Auctions, Equilibria, and Budgets

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Computer Science
in the Graduate School of Duke University
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Abstract

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We design algorithms for markets consisting of multiple items, and agents with budget constraints on the maximum amount of money they can afford to spend. This problem can be considered under two broad frameworks. (a) From the standpoint of Auction Theory, the agents’ valuation functions over the items are private knowledge. Here, a “truthful auction” computes the subset of items received by every agent and her payment, and ensures that no agent can manipulate the scheme to her advantage by misreporting her valuation function. The question is to design a truthful auction whose outcome can be computed in polynomial time. (b) A different, but equally important, question is to investigate if and when the market is in “equilibrium”, meaning that every item is assigned a price, every agent gets her utility-maximizing subset of items under the current prices, and every unallocated item is priced at zero.

First, we consider the setting of multiple heterogeneous items and present approximation algorithms for revenue optimal truthful auctions. When the items are homogeneous, we give an efficient algorithm whose outcome defines a truthful and Pareto-optimal auction. Finally, we focus on the notion of “competitive equilibrium”, which is a well known solution concept for market clearing. We present efficient algorithms for finding competitive equilibria in markets with budget constrained agents, and show that these equilibria outcomes have strong revenue guarantees.
Dedicated to my parents.
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Suppose that a seller has multiple items, and she wants to decide how to allocate these items among multiple “agents” (customers who are interested in buying the items). There are two natural ways to formulate this problem, as described below.

The first approach is to consider an “auction” setting. Here, the agents are usually termed as “bidders” and the seller is termed as “auctioneer”. The bidders report their valuations for the items to the auctioneer, and based on these reported valuations, the auctioneer decides on the allocation of the items and the payments to be made by each bidder. The critical aspect is that a bidder’s valuations for the items are “private knowledge”. Thus, the bidder is willing to manipulate the auction scheme by misreporting her valuations if that results in an outcome that is more favorable to her. A good auction should be robust to such undesirable manipulations.

The second approach is to consider the “market” defined by the set of items and the set of agents. Here, in contrast to an auction, the agents’ valuations for the items are publicly known. The problem is that the valuations of two different agents might require both of them to demand the same item. As a result, the item is over-demanded, and allocating it to any one agent will make the other agent unhappy.
The goal is to set up prices for the items in such a way that reconciles these conflicting interests of various agents and ensures that the market is in “equilibrium”, meaning that the demand equals supply for every item.

There exists a rich literature on these topics [Krishna (2002)]. However, several recent developments such as internet advertising (where the search engines generate a huge amount of revenue by auctioning online ad-slots) have motivated the researchers in Theoretical Computer Science and Artificial Intelligence to revisit the fundamental concepts in auction theory and market equilibrium from an algorithmic perspective [Nisan et al. (2007)]. Here, the problem of designing an auction (or finding a market equilibrium) is “computational”, and the goal is find auction schemes and algorithms for market equilibrium that run in polynomial time. This dissertation presents some results that fall under the purview of this general research agenda.

To be more specific, we investigate the auction (resp. market equilibrium) settings where every bidder (resp. agent) has a “budget constraint”, which specifies the maximum amount of money she can afford to spend. This constraint seems fairly natural for markets that arise in e-commerce applications: Large scale advertising exchanges, keyword search auctions [Dütting et al. (2011); Feldman et al. (2008); Goel et al. (2012)], and Google auctions for TV ads [Nisan (2009)]. The key difficulty with budgets is that the utility of a bidder is equal to her valuation minus price if and only if the price is below the budget constraint, and the utility is $-\infty$ whenever the price exceeds her budget. As a consequence, well-known results such as the VCG auction [Clarke (1971); Groves (1973a); Vickrey (1961)] are no longer directly applicable.

Roadmap for the rest of this chapter. In Section 1.1, we describe the microeconomic concepts that will be used in this thesis. In Section 1.2, we state our results: (1)
Approximately revenue maximizing auctions for heterogeneous items and budget constrained bidders, (2) Pareto-optimal auctions for homogeneous items and budget constrained bidders, and (3) Algorithms for Competitive Equilibria in the presence of budget constrained agents.

1.1 Preliminaries

We review some basic concepts in auction theory and market equilibrium.

1.1.1 Background: Auction Theory

Suppose that multiple bidders are interested in buying a set of items. Each bidder $i$ has a type $t_i \in T_i$. A bidder’s type uniquely determines her valuation function, which specifies her valuation for every subset of items. In an auction, the bidders report their types to the seller. These reported types are called bids. The input bids determine the outcome of the auction, which specifies the subset of items allocated to each bidder, and her payment.

The utility of a bidder is a function of her type and the outcome of the auction. In most settings, the utility functions are considered to be quasilinear, meaning that a bidder’s utility is simply equal to the valuation she receives from the subset of items allocated to her, minus her payment. We make the following assumptions.

1. The type $t_i$ of every bidder $i$ is private knowledge, implying that no one else apart from the bidder knows her type. However, the set of all her possible types (given by $T_i$) is publicly known.

2. The utility functions are public knowledge, that is, everyone knows the utility function of a bidder.

3. The bidders are rational agents, in the sense that each bidder behaves in a way that maximizes her own utility.
4. There are no externalities, meaning that a bidder is only interested in her own allocation and payment, and her utility does not depend on the allocations and payments of the other bidders.

Since a bidder is rational and her type is private knowledge, she is likely to lie to the seller if that results in an increased utility. In other words, her reported bid might not coincide with her private type. To rule out such undesirable behaviors, the seller wants an auction which prevents the bidders from misreporting. The fact that the utility functions are public knowledge makes it possible to design such an auction.

**Definition 1.** (Incentive Compatibility) *Each bidder’s utility is maximized (at a non-negative value) when she reports her true type.*

Throughout the rest of this dissertation, we will use the terms “incentive compatibility” and “truthfulness” interchangeably. We will focus on the important problem of designing incentive-compatible auctions that maximize either the revenue of the auctioneer, or the social welfare.

**Definition 2.** (Revenue) *The revenue from the outcome of an auction is the sum of the payments made by all the bidders.*

**Definition 3.** (Social Welfare) *The social welfare from the outcome of an auction is the sum of the valuations obtained by all the bidders.*

Suppose that our objective is to maximize revenue. First, we note that there is no truthful auction that generates good revenue on every input. Consider the following example. We have only one bidder and one item. In this setting, any deterministic truthful auction will post a price for the item that is independent of the bidder’s

---

1 Equivalently, we can define the social welfare to be the sum of the utilities of all the bidders and the auctioneer. Since the auctioneer’s utility is the total payment made by all the bidders and a bidder’s utility is equal to her valuation minus her own payment, this quantity is the same as the total valuation obtained by all the bidders.
reported type, and make a “take-it-or-leave-it” offer to the bidder. The revenue of
the auction is equal to the posted price if the bidder is willing to accept the offer,
and zero otherwise. Fix any such posted price $p > 0$, and consider an input where
the bidder’s valuation for the item is $0 < v < p$. On this input, the auction generates
zero revenue. However, a different auction that posts a price slightly less than $v$ will
give positive revenue on the same input.

A natural way to circumvent the above difficulty is to take the Bayesian approach.
Here, we assume that the private type $t_i \in T_i$ of each bidder $i$ is drawn from a publicly
known prior distribution, defined over $T_i$. The objective is to design a truthful auction
that maximizes the expected revenue, where the expectation is taken over the random
choices made by the auction, and the prior distributions from which the bidders’ types
are drawn.

Various Kinds of Incentive-Compatibility Constraints

Now, we distinguish between four kinds of incentive-compatibility constraints.

Dominant-strategy incentive-compatibility (DSIC) Fix any bidder $i$. Suppose
that her true type is $t_i$.

- In a universally truthful DSIC mechanism, the utility of bidder $i$ is maximized
  (at a nonnegative value) when she reveals her true type $t_i$, regardless of the
types reported by other bidders and the random choices (if any) made by the
mechanism.

- In a truthful-in-expectation DSIC mechanism, the expected utility of bidder $i$
  (the expectation is over the random choices made by the mechanism) is maxi-
mized (at a nonnegative value) when she reveals her true type $t_i$, regardless of
the types reported by other bidders.
**Bayesian incentive-compatibility (BIC)** Fix any bidder $i$ and suppose that her true type is $t_i$.

- In a *universally truthful BIC mechanism*, the expected utility of bidder $i$ (the expectation is over the prior distributions of the types of the other bidders) is maximized (at a nonnegative value) when she reveals her true type $t_i$, regardless of the random choices made by the mechanism.

- In a *truthful-in-expectation BIC mechanism*, the expected utility of bidder $i$ (the expectation is over the prior distributions of the types of the other bidders and the random choices made by the mechanism) is maximized (at a nonnegative value) when she reveals her true type $t_i$.

Note that if a mechanism is universally truthful DSIC, then the same mechanism satisfies all of the other three notions of incentive compatibility. On the other hand, the union of all the universally truthful DSIC, truthful-in-expectation DSIC, and universally truthful BIC mechanisms is a subset of the set of all truthful-in-expectation BIC mechanisms. Hence, among the four notions described above, universally truthful DSIC (respectively, truthful-in-expectation BIC) is the strongest (respectively, weakest) notion of incentive compatibility.

**Maximizing Revenue: Myerson’s Auction**

In a seminal paper, Myerson (1981) characterized the revenue maximizing auction for a single item. We will now summarize his main result.

There are $n$ bidders and a single item. The private type of a bidder $i$ is identified with her valuation for the item, which is denoted by $v_i$. Although the valuation of a bidder is her private knowledge, it is drawn from a publicly known distribution. To be more specific, each $v_i$ is drawn independently at random from a known distribution with cumulative density function $F_i(z) = Pr[v_i \leq z]$. If $\vec{v} = (v_1 \ldots v_i \ldots v_n)$, then
the notation $\vec{v}_{-i} = (v_1 \ldots v_{i-1} v_{i+1} \ldots v_n)$ will denote the valuation profile of all the bidders excluding bidder $i$. On a similar note, the product distribution $F_1 \times \cdots \times F_n$ will be denoted by $\vec{F}$, and the product distribution corresponding to all bidders except bidder $i$ will be denoted by $\vec{F}_{-i} = F_1 \times \cdots \times F_{i-1} \times F_{i+1} \cdots \times F_n$. The auctioneer asks the bidders to report their valuations (these reports are called bids). Based on the input bids, the auctioneer allocates the single item and the winning bidder is charged a price, perhaps in a randomized manner.

When the input bids are given by $\vec{b} = (b_1 \ldots b_i \ldots b_n)$, let $x_i(b_i, \vec{b}_{-i})$ denote the expected amount of the item obtained by bidder $i$ (or equivalently, the probability that bidder $i$ wins the item), and let $P_i(b_i, \vec{b}_{-i})$ denote her expected payment. Both these expectations are only over the random choices made by the auction. If the private valuation of bidder $i$ is $v_i$, then her expected utility is equal to $v_i x_i(b_i, \vec{b}_{-i}) - P_i(b_i, \vec{b}_{-i})$.

We overload the notation and let $x_i(z) = \mathbb{E}_{\vec{v}_{-i} \sim \vec{F}_{-i}}[x_i(z, \vec{v}_{-i})]$ (resp. $P_i(z) = \mathbb{E}_{\vec{v}_{-i} \sim \vec{F}_{-i}}[P_i(z, \vec{v}_{-i})]$) denote the expected amount of the item allocated to bidder $i$ (resp. her expected payment) when her reported valuation is $z$. Now, these expectations are over (a) the prior distributions of other bidders and (b) the random choices made by the auction.

Myerson showed that in any truthful-in-expectation BIC auction, the expected allocation to a bidder is a non-decreasing function of her bid, and surprisingly, the expected payment is completely characterized by the allocation rule.

**Theorem 1.** An auction is truthful-in-expectation BIC iff for all bidders $i$:

1. The function $x_i(z)$ is monotonically non-decreasing in $z$.
2. For all $v_i$, we have: $P_i(v_i) = v_i x_i(v_i) - \int_{z=0}^{v_i} x_i(z) dz$.

Next, Myerson introduced the notion of virtual valuation. It depends on the
bids’ private valuation and the distribution from which this valuation is drawn.

**Definition 4. (Virtual Valuation)** When bidder $i$ has valuation $v_i$ for the item, her virtual valuation is given by:

$$
\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}
$$

Here, $F_i(z)$ (resp. $f_i(z)$) is the cumulative density function (resp. probability density function) of the prior distribution corresponding to bidder $i$.

Myerson characterized the expected revenue of any truthful-in-expectation BIC auction in terms of the expected virtual valuation of the winning bidder.

**Theorem 2.** The expected revenue of a truthful-in-expectation BIC auction is:

$$
Revenue = E_{\vec{v} \sim \vec{F}} \left[ \sum_{i=1}^{n} \phi_i(v_i) x_i(\vec{v}) \right]
$$

The above theorem implies that in order to maximize her revenue, the auctioneer should try to maximize the expected virtual valuation of the winning bidder. Equivalently, the item should be allocated to the bidder with highest virtual valuation, provided that quantity is non-negative. This allocation rule will be monotone in reported valuations (and hence truthful-in-expectation BIC) if $\phi_i(v_i)$ is a monotonically non-decreasing function of $v_i$, for all bidders $i$. Distributions satisfying this property are called *regular*. The payments will be completely characterized by the allocation rule (see Theorem 1).

In later chapters, sometimes we will need to impose another restriction called *Monotone Hazard Rate* on the prior distributions of the bidders. This restriction is stronger than the regularity assumption in the sense that every Monotone Hazard Rate distribution is regular, but not vice versa.
**Definition 5.** A distribution with a cumulative density function $F_i(z)$ and probability density function $f_i(z)$ is

1. **Regular** iff $\phi_i(z)$ is a monotonically non-decreasing function of $z$.

2. **Monotone Hazard Rate** iff $\frac{1 - F_i(z)}{f_i(z)}$ is a monotonically non-increasing function of $z$.

**Myerson’s auction for regular distributions.** In this auction, the bidders report their valuations for the item. If the highest virtual valuation is negative, then no bidder gets the item; else if the highest virtual valuation is non-negative, then the bidder with the highest virtual valuation wins the item.

**Theorem 3.** When all the bidders’ valuations are drawn from regular distributions, Myerson’s auction generates the maximum expected revenue among all truthful-in-expectation BIC auctions.

In fact, we can show that Myerson’s auction is universally truthful DSIC. Thus, there is no gap between the optimal revenue of a single item universally truthful DSIC auction and that of a single item truthful-in-expectation BIC auction. In the next chapter, however, we will see that this property does not hold in more general settings.

**Maximizing Social Welfare: The VCG Auction.**

For the rest of Section 1.1.1, our focus will be on social welfare. We will describe the celebrated VCG auction [Vickrey (1961); Clarke (1971); Groves (1973b)], which is universally truthful DSIC. In contrast to revenue maximization, where the optimal auction gives maximum revenue in expectation over the prior distributions of the bidders, the VCG auction returns a social welfare maximizing allocation on every input. Furthermore, unlike Myerson’s auction (which works only for a single item),
the VCG auction works even when there are multiple items and the bidders have arbitrary valuation functions defined over the subsets of these items.\footnote{In fact, the VCG auction works for even more general settings (e.g. in the presence of externalities).} We will use the following notations.

There are \( n \) bidders and a set \( J \) of \( m \) items. Each bidder \( i \) has a private type that is identified with her valuation function \( v_i : 2^J \to \mathbb{R} \). The valuation profile of all the bidders is denoted by \( \bar{v} = (v_1, \ldots, v_n) \). Let \( \bar{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \) denote the valuation functions of all bidders except bidder \( i \). The bidders report their private valuation functions and based on this input \( \bar{v} \), the auctioneer outputs an allocation of the items \( \mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n) \) and payments \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n) \). Each bidder \( i \) is allocated all items in the set \( \mathcal{A}_i \) and she is charged a payment of \( \mathcal{P}_i \). Thus, for all \( i, j \in \{1, \ldots, n\} \), we have \( \mathcal{A}_i \cup \mathcal{A}_j \subseteq J \) and \( \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \).

The utility of bidder \( i \) from the resulting outcome is equal to \( v_i(\mathcal{A}_i) - \mathcal{P}_i \). The social welfare of the outcome is given by \( \sum_{i=1}^{n} v_i(\mathcal{A}_i) \).

Let \( SW^*(\bar{v}) \) denote the maximum possible social welfare from any allocation, when the valuation functions of the bidders are given by \( \bar{v} = (v_1, \ldots, v_n) \). Let \( \mathcal{A}^* = (\mathcal{A}_1^*, \ldots, \mathcal{A}_n^*) \) be the allocation vector that attains this maximum value. In other words, we have:

\[
SW^*(\bar{v}) = \max_{\mathcal{A}(\bar{v})} \left\{ \sum_{j=1}^{n} v_j(\mathcal{A}_j(\bar{v})) \right\} = \sum_{j=1}^{n} v_j(\mathcal{A}_j^*(\bar{v}))
\]

Similarly, let \( SW^*(\bar{v}_{-i}) \) be the maximum possible social welfare \textit{ignoring the contribution from bidder }\( i \).

\[
SW^*(\bar{v}_{-i}) = \max_{\mathcal{A}(\bar{v})} \left\{ \sum_{j \neq i} v_j(\mathcal{A}_j(\bar{v})) \right\}
\]

We now define the VCG auction [Vickrey (1961); Clarke (1971); Groves (1973b)].
Definition 6. (VCG Auction) Given the valuation functions reported by the bidders \( \vec{v} = (v_1, \ldots, v_n) \), the VCG auction computes a social welfare maximizing allocation \( A^*(\vec{v}) = (A^*_1(\vec{v}), \ldots, A^*_n(\vec{v})) \). Each bidder \( i \) gets all the items in the set \( A^*_i(\vec{v}) \), and she makes a payment \( P^*_i(\vec{v}) = SW^*(\vec{v}_{-i}) - \sum_{j \neq i} v_j(A^*_j(\vec{v})) \).

Theorem 4. The VCG auction has the following properties.

1. The payment made by each bidder is nonnegative.
2. It always outputs a social welfare maximizing allocation.
3. If a bidder reports her true type, then she gets nonnegative utility.
4. It is universally truthful DSIC.

Proof. Since \( SW^*(\vec{v}_{-i}) = \max_{A(\vec{v})} \left\{ \sum_{j \neq i} v_j(A_j(\vec{v})) \right\} \geq \sum_{j \neq i} v_j(A^*_j(\vec{v})) \), the payment \( P^*_i(\vec{v}) = SW^*(\vec{v}_{-i}) - \sum_{j \neq i} v_j(A^*_j(\vec{v})) \) made by any bidder \( i \) is always nonnegative in the VCG auction. Furthermore, by definition, the VCG auction always outputs a social welfare maximizing allocation \( A^*(\vec{v}) \). These prove the first and second parts of the theorem.

For the rest of the proof, fix the valuation functions reported by all bidders excluding bidder \( i \) and denote them by \( \vec{v}_{-i} \). Let \( v_i \) be the private valuation function of bidder \( i \). To see that the VCG auction satisfies the third property, note that if bidder \( i \) reports the truth, then her utility is given by:

\[
v_i(A^*_i(\vec{v}_{-i}, v_i)) - P^*_i(\vec{v}_{-i}, v_i) = v_i(A^*_i(\vec{v}_{-i}, v_i)) + \sum_{j \neq i} v_j(A^*_j(\vec{v}_{-i}, v_i)) - SW^*(\vec{v}_{-i})
\]

\[
= SW^*(\vec{v}_{-i}, v_i) - SW^*(\vec{v}_{-i}) \geq 0
\]

The inequality holds since maximum social welfare increases when we take into account the contribution from bidder \( i \). This proves the third part of the theorem.
Finally, we verify that the VCG auction is universally truthful DSIC. If bidder $i$ reports a valuation function $v'_i$ when her true valuation function is $v_i$, then her utility is given by $v_i(A^*_i(v_{-i}, v'_i)) - p^*_i(v_{-i}, v'_i)$, whereas the utility she receives by reporting the truth is equal to $v_i(A^*_i(v_{-i}, v_i)) - p^*_i(v_{-i}, v_i)$. We note that:

$$v_i(A^*_i(v_{-i}, v'_i)) - p^*_i(v_{-i}, v'_i) = v_i(A^*_i(v_{-i}, v'_i)) + \sum_{j \neq i} v_j(A^*_j(v_{-i}, v'_i)) - SW^*(\bar{v}_{-i})$$

$$= \sum_{j=1}^{n} v_j(A^*_j(v_{-i}, v'_i)) - SW^*(\bar{v}_{-i})$$

$$\leq SW^*(\bar{v}_{-i}, v_i) - SW^*(\bar{v}_{-i})$$

$$= v_i(A^*_i(v_{-i}, v_i)) + \sum_{j \neq i} v_j(A^*_j(v_{-i}, v_i)) - SW^*(\bar{v}_{-i})$$

$$= v_i(A^*_i(v_{-i}, v_i)) - p^*_i(v_{-i}, v_i)$$

The inequality follows from the fact that the social welfare of the allocation $A^*_i(v_{-i}, v'_i)$ is at most the maximum social welfare $SW^*(\bar{v}_{-i}, v_i)$. Thus, regardless of the valuation functions revealed by other bidders, bidder $i$ gets maximum utility when she reports the truth. 

\[\square\]

1.1.2 Background: Market Equilibrium

A market consists of a set of agents and a set of items. In contrast to Auction Theory, here we assume that the agents’ valuation functions are public knowledge. Intuitively, the market is in equilibrium when it sets up the item-prices in a way which ensures that the demand for every item is equal to its supply. The study of this topic dates back to the nineteenth century, and was pioneered by Walras (1874) and Fisher (1891). Two commonly studied concepts of market equilibrium are “Competitive Equilibrium” and “The Fisher Model” - these are described below.
Competitive Equilibrium with Budget Constrained Agents

There are a set of agents $\mathcal{I}$ and a set of indivisible items $\mathcal{J}$. Agent $i \in \mathcal{I}$ has a valuation $v_i(T)$ for every bundle of items $T \subseteq \mathcal{J}$. Furthermore, this agent has a budget $B_i > 0$ on the price she is willing to pay. The goal is to assign a price $P(j)$ to each item $j \in \mathcal{J}$ and allocate the items among the agents. Under price vector $P$, the utility of an agent $i \in \mathcal{I}$ from a subset of items $T \subseteq \mathcal{J}$ is given by:

$$
u_i(T, P) = \begin{cases} v_i(T) - \sum_{j \in T} P(j) & \text{if } \sum_{j \in T} P(j) \leq B_i; \\ -\infty & \text{otherwise.} \end{cases} (1.1)$$

Thus, an agent’s utility is negative infinity if the total payment exceeds her budget. On the other hand, if the total payment is at most her budget, then her utility is equal to her valuation minus price.

Let an allocation vector be denoted by $A$, where the subset of items obtained by agent $i \in \mathcal{I}$ is $A(i) \subseteq \mathcal{J}$. Since an item can be allocated to at most one agent, we have $A(i) \cap A(i') = \emptyset$ for all $i \neq i'$. Define the demand set $\Gamma_i(P)$ of an agent $i \in \mathcal{I}$ to be a (possibly empty) subset of items that gives her maximum utility, i.e., $\Gamma_i(P) \in \arg\max_{J \subseteq \mathcal{J}} u_i(J, P)$.

**Definition 7.** An allocation vector $A$ and a price vector $P$ constitute an envy-free item-pricing if and only if every agent gets her utility maximizing subset of items:

$$u_i(A(i), P) \geq u_i(\Gamma_i(P), P) \quad \text{for all agents } i \in \mathcal{I}.$$  

**Definition 8.** An allocation vector $A$ and a price vector $P$ are in a competitive equilibrium if and only if

1. The pair $(A, P)$ constitute an envy-free item-pricing (see Definition 7); and

2. Every unallocated item has zero price:

$$P(j) = 0 \quad \text{for all items } j \in \mathcal{J} \setminus \bigcup_{i \in \mathcal{I}} A(i).$$
Remark. Although the above definitions assume that the items are indivisible, exactly analogous definitions can be derived for divisible items.

The special case of this setting where the agents do not have budget constraints (i.e., $B_i = \infty$ for all $i \in I$) has been widely studied. For example, in the absence of budget constraints, the First Welfare Theorem (see Bikhchandani and Ostroy (2002) and Theorem 11.13 in Blumrosen and Nisan (2007)) states that if there exists a competitive equilibrium, then the corresponding allocation maximizes the social welfare of the market.

It is also well known [Gul and Stacchetti (1999)] that a competitive equilibrium is guaranteed to exist under Gross Substitutability, which means that the demand for an item can never drop if we increase the prices of the other items.

**Definition 9.** Consider any two price vectors, one dominated by the other, so that we have $P'_I(j) \geq P_I(j)$ for all items $j \in J$. Furthermore, suppose that $P'_I(j^*) = P_I(j^*)$ for some item $j^* \in J$, and agent $i \in I$ demands the item $j^*$ at price vector $P_I$, i.e. $j^* \in D_i(P_I)$. If the demand function of agent $i$ satisfies gross substitutability, then we must have $j^* \in D_i(P'_I)$.

Under gross substitutability, a simple ascending auction converges to a competitive equilibrium: First, initialize the price of every item at zero. Next, whenever two different agents demand the same item, keep increasing its price until the item gets dropped from one of the agents’ demand-sets.

Unfortunately, if the agents’ demand functions are not gross substitutes, then there are instances of markets without any competitive equilibrium. In such settings, it is natural to consider the relaxed notions of approximate-competitive-equilibrium and almost-envy-free item-pricing, which are defined below.

**Definition 10.** An allocation vector $A$ and a price vector $P$ constitute a $\beta$ approximate-competitive equilibrium ($0 < \beta \leq 1$) if and only if
• For all items \( j \in \mathcal{J} \setminus \bigcup_{i \in \mathcal{I}} A(i) \), we have: \( P(j) = 0 \).

• Furthermore, for all agents \( i \in \mathcal{I} \), we have: \( u_i(A(i), P) \geq \beta \times u_i(\Gamma_i(P), P) \).

Hence, in a \( \beta \) approximate-competitive equilibrium, every unallocated item has zero price, and the utility of every agent is at least \( \beta \) times her maximum possible utility (under the current setting of prices).

**Definition 11.** An allocation vector \( A \) and a price vector \( P \) constitute an almost-envy-free item-pricing if and only if for all agents \( i \in \mathcal{I} \), we have:

\[
  u_i(A(i), P) \geq u_i(\Gamma_i(P), P) - \max_{j \in \mathcal{J}} \{ u_i(\{j\}, P) \}.
\]

We can compare the notions of envy-free item-pricing and almost-envy-free item-pricing as follows. An envy-free item-pricing ensures that the utility received by any agent \( i \in \mathcal{I} \) is \( u_i(\Gamma_i(P), P) \), where \( \Gamma_i(P) \) is her demand-set. The same agent’s utility in an almost-envy-free item-pricing can drop by at most \( u_i(\{j^*\}, P) \), where \( j^* \) is the item that gives her the most utility under the current setting of prices.

*The Fisher Model for Market Equilibrium*

The *Fisher model* [Fisher (1891); Vazirani (2007)] assumes that the items are divisible, the agents have budgets, and they are oblivious to the supplies of the items in the market. Given a setting of item-prices, an agent demands an allocation that maximizes her utility, subject to the only constraint that the total payment is at most her budget. In particular, the demand of a single agent for an item can exceed its supply.\(^3\) The market is in *Fisher equilibrium* if and only if no item is over-demanded, every agent gets her utility-maximizing allocation, and every unallocated item has zero price.

\(^3\) Note that this can never happen in the model described in Section 4.1, where the agents are aware of the supplies.
In the *standard Fisher model*, an agent’s utility is simply equal to the valuation she obtains. Thus, an agent derives no utility from her residual money after the market clears. In contrast, the *quasilinear-Fisher model* assumes that an agent’s utility is equal to her valuation minus payment.

Next, we highlight two crucial differences between the quasilinear-Fisher model and competitive equilibrium with budget limits.

For divisible items and budget constrained agents with additive valuations:

- The demand function satisfies gross substitutability in the quasilinear-Fisher model, whereas the demand functions in competitive equilibria do not.

- The equilibrium pricing is unique in the quasilinear-Fisher model, but it is straightforward to construct instances where this property does not hold for competitive equilibria.

Consider the following illustrative example. There are two divisible items, and three agents with additive valuations and budget constraints. The valuations and budgets of the agents are described in Table 1.1.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Valuation for Item 1</th>
<th>Valuation for Item 2</th>
<th>Budget</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>v_1(1) = 2</td>
<td>v_1(2) = 0</td>
<td>B_1 = 2</td>
</tr>
<tr>
<td>2</td>
<td>v_2(1) = 12</td>
<td>v_2(2) = 12</td>
<td>B_2 = 20</td>
</tr>
<tr>
<td>3</td>
<td>v_3(1) = 0</td>
<td>v_3(2) = 6</td>
<td>B_3 = 6</td>
</tr>
</tbody>
</table>

If we set the item-prices to be \( P(1) = 2 \), \( P(2) = 6 \), and allocate both the items to agent 2, then we get a competitive equilibrium. However, note that this is *not* a quasilinear-Fisher equilibrium, the reason being that the utility of agent 2 per unit price from item 1 (resp. item 2) is \((12 - 2)/2 = 5\) (resp. \((12 - 6)/6 = 1\)). In the quasilinear-Fisher model, this agent is oblivious to the item-supplies. Hence, she
wants to exhaust her whole budget on the most desirable item. As a result, her demand for item 1 is equal to \( B_2 / P(1) = 10 \), but only one copy of item 1 is available.

On the other hand, the above example admits a unique quasilinear-Fisher equilibrium: Set the prices at \( P(1) = P(2) = 10 \), and allocate both the items to agent 2.

Despite these differences, it is easy to observe that a quasilinear-Fisher equilibrium is always a competitive equilibrium for divisible items – we call it a *Fisher-type competitive equilibrium*.

The Eisenberg-Gale Convex Program. In the standard-Fisher model with additive valuations, the equilibrium outcome is captured by an elegant convex program due to Eisenberg and Gale (1959). This Eisenberg-Gale convex program maximizes the objective \( \sum_i B_i \log(t_i) \). Here, the symbol \( B_i \) stands for the budget of agent \( i \), the variable \( x_{ij} \) denotes the fraction of item \( j \) allocated to agent \( i \), and \( t_i = \sum_j v_i(j) x_{ij} \) denotes her total valuation. For every item \( j \), there is a supply constraint \( \sum_i x_{ij} \leq 1 \), which ensures that the allocation is feasible. The price of item \( j \) comes from the dual variable associated with this supply constraint. The KKT conditions guarantee that the optimal dual prices and primal allocation are in a standard-Fisher equilibrium. Alternatively, Devanur et al. (2008) present a combinatorial primal-dual approach for solving this problem.

The Eisenberg-Gale convex program was subsequently generalized to several other valuation functions, such as *homothetic* and *quasi-concave* [Jain et al. (2005)], and *spending constraints* [Birnbaum et al. (2011); Devanur (2004); Vazirani (2010)]. The latter model captures the quasilinear-Fisher model with additive valuations as a special case.
1.2 Roadmap for The Rest of This Dissertation

The main results in this dissertation can be summarized as follows. Chapter 2 considers the problem of designing revenue-optimal truthful auctions when there are heterogeneous items and budget constrained bidders (see Section 1.2.1). Chapter 3 considers the setting when there are multiple homogenous items and bidders have budget limits: Here, the goal is to design a truthful auction that is also Pareto-optimal, meaning that after the outcome of the auction has been implemented, no two agents (including the auctioneer) want to trade an item among themselves (see Section 1.2.2). Finally, in Chapter 4, we present algorithms for finding competitive equilibria in markets consisting of budget constrained agents (see Section 1.2.3).

1.2.1 Revenue-Optimal Auctions for Budget Constrained Bidders

In Chapter 2, we present approximation algorithms for designing revenue optimal incentive compatible auctions in the following setting. There are multiple heterogeneous items, and bidders have arbitrary demand and budget constraints (and additive valuations). Furthermore, the type of a bidder (which specifies her valuation for each item) is private knowledge, and the types of different bidders are drawn from publicly known mutually independent distributions. Thus, the bidders have multidimensional private types.

Our Results

First, we assume that the type of each bidder is drawn from a discrete distribution with polynomially bounded (in terms of the number of items and the number of bidders) support size. This restriction on the type distribution, however, allows the random variables corresponding to a bidder’s valuations for different items to be arbitrarily correlated. In this model, we describe a sequential all-pay auction that is truthful in expectation Bayesian incentive compatible. The outcome of our all-pay
auction can be computed in polynomial time, and its revenue is a 4-approximation to the revenue of the optimal truthful in expectation Bayesian incentive compatible auction.

Next, we assume that the valuations of each bidder for different items are drawn from mutually independent discrete distributions satisfying the monotone hazard rate condition. In this model, we present a sequential posted price auction that is universally truthful and incentive compatible in dominant strategies. The outcome of the auction is computable in polynomial time, and its revenue is a $O(1)$-approximation to the revenue of the optimal truthful in expectation Bayesian incentive compatible auction. If the monotone hazard rate condition is removed, then we show a logarithmic approximation, and we complete the picture by proving that no sequential posted price scheme can achieve a sub-logarithmic approximation. Finally, if the distributions are regular, and if the space of auctions is restricted to sequential posted price schemes, then we show that there is a $O(1)$-approximation within this space.

Our results are based on formulating novel LP relaxations for these problems, and developing generic rounding schemes from first principles.

Previous Work

The topic of designing incentive-compatible revenue maximizing auctions - also known as “Bayesian Mechanism Design” - has been extensively studied in the economics literature [Benoit and Krishna (2001); Brusco and Lopomo (2008); Che and Gale (1996, 2000); Laffont and Robert (1996); Manellia and Vincent (2007); Pai and Vohra (2008); Thanassoulis (2004); Wilson (1997)]. It is easy to see that the revenue-optimal auction can always be computed by encoding the incentive-compatibility constraints in an integer program and maximizing the expected revenue. However, the number of variables (and constraints) in this integer program (or linear program, when randomization is allowed) is exponential in the number of bidders, as there are
variables for the allocations and prices for each scenario of revealed types. Therefore, the key difficulty in Bayesian Mechanism Design is *computational* [Conitzer and Sandholm (2004)]: Can the optimal (or approximately optimal) mechanism be efficiently computed and implemented?

Much of the literature in economics considers the case where the auctioneer has one item (or multiple copies of one item). In the absence of budget constraints, Myerson (1981) presents the characterization of any truthful-in-expectation BIC auction in terms of expected allocation made to a bidder: This allocation must be monotone in the revealed valuation of the bidder (see Section 1.1.1). This yields a linear-time computable optimal revenue-maximizing mechanism. The key issue with budget constraints is that the allocations need to be thresholded in order for the prices to be below the budgets [Che and Gale (1996); Laffont and Robert (1996); Pai and Vohra (2008)]. However, even in this case, the optimal truthful-in-expectation BIC auction follows from a polymatroid characterization that can be solved by the Ellipsoid algorithm and an all-pay condition [Border (2007); Pai and Vohra (2008)]. By *all-pay*, we mean that the bidder pays a fixed amount given his revealed type, regardless of the allocation made. This also yields a universally truthful DSIC mechanism that is an $O(1)$-approximation to the optimal truthful-in-expectation BIC revenue, but the result holds only for homogeneous items.

Investigating auction settings where the bidders have multidimensional types is an important open problem. Towards this end, several researchers have considered restricted scenarios. The most common restriction is on the space of auctions; examples include auctions that are sequential by item and second price within each item [Benoit and Krishna (2001); Elkind and Fatima (2007)], and ascending price auctions [Ausubel and Milgrom (2002); Brusco and Lopomo (2008)]. The goal here is to analyze the improvement in revenue by optimal sequencing, or to study incentive compatibility of commonly used ascending price auctions. However, analyzing
the performance of sequential or ascending price auctions is difficult in general, and there is little known in terms of optimal auctions (or even approximately optimal auctions) in these models.

1.2.2 Pareto-Optimal Auction for Budget Constrained Bidders

In Chapter 3, we consider the problem of designing incentive compatible auctions for multiple homogeneous units of a good. This problem has received significant attention, starting with the work of Goldberg et al. (2001). We focus on the scenario where bidders not only have a private valuation per unit of the good, but also a private budget. The budget constraint is hard; a bidder gets a utility of negative infinity if she has to pay a total price larger than her budget. In this model, no truthful auction can generate good social welfare on every input [Borgs et al. (2005)]. Therefore, the natural problems to consider are designing truthful auctions with (1) large revenue and (2) “Pareto-Optimality”, meaning that no pair of agents (including the auctioneer) can simultaneously improve their utilities by trading with each other. Both these aspects have been considered in previous work [Abrams (2006); Borgs et al. (2005); Dobzinski et al. (2008)].

Previous Work

Based on the random partitioning framework of Goldberg et al. (2001), Borgs et al. (2005) present a truthful auction whose revenue is asymptotically optimal compared to that of the optimal single-price mechanism. Using the same framework, Abrams (2006) gives a different auction that improves this result for a range of parameters (but is not asymptotically optimal).

More recently, Dobzinski et al. (2008) presented the adaptive clinching auction based on the clinching auction of Ausubel (2004). This is an ascending price auction where each bidder maintains a demand, which is the amount of the item she is willing
to buy given the current price and her residual budget. Initially, the total demand of all the bidders is larger than supply. Now, fix any bidder $i$. If the total demand of all the bidders except bidder $i$ is less than the supply of items, then bidder $i$ clinches the difference at the current price. The bidder drops out of the auction if the price exceeds her valuation, and the auction stops when the total demand falls below total supply. This auction defines a differential process for an infinitely divisible good with no closed form solution, except in special cases.

It is not difficult to show that this auction is universally truthful DSIC when the budget constraints are public knowledge. Dobzinski et al. (2008) show that in the public budget setting, it is the only such auction that is Pareto-optimal (PO). They further show that with public budgets, this auction has better revenue properties than the auctions proposed by Abrams (2006) and Borgs et al. (2005). It improves the former by a factor of 4, and like the latter, is asymptotically optimal. However, their main result is negative: This auction is not truthful when the budgets are private knowledge, so that there is no Pareto-optimal and universally truthful DSIC auction in this case.

Our Results

The negative result in Dobzinski et al. (2008) holds when the bidders have private valuations and private budgets, and the auction needs to satisfy two properties:

- Ex-post Pareto-Optimality, meaning that regardless of the random choices made by the auction, its outcome is Pareto-Optimal.

- Universally Truthful DSIC (see Section 1.1.1).

In contrast, the main result in Chapter 3 is to show that for bidders with private valuations and private budgets, there is a mechanism that is (1) Ex-post Pareto-
Optimal and (2) Truthful-in-expectation DSIC. The key to showing this result is to develop a novel structural characterization of the adaptive clinching auction in the case of one infinitely divisible good.

1.2.3 Competitive Equilibria with Budget Limits

In Chapter 4, we consider the problem of finding a competitive equilibrium when the agents have budget constraints and the items are indivisible. In our model, the utility of an agent is equal to her valuation minus payment when the payment is no more than her budget; otherwise her utility is negative infinity. In a competitive equilibrium, every item is assigned a non-negative price, and the items are allocated to the agents in such a way that every agent gets her utility-maximizing bundle, and every unallocated item has price zero (see Section 1.1.2).

