Flow and Jamming of Granular Materials in a Two-dimensional Hopper

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics in the Graduate School of Duke University 2012
Abstract

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Flow in a hopper is both a fertile testing ground for understanding fundamental granular flow rheology and industrially highly relevant. Despite increasing research efforts in this area, a comprehensive physical theory is still lacking for both jamming and flow of granular materials in a hopper. In this work, I have designed and constructed a two dimensional (2D) hopper experiment using photoelastic particles (particles' shape: disk or ellipse), with the goal to build a bridge between macroscopic phenomenon of hopper flow and microscopic particle-scale dynamics. Through synchronized data of particle tracking and stress distributions in particles, I have shown differences between my data of the time-averaged velocity/stress profile of 2D hopper flow with previous theoretical predictions. I have also demonstrated the importance of a mechanical stable arch near the hopper opening (orifice) on controlling hopper flow rheology and suggested a heuristic phase diagram for the hopper flow/jamming transition. Another part of this thesis work is focused on studying the impact of shape of particles on hopper flow. By comparing particle-tracking and photoelastic data for ellipses and disks at the appropriate length scale, I have demonstrated an important role for the rotational degree of freedom of elliptical particles in controlling flow rheology through particle tracking and stress analysis. This work has been supported by International Fine Particle Research Institute (IFPRI).
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List of Abbreviations and Symbols

Symbols

\( d \) or \( d_{avg} \) average diameter of disks: 0.53cm

\( D \) Hopper opening size

\( \dot{M} \) or \( \frac{dM}{dt} \) mass flow rate

\( G^2 \) “G square” to represent the magnitude of stress

\( \tau \) Shear stress.

\( P \) Pressure or normal stress.

\( V \) Velocity or flow rate

\( T \) Temperature

\( \phi \) Packing fraction.

\( \phi_C \) Critical packing fraction.

\( Z \) Contact number.

\( Z_C \) Critical contact number.

\( \theta \) Radial coordinate of hopper system

\( \theta_w \) Hopper wall angle

Abbreviations

FFM Free fall model.
1 Introduction of Granular Materials

1.1 Granular materials: definition and applications

Granular materials, at first glance, are simple. They are just collections of macroscopic particles. By “macroscopic” we mean people can often differentiate individual particles by the naked eye, so manifestly the study of granular materials does not involve quantum mechanics. The size of the particles can range from tiny sand grains to huge asteroids. More common examples include food products such as rice and corn, or building materials such as cement and gravel. See Fig. 1.2(a) and (b).

Although granular materials are common in our daily lives, the technology for handling and controlling granular materials is still poorly developed compared to other technologies. Therefore, studies of granular materials have a lot of practical motivations (Fig. 1.2(c-g)). For instance, studies of granular flow can help more efficiently transport building materials in construction fields, and mix materials in pills evenly in pharmaceutical industries. Studies of frictional properties of granular materials provide a laboratory scale domain to investigate the particle-level effects on earthquake and the formation of sand dunes (Fig. 1.2(c) and (d)). Studies of
Figure 1.1: (a) Robot hand, made by a stretchable bag filled with lightweight "granular materials" (such as coffee), grip a glass cup. (b) Inside view of the hand.

Granular impact experiments, which involve dropping objects into granular media, can help in the design of human landers and defense [6]. A very recent and innovative application uses granular jamming and dilation behavior to build flexible robot hands, shown in Fig. 1.1. The elastic property of the hand can be varied by changing the shape of granular particles chains, much like designing molecules for new materials in chemistry [4].

From a scientific point of view, because of the common solid-like to fluid-like transition behavior of soft matter systems such as gels, foams or colloids, results from studies of granular materials also provide insights for the broader soft matter community. For instance, densely packed suspensions exhibit "Shear thickening" behavior, in which the viscosity $\eta$ defined by $\eta = \tau/\dot{\gamma}$ increases as a function of shear rate or shear stress. Recent progress in understanding granular materials shear-jamming [2] (See Section 1.3.2 for more details of [2].) may inform the shear thickening problem: Both shear jamming and shear thickening may be due to the shear-dilated particles within a confining boundary.
Figure 1.2: Natural and industrial examples of granular materials: (a) M&M chocolates. (b) Rices. (c) Damage caused by a Landslide. (d) Damage caused by an earthquake. (e) Workers levels grit pulled from the wastewater treatment plant before hauling it for disposal. These grit, coming from sand used on icy street in winter, clogged pumps after heavy rain. See “Seattle Times” 04/27/2009 for details. (f) A screw conveyor designed for transporting granular materials. (h) The old time measuring machine: An hourglass.
1.2 Physical challenges for the study of granular materials

Despite of its common presence in daily life, granular materials are unique compared to ordinary materials and physically puzzling under scrutiny. Granular materials can present themselves in solid-like, liquid-like and gas-like states. For instance, a sand pile at rest can be considered as solid, since it can support its own weight without changing shape infinitely. The behavior of a sand storm is more like a gaseous system where particle hardly touch each other. The transitions between these states often involve changes of packing fraction or applied stress. However, traditional thermodynamic approaches (like statistical mechanics) and continuum mechanics (like hydrodynamics) fail to predict the full range of behavior of granular materials. We summarized two major causes of these failures below:

Firstly, granular materials interact dissipatively because of static friction and the inelasticity of collisions. Therefore, after agitation, a granular medium quickly sets to a metastable state and cannot free explore phase space. This fact fails the crucial assumption of ergodicity for statistical mechanics. Also, the inelastic interactions result in clustering of grains in granular flows and gases, making it difficult to define an appropriate length scale for spatially-averaging these systems. In fact, some theoretical works have proposed new framework of granular materials, which follows the spirit of statistical mechanics, but uses different state parameters rather than P,V,T.

Secondly, granular materials may lack a well-defined scale for temporal and spatial averaging. The constitutive equations describing dynamics of an ordinary solid, gas or liquid usually require an averaging process over length scales and and time scales that are much larger than typical microscopic scales and much smaller than the macroscopic system size. For many ordinary granular system, such as coal in a silo, the largest system size is only a few thousand times of the individual coal.
Moreover, the stress in a fluid is usually time-averaged over a mesoscopic time scale that spans an enormous number of collisions of individual molecules. This lack of scale separation makes fluctuations and randomness crucial to granular behavior, such as force networks and jamming of granular flow, but are not described by the traditional approaches.

The notorious granular mixing and segregation problem where, for instance, mixing granular materials by vibration end up with segregation, is another example that is not well understood.

1.3 Jamming of granular materials

A major story for this thesis concerns comparing jamming of hopper flow to jamming of granular materials in quasi-static states. Before we review the two type of jamming of granular materials in quasi-static states, e.g., isotropic jamming and shear jamming, let’s first define what we mean by jamming of granular materials. Jamming of a granular material means that the particles are packed densely enough so they are mutually confined and the whole system of particles has rigidity. A jammed system, compared to an unjammed system, has finite elastic moduli so that it exhibits elastic deformation with small applied stress although it will likely deform plastically when applied stresses exceed the yield threshold. For instance, walking on the beach you will not sink, because sand appears solid, but building a dry-sand castle is almost impossible because they will flow.

In fact, the term “jamming” also applies to other soft particle system that exhibit this fluid-like to solid-like transition under applied stress. For instance, shaving foam or toothpaste flow like fluids when pushed out of a container, but act like solids in our hands. An important and common feature of these jammed systems is that the geometrical structures formed by the particles are disordered, unlike an ordinary crystal lattice solid. In 1998, Nagel and Liu [18] proposed the famous jamming phase
figure that ties different system together, as shown in Fig. 1.3.

![Diagram showing phase transitions in a system](image)

Figure 1.3: Jamming phase diagram proposed by Nagel and Liu [18]. The diagram suggests many disordered soft matter materials are jammed under high density and low applied shear stress, but can unjam under low density and high applied shear stress.

1.3.1 Isotropic jamming

To study jamming, theoreticians often start with the simplest case: frictionless disks/spheres. The notable work by C. S. O’hern et al. [25] provide people more direct physical insights into the jamming diagram, by performing computer simulations of frictionless disks/spheres in 2D/3D isotropic compression under zero temperature. Their choices of the interaction potential are of the following form

\[
V(r_{ij}) = \begin{cases} 
se(1 - r_{ij}/R_{ij})^\alpha/\alpha & \text{for } r_{ij} < R_{ij} \\
0 & \text{for } r_{ij} \geq R_{ij} 
\end{cases}
\]  

(1.1)

Their protocol for isotropic compression is to grow all particle sizes step by step
until the jamming point, which they define as when all particles just begin to touch each other. At higher compressions, there is a nonzero pressure inside the system. A crucial conclusion from their study is the sharp transition point \( J \): under zero shear stress/temperature, the system exhibited non-zero pressure, bulk modulus, and shear modulus as soon as the packing fraction exceeds the critical packing fraction \( \phi_c \). Although \( \phi_c \) varies slightly among members of an ensemble, the distribution of \( \phi_c \) shrank with system size and reach a well defined point around 0.84, the “random-close-packing” fraction.

The above simulation result has then been experimentally verified by the seminal work done at Duke by T. Majmudar and R. Behringer [22]. They performed a 2D biaxial cell experiment (shown in Fig. 1.4) with photoelastic particles, and for the first time experimentally measured the contact forces on individual particles. Their results experimentally demonstrates the critical nature of jamming in a real granular materials with \( \phi_c \) around 0.84. Both pressure \( P \) and contact number \( Z \) show scaling laws after their sudden increase at \( \phi_c \):

\[
\begin{align*}
(P - P_c) &\sim (\phi - \phi_c)^{\beta_P} \quad \text{with } \beta_P \approx 1.1 \\
(Z - Z_c) &\sim (\phi - \phi_c)^{\beta_Z} \quad \text{with } \beta_Z \approx 0.5 \sim 0.6
\end{align*}
\]

where \( Z_c = 4 \) for a 2D system for frictionless particles is the so-called isostatic contact number. The reason is why \( Z_c = 4 \) can be derived from simple mathematical arguments. Assume an N-sphere system in \( d_m \) dimensions. We have \( d_m \) N coordinates of the positions of the particles restricted by geometry \( ZN/2 \) the contacts: At point J, the particles just touched, so the distance between two particles is exactly the sum of their radii. To have at least resolution of these geometrical constraint equations, we need \( Z \leq 2d_m \). On the other hand, we have \( ZN/2 \) contact force variables restricted by \( dN \) equations of force balance, so we need \( Z \geq 2d_m \) hin order to have at least solution for these force balance equations. At the jamming point J, both inequalities
must hold and we get $Z_c = 2d_m$.

Figure 1.4: The 2D biaxial cell apparatus, filled with photoelastic disks. Isotropic compression or expand is achieved by compressing or expanding both red and blue boundaries, while shear is achieved by compressing red bar and expanding blue bar at the same time. Figure adapted from [22].

The above scaling law actually brings up another question: how do jammed granular materials compare to an ordinary solid? In fact, predictions based on effective medium theory, which ignores the disorder, contradicts the actual results of jammed granular materials. For instance, for an ordered state, $Z$ should scale linearly with as $\phi - \phi_c$, while the real case is equation (1.2). The vibration modes of jammed granular materials also show unique properties: for an ordered state, the density of states, $D(\omega)$, scales with the characteristic vibrational frequency $\omega$ as $D(\omega) \sim \omega^{d-1}$, while for weakly jammed states, $D(\omega)$ has a much stronger distribution towards low $\omega$, the so-called Boson peaks. All these facts suggest that the disordered structure and non-affine deformation play crucial roles for properties of weakly jammed granular materials.

