_s^2} + R_m s_i + \left(V_d \sqrt{\frac{2}{\pi}} \right) s_i^{1/2}. \tag{5.59}
\]

Next, we note that if \( \bar{P}_d \) is semi-concave; i.e., if

\[
\frac{\partial^2 \bar{P}_d}{\partial s_i^2} \leq 0, \quad \forall s_i > 0 \tag{5.60}
\]

then it follows that \( \bar{P}_d \) can be overbounded by its first-order Taylor expansion about any positive variance \( s_i^0 \); i.e.,

\[
\bar{P}_d \leq \bar{P}_d^0 + R^0 s_i \tag{5.61}
\]

where

\[
R^0 = \left. \frac{\partial \bar{P}_d}{\partial s_i} \right|_{s_i = s_i^0} \tag{5.62}
\]

\[
\bar{P}_d^0 = \left. \bar{P}_d \right|_{s_i = s_i^0} - R^0 s_i^0 \tag{5.63}
\]

with the equality holding (as well as the slope) when \( s_i = s_i^0 \). The loss model derived in Equation (5.59) is semiconcave because both terms involving \( s_i \) have exponents less than or equal to 1.

For semiconcave loss models, we may conservatively overbound the losses through a summation of a static (i.e., current-independent) loss model and a quadratic (i.e., resistive) loss model. This permits us to nest the above loss model inside the iterative algorithm used to solve Equation (5.48) as follows:

**Step 0:** Initialize the loss model by taking an arbitrary value for \( R > 0 \) and initialize \( S_0 \) by solving the linear energy harvesting problem (i.e., with \( U = V = 0 \)).

**Step 1:** Compute new values for \( V_0 \) and \( U_0 \) using \( S_0 \).
Step 2: Compute $P = \Theta(S_0)$.

Step 3: Compute $K$ from Equation (3.21) and fix $V \leftarrow V_0$ in Equation (5.40) to solve the resultant Lyapunov equation for $S$.

Step 4: Compute the variance of the current, as $s_i^0 = KSK^T$.

Step 5: For the new value of $s_i^0$, compute $R^0$ via Equation (5.62).

Step 6: Set $R \leftarrow R^0$ and $S_0 \leftarrow S$ and return to Step 1.

Convergence of the algorithm is reached when the absolute value of the change in $\bar{P}_{gen}$ between the current and previous iteration is below a certain tolerance. The value of $\bar{P}_{gen}$ at the first iteration can be calculated using Equation (3.22) while the value of $\bar{P}_{gen}$ at any subsequent iteration can be calculated by

$$\bar{P}_{gen} = -\text{tr}\{\left[\frac{1}{2}KTB^T + \frac{1}{2}BK + R^0K^TK\right]S\} - \bar{P}_d^0. \quad (5.64)$$

5.4.3 Simulation Example

To illustrate how the non-quadratic loss model affects the average power generated, we return to the SDOF energy harvester system described in Equation (5.19). The parameters in the loss model are listed in Table 5.2 and we justify these values as follows. We assume that the diodes are standard Silicon with each modeled as possessing a current-independent conduction voltage of 0.7V. The MOSFETs have

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_m$</td>
<td>2.61Ω</td>
</tr>
<tr>
<td>$V_d$</td>
<td>1.4V</td>
</tr>
<tr>
<td>$L$</td>
<td>8.93mH</td>
</tr>
<tr>
<td>$f_s$</td>
<td>33kHz</td>
</tr>
<tr>
<td>$V_S$</td>
<td>80V</td>
</tr>
</tbody>
</table>

Table 5.2: H-bridge electronic parameters for the loss model.
a resistive drain-to-source impedance, with a value of 0.1Ω when gated with a gate-
source voltage of 1.5V. For the inductor, we take the inductance to be that of winding
of the transducer’s coil (i.e., \( L = 8.93 \text{mH} \)). In addition, we know that winding
of the coil has an effective series resistance of 2.41Ω. Thus, the total conductive
loss parameters for the H-bridge are taken to be \( V_d = 2 \times 0.7 = 1.4 \text{V} \) and \( R_m =
2 \times 0.1 + 2.41 = 2.61 \Omega \). Finally, we assume that the H-bridge is connected to a
constant 80V DC voltage source and that the MOSFETs are switching at a frequency
of 33kHz. This is the switching frequency for a S16A8 servo drive from Advanced
Motion Controls [2] used in the experimental validation of the transducer in Chapter
2.

We begin by illustrating the equivalent resistive losses resulting from the non-
quadratic loss model in Equation (5.57) over a range of disturbance bandwidths.
The plot in Figure 5.11 shows the dependence of \( R \) on \( \zeta_a \) for \( \zeta_a \in [0, 1] \). From this
plot we see that \( R \) initially decreases as \( \zeta_a \) increases until it reaches a minimum value.
In this case the minimum value is \( R = 3.62 \Omega \), which occurs at \( \zeta_a = 0.164 \). Once \( R 
\) reaches this minimum value it increases linearly as \( \zeta_a \) increases. This relationship
suggests that there is a specific disturbance bandwidth where the electronics are
operating most efficiently.

![Figure 5.11: Plot of the equivalent resistive losses R versus \( \zeta_a \).](image)
Figure 5.12: Surface plots for $\bar{P}_{\text{gen}}$ as a function of $\{F_c, V_d\}$ for: (a) $\zeta_a = 0.05$, (b) $\zeta_a = 0.2$, and (c) $\zeta_a = 0.7$.

Figure 5.13: Plot of the nondimensional Coulomb friction and diode conduction voltage that results in $\bar{P}_{\text{gen}} \approx 0$, for $\zeta_a$ values of 0.05, 0.1, 0.2, 0.3, 0.5, 0.7 (from top to bottom).
Next, we illustrate how both the Coulomb friction in the electromagnetic transducer and the conduction voltage of the diodes in the H-bridge influence the performance of the energy harvesting system. We plot the $\bar{P}_{gen}$ surface for various combinations of $\{F_c, V_d\}$ and for three disturbance bandwidth values. The plots in Figures 5.12(a)–(c) illustrate the $\bar{P}_{gen}$ surface for $\zeta_a$ values of 0.05, 0.2, and 0.7, respectively. From these plots we see that as $F_c$ and $V_d$ approach 0, the value obtained for $\bar{P}_{gen}$ approaches the performance from the linear energy harvesting example with purely resistive losses. In addition, we see that $\bar{P}_{gen}$ becomes more sensitive to increases $F_c$ and $V_d$ as $\zeta_a$ increases.

If either $V_d$ or $F_c$ is sufficiently large, power generation will effectively become infeasible, and $\bar{P}_{gen} \approx 0$ at the optimum. Levels of $\{V_d, F_c\}$ resulting in this condition are illustrated in Figure 5.13. In this plot, both $F_c$ and $V_d$ are nondimensionalized by using the standard deviation of the disturbance acceleration $\sigma_a$. Furthermore, each of the lines in the plot correspond to a different disturbance bandwidths $\zeta_a$. If the values of $F_c$ and $V_d$ (for a given $\sigma_a$ and $\zeta_a$) result in a point at or above its corresponding line, then $\bar{P}_{gen} \approx 0$. Similarly, if the values of $F_c$ and $V_d$ (for a given $\sigma_a$ and $\zeta_a$) result in a point below its corresponding line, then $\bar{P}_{gen} > 0$. From the plot, we see that the levels of $V_d$ and $F_c$ required for significant power generation are heavily coupled, and thus cannot be considered in isolation.

### 5.5 Summary

This chapter demonstrated the energy harvesting capability of the actively controlled electromagnetic transducer characterized in Chapter 2. We considered a vibratory energy harvesting system consisting of a RTHT experiment in which the transducer is embedded between the ground and moving mass of a simulated SDOF oscillator. The SDOF oscillator is excited at its base by a sinusoidal disturbance of known amplitude and frequency. In addition, the nonlinear Coulomb friction force present in the device...
is included in the dynamics of the system. In order to predict the energy harvested by the system, we derived an analytical expression for the average power generated and use this expression to determine the optimal resistive load and optimal average power generated. Both the analytical model using the experimentally fit parameters and the experimental system showed similar results for the average power generated over the range of resistive loads, and for three levels of mechanical damping in the SDOF oscillator. Furthermore, we were able to accurately predict the optimal load resistances and optimal power generated values in the experimental system.

For the case in which the system is excited by a stochastic disturbance, we extended the results presented in Chapter 3 to include the effects of the nonlinear Coulomb friction force in the control design. The nonlinearities arising in the dynamics of the vibrating system can be statistically linearized if we assume that its response can be approximated by a Gaussian distribution. As such, the second half of this chapter illustrated how to account for the Coulomb friction present in the system, and maximized the average power generation by simultaneously solving two coupled nonlinear algebraic equations. These equations were derived from first principles and an iterative algorithm is proposed to solve for the statistically linearized covariance matrix as well as the optimal feedback gain matrix. For the statistically linearized energy harvesting system with purely resistive losses, the LQG controller generates the same amount of average power as the SA controller in the narrowband limit. This result was observed for the linear energy harvesting system in Chapter 3.

Additionally, this chapter presents a nonlinear loss model for the conductive power dissipated in an H-bridge operated in CCM. The proposed statistical linearization and optimal control algorithm is augmented to include the non-quadratic loss model. However, the algorithm is only guaranteed to converge if the loss model exhibits the semiconcave property, and this property may not hold for all systems. For an H-bridge operating in CCM, we illustrate that the equivalent resistance re-
sulting from the non-quadratic loss model reaches a minimum value at a nontrivial disturbance bandwidth. In addition, for a given disturbance bandwidth, we show the influence of varying levels of Coulomb friction and diode conduction voltage on the performance of the energy harvester. Finally, we illustrate that there are critical levels of Coulomb friction and diode conduction voltage beyond which power generation becomes infeasible, and that the critical values of these two quantities are strongly coupled.
6

Sub-Optimal Nonlinear Controllers for Power-Flow-Constrained Vibratory Energy Harvesters

6.1 Background

This chapter focuses on the derivation of nonlinear controllers for vibratory energy harvesters that are operated using single-directional converters. The main advantage of single-directional converters is that they only require a single MOSFET to be gated during each switching cycle, which reduces the amount of parasitic power consumed by the device. However, these types of converters impose a constraint on the directionality of power-flow, which results in a purely dissipative energy harvester. This motivates the problem of how to maximize the average power generated by a system subject to a constraint on power-flow.

Such systems are very similar to semi-active vibration control problems, in which a damper’s viscosity is varied according to a feedback control algorithm in order to suppress vibratory responses more favorably than any constant-damping system. Margolis [80] and Karnopp [58] first examined control algorithms for semi-active vehi-
cle suspensions. Since then, a number of researchers have investigated the advantages of using active and semi-active control strategies for automotive applications [51]. In addition, semi-active control techniques for magnetorheological (MR) dampers, electrorheological (ER) dampers, and other types of controllable dampers have been widely investigated in the literature to mitigate the effects of seismic excitations on multi-story structures [52, 121]. The most popular semi-active control synthesis for structural control applications is “clipped-optimal” control, which emulates a fully active control law whenever the desired force is instantaneously dissipative [31]. In a study by Ying et al. [133], the optimal semi-active control law for ER and MR dampers is computed numerically from the stationary solution to the stochastic Bellman equation.

A technique for semi-active feedback control which is guaranteed to outperform constant damping was first proposed for a single-device semi-active system by Tseng and Hedrick [123]. Scruggs et al. in [108] extended those results by developing a stochastic performance-guaranteed theory for systems with an arbitrary number of devices and semi-active or regenerative constraints. That paper illustrated the improvement in performance that can be achieved with a performance-guaranteed controller as compared to the clipped-optimal and passive controllers over a wide range of design objectives. Furthermore, it was also shown in [108] that the clipped-optimal controller actually performs worse than the optimal static controller when the suppression of accelerations becomes more important than the suppression of displacements.

The main contribution of this chapter is the development of a power-flow-constrained feedback controller which is analytically guaranteed to outperform the optimal static admittance controller in stationary stochastic response. The technique is analogous to those proposed in [108, 123] and discussed above, but the control objective is different. Those studies sought to minimize a positive-definite quadratic performance
measure (such as mean-square accelerations or strains), while the present study seeks to maximize the average power generation. Furthermore, this study restricts the operating regime of the single-directional power electronic converter to discontinuous conduction in the formulation of the power-flow constraint, resulting in a control problem that is slightly different from semi-active problems. In addition to the synthesis of the performance-guaranteed controller, this chapter also presents the energy harvesting equivalent of the clipped-optimal semi-active control method. The performances of the controllers are compared for both a single-degree-of-freedom (SDOF) resonant oscillator with electromagnetic coupling, as well as a multi-mode piezoelectric bimorph cantilever beam. The work presented in this chapter expands on results obtained by the author in [18].