Previous work [Aggarwal et al. (2009); Ashlagi et al. (2009); Chen et al. (2010)] on competitive equilibria with budgets considers unit-demand agents. Such an agent wants to buy at most one item subject to her budget limit. It is easy to check that this setting satisfies gross substitutability (see Definition 9). Hence, modulo tie-breaking, a competitive equilibrium can be found by extending the ascending auction framework of Demange, Gale and Sotomayor (DGS) [Demange et al. (1986)].

Our focus is to study the solution concept of competitive equilibrium when the agents have budget limits and their demands are not gross substitutes.

Investigating the landscape beyond gross substitutes is challenging, primarily because of two reasons. First, we can no longer hope to apply DGS type approaches. Worse still, even the existence of a competitive equilibrium is not guaranteed. We ask two questions.

1. For general classes of valuation functions, are there efficient algorithms that

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4 In the case of one infinitely divisible good, the auction also needs to be anonymous for the negative result in Dobzinski et al. (2008) to hold, meaning that the auction is symmetric for bidders with identical types; our mechanism can also easily be made anonymous.
always find some competitive equilibrium?

2. What are the revenue properties of the competitive equilibria returned by those algorithms?

As described below, our work mainly deals with two types of valuation functions - additive and concave-combinatorial.

- If an agent \( i \) has additive valuation function, then her total valuation from any subset of items \( S \subseteq J \) is simply equal to \( \sum_{j \in S} v_i(\{j\}) \).

- If an agent \( i \) has concave combinatorial valuation function, then she is interested in a specific subset of items \( S_i \subseteq J \); and she has zero valuation for every other item \( j \in J \setminus S_i \). Furthermore, her marginal valuation within the subset \( S_i \) is non-increasing in total allocation, and depends only on the number of items she wins in \( S_i \).

Remark. It is easy to see that if a budget constrained agent has an additive or concave combinatorial valuation function, then her demand function does not satisfy gross substitutability.

First, we consider markets where the agents are budget constrained and have additive valuation functions. We show that it is NP-hard to decide whether such a market admits a competitive equilibrium. To circumvent this computational hardness, we develop connections to the Fisher model of market clearing (see Section 1.1.2). Specifically, we show that the Fisher-type competitive equilibria have certain nice properties, for they pretend that the agents are oblivious to the supplies of the items. We crucially use these properties for designing a polynomial time algorithm for approximate-competitive equilibrium (see Definition 10). We note that a simple modification to our algorithm makes it return an almost-envy-free item-pricing (see
Definition 11). Though the basic template seems simple, we require several new ideas to handle indivisible items and quasi-linearity. Along the way, we present a new interpretation of the celebrated Eisenberg-Gale convex program.

Next, we consider markets where the agents have budget limits and concave combinatorial valuation functions. Here, we prove that a competitive equilibrium is guaranteed to exist for almost all input instances, and present a polynomial time ascending auction based algorithm for computing such an equilibrium. We also show strong revenue guarantees for our algorithms.
Revenue-Optimal Auctions for Budget Constrained Bidders

The chapter is based on the paper by Bhattacharya et al. (2010a). We present approximation algorithms for computing revenue-optimal incentive-compatible mechanisms. We consider a setting where the bidders have multidimensional private types and budget constraints.

2.1 Our model

There are $n$ bidders and $m$ heterogeneous items. The type of a bidder is an $m$-tuple representing her valuations for each item. The bidders’ types are private knowledge, but they are drawn from mutually independent publicly known prior-distributions. Furthermore, every bidder $i$ has two publicly known constraints: A demand constraint $d_i$ on the maximum number of items she is willing to buy, and a budget constraint $B_i$ on the maximum total price she can afford to pay. In a mechanism, the bidders report their types to the auctioneer, and based on the reported types, the auctioneer computes an allocation of the items and payments. The utility of bidder $i$ is defined as follows: Suppose that she gets a subset $A$ of items where $|A| \leq d_i$, and
pays a total price $P_i$. Let $v_{ij}$ denote the valuation of bidder $i$ for item $j$. If $P_i \leq B_i$, then the utility of bidder $i$ is equal to $\sum_{j \in A} v_{ij} - P_i$. In contrast, if $P_i > B_i$, then her utility is $-\infty$. The revenue of the auctioneer is given by $\sum_i P_i$. The mechanism should be incentive-compatible in that no bidder gains in utility by misreporting her type. For a discussion on various notions of incentive-compatibility, see Section 1.1.1.

A standard assumption in Economics is that a bidder’s valuation for an item is drawn from a distribution satisfying some useful properties. In this chapter, we are interested in the following classes of distributions.

**Definition 12.** Suppose that the valuation of a bidder $i$ for item $j$ is a discrete random variable $v_{ij}$ with integral support $\{1, \ldots, L_{ij}\}$. The distribution of $v_{ij}$ is regular if and only if

$$r - \frac{\Pr[v_{ij} > r]}{\Pr[v_{ij} = r]}$$

is a non-decreasing function of $r \in \{1, \ldots, L_{ij}\}$.

**Definition 13.** Suppose that the valuation of a bidder $i$ for item $j$ is a discrete random variable $v_{ij}$ with integral support $\{1, \ldots, L_{ij}\}$. The distribution of $v_{ij}$ is monotone hazard rate if and only if

$$\frac{\Pr[v_{ij} > r]}{\Pr[v_{ij} = r]}$$

is a non-increasing function of $r \in \{1, \ldots, L_{ij}\}$.

The above notions were described in Chapter 1 (see Definition 5) in the context of continuous distributions. Here, we consider distributions having discrete supports.

If a distribution is monotone hazard rate, then it is regular. Examples of monotone hazard-rate distributions include geometric distributions and uniform distributions. In contrast, if we have $\Pr[v_{ij} \geq r] = 1/r$ for $r = 1, 2, \ldots, L_{ij}$, then the distribution of $v_{ij}$ is regular but not monotone hazard rate. Finally, if we have a
bimodal distribution, such as $\Pr[v_{ij} = 1] = \Pr[v_{ij} = 3] = 4/9$ and $\Pr[v_{ij} = 2] = 1/9$, then it is easy to check that the random variable $v_{ij}$ does not follow a regular distribution.

2.2 Roadmap for the rest of the chapter

Recall that the type of a bidder is an $m$-tuple representing her valuations for each item, and the types of different bidders follow mutually independent public distributions. From the type-distribution of a bidder $i$, we can determine the distribution of the random variable $v_{ij}$, which denotes the valuation of bidder $i$ for item $j$. Our results depend on whether the random variables $\{v_{ij}\}_j$ (denoting the valuations of the same bidder for different items) are correlated or mutually independent. All our results can be viewed as presenting simple characterizations of approximately revenue-optimal mechanisms in these contexts.

2.2.1 Our result in Section 2.3

We consider the following scenario in Section 2.3: For every bidder $i$, the random variables $\{v_{ij}\}_j$ (denoting the valuations of bidder $i$ for different items) can be arbitrarily correlated. However, the type of bidder $i$ is drawn from a discrete distribution with polynomially bounded support size.

*Truthful-in-expectation BIC mechanism.* We present a simple *all-pay* mechanism whose outcome is computable in time polynomial in the input size. The mechanism charges each bidder a fixed price that depends only on her revealed type, while the allocation made to the bidder depends on the reported types of other bidders and the random choices made by the mechanism. The resulting scheme is truthful-in-expectation BIC, and we show that its revenue is a 4-approximation to the revenue of the optimal truthful-in-expectation BIC mechanism (see Theorem 6).
Inapproximability of the optimal universally truthful DSIC mechanism. An all-pay mechanism is unrealistic in several situations, since a bidder is forced to participate even if she obtains a negative utility when the mechanism concludes. A natural question to ask is whether we can compute a universally truthful DSIC mechanism with good revenue properties. We can show that this problem generalizes the problem of unlimited supply unit-demand profit-maximizing envy-free pricing [Guruswami et al. (2005)], as described below.

Consider a single bidder and \( m \) items, and suppose that the bidder’s type is drawn from a uniform distribution over the discrete set \( T = \{ t^{(1)}, \ldots, t^{(n)} \} \) of size \( n \), and that the bidder has unit demand and infinite budget. In this setting, the optimal universally truthful DSIC mechanism will post a price for each item, and it will allow the bidder to pick the item that gives her maximum (nonnegative) utility. This is exactly equivalent to the following instance of the unlimited supply unit-demand envy-free pricing problem [Guruswami et al. (2005)]: We have \( m \) items and \( n \) agents, and the valuation profile of each agent \( i \in \{ 1, \ldots, n \} \) (which gives her valuations for each item) is specified by the type \( t^{(i)} \).

For the unlimited supply unit-demand envy-free pricing problem, the best known polynomial-time algorithm gives a logarithmic approximation, and there is strong evidence that a better polynomial-time approximation is not possible [Briest (2008)]. Consequently, it is highly unlikely that we will be able to design a polynomial-time computable and universally truthful DSIC mechanism with good revenue properties. In Section 2.4, we impose further restrictions on the type-distributions to circumvent this negative result.

2.2.2 Our results in Section 2.4

We make the following assumption throughout Section 2.4: For every bidder \( i \), the random variables \( \{ v_{ij} \}_j \) (denoting the valuations of bidder \( i \) for different items) are
mutually independent. However, in contrast to Section 2.3, we no longer require that the type-distribution of a bidder should have polynomial support.

The assumption mentioned above has been used previously in the literature. The work of Chawla et al. (2007) considers the special case of a single bidder with unit-demand and infinite budget, whose valuations for different items are drawn from mutually independent public distributions. For this problem, Chawla et al. give a constant factor approximation to the revenue maximizing universally truthful DSIC mechanism, by deriving an elegant connection to Myerson’s mechanism. Independently of our work, Chawla et al. (2010) extend the previous result to a setting with multiple bidders. However, this result also crucially requires the unit-demand assumption.

Monotone hazard rates and universally truthful DSIC mechanism. In Section 2.4.2, we further assume that for every bidder $i$ and item $j$, the random variable $v_{ij}$ (denoting the valuation of bidder $i$ for item $j$) is drawn from a distribution that satisfies the monotone hazard-rate (see Definition 13) condition. We give a universally truthful DSIC mechanism, whose outcome is computable in polynomial time, and whose revenue is a constant-factor approximation to the revenue of the optimal truthful-in-expectation BIC mechanism. Our mechanism is a Sequential Posted Price scheme. Any sequential posted-price scheme has the following simple structure: The auctioneer considers the bidders sequentially in arbitrary order, and each bidder is offered a subset of the available items, so that each item in the subset has to be purchased at a pre-computed price, and the bidder herself picks the items she wants to buy under these prices. Hence, we get a constant-factor gap between the revenues of the optimal truthful-in-expectation BIC mechanism and the optimal universally truthful DSIC mechanism (see Theorem 17), which is in sharp contrast with the corresponding negative result [Briest et al. (2010)] when the valuations of a bidder for different
items are drawn from correlated distributions.

Regular distributions and adaptive posted-price mechanisms. In Section 2.4.3, we show that the monotone hazard-rate condition is indeed necessary if we want to design a sequential posted-price mechanism with good revenue properties. Suppose that the monotone hazard-rate condition is slightly relaxed, and we consider the scenario where there is a single bidder and her valuations for different items are drawn from mutually independent regular distributions (see Definition 12). In this case, the optimal universally truthful DSIC mechanism will have a logarithmic gap against the revenue of the optimal sequential posted-price scheme. We prove that this gap is tight by showing the existence of a sequential posted-price mechanism achieving this approximation ratio (see Theorem 19). On a positive note, we prove that for regular distributions, if the space of feasible mechanisms is restricted to those that consider the bidders in some adaptive order and post prices for the items that may depend on the outcomes so far, then there is a $O(1)$-approximation within this space that considers the bidders in an arbitrary but fixed order, and pre-computes the posted prices (see Theorem 24).

2.2.3 Our techniques

If, for every bidder, the valuations for different items are drawn from correlated distributions (see Section 2.3), then the optimal truthful-in-expectation BIC revenue can be bounded from above by a linear program (LP1) that requires the incentive-compatibility, individual-rationality, supply, and demand constraints to hold only in expectation. We construct a truthful-in-expectation BIC all-pay mechanism (see Figure 2.1) that basically implements a rounding scheme on the optimal solution to LP1, losing a constant factor in revenue (see Theorem 6).

As mentioned before, Chawla et al. (2007) consider the Bayesian unit-demand
pricing problem. There are \( m \) heterogeneous items, a single bidder with unit demand, and her valuations \( (v_j \text{ for item } j \in [1, \ldots, m]) \) are drawn from independent distributions. They present an elegant pricing scheme that is a constant approximation to the optimal revenue by upper bounding it using the revenue of Myerson's mechanism in the following setting: There is a single item, \( m \) bidders, and the valuation of each bidder \( j \) follows the same distribution as that of \( v_j \). However, this technique cannot be applied if the unit demand assumption is removed.

In contrast, our approach in Section 2.4 (where the valuations of a bidder for different items are drawn from mutually independent distributions) does not require the unit-demand assumption, and is based on a novel LP relaxation (LPRev) for the problem (see Lemma 8). Unlike the LP relaxation of Section 2.3, and perhaps surprisingly, LPRev does not encode any incentive-compatibility constraints, and our universally truthful DSIC mechanism (see Figure 2.2), which competes against this LP, is in fact a constant approximation to the optimal truthful-in-expectation BIC revenue. One limitation of our approach is that we have to (necessarily) assume that the bidders’ valuations are drawn from montone hazard rate distributions (see Definition 13). In the process of proving our main result (see Theorem 17), we describe a crucial property of monotone hazard rate distributions (see Lemma 13) that can be used to extend the type of results shown by Hartline and Roughgarden (2009). For example, in multi-item settings with only demand constraints, posted-price schemes generate revenue that is a constant factor of the optimal social welfare, assuming monotone hazard rate distributions (see Corollary 18). The LP formulations also generalize the stochastic matching setting in Chen et al. (2009).

2.3 Truthful-in-expectation BIC mechanism

In this section, we consider the problem of approximating the optimal truthful-in-expectation Bayesian incentive-compatible mechanism. We show an all-pay mecha-
nism that is a 4-approximation to optimal revenue.

2.3.1 Notations

There is a set $\mathcal{I}$ of $n$ bidders, and a set $\mathcal{J}$ of $m$ indivisible items. A type $t$ is an $m$-tuple $\langle t(1), t(2), \ldots, t(m) \rangle$. If a bidder $i \in \mathcal{I}$ has type $t_i$, then her valuation for item $j \in \mathcal{J}$ is given by $v_{ij} = t_i(j)$. Every bidder $i \in \mathcal{I}$ has a demand $d_i \geq 1$, which upper bounds the number of items that can be allocated to her, and a budget $B_i$, which specifies the maximum total payment she can make. Both her demand and budget are publicly known. Suppose that she gets a subset $A \subseteq \mathcal{J}$ of items, where $|A| \leq d_i$, and is charged a price of $P$. Her utility $u_i(A, P)$ is given by the following expression.

$$u_i(A, P) = \begin{cases} -\infty & \text{if } P > B_i, \\ \sum_{j \in A} v_{ij} - P & \text{if } P \leq B_i. \end{cases}$$

The type of a bidder $i \in \mathcal{I}$ is private knowledge. Furthermore, it is drawn from a discrete probability distribution $f_i(\cdot)$ with support $\mathcal{T}_i \subseteq \mathbb{R}^m$. For all $t_i \in \mathcal{T}_i$, we have $f_i(t_i) = \Pr[\text{type of bidder } i = t_i]$, and $\sum_{t_i \in \mathcal{T}_i} f_i(t_i) = 1$. The distributions $f_1(\cdot), \ldots, f_n(\cdot)$ are mutually independent and publicly known. The notation $f_{ij}(\cdot)$ denotes the marginal distribution of the valuation of bidder $i \in \mathcal{I}$ for item $j \in \mathcal{J}$:

$$f_{ij}(v) = \Pr[v_{ij} = v] = \sum_{t_i \in \mathcal{T}_i : t_i(j) = v} f_i(t_i).$$

The distribution $f_{ij}(\cdot)$ has support $\mathcal{T}_{ij}$, that is, $\mathcal{T}_{ij} = \{v \in \mathbb{R} : f_{ij}(v) > 0\}$. Note that a bidder’s valuations for different items can be correlated. Specifically, we may have $\Pr[v_{ij} = v \mid v_{ij'} = v'] \neq f_{ij}(v)$ for some bidder $i$, items $j \neq j'$, and $v \in \mathcal{T}_{ij}, v' \in \mathcal{T}_{ij'}$.

In a mechanism, the bidders first report their types. Based on these reported types, the auctioneer computes an allocation of items and payments. The price charged to a bidder $i$ should not exceed her budget $B_i$, and the number of items allocated to bidder $i$ should be at most $d_i$. 

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2.3.2 The problem and the LP-relaxation

We want to find a mechanism that is incentive compatible (and individually rational) in the following sense: Fix any bidder \(i\) and suppose that her (private) type is \(t_i\). Her expected utility, where the expectation is over the distributions of types of other bidders and the random choices made by the mechanism, is maximized (at a nonnegative value) if she reveals her true type \(t_i\). We are interested in a mechanism that (approximately) maximizes the expected revenue, and can be computed in time polynomial in the input size, i.e., in \(n, m\), and \(\max_{i \in I}|T_i|\). Throughout the rest of Section 2.3, we will make the following assumption.

**Assumption 5.** The distributions \(f_1(\cdot), \ldots, f_n(\cdot)\) have polynomial supports.

**Linear programming relaxation** For any feasible mechanism, let \(x_{ij}(t_i)\) denote the probability that bidder \(i\) obtains item \(j\) if her reported type is \(t_i\). Let \(P_i(t_i)\) denote the expected price paid by bidder \(i\) when she reports type \(t_i\). We have the following LP.

\[
\text{Maximize } \sum_{i \in I} \sum_{t_i \in T_i} f_i(t_i) P_i(t_i) \quad (LP1)
\]

\[
\sum_{i \in I} \sum_{t_i \in T_i} f_i(t_i) x_{ij}(t_i) \leq 1 \quad \forall j \in J
\]

\[
\sum_{j \in J} x_{ij}(t_i) \leq d_i \quad \forall i \in I, t_i \in T_i
\]

\[
\sum_{j \in J} t_i(j) x_{ij}(t_i) - P_i(t_i) \geq \sum_{j \in J} t_i(j) x_{ij}(t_i') - P_i(t_i') \quad \forall i \in I, t_i, t_i' \in T_i
\]

\[
\sum_{j \in J} t_i(j) x_{ij}(t_i) - P_i(t_i) \geq 0 \quad \forall i \in I, t_i \in T_i
\]

\[
x_{ij}(t_i) \in [0, 1] \quad \forall i \in I, j \in J, t \in T_i
\]

\[
P_i(t_i) \in [0, B_i] \quad \forall i \in I, t_i \in T_i
\]

The optimal truthful-in-expectation BIC mechanism is feasible for the above constraints. The first constraint encodes the fact that, in expectation, each item is assigned at most once. Now fix any bidder \(i \in I\), and suppose that her true type is
\( t_i \in T_i \). The second constraint encodes the demand: Bidder \( i \) can get at most \( d_i \) items in expectation. The third constraint encodes Bayesian incentive-compatibility: The expected utility of bidder \( i \), when she reports any false type \( t'_i \), cannot be greater than her expected utility when she reports her true type \( t_i \). The fourth constraint encodes individual rationality: If bidder \( i \) reports her true type \( t_i \), then her expected utility is nonnegative. Therefore, the LP1 value is an upper bound on the expected revenue.

**Remark** Fix any bidder \( i \in I \). Note that the first constraint in LP1 holds in expectation over both her own type and the types of other bidders. In contrast, the next three constraints are enforced for every possible type \( t_i \in T_i \), and in these constraints the expectations are taken only over the types of other bidders.

### 2.3.3 The all-pay mechanism

Suppose that the optimal solution to LP1 assigns a value of \( x_{ij}^*(t_i) \) (respectively, \( P_i^*(t_i) \)) to each variable \( x_{ij}(t_i) \) (respectively, \( P_i(t_i) \)). We design an all-pay mechanism (Figure 2.1). The key observation is that the mechanism satisfies the following property: Fix any bidder \( i \in I \) and her reported type \( t_i^* \in T_i \). Furthermore, suppose that every other bidder \( i' \in I \setminus \{i\} \) reveals her true type. In this case, the probability that bidder \( i \) gets any item \( j \in J \), over the randomness introduced by the mechanism and the distributions of types of other bidders, is equal to \( x_{ij}^*(t_i^*)/4 \); bidder \( i \) is charged a fixed payment of \( P_{ij}^*(t_i^*)/4 \). Since both the expected allocation and the payment are scaled down by exactly the same factor [Archer et al. (2003); Lavi and Swamy (2011)], this scheme preserves the Bayesian incentive-compatibility and individual-rationality conditions enforced by the constraints in LP1. Finally, note
All-Pay Mechanism

1. Collect the reported types of all bidders.
   Denote the reported type of bidder $i$ by $t_i^*$. 

2. Find the optimal solution to LP1, and denote the variable values by \( \{x^*_i(t_i), P^*_i(t_i)\} \).

3. Choose an arbitrary but fixed ordering of all bidders and denote it by \(1, 2, \ldots, n\).

4. For all bidders \(i = 1, 2, \ldots, n\), items \(j \in J\) and types \(t_i \in T_i\), let
   \[
   \bar{x}_{ij}(t_i) = \frac{x^*_i(t_i)}{2}, \quad X_{ij} = \sum_{t_i \in T_i} f_i(t_i) \bar{x}_{ij}(t_i), \quad Z_{ij} = \prod_{t'_{i'}=1}^{i-1} (1 - X_{t'_{i'}}).
   \]

5. For all bidders \(i\) and types \(t_i \in T_i\), partition the set of items \(J\) into \(d_i\) disjoint groups such that in each group \(G(i, t_i, k)\), we have \(\sum_{j \in G(i, t_i, k)} \bar{x}_{ij}(t_i) \leq 1\).

6. For \(i = 1, 2, \ldots, n\), initialize \(W_i, S_i \leftarrow \emptyset\);
   For \(k = 1\) to \(d_i\)
     Pick a single item \(j \in G(i, t_i^*, k)\) with probability \(\bar{x}_{ij}(t_i^*)\);
     \(S_i \leftarrow S_i \cup \{j\}\);
     \(Q_i \leftarrow S_i \cup \bigcup_{k'=1}^{i-1} S_{i'}\);
     For all items \(j \in Q_i\)
       \(W_i \leftarrow W_i \cup \{j\}\) with probability \(1/(2Z_{ij})\);
   Bidder \(i\) gets the (random) set \(W_i\), and pays a (fixed) price \(P^*_i(t_i^*)/4\).

**Figure 2.1:** BIC Mechanism for correlated valuations

That the auctioneer’s expected revenue is given by the expression

\[
\sum_{i \in I} \sum_{t_i \in T_i} f_i(t_i) \cdot \frac{P^*_i(t_i)}{4},
\]

which is \(1/4\) times the optimal objective value of LP1. Hence, the mechanism gives a 4-approximation to optimal revenue.

In Step 1 of the All-Pay Mechanism (Figure 2.1), we ask the bidders to reveal their types. The reported type of bidder \(i \in I\) is denoted by \(t_i^*\). In Step 2, we solve (LP1). In Step 3, we order the bidders as \(1, 2, \ldots, n\). In Step 4, we scale down the values of the allocation variables by a factor of 2, and introduce the notations \(\bar{x}_{ij}(t_i), X_{ij}\)
and $Z_{ij}$. Applying the second constraint in LP1, we get $\sum_{j \in J} \tilde{x}_{ij}(t_i) \leq d_i/2$, for all bidders $i \in I$ and types $t_i \in T_i$. In Step 5, we exploit this property while partitioning the set of items $J$ into $d_i$ disjoint groups. Specifically, we employ the following greedy strategy.

- Initialize $V \leftarrow J$ and $k \leftarrow 1$.
- Repeat until $V = \emptyset$
  - Pack maximal number of items from $V$ into the group $G(i, t_i, k)$, so that we have
    \[ \frac{1}{2} \leq \sum_{j \in G(i, t_i, k)} \tilde{x}_{ij}(t_i) \leq 1. \]
  - Set $V \leftarrow V \setminus G(i, t_i, k)$ and $k \leftarrow k + 1.$

In Step 6, we visit the bidders according to the ordering $1, 2, \ldots, n$. While visiting bidder $i$, the notation $S_i$ denotes the *tentative* allocation to bidder $i$, whereas her *actual* allocation is denoted by $W_i$. Each item $j \in J$ is included in $S_i$ with probability $\tilde{x}_{ij}(t^*_i)$. Since the set $S_i$ contains at most one item from each group $G(i, t^*_i, k)$ and there are $d_i$ such groups, we get $|S_i| \leq d_i$, and hence at most $d_i$ items can be allocated to bidder $i$. The notation $Q_i$ denotes a subset of $S_i$, consisting of all the items in $S_i$ that were *not* included in the tentative allocation of any bidder $i' < i$. Each item $j \in Q_i$ is included in $W_i$ with probability $1/(2Z_{ij})$. Bidder $i$ gets the (random) set of items $W_i$ and pays a (fixed) price $P^*_i(t^*_i)/4$. Note that the subsets $Q_1, \ldots, Q_m$ are mutually disjoint by definition, and furthermore we have $W_i \subseteq Q_i$ for all bidders $i = 1, \ldots, n$. Hence, the subsets of items $W_1, \ldots, W_n$ allocated to the different bidders are also mutually disjoint.

**Theorem 6.** The All-Pay Mechanism (Figure 2.1) is truthful-in-expectation BIC, and its revenue is a 4-approximation to the revenue of the optimal truthful-in-expectation
**BIC mechanism.**

**Proof.** Applying the first constraint in LP1, we get $\sum_{i} X_{ij} \leq 1/2$ for all items $j$. The inequality holds since we scaled down the LP variables by a factor of 2. Consequently, we have

$$Z_{ij} = \prod_{i'-i}(1 - X_{ij}) \geq 1 - \sum_{i'-i} X_{ij} \geq \frac{1}{2}.$$ 

This implies $1/(2Z_{ij}) \leq 1$.

Fix a bidder $i$, her reported type $t_i^*$, an item $j$, and suppose that every other bidder $i' \neq i$ reveals her true type. In other words, the reported type of any bidder $i' \neq i$ follows the distribution $f_{i'}(\cdot)$. Hence, for all bidders $i' \neq i$, we have

$$Pr[j \notin S_{i'}] = 1 - Pr[j \in S_{i'}] = 1 - \sum_{t_{i'} \in T_{i'}} f_{i'}(t_{i'}) \tilde{x}_{ij}(t_{i'}) = 1 - X_{i'j}.$$ 

We now show that bidder $i$ gets item $j$ with probability $x_{ij}^*(t_i^*)/4$:

$$Pr[j \in W_i] = Pr[j \in W_i \mid j \in Q_i] \cdot Pr[j \in Q_i]$$

$$= Pr[j \in W_i \mid j \in Q_i] \cdot Pr\left[ j \in S_i \setminus \bigcup_{i'-1} S_{i'} \right]$$

$$= Pr[j \in W_i \mid j \in Q_i] \cdot Pr[j \in S_i] \cdot \prod_{i'=1}^{i-1} Pr[j \notin S_{i'}]$$

$$= \frac{1}{2Z_{ij}} \cdot \tilde{x}_{ij}(t_i^*) \cdot \prod_{i'=1}^{i-1} (1 - X_{i'j})$$

$$= \frac{1}{2Z_{ij}} \cdot \tilde{x}_{ij}(t_i^*) \cdot Z_{ij}$$

$$= \frac{x_{ij}^*(t_i^*)}{4}.$$ 

Equality (2.1) holds since the distributions of types of all the bidders are mutually independent, and we have assumed that every bidder $i' \neq i$ reveals her true type.
By linearity of expectation, the expected welfare of items allocated to bidder \( i \) is given by the expression
\[
\frac{1}{4} \sum_{j \in J} t_i(j) \cdot x_{ij}(t_i^*),
\]
when her true type is \( t_i \). Since bidder \( i \) pays a fixed price \( P_i^*(t_i^*)/4 \), both the expected welfare and expected price are scaled down by a factor exactly 4 relative to the LP values regardless of her revealed type. This preserves the incentive-compatibility constraints in the LP, and makes the scheme be incentive compatible and satisfy individual rationality in expectation over the types of other bidders and the randomness introduced by the mechanism. The theorem follows. \( \square \)

*Remark.* We note that if the objective function is replaced with
\[
\sum_{i \in I} \sum_{j \in J} \sum_{t \in T_i} f_i(t_i) t_i(j) x_{ij}(t_i),
\]
then the resulting scheme gives a 4-approximation to the optimal expected social welfare.

### 2.4 Universally truthful DSIC mechanisms

We described an all-pay mechanism (Figure 2.1) in Section 2.3 that gives constant approximation to optimal revenue. This mechanism, however, is truthful-in-expectation BIC; for example, a bidder has to pay a fixed price even if she does not get any item in a random allocation. In contrast, it is more desirable in practice to implement mechanisms that are universally truthful DSIC and have good revenue properties. Unfortunately, we cannot achieve this goal unless we make additional assumptions on the input, for the following reason: The setting considered in Section 2.3 allows a bidder’s valuations for different items to be drawn from correlated distributions, and it is computationally hard to approximate the revenue-optimal universally truthful
DSIC mechanism [Briest (2008)] under such settings (see Section 2.2.1). We now formally state the assumptions that will be used throughout the rest of Section 2.4 to circumvent this hardness. We use the same notations as in Section 2.3.1.

**Assumption 7.** We assume that the marginal distributions \( \{ f_{ij}(\cdot) \}_{i \in I, j \in J} \) are mutually independent (see Section 2.3.1), and any distribution \( f_{ij}(\cdot) \) has a support \( T_{ij} = \{1, \ldots, L_{ij}\} \), where \( L_{ij} \) is a positive integer. In other words, the valuation of bidder \( i \) for item \( j \) is a positive integer-valued random variable \( v_{ij} \in [1, L_{ij}] \) with probability mass function \( f_{ij}(\cdot) \), and the random variables \( \{ v_{ij} \}_{i \in I, j \in J} \) are mutually independent.

Note that, in contrast to Section 2.3.2, we no longer require the distributions \( f_1(\cdot), \ldots, f_n(\cdot) \) to have polynomial supports. Furthermore, the assumption that the distributions \( f_{ij}(\cdot) \) are defined at positive integer values is made without any loss of generality, since we can discretize continuous distributions in powers of \( (1 + \epsilon) \) and apply the same arguments. This also holds when the values taken by the random variables are not polynomially bounded.

**Sequential posted-price mechanisms.** In this section, our main results are sequential posted-price mechanisms that give good approximations to optimal revenue. Such a mechanism considers the bidders sequentially in arbitrary order, and for each bidder, posts a certain price for each item, and lets the bidder select the subset of items she wants to buy under these prices. Note that any sequential posted-price mechanism is universally truthful DSIC by definition.

**Our results.** We first consider the scenario where the random variables \( v_{ij} \) satisfy the monotone hazard-rate condition (Section 2.4.2). In this case, we present a sequential posted-price mechanism, and show that it is a \( O(1) \)-approximation to the optimal truthful-in-expectation BIC mechanism (Theorem 17). One of the interesting aspects
of this result is that although our mechanism is universally truthful DSIC (which is the strongest notion of incentive compatibility), it still approximates the revenue of the optimal truthful-in-expectation BIC mechanism.

In Section 2.4.3, we relax the monotone hazard rate assumption, and consider the scenario where the random variables $v_{ij}$ are arbitrary integer-valued variables over the domain $[1, L]$, that is, $L_{ij} = L$ for all $i \in \mathcal{I}, j \in \mathcal{J}$. In this case, we present a sequential posted-price mechanism that gives $O(\log L)$-approximation to the revenue of the optimal universally truthful DSIC mechanism, and we also prove that no other sequential posted-price mechanism can achieve an asymptotically better approximation ratio (Theorem 19).

2.4.1 Linear programming relaxation

We start with a simple definition.

**Definition 14.** For all $i \in \mathcal{I}, j \in \mathcal{J}$: Define $\mathcal{V}_{ij} = \min(v_{ij}, B_i/4)$, let its probability mass function be $g_{ij}(\cdot)$, and let $\mathcal{R}_{ij} = \{1, \ldots, |\mathcal{R}_{ij}|\}$ be the range of values the random variable $\mathcal{V}_{ij}$ can possibly take. Since the random variable $v_{ij}$ takes integer values in the range $\{1, \ldots, L_{ij}\}$, we get

$$\Pr[\mathcal{V}_{ij} = r] = g_{ij}(r) = \begin{cases} f_{ij}(r) & \text{if } r < B_i/4, \\ \sum_{v=r}^{L_{ij}} f_{ij}(v) & \text{if } r = B_i/4, \\ 0 & \text{otherwise.} \end{cases}$$

We now describe an LP-relaxation in order to upper bound the revenue of the optimal truthful-in-expectation BIC mechanism.

$$\text{Maximize } \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} r \cdot g_{ij}(r) \cdot x_{ij}(r) \quad \text{(LPRev)}$$
Lemma 8. The optimal value of the linear program (LP Rev) is at least $1/4$ times the revenue of the optimal truthful-in-expectation BIC mechanism. Furthermore, in the LP solution, $x_{ij}(r)$ is a monotonically non-decreasing function of $r$, for all bidders $i \in \mathcal{I}$ and items $j \in \mathcal{J}$.

Proof. Without any loss of generality, we assume that whenever a bidder is allocated a subset of items, the total price she has to pay is distributed amongst the individual items obtained. It is easy to ensure that the expected price on a single item is never greater than the valuation for the item or the overall budget. Given that $V_{ij} = r \in \mathcal{R}_{ij}$, let $x_{ij}(r)$ denote the probability that item $j \in \mathcal{J}$ is allocated to bidder $i \in \mathcal{I}$, and let $p_{ij}(r)$ denote the expected price paid conditioned on obtaining the item. Since $V_{ij} = \min(v_{ij}, B_i/4)$, it is easy to see that

$$p_{ij}(r) \leq \min(v_{ij}, B_i) \leq 4 \cdot \min(v_{ij}, B_i/4) = 4 \cdot r.$$ 

Hence, the revenue of the optimal truthful-in-expectation BIC mechanism can be relaxed as:

Maximize $\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} p_{ij}(r) \cdot g_{ij}(r) \cdot x_{ij}(r)$

$$\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq n_i \quad \forall i \in \mathcal{I} \quad (1)$$
$$\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} p_{ij}(r) \cdot g_{ij}(r) \cdot x_{ij}(r) \leq B_i \quad \forall i \in \mathcal{I} \quad (2)$$
$$\sum_{i \in \mathcal{I}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq 1 \quad \forall j \in \mathcal{J} \quad (3)$$
$$x_{ij}(r) \in [0, 1] \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij} \quad (4)$$
$$p_{ij}(r) \in [0, 4r] \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij} \quad (5)$$
Since the value of $x$ is decreasing in $g$, we have $r$ of the optimal truthful-in-expectation BIC mechanism.

Corollary 9. Fix any bidder $i \in I$, item $j \in J$, and $r_1, r_2 \in R_{ij}$ such that $r_1 < r_2$. Furthermore, let $\phi_{ij}(r)$ be any positive non-decreasing function of $r \in R_{ij}$. If we have $0 < x_{ij}(r_1), x_{ij}(r_2) < 1$, and we preserve $\sum_{r \in R_{ij}} \phi_{ij}(r)g_{ij}(r)x_{ij}(r)$ while increasing $x_{ij}(r_2)$ and decreasing $x_{ij}(r_1)$ for $r_1 < r_2$. In this process, $\sum_{r} g_{ij}(r)x_{ij}(r)$ can never increase (see Corollary 9), preserving all the constraints, which implies the monotonicity.

Proof. Suppose that we increase $x_{ij}(r_2)$ by an amount $\delta_2 > 0$, and we decrease $x_{ij}(r_1)$ by an amount $\delta_1 < 0$. Since $x_{ij}(r)$ remains unchanged for every $r \notin \{r_1, r_2\}$, and the sum $\sum_{r \in R_{ij}} \phi_{ij}(r)g_{ij}(r)x_{ij}(r)$ is preserved, we must have

$$\phi_{ij}(r_1) \cdot g_{ij}(r_1) \cdot \delta_1 = \phi_{ij}(r_2) \cdot g_{ij}(r_2) \cdot \delta_2.$$  

Since $\phi_{ij}(r)$ is a positive non-decreasing function of $r$, we have $\phi_{ij}(r_1) \leq \phi_{ij}(r_2)$. It follows that $g_{ij}(r_1)\delta_1 \geq g_{ij}(r_2)\delta_2$. Hence, we get

$$g_{ij}(r_1) \cdot (x_{ij}(r_1) - \delta_1) + g_{ij}(r_2) \cdot (x_{ij}(r_2) + \delta_2) \leq g_{ij}(r_1) \cdot x_{ij}(r_1) + g_{ij}(r_2) \cdot x_{ij}(r_2).$$

Since the value of $x_{ij}(r)$ remains unchanged for all $r \notin \{r_1, r_2\}$, the expression $\sum_{r \in R_{ij}} g_{ij}(r)x_{ij}(r)$ can never increase.  

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Remark. Due to the presence of budget constraints, \((LPrev)\) bounds only the expected revenue of any truthful-in-expectation BIC mechanism and \textit{not} the expected social welfare, which can be larger by an unbounded amount. However, if the budget constraints are removed, the resulting LP also bounds the optimal social welfare.

2.4.2 Monotone hazard rates

We will present a constant-factor approximation to the optimal truthful-in-expectation BIC mechanism via sequential posted-price schemes, assuming that the random variables \(v_{ij}\) satisfy the monotone hazard-rate condition (Definition 13). Formally, we will make the following assumption throughout Section 2.4.2.

**Assumption 10.** For all \(i \in I\) and \(j \in J\), the random variable \(v_{ij}\) (which denotes the valuation of bidder \(i\) for item \(j\)) satisfies the monotone hazard rate (MHR) condition.

**Claim 11.** If \(X\) is a positive-integer-valued random variable satisfying the monotone hazard-rate condition, then the random variable \(\min(X, a)\) also satisfies the MHR condition, for any integer \(a \geq 1\).

**Proof.** Suppose that the random variable \(X\) is drawn from an MHR distribution with support \(\{1, \ldots, k\}\). If \(a \geq k\), then clearly the random variable \(\min(X, a)\) also satisfies the MHR condition, since it has a support of \(\{1, \ldots, k\}\) and we have

\[
\Pr[\min(X, a) = r] = \Pr[X = r] \quad \text{for all } r \in \{1, \ldots, k\}.
\]

Hence, we will assume that \(a < k\) throughout the rest of the proof. In this case, the random variable \(\min(X, a)\) follows a distribution with support \(\{1, \ldots, a\}\), and we
have
\[
\Pr[\min(X, a) = r] = \Pr[X = r] \quad \text{for all } r \in \{1, \ldots, a - 1\},
\]
(2.2)
\[
\Pr[\min(X, a) > r] = \Pr[X > r] \quad \text{for all } r \in \{1, \ldots, a - 1\},
\]
(2.3)
\[
\Pr[\min(X, a) = a] = \sum_{r=a}^{k} \Pr[X = r],
\]
(2.4)
\[
\Pr[\min(X, a) > a] = 0.
\]
(2.5)

From equations (2.2), (2.3), (2.4) and (2.5), it follows that
\[
\frac{\Pr[\min(X, a) > r]}{\Pr[\min(X, a) = r]} \geq \frac{\Pr[\min(X, a) > r']}{\Pr[\min(X, a) = r']} \quad \text{for all } r, r' \in \{1, \ldots, a\} \text{ such that } r < r'.
\]
(2.6)

Hence, the random variable \( \min(X, a) \) satisfies the MHR condition.