Before we move on to discuss “shear jamming”, let’s finish this section by dis-
cussing simulation results of frictional disks [26]. Both $Z_c$ and $\phi_c$ now become functions of friction coefficient, $\mu$, and decrease from critical values of the frictionless case with increasing $\mu$, as shown in Fig. 1.5. This dependence is due to the undetermined nature of frictions of granular materials, as well as the added constraint of torque balance for the frictional cases. However, although the simulation claims the results are from isotropic compression case, they do not show the complete analyses of stress and fabric tensors. Therefore, it might be possible that the jammed states of frictional disks from simulations actually do not differ too much with the shear jamming states we will talk below.

Figure 1.5: Recent work by O'hern [26] showing for isostatic jamming of frictional disks, the critical packing fraction $\phi_J$ and particle contact number $Z_c$ decrease with increasing $\mu$. The term “CS” and “GA” are their two kinds of friction models. Figure adapted from [26].

1.3.2 Shear jamming

new states on the Liu-Nagel phase diagram, namely the “fragile” states and the “shear jamming” (SJ) states, as shown in Fig. 1.6. These researchers discovered that jamming of system with frictional disks, with packing fraction less than $\phi_J \approx 0.842$, can be induced by applying shearing. Under small applied shear stress, the system transits from unjammed state to fragile state, with strong force network percolating in only one direction. After the applied strain has passed some threshold, the system reaches a shear-jammed state, with strong force network percolating in both the dilatation and compression directions, as illustrated in Fig. 1.7.

![Figure 1.6: The shear jamming phase diagram proposed by R. Behringer and B. Chakraborty [2]. For a system with $\phi$ between $\phi_S$ and $\phi_J$, “fragile” states and “shear jamming” (SJ) states are induced by applying shearing. Figure adapted from [2].](image)

In more detail, their experiment is done using the same apparatus in Fig. 1.4, while performing simple shear by compressing on one side and expanding the other side of the box. To analyze the anisotropy of the force network, they first used the ratio between strong-force network cluster size and the system size to characterize
the percolation in a certain direction, and then used the fabric tensor defined as

\[ R = \frac{1}{N} \sum_{i=j} r_{ij} \| r_{ij} \| \otimes \| r_{ij} \| \]  

(1.3)

to characterize the anisotropy of the contact network. The contact anisotropy is defined as \( R_1 - R_2 \) where \( R_1 \) and \( R_2 \) are the two eigenvectors of this tensor \( R \). Their results show that the anisotropy of the jammed states vanishes as the system approaches point J from below.

Figure 1.7: Left: “fragile” state where the strong force network percolates only in the compression direction. Right: “Shear jamming” state where the strong force network percolates in both compression and dilation directions. Figure adapted from [2].
Experimental Set Up, and Image/Data Processing Techniques

In this chapter I will describe the details of the 2D hopper experiment set up, and various image/data processing techniques that I developed/adapted throughout my PhD career.

2.1 The hopper set up and the synchronized high-speed imaging system

The experiment apparatus entails a quasi two dimensional geometry with a pair of bin sections (top and bottom) connected by a hopper section, as sketched in Fig. 2.1(b). We form this geometry by a pair of Plexiglas sheets separated by aluminum spacers. A pair of aluminum side walls are used to form the wedge-shaped hopper and two horizontal teflon sliding bars are used to open and close the opening of the hopper. Around five thousands of bidisperse photoelastic disks (smaller particle diameter 5mm, bigger particle diameter 6mm, the ratio between their amount are 2:1, so I take the average diameter \( d = (5 \times 2 + 6) / 3 = 5.3 \text{mm} \)), or monodisperse ellipses (minor axis 5mm, major axis 10mm), are put in between these two plexiglas sheets, and we
make use this special property to measure the stress exerted by particles on each other. The principle of photoelasticity will be introduced at Section 2.2. Different kinds of camera and lighting are adopted based on three types of experimental needs, as detailed below.

For the first types of experiment, we measure how the statistics of jamming depends on hopper geometry. We align the hopper vertically and load all particles at top section (top bin and hopper) and initiate the flow by moving the sliding bars outwards. As the particles flow out, they are collected in bottom bin section. We use a Redlake MotioPro high-speed camera to record the flow sequence at 50 fps. Whenever the flow is stopped by particles jamming near the opening, we read the time the materials flow before jamming, by checking 50fps image sequences and we call this time the “survival time”. We then use a single pendulum to strike the outside of the hopper to reinitiate the flow, and repeat the same procedure to record around 100 survival times for each hopper geometry (We vary hopper geometry by varying hopper opening size $D$ and hopper wall angle $\theta$). If the particles empty out from top section, we simply rotate the whole apparatus with its pivot point to transfer materials from bottom bin section to top bin section in order to prepare another experiment. We then fit this pool of survival time to be a hypothetical distribution to extract the fitting parameter as mean survival time $\tau$. Results can be see in Section 4.3.

The second type of experiments involves the usage of two synchronized Red lake high speed cameras. We initiate the hopper flow as the first type of experiment and record the flow at 500 fps with 1024x768 or 1024x1024 resolution. Particle tracking and image analysis are done with Matlab. The two high speed cameras are separated by a beam splitter. One of the cameras (camera A) is used to take raw pictures for particle-tracking purposes and one with optical polarizer (camera B) in front so we can take pictures with photoelastic effect. Two example pictures are shown in Fig. 2.2
Figure 2.1: (a) Sketch of front view and side vide of the hopper set up. (b). The actual hopper apparatus. Only one camera is shown here.
(a) and (b). Since these two cameras are synchronized, we can then overlap these two pictures (Fig. 2.2 (c)) and analyze the correlation between stress and particle kinetics. Details of the overlapping techniques (image registration) and related force network analysis will be discussed in Section 2.3.2.

The third type experiments involves using HD SLR (high definition single-lens reflex) Camera to study the statistics of jamming configurations of hopper. For this type experiments, we collect a zoo of images of hopper jammed states, typically 50 images for each hopper geometry. We than calculate packing fraction $\phi$ and contact number $Z$ for each image in order to compare the hopper jamming with isotropic jamming. Because resolution of these data are crucial to differentiate the minor difference among different states, HD images are needed. Results of this experiment are discussed in Section 4.4.
Figure 2.2: (a): The original Image (from Camera A) for particle tracking purpose. (b): The photoelastic image (from Camera B) for (a), showing stress chain networks. (c): After image alignment, (a) is overlapped on top of (b). The location network (green) are then extracted to represent the stress chain networks.
2.2 Photoelastic techniques and stress calibration

One of the exciting technique that has been used in our group is the photoelastic techniques. By making the particles out of photoelastic materials, we can actually view the stress (unit: force/area) pattern inside those particles. The core idea is that photoelastic material exhibit birefringence under applied stress. Below we will show how to quantify the stress based on the photoelastic image, by following the optical path we used in our experimental system (Fig. 2.3).

![Anatomy of the circular polariscope used in industry for visualization of photoelasticity. Our experiment systems also use the same principle.](image)

Following the optical path, the light source first emits light with intensity $I_0$. We usually put some diffusion plates/paper in front of the light source in order to create uniform intensity. It then enters through a left-hand circular polarizer, which consists of a linear polarizer and a quarter wave retarded plate Q-1. The light coming out from the circular polarizer then has a “fast” and “slow” component because of the $\pi/2$ phase difference created by the quarter-wave plate.

The light then enter throughs particles made of photoelastic materials. The
particle become “birefringence” if the particles are stressed. For a “birefringerence” materials with index of refraction $n_1$ and $n_2$, we know the light coming out will be a with phase difference

$$\alpha = \frac{2\pi(n_1 - n_2)l}{\lambda}$$

(2.1)

where $l$ is the thickness of the material and $\lambda$ is the wavelength of the light.

The crucial feature of photoelastic materials is that the principal planes of optical symmetry for the birefringent material coincide with the eigenvectors of the stress tensor. Assume the stress tensor at a local point of the photoelastic materials has eigenvector $\sigma_1$ and $\sigma_2$. The photoelasticity connects the stress and index of fraction through a stress-optical coefficient $C$: $n_i = C\sigma_i$. Following from equation (2.1), we have

$$\alpha = \frac{2\pi C(\sigma_1 - \sigma_2)l}{\lambda}$$

(2.2)

To observe the refracted lights, we put a right-hand circular polarizer behind the material (and before the camera) so that the intensity of the light going into the camera depends only on the phase separation $\alpha$, but not affected by the relative orientation of the polarizers to the stress eigenvector. The final intensity of the light, as viewed behind the right-hand circular polarizer, is:

$$I = I_0 \sin^2(\alpha/2)$$

(2.3)

The particles we use is made of PSM-4 materials with sensitive photoelasticity. When subject to a moderate amount of stress, the particles show fringe patterns when $\alpha$ varies form 0 to $\pi$ for several cycles inside the particles.

To calibrate the stress based on the intensity of the photoelastic images, we have two approaches. One approach is to solve the true contact forces that reproduce a image most similar to experimental observations. The work of Majmudar [20, 21] has been able to measure, for the first time, the normal and the tangential forces in bulk
granular systems based on the photoelastic image obtained from their experiments. Their calculations are based on the theory that stress resulting from a line force $F$ acting on a semi-infinite half plane is:

$$\sigma_{rr} = -\frac{2F \cos \theta}{\pi r}$$  \hspace{1cm} (2.4)

After some mathematical derivation \([20, 21]\), the final stress inside a particle resulting from several forces acting on the particle is:

$$\sigma_{rr} = -\sum_i \frac{2F_i \cos \theta_i}{\pi r_i} + \sum_i \frac{F_i}{\pi d} \sin(\theta_1 + \theta_2)$$  \hspace{1cm} (2.5)

$$\sigma_{r\theta} = 0$$  \hspace{1cm} (2.6)

They then use the above equations to solve the force inverse problem.

However, the above approach often require a high image resolution in order to clearly differentiate every bright/dark fringe inside the particle. For my case of hopper flow experiment, we need to use high-speed cameras with a low image resolution. So we adopt the second approach— the “$G^2$” technique. This approach is a “rough” technique compared to the first approach, but it still give reasonable estimate of the average pressure inside the region of interest and has been widely used in experiments of our group. The main idea is that the stronger the stress, the more fingers inside the particles, which in turn means a larger image intensity gradient.

Then as a measure of this intensity gradient, the “$G^{2n}$” at one image pixel is defined as

$$G^2_{i,j} = \left( \frac{(I_{i-1,j} - I_{i+1,j})^2}{2} \right) + \left( \frac{(I_{i,j-1} - I_{i,j+1})^2}{2} \right) + \left( \frac{(I_{i-1,j+1} - I_{i+1,j-1})^2}{2\sqrt{2}} \right) + \left( \frac{(I_{i-1,j-1} - I_{i+1,j+1})^2}{2\sqrt{2}} \right)$$  \hspace{1cm} (2.7)

Then the “$G^2$” we use in this thesis is usually averaged over the region of interest:

$$G^2 = \frac{1}{N} \sum_{i,j} G^2_{i,j}$$  \hspace{1cm} (2.8)
Fig. 2.4 (a) shows the calibration apparatus used to quantify the relation between stress and $G^2$. Fig. 2.4 (b) shows the calibration result. In the current range of $G^2$ we have observed in experiments, stress grows linearly with $G^2$. In later parts of this thesis, we will use $G^2$ as a rough measure of the magnitude of stress.