6.2 The Constrained Vibratory Energy Harvesting Problem

6.2.1 Energy Harvesting Objective

Recall that the augmented harvester and disturbance state space representation is

$$\dot{x}(t) = Ax(t) + Bi(t) + Gw(t) \tag{6.1a}$$
$$v(t) = B^T x(t) \tag{6.1b}$$

where we make the same assumptions about \(\{A, B, G\}\) as in Chapter 3. In addition, we assume that the energy harvesting performance can be expressed as the expectation of the power harvested by the transducer minus the expectation of resistive transmission losses in the electronics; i.e.,

$$\bar{P}_{gen} = -E \left\{ \begin{bmatrix} x \\ i \end{bmatrix}^T \begin{bmatrix} 0 & B \\ \frac{1}{2} B^T & \frac{1}{2} R \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix} \right\} \tag{6.2}$$

With the state space representation in Equation (6.1) and the energy harvesting performance in Equation (6.2), we recall from Theorem 1 that the unconstrained
linear-quadratic-Gaussian (LQG) current relationship is $i(t) = Kx(t)$ where

$$K = -\frac{1}{R} B^T (P + \frac{1}{2}I) . \quad (6.3)$$

The matrix $P = P^T < 0$ is the solution to the following nonstandard Riccati equation

$$A^T P + PA - \frac{1}{R} \left( P + \frac{1}{2}I \right) B B^T \left( P + \frac{1}{2}I \right) = 0 . \quad (6.4)$$

If the entire system state is available for feedback, the optimal LQG performance is

$$\bar{P}_{LQG}^{gen} = -G^T P G . \quad (6.5)$$

In addition, we recall that when the electronics implement the static admittance (SA) controller $i(t) = -Y_s v(t)$, the closed-loop system dynamics are

$$\dot{x}(t) = \left[ A - Y_s B B^T \right] x(t) + G w(t) . \quad (6.6)$$

As a direct result of Theorem 3, the energy harvesting performance resulting from the SA controller is

$$P_{SA}^{gen} = -G^T W G \quad (6.7)$$

where $W = W^T < 0$ is the solution to the Lyapunov equation

$$\left[ A - Y_s B B^T \right]^T W + W \left[ A - Y_s B B^T \right] + B \left( -Y_s + Y_s^2 R \right) B^T = 0 . \quad (6.8)$$

Building upon the derivations of the SA and LQG controllers, the main focus of this chapter will be on developing a time-varying admittance to harvest energy. Specifically, we assume that $i(t)$ can be related to $v(t)$ through a controllable, time-varying admittance $Y(t)$; i.e.,

$$i(t) = -Y(t)v(t) \quad (6.9)$$

where the power-flow directionality constraint requires $Y(t) \geq 0$. Subject to this constraint, $Y(t)$ is modulated as a function of feedback measurements. As such, the techniques presented in this chapter are fundamentally nonlinear.
6.2.2 Power-Flow Constraint

In general, the power electronic hardware necessary to achieve the causal limit $P_{gen} = \bar{P}_{gen}^{LQG}$ must be capable of extracting as well as injecting power into the system. Thus, a bi-directional converter such as an H-bridge, which places no restrictions on power directionality, must be used to implement it. When the capabilities of the power electronic converter are restricted, the causal limit is generally unreachable.

As discussed in the background, this chapter assumes the converter to be single-directional. This implies that the controlled current $i(t)$ must be related to $v(t)$ only through a controllable time-varying admittance $Y(t)$, as in Equation (6.9), with restrictions on the range of feasible $Y(t)$ values. More specifically, we investigate the optimal way to implement the relationship in Equation (6.9) using the single-directional buck-boost converter in Figure 6.1, which is used to regulate $Y(t)$. This type of power electronic converter has been demonstrated for piezoelectric energy harvesting applications in [64, 69].

The buck-boost converter in Figure 6.1(a) is controlled via high-frequency pulse-width modulated (PWM) switching of a single MOSFET $Q_1$. In this paper, we assume the converter is operated in discontinuous conduction mode (DCM), described as follows. Prior to the leading edge of each switching cycle, inductor $L$ is demagnetized (i.e., its current is zero). Upon the leading edge, $Q_1$ is gated on, which connects the inductor to the smoothing capacitor $C_R$, inducing the current in $L$ to rise at a rate $\frac{d}{dt}i_L(t) \approx V_R/L$. After a fraction $D$ of the total switching period, $Q_1$ is gated off, causing the inductor current to be routed to the storage bus capacitor $C_S$ and the battery. This causes the inductor current to drop, at a rate $\frac{d}{dt}i_L(t) \approx -V_S/L$. In DCM, the inductor current drops to zero before the end of the switching cycle, and remains so until $Q_1$ is gated on again at the leading edge of the next switching cycle. The inductor current appears as a periodic train of triangular pulses, as illustrated in
Figure 6.1: (a) Energy harvesting transducer interfaced with a full-bridge rectifier and a buck-boost converter. (b) Inductor current $i_L(t)$ with the buck-boost converter operating in DCM.

Figure 6.1(b). During each switching cycle, the total charge transferral is governed by the area underneath the inductor current waveform, which is in turn governed by $D$.

The advantages of operating a buck-boost converter in DCM are two-fold. First, from the point of view of the rectifier voltage $V_R$, the input impedance of the converter looks resistive at frequencies well below the switching frequency of the converter [35]. Second, the input impedance is approximately decoupled from the behavior of storage voltage $V_S$. Indeed, we can find an approximate relationship between the duty cycle and input admittance can be found depending only on inductor $L$ and switching frequency $f_{sw}$; i.e.,

$$D = \sqrt{2YLf_{sw}}$$  \hspace{1cm} (6.10)

As shown in Figure 6.1(a), once a desired admittance $Y$ has been determined,
its signal is passed through a square root block and a gain block $K$ to synthesize $D$. Next, this signal is sent to a comparator along with a sawtooth function, which has a period of $1/f_w$. The output of the comparator is a voltage signal that is then used to gate $Q_1$ at the appropriate duty cycle. In [64, 90], similar approaches were taken in which a static admittance was realized by connecting a passive circuit containing two parallel resistors to the non-inverting input of the comparator. With the configurations proposed in those studies, it is shown that the duty cycle of the converter is approximately equal to the ratio of the two resistors. It would be straightforward to augment that circuit such that it would realize the time-varying admittance $Y(t)$.

The single-directionality of the buck-boost converter in Figure 6.1(a) restricts the flow of power to extraction; i.e., $i(t)v(t) \leq 0$, $\forall t$. Furthermore, an additional constraint must be placed on the circuit such that the operating regime of the buck-boost converter is limited to DCM. Both of these conditions can be enforced by setting $Y(t) \in [0, Y_{max}]$ where $Y_{max}$ corresponds to the duty at which the converter transitions from discontinuous to continuous conduction mode. These two restrictions result in the following power-flow constraint

$$i(t)v(t) + i^2(t)/Y_{max} \leq 0, \quad \forall t.$$  \hspace{1cm} (6.11)

Thus, the constrained energy harvesting problem is to design $Y(t)$ which maximizes the energy harvesting performance in Equation (6.2) subject to the power-flow constraint in Equation (6.11).

6.3 Nonlinear Power-Flow-Constrained Control

This section extends the results presented for the SA controller, in which $Y(t)$ is adjusted in response to feedback measurements of the full system state, to further increase $\bar{P}_{gen}$. Because it is more difficult to find analytical expressions for $\bar{P}_{gen}$ (such as in Equation (6.7)) for time-varying $Y(t)$, we relax the control design objective
to the design of controllers for which sufficient conditions exist to guarantee that \( \bar{P}_{gen} \geq \bar{P}_{gen}^{SA} \). In other words, the feedback controller is guaranteed to generate at least as much power as the optimal SA controller.

6.3.1 Performance-Guaranteed Control

We begin by presenting a theorem that introduces the performance-guaranteed feedback controller. A proof of this theorem can be found in Appendix A.5.

**Theorem 5.** For the linear system in Equation (3.13), let \( Y_s \) be a feasible SA controller. Let \( x(t) \mapsto i(t) \) be any causal mapping, such that the closed-loop system is globally bounded-input bounded-state stable. Then, in stationary response, the energy harvesting performance is

\[
\bar{P}_{gen} = -G^T WG + R \mathbb{E} \left\{ (Fx + Y_s v)^2 - (Fx - i)^2 \right\}
\]

(6.12)

where \( W = W^T < 0 \) is the solution to Equation (6.8) and

\[
F = -\frac{1}{R} B^T (W + \frac{1}{2} I) .
\]

(6.13)

In the above theorem, we see that the expression for \( \bar{P}_{gen} \) is a summation of two terms. The first of these is actually the value of \( \bar{P}_{gen} \) when \( Y(t) = Y_s \). The second term in Equation (6.12) (i.e., the expectation) does not in general have a closed-form. In the formulation of the PG feedback controller, \( Y(t) \) is chosen such that the term in the expectation is positive at every time, thus ensuring a positive expectation and enhanced power generation.

**Theorem 6.** For the system in Equation (6.6), let \( Y_s^\star \) be the optimal SA, and let \( \bar{P}_{gen}^{SA^\star} \) be its performance. Also, let \( W \) and \( F \) be determined by Equations (6.8) and (6.13) with \( Y_s = Y_s^\star \). Then the following control law

\[
Y(t) = \arg \min_{\hat{Y} \in [0, Y_{max}]} \left\{ R(Fx(t) + \hat{Y} v(t))^2 \right\}
\]

(6.14)
adheres to the performance-guaranteed (PG) power generation inequality

\[ \bar{P}_{gen}^{PG} \geq \bar{P}_{gen}^{SA^*}. \]  

(6.15)

Proof. Substitution of Equation (6.9) into Equation (6.12) gives

\[ \bar{P}_{gen}^{PG} = \bar{P}_{gen}^{SA^*} + R \mathcal{E} \left\{ (Fx + Ysv)^2 - (Fx + Yv)^2 \right\}. \]  

(6.16)

Thus, the control law in Equation (6.14) gives performance

\[ \bar{P}_{gen}^{PG} = \bar{P}_{gen}^{SA^*} + R \mathcal{E} \left\{ (Fx + Ysv)^2 - \min_{Y \in [0,Y_{max}]} \left( Fx + Yv \right)^2 \right\}. \]  

(6.17)

The term in the brackets is nonnegative at every time, because \( Y_s \in [0,Y_{max}] \). Thus, its expectation is nonnegative, completing the proof.

The control law in Equation (6.14) may also be expressed directly in terms \( i(t) \).

We begin by defining the unconstrained linear feedback controller \( i_u(t) \) as

\[ i_u(t) = Fx(t). \]  

(6.18)

Then the time-varying admittance is equivalent to the control current \( i(t) \) as

\[ i(t) = \arg \min_{i} \left\{ R \left( \tilde{i}(t) - i_u(t) \right)^2 \right\} \]  

(6.19)

where the feasibility of the \( \tilde{i}(t) \) is determined by the power-flow constraint in Equation (6.11). Thus, \( i(t) \) tracks the unconstrained feedback controller \( i_u(t) \) if it is feasible, and if not \( i_u(t) \) is clipped to the feasible region. For the case in which \( R \) is a scalar, the best way to resolve the clipping action is by formulating \( i(t) \) as a saturation function of \( i_u(t) \); i.e.,

\[ i(t) = \begin{cases} i_u(t) & : i_u(t) v(t) + i_u^2(t)/Y_{max} \leq 0 \\ 0 & : i_u(t) v(t) + i_u^2(t)/Y_{max} > 0 \text{ and } i_u(t) v(t) > 0 \\ -Y_{max} v(t) & : \text{otherwise} \end{cases} \]  

(6.20)

(6.21)
For more complicated problems involving multiple transducers, the clipping action that instantaneously minimizes the tracking error in Equation (6.19) may not be a saturation function. Interested readers should see [108], which presents a more detailed clipping procedure for a system with an arbitrary number of transducers and semi-active or regenerative constraints.

6.3.2 Clipped-Optimal Control

It turns out that the control formulation in Equation (6.20) is identical in form to that of clipped-optimal (CO) control. Replacing the PG gain matrix $F$ with the LQG gain matrix $K$ in Equation (6.18) and updating $i_u(t)$ in Equation (6.20) results in the CO controller. The performance of an energy harvester implementing the CO controller can be expressed as

$$\bar{P}_{gen}^{CO} = -G^TPG - R\mathcal{E}\left\{ (i - i_u)^2 \right\}$$

(6.22)

where $i_u(t) = Kx(t)$. The first term in the summation above is equal to the performance of the optimal LQG controller while the second term is the depreciation in performance due to the fact that $i(t)$ is constrained to a feasible region. The CO controller attempts to minimize the expectation in the second term at every time instant, resulting in the same controller as Equation (6.20) but with $i_u(t)$ formulated as above, as based on the LQG gain matrix.

In other words, the CO controller attempts to get as close as possible to realizing the fully active, unconstrained optimal feedback law. However, there is in general no guarantee for how close it actually gets to doing this; i.e., there is no analytical bound on the second term in Equation (6.22).
6.4 Simulation Examples

6.4.1 Electromagnetic Example

Consider the energy harvester in Figure 6.2(a), consisting of an ideal electromagnetic transducer embedded within a single-degree-of-freedom (SDOF) resonant oscillator. We assume the SDOF oscillator has mass $m$, damping $c$, and stiffness $k$ and the transducer has back-emf motor constant $K_t$. As such, the harvester state space is identical to the representation in Equation (5.17) for the case in which we neglect the Coulomb friction (i.e., $F_c = 0$). For the example considered in this chapter, the parameter values for the SDOF oscillator and electromagnetic transducer are given in Table 5.1. In addition, we assume that the disturbance state space is identical to the representation in Equation (5.18) where $\omega_a = \sqrt{k/m}$ and $\sigma_a = 0.18\text{m/s}^2$. The disturbance parameter $\zeta_a$ and the transmission dissipation $R$ will be treated as adjustable parameters.