Since the random variables \( v_{ij} \) satisfy the MHR condition (Assumption 10), we get the following corollary.

**Corollary 12.** For all \( i \in \mathcal{I} \) and \( j \in \mathcal{J} \), the random variable \( V_{ij} = \min(v_{ij}, B_i/4) \) satisfies the MHR condition.

The next definition is due to Myerson (1981). First, recall that the random variable \( V_{ij} \) has a probability mass function \( g_{ij}(\cdot) \) with support \( \mathcal{R}_{ij} \) (Definition 14).

**Definition 15.** For all \( r \in \mathcal{R}_{ij} \), let \( G_{ij}(r) = \Pr[V_{ij} > r] \). Define the virtual valuation as
\[
\varphi_{ij}(r) = r - \frac{G_{ij}(r)}{g_{ij}(r)} \quad \text{for all } i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}.
\]

The distribution \( g_{ij}(\cdot) \) is said to be regular if and only if we have
\[
\varphi_{ij}(r) \leq \varphi_{ij}(r') \quad \text{for all } r, r' \in \mathcal{R}_{ij} \text{ such that } r < r'.
\]
Clearly, monotone hazard-rate distributions are regular. We now present a crucial lemma for monotone hazard-rate distributions.

**Lemma 13.** If the random variable $V_{ij}$ satisfies the MHR condition, then we have

$$\Pr\left[\varphi_{ij}(V_{ij}) > \frac{V_{ij}}{2}\right] \geq \frac{1}{e^2}.$$ 

**Proof.** Recall that for all $r \in \mathcal{R}_{ij}$, we have $g_{ij}(r) = \Pr[V_{ij} = r]$ and $G_{ij}(r) = \Pr[V_{ij} > r]$. Let the support of the distribution $g_{ij}(\cdot)$ be given by $\mathcal{R}_{ij} = \{1, \ldots, k\}$, where $k$ is a positive integer. Before proceeding any further, note that if $k = 1$, then $G_{ij}(1) = 0$, $\varphi_{ij}(1) = 1$, and the lemma holds since $\Pr[\varphi_{ij}(V_{ij}) > V_{ij}/2] = 1 \geq 1/e^2$. Thus, throughout the rest of the proof, we assume that $k \geq 2$.

Define

$$h_{ij}(r) = \frac{g_{ij}(r)}{G_{ij}(r)} \quad \text{for all } r \in \{1, \ldots, k\}.$$ 

It is easy to see that $\varphi_{ij}(r) = r - 1/h_{ij}(r)$. Hence, we have $\varphi_{ij}(r) \geq r/2$ if and only if $h_{ij}(r) \geq 2/r$. As the random variable $V_{ij}$ satisfies the MHR condition, we infer that $h_{ij}(r)$ is a non-decreasing function of $r$. Finally, we note that $2/r$ is a non-increasing function of $r$, and that $h_{ij}(k) = \infty > 2/k$. These observations lead us to conclude the following:

There is an integer $k^* \in \{1, \ldots, k\}$ such that $\varphi_{ij}(r) > r/2$ if and only if $r \geq k^*$, and $h_{ij}(r) \leq h_{ij}(k^* - 1) \leq 2/(k^* - 1)$ for all $r \in \{1, \ldots, k^* - 1\}$.

Therefore, we have

$$\Pr[\varphi_{ij}(V_{ij}) > V_{ij}/2] = \Pr[V_{ij} \geq k^*].$$ 

If $k^* = 1$, then $\Pr[V_{ij} \geq k^*] = 1 > 1/e^2$, and the lemma holds. Consequently, throughout the rest of the proof, we assume that $k^* \geq 2$. 

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We construct a continuous probability distribution with probability density function \( \hat{g}_{ij}(\cdot) \) and support \([0, k]\) such that

\[
g_{ij}(r) = \int_{t=(r-1)}^{r} \hat{g}_{ij}(t) \, dt \quad \text{for all} \quad r \in \{1, \ldots, k\}.
\]

Next, for every real number \(0 \leq t \leq k\), we define

\[
\hat{G}_{ij}(t) = 1 - \int_{q=0}^{t} \hat{g}_{ij}(t) \, dt \quad \text{and} \quad \hat{h}_{ij}(t) = \frac{\hat{g}_{ij}(t)}{\hat{G}_{ij}(t)}.
\]

Since \( \frac{d}{dt}(\hat{G}_{ij}(t)) = -\hat{g}_{ij}(t) \), it follows that

\[
\int \hat{h}_{ij}(t) \, dt = -\log(\hat{G}_{ij}(t)).
\]

Hence, we have

\[
\exp \left( -\int_{t=0}^{(k^*-1)} \hat{h}_{ij}(t) \, dt \right) = \frac{\hat{G}_{ij}(k^*-1)}{\hat{G}_{ij}(0)} = \hat{G}_{ij}(k^*-1). \tag{2.7}
\]

Note that \( \hat{G}_{ij}(t) \) is a non-increasing function of \( t \). Thus, for all \( r \in \{1, \ldots, (k^*-1)\} \), we have

\[
-\int_{t=(r-1)}^{r} \hat{h}_{ij}(t) \, dt \geq -\frac{1}{\hat{G}_{ij}(r)} \int_{t=(r-1)}^{r} \hat{g}_{ij}(t) \, dt = -\frac{g_{ij}(r)}{\hat{G}_{ij}(r)} = -h_{ij}(r) \geq -\frac{2}{(k^*-1)}.
\]

Summing over \( r = 1, \ldots, (k^*-1) \), we get

\[
-\int_{t=0}^{(k^*-1)} \hat{h}_{ij}(t) \, dt \geq -\sum_{r=1}^{(k^*-1)} \int_{t=(r-1)}^{r} \hat{h}_{ij}(t) \, dt \geq -\sum_{r=1}^{(k^*-1)} \frac{2}{(k^*-1)} \geq -2. \tag{2.8}
\]
To conclude the proof of the lemma, we deduce that

$$\Pr[\varphi_{ij}(V_{ij}) > V_{ij}/2] = \Pr[V_{ij} \geq k^*]$$

$$= 1 - \sum_{r=1}^{(k^*-1)} g_{ij}(r)$$

$$= 1 - \int_{t=0}^{(k^*-1)} \hat{g}_{ij}(t) \, dt$$

$$= \hat{G}_{ij}(k^* - 1)$$

$$= e^{-\left(\frac{(k^*-1)}{\mu} - 1\right) \hat{h}_{ij}(t) \, dt} \quad \text{(by equation (2.7))}$$

$$\geq e^{-2} \quad \text{(by equation (2.8))}.$$ 

\[ \square \]

Note that the bound established is Lemma 13 is tight for exponential distributions. Suppose that $X$ is a continuous random variable drawn from an exponential distribution with rate parameter $\mu$, and let $\varphi(X)$ denote the virtual valuation function of $X$. We get

$$\Pr \left[ \varphi(X) > \frac{X}{2} \right] = \Pr \left[ X - \frac{e^{-\mu X}}{\mu} > X/2 \right] = \Pr \left[ X > \frac{2}{\mu} \right] = \frac{1}{e^2}.$$ 

Recall Definition 14 and Definition 15. The next observation follows from Corollary 12 and Lemma 13.

**Observation 1.** For all bidders $i \in I$ and items $j \in J$, there exists an integer $v^*_ij \in R_{ij}$ such that

$$\Pr[V_{ij} \geq v^*_ij] \geq e^{-2} \quad \text{and} \quad \varphi_{ij}(V_{ij}) > V_{ij}/2 \quad \text{whenever} \quad V_{ij} \geq v^*_ij.$$ 

**Incorporating virtual valuations.**

First, we state a characterization of virtual valuations that was proved by Myerson (1981). We will invoke Lemma 14 multiple times throughout the rest of this chapter.
Lemma 14. For all bidders $i \in \mathcal{I}$ and items $j \in \mathcal{J}$, suppose that the random variable $v_{ij} = \min(v_{ij}, B_i/4)$ follows a discrete distribution $g_{ij}(\cdot)$ with support $\mathcal{R}_{ij}$, and $0 \leq x_{ij}(r) \leq 1$ for all $r \in \mathcal{R}_{ij}$. We have

$$\sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) = \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \left( r \cdot x_{ij}(r) - \sum_{s \in \mathcal{R}_{ij} : s > r} x_{ij}(s) \right).$$

Proof. Recall that $\varphi_{ij}(r) = r - G_{ij}(r)/g_{ij}(r)$, where $G_{ij}(r) = \sum_{s \in \mathcal{R}_{ij} : s > r} g_{ij}(s)$. Hence, we get

$$\sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) = \sum_{r \in \mathcal{R}_{ij}} \left( r \cdot g_{ij}(r) - \sum_{s \in \mathcal{R}_{ij} : s > r} g_{ij}(s) \right) \cdot x_{ij}(r)$$

$$= \sum_{r \in \mathcal{R}_{ij}} r \cdot g_{ij}(r) \cdot x_{ij}(r) - \sum_{r \in \mathcal{R}_{ij}} \sum_{s \in \mathcal{R}_{ij} : s > r} g_{ij}(s) \cdot x_{ij}(r)$$

$$= \sum_{r \in \mathcal{R}_{ij}} r \cdot g_{ij}(r) \cdot x_{ij}(r) - \sum_{r \in \mathcal{R}_{ij}} \sum_{s \in \mathcal{R}_{ij} : s < r} g_{ij}(r) \cdot x_{ij}(s)$$

$$= \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \left( r \cdot x_{ij}(r) - \sum_{s \in \mathcal{R}_{ij} : s < r} x_{ij}(s) \right).$$

This concludes the proof. \qed
Now consider the following linear program obtained from (LPRev).

\[
\text{Maximize } \sum_{i \in I} \sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) \quad (\text{LP2})
\]

\[
\sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq n_i \quad \forall i \in I \quad (1)
\]

\[
\sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) \leq B_i \quad \forall i \in I \quad (2)
\]

\[
\sum_{i \in I} \sum_{r \in R_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq 1 \quad \forall j \in J \quad (3)
\]

\[
x_{ij}(r) \in [0, 1] \quad \forall i \in I, j \in J, r \in R_{ij} \quad (4)
\]

**Lemma 15.** The value of (LP2) is at least \(1/(2e^2)\) times the value of (LPRev).

**Proof.** Let \(\{x_{ij}^*(r)\}\) denote the values taken by the \(x_{ij}(r)\) variables in the optimal solution to (LPRev). For all \(i \in I, j \in J, r \in R_{ij}\), define:

\[
\tilde{x}_{ij}(r) = \begin{cases} 
0 & \text{if } r < v_{ij}^*, \\
x_{ij}(r) & \text{if } r \geq v_{ij}^*.
\end{cases} \quad (2.9)
\]

The optimal objective of (LPRev) is given by

\[
\sum_{i,j} \mathbf{E}_{V_{ij} \sim g_i(\cdot)} [V_{ij} \cdot x_{ij}^*(V_{ij})].
\]

Lemma 8 implies that the function \(V_{ij} \cdot x_{ij}^*(V_{ij})\) is monotonically non-decreasing in \(V_{ij}\). Hence, we get

\[
\mathbf{E}_{V_{ij}} [V_{ij} \cdot x_{ij}^*(V_{ij})] \leq \mathbf{E}_{V_{ij}} [V_{ij} \cdot x_{ij}^*(V_{ij}) \mid V_{ij} \geq v_{ij}^*]
\]

\[
\leq \mathbf{E}_{V_{ij}} [2 \cdot \varphi_{ij}(V_{ij}) \cdot x_{ij}^*(V_{ij}) \mid V_{ij} \geq v_{ij}^*] \quad \text{(Observation 1)}
\]

\[
= \frac{2 \cdot \mathbf{E}_{V_{ij}} [\varphi_{ij}(V_{ij}) \cdot \tilde{x}_{ij}(V_{ij})]}{\mathbf{Pr}[V_{ij} \geq v_{ij}^*]} \quad \text{(by equation (2.9))}
\]

\[
\leq 2e^2 \cdot \mathbf{E}_{V_{ij}} [\varphi_{ij}(V_{ij}) \cdot \tilde{x}_{ij}(V_{ij})] \quad \text{(Observation 1)}
\]

\[
= 2e^2 \cdot \sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot \tilde{x}_{ij}(r).
\]
Summing over all bidders \( i \in \mathcal{I} \) and items \( j \in \mathcal{J} \), we infer:

\[
\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} r \cdot g_{ij}(r) \cdot x^*_ij(r) \leq 2e^2 \cdot \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot \tilde{x}_{ij}(r). \tag{2.10}
\]

Furthermore, we note the following inequalities.

\[
0 \leq \tilde{x}_{ij}(r) \leq x^*_ij(r) \quad \text{for all } i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}, \tag{2.11}
\]

\[
\varphi_{ij}(r) \cdot \tilde{x}_{ij}(r) \leq r \cdot x^*_ij(r) \quad \text{for all } i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}. \tag{2.12}
\]

Equation (2.11) holds by definition, and equation (2.12) holds since \( \varphi_{ij}(r) \leq r \) (Definition 15). Recall that the \( x^*_ij(r) \) values constitute an optimal solution to (LPRev). Hence, equations (2.10), (2.11), and (2.12) imply that the \( \tilde{x}_{ij}(r) \) values constitute a feasible solution to (LP2), having an objective that is within a factor of \( 2e^2 \) of the optimal objective of (LPRev). The lemma follows. \( \square \)

**Lemma 16.** Let \( x^*_ij(r) \) denote the values assigned to the variables in the optimal solution to (LP2). For all \( i \in \mathcal{I}, j \in \mathcal{J} \), we can express the corresponding \( \{x^*_ij(r)\}_{r \in \mathcal{R}_{ij}} \) values as a convex combination of two solutions. The first (respectively, second) solution has variable values \( \gamma^*_ij(r) \) (respectively, \( \lambda^*_ij(r) \)), and an integer \( r^*_ij \in \mathcal{R}_{ij} \) (respectively, \( s^*_ij \in \mathcal{R}_{ij} \)) so that

\[
\gamma^*_ij(r) = \begin{cases} 
0 & \text{if } r < r^*_ij, r \in \mathcal{R}_{ij}, \\
1 & \text{if } r \geq r^*_ij, r \in \mathcal{R}_{ij},
\end{cases}
\]

and

\[
\lambda^*_ij(r) = \begin{cases} 
0 & \text{if } r < s^*_ij, r \in \mathcal{R}_{ij}, \\
1 & \text{if } r \geq s^*_ij, r \in \mathcal{R}_{ij}.
\end{cases}
\]

Suppose that in the convex combination, the first solution has weight \( 0 \leq w_{ij} \leq 1 \) and the second solution has weight \( 1 - w_{ij} \). In this case, we have \( x^*_ij(r) = w_{ij} \cdot \gamma^*_ij(r) + (1 - w_{ij}) \cdot \lambda^*_ij(r) \), and furthermore

\[
\sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x^*_ij(r) = w_{ij} \cdot r^*_ij \cdot \Pr[\mathcal{V}_{ij} \geq r^*_ij] + (1 - w_{ij}) \cdot s^*_ij \cdot \Pr[\mathcal{V}_{ij} \geq s^*_ij]. \tag{2.13}
\]
Proof. Since the variable values \( \{x^*_{ij}(r)\} \) constitute an optimal solution to (LP2), we have \( x^*_{ij}(r) = 0 \) whenever \( \varphi_{ij}(r) \leq 0 \). Furthermore, the virtual valuation \( \varphi_{ij}(r) \) is a non-decreasing function of \( r \in R_{ij} \) (Corollary 12). We now apply Corollary 9 and infer the following: There exists an integer \( r^* \in R_{ij} \) such that

\[
x^*_{ij}(r) = \begin{cases} 
0 & \text{if } r < r^*, \ r \in R_{ij}, \\
1 & \text{if } r > r^*, \ r \in R_{ij}.
\end{cases}
\]

This implies that the optimal solution to (LP2) can be written in the fashion implied by the lemma, with \( r^*_{ij} = r^* \) and \( w_{ij} = x^*_{ij}(r^*) \). To prove equation (2.13), we first note that

\[
\sum_{r \in R_{ij}} g_{ij}(r) \varphi_{ij}(r) x^*_{ij}(r) = w_{ij} \sum_{r \in R_{ij}} g_{ij}(r) \varphi_{ij}(r) \gamma^*_{ij}(r) + (1 - w_{ij}) \sum_{r \in R_{ij}} g_{ij}(r) \varphi_{ij}(r) \lambda^*_{ij}(r).
\]

(2.14)

Let the range of values of the random variable \( \mathcal{V}_{ij} \) be given by \( R_{ij} = \{1, \ldots, k\} \). Recall that \( \gamma^*_{ij}(r) = 0 \) if \( 1 \leq r < r^*_{ij} \) and \( \gamma^*_{ij}(r) = 1 \) if \( r^*_{ij} \leq r \leq k \). Hence, we have

\[
\sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot \gamma^*_{ij}(r) = \sum_{r \in R_{ij}} g_{ij}(r) \left( r \cdot \gamma^*_{ij}(r) - \sum_{s \in R_{ij} : s < r} \gamma^*_{ij}(s) \right) \quad \text{(Lemma 14)}
\]

\[
= \sum_{r = r^*_{ij}}^{k} g_{ij}(r) \left( r - (r - r^*_{ij}) \right) \\
= r^*_{ij} \cdot \Pr[\mathcal{V}_{ij} \geq r^*_{ij}].
\]

Similarly, we can show that

\[
\sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot \lambda^*_{ij}(r) = s^*_{ij} \cdot \Pr[\mathcal{V}_{ij} \geq s^*_{ij}].
\]

Now, the lemma follows from equation (2.14). \( \square \)
Posted-price mechanism and analysis.

The posted-price mechanism is described in Figure 2.2. In Step 1, we select an arbitrary ordering of the bidders. In Step 2, we find the optimal solution to (LP2).

To explain Step 3 and Step 4, we first recall Lemma 16, which has the following simple interpretation. For all bidders \( i \in I \) and items \( j \in J \), the optimal solution to (LP2) treats the pair \( (i, j) \) as a separate entity who is interested only in item \( j \), and who is willing to pay at most \( \nu_{ij} \) (Definition 14) for item \( j \). Furthermore, the pair \( (i, j) \) is offered item \( j \) at a posted price that is chosen randomly from the set \( \{ r_{ij}^*, s_{ij}^* \} \). Note that this is not a feasible mechanism: Since each pair \( (i, j) \) is considered separately, an item may be allocated more than once, or a bidder may exhaust her budget or demand. The LP solution, however, is extremely useful, as it allows us to “uncouple” all the \( (i, j) \) pairs and find the prices \( \{ r_{ij}^*, s_{ij}^* \} \) for each of them.

Motivated by the above interpretation, in Steps 3 and 4, we set \( \tilde{r}_{ij} = r_{ij}^* \) with probability \( w_{ij} \), and with the remaining probability \( (1 - w_{ij}) \), we set \( \tilde{r}_{ij} = s_{ij}^* \). Furthermore, we ensure that these random events corresponding to all possible (bidder, item) pairs are mutually independent. Under the assumption that every \( (i, j) \) pair behaves as a separate entity who is willing to pay at most \( \nu_{ij} \) for an item, let \( \tilde{X}_{ij} \) be the indicator random variable that is set to 1 if she is allocated the item, and let \( \tilde{P}_{ij} \) denote her payment. Note that \( \tilde{P}_{ij} = \tilde{r}_{ij} \) if \( \tilde{X}_{ij} = 1 \), and \( \tilde{P}_{ij} = 0 \) if \( \tilde{X}_{ij} = 0 \). It follows that

\[
E[\tilde{X}_{ij}] = w_{ij} \cdot \Pr[\nu_{ij} \geq r_{ij}^*] + (1 - w_{ij}) \cdot \Pr[\nu_{ij} \geq s_{ij}^*]
\]

\[
= w_{ij} \cdot \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \gamma_{ij}^*(r) + (1 - w_{ij}) \cdot \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \lambda_{ij}^*(r)
\]

\[
= \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \tilde{x}_{ij}^*(r)
\]
Posted-Price Mechanism

1. Choose an arbitrary but fixed ordering of all bidders and denote it by $1, \ldots, n$.

2. Solve LP2, and let $x^*$ denote the value assigned to variable $x_{ij}$, for all $i \in I$, $j \in J$.

3. For each $(i, j)$
   Independently pick one of the two solutions in the convex combination (see Lemma 16) with probability equal to its weight in the combination.

4. Let $\tilde{r}_{ij} \in \{r^*, s^*\}$ denote the threshold in the chosen solution where the allocation function $\tilde{\gamma}_{ij}$ jumps from zero to one.

5. Let $\tilde{Y}_{ij} \leftarrow J$ (Initialize $Q$ to be the set of all items).
   For $i = 1, 2, \ldots, n$
   Initialize $W_i \leftarrow \emptyset$;
   For each item $j \in Q$
   $W_i \leftarrow W_i \cup \{j\}$ iff $Y_{ij} = 1$;
   Only the set of items $W_i$ is offered to bidder $i$;
   Each $j \in W_i$ is offered at a price $\tilde{r}_{ij}$;
   Bidder $i$ buys a subset of items $S_i \subseteq W_i$;
   $Q \leftarrow Q \setminus S_i$.

Figure 2.2: DSIC Mechanism for independent valuations

We can also write:

$$E[\tilde{P}_{ij}] = w_{ij} \cdot r^*_{ij} \cdot \Pr[V_{ij} \geq r^*_{ij}] + (1 - w_{ij}) \cdot s^*_{ij} \cdot \Pr[V_{ij} \geq s^*_{ij}] = \sum_{r \in R_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x^*_{ij}(r).$$

The last equality follows from equation (2.13). Hence, the optimal objective value of (LP2) is given by $\sum_{i,j} E[\tilde{P}_{ij}]$. Furthermore, Constraints 1, 2 and 3 of (LP2) imply that

$$\sum_{j \in J} E[\tilde{X}_{ij}] \leq n_i, \quad \sum_{j \in J} E[\tilde{P}_{ij}] \leq B_i, \quad \text{and} \quad \sum_{i \in I} E[\tilde{X}_{ij}] \leq 1.$$  \hspace{1cm} (2.15)

To summarize, if every pair $(i, j)$ were behaving as a separate entity and we posted the (random) price $\tilde{r}_{ij}$ for the pair $(i, j)$, then the total expected revenue will be equal
to the optimal objective of (LP2), and the demand, budget, and supply constraints will hold in expectation.

There are two difficulties in implementing the above scheme: (1) In reality, the pairs \((i, j)\) do not behave as separate entities. (2) Since the budget, demand and supply constraints hold only in expectation, there may be occasions where one or more of these constraints are violated.

Step 5 circumvents these difficulties. We visit the bidders according to the predetermined ordering. While visiting bidder \(i\), if an item \(j\) is available (i.e., has not been purchased by any bidder \(i' < i\)), then we offer item \(j\) to bidder \(i\) with probability \(1/4\), at a (random) posted price \(\tilde{r}_{ij}\). Intuitively, this step ensures that with constant probability, each pair \((i, j)\) behaves as a separate agent (see the proof of Theorem 17), so that the expected revenue remains within a constant factor of the optimal objective of (LP2). Applying Lemma 8 and Lemma 15, we infer that the revenue of the sequential posted-price mechanism is a \(O(1)\)-approximation to the revenue of the optimal Bayesian incentive-compatible mechanism.

Note that since an item is offered to a bidder only if it has not been purchased by anyone else, and since the bidder herself decides the subset of items she wants to buy at the posted prices, this mechanism is clearly feasible (i.e., satisfy the demand, budget and supply constraints), and universally truthful DSIC.

**Theorem 17.** The posted-price mechanism in Figure 2.2 is a universally truthful DSIC mechanism and its revenue is a \(O(1)\)-approximation to the revenue of the optimal truthful-in-expectation BIC mechanism, when the valuations of a bidder follow product distributions that satisfy the monotone hazard-rate condition.

**Proof.** For all bidders \(i \in \mathcal{I}\) and items \(j \in \mathcal{J}\), let \(X_{ij}\) be the 0/1 random variable denoting whether item \(j\) is taken by bidder \(i\), and let \(P_{ij}\) denote the price at which it is taken (which is 0 if the item is not taken by the bidder). Recall equation (2.15),
where $\tilde{X}_{ij}$ (respectively, $\tilde{P}_{ij}$) is the random variable for the allocation (respectively, payment) corresponding to the pair $(i, j)$, under the assumption that every (bidder, item) pair behaves as a separate entity. Since the probability that any pair $(i, j)$ is considered at all is $1/4$ (see Step 5, Figure 2.2), we infer that $X_{ij} \leq \tilde{X}_{ij}/4$ and $P_{ij} \leq \tilde{P}_{ij}/4$. By linearity of expectation, the inequalities of equation (2.15) imply that

$$
E \left[ \sum_{k \neq j} P_{ik} \right] \leq B_i/4, \quad E \left[ \sum_{k \neq j} X_{ik} \right] \leq d_i/4, \quad \text{and} \quad E \left[ \sum_{k \neq i} X_{kj} \right] \leq 1/4 \quad \forall i \in I, j \in J.
$$

Since the valuation $v_{ij}$ is independent of other valuations, this implies the above statements hold regardless of $v_{ij}$. Now applying Markov’s inequality, we have for all $i, j$:

$$
\Pr \left[ \sum_{k \neq j} P_{ik} \geq 3B_i/4 \right] \leq 1/3, \quad \Pr \left[ \sum_{k \neq j} X_{ik} \geq d_i \right] \leq 1/4, \quad \Pr \left[ \sum_{k \neq i} X_{kj} \geq 1 \right] \leq 1/4.
$$

By union bounds, this implies that with probability at least $1/6$, we have

$$
\sum_{k \neq j} P_{ik} < 3B_i/4, \quad \sum_{k \neq j} X_{ik} < d_i, \quad \text{and} \quad \sum_{k \neq i} X_{kj} = 0.
$$

In this event, item $j$ is offered to bidder $i$ with probability $1/4$ (using one of two random choices of the posted price from Lemma 16). Furthermore, in this event, the bidder will take the item if $v_{ij} \geq \tilde{r}_{ij}$, since $\tilde{r}_{ij} \leq B_i/4$ (so that the bidder has sufficient budget to purchase this item), and the bidder has not exhausted his demand $d_i$. Since the valuation $v_{ij}$ itself is independent of the event that the item $j$ is offered to bidder $i$, this implies that with probability at least $1/6 \times 1/4$, the posted-price mechanism obtains revenue $\sum_r g_{ij}(r) \varphi_{ij}(r)x_{ij}^*(r)$ along each $(i, j)$ (using the definition of the latter quantity from equation (2.13)). By linearity of expectation over all $(i, j)$, we have a $O(1)$-approximation to the objective of (LP2). The theorem follows from Lemma 8 and Lemma 15. \qed
Corollary 18. Under the assumptions of Theorem 17, if there are no budget constraints, then the revenue of the sequential posted-price mechanism is a constant-factor approximation to the optimal social welfare.

Proof. If the Budget Constraint (2) is removed from (LPRev), it is an upper bound on the optimal social welfare that can be obtained by any mechanism. The rest of the analysis remains the same.

One issue with mechanisms where bidders have budget and demand constraints is that given posted prices for the items, the bidder needs to solve a two-dimensional Knapsack problem to determine her optimal bundle, and this is \textsc{np}-hard in general. However, this does not change our results, for the following reasons. First, the analysis of the algorithm in Figure 2.2 simply shows that with constant probability, the bidder solves a trivial Knapsack instance where all items fit into the Knapsack. Another interesting aspect of our mechanism is that the LP relaxations (both (LPRev) and (LP2)) do not encode any incentive-compatibility constraint. Furthermore, in a sequential posted-price mechanism, a bidder $i \in \mathcal{I}$ does not have to report her valuations to the auctioneer. Depending on the publicly known valuation distributions, the choices made by the bidders $i' < i$, and the randomness of the mechanism, bidder $i$ is offered a subset $W_i$ of available items, and she has to pay a posted price $\tilde{r}_{ij}$ for buying any item $j \in W_i$. The subset of items $W_i$ and the posted prices $\{\tilde{r}_{ij}\}$ do not depend on the valuations of bidder $i$. Hence, the scheme remains incentive compatible regardless of the Knapsack heuristic used by the bidder. Therefore, our results hold as long as the bidder uses any reasonable Knapsack heuristic that is maximal, in the sense that the bidder goes on buying more and more items until she exhausts her demand or budget constraints.
2.4.3 General distributions

The key point shown by Theorem 17 is that the revenue of the optimal truthful-in-expectation BIC mechanism is approximated to a constant factor by a simple sequential posted-price mechanism, where the prices are posted for each item and each bidder. This crucially used the monotone hazard-rate condition. The natural question to ask is whether such a simple mechanism is a good approximation for more general classes of distributions. Note that even for one bidder, the optimal universally truthful DSIC mechanism may not be a sequential posted-price scheme. We show below that the optimal universally truthful DSIC mechanism has a logarithmic advantage over sequential posted pricing even for regular distributions (see Definition 15), and that this gap is tight.

**Theorem 19.** Suppose that $v_{ij} \in [1, L]$, are independent for different $(i, j)$ but do not necessarily satisfy the monotone hazard-rate condition. Then, there is a $\Theta(\log L)$ gap between the revenues of the optimal sequential posted-price scheme and the optimal universally truthful DSIC mechanism. The lower bound holds for regular distributions and one bidder.

*Proof.* To show the lower bound, consider the following scenario: There is only one bidder, $m$ items, and no budget or demand constraints. The valuations are i.i.d. for each item $j$, and follow the common distribution with $\Pr[v \geq r] = 1/r$, for $r = 1, 2, \ldots, m$, so that $m = L$. This distribution is regular, since $\varphi(r) = 0$ for $r < m$, and $\varphi(m) = m$. We have $\mathbf{E}[v_{1j}] = \sum_{r=1}^{m} (1/r) = H_m$ for every item $j$, and by Chernoff bounds,

$$\Pr \left[ \sum_{j=1}^{m} v_{1j} \leq (1 - \epsilon) m H_m \right] \leq e^{-\frac{m H_m \epsilon^2}{2m}} = o(1).$$

Therefore, a scheme that sets a price of $m H_m (1 - \epsilon)$ for the bundle of $m$ items sells
the bundle with probability $1-o(1)$, so that the expected revenue is $\Omega(m \log m)$. Any sequential posted-price scheme can extract a revenue of $\max_r r \cdot \Pr[v \geq r] = 1$ from each item, so that the total expected revenue from $m$ items is at most $m \times 1 = m$. This shows a gap of $\Omega(\log m) = \Omega(\log L)$.

The upper bound follows from a standard scaling argument. We start with (LPRev) which upper bounds the optimal achievable revenue. For each $(i, j)$, we group the $r$ values in powers of 2 so that there are $(\log L)$ groups. Let group $G_k$ denote the interval $[2^k, 2^{k+1})$. We find a group $G_{k_{ij}}$ that contributes at least $(1/\log L)$ fraction to the sum $\sum_r r g_{ij}(r) x_{ij}(r)$. Specifically, we require that

$$\sum_{r \in G_{k_{ij}}} r \cdot g_{ij}(r) \cdot x_{ij}(r) \geq (1/\log L) \sum_{r \in R_{ij}} r \cdot g_{ij}(r) \cdot x_{ij}(r).$$

Note that if we use the price $2^{k_{ij}}$ for the pair $(i, j)$, and every (bidder, item) pair behaves as a separate entity, then our expected revenue will be at least $(1/2 \log L)$ times the optimal objective value of (LPRev). Suppose that we modify the Steps 2, 3, and 4 of the algorithm in Figure 2.2: Instead of solving (LP2) and randomly selecting $\tilde{r}_{ij} \in \{r_{ij}^*, s_{ij}^*, \}$, we solve (LPRev) and deterministically set $\tilde{r}_{ij} = 2^{k_{ij}}$. Step 5 of the algorithm is left unchanged. Using similar arguments as in the proof of Theorem 17, it is now easy to see that the resulting sequential posted-price mechanism will give a $\Omega(1/\log L)$-approximation to the optimal revenue.

\[\square\]

Adaptive posted-price schemes.

The above implies that we cannot generalize the result in Section 2.4.2 even to the class of regular distributions (see Definition 12). This is because an $\omega(1)$ gap is introduced in going from (LPRev) to (LP2). However, we can show that if the space of mechanisms is restricted, (LP2) itself is a good relaxation to the optimal
revenue in this space.

In particular, suppose that we restrict the space of mechanisms to be those that are posted price, and sequential by bidder: Depending on the subset of items and bidders left, the mechanism adaptively chooses the next bidder and posts prices for a subset of the remaining items. The bidder, being a utility maximizer, solves a knapsack problem to choose the optimal subset of items. Once this bidder is dealt with, the mechanism again adaptively chooses the next bidder depending on the acceptance strategy of this bidder. Since the optimization problem the bidders need to solve is a two-dimensional Knapsack problem, which is \( \text{NP}- \text{hard} \), we assume that the bidders are monotone optimizers in the following sense.

**Assumption 20.** Suppose that we are given some adaptive posted-price scheme. Consider any pair \((i, j)\), where \(i \in \mathcal{I}, j \in \mathcal{J}\). Fix the valuations \(v_{i,j}\) corresponding to all other (bidder, item) pairs \((i', j') \neq (i, j)\). Also fix the Knapsack heuristics used by all the other bidders \(i' \neq i\). It follows that we can uniquely determine the subset of items that will be offered to bidder \(i\), and the prices at which those items will be offered to bidder \(i\). Under these circumstances, we assume that the quantity of item \(j\) taken by bidder \(i\) is a monotonically non-decreasing function of \(v_{ij}\).

The above assumption holds if the bidder solves Knapsack optimally. Similar to the posted-price scheme in Section 2.4.2, our algorithm itself will allow arbitrary Knapsack heuristics, as long as it satisfies Assumption 20 and the bidder buys all the items if she is not constrained by either demand or budget. Next, we state another assumption that will be used throughout the rest of Section 2.4.3.

**Assumption 21.** The valuations of different bidders for different items follow independent distributions, that is, the random variables \(\{v_{ij}\}_{i \in \mathcal{I}, j \in \mathcal{J}}\) are mutually independent. Furthermore, the distribution \(f_{ij}(\cdot)\), from which the random variable \(v_{ij}\) is drawn, is regular for all \(i \in \mathcal{I}, j \in \mathcal{J}\). 

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Under Assumptions 20 and 21, we will show a \(O(1)\)-approximation to the revenue maximizing adaptive posted-price mechanism. First, we consider the following non-linear program.

Maximize \[ \sum_{i \in I} \sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot 4 \cdot \tilde{p}_{ij}(r) \cdot x_{ij}(r) \] (SEQ)

\[
\sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq d_i \quad \forall i \in I \tag{1}
\]

\[
\sum_{j \in J} \sum_{r \in R_{ij}} g_{ij}(r) \cdot \tilde{p}_{ij}(r) \cdot x_{ij}(r) \leq B_i \quad \forall i \in I \tag{2}
\]

\[
\sum_{i \in I} \sum_{r \in R_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq 1 \quad \forall j \in J \tag{3}
\]

\[
\tilde{p}_{ij}(r) \cdot x_{ij}(r) \leq r \cdot x_{ij}(r) - \sum_{s=1}^{r-1} x_{ij}(s) \quad \forall i \in I, j \in J, r \in R_{ij} \tag{4}
\]

\[
x_{ij}(r) \in [0, 1] \quad \forall i \in I, j \in J, r \in R_{ij} \tag{5}
\]

\[
\tilde{p}_{ij}(r) \in [0, r] \quad \forall i \in I, j \in J, r \in R_{ij} \tag{6}
\]

**Lemma 22.** If the bidders are monotone optimizers, then the objective of the non-linear program (SEQ) upper bounds the revenue of the optimal adaptive posted-price mechanism.

**Proof.** Fix any adaptive posted-price mechanism, and let \(x_{ij}(r)\) denote the probability that bidder \(i \in I\) obtains item \(j \in J\) when \(V_{ij} = r \in R_{ij}\) (Definition 14). Similarly, let \(p_{ij}(r)\) denote the expected price paid by bidder \(i\) for item \(j\), conditioned on the events that \(V_{ij} = r\) and bidder \(i\) obtains item \(j\). Define \(\tilde{p}_{ij}(r) = p_{ij}(r)/4\). It follows that the auctioneer’s revenue is equal to the objective of (SEQ). Constraint 1 of (SEQ) holds since no bidder \(i\) buys more than \(d_i\) items, constraint 2 holds since no bidder pays more than her budget, and constraint 3 holds since no item is allocated more than once.

To interpret constraint 4, fix a bidder \(i \in J\), an item \(j \in J\), and also fix all the valuations \(v_{i'j'}\) corresponding to all other (bidder, item) pairs \((i', j') \neq (i, j)\), and the KNAPSACK heuristics used by every bidder \(i' \neq i\). This uniquely determines the...
subset of items (and their prices) that will be offered to bidder $i$. If the adaptive posted-price mechanism offers item $j$ to bidder $i$, then it posts a unique price $p^*$ for the item. Given these conditions, let $X_{ij}(r) \in [0, 1]$ denote the probability that bidder $i$ takes item $j$ when $V_{ij} = r \in R_{ij}$, and let $P_{ij}(r) = p^* \cdot X_{ij}(r)$ denote the expected price paid by bidder $i$ for item $j$ when $V_{ij} = r \in R_{ij}$. We will show:

$$\frac{P_{ij}(r)}{4} = \left(\frac{p^*}{4}\right) \cdot X_{ij}(r) \leq r \cdot X_{ij}(r) - \sum_{s=1}^{r-1} X_{ij}(s) \quad \text{for all } r \in R_{ij}. \quad (2.16)$$

To verify equation (2.16), we consider two possible cases.

**Case 1:** $p^*/4 > B_i/4$.

In this case, bidder $i$ never takes item $j$ since the price exceeds her budget, implying that $X_{ij}(r) = P_{ij}(r) = 0$ for all $r \in R_{ij}$. Hence, equation (2.16) holds.

**Case 2:** $p^*/4 \leq B_i/4$.

To analyze this case, first recall that we have already fixed both the valuations of bidder $i$ for every item $j' \neq j$, and the subset of items (and their prices) offered to bidder $i$. Next, note that if $V_{ij} = \min(v_{ij}, B_i/4) < B_i/4$, then $v_{ij}$ is also uniquely determined. Hence, we see that $X_{ij}(r)$ is either 0 or 1 for all $r < B_i/4$. However, at $r = B_i/4$, the valuation $v_{ij}$ is not unique and $X_{ij}(r)$ can take a fractional value. Since the bidder is a monotone utility maximizer (Assumption 20), we infer that $X_{ij}(r)$ is a monotone step function of $r$, and this step function has at most one jump. If the step function has no jump at all, then $X_{ij}(r) = 0$ for all $r \in R_{ij}$, and equation (2.16) holds. Consequently, we assume that the step function jumps at $r^* \leq B_i/4$, that is, $X_{ij}(r) = 0$ for all $r < r^*$, and $X_{ij}(r) = X_{ij}(r^*) > 0$ for all $r \geq r^*$. It is easy to see that equation (2.16) holds for all $r < r^*$.