2.3 Image processing and data processing techniques

2.3.1 The Hough transform for locating particles

Traditionally, locating circular particles for granular materials have involved two main approaches: extracting features using “convolution” technique [30] or using the “edge detection” technique to separate particles into individual blobs (Fig. 2.5). However, as will be mentioned in more details later, these methods substantially suffer from the imperfections of experimental images. Factors such as unclear difference of image intensity between particles and background, as well as obstructions and image noise greatly reduce the accuracy of these approaches. Moreover, if a particle is close to the image boundary and is only partially visible, it will not be detected using the traditional approaches.

The Hough transform method, however, can overcome all the difficulties described above. The essence of the Hough transform involves representing a geometric shape by a set of relevant parameters in the Hough space and then looking for local accumulation peaks in the Hough space as recognized objects with a shape of interest. Lines and circles are among the first sets of shapes to be robustly detected by the Hough transform because they require only 2-dimensional (2D) and 3-dimensional (3D) spaces respectively. Detailed algorithms for Linear Hough Transform (LHT) and Circular Hough Transform (CHT) can be found in early literatures [8, 17] and many computer vision textbooks [3]. In contrast, ellipse detection using the Hough Transform is still an active field of research because it requires five parameter for a 2D ellipse while a 5D Hough space requires too much computation. Different tech-
Figure 2.4: (a) The apparatus used for calibration of stress. A “T-shape” bar is put on top of a box of particles. We can then hang 20g weights one by one on the bar to calibrate the Stress vs $G^2$ relationship. (b) Photoelastic image of the calibration. (c) Stress vs $G^2$.

Techniques have been proposed in the past to reduce the dimensionality of the Hough space by ellipse geometry features [36, 23, 5, 35]. The work by Robert McLaughlin...
Figure 2.5: Illustrations of the “edge detection” approach. (a): A sample grayscale picture from granular physics experiments. (b): Edge detection results from (a). (c): Black/White complement picture from (b), with morphology opening operation to clear the picture. Particles are separated into individual blobs so their center positions can be located.

[23] has become a milestone for the recent trend to improve ellipse detection [5]. Because the Hough transform relies on curvatures of edge segments, it is insensitive to image intensity variations and also able to detect particles whose edges are not perfect everywhere.

In this section, I specifically describe how we applied the Hough Transform into the pattern recognition of our experimental images, with both circular- and elliptical-shaped particles. I will show two examples. The first example uses Circular Hough Transform to detect disks with line-shaped obstructions due to special experimental requirements. The second example uses Elliptical Hough Transform from [23] to detect rice-shaped elliptical particles in a hopper/silo from low-resolution images in noisy environments. We will show in detail how the imperfections in images make the traditional pattern finding methods inapplicable, and how our Hough method shows much improved results.

For the case of the CHT, apparatus of the experiment is sketched in Fig. 2.6. The apparatus performs simple shear deformation at constant volume in two dimen-
sions by deforming a rectangular cell into a parallelogram (Fig. 1(a)). The base of the test cell consists of a large number of adjacent narrow slats, on which rest about 1000 particles in two different sizes. The particles are made from photoelastic materials [11] so that they can provide contact force information. Each slat, with width comparable to the particle size, is independently driven at a same shear rate in order to create a homogeneous shear response of the granular materials medium. More details of the set up can be found at [28]. The experiment is a quasi-static experiment in the sense that we can take pictures at the speed of 1 frame per second and then drive the slats with one more step. Thus, we can use a standard digital camera to take high-resolution pictures for this experiment.

Figure 2.6: (a): Sketch of the shear apparatus where particles are driven by underlying slats. (b): The actual apparatus in the laboratory. Contact force of photoelastic particles can be seen by adding a polarizer on top of the particles.

While the set up is innovative and considerable from a physics perspective, the boundary lines of the slats and the material property of the particles cause a difficulty in locating disks by the traditional methods mentioned in Section ???. For instance, since photoelastic materials appear transparent to the lights without a polarizer, slats lying under the particles will cause obstructions and make traditional
“edge detection” approach fail to separate particles correctly, as shown in Fig. 2.7 (a) and (b). Also, transparency makes the particles difficult to differentiate with a background with a single threshold. As shown in Fig. 2.7 (c) and (d), bright spots in the middle of particles will make it almost impossible to establish an accurate threshold to extract “real” peaks corresponding to particles from convolution results. As mentioned previously, images taken for these specific experiments are high resolution. Therefore, if we adjust input parameters very carefully, traditional methods might just give acceptable error rates. However, for some of our high-speed imaging experiments, the low-resolution images and blurring of fast-moving particles’ edges definitely rule out the traditional particle detection approaches. We will show such an example of the high-speed imaging experiments in hopper flow of ellipses.

The above difficulty can be overcome when CHT is adopted in this scenario. Compared to traditional particle detection methods, CHT relies more on the shape of particle edges. It works well with incomplete circles so that some missing edges will not affect the results. Consequentially, CHT is almost unaffected by obstructions and intensity variations across images. In addition, CHT estimates radius information simultaneously, which is useful for experiments using particles of different sizes. The algorithms in the lab used the standard CHT method and the result is shown in Fig. 2.8.

For the case of EHT, we will use the example about how we detect ellipses for hopper flow. Traditional particle detection methods cannot be used here for several reasons. First, the “convolution” method does not work since an ellipse has an orientation. In order to find a perfect match for certain elliptical particles, an ellipse convolution mask needs to be rotated in stages up to $2\pi$. This added computational complexity immediately eliminates “convolution” from potential solutions. Second, the particles are moving so fast, especially near the opening, that their edges are blurred a little. Baking powder will also cause some dirt on the surface of the particles
(a) a small part from the original experimental image. (b): Lines created problems for edge detection. (c): Another part of the original picture, where one particle on the left is so bright that some white spots appear on its surface after thresholding. A small threshold in the convolution space is needed in order to recognize this particle, although a small-threshold will then create the problem of (d). (d): Because the particles in (a) pack are very dense, a fake convolution peak will show up in the middle.

on the acquired images. As shown in Fig. 2.9 (b), the blurring and dirt makes it almost impossible to robustly detect the particles’ edges in the image. For instance, if the intensity gradient threshold for edge detection is set to be too sensitive, too many fake edges created by dirt will cause particle separation to be difficult. On the other hand, if the edges detection threshold is set too weak, the failure to detect the edges between particles will also result in a number of connected particles. The bridges between connected particles, especially for ellipses, are often very wide and
Figure 2.8: Results from CHT showing circular-shaped particles with obstructions and close intensity with background can be successfully detected by CHT.

Figure 2.9: (a): A gray-scale picture of elliptical particles flowing in a hopper. (b): A zoom-in figure of (a) after “edge detection” technique. It shows in the region near the hopper opening where particles flow faster, that some particles cannot be separated for the same edge threshold that works well on slower particles. Also, the dirt on some particle have made some particles’ surface incomplete, thus making further attempt to separate those particles using erosion even more difficult.

so simple image erosion morphology cannot delete the bridge without significantly eroding the body of the real ellipse.

For reasons similar to CHT, Elliptical Hough Transform (EHT) is perfectly suited to tackle the above problems above. We adopt the standard EHT from [23], with
some modifications to better meet our needs. The steps of the algorithm are listed as follows.

1. Use the “edge detection” method to separate the clear particles that have clear edges and locate their centers and orientation by calculating the properties of the separated blobs. This step is used to reduce computational complexity for the EHT because EHT is slower than “edge detection” methods, although more precise.

2. Dig out the detected “clear” particles from the original image, leaving “connected” particles, which are usually clusters of two or three connected particles.

3. Use the edge pixels of these “connect” particles as inputs for EHT.

4. Inside the EHT, we first label all of the edge pixels and find their local curvature by fitting a line with their close edge pixels (close edge pixels are defined as no more than two pixels with the fitting edge pixel).

5. We then start at one edge pixel as reference point p1, and attempt to find the ellipse that p1 is on.

6. We then search the p1 neighborhood window (size: major axis length x major axis length) to randomly choose the other two edge points p2 and p3, which may be on the same ellipse as p1.

7. We then use the method in [23] to fit the ellipse parameter set \((p, q, r_1, r_2, \theta)\) from p1, p2 and p3. Many choices of p1, p2 and p3 will be immediately ruled out by checking basic criterions such as if \(ac - b^2 > 0\) and if \(r_1\) (and \(r_2\)) is close enough to known major (and minor) axis lengths.

8. If the ellipse parameter set passes the basic criterions, we check to see if there are enough edge points surrounding the ellipse. If the number of points near
the fitting ellipse passes threshold 1 (70 pixels in our case), store the fitting parameter in the Hough space. Furthermore, if the number of the points exceeds threshold 2 (90 pixels in our case), we are certain that a true ellipse has been found. So, take all edge pixels close to this ellipse out of the searching pool for speed. Setting two thresholds is necessary because the ellipses are not equally perfect. So, we want to keep all of the candidates from threshold 1 and find the local best fit later. Go to step 10 if threshold 1 has been exceeded.

9. Loop to step 6 and repeat step 50 times. 50 is a parameter that can be adjusted, depending on how the precisely the local curvature can be fitted from neighborhood pixels.

10. Loop to step 5 until all labeled pixels are checked.

11. In the candidate ellipses found from the step 5 to step 10, group all ellipse centers that are close together, and search for local maximum of number of points on the ellipse within that cluster as the best center position for that ellipse. Additional filters like checking if the fitting ellipse covers enough particle area can also help improve the accuracy of the result.

12. Output the parameter sets of the best fit for each cluster.

Results of the EHT are shown in Fig. 2.9 (b). In regards to the computational performance of pictures taken from the hopper experiment, the left “connected” particles are usually around 10 particles, with around 1000 edge pixel points. The time taken to process one image is of the order of 60 seconds and the accuracy is more than 90 percent for 10 test images on a Linux, quad-intel(R) core (TM) i5 CPU 750@2.67GHZ.

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2.3.2 Using image registration techniques to extract force network

The photoelastic image, which usually consists of bright chain-like structures in dark background, gives people a general visual understanding of how the stress distributes in the granular medium and inspires the name “force chains”. However, to incorporate this visual information into physical theory, we need to find a way to quantify the “force chain” structures, e.g., what is the orientation of the force chains, what are the lengths of the force chains and how long can the force chains stay? It might be possible to directly apply “The Line Hough transform” on the photoelastic images to find “chain-like structures” and fit them into line segments. However, the low image resolution and the “force chain” being not perfect linear create difficulty for this approach.

Fortunately, we can make use of our data from synchronized cameras. The idea is to use location networks that connect stressed particles to represent the visual “force chains network”, as shown in Fig. 2.11. In more details, we follow the below steps: (See Fig. 2.2).
1. First, we use image registration techniques to align image from Camera A (See Section 2.1) and the corresponding image from Camera B. To do this, we select matching feature points in the above two images (e.g. the corner of the hopper wall) and use the location of the feature points as inputs to the MATLAB image registration toolbox. We assume the image transformation as affine transformations which only includes translation, rotation and scale. The result from MATLAB toolbox is a 3X3 transformation matrix that then can be used to apply to any other points in the regiserted image.