We assume that the maximum admittance that can be imposed by the electronics (i.e., $Y^{max}$) is $0.05\Omega^{-1}$. This assumption is justified as follows. For an ideal system, $Y^{max}$ must be less than or equal to $1/R$. As such, setting $Y^{max} = 0.05\Omega^{-1}$ is a conservative approximation of $Y^{max}$ for the values of $R$ investigated in this example (except for the limiting case in which $R = 20\Omega$). In a physical system, $Y^{max}$ can easily be calculated using the known values of the power electronic converter (i.e.,...
inductance, switching frequency, etc.). However, designing an actual buck-boost converter in order to calculate $Y_{\text{max}}$ is beyond the scope of this dissertation.

We begin by determining an expression for the PG feedback controller. It turns out that the solution to the Lyapunov equation in Equation (6.8) has a special form for this example; i.e.,

$$W = \begin{bmatrix} W_{22} & 0 & W_{13} & 0 \\ 0 & W_{22} & 0 & W_{24} \\ W_{13} & 0 & W_{33} & 0 \\ 0 & W_{24} & 0 & W_{44} \end{bmatrix}$$  \hspace{1cm} (6.23)

As such, we can explicitly solve for $\{W_{22}, W_{33}, W_{44}, W_{13}, W_{24}\}$ analytically in terms of the harvester and disturbance parameters. For the state vector partitioned analogously to $W$, the corresponding PG gain matrix is

$$F = \begin{bmatrix} 0 & -\frac{K_i}{\sqrt{mR}} (W_{22} + \frac{1}{2}) & 0 & -\frac{K_i}{\sqrt{mR}} W_{24} \end{bmatrix}$$  \hspace{1cm} (6.24)

where

$$W_{22} = \frac{K_i^2 \left( -Y_s^* + Y_{s}^* R \right)}{2 \left( c + K_i Y_s^* \right)} ,$$  \hspace{1cm} (6.25a)

$$W_{24} = \frac{K_i^2 m_s \sqrt{m} \left( -Y_s^* + Y_{s}^* R \right)}{2 \left( c + K_i Y_s^* \right) \left( c + K_i^2 Y_s^* + 2m \zeta a \omega_a \right)} .$$  \hspace{1cm} (6.25b)

Given the expression for $F$ in Equation (6.24) and the fact that $v(t) = K_i \dot{x}(t)$, we can express the PG control law as

$$i(t) = \text{sat}_{i e + i^2/Y_{\text{max}} \leq 0} \left\{ F_v v(t) + F_a a(t) \right\}$$  \hspace{1cm} (6.26)

where $F_v$ and $F_a$ are nonzero components of the PG gain matrix $F$. Recognizing that $i(t) = -Y(t)v(t)$, this implies that the equation to determine the time-varying admittance for this controller directly is

$$Y(t) = -\text{sat}_{[0,Y_{\text{max}}]} \left\{ \begin{bmatrix} F_v + F_a \frac{a(t)}{v(t)} \end{bmatrix} \right\}$$  \hspace{1cm} (6.27)
Figure 6.3: Example simulation using the PG controller for $R = 5\Omega$ and $\zeta_a = 0.5$: (a) $Y(t)$ versus time, (b) $P_S(t)$ versus time, and (c) $i(t)$ versus $v(t)$.

The argument in the brackets consists of a constant term, plus a term that varies with the ratio $a(t)/v(t)$. It is this variable term that is responsible for the improvement in performance over the optimal SA controller. Furthermore, we note that an experimental implementation of this controller only requires feedback measurements of the transducer voltage and disturbance acceleration. A simulation example of the PG controller is provided over a time span of 60s in Figure 6.3. For this example, we illustrate the equivalent time-varying admittance resulting from this controller in Figure 6.3(a) and the power delivered to storage in Figure 6.3(b). The power-flow constraint in Equation (6.11) is successfully enforced for this example, as shown in Figure 6.3(c).

Next, we determine an expression for the CO controller. It turns out that the solution to the Riccati equation in Equation (6.4) has a special form for the augmented state space defined in this example. Thus, we can utilize the decoupling properties of the Riccati equation for energy harvesters adhering to Equation (6.1), which was presented in Chapter 3, in order to determine the CO controller. The admittance
resulting from the CO control law can be written as

\[ Y(t) = -\operatorname{sat}_{[0,Y_{\text{max}}]} \left\{ K_v + K_a \frac{a(t)}{v(t)} \right\} \]

(6.28)

where \( K_v \) and \( K_a \) are the nonzero components of the LQG gain matrix \( K \).

To compare the performances of the PG controller and the CO controller, we plot the ratio of the performance of the SA controller divided by the performances of the PG and CO controllers. This ratio represents the marginal improvement afforded by these controllers over the optimal SA. For reference, we also include the ratio of the performance of the SA controller divided by the performance of the LQG controller. These ratios are plotted over a range of disturbance bandwidths (i.e., \( \zeta_a \in [0,1] \)) and for several transmission dissipation values. Because the performances of the PG and CO controllers do not have closed-form solutions, we simulate the closed-loop system for 3000s at a sample rate of 100Hz in order to calculate an approximate stationary solution for \( \bar{P}_{\text{gen}} \).

In Figure 6.4, we see that the CO controller performs slightly better than the PG controller for all values of \( R \). From this result, we have shown that CO controllers perform well in terms of the energy harvesting performance measure for single-mode systems. For larger values of \( R \), we notice that the PG and CO controllers converge to the same value, which can be seen in Figure 6.4(d). In the narrowband limit (i.e., as \( \zeta_a \to 0 \)) the performance ratios of all of the controllers approach unity. As pointed out in Chapter 3, this is due to the fact that for the system under consideration, the velocity and disturbance acceleration gains are the only gains required for the feedback control laws. As such, \( \dot{x} \) and \( a(t) \) become purely sinusoidal and exactly in phase in the narrowband limit, which means that knowledge of both is redundant. Thus, we have verified that for this simplified case, the optimal \( \bar{P}_{\text{gen}} \) is attained by imposing a static admittance.
Figure 6.4: Ratio of $\bar{P}_{SA}^{gen}/\bar{P}_{gen}$ for different controllers with $\omega_a = \sqrt{k/m}$, for: (a) $R = 1\Omega$, (b) $R = 5\Omega$, (c) $R = 10\Omega$, and (d) $R = 20\Omega$.

Figure 6.5: Comparison of $\bar{P}_{gen}$ for different controllers with varying $Y_{\text{max}}$, for $\zeta_a = 0.2$ and $R = 5\Omega$. 

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Finally, we plot the performance of the various controllers for varying $Y_{\text{max}}$ values in Figure 6.5. In this plot, we set $\zeta_a = 0.2$ and $R = 5\Omega$ and compare the average power generated by the LQG, SA, CO, and PG controllers. As $Y_{\text{max}} \rightarrow 0$, we see that the performance of the SA, PG, and CO controllers approach zero. This limit corresponds to the case where the electronics can only impose an open circuit condition on the transducer, which results in no average power generated by the system. For increasing values of $Y_{\text{max}}$, the CO controller slightly outperforms the PG controller and both the CO and PG controllers outperform the SA controller. The most important result of this analysis is that the performance of the SA, CO, and PG controllers approach constant values as $Y_{\text{max}}$ increases. This implies that there is a critical value of $Y_{\text{max}}$ that must be achieved by the electronics in order to fully realize the potential performance offered by these control laws.

6.4.2 Piezoelectric Example

Next, we consider the bimorph piezoelectric cantilever beam in Figure 6.2(b). Using standard Rayleigh-Ritz techniques to arrive at a finite-dimensional beam model, and imposing classical mechanical damping, the harvester state vector can be partitioned as

$$ x_h(t) = [q_1(t) \quad \dot{q}_1(t) \quad \cdots \quad q_N(t) \quad \dot{q}_N(t) \quad p(t)]^T $$

(6.29)

where \{\{q_k(t), \dot{q}_k(t)\}\} are generalized mechanical position and velocity coordinates of vibratory mode $k$, and $p(t)$ is normalized piezoelectric voltage. Given these coordinates, a realization exists for which

$$ A_h = \begin{bmatrix} \Omega & \Theta \\ -\Theta^T & -1/\tau \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \quad G_h = \begin{bmatrix} N \\ 0 \end{bmatrix} $$

and where the further partitionings are made in modal form; i.e.,

$$ \Omega = \text{diag} \begin{bmatrix} 0 & \omega_k \\ -\omega_k & -2\zeta_k \omega_k \end{bmatrix}, \quad \Theta = \text{col} \begin{bmatrix} 0 \\ [\theta_k] \end{bmatrix}, \quad N = \text{col} \begin{bmatrix} 0 \\ [\eta_k] \end{bmatrix} $$
Table 6.1: Parameter values for the piezoelectric energy harvester.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>241 rad/s</td>
<td>$\theta_3$</td>
<td>375 s$^{-1}$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1510 rad/s</td>
<td>$\eta_1$</td>
<td>$-0.0820 \sqrt{\text{kg}}$</td>
</tr>
<tr>
<td>$\omega_3$</td>
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<td>$\eta_2$</td>
<td>$-0.0454 \sqrt{\text{kg}}$</td>
</tr>
<tr>
<td>$\zeta_1$</td>
<td>0.010</td>
<td>$\eta_3$</td>
<td>$-0.0267 \sqrt{\text{kg}}$</td>
</tr>
<tr>
<td>$\zeta_2$</td>
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<td>$\beta$</td>
<td>1770 $\sqrt{\Omega/\text{s}}$</td>
</tr>
<tr>
<td>$\zeta_3$</td>
<td>0.121</td>
<td>$\sigma_a$</td>
<td>9.81 m/s</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>65.8 s$^{-1}$</td>
<td>$Y_{\text{max}}$</td>
<td>0.01 $\Omega^{-1}$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$-228$ s$^{-1}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the example considered in this paper, the parameters for the piezoelectric bimorph cantilever beam are given in Table 6.1. These correspond to the transducer studied in [113] with the exception of $\tau$, which was assumed to be infinite in that study and is given a finite value here to reflect finite dielectric leakage of the piezoelectric transducer.

We assume that the disturbance state space matrices $A_a$, $B_a$, and $C_a$ are the same as the previous example and the value for $\sigma_a$ is listed in Table 6.1. In addition, the parameters $\zeta_a$, $\omega_a$, and $R$ are treated as adjustable variables. We set $Y_{\text{max}} = 0.01 \Omega^{-1}$, which corresponds to the inverse of the maximum $R$ value investigated in this example. Unlike the previous example, the augmented state space for this example results in LQG, CO, and PG control laws that do not decouple and the entire augmented state vector $x(t)$ is required for feedback. As such, the gain matrices $K$ and $F$ must be computed numerically and do not have any entries that are identically zero.

For this example, we begin by plotting the ratio of the performance of the SA controller divided by the performances of the LQG, CO, and PG controllers. Since the performances of the PG and CO controllers do not have closed-form solutions, we simulate the closed-loop system for 40s at a sample of rate of 10kHz in order
to calculate an approximate stationary solution for $\bar{P}_{gen}$. The plots in Figure 6.6 illustrate these ratios over a range of disturbance bandwidths and for $\omega_a = \omega_1$ (i.e., the passband of the disturbance filter is centered at the first natural frequency of the beam). From these plots we see that the PG controller outperforms the CO controller for all values of $R$. Furthermore, in the narrowband limit as $\zeta_a \to 0$ the ratios are no longer equal to unity because $\bar{P}_{gen}$ is optimized by a passive admittance network consisting of a resistor and an inductor in series.

Next, we plot the same performance ratios for the case where the passband of the disturbance filter is tuned to the second natural frequency of the beam; i.e., $\omega_a = \omega_2$. Plots of the performance ratios for this case with increasing values of $R$ can be seen.
Figure 6.7: Ratio of $\bar{P}_{SA}^{gen}/\bar{P}_{gen}$ for different controllers with $\omega_a = \omega_2$, for: (a) $R = 5\Omega$, (b) $R = 10\Omega$, (c) $R = 50\Omega$, and (d) $R = 100\Omega$.

In Figure 6.7, again, we notice that the PG controller significantly outperforms the CO controller for all values of $R$ and that the ratios are not equal to unity in the narrowband limit. We can conclude that efficient harvesting energy from systems excited at higher modes requires some type of feedback control.

In order to further compare the performance of the CO and PG controllers, the plot in Figure 6.8 illustrates the performance ratios over a range of disturbance passband values; i.e., $\omega_a \in [\omega_1, \omega_3]$. For these plots, we fix $R = 10\Omega$ and plot the
Figure 6.8: Ratio of $\bar{P}_{SA}/\bar{P}_{gen}$ for different controllers with $R = 10\Omega$, for: (a) $\zeta_a = 0.05$, (b) $\zeta_a = 0.1$, (c) $\zeta_a = 0.5$, and (d) $\zeta_a = 1$.