Recall that $p^*/4 \leq B_i/4$. As a consequence, if $V_{ij} = \min(v_{ij}, B_i/4) < p^*/4$, then
\( v_{ij} < p^*/4 < p^* \). Since bidder \( i \) never takes item \( j \) when \( v_{ij} < p^* \), and since \( X_{ij}(r^*) > 0 \), it follows that \( p^*/4 \leq r^* \). To summarize, it remains to verify equation (2.16) only if \( p^*/4 \leq r^* \leq r \leq B_i/4 \). In this situation, we get

\[
\left( \frac{p^*}{4} \right) \cdot X_{ij}(r) = \left( \frac{p^*}{4} \right) \cdot X_{ij}(r^*) \leq r^* \cdot X_{ij}(r^*). \tag{2.17}
\]

Furthermore, we have

\[
r \cdot X_{ij}(r) - \sum_{s=1}^{r-1} X_{ij}(s) = r \cdot X_{ij}(r^*) - (r - r^*)X_{ij}(r^*) = r^* \cdot X_{ij}(r^*). \tag{2.18}
\]

Equation (2.16) follows from equations (2.17) and (2.18). Having verified equation (2.16), we proceed with the proof of the lemma.

Taking expectation over all the remaining valuations \( \{v_{i'j'}\} \), where \( (i',j') \neq (i,j) \), and recalling the definitions of \( x_{ij}(r), p_{ij}(r) \), we see that \( x_{ij}(r) = \mathbb{E}[X_{ij}(r)] \) and \( p_{ij}(r)x_{ij}(r) = \mathbb{E}[P_{ij}(r)] \). Hence, equation (2.16) implies:

\[
\tilde{p}_{ij}(r) \cdot x_{ij}(r) = \frac{p_{ij}(r)}{4} \cdot x_{ij}(r) = \mathbb{E} \left[ \frac{P_{ij}(r)}{4} \right] \leq r \cdot x_{ij}(r) - \sum_{s=1}^{r-1} x_{ij}(s). \tag{2.19}
\]

Thus, constraint 4 of (Seq) holds, and the revenue maximizing adaptive posted-price mechanism is a feasible solution to the nonlinear program (Seq).

**Lemma 23.** If the bidders are monotone optimizers, then (LP2) is a 4-approximation to the revenue of the optimal adaptive posted-price scheme.

**Proof.** First, we take the optimal solution to the non-linear program (Seq). Next, for all \( i \in \mathcal{I}, j \in \mathcal{J} \), let the range of \( \mathcal{V}_{ij} \) be given by \( \mathcal{R}_{ij} = \{1, \ldots, |\mathcal{R}_{ij}|\} \). We do the following:

**For all** \( i \in \mathcal{I}, j \in \mathcal{J} \)

\[
\text{For } r = 1, 2, \ldots, |\mathcal{R}_{ij}|
\]
Increase the value of $\tilde{p}_{ij}(r)$ and decrease the value of $x_{ij}(r)$, ensuring that their product $\tilde{p}_{ij}(r) \cdot x_{ij}(r)$ remains the same, and continue doing this until the corresponding constraint 4 (defined by the tripe $(i,j,r)$) of the non-linear program (SEQ) becomes tight.

This preserves the objective of (SEQ) and all the other constraints. When this process terminates, we get an optimal solution to (SEQ) where the constraint 4 is tight for all $i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}$. Hence, we can replace $\tilde{p}_{ij}(r) \cdot x_{ij}(r)$ by the right hand side of constraint 4, and rewrite the non-linear program (SEQ) as a linear program (LP3).

\[
\text{Maximize } \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot 4 \cdot \left( r \cdot x_{ij}(r) - \sum_{s=1}^{r-1} x_{ij}(s) \right) \quad \text{(LP3)}
\]

\[
\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq d_i \quad \forall i \in \mathcal{I}
\]

\[
\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot (r x_{ij}(r) - \sum_{s=1}^{r-1} x_{ij}(s)) \leq B_i \quad \forall i \in \mathcal{I}
\]

\[
\sum_{i \in \mathcal{I}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq 1 \quad \forall j \in \mathcal{J}
\]

\[
x_{ij}(r) \in [0, 1] \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}
\]

Now, we invoke Lemma 14 and deduce that (LP3) is equivalent to the following linear program.

\[
\text{Maximize } \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} 4 \cdot g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) \quad \text{(LP4)}
\]

\[
\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq d_i \quad \forall i \in \mathcal{I}
\]

\[
\sum_{j \in \mathcal{J}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot \varphi_{ij}(r) \cdot x_{ij}(r) \leq B_i \quad \forall i \in \mathcal{I}
\]

\[
\sum_{i \in \mathcal{I}} \sum_{r \in \mathcal{R}_{ij}} g_{ij}(r) \cdot x_{ij}(r) \leq 1 \quad \forall j \in \mathcal{J}
\]

\[
x_{ij}(r) \in [0, 1] \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, r \in \mathcal{R}_{ij}
\]

Hence, the objective of (LP4) upper bounds the revenue of the optimal adaptive posted-price scheme when the bidders are monotone optimizers. The lemma follows from comparing (LP4) with (LP2).
The final mechanism and analysis are the same as in Section 2.4.2: Solve (LP2), and making use of Assumption 21, decompose the LP solution into a convex combination of posted prices per edge, and sequentially post these prices for every bidder. To complete the analysis, note that Lemma 16 only requires that the distribution $g_{ij}(\cdot)$ be regular. This shows the following theorem:

**Theorem 24.** There is a polynomial time $O(1)$-approximation to the revenue of the optimal adaptive posted-price scheme, when the bidders are monotone optimizers, and the valuations of different bidders for different items are drawn from mutually independent regular distributions.
This chapter is based on the paper by Bhattacharya et al. (2010b). We consider a model where we have multiple homogeneous items, and multiple bidders. Each bidder has a private valuation per unit item, and a private budget. The goal is to design a truthful-in-expectation DSIC mechanism that is ex-post Pareto-Optimal. In other words, regardless of the random choices made by the auction, the following property is satisfied: After the auction’s outcome has been implemented, no pair of agents (including the auctioneer) can simultaneously improve their utilities by trading an item.

The starting point of this work is the “adaptive clinching auction” proposed by Dobzinski et al. (2008) (see Section 3.3). Dobzinski et al. (2008) prove that if the bidders’ budgets are publicly known (but the valuations are private knowledge), then this auction is ex-post Pareto-Optimal and universally truthful DSIC. However, there is no such auction when both the valuations and budgets of the bidders are private knowledge. We show that this negative result can be circumvented if we relax the notion of incentive-compatibility to truthful-in-expectation DSIC (see Section 1.1.1).
3.1 Budget Monotonicity and Randomization

We show an intuitive property of the adaptive clinching auction in the case of divisible items: A bidder cannot gain utility by reporting budget lower than the truth. We term this property Budget Monotonicity. Though this property seems simple, there is no reason to assume it holds: *In fact, this property is false for the adaptive clinching auction in the case of indivisible units* (Theorem 26). The major difficulty in the proof is that the adaptive clinching auction continuously makes allocations at different prices, so that the utility is a complicated function of all the budgets and valuations. In fact, an analysis of this auction is left as an open question\(^1\) in Dobzinski et al. (2008). We show this result by carefully coupling the behavior of two auctions that differ only in the reported budget of one bidder. The proof also establishes several structural results about this auction that are of independent interest. For example, although it is difficult to get closed form expression for the utility of a bidder, our analysis implies the payments and allocations can be computed in polynomial time.

Budget Monotonicity for an infinitely divisible item implies Pareto-optimality fairly directly, since all we need is to prevent a bidder from over-reporting her budget. We can do this in several ways, the simplest being randomization. In the Randomized Extraction Scheme (see Section 3.2), the mechanism simply extracts the whole budget or zero price so that the expected price extracted is equal to the price charged by the deterministic auction. Therefore, if a bidder gets nonzero allocation by over-reporting her budget, then with non-zero probability, she pays her reported budget and her expected utility is \(-\infty\). This scheme can be applied to any deterministic auction to prevent reporting larger budgets than the truth. We show in Section 3.2 that the randomized version preserves Pareto-optimality ex-post, and maintains truthful-in-

\(^1\) Since the focus of Dobzinski et al. (2008) is to prove uniqueness and impossibility, they mainly analyze the auction for two bidders with carefully chosen valuations and budgets. In contrast, we need to develop characterizations for the general case.
expectation DSIC if the deterministic auction was monotone (and charges non-zero price for non-zero allocations).

The monotonicity result for the adaptive clinching auction holds only for one infinitely divisible item. In the case of finitely many indivisible units, we simply run the adaptive clinching auction assuming one infinitely divisible item, and perform a randomized allocation in the end (Corollary 29; also see Abrams (2006); Borgs et al. (2005)). The resulting auction is truthful-in-expectation DSIC, and is also ex-post Pareto-optimal.

Though our mechanism is randomized, the randomness introduced in the price is quite small: *It affects the price charged to only one bidder* (Lemma 35). However, for the randomization to be truthful-in-expectation DSIC, we crucially need the assumption that the utility of a bidder for paying more than her true budget is \(-\infty\). For smoother utility functions, the Budget Monotonicity property can be used in other ways to make the deterministic auction itself truthful and Pareto-optimal: For instance, a standard assumption in spectrum auctions [Brusco and Lopomo (2008)] is that the bidder can be forced to show “proof of funds” for her reported budget (for instance, a bank statement), and this prevents her from over-reporting her budget regardless of her utility function.

*Revenue Properties*

As another consequence of Budget Monotonicity, the revenue properties of the adaptive clinching auction in case of public budgets (shown in [Dobzinski et al. (2008)]) carry over to the randomized version of the auction even with private budgets, and hence, this auction improves the competitive ratio in Abrams (2006) by a factor of 4, and like the auction in Borgs et al. (2005), is asymptotically optimal.

Independently of this work, Hafalir et al. (2009) also consider the problem of designing Pareto-optimal mechanisms with private budgets via randomization, and
present a different mechanism termed Sort-Cut. However, unlike our result, their mechanism is not incentive compatible on valuations, and is hence only Pareto-optimal conditioned on truth-telling by all agents.

Roadmap for the rest of the chapter.

In Section 3.2, we define the notion of Budget Monotonicity. In Section 3.3, we describe the adaptive clinching auction and show some basic properties in the infinitely divisible item case. In Section 3.4, we provide a proof sketch of Budget Monotonicity of this auction. The complete proof is very technical and is presented in Section 3.5.

3.2 Preliminaries

We will mainly consider the case when there is one unit of infinitely divisible item and \( n \) bidders. Bidder \( i \) has a private valuation \( \eta_i \) per unit quantity, and private budget \( \beta_i \). Suppose bidder \( i \) reports valuation and budget \((v_i, B_i)\). The auction is a (randomized) mechanism that (probabilistically) maps the \((\vec{v}, \vec{B})\) into a quantity \( X_i \) the bidder obtains and a total price \( P_i \) the bidder pays; note that these quantities are allowed to be random variables in this chapter. Since there is one unit of the item, we have \( \sum_i X_i \leq 1 \).

The only difference in the case of auctioning \( m \) indivisible copies of the item is that in this case, \( X_i \in \{0, 1, 2, \ldots, m\} \) and \( \sum_i X_i = m \). (The infinitely divisible item case is the limit when \( m \to \infty \).) In the subsequent discussion we will assume one infinitely divisible item unless otherwise stated.

Let \( v_{-i}, B_{-i} \) denote the reported valuations and budgets of bidders other than \( i \).

Bidder \( i \) has the following utility function: If \( P_i > \beta_i \), then his utility is \(-\infty\): this corresponds to the total price exceeding his budget. If \( P_i \leq \beta_i \), then his utility is \( u_i = \eta_i X_i - P_i \).

The goal is to design a randomized auction that satisfies two properties: Incentive
Compatibility (IC) and ex-post Pareto-Optimality (PO). The first condition was defined in Section 1.1.1. We now give a formal definition of the second condition.

**ex-post Pareto-optimality (PO):** We must have (i) $\sum_i X_i = 1$, i.e., the item is completely sold; and (ii) If $X_i > 0$ and $v_j > v_i$, then $P_j = B_j$, i.e., if a bidder gets non-zero quantity then all bidders with higher valuations have exhausted their budgets. This property holds ex-post (regardless of randomization).

In the case of $m$ indivisible units, the Pareto-Optimality condition gets modified as: $\sum_i X_i = m$; further, if $X_i > 0$ and $v_j > v_i$, then $v_i > B_j - P_j$. In both cases, this corresponds to the fact that no pair of agents can improve their utility by trading.

The main focus of this chapter is to understand the behavior of the adaptive clinching auction, which is described in Section 3.3. For a more detailed description (especially for the indivisible units case), see Dobzinski et al. (2008). Their main result is the following:

**Theorem 25** (Dobzinski et al. (2008)). The adaptive clinching auction satisfies ex-post Pareto-Optimality with private budgets and valuations. When the budgets are public knowledge, the auction also satisfies universally truthful dominant strategy incentive-compatibility, and it is the unique auction satisfying universally truthful DSIC and ex-post Pareto-Optimality. Furthermore, there is no auction satisfying these two properties when the budgets are private knowledge.

Our main goal is to show several structural results about this auction, which will culminate in showing that there is indeed a randomized mechanism that is truthful-in-expectation DSIC, and also satisfies ex-post Pareto-optimality, even with private budgets.
3.2.1 Budget Monotonicity and its Consequences

We will show that the adaptive clinching auction with one infinitely divisible item satisfies Budget Monotonicity, which states that a bidder cannot gain by reporting a lower budget.

**Definition 16.** A deterministic auction is Budget Monotone if the following conditions hold for every bidder $i$ regardless of $v_{-i}, B_{-i}$. For reported budget $B_i \in [0, \beta_i]$, where $\beta_i$ is the true budget:

1. The bidder always maximizes utility by reporting $v_i = \eta_i$, where $\eta_i$ is the true valuation.
2. When $v_i = \eta_i$, the utility of the bidder is monotonically non-decreasing in $B_i \in [0, \beta_i]$.

The more interesting condition in the above definition is the second one (the first one following from Dobzinski et al. (2008)). Though Budget Monotonicity is an intuitive property, there is no guarantee that it is satisfied even by reasonable auctions. In fact, quite surprisingly, it does not always hold for the adaptive clinching auction!

**Theorem 26.** In the case of $m = 4$ indivisible units of a item and $n = 3$ bidders, the adaptive clinching auction described in Dobzinski et al. (2008) does not satisfy Budget Monotonicity.

**Proof.** Consider $n = 3$ bidders with the following $(B_i, v_i)$ values: $(B_1, v_1) = (6, 3)$, $(B_2, v_2) = (5, 3)$, and $(B_3, v_3) = (4, 3)$, and suppose these are the true budgets and valuations. It is easy to show that bidders 1 and 2 clinch one unit each at price 2. Bidder 3 obtains zero utility since she can only clinch when price is 3. However, if bidder 3 reports $(3, 3)$, she clinches one unit at price $17/6$ and obtains strictly
positive utility. Therefore, the auction is not monotone. The details are easy to fill
in using the description in Dobzinski et al. (2008).

In sharp contrast, our main result is to show that the adaptive clinching auction
indeed satisfies the Budget Monotonicity property when there is one infinitely divis-
ible item (which is the limiting case of $m$ indivisible items). In particular, our main
theorem is the following:

**Theorem 27 (Budget Monotonicity Theorem).** The adaptive clinching auction
satisfies Budget Monotonicity for one infinitely divisible item.

The key intuitive difference between the divisible and indivisible cases is that in
the former case, there is a nice characterization of bidders receiving non-zero alloca-
tions as those with highest remaining budgets (refer Lemma 32). Budget Monotonic-
ity is equivalent to saying that a bidder cannot gain by under-reporting her budget,
*i.e.*, reporting $B_i < \beta_i$. We now show a simple way to remove the incentive to report
$B_i > \beta_i$.

**Randomized Extraction:** We run the deterministic adaptive clinching auction as
in Dobzinski et al. (2008). The allocation remains the same. However, the price
extraction scheme is randomized as follows. If a bidder reports budget $B_i$ and is sup-
posed to pay $P_i \in [0, B_i]$ according to the deterministic mechanism, then with prob-
ability $P_i/B_i$, we extract her reported budget $B_i$, and with probability $(1 - P_i/B_i)$,
we charge her zero price. Note that this randomization can be applied to any deter-
mminsteric auction where $X_i > 0$ implies $P_i > 0$; the adaptive clinching auction does
satisfy this property. It is now easy to show:

**Theorem 28.** For the case of one infinitely divisible item, the randomized adaptive
clinching auction satisfies truthful-in-expectation DSIC, and is ex-post Pareto-optimal.
Proof. The expected payment of a bidder $i$ after randomization is precisely $P_i \leq B_i$, which is at most the total valuation she receives from the allocation. Thus, her expected utility is nonnegative. To see ex-post Pareto-Optimality, observe that if $P_j = B_j$ before the randomization, the same is true after the randomization. To show truthful-in-expectation DSIC, note that the auction satisfies Budget Monotonicity by Theorem 27, so that for any bidder $i$, we have $v_i = \eta_i$ and $B_i \geq \beta_i$. Furthermore, if bidder $i$ reports a budget $B_i > \beta$ and receives nonzero allocation, then the deterministic adaptive clinching auction charges her a price $P_i > 0$. The randomized auction extracts $B_i$ w.p. $P_i/B_i > 0$ and in this scenario, the utility of the bidder is $-\infty$. Therefore, the bidder will not report $B_i > \beta_i$.

If the allocations generated by the auction for the infinitely divisible case are treated as probabilities of allocation instead (similar to Abrams (2006); Borgs et al. (2005)), the same auction works for the case of indivisible units of the item.

**Corollary 29.** There is a randomized auction satisfying truthful-in-expectation DSIC and ex-post Pareto-Optimality, when there are $m$ indivisible units of the item.

Proof. Run the randomized adaptive clinching auction assuming one infinitely divisible item, with the valuations scaled up by factor of $m$. Modify the allocation step as follows. Suppose the auction should allocate $x_i \in [0, 1]$ to bidder $i$. Choose bidder $i$ with probability $x_i$ and allocate all $m$ units to this bidder.

In the resulting auction, note that the expected utility of a bidder is the same as the utility in the indivisible auction; further, the auction is truthful-in-expectation DSIC by Theorem 28.

To show ex-post Pareto-Optimality, note that the items are completely allocated by the auction. Next, since the infinitely divisible auction satisfies ex-post Pareto-Optimality, if the $m$ units are allocated to bidder $i$, this bidder must have had $X_i > 0$ in the infinitely divisible auction, so that for all $j$ with $v_j > v_i$, we must have
$P_j = B_j$, so that $v_i > B_j - P_j = 0$. Therefore, the auction satisfies Pareto-Optimality regardless of the outcome of randomization.

3.3 The Adaptive Clinching Auction: Infinitely Divisible Case

We now describe the adaptive clinching auction in [Nisan (2009); Ausubel (2004)] in the context of one infinitely divisible item, and show in the next two sections that it satisfies Budget Monotonicity (Theorem 27).

Intuitively, the auction is an ascending price auction. As the price per unit quantity is raised, bidders become inactive because the price has exceeded their valuation. For the remaining (active) bidders, the demand is the amount they are willing to buy given their remaining budget and the current price. Similarly, the supply is the amount of item remaining. For an active bidder, when the supply exceeds the total demand of the other bidders, this bidder clinches the difference at the current price. Since the price is increased continuously and the item is infinitely divisible, the auction defines a differential process.

We describe the clinching auction as a differential process indexed by time $t$, where the price charged per unit quantity increases as time progresses, and the auction continuously allocates (part of) the item and extracts budget. We note that the traditional method is to describe it as a process indexed by the price; however, indexing by time lends itself to an easier analysis. After describing the auction, we present some new observations that characterize its behavior; these will be useful in later sections.

Formally, let $p(t)$ denote the price per unit quantity at time $t \geq 0$. For bidder $i$, let $x_i(t)$ denote the quantity of the item allocated so far to $i$, let $P_i(t)$ denote the price extracted so far from $i$, and let $b_i(t)$ denote the effective budget of the bidder (defined later). Let $S(t) = 1 - \sum_i x_i(t)$ denote the supply of item left with the auctioneer. Initially, $p(0) = P_i(0) = x_i(0) = 0$, $b_i(0) = B_i$, and $S(0) = 1$. In this
section, we will denote the derivative of function $g(t)$ w.r.t. $t$ as $g'(t)$.

Denote the demand of the bidder as $D_i(t) = \frac{b_i(t)}{p(t)}$. If $p(t) < v_i$, this represents the amount of the item bidder $i$ is willing to buy at price $p(t)$. Let $D_{\neg i}(t) = \sum_{j \neq i} D_j(t)$, i.e., the total demand excluding bidder $i$.

**Invariants.** Denote the stopping time of the auction by $f$. In Section 3.3.2, we will show that the adaptive clinching auction satisfies the following invariants for all $t < f$:

**Supply Invariant:** For all bidders $i$, we have $S(t) \leq D_{\neg i}(t) = \sum_{j \neq i} D_j(t)$.

**Clinching Invariant:** $x_i'(t) > 0$ iff both $p(t) < v_i$ (the bidder is active) and $S(t) = D_{\neg i}(t)$.

**Budget Invariant:** If $p(t) < v_i$, $b_i(t) = B_i - P_i(t)$, the true residual budget of the bidder. If $p(t) > v_i$, $b_i(t) = 0$. When $p(t) = v_i$, $b_i(t) \in [0, B_i - P_i(t)]$, and though the demand $D_i(t)$ is well-defined, it will not correspond to any “real” demand, since the bidder will drop out of the auction.

We note that for all $t < f$, we have $S'(t) = -\sum_i x_i'(t)$. Furthermore, we also have $x_i'(t) = -\frac{b_i'(t)}{p(t)}$, since the bidder is being charged price $p(t)$ per unit quantity. The only exception to these conditions is at time $f$ when the auction makes some one-shot allocations; we will define $S(f) = \lim_{t \to f} S(t)$.

**3.3.1 Auction**

In view of the above invariants, each bidder belongs to at least one of the following groups.

**Definition 17.** Define active bidders as $A(t) = \{j|v_j > p(t) \text{ and } b_j(t) > 0\}$; exiting bidders as $E(t) = \{j|v_j = p(t) \text{ and } b_i(t) > 0\}$; and clinching bidders as $C(t) = \{j|j \in A(t) \text{ and } S(t) = D_{\neg i}(t)\}$.
The adaptive clinching auction is now simple to describe, and is described in Figure 3.1. We specify it in terms of the derivatives of the budget, allocation, and prices.

![Adaptive Clinching Auction](image)

**Adaptive Clinching Auction**

(I) **(Stopping Condition)**

If \( \sum_{i \in A(t)} D_i(t) \leq S(t) \) then:

1. At unit price \( p(t) \), preserving the budget constraints allocate:
   - Amount \( \sum_{i \in A(t)} D_i(t) \) to bidders in \( A(t) \).
   - Amount \( S(t) - \sum_{i \in A(t)} D_i(t) \) to bidders in \( E(t) \).

(II) **else if** \( E(t) = \emptyset \) and \( A(t) \neq \emptyset \) **then**:

1. \( p^t(t) = 1 \);
2. For each \( i \in C(t) \) set:
   - \( \ell'_i(t) = -S(t) \);
   - \( P^t_i(t) = S(t) \);
   - \( x^t_i(t) = \frac{\ell'_i(t)}{p(t)} = \frac{S(t)}{p(t)} \).

(III) **else if** \( E(t) \neq \emptyset \) and \( A(t) \neq \emptyset \) **then**:

1. \( p^t(t) = 0 \);
2. For smallest index \( j \in E(t) \) set:
   - \( \ell'_j(t) = -1 \); and
   - \( P^t_j(t) = x^t_j(t) = 0 \).
3. For each \( i \in C(t) \) set:
   - \( \ell'_i(t) = -1 \);
   - \( P^t_i(t) = 1 \); and
   - \( x^t_i(t) = \frac{\ell'_i(t)}{p(t)} = \frac{1}{p(t)} \).

---

\( ^a \) We show in Lemma 33 that the stopping condition is well-defined.

\( ^b \) Fix any ordering of bidders that is independent of the reported \( \vec{v} \) and \( \vec{B} \).

---

**Figure 3.1**: The adaptive clinching auction for one infinitely divisible item.

Given the submitted valuations and budgets, the total allocation \( X_i \) and the total price \( P_i \) can be computed efficiently from the description of the auction; we omit the details. Note that a bidder *clinches* items only when the allocation is made and he is in \( A(t) \). Though the bidder may get some items in Step (I) when he is in \( E(t) \),
we do not consider this clinching, since the bidder gets utility zero from these items (assuming she reports the true valuation).

The key difference between the way we have described the auction and that in Dobzinski et al. (2008) is in Step (III). Here, we have chosen to gradually reduce the budgets of the bidders in \( E(t) \), while if the auction were indexed by price, this step would lead to one-shot allocations. Our method makes the supply \( S(t) \) and the effective budgets \( b_i(t) \) continuous functions. The equivalent formulation of Step (III) in terms of price follows from maintaining the Supply Invariant and stopping condition of the auction (see also Dobzinski et al. (2008)), and is presented below.

**Lemma 30.** If \( t < f \) and \( i \in A(t) \), suppose \( \sum_{j \in A(t), j \neq i} D_j(t) < S(t) \), then bidder \( i \) clinches \( S(t) - \sum_{j \in A(t), j \neq i} D_j(t) \) quantity at price \( p(t) \) in Step (III). When \( t = f \) and \( A(f) \neq \emptyset \), bidder \( i \in A(f) \) clinches a quantity at price \( p(f) \) that exhausts her remaining budget in Step (I).

The above lemma will be critically used in the proof of Budget Monotonicity later. The next theorem simply re-states the positive result in Theorem 25.

**Theorem 31** (Dobzinski et al. (2008)). The adaptive clinching auction satisfies ex-post Pareto-Optimality. Furthermore, for reported budget \( B_i \leq \beta_i \) where \( \beta_i \) is the true budget, the bidder always maximizes utility by reporting the true valuation, \( v_i = \eta_i \).

### 3.3.2 Properties

We will now show some properties of this auction that will be useful later.

**Definition 18.** Define \( b_{\text{max}}(t) = \max_{i \in A(t)} b_i(t) \). Recall that \( f \) as the stopping time of the auction.

We first show that the auction satisfies the invariants. The last two invariants are easy to check: Whenever a bidder \( i \) clinches the item at price \( p(t) < v_i \), we have
\[ b'_i(t) = -p(t)x'_i(t) = -P'_i(t), \] so that \[ b_i(t) = B_i - P_i(t). \] Further, note that if \( i \in E(t), \) the effective budget \( b_i(t) \) of this bidder reduces, so if the price increases beyond \( v_i, \) the effective budget must be identically 0. Therefore, the budget invariant holds. The clinching invariant holds trivially by the description of the auction.

The next result shows the supply invariant, and characterizes the set of bidders that are clinching at any point in time, and a consequent stopping condition based on these bidders.

**Lemma 32.** The following hold for the adaptive clinching auction: (1) If \( C(t) \neq \emptyset, \) then \( C(t) = \{ j \in A(t) | b_j(t) = b_{\text{max}}(t) \}; \) (2) the supply invariant holds for all \( t < f; \) and (3) when a bidder \( i \) drops from \( C(t), \) the auction stops at that time.

**Proof.** First note that for \( t < f, \) the functions \( S(t) \) and \( D_{-i}(t) \) for any \( i \in A(t) \) are continuous. At \( t = 0, \) the former is smaller than the latter. If for all \( t < f, \) \( S(t) < D_{-i}(t), \) then there is nothing to prove. Suppose \( t < f \) be the first time instant when \( S(t) \) becomes equal to \( D_{-i}(t). \) If \( i \notin A(t), \) then \( \sum_{j \in A(t)} D_j(t) \leq D_{-i}(t) = S(t), \) the auction necessarily stops at time \( t, \) that is, \( t = f, \) a contradiction. Thus, bidder \( i \in A(t). \) Furthermore, \( i \) has the largest budget in the set \( A(t), \) since for all \( j \neq i, \) we must have had \( D_{-j}(t) \geq D_{-i}(t) = S(t) \) by the definition of time \( t. \) Therefore, \( b_i(t) = b_{\text{max}}(t) \) Since \( D_{-i}(t) = S(t), \) we have \( i \in C(t). \) For all \( t' < t, \) the set \( C(t) \) is empty by the clinching invariant. We now show that for all subsequent \( t, \) as long as \( i \) has not dropped out, we have \( S'(t) = D'_{-i}(t); b_i(t) = b_{\text{max}}(t); \) and when \( i \) drops out of the auction, the auction stops. This will show all parts of the lemma.

First note that if \( C(t) \) is non-empty, it necessarily has the bidders with highest budget in \( A(t). \) Therefore, if \( i \in C(t), \) then \( b_i(t) \) is necessarily the same as \( b_{\text{max}}(t). \) To show \( i \) clinches continuously, we will show \( S'(t) = D'_{-i}(t) \) when \( i \in C(t), \) so that for all \( t \) as long as bidder \( i \) has not dropped out, \( S(t) = D_{-i}(t), \) and hence bidder \( i \) clinches continuously until she drops out.
Suppose $i \in C(t)$ at some point $t$ and $p(t) < v_i$. We therefore must have $b_i(t) = b_{\text{max}}(t)$ and $S(t) = D_{-i}(t)$. Note that $S(t)$ decreases at the rate of precisely

$$-\sum_{j \in C(t)} x_j^t(t) = c \frac{b_{\text{max}}(t)}{p(t)}$$

if there are $c$ clinching bidders at time $t$.

There are two cases depending on whether $E(t) = \emptyset$ or not. If $E(t) = \emptyset$, then for all $j \in C(t)$:

$$D_j^t(t) = \frac{p(t)b_j^t(t) - b_{\text{max}}(t)p'(t)}{p(t)^2} = -\frac{S(t)}{p(t)} - \frac{b_{\text{max}}(t)}{p(t)^2}$$

Similarly, for $j \in A(t) \setminus C(t)$, we have:

$$D_j^t(t) = -\frac{b_j(t)}{p(t)^2}$$

Note that $S'(t) = c \frac{b_{\text{max}}(t)}{p(t)} = -c \frac{S(t)}{p(t)}$. Since $i \in C(t)$, we have:

$$\frac{d}{dt} (D_{-i}(t)) = -(c - 1) \frac{S(t)}{p(t)} - \frac{D_{-i}(t)}{p(t)}$$

$$= -(c - 1) \frac{S(t)}{p(t)} - \frac{S(t)}{p(t)}$$

$$= -c \frac{S(t)}{p(t)} = S'(t)$$

Suppose now that $E(t) \neq \emptyset$ and $A(t) \neq \emptyset$. Note that $b_k'(t) = -1$ for some $k \in E(t)$. We have: $D_j^t(t) = 0$ for $j \notin C(t) \cup \{k\}$. For $j \in C(t) \cup \{k\}$ we have:

$$D_j^t(t) = \frac{p(t)b_j^t(t) - b_{\text{max}}(t)p'(t)}{p(t)^2} = -\frac{1}{p(t)}$$

$$\Rightarrow \frac{d}{dt} (D_{-i}(t)) = -\frac{c}{p(t)} = -c \frac{b_{\text{max}}(t)}{p(t)} = S'(t)$$

Therefore, $S'(t) = D_{-i}^t(t)$ if $i \in C(t)$, which shows bidder $i$ clinches continuously unless $b_i(t) = 0$ or $p(t) = v_i$. In both cases, $\sum_{j \in A(t)} D_j(t) \leq S(t)$ so that Step (1) kicks in and the auction stops.

\[ 79 \]
The above characterization of $C(t)$ holds only for infinitely divisible items, and is the key reason Budget Monotonicity holds in this case and not in the case of indivisible units. Also note that the auction could stop even if no bidders drop from $C(t)$, but instead, some other set of bidders drop out; therefore, part (3) in the above lemma is a sufficient but not necessary condition for stopping.

The above lemma establishes that for $t < f$, the function $b_{\text{max}}(t)$ is continuous; further, the set $C(t)$, if non-empty, is composed of active bidders $i \in A(t)$ with $b_i(t) = b_{\text{max}}(t)$. We now show that the stopping condition (Step (I)) is well-defined, and relate the prices charged to the stopping condition.

**Lemma 33.** When the auction stops, the bidders in $A(f) \cup E(f)$ have sufficient budget to clinch the quantity $S(f)$ at price $p(f)$.

*Proof.* By definition of the stopping time, for time $t$ approaching $f$ from below, we have $\sum_{i \in A(t)} D_i(t) \geq S(f)$. Note that $A(f) \cup E(f) \subseteq A(t)$. The lemma follows. □

**Lemma 34.** If $i \notin A(t) \cup E(t)$, then $b_i(t) = 0$. If $i \in A(t) \setminus C(t)$, then $b_i(t) = B_i$. Furthermore, if $p(f) < v_i$, then $P_i = B_i$ and $b_i(f) = 0$.

The following lemma shows that the amount of randomness we need to add is small. In particular, we need to randomize the price charged to at most one bidder.

**Lemma 35.** The allocations in Step (I) can be done in a fashion so that when the auction stops, there is at most one bidder $i$ with allocation $X_i > 0$ and price $P_i < B_i$.

*Proof.* At time $f$ the only bidders who can have $P_i \in (0, B_i)$ are bidders in $E(f)$. As $t$ approaches $f$ from below, suppose some bidder in $C(t)$ dropped out causing the auction to stop. At time $t$, the supply invariant holds from the perspective of this bidder who drops out, so that if this bidder is given lowest priority in allocating the remaining supply, the supply exhausts the budget of all bidders except this bidder.
If no clinching bidder drops out or if $C(t) = \emptyset$, then all $i \in E(f)$ had $X_i = 0$, so that the budget can be extracted sequentially from bidders in $E(f)$. This satisfies the lemma.

3.4 The Budget Monotonicity Theorem: Proof Sketch

In this section, we will provide a proof sketch of Theorem 27 for a canonical special case of budgets and valuations; the entire proof is complicated with many cases, and is presented in Section 3.5. The observation that the bidder will always report $v_i = \eta_i$ follows from Dobzinski et al. (2008), re-stated in Theorem 31. We will now show that when $v_i = \eta_i$, a bidder does not gain utility by reporting budget $B_i < \beta_i$, where $\beta_i$ is the true budget. This will complete the proof of Theorem 27, and hence all the results in Section 3.2.1.

3.4.1 Notation

We fix a specific bidder, say Alice, and show monotonicity of her utility with reported budget. We will use sub-script $*$ to denote quantities for this bidder. Let $b_*(t)$ and $v_*$ respectively denote her effective budget at time $t$, and her valuation. Let $P_*(t)$ represent the price extracted from her so far.

For convenience, we will use $t_-$ to denote the limit as $x$ approaches time $t$ from below. Since price increases continuously with time, we can easily replace $p(t_-)$ by $p(t)$ in any algebraic expression. However, if $t$ is the first time instant when the price becomes equal to the valuation of some bidder, then $\{i \mid v_i > p(t)\} \subset \{i \mid v_i > p(t_-)\}$, and so on.

Formula for Utility. For times $t' \leq t'' \leq f$, let $u(t', t'')$ denote the utility gained by Alice as time increased from $t'$ to $t''$. In the computation of utility, we can ignore the contribution from allocation made in Fig. 3.1 when Alice is in $E(f)$, since the allocation is obtained at a price equal to her valuation. If $x_*(t)$ is the fraction of the
item clinched by Alice until time $t$, then, for $t'' < f$ in Steps (II) and (III) of the auction:

\[
\begin{align*}
    u(t', t'') &= \int_{t'}^{t''} (v_a - p(t)) \frac{d}{dt} (x_a(t)) \, dt \\
            &= \int_{t'}^{t''} \frac{(v_a - p(t))}{p(t)} \frac{d}{dt} (b_a(t)) \, dt
\end{align*}
\]  

(3.1)

Moreover, when $t'' = f$, the formula gets modified by the one-shot allocation in Step (I):

\[
\begin{align*}
    u(t', f) &= \frac{(v_a - p(f))}{p(f)} b_a(f) \\
            &+ \int_{t'}^{f} \frac{(v_a - p(t))}{p(t)} \frac{d}{dt} (b_a(t)) \, dt
\end{align*}
\]

Define $u$ as the total utility gained by Alice from the auction.

**Two Auctions.** We let Alice increase her reported budget by an amount $\Delta > 0$, the budgets and valuations of other bidders and Alice’s valuation and true budget remaining the same. Suppose her original reported budget is $B_0^a \leq \beta_a$, and her new reported budget is $B_1^a = B_0^a + \Delta \leq \beta_a$. Denote the former auction (with Alice’s reported budget being $B_0^a$) by LOW and the latter auction by HIGH. We will use superscripts 0 and 1 to denote quantities in these two auctions respectively. Note that for $i \neq$ Alice, we have $B_i^1 = B_i^0$.

We will show the following theorem (proved in Section 3.5), which will imply the proof of Theorem 27. This will also imply all results in Section 3.2.1.

**Theorem 36.** $u^0 \leq u^1$, i.e., Alice’s utility from auction HIGH is at least her utility from auction LOW.
3.4.2 Proof Sketch

The proof of Theorem 36 is very technical and is described in Section 3.5. We outline the basic argument for a special case where the valuations are sufficiently large so that Alice clinches for a finite amount of time in both the auctions. The following definition describes the times at which Alice starts and stops clinching, and the time at which the clinching set becomes nonempty.

**Definition 19.** Let $y^0$ (resp. $y^1$) denote the first time instant at which some bidder enters the clinching set in auction Low (resp. High), that is, $C^0(t)$ (resp. $C^1(t)$) becomes nonempty. Similarly, define $q^0$ (resp. $q^1$) to be the first time when Alice enters the clinching set in auction Low (resp. High).

**Simplifying Assumptions.** The valuations of the bidders are sufficiently large so that Alice clinches in both auctions, i.e., $y^0 \leq q^0 < f^0$ and $y^1 \leq q^1 < f^1$. Alice has the minimum valuation amongst all the bidders, so that no other bidder drops out before Alice, and by Lemma 32, the auction stops when Alice drops out, so that $f^0 = f^1 = f$ and $p(f) = v_\star$. Moreover, the highest budgeted bidder (say bidder 1) has larger budget than Alice in both the auctions Low and High, that is, $B_1 > B^1_\star = B^0_\star + \Delta$.

We now track the two auctions simultaneously as time increases from zero. First note that since Step (III) in Figure 3.1 is never executed, and price increases at rate 1 in Step (II), as long as both auctions run, the prices in the two auctions are coupled as time progresses, and further, the set $A(t)$ is the set of all bidders. Therefore, we can use price and time interchangeably. Since Alice cannot gain any utility after dropping out, we have the following for the utilities of Alice in either auction: $u^0 = u^0(q^0, f_-)$ and $u^1 = u^1(q^1, f_-)$. The main ingredient in the proof is to show the following relation, which implies that though Low starts clinching before High, Alice starts clinching in High earlier than when she starts clinching in Low:
Lemma 37. $y^0 < y^1 \leq q^1 \leq q^0 < f$.