2. After aligning images from the two cameras, we are now ready to find the “location network”. We first detect “geometry contacts” based purely on the geometrical information of particle locations, e.g., considering two particles in contact if the distance between their centers smaller than sum of of their radiuses. However, not all of these “geometrical contacts” are true “force contact”. To find true “force contacts”, we calculate “$G^2$” at a neighborhood of these “geometrical contacts” locations, and pick true “force contacts” only
when its neighborhood-averaged “$G^2$” exceeds a predefined threshold.

3. After locating the “force contacts”, we can then connect the centers of the particles that form the force contacts to obtain a line segment that mimic the stress chain. These piece-wise line segments then compose the final “location network”, which represents the visual “force chains network”. The advantage of “location network” is that it is composed of these well-defined line segments, so it is easy to be quantified.

2.3.3 Video making techniques

It is often needed in our research to make a movie from raw/processed images to gain intuitive impressions of the dynamics of the system, so that we can know what physics we want to quantify. Some examples are listed below:

1. We download sequences of images of hopper flow from Redlake camera. The image sequence often include both the initial jam states and the final jammed states, for which images are basically identical. To avoid processing identical images and improve efficiency, we want to know in which frame the hopper flow starts and in which frame the hopper flow stops before processing images. Therefore, it is useful to make a video out of the raw images quickly from command line. To do that, we make use of ffmpeg software installed on our physics linux machine, and write linux shell scripts to call the ffmpeg software to make the movie. A sample bash script to run the ffmpeg software can be found at Appendix A.

2. The second application involves tracking particles that eventually jam the hopper “backwards” and mark them with different color by computer, so that we can visualize how these ”arch particles” come together to stop the granular flow. An sample video can be found at http://www.youtube.com/watch?feature=
player_embedded (This video helped me win 1st prize of American Physical Society: Gallery of Topical Group on Statistical and Nonlinear Physics.) As we can see from the video, these arch particles are from highly disparate regions with no obvious correlation to each other.

3. The third application involves making movies with split screens so we can simultaneously see dynamics of different systems. We used “i-movie”, the free software on Mac, to achieve this goal. A detailed instruction can be found at http://www.youtube.com/watch?v=4lAbKYuw4nc&feature=BFa&list=PLD5869EFD8B93BB59.

The video teaches how to add one picture frame in another picture frame in i-movie so we can realize the goal of splitting windows in a movie.

2.4 Coarse grain computation to calculate continuum fields based on discrete particle-scale data

From particle tracking data of granular flow, we can obtain velocity of individual particle. However, to compare experimental data with continuum mechanics theory or some other mesoscopic models, we need to find a way to interpolate these discrete data into smooth continuous fields. To achieve this goal, we adopted the “coarse-grain” techniques introduced in the work of Zhang et al. [37]. The core idea is to use a “coarse-grain” function to distribute the physical quantity of each individual particle into the whole space. Since the closer the space point is to the particle, the more contribution of this particle can have, the coarse-grain function usually has its peak value at the particle location and gradually decreases to zero as it moves far from the particle center. Moreover, the and has to be normalized over the whole space.

In this thesis, we use a two-dimensional Gaussian function as our coarse-grain function $\psi$, with its width denoted as $w$: $\psi(r) = \frac{1}{\pi w^2} e^{-\frac{r^2}{2w^2}}$. Fig. 2.12 shows how we
determine the value of $w$.
Mathematically, we then have:

For density field $\rho(r, t)$:

$$\rho(r, t) = \sum_i m_i \psi(r - r_i(t))$$

(2.9)

For the momentum field:

$$p(r, t) = \sum_i m_i v_i(t) \psi(r - r_i(t))$$

(2.10)

The velocity field is then equal to

$$V(r, t) = \frac{p(r, t)}{\rho(r, t)}$$

(2.11)

Note that the stress field can also be derived by calculating the time derivative of the momentum field. However, since the resolution of my experimental images is usually not high enough to solve the force inverse problem, we use the “$G^2$” techniques instead for our stress field. See Section 2.2 for details.

An illustration of the above process is shown in Fig. 2.13

2.5 Parallel computing resources at Duke University: Dscr and Condor

The above coarse grain techniques require a lot of computational efforts since we are computing the field (velocity field, density field etc.) at every pixel of every image. Running the program on one computer is extremely slow. There are two kinds of high-performance computer clusters on Duke campus. One is the non-free Duke Shared Cluster Resource (DSCR) that use high-performance GPU machines, majorly supporting Duke theoretical and computational groups research. Fortunately, Duke Physics recently has collaborated with DSCR group members to installed “Condor” in all of our departmental machines.
Figure 2.12: Coarse-grained density vs $w$ of the coarse-graining function $\psi$ at 5 different locations. From around $w = d_{\text{avg}}$ to $w = 3d_{\text{avg}}$, the density reaches a plateau and is nearly $w$-independent. To keep the detailed feature, we choose $w = d_{\text{avg}}$ as the width of the two dimensional Gaussian coarse-graining function.

Figure 2.13: (a) The original image from Camera A. (b) Coarse-grained density field of (a).
Condor is an open source high-throughput computing software framework for distributed parallelization of computationally intensive tasks. It can be used to manage workload on a dedicated cluster of computers. Condor is developed by the Condor team at the University of Wisconsin and Madison and is freely available for use. A sample submission file can be found in Appendix B.
3.1 Introduction

When hopper flow opening size $D$ is equal/larger than 6 times of the particle diameter $d$, the hopper flow can usually last around several seconds and develop a relatively steady flow profile before jamming. The purpose of this chapter is to gain a general understanding of the time-averaged behavior of hopper flow in this Chapter before we discuss jamming and fluctuations of hopper flow in Chapter 4 and Chapter 5.

We will start by reviewing previous experimental and theoretical studies in Section 3.2 and Section 3.3. We will then compare our experimental data with what previous theoretical studies would suggest. The comparison suggests a more comprehensive theory is needed to fully describe the experimental data.
3.2 Background: continuum mechanics models

3.2.1 Continuum models and the Radial solution

Hopper Flow, as a typical example of granular flow, is a fertile testground for understanding fundamental granular flow rheology and is industrially highly relevant. The majority of earlier studies of this topic in the last century were conducted by chemical engineers or agriculture engineers, who were concerned about storage and transportation of granular materials. These researchers tended to consider granular materials as a continuum and to apply the principles of soil mechanics to address these problems. In this section I will give a brief review of those continuum models of hopper flow, derive one of the commonly used solutions: the Jenike radial solution, and point out some flaws of continuum models in Section 2.1.2. Note I will restrict all the analysis to 2D cases because our experiments are 2D. Similar approaches also apply to 3D cases.

To start, I need to introduce the concept of stress—the foundation of continuum mechanics. Consider a 2D infinite small rectangular material element, as sketched in Fig. 3.1. The normal directions of its boundary planes (1D line in 2D case) are chosen as the x-y axes. The stress tensor for this choice of axes is then defined as:

\[
\sigma_t = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{pmatrix}
\]

where the normal stress \(\sigma_{xx} / \sigma_{yy}\) are the normal force per unit area acting on the surface plane whose normal direction is along x axis / y axis and pointed into the material. The normal stresses are chosen to be positive if they are pointed along the normal directions. Because granular materials are usually non-cohesive and are only jammed useless compressive stress. The shear stresses \(\sigma_{xy} / \sigma_{yx}\) are chosen as the tangential force per unit area acting on the surface plane whose normal direction is along x axis / y axis. The positive direction of the shear stress is chosen as 90
degrees rotated clockwise from the normal direction of the plane it acts on. Under the condition of zero rotation, \( \sigma_{xy} = -\sigma_{yx} \).

![Figure 3.1: Definition of Stresses.](image)

The stress tensor defined above will depend on the coordinate system we choose, i.e., the magnitude of the stresses will change with the plane they are acting on. For convenience, it is common to use the two invariants of the matrix, the mean pressure, \( \sigma \), and maximum shear stress, \( \tau \), to express the stresses at a certain point of the material. This idea is best illustrated by Mohr circle, as shown in Fig. 3.2. The key results from Mohr circles are:

1. Denote the eigenvalues of the stress tensor, \( \sigma \), as the two principle stresses \( \sigma_1, \sigma_2 \);
2. The mean pressure \( \sigma = (\sigma_1 + \sigma_2)/2 \); The maximum shear stress \( \tau = (\sigma_1 - \sigma_2)/2 \);
3. For stress on the plane inclined at an angle \( \gamma \) anti-clockwise to the \( \sigma_1 \) axis (for instance, x-plane in Fig. 3.2):

   \[
   \text{Normal Stress} = \sigma + \tau \cos 2\gamma; \quad \text{Shear Stress} = \tau \sin 2\gamma
   \]  
   \[ (3.1) \]
Another important assumption of the continuum model is the Mohr-Coulomb yield criterion. It assumes that when the granular material is deforming, the Mohr circle of every point of the materials is tangential to the Coulomb yield line: $N = \mu T$. This is the so-called “incipient yield everywhere”. As we will see later, this is a strong assumption which helps complete the equations groups of continuum models, but, it also has intrinsic flaws that do not necessarily match with real experimental data. Under this assumption, and with the definition of internal frictional angle $\mu = \tan(\phi^*)$, where $\phi^*$ is a material property constant, we now have an equation to relate $\tau$ and $\sigma$:

$$\tau = \sigma \sin \phi^*$$  \hspace{1cm} (3.2)  

Similarly to fluid mechanics, we now can write down the below balance laws below in polar coordinates: (See Fig. 3.3 for the polar coordinate system used throughout this thesis)
Mass Balance, considering density $\rho$ to be a constant:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}) \Rightarrow \frac{1}{r} \frac{\partial (rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0 \quad (3.3)$$

Momentum balance:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla \cdot \sigma + \rho \mathbf{b}$$

(r component):

$$\rho(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}) + \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho g \cos \theta = 0 \quad (3.4)$$

(θ component):

$$\rho(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}) + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} - \rho g \sin \theta = 0 \quad (3.5)$$

Note for a hopper, we choose polar coordinates because of the intrinsic cylindrical symmetry of the geometry. We can now express the stress components through (3.1), where the x and y axes are identified with the circumferential and radial directions:

$$\sigma_{\theta\theta} = \sigma + \tau \cos 2\gamma; \quad \sigma_{\gamma\theta} = \sigma_{\theta\gamma} = -\tau \sin 2\gamma; \quad \sigma_{rr} = \sigma - \tau \cos 2\gamma;$$

Note that $\gamma$ now is defined as the orientation of the $\sigma_1$ axis relative to the circumferential direction.

With equation (3.2) there relating $\sigma$ with $\tau$, we have the three equations (3.3), (3.4) and (3.5). But there is still one equation left with four undetermined variable: $v_r, v_\theta, \sigma, \gamma$. So the dynamics are still not solvable. What is missing is a constitutive relation that relates stress with velocity. One common assumption of this relation is the so called “coaxial” flow rule, which basically states that the principle axe of the stress and the rate of the strain tensor are aligned with each other. The idea is illustrated in Fig. 3.4, and its mathematical expression in current context is:

$$\sin 2\gamma(\frac{\partial v_r}{\partial r} - \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r}) - \cos 2\gamma(\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}) = 0 \quad (3.6)$$
Figure 3.3: The polar coordinate system of hopper used for this thesis. The origin point O is at the intersection of the hopper walls.