Figure 6.9: Comparison of $\bar{P}_{gen}$ for different controllers with varying $Y_{max}$, for $\zeta_a = 0.2$ and $R = 5\Omega$: (a) $\omega_a = \omega_1$, and (b) $\omega_a = \omega_2$. 
performance ratios for $\zeta_a$ values of 0.05, 0.1, 0.5, and 1. It is important to point out in Figure 6.8(a) that there is a range of $\omega_a$ values for which $\bar{P}_{gen}^{SA}/\bar{P}_{gen}^{CO} > 1$. In this range, the CO controller actually performs worse than the optimal SA controller. Furthermore, we see that the performance of the PG controller is much better than the performance of the CO controller in range $\omega_a \in [\omega_2, \omega_3]$, for all values of $\zeta_a$. We can therefore conclude that the PG controller is able to harvest more energy from multi-mode systems, especially when the passband of the disturbance is centered at the higher natural frequencies of the system.

Finally, we plot the performance of the various controllers for varying $Y_{max}$ values in Figure 6.9. In this plot, we set $\zeta_a = 0.2$ and $R = 5\Omega$ and compare the average power generated by the LQG, SA, CO, and PG controllers. The plot in Figure 6.9(a) is for $\omega_a = \omega_1$ while the plot in Figure 6.9(b) is for $\omega_a = \omega_2$. As $Y_{max} \to 0$, we see that the performance of the SA, PG, and CO controllers approach zero for both plots. However, unlike the previous example, we see that the CO controller performs slightly better than the PG controller for increasing values of $Y_{max}$. Again, we see that the performance of the SA, CO, and PG controllers approach constant values as $Y_{max}$ increases.

6.5 Summary

In the application of power-flow-constrained feedback control laws to maximize power generation, the increased complexity of such technology must be justified by some assurance that they can outperform simpler passive techniques. Toward this end, this chapter presented a PG control design technique which results in consistent performance trends under the power generation performance measure. The main result of this analysis is that PG controllers can be designed to outperform the optimal performance achievable by the SA controller. However, there is in general no analytical expression for the margin of improvement. By comparison, CO controllers
do not possess theoretical bounds on performance and, in some cases, can perform worse than the optimal SA controller.

The controllers presented in this chapter were simulated for both an electromagnetic energy harvester as well as a piezoelectric energy harvester. For the electromagnetic example, the CO controller generated slightly more average power than the PG controller. Thus, we can conclude that for single-mode systems, using a CO controller is advantageous in terms of the energy harvesting performance. However, this result is in contrast to the piezoelectric example, which consisted of a cantilever beam with three distinct modes. For multi-modal energy harvesting systems, PG controllers are much more effective at generating power than CO controllers, especially for scenarios in which the disturbance passband is centered at frequencies that are greater than the first natural frequency of the harvester. Finally, this chapter showed that the performances of the SA, CO, and PG controllers approach constant values as the upper bound on the maximum feasible admittance of the electronics increases. This result is important because of its implications on the design of power electronic converters to realize a given controller.
7.1 Background

In this chapter, we focus on the derivation of numerical controllers for vibratory energy harvesters with power-flow constraints. As pointed out in Chapter 5, there are numerous scenarios in which accounting for the mechanical and electrical nonlinearities in the harvester is paramount to maximizing the potential power generation. In addition, Chapter 6 addressed the benefits of controlling vibratory energy harvesters with a single-directional DC/DC converter. However, the approaches taken in those chapters rely on statistical approximations, and thus result in control laws that are inherently sub-optimal. Unlike those approaches, the work presented in this chapter formulates optimal control laws for systems with nonlinearities and power-flow constraints using stochastic Hamilton-Jacobi theory.

The necessary condition for optimality of any controllable dynamic system can be obtained through the Hamilton-Jacobi equation (HJE). The derivation of this
optimality principle was first presented by Bellman [10] and is commonly referred to as the dynamic-programming approach to solving optimal control problems. Determining a solution to the HJE is often very challenging, because it is a nonlinear partial differential equation. In general, except for the classical linear-quadratic case, closed-form analytic solutions to the HJE do not exist. However, it is possible to obtain a numerical solution to the HJE using time stepping schemes coupled with spatial discretizations. For example, finite element and finite volume methods have been effectively applied to solve the HJE [12, 125], but these methods require high amounts of computational power to converge to solutions making them impractical for problems with multiple dimensions.

Pseudospectral (PS) methods have recently been proposed [13, 17, 40] to overcome the numerical challenges associated with finite element and finite volume methods and can efficiently solve partial differential equations. The PS method relies on a discretization scheme in which collocation points are chosen based on accurate quadrature rules whose basis functions are typically Chebyshev or Lagrange polynomials. Unlike finite element and finite volume methods, the polynomials used in the PS method are defined over the entire spatial domain, instead of over subdomains or elements. Furthermore, it has been shown that the PS method can achieve similar accuracy with fewer grid points and less computational memory, as compared to finite element and finite difference methods [13].

Recently, several researchers have applied various PS methods to solve a wide range of optimal control problems [32, 33, 37, 38, 45, 114, 115]. The studies by Elnagar et al. [32, 33] are some of the first to use a PS discretization scheme to solve these types of problems. Those studies obtained the optimal controller for the minimum time problem with nonlinear dynamics using the proposed PS method and compared the optimal controller to other numerical solutions. A subsequent study by Fahroo and Ross [38] applied the PS method to infinite-horizon optimal
control problems (e.g., the linear-quadratic-regulator problem). For systems excited by stochastic disturbances, the studies by Song and Dyke [114, 115] propose a PS method combined with a successive approximation algorithm to obtain a discretized control manifold. The combined algorithm proposed in those studies is shown to converge to the optimal solution for a wide range of classical nonlinear dynamical systems.

In this chapter, we extend the results presented in [114, 115] to the stochastic energy harvesting problem. The control objective used in this chapter includes the nonlinear diode drop losses in the power electronics. Including this term results in a more realistic approximation of the actual transmission losses that occur in energy harvesting converters. Additionally, we consider a mixed state-control constraint that results in an instantaneously dissipative controller (i.e., the directionality of power-flow is limited to extraction). Determining numerical controllers for systems with mixed state-control constraints was first investigated by Gong et al. [45]. However, the constraints used in that study were saturation limits on the states and controls.

7.2 The Stochastic Hamilton-Jacobi Equation

Consider the following generic single-input stochastic differential equation of the form

$$\mathbf{d}\mathbf{x} = \{\mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t)u(t)\} \, dt + \mathbf{h}(\mathbf{x}, t)dw$$  \hspace{1cm} (7.1)

where $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$ characterizes the system dynamics, $\mathbf{g}(\mathbf{x}, t) \in \mathbb{R}^n$ is the coefficient matrix for the control input, and $\mathbf{h}(\mathbf{x}, t) \in \mathbb{R}^n$ is the coefficient matrix for the disturbance input. In addition, we have that $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}$ is the control input, and $w(t) \in \mathbb{R}$ is a Weiner process normalized such that $dw/dt$ is white noise with unit intensity. We can think of $\mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t)u(t)$ as a drift term and $\mathbf{h}(\mathbf{x}, t)$ as a diffusion term. It should be noted that if $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$ are linear in $\mathbf{x}(t)$, and $\mathbf{g}(\mathbf{x}, t)$ and $\mathbf{h}(\mathbf{x}, t)$ are independent of $\mathbf{x}(t)$, then Equation (7.1) is a linear
system, which is similar to the systems examined in previous chapters.

In order to determine a controller for the system in Equation (7.1), we first define the bounded cost functional as

\[ J = \mathcal{E} \left\{ \int_0^\infty \exp\left[ -\beta \tau \right] L(x, u, \tau) \, d\tau \right\} < \infty \]  

(7.2)

where \( \beta > 0 \) is the exponential discount factor, which ensures \( J \) is bounded, and \( L(x, u, t) \) is the Lagrangian. The value of \( \beta \) should be chosen to be sufficiently small such that the initial condition of \( x(t) \) has a negligible impact on the value of \( J \). In other words, the value of \( \beta \) is small enough such that \( L(x, u, t) \) reaches stationarity much faster than the exponential decay terms. Like any optimal control problem, the goal is to solve for \( u(t) \) that minimizes \( J \) subject to the constraint in Equation (7.1). We assume the controller can be determined by the feedback relationship \( u(t) = \phi(x, t) \), where the function \( \phi(x, t) \) can be nonlinear.

The necessary condition for optimality of the control problem stated above can be written in terms of the stochastic Hamilton-Jacobi equation (HJE). To accomplish this, we first define the cost-to-go function as

\[ V(x, t) = \min_{u(\tau), \tau \in [t, \infty]} \left[ \mathcal{E} \left\{ \int_t^\infty \exp\left[ -\beta \tau \right] L(x, u, \tau) \, d\tau \mid x(t) = x \right\} \right]. \]  

(7.3)

The cost-to-go function is also referred to as the value function, which means that it is the minimum cost associated with starting at a given \( t \in [0, \infty) \) and any state \( x(t) \). Because the cost functional is defined on the infinite-time horizon, the resulting stationary HJE will be time invariant. From the stochastic optimality principle [30], the HJE can be expressed as

\[ \beta V^*(x) = \min_u \left[ V^*_x(x)^T (f(x) + g(x)u) + L(x, u) + \frac{1}{2} \text{tr} \left\{ h(x) h^T(x) V^*_x(x) \right\} \right] \]  

(7.4)

where \( V^*(x) \) is the value function corresponding to the optimal control \( u^* \), and \( V^*_x(x) \) and \( V^*_xx(x) \) are the first and second partial derivatives of \( V^*(x) \) with respect to \( x \). A
more detailed derivation of the infinite-time horizon HJE can be found, for example, in [39].

In general, the optimal feedback control law $u^\star$ depends on the choice of the Lagrangian. For the case where the Lagrangian is chosen to be the following quadratic function

$$
\mathcal{L}(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}
$$

(7.5)

where $Q - \frac{1}{R}SS^T \succeq 0$ and $R > 0$, then we have that the optimal controller can be defined as

$$
u^\star(x) = -\frac{1}{2R} \left( g^T(x)V^\star_x(x) + 2S^T x \right).
$$

(7.6)

If we substitute the relationship for $u^\star(x)$ in Equation (7.6) into Equation (7.4), then we get

$$
\beta V^\star(x) = V^\star_x(x)f(x) + x^T \left( Q - \frac{1}{R}SS^T \right) x + \frac{1}{2} \text{tr} \left\{ h(x)h^T(x)V^\star_{xx}(x) \right\}
$$

$$
- \frac{1}{4R}g^T(x)V^\star_x(x)V^\star_{xx}(x)g(x) - \frac{1}{R}g^T(x)V^\star_x(x)S^T x.
$$

(7.7)

Equation (7.7) is the stationary HJE that must be solved to obtain the optimal value function $V^\star(x)$. Once the optimal value function has been determined, the optimal controller $u^\star(x)$ can be obtained using Equation (7.6).

Given the quadratic Lagrangian in Equation (7.5), and if the system in Equation (7.1) is linear and independent of the states, then the resulting HJE can be reduced to an algebraic Riccati equation. However, in general, there is no closed form solution to Equation (7.7). There are a number of technical challenges associated with solving the HJE, which is a second order nonlinear partial differential equation. Numerical grid-based techniques such as finite element or finite difference methods suffer from the “curse-of-dimensionality,” which refers to the exponential scaling of computational effort required to solve problems with linearly increasing dimensions.
We therefore implement the combined successive approximation and pseudospectral method proposed by Song and Dyke [115] to solve the HJE. The successive approximation method proposed in that study is summarized by following:

**Step 0**: Start with an initial stabilizing controller $u^0(x)$, such that the solution of $x$ in Equation (7.1) is Lyapunov stable and $J$ is finite.

**Step 1**: Given $u^0(x)$, solve the following equation for $V(x)$

$$\beta V(x) = V_x^T(x) \left(f(x) + g(x)u^0(x)\right) + \mathcal{L}(x, u^0) + \frac{1}{2} \text{tr} \left\{ h(x)h^T(x)V_{xx}(x) \right\}. \quad (7.8)$$

**Step 2**: Compute the new controller as

$$u(x) = -\frac{1}{2R} \left( g^T(x)V_x(x) + 2S^T x \right). \quad (7.9)$$

**Step 3**: Update $u^0(x) \leftarrow u(x)$ and return to Step 1.

It has been shown by Wang and Saridis [124] that the solution for $V(x)$ from one iteration to the next uniformly decreases. As such, the successive approximation method monotonically converges to the optimal value function $V^*(x)$ as the number of iterations approaches infinity.

### 7.3 The Pseudospectral Method

We now present a brief summary of the pseudospectral (PS) method from [115], which will be used in conjunction with the successive approximation method to solve the stationary HJE in Equation (7.4). The PS method is a numerical technique that can be used to approximate the solution to a partial differential equation over an entire computational domain. Compared to the polynomial decay rate of similar finite element and finite difference methods, the approximation error of the PS method decays at an exponential rate. As a consequence of the higher accuracy of the PS
method, the required number of grid points and subsequent number of degrees of freedom can be minimized to reduce the computational effort.