Proof. First note that since bidder 1 has the maximum initial budgets in both the auctions, by Lemma 32, it must be the case that bidder 1 starts clinching in LOW at time $y^0$, and in HIGH at time $y^1$. Beyond time $y^0$ in LOW, the quantity $b^0_{\text{max}}(t)$ decreases with time at rate equal to the supply, $S^0(t)$. Since $S^0(t) \leq 1$, the rate of decrease is at most 1, which easily implies:

$$b^0_{\text{max}}(t) \geq b^0_{\text{max}}(y^0) + (y^0 - t) \quad (3.2)$$

Next note that $b^0_{\text{max}}(y^0) = B_1 > B_1^1 = B^0_s + \Delta$. Combining the above relations, we have $b^0_{\text{max}}(y^0 + \Delta) > B^0_s$ and hence, by the Clinching Invariant, Alice is not clinching at time $y^0 + \Delta$, so that by definition, $q^0 > y^0 + \Delta$. It is straightforward to see that $y^1 = y^0 + \Delta$, since the total of the budgets of bidders 2, 3, …, $n$ differs by exactly $\Delta$, and hence the Clinching Invariant kicks in $\Delta$ time later. Therefore, $q^0 > y^0 + \Delta = y^1$.

We will next show that $q^1 \in [y^1, q^0]$. Note that both auctions are clinching beyond $y^1$. Using Equation (3.2) at $t = y^1$, we must have $B_1 = b^1_{\text{max}}(y^1) = b^0_{\text{max}}(y^0) \leq b^0_{\text{max}}(y^1) + \Delta$. Next, we use the observation (Lemma 32) that the clinching sets $C(t)$ in the two auctions are related to the value $b_{\text{max}}(t)$, which decreases at rate $S(t)$. The clinching set if non-empty is precisely $\{i | b_i(t) = b_{\text{max}}(t)\}$. Therefore the auction with the larger $b_{\text{max}}(t)$ has a smaller clinching set. Using this, we show that in the auction with larger $b_{\text{max}}(t)$, this value decreases at a faster rate. Specifically, if $b^1_{\text{max}}(t) \geq b^0_{\text{max}}(t)$ for $t \geq y^1$, then for all bidders $i$, $b^1_i(t) \geq b^0_i(t)$, and hence, the demands are larger in auction HIGH. By the Clinching Invariant, this implies the supply $S(t)$ is larger in auction HIGH, and hence, the rate of decrease of $b_{\text{max}}(t)$ is larger.

$$b^1_{\text{max}}(t) \geq b^0_{\text{max}}(t) \Rightarrow \frac{d}{dt} (b^1_{\text{max}}(t)) \leq \frac{d}{dt} (b^0_{\text{max}}(t)) \leq 0; \quad \forall t \geq y^1$$
Since $b_{\text{max}}(y^1) \leq b_{\text{max}}(y^1) + \Delta$, we must have for all $t < f$: $b_{\text{max}}^1(t) \leq b_{\text{max}}^0(t) + \Delta$. Specifically, $b_{\text{max}}^1(q^0) \leq b_{\text{max}}^0(q^0) + \Delta = B^0_a + \Delta = B^1_a$. Thus, $q^1 \leq q^0$, which completes the proof.

This shows Alice is clinching in both auctions beyond $q^0$, which helps us relate her utilities. Observe by Eq. (3.1) that the utilities are related to the rate of decrease of $b_{\text{max}}(t)$. Using a similar argument to the above:

$$b_{\text{max}}^1(t) \geq b_{\text{max}}^0(t) \iff \frac{d}{dt} b_{\text{max}}^1(t) \leq \frac{d}{dt} b_{\text{max}}^0(t) \leq 0; \forall t \geq q^0$$

We consider two cases. First, if $b_{\text{max}}^1(q^0) \geq b_{\text{max}}^0(q^0)$, then we have $\frac{d}{dt} (b_{\text{max}}^1(t)) \leq \frac{d}{dt} (b_{\text{max}}^0(t))$, and $b_{\text{max}}^1(t) \geq b_{\text{max}}^0(t)$ for all $t \geq q^0$. Applying Equation 3.1, we have:

$$u^1 \geq \int_{q^0}^{q^1} -\frac{(v_a - p(t))}{p(t)} \frac{d}{dt} (b_{\text{max}}^1(t))$$

$$\geq \int_{q^0}^{q^1} -\frac{(v_a - p(t))}{p(t)} \frac{d}{dt} (b_{\text{max}}^0(t))$$

$$= u^0$$

If $b_{\text{max}}^1(q^0) < b_{\text{max}}^0(q^0)$, then we need to take into account the utility gained by Alice in HIGH during the time interval $q^1 \leq t \leq q^0$ to complete the proof; details are in Section 3.5.

The reason the general case is complicated is that we need to take care of two tricky issues: (1) One of the auctions can stop due to bidders dropping out. We need to account for this event in several of the proofs. (2) If $q^0 = f^0$, then Alice obtains only a one-shot allocation in Step (I) of the auction. In this case, we have an explicit formula for the utility of Alice in auction LOW, and we essentially argue that the auction HIGH derives at least that much utility at price $p(q^0)$. This shows Theorem 36 already holds in this case.
3.5 Proof of Theorem 36

In this section, we describe the proof of Theorem 36 in its full generality. We note that by Theorem 31, the reported valuations are always the truth, meaning that \( v_i = \eta_i \) for all bidders \( i \).

Recall that bidder Alice increases her budget by a quantity \( \Delta \) in auction \textsc{High} as compared to auction \textsc{Low}. Also recall that the subscript * is used to denote quantities for Alice, and the superscripts 1, 0 to denote quantities in auctions \textsc{High} and \textsc{Low} respectively. We will define the following starting and stopping times.

**Definition 20.** Let \( y^0 \) (resp. \( y^1 \)) denote the first time instant at which some bidder enters the clinching set in auction \textsc{Low} (resp. \textsc{High}), that is \( C^0(t) \) (resp. \( C^1(t) \)) becomes nonempty. Similarly, define \( q^0 \) (resp. \( q^1 \)) to be the first time instant when Alice enters the clinching set in auction \textsc{Low} (resp. \textsc{High}). If the required event does not happen, define these as \( f^0 \) (resp. \( f^1 \)).

Recall that we will use \( t_- \) to denote the limit as \( x \) approaches time \( t \) from below. Since price increases continuously with time, we can easily replace \( p(t_-) \) by \( p(t) \) in any algebraic expression. However, if \( t \) is the first time instant when the price becomes equal to the valuation of some bidder, then \( \{ i \mid v_i > p(t) \} \subseteq \{ i \mid v_i > p(t_-) \} \), and so on.

### 3.5.1 Assumptions

We now show that the theorem is straightforward if some assumptions do not hold. First, note that if \( p^0(q^0) = v_* \), Alice receives zero utility in \textsc{Low}, and Theorem 36 is trivially true. Thus, we must have:

**Assumption 1.** Alice receives non-zero utility in auction \textsc{Low}. In other words, \( u^0 > 0 \) and \( p^0(q^0) < v_* \).
Using this assumption, we show that the prices in the two auctions are coupled. Let $p^0(t)$ and $p^1(t)$ denote the prices at time $t$ in Low, High respectively.

**Claim 38.** For all $t \leq \min(f^0, f^1)$, $p^0(t) = p^1(t)$

*Proof.* At the beginning, $p^0(0) = p^1(0) = 0$. Simultaneously follow auction Low and High as time increases from zero. When the price is not equal to the valuation of any bidder, both $p^0(t)$ and $p^1(t)$ are increasing at unit rate. When the price $p$ hits the valuation of some bidder(s), two cases may occur. If the set $\{ i \mid v_i = p \}$ has nonempty intersection with $C^0(t_{-})$ (resp. $C^1(t_{-})$), then auction Low (resp. High) necessarily stops at that time $t = \min(f^0, f^1)$. Otherwise, if none of the bidders with valuation equal to $p$ belonged to $C^0(t_{-}) \cup C^1(t_{-})$, then price remains equal to $p$ in both the auctions for exactly $\sum_{i : v_i = p} B_i$ amount of time. Note that in the later case, Alice cannot have a valuation equal to $p$, else she receives zero utility in both the auctions and Theorem 36 is trivially true.

From now on, we will use $p(t)$ to denote both $p^1(t)$ and $p^0(t)$. A direct consequence of the above proof is the following, whose proof is simple and omitted. Note that $E(t)$ are coupled since the auctions do not stop (so that all bidders in $E(t)$ could not have been clinching), and Step (III) reduces the budgets of these bidders in a fixed order.

**Corollary 39.** For all $t < \min(f^0, f^1)$, $A^0(t) = A^1(t)$. Further, $E^0(t) = E^1(t)$.

We will now show another assumption whose violation easily implies Theorem 36.

**Assumption 2.** Auction High stops at a time that is strictly greater than the price at which Alice starts to clinch in auction Low, that is $f^1 > q^0$.

**Claim 40.** If Assumption 2 is violated, Theorem 36 is true.
Proof. Suppose $f^1 \leq q^0$. Clearly, $q^0 \leq f^0$. From Assumption 1, $p(f^1) \leq p(q^0) < v_s$.

From the Lemma 34, $P_s^1 = B_s^1$. In auction HIGH, Alice receives at least $B_s^1 \frac{f^1}{p(f^1)}$ fraction of the item at an average unit price that is at most $p(f^1)$. That is,

$$u^1 \geq \frac{v_s - p(f^1)}{p(f^1)} B_s^1$$

However, in auction LOW, she can receive at most $\frac{B_s^0}{p(q^0)}$ fraction of the item at an average unit price that is at least $p(q^0)$. That is,

$$u^0 \leq \frac{v_s - p(q^0)}{p(q^0)} B_s^0$$

Since $p(f^1) \leq p(q^0)$ and $B_s^1 > B_s^0$, we get $u^1 \geq u^0$. This implies Theorem 36. \qed

We will use Assumptions 1 and 2 several times throughout the rest of Section 3.5.

3.5.2 The Canonical Case: Alice Enters Set $C(t)$ in Auction LOW, that is $q^0 < f^0$

The argument consists of two stages. First we relate the times at which Alice starts to clinch in either auction, in particular, we show that Alice starts clinching in HIGH no later than in LOW. This statement is critically used in the next stage of our proof, where we compare the utilities gained by Alice in the two auctions, and show that her utility from LOW is at most her utility from HIGH.

Lemma 41 (Structure Lemma). The starting and stopping times in HIGH and LOW are related as:

$$y^0 \leq y^1 \leq q^1 \leq q^0 < \min(f^0, f^1)$$

Since $q^0 < f^0$, Assumption 2 immediately implies the last inequality. Most important part of the above lemma is the claim that $q^1 \leq q^0$, i.e., Alice joins the clinching set no later in HIGH than in LOW.
Proof of the Structure Lemma

Let bidder 1 have the largest budget among all active bidders excluding Alice at the time when auction Low starts clinching. We first present a high-level idea of the proof. At any point in time, the set of clinching bidders, if non-empty, is the set of active bidders $i$ with $b_i(t) = b_{\text{max}}(t)$; furthermore, once the auction starts clinching, $b_{\text{max}}(t)$ decreases continuously. We therefore relate the evolution of $b_{\text{max}}(t)$ in the two auctions, and show the time $t$ at which $b_{\text{max}}(t) = B_s^0$ in auction Low is at least the time $t$ at which $b_{\text{max}}(t) = B_s^1$ in auction High. We use the following observations about the curves $b_{\text{max}}^0(t)$ and $b_{\text{max}}^1(t)$ in auctions Low and High respectively:

1. The curves have downward slope at most 1, and are parallel.

2. Auction High starts clinching at most $\gamma$ time after Low starts clinching, where $\gamma = \min(B_1, B_s^*) - B_s^0$. In particular, High starts clinching before time $q^0$; and

3. If $b_{\text{max}}^0(t) \leq b_{\text{max}}^1(t)$, then $b_{\text{max}}^1(t)$ decreases at a faster rate once both auctions are clinching.

Using these observations, the proof is simple geometry with two cases depending on whether $B_1 \leq B_s^1$ or $B_1 \geq B_s^1$, i.e., whether or not Alice has the highest budget in auction High. We now present the proof in the following sequence of claims.

**Claim 42.** $y^0 \leq y^1$. Furthermore, for all $t \leq y^0$, we have the following: If bidder $i$ is not Alice, then $b_{i}^0(t) = b_i^1(t)$. In particular, $b_{i}^0(t) = b_i^1(t) = B_i$ when $p(t) < v_i$. For Alice, $b_{a}^0(t) = B_s^0 < b_{a}^1(t) = B_s^1$.

**Proof.** As time $t$ increases gradually from $t = 0$, as long as no bidder is clinching in either auction (i.e., $C^0(t) = C^1(t) = \emptyset$), the current budget $b_i(t)$ of every active bidder $i$ equals her original budget $B_i$. Furthermore, Step (III) reduces the budgets of exiting bidders in a fixed order. We conclude that the current budget of every
bidder other than Alice remains the same across the two auctions, and the current budget of Alice is greater in auction High than in Low. Thus, from the perspective of any bidder (including Alice), the total demand of the other bidders is no less in auction High than in auction Low. In particular, it implies Low starts clinching no later than High, that is, \( y^0 \leq y^1 \).

**Lemma 43.** If Alice is the first bidder to join the clinching set \( C_0(t) \) in auction Low, i.e., if \( y^0 = q^0 \), then the Structure Lemma holds.

**Proof.** Suppose \( y^0 = q^0 \). From Alice’s perspective, in both the auctions, total demand of other bidders is exactly equal to the initial supply at time \( t = y^0 \) (Claim 42). Thus, Alice joins the clinching set in High at the same time instant as in Low. Hence \( y^0 = y^1 = q^1 = q^0 \) and the Structure Lemma holds.

For the rest of Section 3.5.2, we assume that Alice is not the first bidder to start clinching in Low, so that \( y^0 < q^0 \). Therefore bidder 1 with \( B_1 > B^0_\star \) starts clinching in Low at time \( y^0 \). So far, we have the following inequalities:

\[
y^0 < q^0 < \min\left(f^0, f^1\right),
\]

\[
y^0 \leq y^1 \text{ and } B_1 > B^0_\star
\]

(3.3)

The above implies \( b_{\max}^0(y^0) = B_1 = B^0_\star + \delta \) for some \( \delta > 0 \). Note that the active sets \( A(t) \) in the two auctions are identical at any point in time (Corollary 39). Applying Lemma 32 and Assumption 1, we have that both Alice and bidder 1 are active during the time interval \( y^0 \leq t \leq q^0 \). Furthermore, for all \( t \in [y^0, q^0] \), bidder 1 belongs to the clinching set in Low, and has the maximum budget amongst all the active bidders, that is, \( b_{\max}^0(t) = b_1^0(t) \).

**Claim 44.** In auctions Low and High, \( b_{\max}(t) \) decreases at a rate at most one, i.e.,

\[
\frac{d}{dt}(b_{\max}(t)) \in [-1, 0].
\]
Proof. If the clinching set $C(t)$ is empty, then $\frac{d}{dt}(b_{\text{max}}(t)) = 0$. Otherwise, by Lemma 32, either $\frac{d}{dt}(b_{\text{max}}(t)) = -1$ in Step (III), or $\frac{d}{dt}(b_{\text{max}}(t)) = -S(t)$ in Step (II). Since $S(t) \leq 1$, the claim follows.

Claim 45. Recall $\delta = B_1 - B^0_*$. In auction Low, Alice starts clinching at least $\delta$ later than the time instant at which bidder 1 starts clinching, i.e., $q^0 \geq y^0 + \delta$.

Proof. As time increases beyond $y^0$, by Lemma 32, Alice starts clinching in Low when $b^0_{\text{max}}(t)$ becomes equal to Alice’s reported budget $B^0_*$. Since $b^0_{\text{max}}(y^0) = B_1 = B^0_* + \delta$, and since $b^0_{\text{max}}(t)$ decreases at a rate at most one, we have the claim.

Claim 46. Suppose bidder 1 has higher initial budget than Alice in High i.e., $B_1 > B^1_*$, then we have $y^1 = y^0 + \Delta < q^0$.

Proof. Since $B^0_* + \delta = B_1 > B^1_* = B^0_* + \Delta$, we have $\Delta < \delta = B_1 - B^0_*$. Thus, the inequality $y^0 + \Delta < q^0$ follows from Claim 45. Applying the Clinching Invariant and Claim 42 in High, we have at $t = y^0$:

$$p(t) = p(t)S^0(t) = b^0_{-1}(t) = b^1_{-1}(t) - \Delta$$

As time increases beyond $y^0$, as long as there is no clinching in High, either the LHS increases at rate 1 in Step (II) or the RHS decreases at rate one in Step (III). In either case, at time $t = y^0 + \Delta$, we must have $p(t) = b^1_{-1}(t)$, which implies $y^1 = y^0 + \Delta$.

Claim 47. Suppose bidder 1 has higher initial budget than Alice in auction High i.e., $B_1 > B^1_*$. If $b^1_{\text{max}}(t) \geq b^0_{\text{max}}(t)$ at some time instant $t \in [y^1, q^0)$, then $\frac{d}{dt}(b^1_{\text{max}}(t)) \leq \frac{d}{dt}(b^0_{\text{max}}(t)) \leq 0$.

Proof. Note that clinching set is nonempty in both auctions in this time range. The active and exiting sets, $A(t)$ and $E(t)$, in the two auctions are coupled (Corollary 39). If the existing set $E(t)$ is nonempty, then $b^0_{\text{max}}(t)$ and $b^1_{\text{max}}(t)$ are each decreasing at
rate one in Step (III), and the claim is true. For rest of the proof, assume $E(t)$ is empty.

We first show that compared to auction Low, every active bidder has larger remaining budget in auction High, i.e., $b^1_i(t) \geq b^0_i(t)$. Since $b^1_{\text{max}}(t) \geq b^0_{\text{max}}(t)$, the statement is clearly true for all bidders who are clinching either in High or in Low. For all other active bidders, the current budget $b_i(t)$ equals the reported budget $B_i$. Since Alice reports a higher budget in High and every other bidder reports the same budget in the two auctions, the statement is valid even for active bidders who are not clinching in both auctions.

Since Alice reports a lower budget than bidder 1 in High, bidder 1 is clinching in High the time range $[y^1, q^0]$. Considering the Supply Invariant from the perspective of bidder 1, we conclude that

$$S^1(t) - S^0(t) = \sum_{i \neq 1} \frac{b^1_i(t)}{p(t)} - \sum_{i \neq 1} \frac{b^0_i(t)}{p(t)}$$

$$= \sum_{i : \text{i} \in A(t), i \neq 1} \frac{b^1_i(t) - b^0_i(t)}{p(t)} \geq 0 \tag{3.4}$$

The above holds since the exiting set $E(t)$ is empty and since all active bidders have larger remaining budget $b_i(t)$ in High. The claim follows immediately from Step (II) of Figure 3.1.

\[\square\]

**Lemma 48 (Lemma 41).** The starting and stopping times in High and Low are related as:

$$y^0 \leq y^1 \leq q^1 \leq q^0 < \min(f^0, f^1)$$

**Proof.** All we need to show is $q^1 \leq q^0$. We split the proof into cases depending on whether Alice has the highest budget in High or not.
Case 1. $B_1^1 \geq B_1$: At time $t = y^0$, in both auctions the current budget of every active bidder equals her initial budget (Claim 42). Since Alice reports a higher budget than bidder 1 in High, Alice has the highest budget amongst all the active bidders in High at time $t = y^0$. By Assumption 1 and Claim 45, Alice is active during the time interval $[y^0, q^0] \supseteq [y^0, y^0 + \delta]$. As time increases beyond $y^0$, as long as Alice is active, no other bidder can start clinching before Alice in auction High (Lemma 32). Considering the Clinching Invariant for auction Low, at time $t = y^0$,

$$p(t) = p(t)S^0(t) = b^0_{-1}(t) = b^1_{-\text{Alice}}(t) - \delta$$

The last equality follows from Claim 42. In auction High, in the time range $[y^0, y^0 + \delta]$, either the LHS increases at rate 1 in Step (II) or the RHS decreases at rate 1 in Step (III), so that Alice must start clinching at $y^1 = q^1 = y^0 + \delta$. Combining this with Claim 45, we have the proof.

Case 2. $B_1 > B_1^1$: Since Alice’s reported budget in High is less than that of bidder 1, bidder 1 has the maximum budget amongst active bidders in High at time $t = y^0$ (Claim 42). By Lemma 32, at every time instant $t \in [y^0, q^0]$, we have $b^1_{\text{max}}(t) = b^1_1(t)$. Therefore, during the interval $[y^0, q^0]$, in both auctions, $b_{\text{max}}(t)$ is equal to the current budget of bidder 1. We simultaneously track $b_{\text{max}}(t)$ of the two auctions in this time range (see Figure 3.2). At time $t = y^0$, both $b^0_{\text{max}}(t)$ and $b^1_{\text{max}}(t)$ are equal to $B_1$ (initial budget of bidder 1). Bidder 1 starts clinching in Low at the same time instant. Thus, $b^1_{\text{max}}(t)$ starts decreasing continuously as $t$ increases beyond $y^0$. However, $b^1_{\text{max}}(t)$ decreases below $B_1$ only after $t$ goes past the value $y^1$ (note that $y^1 \geq y^0$ by Claim 42). Claim 46 shows that $y^1$ and $y^0$ differs by exactly $\Delta = B_1^1 - B_0^1$ amount. In particular, since $y^1 \leq q^0$, $b^1_{\text{max}}(t)$ starts decreasing before Alice enters the clinching set in Low. Applying Claim 44, at time time $t = y^1$, the vertical distance between the curves $b^0_{\text{max}}(t)$ and $b^1_{\text{max}}(t)$ is no more than $\Delta$. Claim 47 implies that
Figure 3.2: Proof of the Structure Lemma for the case $B_1 > B_{s1}$. Note that HIGH starts clinching at most $\Delta = B_{s1} - B_s^0$ time later, and beyond this point, for any $t$, $b_{max}^1(t)$ decreases at least as fast as $b_{max}^0(t)$.

in the time range $y^1 \leq t < q^0$, whenever the curve $b_{max}^1(t)$ lies above $b_{max}^0(t)$, the former reduces at a larger rate. We thus have $b_{max}^1(q^0) \leq b_{max}^0(q^0) + \Delta$. Since Alice starts clinching in LOW at time $q^0$, $b_{max}^0(t)$ equals her reported budget $B_s^0$ at that time instant. In other words, $b_{max}^1(q^0) \leq B_s^0 + \Delta = B_{s1}^1$, and Alice must have joined the clinching set in HIGH no later than $q^0$. This completes the proof.

Figure 3.2 illustrates the geometric intuition behind the above proof in the case $B_1 > B_{s1}$.

Relating the Utilities in Auctions Low and High

By Lemma 32, Alice clinches in LOW (resp. HIGH) throughout the time interval $q^0 \leq t < f^0$ (resp. $q^1 \leq t < f^1$). During the next phase of our proof, we simultaneously track the two auctions as time increases from $q^0$ to $\min(f^0, f^1)$ and show that $b_{max}(t)$ of one of the auctions dominates the other. This helps us compare the utilities gained by Alice during this phase, including the utilities from one-shot allocations at the stopping times.

Define $f_{min} = \min(f^0, f^1)$. Note that the Structure Lemma implies $\max(y^0, y^1) <
\( f_{\min} \). In particular, by Lemma 32, Alice clinches in auction High (resp. Low) during the time interval \( q^1 \leq t < f_{\min} \) (resp. \( q^0 \leq t < f_{\min} \)). We will need the following two claims. The proofs are similar to that of Claim 47. By the Structure Lemma, Alice is clinching in both auctions when \( q^0 \leq t < f_{\min} \), and we only need to replace bidder 1 by Alice in Equation 3.4.

**Claim 49.** If \( b_{\max}^1(t) \geq b_{\max}^0(t) \) at some time \( q^0 \leq t < f_{\min} \), then \( \frac{d}{dt} (b_{\max}^1(t)) \leq \frac{d}{dt} (b_{\max}^0(t)) \). Therefore, if \( b_{\max}^1(q^0) < b_{\max}^0(q^0) \), then for all \( t \in [q^0, f_{\min}] \), \( b_{\max}^1(t) \leq b_{\max}^0(t) \).

**Claim 50.** If \( b_{\max}^1(t) \leq b_{\max}^0(t) \) at some time \( q^0 \leq t < f_{\min} \), then \( \frac{d}{dt} (b_{\max}^1(t)) \leq \frac{d}{dt} (b_{\max}^0(t)) \). Therefore, if \( b_{\max}^1(q^0) \geq b_{\max}^0(q^0) \), then for all \( t \in [q^0, f_{\min}] \), we have \( b_{\max}^1(t) \geq b_{\max}^0(t) \).

**Case 1:** \( b_{\max}^1(q^0) \geq b_{\max}^0(q^0) \)

We will now prove Theorem 36 in two cases. We will first prove Theorem 36 under the assumption that \( b_{\max}^1(q^0) \geq b_{\max}^0(q^0) \).

**Lemma 51.** If \( b_{\max}^0(q^0) \leq b_{\max}^1(q^0) \), then \( u^0 \leq u^1 \).

**Proof.** We first show \( f^0 \leq f^1 \). Suppose \( f^0 > f^1 \). If some bidder \( i \) other than Alice is clinching in High just before time \( f^1 \), then \( B_i \geq b_{\max}^1(f^1) = b_{\max}^0(f^1) \geq b_{\max}^0(f^1) \). Thus, \( i \in C^0(f^1) \). Also note that Alice \( \in C^0(f^1) \cap C^1(f^1) \). Thus, \( C^1(f^1) \subseteq C^0(f^1) \).

If auction High stops at time \( f^1 \) because some bidder in \( C^0(f^1) \) drops out, then Low will also stop at \( f^1 \), a contradiction. Thus, assume none of the bidders with valuation equal to \( p(f^1) \) is in the set \( C^0(f^1) \). All those bidders will retain their initial budgets in both the auctions till time \( f^1 \). Therefore the excess-demand (that is, \( \sum_{k : v_k > p(t)} D_k(t) - S(t) \)) will reduce by the same quantity in both the auctions at time \( t = f^1 \). Now, by the clinching invariant, the difference between the excess
demands between auctions HIGH and LOW at time \( f_1 \) is precisely \( \frac{b_1^1(f_1^1) - b_0^0(f_0^0)}{p(f_1^1)} \), so that since \( b_0^0(f_1^1) \leq b_1^1(f_1^1) \), the excess-demand in LOW is less than that of HIGH. Since the excess demand in HIGH becomes non-positive at \( t = f_1 \) (stopping condition), we conclude that excess-demand in LOW will become non-positive at time \( t = f_1 \) so that LOW will stop at that time, again a contradiction.

We thus have \( q^1 \leq q^0 < f_1 \) and \( b_0^0(t) = b_1^0(t) \leq b_1^1(t) = b_1^1(t) \) in the interval \( q^0 \leq t < f_0 \). Applying Claim 49 and Equation 3.1,

\[
\begin{align*}
 u^0(q^0, f_0^0) &= \int_{q^0}^{f_0^0} \left( v_i^* - \frac{p(t)}{p(t)} \right) \frac{d}{dt} \left( b_0^0(t) \right) dt \\
 &\leq \int_{q^0}^{f_0^0} \left( v_i^* - \frac{p(t)}{p(t)} \right) \frac{d}{dt} \left( b_1^1(t) \right) dt \\
 &= u^1(q^0, f_0^0)
\end{align*}
\]

Since auction LOW stops at time \( f_0 \) and Alice \( \in C^0(f_0^0) \), we can bound the utility of Alice from the final one shot allocation in LOW as

\[
u^0(f_0^0) \leq \frac{(v_i^* - p(f_0^0))}{p(f_0^0)} b_0^0(f_0^0)
\]

Assume \( v_i^* > p(f_0^0) \), else we are already done. In this case, since Alice \( \in C^0(f_0^0) \cap C^1(f_0^0) \), we have

\[
S^0(f_0^0) - \sum_{i \neq \text{Alice}} D_i^0(f_0^0) \\
= S^1(f_0^0) - \sum_{i \neq \text{Alice}} D_i^1(f_0^0) \\
= 0
\]

Following the proof of Claim 47, we have \( b_i^0(f_0^0) \leq b_i^1(f_0^0) \) for all bidders \( i \) with
\( v_i = p(f^0) \). Therefore,
\[
S^0(f^0) - \sum_{i \neq Alice, v_i > p(f^0)} D_i^0(f^0) \\
\leq S^1(f^0) - \sum_{i \neq Alice, v_i > p(f^0)} D_i^1(f^0)
\]

Since auction Low stops at time \( f^0 \), we have:
\[
\frac{b^0_v(f^0)}{p(f^0)} \leq S^0(f^0) - \sum_{i \neq Alice, v_i > p(f^0)} D_i^0(f^0) \\
\leq S^1(f^0) - \sum_{i \neq Alice, v_i > p(f^0)} D_i^1(f^0)
\]

Thus, by Lemma 30, in auction High, Alice gets at least \( \frac{b^0_v(f^0)}{p(f^0)} \) fraction of the item at price \( p(f^0) \), and hence:
\[
u^1(f^0, f^1) \geq \frac{(v_* - p(f^0))}{p(f^0)} b^0_v(f^0) \geq u^0(f^0)
\]

Therefore, \( u^0 = u^0(q^0, f^0) + u(f^0) \leq u^1(q^0, f^0) + u^1(f^0, f^1) = u^1 \). This completes the proof.

\( \square \)

Case 2: \( b^1_{\text{max}}(q^0) < b^0_{\text{max}}(q^0) \)

We will now prove Theorem 36 for the case when \( b^1_{\text{max}}(q^0) < b^0_{\text{max}}(q^0) \); this will complete its proof assuming \( q^0 < f^0 \).

We now show a sequence of claims bounding the utility obtained in various phases of the auction.

Claim 52. If \( b^1_{\text{max}}(q^0) < b^0_{\text{max}}(q^0) \), then for all \( t \in [q^0, f_{\text{min}}) \):
\[
u^0(q^0, t) \leq u^1(q^0, t) + \frac{(v_* - p(q^0))}{p(q^0)} \left( b^0_{\text{max}}(q^0) - b^1_{\text{max}}(q^0) \right)
\]
\[
- \frac{(v_* - p(q^0))}{p(q^0)} \left( b^0_{\text{max}}(t) - b^1_{\text{max}}(t) \right)
\]

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Proof. By Claim 50 and Claim 49, \( \frac{d}{dt} (b_{\max}^1(t) - b_{\max}^0(t)) \geq 0 \). Now applying Equation 3.1,

\[
\begin{align*}
 u^0(q^0, t) - u^1(q^0, t) \\
= \int_{q^0}^t \frac{(v_* - p(t))}{p(t)} \left( \frac{dt}{b_{\max}^0(t)} \right) dt \\
+ \int_{q^0}^t \frac{(v_* - p(t))}{p(t)} \left( \frac{dt}{b_{\max}^1(t)} \right) dt \\
= \int_{q^0}^t \frac{(v_* - p(t))}{p(t)} \left( \frac{dt}{b_{\max}^1(t) - b_{\max}^0(t)} \right) dt \\
= \frac{(v_* - p(q^0))}{p(q^0)} \left[ b_{\max}^1(t) - b_{\max}^0(t) \right]_{q^0}^t \\
\end{align*}
\]

The claim follows. \( \square \)

Claim 53. If \( b_{\max}^1(q^0) < b_{\max}^0(q^0) \), then:

\[
u^1(q^1, q^0) \geq \frac{(v_* - p(q^0))}{p(q^0)} \left( b_{\max}^0(q^0) - b_{\max}^1(q^0) \right)
\]

Proof. Consider auction High. Alice starts to clinch at time \( q^1 \). As the price increased from \( p(q^1) \) to \( p(q^0) \), her budget decreased by an amount \( b_{\max}^1(q^1) - b_{\max}^0(q^0) \). The price was always less than \( p(q^0) \) during this interval; thus she gets at least \( (1/p(q^0)) (b_{\max}^1(q^1) - b_{\max}^1(q^0)) \) fraction of the item at an average unit price that is at most \( p(q^0) \). We get

\[
u^1(q^1, q^0) \geq \frac{(v_* - p(q^0))}{p(q^0)} \left( b_{\max}^1(q^1) - b_{\max}^0(q^0) \right)
\]

By definition, \( b_{\max}^1(q^1) = B^1_* > B^0_* = b_{\max}^0(q^0) \), and the claim is proved. \( \square \)
Claim 54. If $b^1_{\text{max}}(q^0) < b^0_{\text{max}}(q^0)$, then for all $t \in [q^0, f_{\text{min}})$:

$$u^0(q^0, t) \leq u^1(q^1, t) - \frac{(v_* - p(t))}{p(t)} (b^0_{\text{max}}(t) - b^1_{\text{max}}(t))$$

Proof. Applying Claim 52, 53, we get

$$u^0(q^0, t) \leq u^1(q^0, t) + u^1(q^1, q^0) - \frac{(v_* - p(q^0))}{p(q^0)} (b^0_{\text{max}}(t) - b^1_{\text{max}}(t))$$

$$= u^1(q^1, t) - \frac{v_* - p(q^0)}{p(q^0)} (b^0_{\text{max}}(t) - b^1_{\text{max}}(t))$$

$$\leq u^1(q^1, t) - \frac{v_* - p(t)}{p(t)} (b^0_{\text{max}}(t) - b^1_{\text{max}}(t))$$

\[ \square \]

Lemma 55. If $b^1_{\text{max}}(q^0) < b^0_{\text{max}}(q^0)$, then $u^0 \leq u^1$.

Proof. Similar to the proof of Lemma 51, it can be shown that $f^1 \leq f^0$. Putting $t = f^1_{\text{min}}$ in Claim 54,

$$u^0(q^0, f^1_{\text{min}}) + \frac{(v_* - p(f^1))}{p(f^1)} b^0_{\text{max}}(f^1_{\text{min}})$$

\[ \leq u^1(q^1, f^1_{\text{min}}) + \frac{(v_* - p(f^1))}{p(f^1)} b^1_{\text{max}}(f^1_{\text{min}}) \] (3.5)

If $p(f^1) = v_*$, then $f^0 = f^1$ and Alice receives zero utility from the final one-shot allocations in both the auctions. Note that by Claim 49, we have $b^1_{\text{max}}(f^1_{\text{min}}) \leq b^0_{\text{max}}(f^1_{\text{min}})$. Thus, $u^0 = u^0(q^0, f^1_{\text{min}}) \leq u^1(q^1, f^1_{\text{min}}) = u^1$ and the lemma is true.

Now suppose $p(f^1) < v_*$. Alice’s utility from the final one-shot allocation in High is given by:

$$u^1(f^1) = \frac{(v_* - p(f^1))}{p(f^1)} b^1_{\text{max}}(f^1_{\text{min}})$$

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On the other hand, in Low, during the time interval $f^1 \leq t \leq f^0$, Alice can get at most $\frac{b_0^0(f^1)}{p(f^1)}$ fraction of the item at an average unit price that is at least $p(f^1)$. Thus,

$$u^0(f^1, f^0) \leq \frac{(v_* - p(f^1))}{p(f^1)}b_0^0(f^1)$$

Adding this to Equation 3.5, we get

$$u^0 = u^0(q^0, f^1) + u^0(f^1, f^0) \leq u^1(q^1, f^1) + u^1(f^1) = u^1$$

This completes the proof. \qed

The proof of Theorem 36 for the case when $q^0 < f^0$ now follows from Lemmas 41, 51 and 55.

3.5.3 The Special Case: Alice Never Enters $C(t)$ in Auction Low, that is $q^0 = f^0$

In this section, we prove Theorem 36 when $q^0 = f^0$, that is, in auction Low, Alice receives all her utility from the final one shot allocation in Step (I) of Figure 3.1. We will consider three mutually exclusive and exhaustive cases corresponding respectively to Low stopping: (i) before any bidder starts clinching; (ii) after some bidder starts clinching, but before any bidder starts clinching in High; and (iii) after some bidder starts clinching in High. We first show the following claim which gives a closed form expression for the utility gained by Alice in Low.

**Claim 56.** In auction Low, Alice only receives a one shot allocation of $\frac{B^q_0}{p(q^q)}$ at price $p(q^0)$, and her utility is given by $u^0 = \frac{(v_* - p(q^0))}{p(q^0)}B^q_0$. Furthermore, in this case,

$$\sum_{i : v_i = p(q^0)} b^0_i(q^0) \geq b^0_{\max}(q^0)$$
Proof. Consider auction LOW. Since $q^0 = f^0$, Assumption 1 implies $p(f^0) < v_*$, so that by Lemma 34, Alice’s budget is extracted completely at price $p(q^0)$. The first part of the claim follows.

To see the second part, first note that Supply Invariant holds just before the auction stops. At time $q^0$, from the perspective of the active bidder with highest remaining budget ($b_{\text{max}}(q^0)$), total demand of other active bidders is no less than the available supply ($S_0(q^0)$). In other words:

$$\sum_{i \in A(q^0)} \frac{b_i(q^0)}{p(q^0)} \geq S_0(q^0) + \frac{b_{\text{max}}(q^0)}{p(q^0)}$$

At time $q^0$, the auction stops because total demand of all the active bidders is no more than available supply, so that $\sum_{i \in A(q^0)} \frac{b_i(q^0)}{p(q^0)} \leq S_0(q^0)$. Thus, total demand of active bidders drop by at least $\frac{b_{\text{max}}(q^0)}{p(q^0)}$ as time changes from $q^0$ to $q^0$. This abrupt decrease in total demand is caused by the set of exiting bidders (that is, bidders with $v_i = p(q^0)$). Therefore, we get: $\sum_{i : v_i = p(q^0)} b_i(q^0) \geq b_{\text{max}}(q^0)$, completing the proof.

Case 1: $y^0 = q^0$

We first consider the case where LOW stops before any bidder starts clinching. We have $y^0 = q^0 = f^0$. Using an argument similar to the proof of Claim 42, it can be shown that LOW starts clinching no later than HIGH, that is, $y^0 \leq y^1$. Furthermore, just before LOW starts clinching (at time $y_0^1$), the remaining budget of every active bidder equals her reported budget. In particular, every bidder $i$ other than Alice has the same remaining budget across the two auctions, that is, $b_i^0(y_0^1) = b_i^1(y_0^1)$. For Alice, $b_a^1(y_0^1) = b_a^0(y_0^1) + \Delta$. Also note that $S^0(y_0^1) = S^1(y_0^1) = 1$. Since auction
Low stops at time $y^0$, we must have:

$$\sum_{i : v_i > p(y^0)} \frac{b_i^1(y^0)}{p(y^0)} = \frac{\Delta}{p(y^0)} + \sum_{i : v_i > p(y^0)} \frac{b_i^0(y^0)}{p(y^0)}$$

$$\leq \frac{\Delta}{p(y^0)} + S^0(y^-)$$

$$= \frac{\Delta}{p(y^0)} + S^1(y^-)$$

Comparing the LHS and the RHS, we see that in auction \text{HIGH},

$$\frac{B^0_A}{p(y^0)} + \sum_{i : v_i > p(y^0), i \neq \text{Alice}} \frac{b_i^1(y^-)}{p(y^0)} \leq S^1(y^-)$$

Thus, Alice receives at least $\frac{B^0_A}{p(y^0)}$ fraction of the item at unit price $p(y^0)$ in auction \text{HIGH}. Since $y^0 = q^0$, Claim 56 implies her utility from \text{HIGH} is no less than her utility from \text{LOW}.