Figure 3.4: The Coaxial Flow Rules. Figure adapted from [[15]].

Equipped with this flow rule and appropriate boundary conditions of stress/velocity, we finally have a complete group of equations: (3.3)-(3.6). Because of difficulty for directly solving the equations, there have been different solutions developed with different simplifying assumptions. A famous one is the so called the radial solu-
tion developed by Jenike reference, which omits the velocity term in the momentum balance equation. Therefore it is not valid near the hopper opening, where the acceleration is large. But it gives a meaningful physical picture that matches with some experimental data in an average sense. The radial solutions are:

\[ v_r = -f_0(\theta)/r; v_\theta = 0; \sigma = b_0(\theta)r, \gamma = g_0(\theta); \]  

(3.7)

The numerical solution of the profile of \( n(\theta) \), \( g(\theta) \) and \( \gamma \), dervied from solving equation groups (3.3) to (3.5) with (3.7) plugged in, are shown in Fig. 3.5 (a),(b) and (c), adapted from [27].

![Flow through Hoppers](image)

Figure 3.5: Radial Solution Profile. Figure adapted from [27]
3.2.2 Flaws of continuum models

Because of the natural discreteness and randomness of granular materials, and mathematical problems, the assumptions made in previous models have intrinsic flaws and they do not always match with experimental data. The work by [[15]] has given a comprehensive review of these flaws. We summarize the key points below from our perspective.

First, careful mathematical analysis by mathematician David Schaeffer [29] has suggested there exist instability, ill-posedness and complex nonlinearity in the continuum models of granular materials. The appearance of ill-posedness and instability suggest little numerical errors in numerical simulations will be amplified beyond control, thus invalidating the computation. As the Schaeffer states on his website: “However, I came to believe that the lack of well-posed governing equations was the major obstacle to progress in the field, and I believe that finding appropriate constitutive relations is a task better suited for physicists than mathematicians, so I reluctantly moved on.”. The numerical solution provide by work [13], based on the Mohr-Cołumb yield criterion, ends up with complex stress discontinuities (and hence the associated velocity filed) that do not match with experimental data, which usually show smooth velocity/stress field.

Second, the assumption of the coaxial flow rules also shows problems after careful examination. For a slow, dense flow in the silo geometry, the major principal stress is everywhere vertical, so coaxiality requires the material to stretch horizontally. This picture clearly contradicts the practical situation, where the materials converge and exit through the silo orifice. This problem has traditionally been handled by imposing a sudden exchange of directions of principal stresses that occurs once the orifice opens, such that stress from the walls($\sigma_{xx}$) drives the flow, not gravity($\sigma_{yy}$). This wall-driven flow is the so-called “passive state”, see Fig. 3.6. However, this solution
predicts that the only non-stagnant regions in the silo are two narrow channels which converge on the silo opening along angles at 45 degree from the vertical wall. This picture is still contradictory to experimental flow profiles, as shown in Section 3.4.

![Diagram of stress chains](image)

**Figure 3.6:** Major principal stress chains in a quasi-2D silo for the active case and passive case. Figure adapted from [[15]].

Third, as mentioned in Section 3.2.1, the Mohr-Coulomb yield criterion assumes the granular material is at incipient failure ($\tau = \sigma \sin \phi^*$) everywhere when flowing. However, this assumption seems to be too strong. For instance, in hopper/silo flow, the materials are nearly stagnant in the region near the wall and far above the opening, while accelerating almost in free fall in the region near the opening. Thus, the relation between mean pressure and shear stress should be different in these regimes, rather than following a uniform "incipient yielding" criterion. In fact, as we will see later in Section 3.3.1, the inertial number theory suggests the effective friction coefficient (ratio between mean pressure and maximum stress) actually depends on the shear rate of the materials, which is not a material property constant anymore.

Last, but maybe most importantly, is the nature of granular materials: randomness and discreteness, are often crucial for the unique properties of granular materials. On the one hand, the prerequisite for a material to be considered as a continuum usually requires that there is a good separation of microscopic and macroscopic scales of
the system, so that if we define the length scale of the volume element as $L$, then we should have $d \ll L \ll L_s$. However, this is not the case for a lot of practical granular systems. On the other hand, this random and discrete nature will make fluctuations so important for a lot of practical granular problem, such as hopper jamming or granular stick-slip behaviors. Therefore, whether or not granular materials can be regarded as continua depends on the particular case and needs careful consideration.

3.2.3 The Janssen model and the Beverloo equation

Besides the above discussion of the comprehensive continuum model, there are some simplified continuum models which only focus on certain aspects of granular materials. Although simple, these models actually suggest meaningful physical interpretations of hopper flow, and thus play crucial roles in the design of modern experiments.

German engineer H. A. Janssen used the method of differential slices to derive an analytical solution of stress distributions in a silo. Consider a granular material in a 2D rectangular silo. For a horizontal slice shown in Fig. 3.7, Janssen assumed that the stress does not vary horizontally, and considered stress balance for this whole horizontal slice:

$$(\sigma_{yy}(y + \delta y) - \sigma_{yy}(y))L = \rho g \delta y L - 2 \sigma_{xy} \delta y. \tag{3.8}$$

For materials near the wall, Janssen assumed the Coulomb law of friction:

$$\sigma_{xy} = \mu_w \sigma_{xx} \tag{3.9}$$

Janssen further made an important assumption that the $\sigma_{xx}$ and $\sigma_{yy}$ are the two principle stresses. From the Mohr circle, we see this leads to:

$$\frac{\sigma_{yy}}{\sigma_{xx}} = \frac{\sigma + \tau}{\sigma - \tau} = \frac{1 + \sin \phi^*}{1 - \sin \phi^*} = K \tag{3.10}$$

$K$ is known as the Janssen constant. Now plug equations (3.9) and (3.10) into
equation (3.8), and rewrite it as a differential equation:

\[
\frac{d\sigma_{yy}}{dy} = -\frac{2K\mu_w}{L}\sigma_{yy} + \rho g \tag{3.11}
\]

the solution is an exponential form that has characteristic length scale \( \lambda = \frac{L}{2K\mu_w} \):

\[
\sigma_{yy} = Q_0 e^{-\frac{y}{\lambda}} + \rho g \lambda (1 - e^{-\frac{y}{\lambda}}) \tag{3.12}
\]

\( Q_0 \) is an additional overload on top of the gounder materials. The physical interpretation of the Janssen model is that due to frictions the wall carries the weight of the material and the overload, so that the “pressure” does not propagate to the materials below after a certain depth, an effect below as “screening”. The Janssen effect qualitatively matches with numerous experiments [24], but it also has shown underestimates the actual value of the stress at the bottom. In fact, the assumption that the \( \sigma_{xx} \) and \( \sigma_{yy} \) are principle stresses is problematic in the sense that \( \sigma_{xy} \) is not zero. Indeed, the experiment done by John Wambaugh [34] with photoelastic disks contained in a 2D silo shows for small overloads that the pre-existing contact force network propagates deep into the the pile, contrary to the classical Janssen picture.
The reason why we review Janssen model here is that it serves as the foundation for one of the well-known equations for predicting the hopper flow rate at the exit, namely the Beverloo equation. Based on experiments with sand, Beverloo et al. suggested that [1] the mass flow rate from a hopper has the following law:

$$\frac{dM}{dt} = 0.58 \rho \sqrt{g(D - 1.4d_p)^{5/2}}$$  \hspace{1cm} (3.13)

Here, D is the diameter of the circular exit of the hopper, $d_p$ is the average particle diameter, $\rho$ is the packing fraction. The term with $d_p$ can be interpreted in terms of an shear zone adjacent to the edge of the orifice, where there are fewer particles. The 5/2-power scaling law between mass flow rate and the opening size has been substantiated in numerous studies [24].

A simple interpretation of this equation is based on the concept of a “free-fall arch”, see Fig. 3.8. In this scenario the arch is a surface spanning the exit at a height approximately D away from the exit. Above the surface, the particles behave like a solid and interact with each other. Below this surface, the particles are not in rigid contact and undergo a free fall with acceleration g. Then the exit speed of particles at the exit will be:

$$v \sim (gD)^{1/2}$$  \hspace{1cm} (3.14)

For a 3D system, the mass flow rate $\frac{dM}{dt} = \rho v A$, where $A \sim (D)^2$. We then obtain the Beverloo scaling law: $\frac{dM}{dt} \sim D^{5/2}$. This “free-fall arch” picture will be used as a theoretical guide for our jamming statistics study in Section 4.3.

3.3 Background: mesoscopic models of granular flow

3.3.1 The inertial number theory

Noted in Section 3.2.2, the constitutive relations that relate stress with strain rate and normal stress with shear stress have intrinsic flaws at least in part due to the
discrete nature of granular materials. For this reason, a new rheology of granular flow has been the subject of intensity scrutiny during the past 30 years. The “inertial number” theory, developed recently, can be considered as a milestone that greatly advance’s people’s understanding of this field. It basically use a dimensionless number $I$ to clarify different flow regimes and sets the tone for local rheology.

It was started by experiments and numerical simulations done by O. Pouliquen et al. [14] and F. Cruz et al. [7], where they basically find for a simple flow geometry (e.g., silo, planar shear, inclined plane) that the velocity/packing fraction profiles with different system parameters (e.g., system size, particle size, frictional coefficient, boundary pressure) collapse on whom their corresponding inertial number $I$, defined below, are the same:

$$I = \frac{\dot{\gamma}d}{\sqrt{\frac{P}{\rho_p}}}$$

(3.15)

Here, $\dot{\gamma}$ is the shear rate (velocity spatial gradient), $P$ is the confining pressure, $\rho_p$ is density of the particles, and $d$ is particle size.
In particular, F. Cruz performed MD simulation on a simple planar shear geometry and obtained the stress components from MD simulations in forms of a sum of a “contact” stress tensor and a “Reynolds” stress tensor (fluctuations), see [7] for details:

$$\sigma_{ij} = \sigma_{ij}^c + \sigma_{ij}^f \quad (3.16)$$

Fig. 3.9 shows the dependence of \(\mu\) from F. the Cruz et al. MD simulations results. O. Pouliquen then proposed the following analytical form to fit the experimental data:

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1} \quad (3.17)$$

It says that the effective frictional coefficient changes from a minimum value for very low \(I\) to an asymptotic value \(\mu_2\) when \(I\) increases. Pouliquen then use this relation to numerically solve for the flow profile, and achieved good agreement between experimental data and numerical solutions. He also extended the flow rule generated to a 3D case, by making use of the second invariants of the strain rate tensor, \(\|\gamma_{ij}\| = (0.5\gamma_{ij}\gamma_{ij})^{0.5}\), and definition of the shear stress tensor:

$$\tau_{ij} = \frac{\mu(I)P}{\|\gamma_{ij}\|} \dot{\gamma}_{ij} \quad (3.18)$$

The physical understanding of the inertial number, \(I\), is a relation between the microscopic rearrangement time \(t_{micro}\) to the macroscopic time \(t_{macro}\):

$$t_{micro} = \frac{d}{\sqrt{\frac{P}{\rho_p}}}$$

is estimated by a simple free-fall of the particle of diameter \(d\) and density \(\rho_p\) under a force \(Pd^2\) over a distance \(d\); \(t_{macro} = 1/\dot{\gamma}\) is the inverse of the shear rate representing the mean deformation to move a particle from one hole to the next. When \(I \ll 1\), the materials are in a slow, dense flow regime. When \(I \gg 1\), the materials is agitated, dilute flow regime, like a granular gas. Cruz et al. [7] suggests two possible interpretations of the relation between \(\mu\) and \(I\): First, most of contacts become
dissipative sliding collisions when the system transitions to the collisional regime of big \( I \); second, the contact network becomes more anisotropic when the system is in a collisional regime. Both of these effects can result in increased effective friction.