The PS method used in this chapter is based on interpolation functions collocated on Chebyshev nodes, which are distributed over the interval \( \xi \in [-1, 1] \). To accommodate the use of an arbitrary computation domain \( x \in [x_0, x_f] \), we use the following affine transformation

\[
x(\xi) = \frac{(x_f - x_0)\xi + (x_f + x_0)}{2}, \quad \xi \in [-1, 1]
\]  

(7.10)

where we define the slope constant as

\[
m = \frac{dx}{d\xi} = \frac{x_f - x_0}{2}.
\]

(7.11)

The solution to the HJE can be approximated by a truncated polynomial \( V_{N-1}(x) \) of order \( N - 1 \) with \( N \) interpolation nodes. For the single-dimensional case, this approximation can be expressed as

\[
V_{N-1}(x) = \sum_{i=1}^{N} v_i \psi_i(x)
\]

(7.12)

where \( \psi_i(x) \) is a set of polynomial basis functions and \( v_i \) is the corresponding interpolation coefficient. In this chapter, Lagrange polynomial interpolation functions are used and the interpolation nodes are chosen at Chebyshev nodes; i.e.,

\[
\xi_k = \cos \left( \frac{k\pi}{N} \right), \quad k = 1..N.
\]

(7.13)

Using the Lagrange polynomial functions and the Chebyshev nodes, we can approximate partial derivative of Equation (7.12) at node \( x_i \) for \( i \in [1, N] \) as

\[
V_{N-1}^{(\ell)}(x_i) = \sum_{j=1}^{N} D_{i,j}^{(\ell)} v_j, \quad i = 1..N
\]

(7.14)
where the superscript (·) indicates the order of the derivative and where \(D^{(\ell)}_{i,j}\) is the differentiation matrix defined on \([x_0, x_f]\). The differentiation matrix is derived, for example, in [17], and its entries are given as

\[
D^{(\ell)}_{i,j} = \frac{1}{m} \left\{ \begin{array}{ll}
c_i(-1)^{i+j} & : \ i \neq j \\
c_j(x_i - x_j) & : \ i = j \neq 1, N \\
\frac{1}{x_i} & : \ i = j = 1 \\
\frac{21 - x_i^2}{2(N-1)^2 + 1} & : \ i = j \neq 1 \\
\frac{2(N-1)^2 + 1}{6} & : \ i = j = N
\end{array} \right.
\] (7.15)

where

\[
c_i = \left\{ \begin{array}{ll}
1 & : \ 2 \leq i \leq N-1 \\
2 & : \ i = 1, N
\end{array} \right.
\] (7.16)

Equation (7.14) can be written in vector form as

\[
V^{(\ell)}_{N-1} = D^{(\ell)} v = (D^{(1)})^\ell v
\] (7.17)

where \(V^{(\ell)}_{N-1}\) and \(v\) are the column vector forms of \(V^{(\ell)}(x_i)\) and \(v_i\), respectively.

Using the vector relationship for the value function in Equation (7.17), we can discretize the original HJE in Equation (7.7) over the computational domain \(x \in [x_0, x_f]\). The discretized HJE for the \(i\)-th Chebyshev node \(x_i\) can be written as

\[
\frac{1}{4R} v^T D_i^T g g^T D_i v + \beta I_i v - f^T D_i v - \frac{1}{2} \text{tr}\left\{ h h^T D_i^{(2)} v \right\} 
+ \frac{1}{R} g^T D_i v S^T x_i = x_i^T \left( Q - \frac{1}{R} S S^T \right) x_i
\] (7.18)

where the subscript \(i\) indicates the \(i\)-th row of the corresponding matrix. It should be noted that \(f, g, h,\) and \(x_i\) are all scalars for the single-dimensional case. Setting up similar discretized equations for the remaining Chebyshev nodes results in a system of \(N\) equations where the unknowns to be solved are the \(N\) values of the vector \(v\). However, as pointed out in [115], solving Equation (7.18) is challenging for a large number of nodes because the first term is quadratic in \(v\).
7.3.1 Iterative Algorithm

Because of the challenges associated with explicitly solving Equation (7.18), we combine the PS method with the successive approximation algorithm to solve the discretized HJE.

**Step 0**: Start with an initial stabilizing truncated controller \( u^0_{N-1}(x) \), which can be found as

\[
u^0_{N-1}(x_i) = \sum_{i=1}^{N} u_i^0 \psi_i(x)
\]  

(7.19)

where \( u_i \) is the nodal value of the chosen initial control at the \( i \)-th Chebyshev node. For this chapter, we assume that \( u^0 \) is the optimal controller of the corresponding linear system.

**Step 1**: Given \( u^0 \), set up \( N \) linear equations for \( i \in [1, N] \) of the form

\[
\left(f^T D_i + (u_i^0)^T g^T D_i - \beta I_i\right) v_i + \frac{1}{2} \text{tr}\left\{hh^T D_i^{(2)} v_i\right\} = -L(x_i, u_i^0)
\]  

(7.20)

where \( D_i \) and \( I_i \) are the \( i \)-th rows of \( D \) and \( I \), respectively. Because the resulting system of equations is linear in \( v \), we can compute \( v \) using any linear equation solver.

**Step 2**: Compute the values of the updated discretized controller on the \( i \)-th nodes as

\[
u_i = -\frac{1}{2R} \left( g^T D_i v + 2S^T x_i \right).
\]  

(7.21)

**Step 3**: Update \( u^0 \leftarrow u \) and return to Step 1.

Convergence of the combined PS and successive approximation algorithm is obtained when the norm of the difference between discretized controllers at the present
and previous iterations is below a user specified tolerance. The convergence of the combined algorithm is discussed in [115].

It is straightforward to extend the combined PS and successive approximation algorithm for systems with multiple dimensions. The main difference between the single- and multi-dimensional cases is in the derivation of the multi-dimensional differentiation matrix. Instead of expressing the truncated polynomial $V_{N-1}$ as a tensor product, the analysis can be simplified by combining the grid points for each dimension into a single vector. As such, the multi-dimensional differentiation matrix can be assembled using components of the single-dimensional differentiation matrices for each node in each dimension. Interested readers should see [115] for a more detailed derivation of the multi-dimensional differentiation matrix.

7.4 Power-Flow-Constrained Vibratory Energy Harvesting

In this section, we present the power-flow-constrained energy harvesting problem in terms of the stochastic HJE. Specifically, the energy harvester we will investigate consists of a SDOF oscillator with electromagnetic coupling, which was first examined in Chapter 3. We assume the SDOF oscillator has mass $m$, damping $c$, and stiffness $k$, and the transducer has back-emf motor constant $K_t$. However, we assume that a single-directional power electronic converter (such as the one in Figure 1.3(b)) is used to control the transducer, which limits the directionality of power-flow to extraction. Recall that the nondimensional harvester state space for this system can be written as

$$\dot{x}_h(t) = A_h x_h(t) + B_h i(t) + G_h a(t)$$

$$v(t) = B_h^T x_h(t)$$

where

$$A_h = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
and where \( d = c/\sqrt{mk} \) is the damping ratio. The particular characteristics of the disturbance will be discussed in the following subsections.

For this example, we consider a conservative approximation of the losses in the power electronics. We define the energy harvesting performance measure as

\[
\bar{P}_{\text{gen}} = -\mathcal{E}\{iv + Ri^2 + V_d|i|\}
\]  

(7.23)

where \( R \) and \( V_d \) are the resistive and diode losses, respectively. This loss model corresponds to a power electronic converter consisting of a bridge rectifier connected to a PWM-switched controllable DC/DC converter (e.g., Figure 1.3(b)). We assume the bridge rectifier is implemented actively, by gating MOSFETs to mimic diodes. Parasitic losses in an active rectifier are minimal because the MOSFETs only need to be gated when the direction of the current changes. The advantage of an active rectifier is that its conductive dissipation is quadratic in current, thus resembling a resistor. This is in contrast to a passive diode rectifier, which has a finite conduction voltage threshold. Because of this difference, even though active rectifiers require gating, their overall efficiency can be higher when the transducer current is very low.

In addition, we impose the power-flow directionality constraint on the transducer current as in Equation (6.11), where we assume \( Y_{\text{max}} = 1/R \).

Thus far, we have accounted for nonlinearities in the harvester dynamics and non-quadratic loss models using the statistical linearization procedure in Chapter 5. Such approximations assume the feedback controller is a linear combination of the states. In addition, the nonlinear controllers for power-flow-constrained energy harvesters in Chapter 6 are synthesized from the unconstrained controllers. Both methods result in performances that are sub-optimal. However, using the combined PS and successive approximation method, we can compute numerical controllers that satisfy the necessary conditions for optimality, for systems with non-quadratic performance measures and mixed state-control constraints. Additional nonlinearities
in the harvester dynamics (e.g., Coulomb friction) can easily be included in the control design with only slight modifications to the PS algorithm.

In terms of the formulation from Section 7.2, the value function for the constrained energy harvesting problem is

\[
V(x, t) = \min_{i(\tau), \tau \in [t, \infty)} \left[ \mathcal{E} \left\{ \int_t^\infty \exp[-\beta \tau](x^T(\tau)B i(\tau) + \dot{i}^2(\tau)R + |i(\tau)| V_d) \, d\tau \mid x(t) = x \right\} \right]
\]

and the corresponding stationary HJE is

\[
\beta V^*(x) = \min_{i(x)} \left[ x^T B i(x) + \dot{i}^2(x) R + |i(x)| V_d + (Ax + Bi(x))^T V^*_x(x) + \frac{1}{2} \text{tr}\{GG^T V^*_xx(x)\} \right].
\]

The goal is to find a feedback controller for the form \( i(x) = \phi(x) \) that minimizes the expression above.

It turns out we can find a closed form solution for the unconstrained energy harvesting controller with the performance defined in Equation (7.23). Using the expression for the optimal control law in Equation (7.6), the unconstrained HJE in Equation (7.25) is minimized by

\[
i_u(x) = \begin{cases} 
-\frac{1}{2R} \left( B^T(V_x(x) + x) + V_d \right) : B^T(V_x(x) + x) + V_d < 0 \\
-\frac{1}{2R} \left( B^T(V_x(x) + x) - V_d \right) : B^T(V_x(x) + x) - V_d > 0 \\
0 : \text{otherwise}
\end{cases}
\]

As such, we have that \( \phi(x) \) can be defined as

\[
\phi(x) = \arg \min_{x^T B i + \dot{i}^2 / Y_{\text{max}} \leq 0} |i(x) - i_u(x)|.
\]
We can determine an expression for $i^*(x)$ that minimizes the above equation using a similar clipping procedure that was described in Section 6.3.1. Thus, we have that the optimal controller $i^*(x) = \phi^*(x)$ is

$$i^*(x) = \text{sat}_{x^TB_iu + i^2_u/Y_{\text{max}}} \{i_u(x)\}$$  

(7.28)

where the sat\{·\} function is defined in Equation (6.20). The corresponding optimal PS performance can be expressed as

$$\bar{P}^\text{PS}_{\text{gen}} = -\lim_{T \to \infty} \mathcal{E} \left\{ \frac{1}{T} \int_0^T \mathcal{L}(x, u^*, t) \, dt \mid x(0) = 0 \right\}$$  

(7.29)

which is equivalent to

$$\bar{P}^\text{PS}_{\text{gen}} = -\lim_{\beta \to 0} \beta V^*(0).$$  

(7.30)

If $\beta$ is made sufficiently small compared to the open-loop dynamics of $A$ (i.e., $\beta < \frac{1}{100} \min_i |\Re(\lambda_i(A))|$), then the PS performance can be approximated as $\bar{P}^\text{PS}_{\text{gen}} = -\beta V^*(0)$.

### 7.5 Simulation Examples

#### 7.5.1 White Noise Disturbance

The first example we consider is the case where the SDOF oscillator in Figure 6.2(a) is excited by a broadband white noise disturbance (i.e., $a(t) = w(t)$). For this example, the augmented state space matrices are $A = A_h$, $B = B_h$, and $G = G_h$. Because the augmented state space has just two degrees-of-freedom, we are able to visually examine the effects of the diode losses and the power-flow constraint on the discretized value function and control manifold. We set the discount factor as $\beta = 0.01$ and compute the optimal value function and corresponding optimal control manifold using a discretized $11 \times 11$ grid on the computational domain $[-5,5] \times [-5,5]$. 

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We begin by providing an example that illustrates the convergence of the combined PS and successive approximation algorithm. The plot in Figure 7.1 shows the norm of the discretized control manifold monotonically decreasing until the algorithm converges in 6 iterations. For this example, the convergence tolerance was set at $10^{-6}$ and the initial guess for the algorithm was chosen to be the LQG solution for the case without diode losses (i.e., $V_d = 0$). In general, the algorithm converges in 5–20 iterations for all of the examples considered in this chapter.