\textit{Case 2: } $y^0 < q^0 \leq y^1$

We next consider the case where \text{LOW} stops after some bidder starts clinching, but before any bidder starts clinching in \text{HIGH}. Let $A$ stands for “Alice”. Suppose $y^0 < q^0 \leq y^1$. Since \text{LOW} stops at $q^0$,

$$\sum_{i : v_i > p(q^0), i \neq A} \frac{b_i^0(p(q^0))}{p(q^0)} + \frac{B^0_A}{p(q^0)} \leq S^0(q^-)$$
For all bidders $i$ with $v_i > p(q^0)$, $i \neq A$, we have $b_i^1(q^0) = B_i$, otherwise $y^1 < q^0$.

Since in Low, clinching bidders clinched at price at most $p(q^0)$, we have

$$\sum_{i : p(q^0)} B_i - B_i^0(q^0) \leq 1 - S^0(q^0)$$

$$\Rightarrow \sum_{i : p(q^0)} \frac{b_i^1(q^0)}{p(q^0)}$$

$$= \sum_{i : p(q^0)} \frac{B_i}{p(q^0)}$$

$$\leq 1 + \sum_{i : p(q^0)} \frac{b_i^0(q^0)}{p(q^0)} - S^0(q^0)$$

For all bidders $i$ with $v_i > p(q^0)$, $i \neq A$, we have $b_i^1(q^0) = b_i^0(q^0) = B_i$. It follows that

$$\sum_{i : v_i > p(q^0), i \neq A} \frac{b_i^0(q^0)}{p(q^0)} + \frac{B^0_a}{p(q^0)}$$

$$\leq 1 + \sum_{i : v_i > p(q^0), i \neq A} \frac{b_i^0(q^0)}{p(q^0)} + \frac{B^0_a}{p(q^0)} - S^0(q^0)$$

$$\leq 1 = S^1(q^0)$$

Therefore, by Lemma 30, Alice clinches at least $B^0_a/p(q^0)$ quantity in High at price $p(q^0)$, so that Theorem 36 holds.

Case 3: $y^0 \leq y^1 < q^0$

We finally consider the case where Low stops after some bidder starts clinching in both auctions. We have $y^0 \leq y^1 < q^0 = f^0 < f^1$ (see Assumption 2). Since the second inequality is strict, we get $C^0(q^0), C^1(q^0) \neq \emptyset$.

Following an argument exactly similar to the one outlined in Section 3.5.2, we have $b_{\text{max}}^1(y^1) \leq b_{\text{max}}^0(y^1) + \Delta$, and whenever $b_{\text{max}}^1(t) \geq b_{\text{max}}^0(t)$, the former reduces at
a larger rate. Therefore:
\[ b_{\text{max}}^1(q^0) \leq b_{\text{max}}^0(q^-) + \Delta \]  

(3.6)

We will first show that \( q^1 < q^0 \), else Theorem 36 is true. Suppose \( q^1 \geq q^0 \). By Claim 56, Alice gets a one shot allocation of \( B_s^0/p(q^0) \) at stopping price \( p(q^0) \). We will show that Alice will also get at least \( B_s^0/p(q^0) \) at the same price in \textsc{high}. Now, since \( q^0 \leq q^1 \) and \( q^0 < f^1 \), we must have:

\[
D_{-\text{Alice}}^1(q^0) + \frac{B_s^1}{p(q^0)} = \sum_k D_k^1(q^0)
\]

\[
= S^1(q^0) + \frac{b_{\text{max}}^1(q^0)}{p(q^0)}
\]

\[
\Rightarrow S^1(q^0) - D_{-\text{Alice}}^1(q^0) = -\frac{b_{\text{max}}^1(q^0)}{p(q^0)} + \frac{B_s^1}{p(q^0)}
\]

Since auction \textsc{low} stops at time \( q^0 \), by Claim 56, we must have

\[
\sum_{i : v_i = p(q^0)} b_i^0(q^-) \geq b_{\text{max}}^0(q^-)
\]

Note that a bidder with valuation equal to \( p(q^0) \) can never be in \( C_1^1(q^0) \), otherwise we will have \( q^0 = f^1 \), a contradiction. Also note by Claim 56 that Alice does not have valuation equal to \( p(q^0) \). Thus, for all bidders \( i \), if \( v_i = p(q^0) \), then \( b_i^1(q^0) = B_i^1 = B_i^0 \geq b_i^0(q^-) \). That is,

\[
\sum_{i : v_i = p(q^0)} b_i^1(q^0) \geq \sum_{i : v_i = p(q^0)} b_i^0(q^-) \geq b_{\text{max}}^0(q^-)
\]

It follows that

\[
S^1(q^0) - \sum_{j \in A(q^0), j \neq \text{Alice}} D_j^1(q^0)
\]

\[
\geq -\frac{b_{\text{max}}^1(q^0)}{p(q^0)} + \frac{B_s^1}{p(q^0)} + \frac{b_{\text{max}}^0(q^-)}{p(q^0)}
\]

\[
\geq \frac{B_s^1 - \Delta}{p(q^0)} = \frac{B_s^0}{p(q^0)}
\]
The final inequality follows from Equation (3.6). Therefore, by Lemma 30, Alice clinches at least $B^0_u/p(q^0)$ at price $p(q^0)$ in HIGH to maintain the Supply Invariant, and Theorem 36 is true.

Therefore, if Theorem 36 is not already true, we must have: $y^0 \leq y^1 < q^0 = f^0 < f^1$. In particular, Alice is clinching in HIGH during the time interval $q^1 \leq t < f^1$. Furthermore, we have $C^0(q^0), C^1(q^0) \neq \emptyset$ and Alice $\in C^1(q^0)$. Similar to the argument above, we must have:

$$\sum_{i : v_i = p(q^0)} b_i^1(q^0) \geq \sum_{i : v_i = p(q^0)} b_i^0(q^0) \geq b_{\max}^0(q^0)$$

Since Alice $\in C^1(q^0)$, we have:

$$S^1(q^0) - D^1_{\text{Alice}}(q^0) = 0$$

$$\Rightarrow S^1(q^0) - \sum_{j \in A(q^0), j \neq \text{Alice}} D^1_j(q^0) \geq \frac{b_{\max}^0(q^0)}{p(q^0)}$$

Therefore, in HIGH, by Lemma 30, Alice gets at least $\frac{b_{\max}^0(q^0)}{p(q^0)}$ fraction at unit price $p(q^0)$. Also note that Alice reduced her budget from $B^1_u$ to $b_{\max}^1(q^0)$ during the time interval $q^1 \leq t < q^0$. Thus, in this time interval, she clinched at least $\frac{(B^1_u - b_{\max}^1(q^0))}{p(q^0)}$ at an average unit price that is at most $p(q^0)$. Thus, we conclude:

$$u^1 \geq \frac{v_* - p(q^0)}{p(q^0)} \left( B^1_u - b_{\max}^1(q^0) + b_{\max}^0(q^0) \right)$$

$$\geq \frac{v_* - p(q^0)}{p(q^0)} \left( B^1_u - \Delta \right) = u^0$$

The final inequality follows from Equation (3.6). This implies Theorem 36.
This chapter is based on the paper by Bhattacharya et al. (2012). We focus on “competitive equilibrium” [Bikhchandani and Ostroy (2002); Blumrosen and Nisan (2007); Gul and Stacchetti (1999)] - a widely used solution concept for market clearing. We have a market consisting of multiple items, and multiple agents are interested in buying these items. The agents’ valuation functions are public knowledge. In this market, the value of money is exogenously determined, and the agents value the items in units of money. As a result, an agent’s utility is simply equal to her total valuation for the items she gets, minus her total payment. The agent is aware of the supply of each item, and given a setting of item-prices, she demands a bundle of items that gives her maximum utility. In a competitive equilibrium, every agent gets her utility-maximizing bundle and every unallocated item has zero price (see Section 1.1.2).

This solution concept is used to analyze the markets concerned with e-commerce applications: Large scale advertising exchanges, keyword search auctions [Dütting et al. (2011); Feldman et al. (2008); Goel et al. (2012)], and Google auctions for
TV ads [Nisan (2009)]. Indeed, the agents here have exogenously defined value for money, and are aware of the item-supplies. Furthermore, in these markets, the agents have budget constraints on the maximum amount of money they can afford to pay. This motivates the study of competitive equilibria in presence of budget limits.

We propose efficient algorithms for finding competitive equilibria in markets where the agents are budget constrained, and show that the solutions returned by our algorithms have strong revenue guarantees.

4.1 Our Model

Throughout the rest of this chapter, we will use the notations and terminologies introduced in Section 1.1.2. We begin by defining four types of valuation functions.

4.1.1 Valuation Functions

The valuation functions described below are fairly natural and widely studied in other related contexts [Nisan (2009); Fiat et al. (2011); Dobzinski et al. (2008); Feldman et al. (2012)]. For instance, the Google TV auction considers additive valuations [Nisan (2009)], while keyword search auctions are typically multi-unit [Goel et al. (2012); Dobzinski et al. (2008); Feldman et al. (2008, 2012)].

For simplicity, we slightly abuse the notation and let $v_{i,p}(j)$ (instead of $v_{i,p}([j])$) represent the valuation of agent $i \in I$ for item $j \in J$.

**Additive Valuations.** The valuation of an agent $i \in I$ for a subset of items $J \subseteq J$ is given by: $v_i(J) = \sum_{j \in J} v_i(j)$.

**Concave Combinatorial Valuations.** Such an agent $i \in I$ is interested in a specific subset of items $S_i \subseteq J$; and she has zero valuation for every other item $j \in J \setminus S_i$. Her marginal valuation within the subset $S_i$ is non-increasing in total allocation, and depends only on the number of items she wins in $S_i$. Specifically, if
agent $i$ gets a subset of items $J \subseteq \mathcal{J}$ and $|J \cap S_i| = k$, then her valuation is given by: $v_i(J) = \sum_{t=1}^{k} \alpha_i(t)$, where $\alpha_i(1) \geq \cdots \geq \alpha_i(k) \geq \cdots \geq \alpha_i(|S_i|) \geq 0$.

**Single Valued Combinatorial Valuations.** A special case of concave-combinatorial valuations where $\alpha_i(1) = \cdots = \alpha_i(|S_i|)$. Hence, the valuation is proportional to the number of items received from $S_i$.

**Multi-Unit Valuations.** This, in turn, is a special case of single-valued-combinatorial valuation, where each agent is interested in every item: For all $i \in \mathcal{I}$, we have that $S_i = \mathcal{J}$.

**Remark.** It is easy to see that if a budget constrained agent has any of the above four valuation functions, then her demand function is not gross substitutes.

### 4.1.2 Optimal-Revenue Benchmark

We will present algorithms for competitive equilibria that compete against a very natural revenue benchmark. Given an allocation of the items, consider the least upper bound on revenue that can be obtained using the only constraint that every agent should receive non-negative utility. Specifically, the benchmark corresponding to an allocation $A$ is defined as:

$$\text{OPT}(A) = \sum_{i \in \mathcal{I}} \min \{ v_i(A(i)), B_i \}$$  \hspace{1cm} (4.1)

Now, the **optimal-revenue benchmark** is defined as:

$$\text{OPT} = \max_A \{ \text{OPT}(A) \}$$  \hspace{1cm} (4.2)

### 4.1.3 Pseudo-Competitive Equilibrium

We start with a simple illustrative example.

**Example 1.** A market consists of one indivisible item, and two agents with valuations 10 and 20 for the item. Both the agents have a budget of 5.
In Example 1, if the item is priced above 5, then it remains unallocated. On the other hand, if the item is priced at or below 5, then both the agents strictly want the item. Hence, this market does not admit any competitive equilibrium. Note however that this example is very “brittle”, in the sense that we can get around it by slightly perturbing the agents’ budgets. More precisely, if the budget of agent 1 were $5 + \epsilon$, then we could allocate the item at price $5 + \epsilon$ to her, resulting in a competitive equilibrium. This leads us to the notion of a pseudo-competitive equilibrium, which essentially rules out such degenerate corner cases.

In a pseudo-competitive equilibrium, every agent is penalized by an infinitesimally small amount if she tries to switch from her equilibrium allocation. Let the symbol $P + \epsilon$ denote the price vector which assigns a price of $P(j) + \epsilon$ to every item $j \in J$. At the price vector $P$, the pseudo-utility of an agent $i \in I$ from a subset of items $T \subseteq J$ is given by:

$$u_i^+(T, P) = \lim_{\epsilon \to 0+} u_i(T, P + \epsilon)$$

**Definition 21.** The allocation $A$ and pricing $P$ are in a pseudo-competitive equilibrium if and only if every unallocated item is priced at zero, and the utility received by each agent is at least her maximum possible pseudo-utility. Thus, we require that:

$$P(j) = 0 \quad \text{for all items } j \in J \setminus \bigcup_{i \in I} A(i);$$

$$u_i(A(i), P) \geq \max_{J \subseteq J} \{u_i^+(J, P)\} \quad \text{for all agents } i \in I.$$ 

To appreciate the usefulness of this concept, consider the following question: Is a competitive equilibrium guaranteed to exist in markets where the agents are budget constrained and have concave-combinatorial valuation functions? Example 1 rules out any such guarantee. In Section 4.4, however, we show that all such markets allow for a pseudo-competitive equilibrium. This result, coupled with Definition 21, implies that almost all instances of such markets do in fact admit a competitive equilibrium.
In other words, the bad instances (such as Example 1) are of measure zero. In contrast, it is well known that if the agents have additive valuation functions and budget limits, then the market may not even admit a pseudo-competitive equilibrium.

4.2 Our Results

Our results for additive valuations (see Section 4.3) are obtained by extending the Eisenberg-Gale convex program (see Section 1.1.2), and rounding the fractional outcome to an integral one. In contrast, our results for concave-combinatorial valuations (see Section 4.4) are obtained by extending the primal-dual framework of Devanur et al. (2008).

4.2.1 Our Results for Additive Valuations

Computational Hardness of Pseudo-Competitive Equilibrium. In this setting, there are known examples [Nisan (2009)] of markets which do not admit any pseudo-competitive equilibrium (see Table 4.1). Hence, we focus on the computational problem: Given a market, decide whether it admits a pseudo-competitive equilibrium. Note that given the price vector, an agent has to solve a Knapsack problem to find her optimal bundle of items, which itself is weakly NP-hard. Thus, we restrict ourselves to instances where the optimization problem faced by every agent is easy to solve. We show that even on these restricted instances, it is strongly NP-hard to decide if a market admits any pseudo-competitive equilibrium (Theorem 60). Our proof is based on converting the non-existence example in [Nisan (2009)] into a hardness gadget. We next ask the question: Given an allocation of the items to the agents, decide whether there are item-prices that support this allocation in a pseudo-competitive equilibrium. We show that this problem is also NP-hard (Theorem 66).

Both of the above hardness results also apply to the case where the agents have a demand constraint, and are interested in at most 3 items, and the goal is to decide
the existence of an exact competitive equilibrium. This shows that the algorithms for the unit-demand agents with budget limits [Ashlagi et al. (2009); Chen et al. (2010)] cannot be extended much further.

Convex Program for Fisher-Type Competitive Equilibrium. We note that the algorithms in [Birnbaum et al. (2011); Vazirani (2010)] can be used to compute a quasilinear-Fisher equilibrium (see Section 1.1.2). We augment these results (see Section 4.3.2) by showing that a Fisher-type competitive equilibrium can also be found via a different, more intuitive, convex program. Our program is obtained by simply changing the objective function of the Eisenberg-Gale convex program. Instead of optimizing the objective $\sum_i B_i \log(t_i)$, where $t_i$ is the total valuation accrued by agent $i$ and $B_i$ is her budget, the new objective optimizes $\sum_i f_i(t_i)$. Here, the function $f_i(t_i)$ is equal to $t_i$ if $t_i \leq B_i$, and equal to $B_i (\log(t_i/B_i) + 1)$ if $t_i \geq B_i$. This approach generalizes to all homogeneous log-concave valuation functions. We term the function $f_i(t_i)$ as the budget aware social welfare of agent $i \in I$. As explained below, this extends the first welfare theorem to budget constrained agents, and yields a simple interpretation of the Eisenberg-Gale convex program when applied to quasilinear utilities.

Budget Aware Social Welfare. The notion of a competitive equilibrium is a “local” one: It only guarantees that no two agents envy each other, and that any unallocated item has zero price. In the absence of budget limits (i.e., with infinite budgets), the first welfare theorem [Bikhchandani and Ostroy (2002); Blumrosen and Nisan (2007)] gives a very intuitive “global” characterization of the same concept. It states that any competitive equilibrium maximizes the social welfare, which is defined as the sum of the valuations obtained by all the agents. We ask: Can we generalize the first welfare theorem when the agents have budget limits?

Our convex program shows that a Fisher-type competitive equilibrium, in the
presence of budget constrained agents, maximizes the benchmark Budget Aware Social Welfare (BASW): \( \sum_i f_i(t_i) \). This benchmark coincides with traditional social welfare when the agents have infinite budgets, thereby extending the first welfare theorem [Blumrosen and Nisan (2007)]. In general, BASW is at most the social welfare \( \sum_i t_i \), and at least the optimal-revenue benchmark \( \sum_i \min(B_i, t_i) \). When the budgets are relatively small compared to the valuations, the benchmark \( \sum_i t_i \) might be impossible to attain, whereas the benchmark \( \sum_i \min(B_i, t_i) \) might be a gross underestimate. BASW lies in between these two extremes.

The standard-Fisher model is “scale invariant” with respect to the agents’ valuations, meaning that the equilibrium outcome remains unchanged if an agent’s valuation for every item is scaled up by exactly the same factor. Furthermore, if the agents’ valuations are very large compared to their budgets, then the Fisher-type competitive equilibrium coincides with the standard-Fisher equilibrium. Thus, we can solve the standard-Fisher model as follows: First, scale up the valuations of all the agents by a sufficiently large factor, and then apply our convex program. This shows that our convex program generalizes the Eisenberg-Gale convex program.

Approximate Competitive Equilibrium. As noted above, finding a pseudo-competitive equilibrium is NP hard in this setting. A natural way to circumvent this computational hardness is to focus on the relaxed notion of an approximate competitive equilibrium (see Definition 10).

In Section 4.3.3, we give an efficient algorithm that finds a \((1 - 2\gamma)\)-approximate competitive equilibrium in polynomial time, where \( \gamma = \max_{i,j} (v_i(j)/B_i) \), i.e., the approximation is good if an agent’s valuation for a single item is small compared to her budget.\(^1\) This result is obtained in two steps: We first find any Fisher-type equi-

\(^1\) For example, typically an advertiser’s valuation for a single advertisement opportunity to a user is very small compared to the advertiser’s overall budget.
librium for divisible items and next convert this into an approximate equilibrium for indivisible items. The second step uses the rounding algorithm of Shmoys and Tardos [Shmoys and Tardos (1993)] for the generalized assignment problem (GAP) and is crucially based on finding a Fisher type equilibrium in the first step. Technically, the interesting aspect here is the first step.

We also note that a simple modification to our algorithm makes it return an almost-envy-free item-pricing (see Definition 11).

Revenue Properties. We show that the revenue of our approximate competitive equilibrium (or equivalently, the almost-envy-free item-pricing) is at least \(1/2 - \gamma\) times the optimal-revenue benchmark.

4.2.2 Our Results for Concave Combinatorial Valuations

If the items are divisible, then the concave-combinatorial valuation function is homothetic and quasi-concave, and Jain et al. (2005) showed how to obtain a convex programming based market clearing solution in the standard-Fisher model under these conditions. Their framework is based on monotonically transforming the valuation function to make it homogeneous. Since an agent’s utility is simply equal to her valuation in the standard-Fisher model, such a transformation does not affect the set of equilibrium outcomes. Unfortunately, this framework cannot be applied in our setting: A monotone transformation of the valuation function might alter the set of competitive equilibria in the presence of budget constrained agents (as well as the set of quasilinear-Fisher equilibria).

Pseudo-Competitive Equilibrium. In spite of the (apparent) failure of convex programs, we present an ascending auction that runs in polynomial time and always returns a pseudo-competitive equilibrium. Our algorithm builds upon the combinatorial algorithm of Devanur et al. (2008) for the standard-Fisher model with additive
valuations. Their algorithm is a primal dual ascending price method applied to the KKT conditions of the Eisenberg-Gale convex program. Extending their algorithm to competitive equilibria is not straightforward, posing three significant issues: Our valuations are concave (and not additive); the items are indivisible; and our utilities are quasi-linear. Though our basic template is the same as Devanur et al. (2008), we need to define a demand graph in a different way, and more importantly, their approach to computing allocations from the tight sets does not achieve a competitive equilibrium. We instead use a different iterated augmenting path technique to compute allocations, which we believe is novel in our context. We leave it as an open question to achieve the same results via convex programming.

Revenue Properties. For concave-combinatorial valuations, there are instances (see Example 2) where no pseudo-competitive equilibrium generates good revenue. Nevertheless, we can prove that for the subclass of single-valued-combinatorial valuations, the revenue of the pseudo-competitive equilibrium returned by our algorithm is at least 1/3 times the optimal-revenue benchmark. We will show that this result extends the recent work by Feldman et al. (2012). Towards this end, we need to introduce the concept of an “envy-free bundle pricing”.

Definition 22. An envy-free bundle pricing specifies the subset of items $A(i) \subseteq J$ received by every agent $i \in I$ and her total payment $p_i$, subject to the condition that no agent envies the allocation and payment of another agent. To be more specific, for any two distinct agents $i \neq i'$, we have:

Either $p_{i'} > B_i$ or $v_i(A(i)) - p_i \geq v_i(A(i')) - p_{i'}$

Consider any envy-free item pricing $(A, P)$ (see Definition 7). Clearly, if we allocate the subset of items $A(i) \subseteq J$ to every agent $i \in I$ and charge her a payment of $\sum_{j \in A(i)} P(j)$, then we get an envy-free bundle pricing, which is essentially the same
as the original envy-free item pricing \((A, P)\). Thus, we have: (a) Every competitive equilibrium is an envy-free item pricing (see Definitions 7, 8), and (b) Every envy-free item pricing is an envy-free bundle pricing. This leads us to equation 4.4.

Let the symbols \(\text{OPT}_{\text{Comp-Eq}}, \text{OPT}_{\text{EF-Item}}\) and \(\text{OPT}_{\text{EF-Bundle}}\) denote the maximum possible revenue from any competitive equilibrium, envy-free item pricing and envy-free bundle pricing respectively. Recall that the symbol \(\text{OPT}\) stands for the optimal-revenue benchmark. We have:

\[
\text{OPT}_{\text{Comp-Eq}} \leq \text{OPT}_{\text{EF-Item}} \leq \text{OPT}_{\text{EF-Bundle}} \leq \text{OPT}
\]

(4.4)

The work by Feldman et al. (2012) considers the following setting: A market of indivisible items, and budget constrained agents with multi-unit valuation functions. Here, they show that \(\text{OPT}_{\text{EF-Item}}\) can be arbitrarily low compared to \(\text{OPT}_{\text{EF-Bundle}}\). On a positive note, they give a polynomial-time algorithm that returns an envy-free bundle pricing whose revenue is at least \((1/2) \times \text{OPT}_{\text{EF-Bundle}}\).

As an immediately corollary of the above work, we see that even for markets where the agents have multi-unit valuation functions and budget limits, the quantity \(\text{OPT}_{\text{Comp-Eq}}\) can be arbitrarily small compared to \(\text{OPT}\) (see equation 4.4). In contrast with this negative result, we show the following strong positive result: For almost all instances of such markets, we have that \(\text{OPT}_{\text{Comp-Eq}} \geq (1/3) \times \text{OPT}\). Equivalently, given any instance with large gap between \(\text{OPT}_{\text{Comp-Eq}}\) and \(\text{OPT}\), the gap reduces to 3 on a slight perturbation of the input. In fact, this holds for the more general class of budget constrained agents with single-valued-combinatorial valuation functions. This result follows directly from the fact that the revenue of the pseudo-competitive equilibrium returned by our algorithm is always within a factor of 3 of the optimal-revenue benchmark.

To summarize, we compare our results with the work of Feldman et al. (2012).

1. They consider the class of multi-unit valuation functions, whereas we consider
the (more general) class of single-valued-combinatorial valuation functions.

2. Their algorithm returns an envy-free bundle pricing, whereas our algorithm returns a competitive equilibrium (which is a stronger solution concept).

3. Their algorithm competes against the revenue benchmark $\text{OPT}_{\text{EF-Bundle}}$, whereas we compete against the (stronger) revenue benchmark $\text{OPT}$.

4. They achieve an approximation ratio of 2, whereas we achieve a (slightly weaker) approximation ratio of 3.

5. Their result holds for all input instances, whereas our result holds for almost all input instances.

4.3 Additive Valuations

We consider markets of indivisible items and budget constrained agents with additive valuations. In such markets, there is no guarantee that a pseudo-competitive equilibrium will exist. Consider the following example.

There are 3 indivisible items, and 3 agents with additive valuations and budget limits. The agents’ valuations and budgets are given in Table 4.1. Nisan (2009) showed that this instance does not admit any pseudo-competitive equilibrium. For the sake of completeness, we reproduce his argument.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Item 1</th>
<th>Item 2</th>
<th>Item 3</th>
<th>Budget</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_1(1) = 5$</td>
<td>$v_2(2) = 5$</td>
<td>$v_3(3) = 0$</td>
<td>$B_1 = 6$</td>
</tr>
<tr>
<td>2</td>
<td>$v_2(1) = 4$</td>
<td>$v_2(2) = 4$</td>
<td>$v_3(3) = 8$</td>
<td>$B_2 = 9$</td>
</tr>
<tr>
<td>3</td>
<td>$v_3(1) = 0$</td>
<td>$v_3(2) = 0$</td>
<td>$v_3(3) = 7$</td>
<td>$B_3 = 7$</td>
</tr>
</tbody>
</table>

Suppose that this instance admits a pseudo-competitive equilibrium $(\mathbf{A}, \mathbf{P})$. Since $v_i(1) = v_i(2)$ for every agent $i \in \{1, 2, 3\}$, item 1 and item 2 gets the same price, that
is, we have: \( P(1) = P(2) = q \) (say). We now consider two mutually exclusive and exhaustive cases.

**Case 1.** Item 3 goes to agent 3. Hence, \( P(3) \leq 7 \).

In this case, agent 2 has pseudo utility \( u_2^+(\{3\}, P) \geq 1 \) for item 3. Since this agent is not getting item 3, we deduce that her utility from the subset of items \( \{1, 2\} \) is at least 1, that is: \( u_2(\{1, 2\}, P) \geq u_2^+(\{3\}, P) \geq 1 \). This can only happen if \( P(1) = P(2) = q \leq 3.5 \). But in that case, there is no way to allocate the first two items which will ensure that agent 1 and agent 2 do not envy each other.

**Case 2.** Item 3 goes to agent 2. Since agent 3 does not complain about this, we must have: \( P(3) \geq 7 \). Note that without any loss of generality, we can assume: \( P(3) = 7 \) (this only makes agent 2 more happy).

In this case, agent 2 derives a utility of 1 from item 3: \( u_i(\{3\}, P) = 1 \). After purchasing item 3, agent 2 has a remaining budget of \( B_2 - P(3) = 2 \). If she is buying any other item, then we must have: \( P(1) = P(2) = q \leq 2 \). But in that case, agent 1 will envy agent 2 if agent 2 gets any of the items in \( \{1, 2\} \). We conclude that in the allocation vector \( A \), we have: \( A(1) = \{1, 2\}, A(2) = \{3\} \) and \( A(3) = \emptyset \).

Thus, we get: \( u_2(A, P) = 1 \geq u_2^+(\{1, 2\}, P) \). Now, we note that \( u_2^+(\{1, 2\}, P) \leq 1 \) only if \( P(1) = P(2) = q \geq 3.5 \). But in that case, total payment of agent 1 is equal to 7, which exceeds her budget. Hence, agent 1 gets negative utility. A contradiction. Thus, we conclude that the market described by Table 4.1 does not admit any pseudo-competitive equilibrium.

### 4.3.1 Pseudo-Competitive Equilibrium for Indivisible Items

The example in Table 4.1 raises a natural question: Can we decide (in polynomial time) if a given market admits a pseudo-competitive equilibrium? Formally, we use the notation \( \text{Pseudo-Comp-Eq}(\mathcal{I}, \mathcal{J}) \) to represent an instance of this problem.
Every agent $i \in \mathcal{I}$ has an additive valuation function $v_i$, and a budget $B_i$. Items are indivisible. We want to decide whether the instance admits a pseudo-competitive equilibrium. We show (see Theorem 60) that this problem is strongly NP hard. The hardness result holds even if the optimization problems faced by the agents (that of finding their utility-maximizing bundles given the price vector) are easy to solve.

We give a reduction from the problem of 3-Partition, which is described below.

**3-Partition.** In an instance of the 3-PARTITION problem, we are given a multiset $T$ of $3n$ positive integers:

$$T = \{\theta_1, \ldots, \theta_{3n}\}.$$  

Furthermore, we have $B/4 \leq \theta_k \leq B/2$ for all $k$, where $B = (\sum_{k=1}^{3n} \theta_k) / n$ is also a positive integer. The objective is to decide whether $T$ can be partitioned into $n$ subsets $T_1, \ldots, T_n$ so that the sum of the elements in each subset is exactly equal to $B$. Since all the elements of $T$ lie between $B/4$ and $B/2$, each subset will contain exactly three elements. We denote an instance of this problem by $\text{PARTITION}(\{\theta_1 \ldots \theta_{3n}\}, B)$.

The 3-Partition problem is strongly NP hard Garey and Johnson (1979).

**The Reduction.** Given an instance $\text{PARTITION}(\{\theta_1 \ldots \theta_{3n}\}, B)$, we construct an instance $\text{PSEUDO-COMP-EQ}(\mathcal{I}, \mathcal{J})$, as follows.

We create $3n$ items, where item $j \in \{1, \ldots, 3n\}$ corresponds to the integer $\theta_j$. We create $n+1$ agents $\{1, \ldots, n+1\}$. Furthermore, we create a gadget which corresponds (modulo a scaling factor) to the market described in Table 4.1. We denote the set of agents (resp. items) in the gadget by $\{i_1, i_2, i_3\}$ (resp. $\{j_1, j_2, j_3\}$). To summarize, in the instance of the PSEUDO-COMP-EQ problem we construct, the set of agents (resp. items) is given by $\mathcal{I} = \{1, \ldots, n+1, i_1, i_2, i_3\}$ (resp. $\mathcal{J} = \{1, \ldots, 3n, j_1, j_2, j_3\}$).

The valuations and the budget of the agents in $\{1, \ldots, n+1\}$ are described below.

- For all $i \in \{1, \ldots, n\}$:
– Agent $i$ has budget $B_i = B$.

– For all items $j \in \{1, \ldots, 3n\}$, we have $v_i(j) = \theta_j$.

– For all items $j \in \{j_1, j_2, j_3\}$, we have $v_i(j) = 0$.

• Agent $n+1$ has budget $B_{n+1} = 7B$. Furthermore,

  – For all items $j \in \{1, \ldots, 3n\}$, we have $v_{n+1}(j) = \theta_j$.
  
  – For item $j_3$, we have $v_{n+1}(j_3) = 7B$.
  
  – For all items $j \in \{j_1, j_2\}$, we have $v_i(j) = 0$.

The valuations and the budget of the agents in $\{i_1, i_2, i_3\}$ are described below.

• Agent $i_1$ has budget $B_{i_1} = 6B$.

  – Her valuations for the items in the gadget are given by: $v_{i_1}(j_1) = v_{i_1}(j_2) = 5B$ and $v_{i_1}(j_3) = 0$.
  
  – She has zero valuation for every other item. For all $j \in \{1 \ldots 3n\}$, we have that $v_{i_1}(j) = 0$.

• Agent $i_2$ has budget $B_{i_2} = 9B$.

  – Her valuations for the items in the gadget are given by: $v_{i_2}(j_1) = v_{i_2}(j_2) = 4B$ and $v_{i_1}(j_3) = 7B$.
  
  – She has zero valuation for every other item. For all $j \in \{1 \ldots 3n\}$, we have that $v_{i_2}(j) = 0$.

• Agent $i_3$ has budget $B_{i_3} = 7B$.

  – Her valuations for the items in the gadget are given by: $v_{i_3}(j_1) = v_{i_3}(j_2) = 0$ and $v_{i_3}(j_3) = 7B$.
She has zero valuation for every other item. For all $j \in \{1 \ldots 3n\}$, we have that $v_i(j) = 0$.

**Lemma 57.** Consider the instance PSEUDO-COMP-EQ($I, J$) described above. If it admits any pseudo-competitive equilibrium, then item $j_3$ has to be allocated to agent $(n+1)$ at price $7B$.

**Proof.** If the item $j_3$ is left unallocated, then it will have zero price, and as a result, it will be demanded by agent $i_3$. Thus, we conclude that the item $j_3$ has to be allocated to some agent.

Suppose that the item is not assigned to agent $(n+1)$. In that case, the gadget, which consists of the agents $\{i_1, i_2, i_3\}$ and items $\{j_1, j_2, j_3\}$, can never be in a pseudo-competitive equilibrium (see the discussion following Table 4.1).

Now, if the item is priced above $7B$, then agent $n+1$ will not purchase the item (since $B_{n+1} = v_{n+1}(j_3) = 7B$), and if the item is priced below $7B$, then the agent $i_3$ will demand it. Hence, the item has to be priced exactly at $7B$. \qed

**Lemma 58.** If the problem-instance PSEUDO-COMP-EQ($I, J$) admits a pseudo-competitive equilibrium $(A, P)$, then the problem-instance PARTITION($\{\theta_1, \ldots, \theta_{3n}\}, B$) admits a valid 3-Partition.

**Proof.** In $(A, P)$, item $j_3$ goes to agent $n+1$, and it has price $P(j_3) = 7B$ (see Lemma 57). In other words, agent $n+1$ spends all of her budget on item $j_3$, and she receives zero utility. Thus, her pseudo utility from every other item has to be non-positive: $u^+_{n+1}({\{j\}, P}) \leq u_{n+1}(A, P) = 0$, for all $j \neq j_3$. We conclude that $P(j) \geq v_{n+1}(j) = \theta_j$, for every item $j \in \{1, \ldots, 3n\}$. On the other hand, we must have $P(j) \leq \max_i v_i(j) = \theta_j$, for all $j \in \{1 \ldots 3n\}$. Therefore, we get:

$$P(j) = \theta_j, \text{ for all items } j \in \{1, \ldots, 3n\}.$$
Note that the agents in \( \{i_1, i_2, i_3\} \) have zero valuations for the items in \( \{1, \ldots, 3n\} \). We infer that the pseudo-competitive equilibrium distributes all the items in \( \{1, \ldots, 3n\} \) amongst the agents in \( \{1, \ldots, n\} \). Since every such item \( j \) is priced at \( \theta_j \) and every such agent \( i \) has a budget \( B \), we must have:

\[
\sum_{j \in A(i)} \theta_j \leq B, \text{ for all agents } i \in \{1, \ldots, n\}.
\]

Therefore, the collection of subsets \( \{A(1), \ldots, A(n)\} \) gives us a valid 3-Partition for the instance \( \text{PARTITION}(\{\theta_1, \ldots, \theta_{3n}\}, B) \).

**Lemma 59.** If the problem-instance \( \text{PARTITION}(\{\theta_1, \ldots, \theta_{3n}\}, B) \) admits a valid 3-Partition \( \{T_1, \ldots, T_n\} \), then the problem-instance \( \text{PSEUDO-COMP-EQ}(\mathcal{I}, \mathcal{J}) \) admits a pseudo-competitive equilibrium.

**Proof.** We construct the following allocation \( A \).

- For all agents \( i \in \{1, \ldots, n\} \), we have \( A(i) = T_i \).
- Agent \( n + 1 \) takes item \( j_3 \). We have \( A(n + 1) = \{j_3\} \).
- Agent \( i_1 \) takes item \( j_1 \). We have \( A(i_1) = \{j_1\} \).
- Agent \( i_2 \) takes item \( j_2 \). We have \( A(i_2) = \{j_2\} \).
- Agent \( i_3 \) does not get any item. We have \( A(i_3) = \emptyset \).

We construct the following pricing.

- For all items \( j \in \{1, \ldots, 3n\} \), we have \( P(j) = \theta_j \).
- We have \( P(j_1) = P(j_2) = 4B \), and \( P(j_3) = 7B \).

It is easy to check that \( (A, P) \) is a competitive equilibrium (and hence a pseudo-competitive equilibrium) for the problem-instance \( \text{PSEUDO-COMP-EQ}(\mathcal{I}, \mathcal{J}) \).
Theorem 60 follows from Lemmas 58 and 59.

**Theorem 60.** The problem PSEUDO-COMP-EQ(\mathcal{I}, \mathcal{J}) is strongly NP hard, even for the families of instances where every agent can easily identify her utility-maximizing bundle given the price vector.

**Agents with multi unit-demands.** The above reduction holds even if we assume that every agent in \mathcal{I} can purchase at most 3 items. In other words, this shows that the positive result for the unit-demand agents with budget limits Ashlagi et al. (2009); Chen et al. (2010) cannot be extended even to settings where the agents have small constant demand constraints.

Given the above hardness result, one might ask if the problem becomes any easier when we fix the allocation of the items. Formally, let EQ-PRICING(\mathcal{I}, \mathcal{J}, A) denote an instance of the problem. The set of agents (resp. items) is given by \mathcal{I} (resp. \mathcal{J}). We want to decide if there are item-prices that support allocation A in a pseudo-competitive equilibrium. We show that this problem is also NP hard. We give a reduction from the Monotone 3-SAT problem, which is described below.

**Monotone 3-SAT.** We let MONOTONE(U, C) denote an instance of this problem. It consists of a set U of n variables, and a set C of m clauses. The set of clauses C can be partitioned into two subsets C^+ and C^- . Every clause c^+ ∈ C^+ (resp. c^- ∈ C^-) is a disjunction of three positive (resp. negative) literals. In other words, every positive clause c^+ ∈ C^+ can be written as: c^+ = x \lor y \lor z, where x, y, z ∈ U. Similarly, every negative clause c^- ∈ C^- can be written as: c^- = \lnot x \lor \lnot y \lor \lnot z, where x, y, z ∈ U. The problem is to decide if there is a boolean assignment T : U → \{0, 1\} that satisfies every clause in C = C^+ ∪ C^- . This problem is NP complete Garey and Johnson (1979).
Notation. We slightly abuse the notation, and write \( x \in c^+ \) (resp. \( x \in c^- \)) whenever the positive clause \( c^+ \in C^+ \) (resp. negative clause \( c^- \in C^- \)) contains the variable \( x \in U \).

Thus, if a positive clause is given by: \( c^+ = x \lor y \lor z \), then we write: \( x \in c^+, y \in c^+ \) and \( z \in c^+ \). Similarly, if a negative clause is given by: \( c^- = \neg x \lor \neg y \lor \neg z \), then we write: \( x \in c^-, y \in c^- \) and \( z \in c^- \).