The inertial number theory has limitations when describing the transition to quasi-static flows where the shear rate vanishes, and fails to predict shear bands in some configurations. Recent studies by Bazant [16] which proposed a “non-local” flow rheology has made some improvements to these flaws.

3.3.2 The spot model

Another mesoscopic model that uses some tempo-spatial average but also keeps the random/discrete nature of granular flow is the “spot mode” proposed by K. Karmin and M. Z. Bazant et al. [15]. The basic idea is that particles flow is caused by opposing the corresponding ”diffusing” spot. This idea is inspired by the facts that a variety of mean flow profile of dense granular flow resemble solutions of diffusion equation, and particles movement in granular medium are correlated with their neighboring...
particles. The later has been observed in a lot of experimental and numerical studies, and inspires technique terms such as ”granular cluster” or ”cage breaking”. (In fact, heterogeneity are very important to phase transition of disordered systems). The size of the spot, is then approximately $3d \sim 5d$ based on experimental observations. The idea is illustrated in Fig. 3.10.

**Figure 3.10:** (a) Illustration of the opposite motion of a spot and a cluster of granular particles. (b) “Spot” model applications to Silo drainage. Spots are injected at the orifice and perform random walk upwards, causing downward motion of clusters of particles. Figure adapted from [15].

To examine the model, the researchers performed numerical simulations based the spot model to the silo drainage case and show good agreements between the simulations and the experiment results. They then proceed further to analytically solve the flow profiles, as explained below: First, they consider the ”diffusion” of spots as random walker, and they then have the follow stochastic equation, the Fokkere-Planck drift-diffusion equation, to describe the probability density (or concentration)
of spots, $\rho_s(r,t)$:

$$\frac{\partial \rho_s}{\partial t} + \frac{(D_1^\alpha \rho_s)}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} (D_2^{\alpha\beta} \rho_s)$$  \hspace{1cm} (3.19)

The mean drift velocity of particles $u = u^\alpha$ is then calculated as:

$$u^\alpha = -\int dr' w(r, r') [D_1^\alpha (r', t) \rho_s(r', t) - \frac{\partial}{\partial x_\beta} [D_2^{\alpha\beta} (r', t) \rho_s(r', t)]]$$  \hspace{1cm} (3.20)

Here, $w(r, r')$ represents the spot “influence function” specifying how much a particle at $r$ moves in response to a drift spot at $r'$ - the nonlocal feature. The first term on the right hand of equation is a particle drift velocity. The second term, which depends on spot diffusion tensor, is the so called “noise-induced drift”.

To simplify the above equations, they make strong assumptions about $D_1$ and $D_2$, based on their experimental observations:

$$\|D_1\| = L_s/\Delta t, \quad D_2 = L_s^2/2\Delta t$$  \hspace{1cm} (3.21)

Note that they use a same time scale $\delta t$ for $D_1$ and $D_2$ based on the picture that the characteristic length scale of the random walk spot is the spot size $L_s$. By plug in the above assumption, the equation (3.19) and the equation (3.20) are then simplified as:

$$\nabla \cdot (d_s \rho_s) = \frac{L_s}{2} \nabla^2 \rho_s$$  \hspace{1cm} (3.22)

$$u = \frac{L_s}{\Delta t} \int dr' w(r, r') (d_s (r') \rho_s(r') - \frac{L_s}{2} \nabla \rho_s(r'))$$  \hspace{1cm} (3.23)

Here, $d_s$ is the unit spot drift direction, determined by the below flow rule.

In order to connect the stress field with the spot model, they also suggest a flow rule that connect the spot drift direction to the incipient yield line determined by Mohr-Columb continuum mechanics. The spot drift direction is obtained by projecting the net force on the material cell onto the slip lines and then averaging them. The mathematical expression can be found at [15].

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With this flow rule equipped, they can then fully solve the flow profile based on different boundary conditions. The resulting profiles, for silo case, is a Gaussian function:

\[ v(x, z) \sim \frac{e^{-x^2/2\sigma_v^2(z)}}{\sqrt{2\pi\sigma_v^2(z)}} \]  

(3.24)

3.4 The steady flow profile

3.4.1 Compare time-averaged flow profile with the radial solutions

We compare our experimental data to data from previous experimental and theoretical studies in this sample line. We obtain images at 500 fps from our experimental set up discussed in Section 2.1. We then process the images to locate the centers and sizes of the particles during the flow via the Hough Transform methods in Section 2.3.1. To obtain the steady flow profile, we first use the method mentioned in Section 2.4 to coarse grain the particle tracking data for every image, and then time average them.

A typical example of the steady flow profile is shown in Fig. 3.11. The flow profile exhibits typical plug zones, stagnant zones and funnel zones as observed by previous studies. Recall that a representative result suggested by continuum approaches is the so-called “radial solution” proposed by Jenike, introduced at Section 3.2.1. To compare our flow profile with the radial solution, we transferred the x-y coordinate system into polar coodinates (See Fig. 3.3 for the definition of the polar coordinates).

In Fig. 3.12(a), we plot a loglog plot of velocity y compone along two \( \theta \) line to check the power law dependence. As we can see, the velocity r component at the center line \( (\theta = 0^\circ) \) does match with the radial solution: \( V_r \sim r^{-1} \), while the velcotiy near hopper wall \( (\theta = 28^\circ) \) decreases faster than \( \frac{1}{r} \): \( V_r \sim r^{-1.27} \).

The above observation is also evidenced in Fig. 3.12(b). There, I multiply the whole velocity field r component by their corresponding value of r. If \( V_r \) is indeed the form \( V_r = -f_0(\theta)/r \), the transformed velocity field should all collapse into a
single curve \( f_0(\theta) \). However, the deviation from a single curve in Fig. 3.12(b) shows that the “radial solution” only fits well for central region not near the opening. For regions near the opening or near the hopper wall, the velocity profiles cannot be simply fitted with “radial solution”.

![Figure 3.11: Time-averaged coarse-grained velocity spatial field y component for D=8d (First coarse-grain every image, then time-average over the entire image sequences). White dash lines are where we sample velocity points to generate Fig. 3.12(a). For the rest of this chapter, d is the average particle diameter.](image)

Moreover, the continuum approaches suggest the discharge rate (number of particles flowing out hopper per second) \( \dot{M} \) depend on the height of materials remaining in the hopper, although staying more or less a constant when the height of materials are much bigger than the hopper opening size. This is inconsistent with what we have observed in our experiment. As shown in Fig. 3.13, the velocity fluctuates around a constant value from the beginning to the end of flow, except a little transition at the very beginning or end of the flow.
Figure 3.12: (a): loglog plot of $V_r$ along two $\theta$ line: $V_r(\theta = 0) \sim r^{-1}$, $V_r(\theta = 28^\circ) \sim r^{-1.27}$. (b): $rV_r$ vs $\theta$. at different $r$ (unit: d). Note the deviations of data of $r=8d$ and data of $V_r$ near the wall from the central collapse curves.
3.4.2 Compare time-averaged flow profile with “Free-Fall” arch model

The above long-duration stability of hopper flow in the time-averaged sense suggests some other approaches are needed instead of the continuum approach. The free-fall arch theory assumes the discharge rate only depends on the particle dynamics in regions near the hopper opening, see Section 3.2.3. So this model does not conflict with the long-duration stability, and also gives the well-established Beverloo scaling of $\dot{M} \sim D^{5/2}$. We observe the same scaling $\dot{M} \sim D^{3/2}$, in our 2D experiment by two different approaches, as shown in Fig. 3.14.

However, the free-fall arch model (FFM) is just a simple scaling theory that results in abrupt variations of both the velocity field and the density field near the free-fall arch. For instance, the model says that the particles gain little momentum above the free-fall arch, while accelerating almost freely under gravitation below the arch. We see from Fig. 3.12 (b) that the velocity in the centerline exhibits smooth
transition, not exactly matching what FFM would suggest. Even more interesting is the density profile results, shown in Fig. 3.15. The density does fall down near the opening during the flow, but the overall profile seems not to scale with the hopper opening size, in contrast to what FFM would suggest: the height of the free-fall arch (which separates a densely packed region and a fluidized region) should scale with the hopper opening size.

The above experimental observations suggest new theoretical models are needed to fully predict the steady flow profile of hopper flow.
Figure 3.15: (a): Time-averaged coarse-grained density spatial field for \( D = 8d \), \( \theta_w = 60^\circ \). (b): Density as a function of height sampled at the center line of the hopper, for different hopper opening size \( D \) with hopper wall angle \( \theta_w = 60^\circ \). (Symbol is bigger than errorbar). We see although there seems to be a transition point that densify falls down from a high value, the heights of the transition points do not scale with \( D \) obviously.
3.5 The time-averaged stress profile

Similar to what we have done in Section 3.4.1, we can also compare our experimental stress profile with the radial solutions: \( \sigma = b_0(\theta)r \). Fig. 3.16 (a) shows the time-averaged stress profile for \( D=8d \). The \( G^2 \) is used as a measure of the mean pressure \( \sigma \), as introduced in Section 2.2. The profile suggests low stress near the opening, and strong stresses near the wall, which qualitatively matches with the radial solution. However, the profile is far less smooth compared to what the radial solution would suggest. From Fig. 3.16(a), We can clearly see the trajectories of the linear structure of long-lasting force chains moving down with the flow.

Fig. 3.16(b) shows log\((G^2)\) vs log\((r)\) along the center line. The linear fit suggests \( G^2(\theta = 0) \sim r^3 \) for our case. Therefore the stress profile seems not to match with what the radial solution would suggest.
Figure 3.16: (a): The time-averaged stress profile. (b): $\log(G^2)$ vs $\log(r)$. 

$log(G) = 3 \times log(r) - 8.2$
Project 2: Jamming of 2D Hopper Flow of Disks

4.1 Introduction

In Chapter 3, I have studied the time-averaged flow/stress profile of hopper flow. However, one of the distinguish features of granular flow is its fluctuations due to the discrete and random nature of granular materials. The fluctuations can be so noticeable that hopper jamming frequently happens when the opening size is small and becomes industrially relevant. Study of hopper jamming is also scientifically interesting because it is often considered as a typical example of granular materials in jammed state, as discussed in Section 1.3. This chapter will be dedicated to studying the physical mechanism of hopper jamming.

In Section 4.2, I review some previous studies of hopper jamming. Researches start from macroscopic statistics [39, 38, 32, 33], and progress to particle-scale study [19, 12, 9, 10] recently. The experimental studies only probe the role of boundary forces, and considers the jamming transition as a critical transition when decreasing the hopper opening size. However, this is not true since the jamming of hopper flow as seen from the present work, is a stochastic phenomenon occurring for
a range of opening sizes in our study.

In Section 4.3, we start by proposing the stochastic Poisson process of jamming of hopper flow, based on well-established Janssen picture. We use our experimental data to confirm this assumption.