We provide example plots of the optimal value function $V^*(x)$ and the optimal control manifold $i^*(x)$ in Figure 7.2. The states “$x_1$” and “$x_2$” correspond to the nondimensional displacement and velocity of the harvester, respectively. We notice the distribution of Chebyshev nodes resembles an approximately quadratic clustering near the boundaries of the grid domain. Such a distribution is more suitable for approximating higher order polynomials than an equidistant node distribution. In addition, it should be noted that the power-flow constraint in Equation (6.11) is never violated for this controller. This is because the optimal control law is just voltage feedback for broadband energy harvesting, which always results in a purely dissipative system. The addition of diode losses to the performance measure in

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Equation (7.23) does not change the fact that voltage feedback is still optimal for broadband energy harvesting. However, the controller is no longer proportional to the voltage, but is instead a nonlinear function of $x_2$.

To better illustrate the properties of the numerical controller, we simulate the closed-loop system for 200 seconds. The plot in Figure 7.3(a) shows the power delivered to storage $P_S(t)$ while the plot in Figure 7.3(b) shows the optimal current and voltage relationship. From the plot in Figure 7.3(b), the optimal control law can be approximated by a sum of a linear term and a cubic term in $v(t)$. In other words, an approximate realization of the optimal control law is

$$i^*(t) = -Y \frac{v^3(t)}{v_0^2 + v^2(t)}$$

(7.31)

where $Y > 0$ and $v_0 > 0$ depend on the system parameters. It would be straightforward to determine the values for $Y$ and $v_0$ which best fit the data in Figure 7.3(b) using a least squares algorithm.

Next, we illustrate the effects of the electronic loss parameters on the energy harvesting performance when the electronics implement both the static admittance (SA) and PS controllers. Since the dynamics of the harvester are linear, we can
optimize the SA controller using the line search procedure discussed in Chapter 3. From the energy harvesting performance in Equation (7.23), and when the electronics implement the SA control law $i(t) = -Y_s v(t)$, the average power generated is

$$
\bar{P}_\text{SA}^{\text{gen}} = (Y_s - Y_s^2 R)B^T S B - Y_s \frac{1}{\pi} \sqrt{2} V_d \sqrt{B^T S B} \sqrt{2} \pi V_d \right)^{1/2} \tag{7.32}
$$

where $S = \mathcal{E}\{xx^T\}$ is the closed-loop covariance matrix. The second term on the right hand side of the above equation can be obtained from the third term on the right hand side of Equation (5.59), where the variance of the current is defined as $s_i = Y_s^2 B^T S B$

The plot in Figure 7.4(a) shows $\bar{P}_\text{gen}^{\text{PS}}$ as a function of $\{R, V_d\}$ while the plot in Figure 7.4(b) shows $\bar{P}_\text{gen}^{\text{PS}}$ as a function of $\{R, V_d\}$. The optimal PS performance was calculated using Equation (7.30) while the optimal SA performance was calculated using Equation (7.32) with $Y_s = Y_s^*$. From these plots, the energy harvesting performance for both controllers decreases as $R$ and $V_d$ are increased. We also plot the performance ratio as a function of $\{R, V_d\}$ in Figure 7.5. In this figure we see that $\bar{P}_\text{gen}^{\text{SA}}/\bar{P}_\text{gen}^{\text{PS}} = 1$ as $V_d \to 0$, because the controller that optimizes $\bar{P}_\text{gen}$ is a static admittance for the broadband energy harvesting problem.
7.5.2 Filtered White Noise Disturbance

Next, we consider the case in which the SDOF oscillator is excited by filtered white noise. Specifically, we will model the disturbance as a first-order lowpass filter, which can be represented by the nondimensionalized disturbance state space

\[ \dot{x}_a(t) = -\omega_c x_a(t) + \sqrt{2\omega_c} \, w(t) \quad (7.33a) \]
\[ a(t) = x_a(t) \quad (7.33b) \]

where \( \omega_c \) is the nondimensional cutoff frequency of the lowpass filter. The term preceding \( w(t) \) ensures the standard deviation of the acceleration is equal to unity regardless of the value chosen for \( \omega_c \). To better illustrate the properties of the lowpass filter disturbance model, we plot the magnitude and phase of \( G_{aw}(s) \) for three values of \( \omega_c \) in Figure 7.6. Combining the harvester and disturbance dynamics results in the augmented state space in Equation (7.22), with appropriate definitions for \( \{ A, B, G \} \).

Because the optimal value function \( V^*(x) \) and optimal control manifold \( i^*(x) \) are now a function of three states, we cannot illustrate them graphically. Instead, we simulate the closed-loop system for 200 seconds in which we set \( d = 0.1, R = 0.5, V_d = 0.2, \) and \( \omega_c = 1 \). For this example, we choose a discount factor of \( \beta = 1e^{-3} \)

\[ \text{Figure 7.4: Surface plot of } \bar{P}_{gen} \text{ as a function of } \{ R, V_d \} \text{ for } d = 0.1: (a) } \bar{P}_{gen}^{SA}, \text{ and (b) } \bar{P}_{gen}^{PS}. \]
and compute the optimal control manifold using a discretized $7 \times 7$ grid on the computational domain $[-5, 5] \times [-5, 5] \times [-5, 5]$. The plot in Figure 7.7(a) shows the power delivered to storage $P_S(t)$ while the plot in Figure 7.7(b) shows the optimal current and voltage relationship. We see that the power-flow constraint in Equation (6.11) is satisfied in Figure 7.7(b).

Next, we compare the performance of the PS controller to the performance of the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.5}
\caption{Performance ratio surface as a function of $\{R, V_d\}$ for $d = 0.1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.6}
\caption{Magnitude and phase of the disturbance lowpass filter $G_{aw}(j\omega)$.}
\end{figure}
SA controller. For reference, we also compute the performance of the unconstrained state feedback LQG controller. The diode losses are accounted for in the design of the LQG controller using the iterative procedure discussed in Section 5.4.2. Therefore, we plot the performances of the SA, PS, and LQG controllers as a function of $\{\omega_c, R\}$ and for three increasing values of $V_d$. In addition, to illustrate the improvement in performance achieved by the PS controller, we plot the performance ratios $\bar{P}_{SA} / \bar{P}_{PS}$ and $\bar{P}_{PS} / \bar{P}_{LQG}$ over the same domain. We note that the PS algorithm is unable to converge to a solution as both $\omega_c$ and $R$ approach 0. Therefore, those parameters were limited to be within the range $[0.1, 10]$.

We first consider the case in which we have purely quadratic losses (i.e., $V_d = 0$). The plots in Figures 7.8 and 7.9 illustrate the controller performances and performance ratios, respectively, for $V_d = 0$. From the plot in Figure 7.9(a), the SA and PS performances are almost identical except for when $\omega_c$ is near unity and $R \to 0$. Furthermore Figure 7.9(b) clearly shows that the unconstrained LQG controller significantly outperforms the PS controller for all values of $\omega_c$ and $R$.

Now, we modify our example to include two nonzero values for the diode losses. The plots in Figures 7.10 and 7.11 show the case where $V_d = 0.2$ while the plots in Figures 7.12 and 7.13 show the case where $V_d = 0.4$. From these two cases, we
Figure 7.8: \( \bar{P}_{\text{gen}} \) surface plots as a function of \( \{\omega_c, R\} \) for \( d = 0.1 \) and \( V_d = 0 \): (a) \( \bar{P}^{SA}_{\text{gen}} \), (b) \( \bar{P}^{PS}_{\text{gen}} \), and (c) \( \bar{P}^{LQG}_{\text{gen}} \).

Figure 7.9: Performance ratios as a function of \( \{\omega_c, R\} \) for \( d = 0.1 \) and \( V_d = 0 \): (a) \( \bar{P}^{SA}_{\text{gen}} / \bar{P}^{PS}_{\text{gen}} \), and (b) \( \bar{P}^{PS}_{\text{gen}} / \bar{P}^{LQG}_{\text{gen}} \)
Figure 7.10: $\bar{P}_{gen}$ surface plots as a function of $\{\omega_c, R\}$ for $d = 0.1$ and $V_d = 0.2$: (a) $\bar{P}_{gen}^{SA}$, (b) $\bar{P}_{gen}^{PS}$, and (c) $\bar{P}_{gen}^{LQG}$.

Figure 7.11: Performance ratios as a function of $\{\omega_c, R\}$ for $d = 0.1$ and $V_d = 0.2$: (a) $\bar{P}_{gen}^{SA}/\bar{P}_{gen}^{PS}$, and (b) $\bar{P}_{gen}^{PS}/\bar{P}_{gen}^{LQG}$.
Figure 7.12: $\bar{P}_{gen}$ surface plots as a function of $\{\omega_c, R\}$ for $d = 0.1$ and $V_d = 0.4$: (a) $\bar{P}_{gen}^{SA}$, (b) $\bar{P}_{gen}^{PS}$, and (c) $\bar{P}_{gen}^{LQG}$.

Figure 7.13: Performance ratios as a function of $\{\omega_c, R\}$ for $d = 0.1$ and $V_d = 0.4$: (a) $\bar{P}_{gen}^{SA}/\bar{P}_{gen}^{PS}$, and (b) $\bar{P}_{gen}^{PS}/\bar{P}_{gen}^{LQG}$
obtain the interesting result that there are regions in the \( \{ \omega_c, R \} \) space in which the PS controller outperforms the unconstrained LQG controller. This is because the unconstrained LQG controller is restricted to linear feedback, which is only proven to be optimal when diode losses are zero, whereas the PS controller is nonlinear in the system states. As \( V_d \) increases, much more power is generated by the PS controller as compared to the unconstrained LQG controller for values of \( \omega_c \) away from unity. However, as shown in Figure 7.11(b), the unconstrained LQG slightly outperforms the PS controller for values of \( \omega_c \) near unity and as \( R \to 0 \).

### 7.6 Summary

The purpose of this chapter has been to derive optimal numerical feedback controllers for energy harvesters with power-flow constraints. Determining controllers for constrained systems requires a solution to the stochastic HJE, which is challenging because it is a nonlinear partial differential equation. Thus, we present an iterative approach to compute the discretized control manifold using the combined PS and successive approximation algorithm from [115]. However, unlike the examples considered in that paper, we explored the ability of the algorithm to handle the mixed state-control power-flow constraint from Chapter 6.

Control manifolds for a power-flow-constrained SDOF energy harvester with electromagnetic coupling were computed for two disturbance scenarios. We also considered a non-quadratic loss model, which takes resistive and diode losses into account. In the first scenario, we considered the case in which the energy harvester is excited by a white noise disturbance. For that scenario, we compared the performance of the PS controller to the performance of the SA controller over a range of \( R \) and \( V_d \) values, and showed that the PS controller always outperforms the SA controller. Next, we compared the performances of the PS, SA, and unconstrained LQG controller for the case in which the energy harvester is excited by a lowpass-filtered white noise
disturbance. From the performance ratios computed for that scenario, we obtained the interesting result that the PS controller outperforms the unconstrained LQG controller for certain \( \{\omega_c, R\} \) regions and when \( V_d > 0 \). For higher values of \( V_d \), the PS controller always outperformed the unconstrained LQG controller over the entire \( \{\omega_c, R\} \) domain.

There are two drawbacks to the PS control approach. The first drawback relates to the synthesis of the control input from the discretized control manifold. Because the discretized control manifold is a function of the grid nodes, an approximate value of the control input could be computed from a polynomial expansion of the states at each node. This approach is very computationally demanding, especially for grids with higher resolution, and it may not be feasible to implement a control algorithm of this type in a physical system in real-time. Second, because of the limitation on computational power required for the PS algorithm to converge, the computational domain must be made increasingly coarse as the number of states increases, which results in a less accurate controller. Thus, the PS control approach may not be suitable for systems with multiple modes (e.g., piezoelectric energy harvesters).
8

Summary, Conclusions, and Future Work

8.1 Summary and Conclusions

The research presented in this dissertation explored the benefits of formulating the stochastic vibratory energy harvesting problem in terms of feedback control theory. One of the main challenges encountered in such systems concerns the power electronic hardware used to regulate power extraction. The hardware must be capable of bi-directional power-flow in order to obtain the theoretically-optimal bound on power generation. In the first part of this dissertation, we established an experimentally validated model of an electromagnetic transducer to demonstrate the energy harvesting capability of an actively-controlled system. Next, we derived optimal state feedback controllers for ideal vibratory energy harvesters, and augmented this theory to account for competing objectives, nonlinearities in the harvester dynamics, and non-quadratic transmission loss models. Following that analysis, we presented two techniques to account for power-flow constraints imposed on feedback controllers by single-directional power electronic converters.

In Chapter 2, we developed a predictive model to characterize the dynamics of an
electromagnetic transducer in order to provide a basis by which actively-controlled vibratory energy harvesters could be theoretically studied. Using an idealized approximation of this model (i.e., neglecting the nonlinearities) and an idealized model of a piezoelectric bimorph cantilever beam, Chapter 3 investigated the potential for enhanced energy harvesting performance from stochastic disturbances using state feedback control. The main outcome of that approach was that energy harvesters can be passively tuned such that the optimal current relationship only requires half the states for feedback. Motivated by this result, we then presented a fixed-structure feedback optimization technique to maximize performance when certain system states are fixed in the design. These results have important implications for energy harvesting applications in which the performance of static feedback controllers can be improved by feeding back additional states (e.g., disturbance acceleration), and where the complexity of the feedback implementation must be kept to a minimum.