The reduction. Given a problem-instance \( \text{MONOTONE}(U, C) \), we construct the following problem-instance \( \text{Eq-Pricing}(I, J, A) \).

- For every variable \( x \in U \), create an agent \( i_x \in I \) and an item \( j_x \in J \).
- For every positive clause \( c^+ \in C^+ \), create an agent \( i_{c^+} \in I \). Next, create three items, one for every variable contained in the clause, as follows. For all \( x \in U \), if \( x \in c^+ \), then we have an item \( j_{x,c^+} \in J \).
- For every negative clause \( c^- \in C^- \), create an agent \( i_{c^-} \in I \). Next, create three items, one for every variable contained in the clause, as follows. For all \( x \in U \), if \( x \in c^- \), then we have an item \( j_{x,c^-} \in J \).

The budgets of the agents in \( I \), and their valuations for the items in \( J \) are described below.

- For all \( x \in U \), the agent \( i_x \in I \) has budget \( B_{i_x} = 10 \), and her valuations are as follows.
  - She has valuation = 10 for the item \( j_x \), that is, \( v_{i_x}(j_x) = 10 \).
  - For every clause \( c^+ \in C^+ \) containing variable \( x \) (i.e. \( x \in c^+ \)), we have \( v_{i_x}(j_{x,c^+}) = 10 \).
  - For every clause \( c^- \in C^- \) containing variable \( x \) (i.e. \( x \in c^- \)), we have \( v_{i_x}(j_{x,c^-}) = 2 \).
She has zero valuation for every other item.

- For all $c^+ \in C^+$, the agent $i_{c^+} \in \mathcal{I}$ has budget $B_{i_{c^+}} = 23$, and her valuations are as follows.
  
  - She has valuation $= 10$ for every item $j_{x,c^+}$: that is, $v_{i_{c^+}}(j_{x,c^+}) = 10$ for all variables $x \in c^+$.
  
  - She has zero valuation for every other item.

- For all $c^- \in C^-$, the agent $i_{c^-} \in \mathcal{I}$ has budget $B_{i_{c^-}} = 5$, and her valuations are as follows.
  
  - She has valuation $= 5$ for every item $j_{x,c^-}$: that is, $v_{i_{c^-}}(j_{x,c^-}) = 5$ for all variables $x \in c^-.$
  
  - She has zero valuation for every other item.

The agents have additive valuations. The items are indivisible. The allocation $\mathbf{A}$ is described below.

- The agent $i_x \in \mathcal{I}$ gets only the item $j_x$. Thus, we have: $\mathbf{A}(i_x) = \{j_x\}$ for all $x \in U$.

- We have: $\mathbf{A}(i_{c^+}) = \cup_{x \in c^+} \{j_{x,c^+}\}$, for every positive clause $c^+ \in C^+$.

- We have: $\mathbf{A}(i_{c^-}) = \cup_{x \in c^-} \{j_{x,c^-}\}$, for every negative clause $c^- \in C^-$.

The $\text{Eq-Pricing}(\mathcal{I}, \mathcal{J}, \mathbf{A})$ problem-instance asks whether there is a price vector $\mathbf{P}$ that supports the given allocation $\mathbf{A}$ in a pseudo-competitive equilibrium.

**Lemma 61.** If a price vector $\mathbf{P}$ supports the allocation $\mathbf{A}$ in a pseudo-competitive equilibrium, then we have: $\mathbf{P}(j_x) \leq \mathbf{P}(j_{x,c^+})$, for all $x \in U$ and all $c^+ \in C^+$ where $x \in c^+.$
Proof. Fix any variable \( x \in U \), and any positive clause \( c^+ \in C^+ \) where \( x \in c^+ \). Note that under the allocation \( A \), agent \( i_x \in \mathcal{I} \) only gets item \( j_x \in \mathcal{J} \), and she has the same valuation for both the items \( j_x \) and \( j_{x,c^+} \). Therefore, if \((A,P)\) is a pseudo-competitive equilibrium, then we must have: \( P(j_x) \leq P(j_{x,c^+}) \). \( \square \)

**Corollary 62.** If a price vector \( P \) supports the allocation \( A \) in a pseudo-competitive equilibrium, then without any loss of generality \( P(j_{x,c_1^+}) = P(j_{x,c_2^+}) = p_{x^+} \) (say), for all \( x \in U \) and all \( c_1^+, c_2^+ \in C^+ \) such that \( x \in c_1^+ \) and \( x \in c_2^+ \).

**Proof.** Lemma 61 states that:

\[ P(j_x) \leq \min \left( P(j_{x,c_1^+}), P(j_{x,c_2^+}) \right). \]

If we have \( P(j_{x,c_1^+}) < P(j_{x,c_2^+}) \), then w.l.o.g. we can reduce the price of item \( j_{x,c_2^+} \) until it becomes equal to \( P(j_{x,c_1^+}) \). This does not violate the budget constraint of the agent \( i_{c_2^+} \) (who gets item \( j_{x,c_2^+} \)), and no other agent complains about the decrease in price. \( \square \)

**Lemma 63.** If a price vector \( P \) supports the allocation \( A \) in a pseudo-competitive equilibrium, then w.l.o.g. we have: \( P(j_{x,c_1^-}) = P(j_{x,c_2^-}) = p_{x^-} \) (say), for all \( x \in U \) and all \( c_1^-, c_2^- \in C^- \) where \( x \in c_1^- \) and \( x \in c_2^- \).

**Proof.** The proof is similar to that of corollary 62, and omitted. \( \square \)

**Lemma 64.** Suppose that there is a price vector \( P \) that supports the allocation \( A \) in a pseudo-competitive equilibrium. In this case, the instance MONOTONE\((U,C)\) admits a satisfying assignment.

**Proof.** Given the price vector \( P \), we construct the following boolean assignment \( T : U \rightarrow \{0,1\} \).
• If a variable does not appear in any negative clause, then set its value to 1: For all \( x \in U \), if \( x \notin c^- \) for all \( c^- \in C^- \), then \( T(x) = 1 \).

• Consider any variable \( x \in U \) that appears in at least one negative clause \( c^- \in C^- \) (i.e. \( x \in c^- \)). According to Lemma 63, we have: \( P(j_{x,c^-}) = p_{x^-} \). Set the value of \( x \) to 1 iff \( p_{x^-} \geq 2 \). Therefore:

\[
T(x) = 1 \text{ if } p_{x^-} \geq 2, \text{ and } T(x) = 0 \text{ if } p_{x^-} < 2.
\]

We now show that \( T \) is a satisfying assignment. We consider two mutually exclusive and exhaustive cases.

**Case 1.** Consider any positive clause \( c^+ = x \lor y \lor z \in C^+ \), where \( x, y, z \in U \).

For the sake of contradiction, assume that the clause \( c^+ \) is not satisfied by the assignment \( T \). Hence, we have: \( T(x) = T(y) = T(z) = 0 \). It implies the following.

• There is a negative clause \( c^-_x \in C^- \) that contains the variable \( x \) (i.e. \( x \in c^-_x \)); otherwise we would have had \( T(x) = 1 \).

• Similarly, there is a negative clause \( c^-_y \in C^- \) that contains the variable \( y \) (i.e. \( y \in c^-_y \)), and a negative clause \( c^-_z \in C^- \) that contains the variable \( z \) (i.e. \( z \in c^-_z \)).

From the way the assignment \( T \) has been constructed, we infer that: \( P(j_{x,c^-_x}) < 2 \). Thus, agent \( i_x \in I \) gets positive utility from item \( j_{x,c^-_x} \in J \). Since the agent \( i_x \) only gets item \( j_x \in J \) under allocation \( A \), and the price vector \( P \) supports the allocation \( A \) in a pseudo-competitive equilibrium, we get:

\[
P(j_x) + P(j_{x,c^-_x}) \geq B_{i_x}.
\]

Since \( B_{i_x} = 10 \) and \( P(j_{x,c^-_x}) < 2 \), we get: \( P(j_x) > 8 \). Since \( P(j_{x,c^+}) \geq P(j_x) \) (Lemma 61), we deduce:

\[
P(j_{x,c^+}) > 8. \tag{4.5}
\]
Similarly, we conclude that:

\[ P(j_{y,c^+}) > 8, \text{ and } P(j_{z,c^+}) > 8. \]  \hfill (4.6)

Now, under allocation \( A \), agent \( i_{c^+} \) gets the three items \( j_{x,c^+}, j_{y,c^+} \) and \( j_{z,c^+} \). Equation (4.5) and equation (4.6) imply that the total payment of the agent \( i_{c^+} \) (under \( (A, P) \)) exceeds 24, whereas her budget is \( B_{i_{c^+}} = 23 \). This contradicts the assumption that \( (A, P) \) is a pseudo-competitive equilibrium.

Hence, we conclude that the assignment \( T \) satisfies every positive clause \( c^+ \in C^+ \).

**Case 2.** Consider any negative clause \( c^- = \neg x \lor \neg y \lor \neg z \in C^- \), where \( x, y, z \in U \).

For the sake of contradiction, assume that the clause \( c^- \) is not satisfied by the assignment \( T \). Hence, we have: \( T(x) = T(y) = T(z) = 1 \). Obviously, each of the variables \( x, y, z \) belong to at least one negative clause. From the way the assignment \( T \) was constructed, we infer that:

\[ P(j_{x,c^-}) \geq 2, P(j_{y,c^-}) \geq 2 \text{ and } P(j_{z,c^-}) \geq 2 \]  \hfill (4.7)

Under allocation \( A \), the agent \( i_{c^-} \in I \) gets all the three items \( j_{x,c^-}, j_{y,c^-} \) and \( j_{z,c^-} \). Equation (4.7) states that her payment is at least 6 under the price vector \( P \), whereas her budget is \( B_{i_{c^-}} = 5 \). This contradicts our assumption that \( (A, P) \) is a pseudo-competitive equilibrium.

Hence, we conclude that the assignment \( T \) satisfies every negative clause \( c^- \in C^- \).

The Lemma follows. \( \square \)

**Lemma 65.** Suppose that the instance \( \text{MONOTONE}(U,C) \) admits a satisfying assignment \( T \). In this case, there is a price vector \( P \) that supports the allocation \( A \) in a pseudo-competitive equilibrium.

**Proof.** Given the satisfying assignment \( T \), we construct the following price vector \( P \).
• For all variables $x \in U$ such that $T(x) = 1$:
  
  - Set $P(j_x, c^+) = 5$ for all clauses $c^+ \in C^+$ that contain the variable $x$ (i.e. $x \in c^+$).
  
  - Set $P(j_x, c^-) = 2$ for all negative clauses $c^- \in C^+$ that contain the variable $x$ (i.e. $x \in c^-$).

• For all variables $x \in U$ such that $T(x) = 0$:
  
  - Set $P(j_x, c^+) = 9$ for all clauses $c^+ \in C^+$ that contain the variable $x$ (i.e. $x \in c^+$).
  
  - Set $P(j_x, c^-) = 1$ for all negative clauses $c^- \in C^+$ that contain the variable $x$ (i.e. $x \in c^-$).

It is easy to verify that the price vector $P$ supports the allocation $A$ in a pseudo-competitive equilibrium. 

Theorem 66 follows from Lemma 64 and Lemma 65.

**Theorem 66.** The problem $\text{EQ-Pricing}(I, J, A)$ is NP hard, even for the families of instances where every agent can easily identify her utility-maximizing subset of items given the price vector.

### 4.3.2 Competitive Equilibrium for Divisible Items

To circumvent the computational hardness (see Section 4.3.1) and to guarantee existence, we focus on the problem of finding a competitive equilibrium for divisible items. Let $\text{Frac-Comp-Eq}(I, J, \{v_i, b_i\})$ denote an instance of the problem. Every agent $i \in I$ has an additive valuation function $v_i$, and a budget $b_i$. Items are divisible. We want to find a competitive equilibrium.
Consider the following function.

\[ f_i(t_i) = \begin{cases} t_i & \text{if } t_i \leq b_i; \\ b_i \log \left( (e/b_i) \times t_i \right) & \text{otherwise.} \end{cases} \quad (4.8) \]

The function \( f_i(t_i) \) is continuous and concave, and it defines the objective of the convex program described below.

\[
\begin{align*}
\text{Maximize} & \quad \sum_{i \in I} f_i(t_i) \\
\text{s.t.} & \quad t_i = \sum_{j \in J} v_i(j) \cdot x_{ij} \quad \forall i \in I \\
& \quad \sum_{i \in I} x_{ij} \leq 1 \quad \forall j \in J \\
& \quad x_{ij} \geq 0 \quad \forall i \in I, j \in J
\end{align*}
\quad (4.9-4.12)\]

The variable \( x_{ij} \) gives the fraction of item \( j \in J \) received by agent \( i \in I \). Constraint (4.10) sets \( t_i \) to be the total valuation obtained by agent \( i \in I \), and constraint (4.11) ensures that no more than one unit of an item is allocated. The symbol \( \lambda_{ij} \) (resp. \( \lambda_{ij} \)) will stand for the Lagrangian multiplier associated with constraint (4.11) (resp. constraint (4.12)). The multiplier \( P_j \) will give the price of item \( j \in J \). The Lagrangian dual function of the convex program (4.9) is given by:

\[
\mathcal{L} = \sum_{i \in I} f_i \left( \sum_{j \in J} v_i(j) \cdot x_{ij} \right) + \sum_{j \in J} \left( 1 - \sum_{i \in I} x_{ij} \right) \cdot \lambda_{ij} + \sum_{i \in I, j \in J} x_{ij} \cdot \lambda_{ij} \quad (4.13)
\]

The Lagrangian dual problem is defined as follows.

\[
\begin{align*}
\text{Minimize} & \quad \mathbf{P}(j) \geq 0, \lambda_{ij} \geq 0 \\
\text{Maximize} & \quad \mathbf{X}_{ij} \geq 0 \quad \mathcal{L}
\end{align*}
\quad (4.14)
\]

Let \( \{x_{ij}^*, P^*(j), \lambda_{ij}^*\} \) be an optimal solution to the Lagrangian dual problem of (4.9).
The **utility per unit price** of agent $i$ from item $j$ is defined as: $u_i^j = (v_i(j) - P^*(j))/P^*(j)$. Let $u_i$ denote the agent’s maximum utility per unit price from any item, and let $J_i^{\text{max}} = \{j \in J : u_i^j = u_i^{\text{max}}\}$ denote the subset of items where this maximum is attained. We partition the set of agents $I$ into three groups $I^-$, $I^0$ and $I^+$, depending on whether the agent’s maximum utility per unit price is negative, zero, or positive.

$$I^- = \{i \in I : u_i^{\text{max}} < 0\}$$
$$I^0 = \{i \in I : u_i^{\text{max}} = 0\}$$
$$I^+ = \{i \in I : u_i^{\text{max}} > 0\}$$

**Assumption 3** (No Dummy Items). For every item $j \in J$, there is some agent $i \in I$ such that $v_i(j) > 0$.

As mentioned above, the optimal solution to the dual problem (4.14) is given by $\{x_{ij}^*, P^*(j), \lambda_{ij}^*\}$. For all agents $i \in I$, define $t_i^* = \sum_{j \in J} v_i(j) \cdot x_{ij}^*$. Strong duality implies that $\{t_i^*, x_{ij}^*\}$ is an optimal solution to the primal problem (4.9). We now invoke the KKT conditions. First, the partial derivative of the function $L$ (Equation 4.13), with respect to $x_{ij}$, is zero at the optimal solution $\{x_{ij}^*, P^*(j), \lambda_{ij}^*\}$ to the dual problem (4.14). Second, if $x_{ij}^* > 0$, then $\lambda_{ij}^* = 0$. Finally, recall that $\lambda_{ij}^*$ is always non-negative.

Putting all the above facts together, we infer that for all $i \in I$ and $j \in J$:

$$\begin{align*}
\text{If } t_i^* < b_i \text{ and } x_{ij}^* > 0, & \text{ then } v_i(j) = P^*(j). \\
\text{If } t_i^* < b_i \text{ and } x_{ij}^* = 0, & \text{ then } v_i(j) \leq P^*(j). \\
\text{If } t_i^* > b_i \text{ and } x_{ij}^* > 0, & \text{ then } v_i(j) = (t_i^*/b_i) \cdot P^*(j). \\
\text{If } t_i^* > b_i \text{ and } x_{ij}^* = 0, & \text{ then } v_i(j) \leq (t_i^*/b_i) \cdot P^*(j).
\end{align*}$$

**Lemma 67.** The allocation $\{x_{ij}^*\}$ ensures that there is no surplus of items. Further-
more, the price of every item is positive. Hence, we have:

\[ \sum_{i \in I} x^*_i = 1, \quad \text{and} \quad P^*(j) > 0 \quad \text{for all } j \in J. \]

**Proof.** If \( P^*(j) = 0 \), then equation (4.16) implies that \( v_i(j) = 0 \) for agents \( i \in I \), violating Assumption 3. Hence, we conclude that \( P^*(j) > 0 \). Now, invoking complementary slackness, we see that constraint (4.11) must be tight. The lemma follows.

Fix any agent \( i \in I \) and any item \( j \in J \) such that \( x^*_{ij} > 0 \). For all items \( j' \in J \), we have that:

\[ \frac{v_i(j)}{P^*(j)} \geq \max \left\{ 1, \frac{v_i(j')}{P^*(j')} \right\} \quad (4.17) \]

This can be verified by rearranging the terms of equation (4.16) and noting that the price of the item \( P^*(j) > 0 \) (see Lemma 67). The next claim is a simple (and intuitive) reinterpretation equation (4.17).

**Claim 68.** The allocation \( \{x^*_i\} \) and the pricing \( P^* \) ensure the following property: An agent \( i' \in I \) gets positive fraction of an item \( j' \in J \) only if the item \( j' \) gives her maximum utility per unit price (that is, \( j' \in J^\text{max} \)) and this quantity is non-negative (that is, \( u^{\text{max}}_{j'} \geq 0 \)).

Claim 68 is used to prove the next lemma.

**Lemma 69.** Fix the fractional allocation \( \{x^*_i\} \) and the price vector \( P^* \). We have:

1. Every agent \( i \in I^- \) receives zero allocation and makes zero payment.

2. Every agent \( i \in I^0 \) receives positive fraction of an item \( j \in J \) only if \( j \in J^\text{max}_i \).

The total payment of agent \( i \) does not exceed \( b_i \). Hence, we have: \( \sum_{j \in J} x^*_i \cdot P^*(j) \leq b_i \).
3. An agent \( i \in \mathcal{I}^+ \) receives positive fraction of some item \( j \in \mathcal{J} \) only if \( j \in J_{i}^{\max} \). The total payment of agent \( i \) is exactly equal to \( b_i \). Hence, we have:

\[
\sum_{j \in \mathcal{J}} x_{ij}^* \cdot P^*(j) = b_i.
\]

Proof. We consider three mutually exclusive and exhaustive cases.

Case 1. (Agent \( i \in \mathcal{I}^- \))

By definition, the maximum utility per unit price of agent \( i \) is negative. Claim 68 states that such an agent receives zero allocation. Hence, she pays zero price, and her utility is zero.

Case 2. (Agent \( i \in \mathcal{I}^0 \))

Claim 68 states the agent receives positive fraction of an item \( j \in \mathcal{J} \) only if \( j \in J_{i}^{\max} \). Since her utility per unit price from any item \( j \in J_{i}^{\max} \) is zero, her total payment is equal to her total valuation \( t_{i}^* \).

If it were true that \( t_{i}^* > b_i > 0 \), then the agent would get positive fraction of at least one item, and according to equation (4.16), her utility per unit price from that particular item would be \( u_{i}^{\max} = (t_{i}^* / b_i) - 1 > 0 \), contradicting the assumption that \( i \in \mathcal{I}^0 \). Hence, we infer that \( t_{i}^* < b_i \), and her total payment is:

\[
\sum_{j \in \mathcal{J}} x_{ij}^* \cdot P^*(j) = t_{i}^* < b_i
\]

Case 3. (Agent \( i \in \mathcal{I}^+ \))

Claim 68 states that the agent gets positive fraction of an item \( j \in \mathcal{J} \) only if \( j \in J_{i}^{\max} \). Hence, if we had \( t_{i}^* < b_i \), then according to equation (4.16), her maximum utility per unit price would be non-positive, contradicting the assumption that \( i \in \mathcal{I}^+ \). We conclude that \( t_{i}^* > b_i \).
Now, we again invoke equation (4.16) and write: 
\((b_i/t_i^*) \cdot v_i(j) \cdot x_{ij}^* = P^*(j) \cdot x_{ij}^*\).

Summing both sides of this equality over all items \(j \in J\), we infer that:

\[ b_i = \sum_{j \in J} P^*(j) \cdot x_{ij}^* \]

In other words, the total payment of the agent is exactly equal to \(b_i\). \(\square\)

Lemma 70 follows from Lemma 67, Claim 68 and Lemma 69.

**Lemma 70.** Fix the allocation \(\{x_{ij}^*\}\) and the pricing \(\{P^*(j)\}\). We have: 1) Every item is completely allocated. 2) An agent \(i \in I\) gets non-zero fraction of an item \(j \in J\) only if \(j \in J_i^{\max}\) and \(v_i^{\max} \geq 0\). 3) The payment of each agent \(i \in I\) is at most \(b_i\). 4) The payment of each agent \(i \in I^+\) is exactly equal to \(b_i\).

In Lemma 70, the second and the third conditions imply that every agent \(i \in I^- \cup I^0\) gets zero utility. The second and the fourth conditions imply that every agent \(i \in I^+\) spends her entire budget only on those items that give her maximum (non-negative) utility per unit price. Hence, the allocation \(\{x_{ij}^*\}\) ensures that every agent in \(I\) gets her maximum possible utility, under the price vector \(P^*\). Furthermore, every item in \(J\) is completely allocated. These observations lead us to the following theorem.

**Theorem 71.** The problem \textsc{Frac-Comp-Eq}(\(I, J, \{v_i, b_i\}\)) can be solved using the convex program (4.9).

### 4.3.3 Approximate-Competitive Equilibrium for Indivisible Items

In this section, we present a poly-time algorithm that returns an approximate competitive equilibrium (see Definition 10) for indivisible items, provided an agent’s valuation for a single item is always less than half of her overall budget.
Assumption 4. We have: $0 \leq \gamma < 1/2$, where

$$\gamma = \max_{i \in I, j \in J} \left\{ \frac{v_i(j)}{B_i} \right\}$$

Our algorithm works in three steps. First, we recall that the convex program 4.9 returns a competitive equilibrium when the items are divisible (see Theorem 71). Next, we scale down the budget of every agent by $(1 - \gamma)$ and solve the convex program 4.9. Finally, we use the GAP rounding technique [Shmoys and Tardos (1993)] to transform the fractional allocation $\{x_{ij}^*\}$ (returned by the convex program) into an integral one $\{X_{ij}\}$, and show that it results in a $(1 - 2\gamma)$-approximate competitive equilibrium for indivisible items.

The algorithm is described in Figure 4.1. The next lemma lists some nice properties of the GAP rounding.

**Figure 4.1**: Approximate-competitive equilibrium for indivisible items.

**Lemma 72.** The GAP rounding [Shmoys and Tardos (1993)] converts the fractional allocation $\{x_{ij}^*\}$ into an integral allocation $\{X_{ij}\}$. Here, $X_{ij} = 1$ if item $j$ goes to agent $i$, and zero otherwise. The following properties are maintained.
1. For all items $j \in J$, we have that $\sum_{i \in I} X_{ij} = \sum_{i \in I} x^*_{ij} = 1$.

2. For all agents $i \in I$ and items $j \in J$, we have $X_{ij} = 1$ only if $x^*_{ij} > 0$.

3. For all agents $i \in I$, we have:

$$\sum_{j \in J} x^*_{ij} \cdot P^*(j) - \max_{j \in J} \{v_i(j)\} \leq \sum_{j \in J} X_{ij} \cdot P^*(j) \leq \sum_{j \in J} x^*_{ij} \cdot P^*(j) + \max_{j \in J} \{v_i(j)\}$$

Lemma 72 can be interpreted as follows. The GAP rounding ensures that an agent receives an item in the integral allocation only if she obtained nonzero fraction of the same item in the fractional allocation. As an important consequence, the scheme preserves the property described in Claim 68.

**Claim 73.** The allocation $\{X_{ij}\}$ and the pricing $P^*$ satisfy the following property: An agent $i' \in I$ receives an item $j' \in J$ only if the item gives her maximum utility per unit price (that is, $j' \in J^\text{max}_i$) and this quantity is non-negative (that is, $u^\text{max}_i \geq 0$).

Lemma 72 has another serious implication. As the fractional allocation is transformed into an integral one (keeping the price vector fixed at $P^*$), it gives tight bounds on the net change in an agent's total payment. The following claims quantify this intuition.

**Claim 74.** Under the allocation $\{X_{ij}\}$ and the pricing $P^*$, the total payment of every agent $i' \in I^0 \cup I^+$ is at most her budget.

**Proof.** We write:

$$\sum_{j \in J} X_{ij} \cdot P^*(j) \leq \sum_{j \in J} x^*_{ij} \cdot P^*(j) + \max_{j \in J} \{v_i(j)\}$$

$$\leq b_{i'} + \gamma \cdot B_{i'}$$

$$= (1 - \gamma) \cdot B_{i'} + \gamma \cdot B_{i'} = B_{i'}$$
The first inequality follows from Lemma 72. The second inequality follows from Lemma 69. \( \square \)

**Claim 75.** Under the allocation \( \{X_{ij}\} \) and the pricing \( P^* \), the total payment of every agent \( i' \in \mathcal{I}^+ \) is at least \((1 - 2\gamma)\) fraction of her budget.

**Proof.** We write:

\[
\sum_{j \in \mathcal{J}} X_{ij} \cdot P^*(j) \geq \sum_{j \in \mathcal{J}} x^*_j \cdot P^*(j) - \max_{j \in \mathcal{J}} \{v_\varphi(j)\} = b_i - \max_{j \in \mathcal{J}} \{v_\varphi(j)\} \geq (1 - \gamma) \cdot B_\varphi - \gamma \cdot B_\varphi = (1 - 2\gamma) \cdot B_\varphi
\]

The first inequality follows from Lemma 72. The second equality follows from Lemma 69. \( \square \)

Fix the allocation \( \{X_{ij}\} \) and the price vector \( P^* \). Lemma 72 states that every item is allocated to some agent. Claim 73 and Claim 74 imply that every agent in \( \mathcal{I}^- \cup \mathcal{I}^0 \) receives zero utility, which clearly is the maximum possible utility for such an agent under the given setting of prices.

Now consider any agent \( i' \in \mathcal{I}^+ \). According to Claim 74 and Claim 75, her total payment is at least \((1 - 2\gamma)B_\varphi\) and at most \(B_\varphi\); and recall that she spends her money only on those items that give her maximum utility per unit price (see Claim 73). Hence, her actual utility from the allocation \( \{X_{ij}\} \) is no less than \((1 - 2\gamma)\) fraction of the maximum utility she can expect to get under the pricing \( P^* \). We conclude that the algorithm in Figure 4.1 returns a \((1 - 2\gamma)\)-approximate competitive equilibrium.

**Theorem 76.** The algorithm in Figure 4.1 returns a \((1 - 2\gamma)\) approximate equilibrium \((\{X_{ij}\}, P^*)\) where: (1) Each agent \( i \in \mathcal{I}^+ \) pays at least the amount \((1 - 2\gamma)B_i\). (2) Each agent \( i \in \mathcal{I}^- \cup \mathcal{I}^0 \) gets zero utility.
Next, we show that our approximate competitive equilibrium generates good revenue. The symbol $\text{Rev}$ will denote the revenue from $\{X_{ij}\}, P^*$. The symbol $\text{Opt}$ will denote the optimal-revenue benchmark (see Section 4.1.2). The notation $\text{Opt}^+$ (resp. $\text{Opt}^{-,0}$) will represent the total contribution towards $\text{Opt}$ from the agents in $\mathcal{I}^+$ (resp. $\mathcal{I}^- \cup \mathcal{I}^0$). The outcome $\{X_{ij}\}, P^*$ extracts at least $(1 - 2\gamma)$ fraction of the budget of every agent in $\mathcal{I}^+$, which in turn, upper bounds her contribution towards $\text{Opt}$. Thus, we get:

$$\text{Rev} \geq (1 - 2\gamma) \cdot \text{Opt}(\mathcal{I}^+)$$  \hspace{1cm} (4.18)

The outcome $\{X_{ij}\}, P^*$ gives zero utility to every agent in $\mathcal{I}^- \cup \mathcal{I}^0$. Hence, the price $P^*(j)$ of every item $j \in \mathcal{J}$ is at least the valuation $v_i(j)$ of any agent $i \in \mathcal{I}^- \cup \mathcal{I}^0$. Since every item is allocated, we have: $\text{Rev} = \sum_{j \in \mathcal{J}} P^*(j)$. We conclude that:

$$\text{Rev} = \sum_{j \in \mathcal{J}} P^*(j) \geq \sum_{j \in \mathcal{J}} \max_{i \in \mathcal{I}^- \cup \mathcal{I}^0} \{v_i(j)\}$$

$$\geq \text{Opt}^{-,0}$$

$$\geq (1 - 2\gamma) \text{Opt}^{-,0}$$ \hspace{1cm} (4.19)

Theorem 77 follows if we add up equations (4.18), (4.19) and note that $\text{Opt} = \text{Opt}^+ + \text{Opt}^{-,0}$.

**Theorem 77.** The revenue of the approximate competitive equilibrium $\{X_{ij}\}, P^*$ is always within a factor $(1/2 - \gamma)$ of the optimal-revenue benchmark.

### 4.4 Concave Combinatorial Valuations

Consider any market of indivisible items, and budget constrained agents with concave-combinatorial valuations. We present a poly-time algorithm (see Figure 4.2) which always returns a pseudo-competitive equilibrium. The algorithm is based on an
ascending auction framework, and has the important property that all the items allocated to the same agent are assigned the same price.

First, we will present the underlying intuition behind the algorithm. We use the notations introduced in Section 4.1.1.

Fix any subset of items $J \subseteq J$, a price $q \geq 0$, and an agent $i \in I$. Now, if an agent $i \in I$ wants to find her utility-maximizing bundle $J^* \subseteq J$, then she will pick as many items as possible from the subset $S_i \cap J$, until her marginal utility from the next item becomes negative, or she runs out of her budget $B_i$. Her demand is defined to be the number of items in such an optimal bundle $J^*$.

**Definition 23.** The demand of an agent $i \in I$, w.r.t. the subset $J \subseteq J$ and price $q$, is defined as:

$$D_i(J, q) = \max \left\{ k \in \{1, \ldots, |J \cap S_i|\} : \alpha_i(k) \geq q, kq \leq B_i \right\}$$

**Definition 24.** An agent’s pseudo demand, w.r.t. the subset $J \subseteq J$ and price $q$, is given by:

$$D_i^+(J, q) = \lim_{\epsilon \to 0^+} D_i(J, q + \epsilon)$$

Hence, it is the limiting value of her demand at a slightly higher price $q + \epsilon$.

We now make the following assumption.

**Assumption 5 (No Dummy Items).** For all items $j \in J$, there is at least one agent $i \in I$ such that $j \in S_i$.

If there is an item for which every agent has zero valuation, then we can easily remove that item from the market. Similarly, we can delete any agent who has zero budget. The above assumption, therefore, holds without any loss of generality.

Our algorithm will use the concepts of a demand graph and a pseudo demand graph, which are described below.
The (Pseudo) Demand Graph. Suppose that each item in \( J \subseteq \mathcal{J} \) has been assigned the same price \( q \geq 0 \). Fix any subset of agents \( I \subseteq \mathcal{I} \). The demand graph \( G(I, J, q) \) and the pseudo demand graph \( G^+(I, J, q) \) are weighted directed graphs defined on the same set of nodes \( V(I, J, q) = V^+(I, J, q) = \{s, t\} \cup I \cup J \). We designate the node \( s \) (resp. node \( t \)) as the source (resp. sink). The two graphs have the same edge-set \( E(I, J, q) = E^+(I, J, q) \), but the weights on these edges can vary from one graph to another.

- For all agents \( i \in I \), there is a directed edge \((s, i)\) having a weight of \( D_i(J, q) \) (resp. \( D_i^+(J, q) \)) in the demand graph \( G(I, J, q) \) (resp. pseudo demand graph \( G^+(I, J, q) \)).

- For all \( i \in I \) and \( j \in J \cap S_i \), there is a directed edge \((i, j)\) having infinite weights in both the graphs.

- For all \( j \in J \), there is a directed edge \((j, t)\) having the same weight of 1 in both the graphs.

Interpreting the (pseudo) demand graph. Any integral \( s-t \) flow in graph \( G(I, J, q) \) (resp. \( G^+(I, J, q) \)) defines an allocation of the items in \( J \) to the agents in \( I \): Agent \( i \in I \) gets item \( j \in J \cap S_i \) iff unit flow is routed through the edge \((i, j)\). The amount of flow through the edge \((s, i)\) gives the total number of items allocated to agent \( i \in I \). An item \( j \in J \) remains unallocated iff the flow through the edge \((j, t)\) is zero. The value of maximum \( s-t \) flow in the graph \( G(I, J, q) \) (resp. \( G^+(I, J, q) \)) is given by \( f(I, J, q) \) (resp. \( f^+(I, J, q) \)).

Definition 25. The graph \( G(I, J, q) \) (resp. \( G^+(I, J, q) \)) is feasible iff it admits a \( s-t \) flow of value \(|J|\).
Definition 26. A demand graph $G(I, J, q)$ is tight iff the demand graph $G(I, J, q)$ is feasible but the pseudo-demand graph $G^+(I, J, q)$ is not feasible, i.e., iff $f(I, J, q) = |J|$ and $f^+(I, J, q) < |J|$.

Outline of our algorithm (Figure 4.2). Initially, every item has price $q = 0$ and $f(I, J, 0) = |J| < \sum_{i \in I} D_i(J, 0)$. Thus, we can allocate all the items, but the price is too low and some agents’ demands remain unsaturated.

We increase the price $q$. As a result, the demands of the agents keep decreasing. First, let us consider a very simple special case. Suppose that we find a price where

$$f(I, J, q) = |J| = \sum_{i \in I} D_i(J, q)$$

Here, we can get a competitive equilibrium by setting a uniform price of $q$ across all the items, and returning the allocation characterized by the maximum $s - t$ flow in $G(I, J, q)$. This will exhaust the supply of every item, and at the same time saturate the demand of every agent.

More generally, we start the first iteration of our algorithm by setting $J_1 = J$, $I_1 = I$, and we stop the price rise the moment we get: $f(I_1, J_1, q) = |J_1| > f^+(I_1, J_1, q)$. In other words, we can allocate every item at price $q$, but if we raise the price any further, then some items will remain unallocated. To be precise, the price rise stops when the demand graph $G(I_1, J_1, q)$ becomes tight, meaning that $G(I_1, J_1, q)$ is feasible but $G^+(I_1, J_1, q)$ is infeasible (see Definitions 25, 26).

To continue with our algorithm, suppose that the demand graph $G(I_1, J_1, q)$ becomes tight at $q = q_1$. We identify a critical subset of items $J_1^* \subseteq J_1$, which is preventing us from raising the price beyond $q_1$. Let $I_1^* = \{i \in I_1 : S_i \cap J_1^* \neq \emptyset\}$ denote the subset of all agents interested in $J_1^*$. We want to throw out the agents in $I_1^*$ and the items in $J_1^*$, and raise the price of the remaining items. Three properties are maintained.
1. The pseudo demand graph $G^+(I_1 \setminus I_1^*, J_1 \setminus J_1^*, q_1)$ is feasible.

2. No agent $i \notin I_1^*$ is interested in an item from $J_1^*$, that is, $S_i \cap J_1^* = \emptyset$ for all $i \notin I_1^*$ and $j \in J_1^*$.

3. In the demand graph $G(I_1^*, J_1^*, q_1)$, we can find an $s-t$ flow that allocates all the items, and gives at least $D_i^+(J_1, q_1)$ items to every agent $i \in I_1^*$. We distribute the items in $J_1^*$ amongst the agents in $I_1^*$, at uniform price $q$ and in accordance with the allocation $A_{I_1^*, J_1^*}$ characterized by this $s-t$ flow.

The first condition enables us to increase the price of the items in $J_1 \setminus J_1^*$. The second condition ensures that no agent $i \in I_1 \setminus I_1^*$ complains about this, simply because she does not care about the items in $J_1^*$. The third condition ensures that the agents in $I_1^*$ are also happy, modulo tie-breaking. This holds since the utility of any agent $i \in I_1^*$ is lower bounded by: $u_i(A_{I_1^*, J_1^*}(i), q_1) \geq \max_{J \subseteq J_1} \{u_i^+(J, q_1)\}$, and the price of the items in $J_1 \setminus J_1^*$ can only increase during the course of the algorithm. We will say that the allocation $A_{I_1^*, J_1^*}$ is proper w.r.t. the demand graph $G(I_1, J_1, q)$ (see Definition 27).

**Definition 27** (Proper Allocation). Consider a tight demand graph $G(I, J, q)$, a nonempty subset of agents $\emptyset \subset I^* \subseteq I$, and a nonempty subset of items $\emptyset \subset J^* \subseteq J$. An allocation $A_{I^*, J^*}$ is called proper iff:

- For all $i \in I \setminus I^*$, we have that $S_i \cap J^* = \emptyset$. Agents in $I \setminus I^*$ are not interested in any item $j \in J^*$.

- The pseudo demand graph $G^+(I \setminus I^*, J \setminus J^*, q)$ is feasible. It allows us to raise the price beyond $q$, after removing the agents in $I^*$ and the items in $J^*$.

- For all $i \in I^*, T \subseteq J$, we get $u_i(A_{I^*, J^*}(i), q) \geq u_i^+(T, q)$. In other words, agents in $I^*$ are happy.
No item \( j \in J^* \) remains unallocated: We have that
\[
\sum_{i \in I^*} A_{I^*, J^*}(i) = J^*.
\]

**Remark.** Throughout the rest of this chapter, if all the items in \( T \subseteq J \) are assigned the same price \( q \), then we slightly abuse the notation (see Equation (4.3)). We use the symbol \( u_i(T, q) \) (resp. \( u_i^+(T, q) \)) to denote the utility (resp. pseudo utility) of agent \( i \in I \) from the subset of items \( T \).

---

**The Ascending Auction**

**INPUT:** Set of agents \( I \) and indivisible items \( J \).

Agent \( i \in I \) has valuation function \( v_i \) and budget \( B_i > 0 \).

1. \( I_1 \leftarrow I, J_1 \leftarrow J, q_0 \leftarrow 0, \) and \( k \leftarrow 1 \).
2. **WHILE** \( J_k \neq \emptyset \)
3. Find the price \( q_k > q_{k-1} \) where the demand graph \( G(I_k, J_k, q_k) \) becomes tight.
4. Find a proper allocation \( A_{I^*_k, J^*_k} \) in graph \( G(I_k, J_k, q_k) \).
5. FOR every agent \( i \in I^*_k \): \( \tilde{A}_{I, J}(i) \leftarrow A_{I^*_k, J^*_k}(i) \).
6. FOR every item \( j \in J^*_k \): \( \tilde{P}(j) \leftarrow q_k \).
7. \( I_{k+1} \leftarrow I_k \setminus I^*_k, \) \( J_{k+1} \leftarrow J_k \setminus J^*_k \).
8. \( k \leftarrow k + 1 \).