In Section 4.4, we study the connections between hopper jamming and the jamming that physicists have studied from the perspective of statistical mechanics. By calculating the density, the mean coordination number, as well as temporal density/load fluctuations, we conclude the differences between hopper jamming and “Isotropic jamming”, and propose a new phase diagram for hopper jamming.

In Section 4.5, we continue the discussions by experimentally proving that a mechanically stable arch near the opening is able to dissipate a large portion of the kinetic energy of hopper flow and, therefore, is crucial for hopper jamming.

4.2 Background: study of hopper jamming statistics

4.2.1 The self-avoiding random walker model

In the work by Kiwing To et al. [32, 33], the researchers have extensively studied the statistics of jamming probability of hopper flow. They first measured the dependence of Jamming probability on hopper opening size and hopper wall angle in their 2D hopper experiments of stainless steel disks. They then proposed a theoretical model to fit the jamming probability. The core idea of this model is to consider the probability of jamming of the hopper as the probability to form an arch near the opening. They then calculate the probability forming the arch based a self-avoiding random walker model: They consider the vectors \( (r_1, \ldots, r_{n-1}) \) that from the arch near the opening as a trajectory of a random walker going from left to right with the follow constraints (the parameters are labeled in Fig. 4.1):
\( \pi/2 > \theta_i > -\pi/2 \)  \hspace{1cm} (4.1)
\(
\theta_1 > ... > \theta_i > ... > \theta_{n-1}
\)  \hspace{1cm} (4.2)
\(x - 1 > D/d, \text{where } x = X/d\)  \hspace{1cm} (4.3)

The spirit of random walker model is also reflected by uniformly distributed angle \(\theta\).

They then integrate from leftmost particle to rightmost particle to obtain the probability distribution function for an arch of \(n\) disks to have a horizontal displacement \(x\) and:

\[
a_n(x) = A_n \int_{-\pi/2}^{\pi/2} f_1(\theta_1)d\theta_1 \cdots \int_{\beta_n-1}^{\theta_n-2} f_{n-1}(\theta_{n-1})d\theta_{n-1} \delta(x - \sum_{i=1}^{n-1} \cos \theta_i) \hspace{1cm} (4.4)
\]

The probability that an arch has \(n\) disks is then defined as: \(j_n(d) = \int_{d-1}^{\infty} a_n(x)dx\)

The above two integrations of \(a_n(x)\) and \(j_n(d)\) can be done numerically. Finally, they obtain the theoretical jamming probability:

\[
J(d) = \sum_{n=2}^{\infty} g_d(n)j_n(d) \hspace{1cm} (4.5)
\]

where \(g_d(n)\) is the fraction of arches with \(n\) disks obtained from their experiments.

In the later work done by the same research group above[33], they use Markov process to describe hopper jamming process and conclude that \(J(d) = 1 - e^{-(m-n_0)Ae^{-Bd^2}}\).

Another work regarding the macroscopic statistics of hopper jamming to is the work done by Spanish scientist Iker Zuriguel et al. [39, 38]. One key conclusion of their work is the existence of a critical radius of the opening, beyond which no jamming will occur. However, we argue that this result may be due to their experiment is 3D. The 3D nature makes to the system more difficult to jam.
4.3 The stochastic process of jamming of hopper flow — the Poisson process

With some understanding of the time-averaged flow profile, we focus on the interruption of hopper flow in this section, i.e., jamming of hopper flow. Since jamming is a random event even though the geometry of the hopper is fixed, we want to know what kind of stochastic process is involved, and therefore to predict when the jamming will happen during hopper flow.

We propose that the jamming of hopper should follow a Poisson stochastic process, based on the same Janssen picture that is used for developing the Beverloo equation. (See Section 3.2.3). The Beverloo equation is a robust scaling law that has been tested and verified by various previous studies. We assume that over a short time, dt, the probability of a jam occurring is proportional to dt: \( P_j = dt/\tau_c \), where the parameter \( \tau_c \) is a characteristic time depending on material properties and hopper geometry, but not on the height of the material that has not yet exited.
through the hopper. This assumption on $\tau_c$ is inspired by the Beverloo picture.

To develop the consequences of this approach further, we will now consider the probability that a flow has survived for a time $t$ without a jam, $P_s(t)$. If we know $P_s(t)$, then $P_s(t + dt) = P_s(t)(1 - dt/\tau_c)$, which then yields

$$P_s(t) = \exp(-t/\tau_c). \tag{4.6}$$

Note that $P_s(0) = 1$. The probability that a jam has occurred before $t$ is $P_j(t) = 1 - P_s(t)$.

We then perform the first type of experiment described in Section 2.1 to test the hypothesis. We collect around 100 samples of survival times for each hopper geometry. We then round the survival times into their proximal bins in order to generate its histogram. The frequency at which the hopper flow jams at time $t$ can now be obtained by counting the accumulated number of events in each bin. The temporal resolution of the survival time is 0.02 seconds and the width of each bin is 0.5 seconds. The distribution of $P_s(t)$, is then defined as:

$$P_s(t_0) = \frac{n(t \geq t_0)}{n(0 \leq t \leq \infty)}. \tag{4.7}$$

As one example, Fig. 4.2 shows the resulting distribution of $P_s(t)$, the probability that a flow can survive longer than time $t$, for opening size $D=3.3\text{cm}=6d$. $P_s(t)$ is described well by a decaying exponential curve and hence by a characteristic time, $\tau_c$. For all other opening sizes, we also obtained this exponential trend. This observation gives considerable confidence that our probabilistic approach is accurate. There is nothing is this approach that restricts it to 2D, and we expect that similar results might for fully 3D flows.

In Fig. 4.3, we plot $\tau_c$ as a function of $D/d$ for nine different opening sizes, ranging from 5d to 7.5d with 0.2d intervals. As shown in Fig. 4.3, $\tau_c$ grows much faster than a linear in $D/d$, and can be fitted exponentially based on our current
range of opening size. The impact of hopper wall angle \( \theta_w \) on jamming probability is much weaker compared to opening size, as shown in Fig. 4.3, where lines of different color correspond to different hopper wall angle \( \theta_w \), do not show any clear differences. The differences between the effects of hopper opening size and hopper wall angle can be understood by assuming that the jamming of hopper flow is mainly due to the formation of a stable arch near the opening at jamming, which will be further discussed in Section 4.5.

![Figure 4.2: Probability distribution of survival time of D=2.9cm, \( \theta = 45^\circ \), fitted by an exponential function.](image)

4.4 Phase diagram of the hopper flow/jamming transition

In the last section, we focused on the stochastic process of jamming for hopper flow. The next step is to study the jamming phenomenon more from a physics perspective. As mentioned in Section 1.3, because of the common features of a transition from fluid-like to solid-like behaviors for a range of soft matter systems, including granular materials, foams, colloids etc., unjamming-jamming has been the subject of intense
scrutiny in the past several years. The goal of this section is to see if we can map the hopper flow/jamming states onto the recent “shear jamming” phase diagram suggested by Bi et al. [2], which is a further exploration of the phase space of the well known “isotropic jamming” phase diagram proposed by Liu and Nagel [18]. The idea of using a phase diagram is similar to defining the thermodynamic state of an ordinary material in terms of system parameters such as pressure ($P$), temperature ($T$), and volume ($V$) coordinate system.

Studies of the isotropic jamming and shear jamming transitions of granular systems have shown that: for frictionless disks in a 2D system, the critical packing fraction $\phi_c$ is around 0.84, and the critical mean contact number $Z_c$ is 4 (cite Liu, Ohern work) ; for frictional disks, $\phi_c$ ranges from 0.75 to 0.8 and $Z_c$ ranges from 3 to 4 depending on the specific frictional coefficients and the preparation protocols (cite Behringer,Martin work) . To compare the corresponding parameters of hopper jamming to the above results, we collect images when the hopper flow is jammed
using the high-definition DSCR camera, and we study the statistics of \( \phi \) and \( z \) for this system (see experimental type C in Section 2.1). A sample image from such a hopper jamming collections is shown in Fig. 4.4(a). We then calculate packing fraction and average contact number of materials inside the observation window (the red trapezoid shape window in Fig. 4.4(a)) of each jammed case. To calculate the packing fraction here, we count the total particle area inside the window and then divide by the window area to obtain the packing fraction: 

\[
\phi = \frac{\text{Particle Area}}{\text{Window Area}}.
\]

To calculate the mean contact number here, we use the geometrical euclidean criterion to detect possible force contacts, that is, we consider two particles in contact if the distance between these the centers of these two particles are close to the sum of their radii. We then consider an average over all particles inside the observation window as \( Z \) in the hopper: 

\[
Z = \frac{\text{Number of Possible Force Contacts}}{\text{Number of Particles}}.
\]

For an example, see Fig. 4.4(b). We do not attempt to detect true force-transmitting contacts using a photoelastic criterion (See Majmudar’s work [21], since photoelastic response can be relatively low for small forces in our hopper system, as discussed in Section 2.2. The analysis of \( Z \) here is meant to provide an upper limit for the number of force contacts. By contrast, we measure \( \phi \) much more precisely with an absolute error of about 0.01.

Figure. 4.5 (a) shows the results of the above ensemble fluctuations of \( \phi \) and \( Z \) from the collections of jamming configurations. We see that the packing fraction for jammed hopper fluctuates around 0.75, while the contact number \( Z \) fluctuates around 4. The value of the packing fraction suggests that hopper jamming falls into the regime of shear jamming, where the anisotropic force network created by shearing can jam the system at a low packing fraction. Figure. 4.5 (b) shows a typical photoelastic image of the jammed hopper. We see that the force networks are indeed anisotropic, with strong chains spanning the horizontal directions of the hopper while weak chains expand vertically to support the load.

However, the story of the hopper flow-jamming transition may not be totally the
Figure 4.4: (a): A sample image of hopper jammed situation taken by DSCR camera. Red trapezoid showing Region of Interest for density/contact number calculation. (b): Contacts are marked by the red cross. Averaging the number of these contacts by the number of particles give us an upper limit of the contact number $Z$.

same as an ordinary fragile-shear jamming-yield transition studied by Bi et al. [2].

We examine the temporal fluctuations of the relevant quantities by collecting images
Figure 4.5: (a) $\phi$ (red dot) fluctuates around 0.75 for different jamming hopper configurations. $Z$ (blue cross) fluctuates around 4 for different jamming hopper configurations. $Z$ also shows a positive correlation with $\phi$. Red circles are data from the initial loading. (b) A typical photoelastic image of the jammed hopper showing the anisotropic force network.
from experiments of type 3 in Section 2.1, using the same window-averaging method as the above analysis, in order to measure $\phi$ and $Z$ during flow. The window-averaged pressure, as a measure of the overall pressure inside the window, is calculated through the “$G^2$” technique, as discussed in Section 2.2. Note here that we cannot directly measure the shear stress in the system, since the high-speed image resolution is not high enough to extract the contact force. However, we argue the pressure measured here is a reasonable estimate of the shear stress in the system, based on the results of Zhang et al. [37], where $\tau_c \sim \mu P$. Fig. 4.6(a) shows a typical example of temporal fluctuations for the window-averaged packing fraction as well as the corresponding pressure. Note in the final jamming state where $\phi$ (or $P$) is more or less a constant (small fluctuations are brought by particle-tracking noise or alternate current frequency) with high magnitude. However, during the flow state, there also exists large $\phi$ (or $P$) that is sometimes even larger than the final jamming $\phi$ (or $P$). Two snapshots of a high $\phi$ (or $P$) and a low $\phi$ (or $P$) are shown in Fig. 4.6(b) and (c) respectively. Fig. 4.6(c) shows a hopper state that is unjammed and can flow easily, while Fig. 4.6(b) shows a hopper state that may be considered as “initially” shear jammed, but that become unstable under the particle momentum impulse from above.