Building upon the theory presented in Chapter 3, the next two chapters investigated systems with competing objectives, nonlinearities in the harvester dynamics, and non-quadratic transmission loss models. Chapter 4 developed voltage feedback controllers for actively-controlled systems that can be used to balance the energy harvesting objective with other structural control objectives. Specifically, we showed that the multi-objective control approach can simultaneously suppress the response of a structure and generate a nominal amount of power, which could be stored or immediately re-used by the system itself. Such controllers could be used to improve the performance of tuned mass dampers in tall buildings or vehicle suspensions, which are traditionally designed to dissipate power. Next, Chapter 5 further extended the theory in Chapter 3 to account for nonlinearities in the harvester dynamics as well as non-quadratic transmission loss models using a statistical linearization approach. Including these nonlinearities allowed us to determine critical levels of Coulomb friction
below which power generation is effectively zero.

The final two chapters of this dissertation explored two methods for designing feedback controllers for energy harvesters that are controlled with single-directional power electronic drives. The first method, presented in Chapter 6, involved a sub-optimal nonlinear feedback control technique which is analytically guaranteed to outperform the optimal static admittance controller. We also presented the energy harvesting equivalent of the clipped-optimal semi-active control method. Comparison of the two controllers using the two idealized examples revealed that the clipped-optimal controller is effective at generating power for single-mode systems (e.g., the SDOF oscillator with electromagnetic coupling), while it is much more beneficial to implement the performance-guaranteed controller in systems with multiple modes (e.g., the piezoelectric bimorph cantilever beam). The second method, presented in Chapter 7, used the pseudospectral method to derive numerical controllers which satisfy the necessary conditions for optimality. We found that the pseudospectral controller outperforms the unconstrained linear-quadratic-Gaussian controller when non-quadratic power losses (i.e., diode drop) were included in the performance objective.

8.2 Future Work

Building upon the research presented in this dissertation, we now present three areas of future work in the development of feedback controllers for vibratory energy harvesters.

8.2.1 Control Implementation

Validating the performance of the feedback controllers presented in Chapters 3–7 requires further testing on an experimental system. For example, implementing state feedback controllers using the experimental system from Chapter 2 is not straight-
forward because the transducer is three-phase. Control of such systems first requires the three measured line voltages to be converted to quadrature and direct voltages. Next, the quadrature control current can be synthesized from the quadrature voltage, additional (measured or estimated) system states, and the state feedback gain. Finally, the three line control currents can be obtained from the quadrature control current, where the direct control current is maintained at zero. This process is difficult to perform in real-time because the conversion from line coordinates to quadrature and direct coordinates (and vice versa) is computationally demanding. Therefore, additional research is required to develop and test controllers for three-phase systems that require less computational power. One possible way to reduce the computational power is to estimate the quadrature voltage from the velocity of the device.

8.2.2 Power Electronic Converter Design and Testing

As discussed in Chapter 2, the three-phase motor drive used to control the experimental device has several drawbacks. Primarily, operating the drive requires a high level of quiescent power, which is the result of an inefficient step-down converter used to reduce the supply power to logic power. In addition, the drive has a maximum bus voltage rating of 80V and the switching frequency is not adjustable. Effective testing of the controllers discussed in this dissertation on the experimental system in Chapter 2 would require the design and fabrication of custom power electronic converters. Such converters should be capable of handling higher bus voltages and a range of switching frequencies. Because the back-emf voltages of the device must be below the bus voltage of the drive, higher bus voltages would allow for testing at higher device velocities. It has been shown in [89, 102] that varying the bus voltage and switching frequency of a converter can be used to further optimize the energy harvesting performance. Finally, additional experimental testing should be
conducted to verify the accuracy of the transmission loss models.

8.2.3 Multivariable Control Synthesis

The work presented in this dissertation assumed only a single transduction system regulates power-flow in an energy harvester, and that the system is excited by a single exogenous disturbance. However, the control synthesis could be extended to more general systems consisting of an arbitrary number of transducers and an arbitrary number of disturbances. The studies by Scruggs [98, 99] presented state and voltage feedback controllers for multivariable energy harvesters, but the theory in those papers is limited to systems with linear harvester dynamics. Designing controllers for arbitrary nonlinear systems could be accomplished by extending the statistical linearization approach presented in Chapter 5. In addition, the nonlinear control design for systems with power-flow constraints could be extended to account for multiple transducers. Both of these extensions would be applicable, for example, to ocean wave energy harvesting systems where many spatially-distributed, dynamically-coupled transducers are used to extract energy from the waves. It might also have application in aerospace applications in which several spatially-distributed transducers are used to harvest energy from fluttering wings and panels.
Appendix A

Proofs

A.1 Proof of Theorem 1

We present a brief synopsis of the proof for this theorem, which can be found in more detail in [99]. To begin, we partition $P = P^T$ into a general $2 \times 2$ block matrix such that

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \quad (A.1)$$

Next, we substitute the relationship for $P$ as well as the augmented block matrix relationships for $A$ and $B$ into Equation (3.20) to obtain three new equations; i.e.,

$$A_h^T P_1 + P_1 A_h - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T (P_1 + \frac{1}{2} I) = 0 \quad (A.2a)$$

$$A_h^T P_2 + P_1 G_c C_a + P_2 A_a - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T P_2 = 0 \quad (A.2b)$$

$$A_a^T P_3 + P_3 A_a + C_a^T G_h P_2 + P_2^T G_h C_a - \frac{1}{R} P_2^T B_h B_h^T (P_1 + \frac{1}{2} I) = 0. \quad (A.2c)$$
We can also partition $\mathbf{K} = [\mathbf{K}_1 \quad \mathbf{K}_2]$ such that

\[
\mathbf{K}_1 = -\frac{1}{R} \mathbf{B}_h^T (\mathbf{P}_1 + \frac{1}{2} \mathbf{I}) , \quad (A.3a)
\]

\[
\mathbf{K}_2 = -\frac{1}{R} \mathbf{B}_h^T \mathbf{P}_2 . \quad (A.3b)
\]

Since $\tilde{\mathbf{P}}_{gen}$ is an indefinite quadratic form, we must show that the solutions to the three equivalent equations in Equations (A.2a)–(A.2c) exist and that they result in a stabilizing control law (i.e., $\mathbf{A} + \mathbf{B}\mathbf{K} < 0$).

If we define $\bar{\mathbf{P}}_1$ as

\[
\bar{\mathbf{P}}_1 = \mathbf{P}_1 + \frac{1}{2} \mathbf{I} \quad (A.4)
\]

then Equation (A.2a) is equivalent to the standard Riccati equation

\[
-\frac{1}{2} [\mathbf{A}_h + \mathbf{A}_h^T] + \mathbf{A}_h^T \bar{\mathbf{P}}_1 + \bar{\mathbf{P}}_1 \mathbf{A}_h - \frac{1}{R} \bar{\mathbf{P}}_1 \mathbf{B}_h \mathbf{B}_h^T \bar{\mathbf{P}}_1 = 0 . \quad (A.5)
\]

This equation is guaranteed to have a unique, stabilizing, and positive definite solution if the following three conditions hold: $-\frac{1}{2} [\mathbf{A}_h + \mathbf{A}_h^T] \succeq 0$, $(\mathbf{A}_h, -\frac{1}{2} [\mathbf{A}_h + \mathbf{A}_h^T])$ is observable, and $\mathbf{A}_h$ is asymptotically stable. All three of these conditions are implied by the fact that $H_i(s)$ is WSPR and that the harvester state-space is self-dual. Thus, we have that a unique $\mathbf{P}_1$ exists, which is guaranteed to stabilize $\mathbf{A}_h + \mathbf{B}_h \mathbf{K}_1$. As show in [99], the other components of $\mathbf{P}$ can be shown to be stabilizing once a stabilizing $\mathbf{P}_1$ is found. Finally, it is a standard result that the performance of any causal feedback law is related to the optimal performance through Equation (3.22).
A.2 Proof of Theorem 2

We start by defining the unique, stabilizing solution to the Riccati equation in Equation (3.20) as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \tag{A.6}$$

where

$$P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad P_2 = \begin{bmatrix} P_{13} & P_{14} \\ P_{23} & P_{24} \end{bmatrix}, \quad P_3 = \begin{bmatrix} P_{33} & P_{34} \\ P_{34}^T & P_{44} \end{bmatrix}.$$

Thus, for the augmented state space system in Equation (3.13), the Riccati equation can be expanded into three coupled equations in $P_1$, $P_2$, and $P_3$; i.e.,

$$P_1 A_h + A_h^T P_1 - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T (P_1 + \frac{1}{2} I) = 0, \tag{A.7a}$$

$$P_1 G_h C_a + P_2 A_a + A_a^T P_2 - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T P_2 = 0, \tag{A.7b}$$

$$P_2^T G_h C_a + C_a^T G_h^T P_2 + P_3 A_a + A_a^T P_3 - \frac{1}{R} P_2^T B_h B_h^T P_2 = 0. \tag{A.7c}$$

As discussed in [99], if the harvester is WSPR, then Equation (A.7a) has a unique solution $P_1$ which is stabilizing (i.e., for which $A_h - \frac{1}{R} B_h^T (P_1 + \frac{1}{2} I)$ is asymptotically stable), and for which $P_1 + \frac{1}{2} I > 0$ is a closed-loop Lyapunov matrix for the free response of the controlled harvester (i.e., with $a(t) = 0$). It only remains to show that the decoupled solution above is in fact the stabilizing one. Defining $\bar{P}_1 = P_1 + \frac{1}{2} I$, Equation (A.7a) becomes

$$-\frac{1}{2} (A_h + A_h^T) + P_1 A_h + A_h^T P_1 - \frac{1}{R} P_1 B_h B_h^T P_1 = 0. \tag{A.8}$$

For the case where $A_h$ and $B_h$ satisfy Equation (3.23), then it is straight-forward to verify that a solution to Equation (A.8) is $\bar{P}_{11} = \bar{P}_{22} = \bar{P}_{22}^T$ and $\bar{P}_{12} = 0$ where $\bar{P}_{22}$ can be found by solving the Riccati equation

$$\frac{1}{2} (D + D^T) + P_{22} (-D) + (-D)^T P_{22} - \frac{1}{R} P_{22} B_1 B_1^T P_{22} = 0 \tag{A.9}$$
which is equivalent to Equation (3.26a) for $\mathbf{P}_1$. Now, it is a fact that $(-\mathbf{D}, \mathbf{D} + \mathbf{D}^T)$ is observable if $(\mathbf{A}_h, \mathbf{A}_h + \mathbf{A}_h^T)$ is. This follows from the fact that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}_h$ with eigenvalue $\eta$ with $\Re\{\eta\} < 0$, then it must have the structure $\mathbf{v} = [v_1^T \quad \eta v_1^T]^T$, which, in turn, implies that $v_1$ is an eigenvector of $-\mathbf{D}$, with associated eigenvalue $\eta_1 = \eta + 1/\eta$. If $(-\mathbf{D}, \mathbf{D} + \mathbf{D}^T)$ were unobservable, that would require that at least one such $v_1$ satisfy $(\mathbf{D} + \mathbf{D}^T)v_1 = 0$, but this would also require that the corresponding $\mathbf{v}$ satisfy $(\mathbf{A}_h + \mathbf{A}_h^T)v = 0$, which violates the WSPR assumption. By contradiction, we conclude that $(-\mathbf{D}, \mathbf{D} + \mathbf{D}^T)$ is observable. Furthermore, in the process we have shown that $-\mathbf{D}$ is asymptotically stable, because if $\Re\{\eta\} < 0$ then $\Re\{\eta_1\} = \Re\{\eta\} + \Re\{1/\eta\} < 0$. But if $-\mathbf{D}$ is asymptotically stable, $(-\mathbf{D}, \mathbf{D} + \mathbf{D}^T)$ is observable, and $R > 0$, then Equation (A.9) is a standard Riccati equation with a unique stabilizing solution $\mathbf{P}_{22} = \mathbf{P}_{22}^T > 0$ [11]. Consequently, we have that there is a corresponding unique stabilizing solution $\mathbf{P}_{22} = \mathbf{P}_{22}^T$ to Equation (3.26a), and that the corresponding decoupled solution for $\mathbf{P}_1 = \mathbf{P}_1^T$ in Equation (A.7a) is the stabilizing solution.