**OUTPUT** the allocation vector \( \tilde{A}_{I, J} \) and price vector \( \tilde{P} \).

**Figure 4.2:** Finding a Pseudo-competitive equilibrium.

**Theorem 78.** The algorithm in Figure 4.2 finds a pseudo competitive equilibrium in polynomial time.

The proof of Theorem 78 appears in Section 4.4.1.
Comparison with the Devanur et al. (2008) algorithm. We will show that the algorithm in Figure 4.2 finds a pseudo-competitive equilibrium in polynomial time. Before proceeding, we note that our ascending auction is inspired by the primal-dual framework for the standard-Fisher model with additive valuations [Devanur et al. (2008)]. However, there are some important differences between the two algorithms.

In the demand graph considered by Devanur et al., the weight of an edge \((s, i)\), which is coming out of the source node, is equal to the agent’s budget. Similarly, the weight of an edge \((j, t)\), which is going into the sink, is equal to the item’s price per unit. This demand graph is not applicable if the agents have concave-combinatorial valuations, for the agents’ utilities are no longer additively separable across the items. Instead, our demand graph assigns a weight of \(D_i(J, q)\) to the edge \((s, i)\), and a weight of 1 to the edge \((j, t)\).

Since the items are indivisible, the agents’ demands (and hence the edge weights) change in discrete jumps with the increase in price. This helps us achieve polynomial running time. On the other hand, \textit{we are forced to consider an additional pseudo demand graph to deal with the indivisibility of the items.}

The algorithm by Devanur et al. terminates when the maximum \(s-t\) flow in the demand graph saturates the budget of every agent and the supply of every item, so that: \(\sum_{j \in J} P(j) = \sum_{i \in I} B_i\). At this point, an allocation characterized by arbitrary maximum \(s-t\) flow clears the market. To contrast it with the way we allocate the items, consider the instant when: (1) we have found the price \(q_1\) so that this demand graph \(G(I_1, J_1, q_1)\) is tight, and (2) identified the critical subset of items \(J^*_1\) and agents \(I^*_1\). Here, if we simply take an arbitrary maximum \(s-t\) flow in \(G(I^*_1, J^*_1, q_1)\), we can no longer guarantee that it will ensure a pseudo-competitive equilibrium. In our algorithm, this issue is addressed in the following manner.

We first find a maximum \(s-t\) flow \(f^+\) in the pseudo-demand graph \(G^+(I^*_1, J^*_1, q_1)\). This saturates the weight \(D^+_i(J^*_1, q_1)\) of every edge \((s, i)\), where \(i \in I^*_1\). Next, we
augment this flow $f^+$ along simple $s-t$ paths to get a particular maximum $s-t$ flow $f$ in the graph $G(I^*_s, J^*_t, q_1)$. This transformation ensures that the flow routed through an edge $(s, i)$ is never decreased. Thus, in the allocation characterized by the flow $f$, every agent $i \in I^*_s$ gets at least $D^+_i(J^*_t, q_1)$ items. Next, we show that $D^+_i(J^*_t, q_1) = D^+_i(J^*_t, q_1)$, and we use this property while proving that the outcome of our algorithm is a pseudo-competitive equilibrium.

**Revenue.** There are instances where the revenue of all pseudo-competitive equilibria falls far short of the optimal-revenue benchmark. Consider the following example.

**Example 2.** We have two indivisible items, and two unit-demand agents.

- Agent 1 has budget $B_1 = L$. Her valuations for the items are given by: $v_1(1) = v_1(2) = v_1(\{1, 2\}) = L$.

- Agent 2 has budget $B_2 = \epsilon$. Her valuations for the items are given by: $v_1(1) = v_1(2) = v_1(\{1, 2\}) = \epsilon$.

Clearly, the agents in the above example have concave-combinatorial valuation functions. It is easy to verify that any pseudo-competitive equilibrium will have to set prices $P(1) = P(2) \leq \epsilon$, and will have to allocate one item to each of the agents. As a result, the revenue from any pseudo-competitive equilibrium will be at most $2\epsilon$. On the other hand, the optimal-revenue benchmark is equal to $\text{OPT} = L$. It simply allocates one item to agent 1 at price $L$.

Therefore, we focus on the special case of single-valued-combinatorial valuations. Here, our algorithm ensures that either an agent spends half of her budget, or the price of every item is larger than her valuation for that item. These observations lead us to the next theorem.
**Theorem 79.** If the agents have single-valued-combinatorial valuations, then the algorithm in Figure 4.2 returns a pseudo-competitive equilibrium whose revenue is at least $1/3$ times the optimal-revenue benchmark.

The proof of Theorem 79 appears in Section 4.4.3.

**4.4.1 Correctness of Algorithm: Proof of Theorem 78**

For the correctness of our algorithm (Figure 4.2), we require that a tight demand graph should admit a proper allocation. The next important theorem (whose proof appears in Section 4.4.2) resolves this issue.

**Theorem 80.** In any tight demand graph $G(I, J, q)$, we can find a specific proper allocation $A_{I^*, J^*}$ in polynomial time. It ensures that $\emptyset \neq I^* \subseteq I$, $\emptyset \neq J^* \subseteq J$, and $D_i^+(J, q) < |S_i|$ for all agents $i \in I^*$.

Our algorithm (Figure 4.2) works in the following manner. Suppose that the **WHILE** loop (Steps 3-8) is entering into $k^{th}$ iteration. The previous iterations define mutually exclusive and non-empty subsets of items $J_1^* \ldots J_{k-1}^* \subseteq J$, along with mutually exclusive and non-empty subsets of agents $I_1^* \ldots I_{k-1}^* \subseteq I$. During iteration $l \in \{1 \ldots k - 1\}$, the items in $J_l^*$ are given to the agents in $I_l^*$, at uniform price $q_l$ and according to the allocation $A_{I_l^*, J_l^*}$. The algorithm defines $I_l$ (resp. $J_l$) to be the subset of agents (resp. items) who were not allocated till the $(l - 1)^{th}$ iteration.

$$I_l = I \setminus \{I_1^* \cup \cdots \cup I_{l-1}^*\}, \quad \text{and} \quad J_l = J \setminus \{J_1^* \cup \cdots \cup J_{l-1}^*\}$$

for all $l \in \{1 \ldots k\}$.

The algorithm ensures that the prices are increasing, i.e.,

$$0 = q_0 < q_1 < \cdots < q_{k-1}.$$

Furthermore, it guarantees that for all $l \in \{1, \ldots, k\}$:

The pseudo demand graph $G^+(I_l, J_l, q_{l-1})$ is feasible.  \hspace{1cm} (4.20)
All the above properties can be proved by induction on \( k \) (Assumption 5 guarantees that the statements hold for \( k = 1 \)).

Equation (4.20) ensures that there is a unique price \( q = q_k > q_{k-1} \) where the demand graph \( G(I_k, J_k, q) \) becomes tight, thereby validating Step 3. As the price is increased, the (pseudo) demands of the agents drop in discrete jumps. Hence, Step 3 can be implemented in polynomial time. Since the demand graph \( G(I_k, J_k, q_k) \) is tight, we can efficiently find a proper allocation \( A_{I_k^*, J_k^*} \) where the subsets \( I_k^* \subseteq I_k \) and \( J_k^* \subseteq J_k \) are non-empty (see Theorem 80). Thus, every iteration allocates at least one item, the algorithm runs for at most \( |J| \) iterations, and it terminates in polynomial time.

**Lemma 81.** Consider any iteration \( l \) of the **While** loop in Figure 4.2. For all \( i \in I_l^* \), we have that \( S_i \subseteq J_l \).

**Proof.** Fix any item \( j \in J \setminus J_l \). Clearly, item \( j \) was allocated in an iteration prior to \( l \): For some \( l' < l \), we have that \( j \in J_{l'}^* \). Since \( i \in I_l^* \) and \( l' < l \), agent \( i \) did not get any item during iteration \( l' \). We conclude that \( i \in I_l \setminus I_{l'}^* \). The allocation \( A_{I_{l'}^*, J_{l'}^*} \) is proper w.r.t. the graph \( G(I_{l'}, J_{l'}, q_{l'}) \), implying that \( j \notin S_i \) (see Definition 27). This holds for every item \( j \in J \setminus J_l \). Hence, we get: \( S_i \subseteq J_l \). \qed

We have already noted that the algorithm runs in polynomial time, and halts only when it has allocated all the items (Step 2). For the rest of the proof, fix any agent \( i \in I_l^* \) and any subset of items \( T \subseteq J_l \). This agent gets the items in \( A_{I_l^*, J_l^*} \) at uniform price \( q_l \). We will show:

\[
u_i(A_{I_l^*, J_l^*} (i), q_l) \geq u_i^+(T, \tilde{P}) \tag{4.21}
\]
The allocation $A_{I^*}^{J^*}$ is proper w.r.t. the graph $G(I, J, q)$. Definition 27 implies the following.

$$u_i(A_{I^*}^{J^*}(i), q_t) \geq u_i^+(T \cap J, q) \geq u_i^+(T \cap J, \tilde{P}) \quad (4.22)$$

The last inequality holds since $\tilde{P}(j) \geq q_t$ for all $j \in J_l$.

Since $S_i \subseteq J_l$ (see Lemma 81), we deduce that: $u_i^+(T \setminus J_l, \tilde{P}) \leq 0$. Now, applying equation (4.22), we get:

$$u_i(A_{I^*}^{J^*}(i), q_t) \geq u_i^+(T \cap J_l, \tilde{P}) \geq u_i^+(T \cap J_l, \tilde{P}) + u_i^+(T \setminus J_l, \tilde{P}) = u_i^+(T, \tilde{P})$$

This concludes the proof of Theorem 78, provided we assume that Theorem 80 holds.

In Section 4.4.2, we will prove Theorem 80.

4.4.2 Finding a Proper Allocation: Proof of Theorem 80

Suppose that the demand graph $G(I, J, q)$ is tight (see Assumption 6). We will show that such a demand graph admits a proper allocation, and it can be computed in polynomial time.

**Assumption 6.** The demand graph $G(I, J, q)$ is tight.

**s–t Cuts** Fix a subset of agents $X \subseteq I$, a subset of items $Y \subseteq J$, and consider the pseudo demand graph $G^+(X, Y, q)$. In this graph, every subset of nodes $V \subseteq X \cup Y$ defines a $s–t$ cut. This cut will be denoted by the ordered pair $(s, V)$, indicating that the nodes in $V$ belong to the side of $s$, whereas the rest of the nodes belong to the side of $t$. The value of the cut is denoted by $\Delta^+_{XYq}(V)$. Clearly, the quantity $\Delta^+_{XYq}(V)$
is equal to the total weights of the directed edges in $G^+(X, Y, q)$ that originate from $\{s\} \cup V$ and terminate in $({\{t\} \cup X \cup Y}) \setminus V$.

**Definition 28.** Fix a pseudo demand graph $G^+(X, Y, q)$, and consider any subset of nodes $V \subseteq X \cup Y$. The cut $(s, V)$ is a maximal minimum $s - t$ cut if and only if it satisfies the following conditions.

- For all $V' \subseteq X \cup Y$, we have that $\Delta_{XYq}^+(V') \geq \Delta_{XYq}^+(V)$. In other words, the cut $(s, V)$ is a minimum $s - t$ cut in $G^+(X, Y, q)$.
- For all $v \in (X \cup Y) \setminus V$, we have that $\Delta_{XYq}^+(V \cup \{v\}) > \Delta_{XYq}^+(V)$.

The algorithm is described in Figure 4.3. We first compute a maximal minimum $s - t$ cut $(s, I^s \cup J^s)$ in the pseudo demand graph $G^+(I, J, q)$. This partitions the set of agents $I$ (resp. items $J$) into two mutually exclusive subsets $I^s$ and $I^t$ (resp. $J^s$ and $J^t$).

Corollary 84 states that the pseudo demand graph $G^+(I^t, J^t, q)$ is non-empty. Our algorithm finds an allocation $A_{I^t,J^t}$ of the items in $J^t$ to the agents in $I^t$. Clearly, all the steps in Figure 4.3 can be implemented in polynomial time. It remains to show that the algorithm returns the correct answer.

The rest of Section 4.4.2 is organized as follows. Recall Definition 27. In order to show that the allocation $A_{I^t,J^t}$ is proper in the demand graph $G(I, J, q)$, we have to prove the following properties.

1. No agent in $I^s$ wants an item from $J^t$. Furthermore, the subsets $I^t$ and $J^t$ are non-empty. These are shown in Corollary 83 and Corollary 84.
2. The pseudo demand graph $G^+(I^s, J^s, q)$ is feasible. This is shown in Lemma 86.
3. Under the allocation $A_{I^t,J^t}$, the utility received by every agent $i \in I^t$ is at least her pseudo utility from any subset $T \subseteq J$, provided all the items are priced at $q$. Furthermore, every item in $J^t$ is allocated. These are shown in Lemma 91.
Finding a Proper Allocation

**Input:** A tight demand graph $G(I, J, q)$.

1. Find a maximal minimum $s-t$ cut $(s, I^s \cup J^s)$ in the pseudo-demand graph $G^+(I, J, q)$. The cut partitions the set of agents $I$ (resp. set of items $J$) into two subsets $I^s \subseteq I$ and $I^t = I \setminus I^s$ (resp. $J^s \subseteq J$ and $J^t = J \setminus J^s$).

2. Compute a maximum $s-t$ flow $f^+$ in the pseudo-demand graph $G^+(I^t, J^t, q)$.

3. The flow $f^+$ is a valid $s-t$ flow in the demand graph $G(I^t, J^t, q)$, i.e., it respects the weight of every edge in $G(I^t, J^t, q)$.

4. In the demand graph $G(I^t, J^t, q)$, find a maximum $s-t$ flow $f$, by augmenting the flow $f^+$ along simple paths. Let $A_{I^t, J^t}$ be the allocation characterized by the flow $f$.

**Output:** The allocation $A_{I^t, J^t}$.

**Figure 4.3:** Algorithm for computing a proper allocation in a tight demand graph.

Finally, in order to prove Theorem 80, we need to show that for all agents $i \in I^t$, we have: $D_i^+(J, q) < |S_i|$. This is shown in Claim 89. This claim will be used while proving the revenue property of our algorithm, for the special case of single-valued-combinatorial valuations (see Section 4.4.3).

**Properties of $s-t$ Cuts**

Fix a collection of agents $X \subseteq I$, items $Y \subseteq J$, and consider the pseudo demand graph $G^+(X, Y, q)$. The next claim specifies the value of any $s-t$ cut in $G^+(X, Y, q)$.

**Claim 82.** Consider any $s-t$ cut $(s, X^s \cup Y^s)$ in $G^+(X, Y, q)$. It partitions the set $X$ (resp. $Y$) into two mutually exclusive subsets $X^s$ and $X^t = X \setminus X^s$ (resp. $Y^s$ and $Y^t = Y \setminus Y^s$). Recall that the value of this $s-t$ cut is denoted by $\Delta_{X,Y,q}^+(X^s \cup Y^s)$.
• If the graph $G^+(X,Y,q)$ contains some edge from $X^s \times Y^t$, then we have
  \[ \Delta_{XYq}^{+}(X^s \cup Y^s) = \infty. \]

• Otherwise, we have
  \[ \Delta_{XYq}^{+}(X^s \cup Y^s) = \sum_{i \in X^t} D_i^+(Y,q) + |Y^s|. \]

Proof. Suppose that the graph $G^+(X,Y,q)$ contains some edge $(i_s,j_t) \in X^s \times Y^t$. The edge $(i_s,j_t)$ has infinite weight, and it contributes towards the $s-t$ cut $(s,X^s \cup Y^s)$. Hence, we get:
  \[ \Delta_{XYq}^{+}(X^s \cup Y^s) = \infty \]

Next, suppose that the graph $G^+(X,Y,q)$ does not contain any edge $(i_s,j_t) \in X^s \times Y^t$. Let $E'$ denote the set of edges in $G^+(X,Y,q)$ that contribute towards the $s-t$ cut $(s,X^s \cup Y^s)$. We note:
  \[ E' = \{(s,i) : i \in X^t\} \cup \{(j,t) : j \in Y^s\} \]

In the graph $G^+(X,Y,q)$, every edge $(s,i)$ has weight $D_i^+(Y,q)$ and every edge $(j,t)$ has unit weight. Summing over the weights of all the edges in $E'$, we get:
  \[ \Delta_{XYq}^{+}(X^s \cup Y^s) = \sum_{i \in X^t} D_i^+(Y,q) + |Y^s| \]

This concludes the proof of the claim. \qed

Corollary 83. The pseudo demand graph $G^+(I,J,q)$ does not contain any edge from $I^s \times J^t$. Furthermore, the value of the minimum $s-t$ cut in $G^+(I,J,q)$ is given by:
  \[ \Delta_{I,Jq}^{+}(I^s \cup J^s) = \sum_{i \in I^t} D_i^+(J,q) + |J^s| \]

Proof. In the pseudo demand graph $G^+(I,J,q)$, the $s-t$ cut $(s,I \cup J)$ has value $|J|$. Hence, the value of the minimum $s-t$ cut is at most $|J|$, which is a finite quantity. The corollary follows from Claim 82. \qed
Corollary 84. The subset of items $J_t$ and the subset of agents $I_t$ are non-empty.

Proof. If $J_t = \emptyset$ then $J_s = J$, and Corollary 83 implies that the minimum $s - t$ cut in $G^+(I, J, q)$ is at least $|J|$. Hence, the pseudo demand graph $G^+(I, J, q)$ is feasible. This contradicts Assumption 6. We conclude that the subset of items $J^t$ is non-empty.

Consider any item $j \in J_t$. Since the demand graph $G(I, J, q)$ is feasible (see Assumption 6), there is at least one agent $i \in I$ who is interested in item $j$, that is, $j \in S_i$. By Corollary 83, such an agent cannot belong to the subset $I^s$. Hence, we have that $i \in I^t$, which implies that the set $I^t$ is also non-empty. 

Pseudo-Demand Graph $G^+(I^s, J^s, q)$ is Feasible

Fix any subset of items $Q \subseteq J^s$. The next claim shows that sufficiently many agents in $I^s$, with large enough pseudo demands, are interested in $Q$. This property will be crucial in proving Lemma 86.

Claim 85. For every subset of items $Q \subseteq J^s$ and subset of agents $R = \{i \in I^s : S_i \cap Q \neq \emptyset\}$, we have:

$$\sum_{i \in R} D^+_i(J^s, q) \geq |Q|$$

Proof. Define $I' = I^s \setminus R$ and $J' = J^s \setminus Q$. In the graph $G^+(I, J, q)$, we focus on the $s - t$ cut $(s, I' \cup J')$.

By definition, the graph $G^+(I, J, q)$ does not contain any edge from $I' \times Q$. Since $I' \subseteq I^s$, Corollary 83 implies that the same graph does not contain any edge from $I' \times J'$. Applying Claim 82, we get:

$$\Delta^+_{I,J,q}(I' \cup J') = \sum_{i \in I' \cup R} D^+_i(J, q) + |J'| \quad (4.23)$$

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Since $\Delta^+_{I,J,q}(I^* \cup J^*)$ is the minimum value of a $s-t$ cut in $G^+(I,J,q)$, we infer that:

$$\Delta^+_{I,J,q}(I^* \cup J^*) \geq \Delta^+_{I,J,q}(I^* \cup J^*)$$

or,

$$\sum_{i \in I^* \cup R} D_i^+(J,q) + |J'| \geq \sum_{i \in I^*} D_i^+(J,q) + |J^*|$$

or,

$$\sum_{i \in R} D_i^+(J,q) \geq |Q|$$

(4.24)

The second inequality follows from Equation 4.23 and Corollary 83. The last inequality holds since $|J'| = |J^*| - |Q|$.

Corollary 83 states that no agent in $I^*$ wants an item from $J' = J \setminus J^*$. Hence, we get:

$$D_i^+(J,q) = D_i^+(J^*,q) \quad \text{for all } i \in R \subseteq I^*.$$  

(4.25)

The claim follows from Equation 4.24 and Equation 4.25.

Lemma 86. Pseudo-demand graph $G^+(I^*,J^*,q)$ is feasible.

Proof. We apply Hall’s Marriage Theorem on Claim 85, and draw the following conclusion. There is an allocation of the items in $J^*$ to the agents in $I^*$ where:

- Every item $j \in J^*$ is allocated. Furthermore, the item goes to an agent $i \in I^*$ only if $j \in S_i$.

- Every agent $i \in I^*$ gets at most $D_i^+(J^*,q)$ items.

Let $f^*$ be the $s-t$ flow corresponding to the above allocation. Clearly, the flow $f^*$ is valid in the pseudo demand graph $G^+(I,J,q)$, in the sense that it respects the weight of every edge. Since every item in $J^*$ is allocated, the flow $f^*$ has value $|J^*|$. We conclude that the pseudo demand graph is feasible.
Proper Allocation of Items in $J^t$ to Agents in $I^t$

The first claim in this section is very similar to Claim 85. It states the following. Consider any subset of agents $R \subseteq I^t$. These agents are interested in sufficiently many items from the subset $J^t$, and their pseudo-demands are not large enough.

**Claim 87.** Fix any subset of agents $R \subseteq I^t$, and let $Q = \cup_{s \in R} \{ S_i \cap J^t \}$ denote the subset of items in $J^t$ for which they have non-zero valuations. We have:

$$\sum_{i \in R} D_i^+(J, q) \leq |Q|$$

**Proof.** In the graph $G^+(I, J, q)$, we focus on the $s-t$ cut $(s, R \cup I^s \cup Q \cup J^s)$.

By definition, no agent in $R$ wants an item from $J^t \setminus Q$. Corollary 83 states that no agent in $I^s$ wants an item from $J^t$. Hence, the pseudo demand graph $G^+(I, J, q)$ does not contain any edge from $(I^s \cup R) \times (J^t \setminus Q)$. We can therefore invoke Claim 82 and conclude that:

$$\Delta_{I,J,q}^+(R \cup I^s \cup Q \cup J^s) = \sum_{i \in I^t \setminus R} D_i^+(J, q) + |Q| + |J^s|$$  \hspace{1cm} (4.26)

Since $\Delta_{I,J,q}^+(I^s \cup J^s)$ is the minimum value of any $s-t$ cut in the graph $G^+(I, J, q)$, we get:

$$\Delta_{I,J,q}^+(I^s \cup J^s) \leq \Delta_{I,J,q}^+(R \cup I^s \cup Q \cup J^s)$$

or,

$$\sum_{i \in I^s} D_i^+(J, q) + |J^s| \leq \sum_{i \in I^t \setminus R} D_i^+(J, q) + |Q| + |J^s|$$

or,

$$\sum_{i \in R} D_i^+(J, q) \leq |Q|$$

The second inequality follows from Corollary 83 and Equation 4.26. The last inequality holds since $|I^t| = |R| + |I^t \setminus R|$.

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Proof of the next claim closely parallels the proof of Lemma 86. It shows that the maximum $s-t$ flow in the pseudo demand graph $G^+(I^t, J^t, q)$ equals the total weight of all the edges coming out of node $s$. In contrast, Lemma 86 showed that the maximum $s-t$ flow in the pseudo demand graph $G^+(I^s, J^s, q)$ equals the total weight of all the edges going into node $t$.

**Claim 88.** Any maximum $s-t$ flow in graph $G^+(I^t, J^t, q)$ sends $D_i^+(J^t, q)$ units of flow through the edge $(s, i)$, for all agents $i \in I^t$. In other words, it saturates the weight of every edge coming out of the source.

**Proof.** Using Claim 87 and Hall’s Marriage Theorem, we infer that there is an allocation of the items in $J^t$ to the agents in $I^t$ where:

- Every item $j \in J^t$ is allocated. Furthermore, the item goes to an agent $i \in I^t$ only if $j \in S_i$.

- Every agent $i \in I^t$ gets exactly $D_i^+(J^t, q)$ items.

We follow the notation in Figure 4.3 (see Step 2), and use the symbol $f^+$ to denote the $s-t$ flow characterized by the above allocation. Clearly, this flow is valid in graph $G^+(I^t, J^t, q)$, in the sense that it respects the weight of every edge. Furthermore, it saturates the weight $D_i^+(J^t, q)$ of every edge $(s, i)$ coming out of the source node $s$. Hence, it is a maximum $s-t$ flow in the graph $G^+(I^t, J^t, q)$, and any other maximum $s-t$ flow will also satisfy the same condition. $\square$

As we will see, the allocation $A_{I^t,J^t}$ ensures that every agent $i \in I^t$ gets at least $D_i^+(J, q)$ items. The previous claim, however, provides an apparently weaker lower bound of $D_i^+(J^t, q)$. Claim 90 will resolve this issue, and show that $D_i^+(J^t, q) = D_i^+(J, q)$ for all agents $i \in I^t$. To prove this result, we will use the next claim.
Claim 89. For all agents $i \in I^t$, we have that:

$$D_i^+ (J, q) < |S_i \cap J^t|.$$  

Proof. Fix an agent $i' \in I^t$, and for the sake of contradiction, suppose that $D_i^+ (J, q) \geq |S_{i'} \cap J^t|$. In the pseudo demand graph $G^+(I, J, q)$, we focus the $s-t$ cut $(s, I^s \cup \{i'\} \cup J^s \cup \{S_{i'} \cap J^t\})$.

Corollary 83 states that no agent in $I^s$ wants an item from $J^t$. Furthermore, by definition, the agent $i'$ does not want any item from $J^t \setminus \{S_{i'} \cap J^t\}$. Thus, we infer that the graph $G^+(I, J, q)$ does not contain any edge from $(I^s \cup \{i'\}) \times (J^t \setminus \{S_{i'} \cap J^t\})$.

Applying Claim 82, we get:

$$\Delta^+_{I,J,q} (I^s \cup \{i'\} \cup J^s \cup \{S_{i'} \cap J^t\}) = \sum_{i \in I^t \setminus \{i'\}} D_i^+ (J, q) + |J^s \cup \{S_{i'} \cap J^t\}|$$

$$= \sum_{i \in I^t} D_i^+ (J, q) - D_{i'}^+ (J, q) + |J^s| + |S_{i'} \cap J^t|$$

$$= \Delta^+_{I,J,q} (I^s \cup J^s) + |S_{i'} \cap J^t| - D_{i'}^+ (J, q)$$

$$\leq \Delta^+_{I,J,q} (I^s \cup J^s)$$

The third equality follows from Corollary 83. The last inequality follows from our assumption: $D_{i'}^+ (J, q) \geq |S_{i'} \cap J^t|$, and it contradicts the fact that $(s, I^s \cup J^s)$ is the maximal minimum $s-t$ cut in the pseudo demand graph $G^+(I, J, q)$ (see Definition 28 and Figure 4.3). We conclude that the claim holds for all agents $i' \in I^t$.  

Claim 90. For all agents $i \in I^t$, we have that

$$D_i^+ (J, q) = D_i^+ (J^t, q)$$

Proof. Claim 89 has a simple interpretation: If the items in $J$ are priced at $q$, then we can construct a pseudo utility-maximizing bundle $J^* \subseteq J$ for agent $i \in I^t$ by taking
sufficiently many elements from the set $S_i \cap J^t$. To be more specific, we have: $J^s \subseteq (S_i \cap J^t) \subseteq J^t \subseteq J$. Thus, the subset $J^s$ also maximizes the pseudo utility of agent $i \in I^t$ amongst the collection of items $J^t$. It follows that $D_i^+(J, q) = D_i^+(J^t, q)$. □

We are now ready to prove two key properties of our algorithm.

**Lemma 91.** The allocation $A_{I^t, J^t}$ returned by the algorithm in Figure 4.3 has the following properties.

- Every item $j \in J^t$ is allocated.
- For all agents $i \in I^t$ and for all subsets $T \subseteq J$, we have: $u_i(A_{I^t, J^t}(i), q) \geq u_i^+(T, q)$.

**Proof.** We compute a maximum $s-t$ flow $f^+$ (Step 2, Figure 4.3) in the pseudo demand graph $G^+(I^t, J^t, q)$, which routes $D_i^+(J^t, q)$ units of flow (Claim 88) through the edge $(s, i)$, for all agents $i \in I^t$.

Since the pseudo demand of an agent is at most her demand, the flow $f^+$ is also a valid $s-t$ flow in the demand graph $G(I^t, J^t, q)$. Therefore, we can construct a maximum $s-t$ flow $f$ in $G(I^t, J^t, q)$ by augmenting the flow $f^+$ along simple paths (Step 4, Figure 4.3). Clearly, such a flow $f$ will route at least $D_i^+(J^t, q)$ units and at most $D_i(J^t, q)$ units of flow through every edge $(s, i)$.

As the demand graph $G(I, J, q)$ is feasible (see Assumption 6) and it does not contain any edge from $(I \setminus I^t) \times J^t$ (see Corollary 83), we conclude that the graph $G(I^t, J^t, q)$ is also feasible. Since $f$ is a maximum $s-t$ flow in the feasible demand graph $G(I^t, J^t, q)$, it must have a value of $|J^t|$. Let $A_{I^t, J^t}$ denote the allocation corresponding to the flow $f$ (Step 4, Figure 4.3). We conclude that this allocation has the following properties.

- Every item in $J^t$ is allocated. This holds since the corresponding $s-t$ flow $f$ has value $|J^t|$. 

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• An agent gets only those items she is interested in. This holds since the demand graph \( G(I^t, J^t, q) \) has an edge \((i, j)\) only if \( j \in S_i \), and \( f \) is a valid flow in \( G(I^t, J^t, q) \).

\[
\mathbf{A}_{I^t, J^t}(i) \subseteq S_i \cap J^t \quad \text{for all agents } i \in I^t. \tag{4.27}
\]

• Every agent \( i \in I^t \) gets at least \( D_i^+(J_i, q) = D_i(J_i, q) \) items (Claim 90) and at most \( D_i(J^t_i, q) \) items.

\[
D_i^+(J_i, q) \leq |\mathbf{A}_{I^t, J^t}(i)| \leq D_i(J^t_i, q) \quad \text{for all } i \in I^t. \tag{4.28}
\]

Fix any agent \( i \in I^t \), and let \( T' \subseteq \mathbf{A}_{I^t, J^t}(i) \) be a subset of items of size \( D_i^+(J_i, q) \). Clearly, the subset \( T' \) maximizes the pseudo utility of the agent at price \( q \), amongst all subsets of items \( T \subseteq T' \). Hence, we have:

\[
\begin{align*}
\mu_i(\mathbf{A}_{I^t, J^t}(i), q) & \geq \mu_i(T', q) \quad \text{(Equations 4.27, 4.28)} \\
& \geq \mu_i^{+}(T', q) \geq \mu_i(T', q) \quad \text{for all } T \subseteq J.
\end{align*}
\]

This concludes the proof. \( \square \)

If we set \( I^* = I^t \) and \( J^* = J^t \), then Theorem 80 follows from Corollary 83, Corollary 84, Lemma 86, Claim 89 and Lemma 91.

4.4.3 Revenue Guarantee: Proof of Theorem 79

In this section, we slightly abuse the notation and say that an agent \( i \in \mathcal{I} \) has the same valuation \( v_i \) for every item in her interest set \( S_i \). Her total valuation from any subset \( T \subseteq \mathcal{J} \) is equal to: \( v_i(T) = v_i \times |S_i \cap T| \).

\textbf{Claim 92.} Fix any agent \( i \in I_i^* \) who was considered during the \( l^{th} \) iteration of the \textbf{While} loop in Figure 4.2. In the pseudo-competitive equilibrium returned by our algorithm, this agent receives at least \( D_i^+(J_i, q_l) \) items from her interest set \( S_i \), and she pays a price \( q_l \) for every item allocated to her.
Proof. Recall the algorithm in Figure 4.3 that finds the proper allocation $A_{I^* J^*}$ in the graph $G(I, J, q_i)$. It first computes a maximum $s - t$ flow $f^+$ in the pseudo demand graph $G^+(I^*, J^*, q_i)$, and then augments it along simple $s - t$ paths to get a maximum $s - t$ flow $f$ in the demand graph $G(I^*, J^*, q_i)$. This flow $f$ characterizes the allocation $A_{I^* J^*}$.

According to Claims 88 and 90, the flow $f$ routes $D_{I^* J^*}$ units of flow through every edge $(s, i)$, where $i \in I^*$. Since the flow $f$ is obtained by augmenting the flow $f^+$, it routes at least $D_{I^* J^*}$ units of flow through the same edge. Thus, the allocation $A_{I^* J^*}$ gives at least $D_{I^* J^*}$ items to agent $i \in I^*$. Clearly, the agent gets an item $j \in J^*$ only if $j \in S_i$. Since we have $S_i \subseteq J^*$ (Lemma 81), the claim follows.

Consider an agent $i \in I^*$ as in Claim 92. Theorem 80 states that $D_{I^* J^*} < S_i$. Thus, the price $q_i$ is high enough to ensure that the agent’s pseudo demand is strictly less than the number of items she is interested in. This can happen only if either $q_i \geq v_i$ or $q_i \times D_{I^* J^*} = B_i$.

Case 1. $q_i \geq v_i$. Since the algorithm raises the price in successive iterations, we infer that $\hat{P}(j) \geq q_i \geq v_i$ for all items $j \in S_i$ (see Lemma 81).

Case 2. $q_i \times D_{I^* J^*} = B_i$ and $q_i < v_i$. Here, if $D_{I^* J^*} = 0$, then $D_{I^* J^*} = 1$ and $q_i = B_i$. Using an argument that is analogous to the previous case, we conclude that $\hat{P}(j) \geq B_i$ for all items $j \in S_i$. In contrast, if $D_{I^* J^*} > 0$, then $D_{I^* J^*} \geq D_{I^* J^*} - 1$; and Claim 92 implies that the total payment made by the agent is at least $q_i \times D_{I^* J^*} \geq B_i/2$. The next lemma summarizes all these observations.

Lemma 93. In the pseudo-competitive equilibrium returned by our algorithm (Figure 4.2), every agent $i \in I$ either spends half of her budget, or the price of every item $j \in S_i$ is set at $\hat{P}(j) \geq \min(v_i, B_i)$.

Following the above lemma, we partition the set of agents $I$ into two subsets
\( \mathcal{I}_1 \) and \( \mathcal{I}_2 \). If an agent simultaneously belongs to the two subsets, the tie is broken arbitrarily.

\[
\mathcal{I}_1 = \left\{ i \in \mathcal{I} : \sum_{j \in \mathcal{S}_i} \check{P}(j) \geq B_i/2 \right\}
\]

\[
\mathcal{I}_2 = \left\{ i \in \mathcal{I} : \check{P}(j) \geq \min(v_i, B_i) \text{ for all } j \in \mathcal{S}_i \right\}
\]

Let \( \text{REV} \) denote the revenue obtained by our algorithm, and let \( \text{OPT} \) denote the optimal-revenue benchmark. Furthermore, let \( \text{OPT}_1 \) (resp. \( \text{OPT}_2 \)) denote the contribution towards \( \text{OPT} \) by the agents in \( \mathcal{I}_1 \) (resp. \( \mathcal{I}_2 \)). In the pseudo-competitive equilibrium \((\check{A}_{\mathcal{I},\mathcal{J}}, \check{P})\), every agent \( i \in \mathcal{I}_1 \) spends at least \( B_i/2 \) amount of money, whereas her contribution towards \( \text{OPT} \) is upper bounded by her budget \( B_i \). Summing over all agents in \( \mathcal{I}_1 \):

\[
2 \times \text{REV} \geq \text{OPT}_1 \quad (4.29)
\]

For every agent \( i \in \mathcal{I}_2 \), our algorithm (Figure 4.2) sets a price \( \check{P}(j) \geq \min(v_i, B_i) \) for all items \( j \in \mathcal{S}_i \). Thus:

\[
\text{REV} = \sum_{j \in \mathcal{J}} \check{P}(j) \geq \sum_{j \in \mathcal{J}} \max_{i \in \mathcal{I}_2, j \in \mathcal{S}_i} \{ \min(v_i, B_i) \} \geq \text{OPT}_2 \quad (4.30)
\]

Theorem 79 follows if we add up equations (4.29), (4.30) and note that \( \text{OPT} = \text{OPT}_1 + \text{OPT}_2 \).
Concluding Remarks

In this dissertation, we designed algorithms for the problem of selling multiple items to budget constrained agents. We considered this problem within the frameworks of auction theory and market equilibrium.

In Chapter 2, our focus was on maximizing the seller’s revenue in an auction setting where the items are heterogenous, and the bidders have arbitrary budgets and demand constraints, and additive valuations. We assumed that the budget and demand constraint of every bidder are publicly known. However, a bidder’s valuations for the items are private knowledge, which are drawn from public distributions. In this model, we presented approximation algorithms for designing revenue-optimal incentive compatible auctions. Here, the bidders’ private valuation functions are multi-dimensional, and characterizing the revenue-optimal truthful auctions in such settings constitutes an important research agenda in microeconomics.

In Chapter 3, we considered a scenario where the items are homogeneous and the bidders have private budgets and additive valuations. Here, no incentive-compatible auction can generate good social welfare. Consequently, we focused on a relaxed notion of Pareto-Optimality. We presented an adaptive clinching auction, and showed
that it is truthful-in-expectation DSIC and that its outcome is always Pareto-optimal. Very recently, Goel et al. (2013) have extended this work to the online setting. In their model, there is an infinitely divisible item with online supply. Whenever some quantity of this item becomes available for sale, the auctioneer needs to distribute it among the bidders and decide on their payments. These decisions are irrevocable and have to be made on the fly. First, Goel et al. (2013) prove that the offline adaptive clinching auction for a divisible item is “supply monotone”. In other words, if we increase the quantity of the item to be auctioned, then the allocation and payment of a bidder can never decrease. Next, in the online supply model, they use this property to design an auction that is incentive-compatible and Pareto-optimal. A natural and important open problem is to extend their work to more general settings. For example, it will be interesting to give a truthful auction in the online supply model when the bidders have budget constraints and single-valued-combinatorial valuation functions [Fiat et al. (2011)].

In Chapter 4, we designed polynomial-time algorithms for competitive equilibria. This problem is fairly well-understood in the absence of budget constraints. For example, there is a nice characterization theorem, which states that (under some natural assumptions) a market admits a competitive equilibrium if and only if the demand functions of the agents are gross substitutes. To contrast this with our results, note that we considered precisely those markets where the agents have budget limits and their demand functions are not gross substitutes. We focused on the concept of a pseudo-competitive equilibrium, which essentially coincides with the notion of a competitive equilibrium for almost all instances, and at the same time can be very handy in ruling out degenerate corner cases. Surprisingly, in the presence of budget constraints, gross substitutability is no longer a necessary condition for guaranteeing the existence of a pseudo-compatible equilibrium. To see this, recall that when the agents have concave-combinatorial valuation functions, our algorithm
in Section 4.4 always finds a pseudo-competitive equilibrium. On the other hand, a pseudo-competitive equilibrium may not exist if the agents have additive valuation functions. It will be interesting to give a neat characterization of the markets which admit a pseudo-competitive equilibrium.
Bibliography


Biography

Sayan Bhattacharya was born in Mumbai, India on 24th February, 1987. He obtained his B. E. in Computer Science and Engineering from Jadavpur University, India, and his PhD in Computer Science from Duke University, USA.