Now, as shown in [99], the other components of $\mathbf{P}$ are found uniquely, once $\mathbf{P}_1$ is found. Using the decoupled result obtained for $\mathbf{P}_1$ and substituting $\mathbf{A}_u$ and $\mathbf{C}_u$ as defined in Equation (3.24) into Equation (A.7b) results in $\mathbf{P}_{14} = \mathbf{P}_{23} = 0$. In addition, we have that $\mathbf{P}_{13} = \mathbf{P}_{24}$ where $\mathbf{P}_{24}$ can be found by solving Equation (3.26b). With $\mathbf{P}_1$ and $\mathbf{P}_2$ solved and the decoupled expression for $\mathbf{P}_2$ substituted into Equation (A.7c), we have that $\mathbf{P}_{34} = 0$ and $\mathbf{P}_{33} = \mathbf{P}_{44}$ where $\mathbf{P}_{44}$ is the solution to Equation (3.26c). See [99] for the proof that $\mathbf{P}_3 < 0$. Substituting the decoupled solution for $\mathbf{P}$ into Equation (3.21) gives Equation (3.27) immediately. Finally, substituting the decoupled form for $\mathbf{P}$ into the expression for the upper bound on the average power generated results in Equation (3.28).
A.3 Proof of Theorem 3

We begin by substituting Equation (3.35) into the gradient expression in Equation (3.38) to arrive at

$$\text{tr} \left\{ \left( \frac{\partial \tilde{P}}{\partial \tilde{K}} \right) GG^T \right\} = 0 . \quad (A.10)$$

By taking the partial derivative of each side of Equation (3.36), we get a Lyapunov equation for $\frac{\partial \tilde{P}}{\partial \tilde{K}}$; i.e.,

$$\left[ A + B\tilde{K}C \right]^T \left( \frac{\partial \tilde{P}}{\partial \tilde{K}} \right) + \left( \frac{\partial \tilde{P}}{\partial \tilde{K}} \right) \left[ A + B\tilde{K}C \right]$$

$$+ \left\{ \left( \frac{\partial}{\partial \tilde{K}} \left[ A + B\tilde{K}C \right]^T \right) \tilde{P} + \tilde{P} \left( \frac{\partial}{\partial \tilde{K}} \left[ A + B\tilde{K}C \right] \right) + \frac{\partial}{\partial \tilde{K}} \tilde{Q}(\tilde{K}) \right\} = 0 . \quad (A.11)$$

Next, we evaluate the last partial derivative in the above expression, which can be expressed as

$$\frac{\partial}{\partial \tilde{K}} \tilde{Q}(\tilde{K}) = \frac{\partial}{\partial \tilde{K}} \left[ \frac{1}{2} C^T \tilde{K} B^T + \frac{1}{2} B\tilde{K}C + RC^T \tilde{K}^T \tilde{K}C \right] \quad (A.12)$$

$$= \frac{1}{2} C^T B^T + \frac{1}{2} BC + RC^T \tilde{K}^T C + RC^T \tilde{K} C . \quad (A.13)$$

To summarize the above results, we have that the optimal PSF gain matrix $\tilde{K}$ must satisfy

$$\text{tr}\{WGG^T\} = 0 \quad (A.14)$$

where $W = \frac{\partial \tilde{P}}{\partial \tilde{K}}$ obeys the Lyapunov equation

$$A_{cl}^T W + WA_{cl} + T = 0 \quad (A.15)$$

and where

$$A_{cl} = A + B\tilde{K}C \quad (A.16)$$

$$T = C^T B^T \tilde{P} + \tilde{P} BC + \frac{1}{2} C^T B^T + \frac{1}{2} BC + RC^T \tilde{K}^T C + RC^T \tilde{K} C . \quad (A.17)$$

Now, we introduce the following lemma to help simplify our analysis.
Lemma 1. For any positive-definite, symmetric matrices $X$ and $T$, and any asymptotically stable $A_{cl}$ we have that

$$tr\{WX\} = tr\{ST\}$$

(A.18)

where $W = W^T > 0$ and $S = S^T > 0$ obey the dual Lyapunov equations

$$A_{cl}^T W + WA_{cl} + T = 0$$

(A.19)

$$A_{cl} S + SA_{cl}^T + X = 0 .$$

(A.20)

Proof. Note that if $A_{cl}$ is asymptotically stable, then $W$ and $S$ obey

$$W = \int_0^\infty \exp [A_{cl}^t T] \exp [A_{cl}t] \, dt$$

(A.21)

$$S = \int_0^\infty \exp [A_{cl}t] X \exp [A_{cl}^t T] \, dt .$$

(A.22)

Consequently, we have that

$$tr\{WX\} = tr \left\{ \int_0^\infty \exp [A_{cl}^t T] \exp [A_{cl}t] \, dt \, X \right\}$$

(A.23)

$$= tr \left\{ \int_0^\infty \exp [A_{cl}t] X \exp [A_{cl}^t T] \, dt \, T \right\}$$

(A.24)

$$= tr\{ST\}$$

(A.25)

Thus, if we make the substitution $X \leftarrow GG^T$, we have that

$$\frac{\partial \bar{P}_{gen}}{\partial K} = -tr\{WX\}$$

(A.26)

$$= -tr \left\{ S \left[ C^T B^T \bar{P} + \bar{P}BC + \frac{1}{2} C^T B + \frac{1}{2} BC + RC^T \bar{K} + RC^T \bar{K} \right] \right\}$$

(A.27)

where $W$ obeys Equation (A.15), $S$ obeys Equation (A.20), and $P$ obeys Equation (3.36). Finally, turing through some algebra and using the commutivity property of
matrix multiplication in side of the trace, we have that Equation (A.27) is equivalent to

\[
\frac{\partial \tilde{P}_{gen}^{PSF}}{\partial \tilde{K}} = - \left( B^T \tilde{P} + \frac{1}{2} B^T + R \tilde{K} C \right) S C^T - C S \left( B^T \tilde{P} + \frac{1}{2} B^T + R \tilde{K} C \right)^T
\]  

\begin{equation}
= - 2 \left( B^T \tilde{P} + \frac{1}{2} B^T + R \tilde{K} C \right) S C^T.
\end{equation}

(A.28)

(A.29)
A.4 Proof of Theorem 4

If Equation (5.31) is unstable, this implies the existence of an eigenvalue \( \lambda > 0 \), and a corresponding eigenvector \( S_u \neq 0 \), such that

\[
\lambda S_u = A_{cl} S_u + S_u A_{cl}^T - \sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} \left[ \text{FCS} + \text{SC}^T \text{F}^T \right]. \tag{A.30}
\]

Because we assume take asymptotic stability of \( A_{cl} \) for granted, the solution \( S_u \) to the above satisfies

\[
S_u = \int_0^\infty \exp \left[ (A_{cl} - \frac{1}{2} \lambda I) t \right] \left[ -\sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} \left[ \text{FCS} + \text{SC}^T \text{F}^T \right] \right] dt \tag{A.31}
\]

\[
= -\sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} \int_0^\infty \exp [A_{cl} t] [\text{FCS} + \text{SC}^T \text{F}^T] \exp [A_{cl}^T t] e^{-\lambda t} dt \tag{A.32}
\]

\[
= -\sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} W \tag{A.33}
\]

where \( W \) is the solution to the Lyapunov equation

\[
A_{cl} W + W A_{cl}^T - \lambda W + \text{FCS} + \text{SC}^T \text{F}^T = 0. \tag{A.34}
\]

Consequently, we have that

\[
\text{CS}_u \text{C}^T = -\sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} \text{CWC}^T \tag{A.35}
\]

or, rearranging,

\[
\text{CS}_u \text{C}^T \left( 1 + \sqrt{\frac{1}{2\pi (\text{CSC}^T)^{3/2}}} \right) = 0. \tag{A.36}
\]

Through duality, this is equivalent to

\[
\text{CS}_u \text{C}^T \left( 1 + \sqrt{\frac{2}{\pi (\text{CSC}^T)^{3/2}}} \text{tr} \{\text{FC} \text{ST} \lambda\} \right) = 0 \tag{A.37}
\]
where $T_\lambda$ is the solution to

$$A_{cl}^T T_\lambda + T_\lambda A_{cl} - \lambda T_\lambda + C^T C = 0. \quad (A.38)$$

We know $T_\lambda \geq 0$ because $C^T C \geq 0$ and $A_{cl} - \frac{1}{2} \lambda I$ is asymptotically stable for any $\lambda > 0$. Furthermore, we know that $T_\lambda \to 0$ as $\lambda \to \infty$. Consequently, in Equation (A.37), the solution $CS_u C^T = 0$ is unique for all $\lambda > 0$ if it can be shown that

$$\frac{|\text{tr} \{FCST_\lambda\}|}{(CST^T)^{3/2}} < \frac{\sqrt{\pi}}{2}, \quad \forall \lambda > 0 \quad (A.39)$$

because this implies that the multiplier in the parentheses in Equation (A.37) can never be zero for $\lambda > 0$. But

$$|\text{tr} \{FCST_\lambda\}| = |CST_\lambda F| \quad (A.40)$$

$$\leq (CST_\lambda SC^T)^{1/2} (F^T T_\lambda F)^{1/2} \quad (A.41)$$

where we have used the observation that $T_\lambda \geq 0$, and the Cauchy-Schwarz inequality. Thus, a conservative criterion for the uniqueness of the $CS_u C^T = 0$ solution is

$$\frac{(CST_\lambda SC^T)^{1/2} (F^T T_\lambda F)^{1/2}}{(CST^T)^{3/2}} < \frac{\sqrt{\pi}}{2}, \quad \forall \lambda > 0 \quad (A.42)$$

Now, we observe that because $T_\lambda \geq 0$ and $A_{cl}$ is stable, it is the case that if $\lambda_1 < \lambda_2$, then $T_{\lambda_1} \geq T_{\lambda_2}$. Consequently, the above bound is most tight at $\lambda = 0$, which gives the condition shown in Equation (5.32). Assuming this condition is satisfied, the unstable eigenmode $S_u$ must have $CS_u C^T = 0$. Thus, from Equation (A.30), it must satisfy

$$[A_{cl} - \frac{1}{2} \lambda I] S_u + S_u [A_{cl} - \frac{1}{2} \lambda I]^T = 0 \quad (A.43)$$

which, because $A_{cl} - \frac{1}{2} \lambda I$ is asymptotically stable for all $\lambda > 0$, has the unique solution $S_u = 0$. But $S_u \neq 0$ in order to be an eigenmode, and thus we arrive at a contradiction. $\square$
A.5 Proof of Theorem 5

We begin by multiplying Equation (6.8) by $x^T(t)$ on the left and $x(t)$ on the right, taking the expectation, and subtracting from Equation (6.2), which gives

$$P_{gen} = -\mathcal{E}\left\{x^T \left[ A - YcBB^T \right]^T W x + x^T W \left[ A - YcBB^T \right] x \right\}$$

$$- \mathcal{E}\left\{x^TB (-Yc + Yc^2R) B^T x \right\} - \mathcal{E}\left\{\frac{1}{2}B^T xi \right\} - \mathcal{E}\left\{\frac{1}{2}x^T Bi \right\} - \mathcal{E}\left\{Ri^2 \right\}$$

(A.44)

Some rearranging and use of Equation (6.13) results in

$$P_{gen} = -\mathcal{E}\left\{ \left[ Ax + Bi \right]^T W x + x^T W \left[ Ax + Bi \right] \right\} + R\mathcal{E}\left\{ (Fx + Yc)^2 - (Fx - i)^2 \right\}$$

(A.45)

To show that the first term on the right-hand side of Equation (A.45) is equal to $-G^T W G$, we first define $\psi = x^T(t) W x(t)$. Then, adopting the Itô convention of stochastic calculus [94], we have that

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \left( \nabla_x \psi \right) (Ax(t) + Bi(t)) + \left( \nabla_x \psi \right) Gw(t) + \frac{1}{2}G^T \left( \nabla_x \otimes \nabla^T_x \psi \right) G$$

(A.46)

where $\nabla_x$ is the gradient with respect to the variable $x$, and $\nabla_x \otimes \nabla^T_x$ is the Hessian. Evaluating these for our definition of $\psi$, and taking expectations of both sides, gives

$$\mathcal{E}\left\{ \frac{d}{dt} x^T W x \right\} = \mathcal{E}\left\{ \left[ Ax + Bi \right]^T W x + x^T W \left[ Ax + Bi \right] \right\}$$

$$+ \mathcal{E}\left\{ x^T W Gw + G^T W xw \right\} + G^T W G$$

(A.47)

In stationarity the left-hand side of Equation (A.47) is 0. Furthermore, by the Itô convention, the second-to-last term on the right-hand side of Equation (A.47) is also 0 because the mapping $x(t) \mapsto i(t)$ is causal.

□
Bibliography


Biography

Ian Lerner Cassidy was born on March 6, 1986 in New York, NY. He received both his B.S.E. and M.S. degrees in civil and environmental engineering from Duke University in Durham, NC in 2008 and 2011, respectively. His Masters thesis was completed under the advisement of Prof. Jeffrey Scruggs and was entitled “Modeling and Control of an Electromechanical Transducer for Vibratory Energy Harvesting Applications.” In all, Ian has authored or co-authored five journal articles and six conference papers through the course of his Ph.D. research. In addition, he was a finalist in the Best Student Paper Competition at the 2011 ASME SMASIS Conference, a runner-up in the Best Student Paper Competition at the 2012 SPIE Smart Structures/NDE Conference, and won the 2012 Senol Utku Award for the best peer-reviewed pre-Ph.D. journal paper (awarded by Duke University). Following his Ph.D., Ian has accepted a post-doctoral research position at Duke University starting in September 2012.