Therefore, a pair of “window-averaged” packing fraction and shear stress cannot determine whether the hopper system is jammed or not. Since for each jammed state we see a stable arch-shaped force chain at the opening, we argue that the mechanical stability of the configurations of the particles inside the hopper also plays a crucial role in deciding if the hopper is jammed or not. Based on the above results, a new phase diagram of the hopper system is heuristically suggested in Fig. 4.7. The hopper phase diagram states that hopper jamming needs both an “initially” shear jammed state and a mechanically stable arch of particles near the hopper opening that can withstand the particles’ momentum impulse at that moment.

The physical origin of this additional feature of the hopper phase diagram may
be due to the size effect of the jammed hopper system. For a jammed hopper system, the opening size is usually only several particle-diameter wide, while the conventional granular systems people studies are usually large enough so that the size effects can be ignored.
Figure 4.7: A hopper flow-jamming phase diagram. Insets are representative sketches of various regimes. Packing fraction cannot be too low for a jammed state since particles need to touch each other at least, as can be seen from the difference between sketch (b) and sketch (c). Meanwhile, a “shear” jammed state may have different mechanical stability, depending on the mechanical strength of that jammed state as well as the inertial load brought by particles’ momentum impulse, as can be seen from the difference between sketch (a) and sketch (b).

4.5 The role of a mechanical stable arch near the opening

While a mean packing fraction or pressure does not show any critical behavior relevant to jamming transition, we may still ask if the system exhibits any precursor to jamming. In other words, are there any variation of the system dynamics comparable to the variation of the jamming probability across the opening sizes from 5d to 8d, where jamming probability has changed dramatically?

To address this question, we compared the window-averaged velocity fluctuations, \( \Delta V \), with different hopper opening size \( D \) (6d-14d) for hopper angle fixed at 60
degrees. $\Delta V$ is obtained by calculating standard deviation of time sequence of the window-averaged coarse-grained velocity. Mathematically, we have (for simplicity, we only consider $y$ component below):

$$V_{i,j} = y \text{ component of coarse-grain Velocity filed at pixel (i,j)}$$

$$V_{\text{win}} = \text{Spatially averaging } V_{i,j} \text{ over the observation window}$$

$$\delta V = V_{\text{win}} - <V_{\text{win}} >$$

$$\Delta V = \text{Standard deviation of the time sequences of } V_{\text{win}} = \sqrt{<\delta V^2> }$$

We focus on velocity fluctuations because jamming of hopper flow essentially refers to velocity of flow dropping to zero. Note we need to smooth the velocity data in order to get rid of tracking noise, as stated in Section 2.3.1. As shown in Fig. 4.8(a), $\Delta V$ increases with decreasing opening size. The source of increasing velocity fluctuations can be better understood by plotting the distributions of velocity fluctuations for $D=6d$ and $D=14d$, sampled from a same-duration segment of “quasi-steady” flow period respectively. As shown in the Fig. 4.8(b): the probability distribution of $\delta V$, the increasing velocity fluctuation $\Delta V$ comes from the appearance of larger $\delta V$.

Equipped with synchronized data of velocity and stress, the origin of the above negative tail can be further understood from dynamics. The pressure fluctuations we measured has a negative correlation with velocity fluctuations, as shown in Fig. 4.9.

Considering density fluctuations is only around 3 percents of mean density, the negative tail of velocity distribution should be directly related to significant momentum lost to the hopper side walls. Combined with results of stress geometry and correlations between stress and velocity fluctuations from Fig. 4.9, we propose that these significant momentum lost (large $\delta V$) should be caused by appearance of intermittent force chain arches across the hopper. To prove this point, we make use of our stress data by considering stress signal spike simultaneously rising in the mid-
Figure 4.8: (a) $\Delta V$ against $D$. (b) Comparison of pdf (probability density function) of velocity fluctuations of $D=6d$ and $D=14d$. Note $\delta V$ for $D=6d$ has a broader distribution than $\delta V$ for $D=14d$, which creates the trend of (a).

dele/left/right windows (labeled in Fig. 4.10(a)) as appearance of intermittent force chain arches near opening. In Fig. 4.10(b), we plot the velocity data (the same data sample as Fig. 4.8(b)) versus frame number. On the same figure synchronized stress signals in middle/left/right windows of hopper are also plotted. The threshold to differentiate stress spikes with background noise is set to be the turning point (2 for StressMiddle, 3 for StressLeft and StressRight) from the distribution of stress signals.
Figure 4.9: (a) Synchronized time sequence data of velocity fluctuations and stress fluctuations. (b) Window-averaged stress fluctuations vs velocity fluctuations showing a negative correlation between these two quantities.
(See Fig. 4.10(a)). We also labeled the detected force chain arches in Fig. 4.10(b). We see that at frame number 875, the system exhibit significant momentum loss contributing to the negative tail in Fig. 4.8(b), while at around same frame, we see simultaneous increase of stress signals in all three windows, representing appearance of arch-shape force chains across hopper near opening. Also note in Fig. 4.10(b) is that not every arch can cause significant velocity drops. However, based on the above observations, we could conclude that by decreasing hopper opening size, the geometry effect has been localized to hopper opening, increasing the chances to form arch-shape force chains to halt the flow, and eventually form a stable chains in final jamming state. Indeed, shown in Fig. 4.11, we see that the frequency of appearance of arch-shape force chains increases with decreasing opening size.

The Last but not the least of this section is that even velocity fluctuations decreases about 50 percent from $D=5d$ to $D=8d$, it is very little compared to variation of jamming probability, decreasing more than several hundred percent. One explanation of this is that the jamming transition of hopper flow is a chain effect, i.e., increased velocity fluctuations reduces the time that particles can free fall, then flow velocity will decrease, then the inertial effect will be small, so the arch formed near opening in small velocity field is easier to withstand the momentum from above and is more probable to stay there forever to jam the flow. Also, in the jammed state, there are collaborative force chains higher in the hopper, which accompanies the arch near the opening, to support the weight of the materials. Decrease the opening size shrinks the width across the hopper section and increases the chance to form these collaborative force chains.
Figure 4.10: (a) Probability distribution of stress during quasi-steady flow period, used to decide the threshold to differentiate stress peaks with background noise. Inset shows how we choose middle/left/right observation window to help detect arch-shape stress chains across hopper near opening. (b) Synchronized velocity fluctuations data and stress data versus frame number at 500fps. At frame number 875, the significant velocity drop that contribute to the negative tail of Fig. 4.8(b), is caused by a arch-shape stress chain. Dashed line is the threshold to differentiate stress peaks.
Figure 4.11: $f$-arch (frequency of appearance of arch-shape stress chains) as a function of normalized opening size $D/d$. 
5.1 Introduction

In previous sections, we focused on hopper flow of disks because of the simple geometrical nature of disks. However, in reality, granular particle shapes are often nonspherical. For instance, rice or m&m chocolates may be thought of as ellipsoids, while sands have various irregular shapes. Therefore it is both scientifically interesting and industrially important to explore the impact of the shape of particles on regulating flow rheology as well as jamming mechanism.

One of the simplest directions to explore the impact of shape is to change disks to ellipses, that is, by changing the aspect ratio. In this chapter, we will discuss the results of hopper flow of ellipses from our 2D hopper experiment. We will first compare the mean flow and stress profiles of ellipses to disks, and then compare the magnitude of discharge rate and jamming probability of ellipses to disks. We will then use the synchronized particle-tracking and stress data to analyze the impact of additional orientational freedom of ellipses on regulating ellipse flow rheology.
5.2 Compare mean flow and stress profiles of ellipses to disks

To compare the mean flow and stress profiles of hopper flow of ellipses to disks, I start with the discharge rate at the opening. I measure the discharge rate by simply dividing the total mass by the time taken to empty the total materials. As shown in Fig. 5.1, if we consider the lengthscale of the ellipses to be \( d_{\text{minor}} \) (5mm), which is approximately equal to the lengthscale of the disks (d=5.3mm), we clearly see that ellipse discharge slower than disks. If we rescale the length scale of the ellipses to be \( d_{\text{major}} \) (10mm), then the data from ellipses and disks collapse, as shown in Fig. 5.1 (b). Therefore, we can propose the concept of an effective opening size \( D_{\text{eff}} \), at which the disks flow or jam that was the same form for ellipses. For the current aspect ratio of ellipses, \( A_p = 2 \), we have

\[
D_{\text{eff}} = D / A_p = D / 2.
\]

![Figure 5.1: Discharge rate vs normalized opening sizes of disks and ellipses. (Hopper wall angle \( \theta_w = 60^\circ \)). For Disks, \( l_s = d \); for Ellipses1, \( l_s = d_{\text{major}} \); for Ellipse2: \( l_s = d_{\text{minor}} \).](image)

The second comparison concerns the mean flow profile. Similarly to what we have
done for disks, we can use coarse-graining techniques to yield a continuum descriptions from the discrete particle-tracking data. Note that to include the anisotropic nature of ellipses, we modify the form of the coarse grain function—a 2D circular gaussian function to an elliptical gaussian function, illustrated in Fig. 5.2. The mathematical form is written as below:

$$f(x, y) = A exp\left(-\left(\frac{(x^x - x_c)^2}{\sigma_x} + \frac{(y^y - y_c)^2}{\sigma_y}\right)\right)$$ (5.1)

where $x^x = x \cos(\theta) + y \sin(\theta)$ and $y^y = x \cos(\theta) - y \sin(\theta)$ are the transformed coordinates that rotate ellipses to the x-y axes. In Section 3.4, I compared the flow profiles for disks to the Janike radial solutions and concluded that in the center regions, while near the wall. Here, I performed the same analysis on ellipse flow profile, see Fig. 5.3.

**Figure 5.2:** An example of ellipse coarse grain function, titled -75 degree with image x-axis based on the orientation of its corresponding elliptical particle.

5.3 Compare Jamming of ellipses to disks

Section 5.2 has given us a general understanding of the mean flow properties of ellipses. In this section I will then compare jamming properties of ellipse flow to disk
5.3.1 Compare jamming probability of ellipse to disks

As shown in Fig. 5.1, the discharge rate of ellipse flow still scale as a power law of the opening size D with the exponent that is consistent with 3/2, so we would expect that the free-fall arch concept still applies to ellipses, and that the stochastic process of ellipse jamming still can be modeled as the Poisson process. Indeed, using the same approaches introduced in Section 4.3, we obtain the same kind of exponential decay curve, as typified by Fig. 5.4. Similarly, we can extract a characteristic time $\tau_c$ for each fit (Note $\tau_c$ represents the inverse of the jamming probability). In Fig. 5.5, we compare $\tau_c$ for ellipses and $\tau_c$ for disks. As we have seen for the flow rate, if we consider the length scale of the ellipses to be $d_{\text{minor}}$, we see that ellipses jam more easily than disks. If we rescale the lengthscale of the ellipses to be $d_{\text{major}}$, then the data of ellipses and disks nearly collapse.

So what is the appropriate length scale we should use to normalize the opening flow.

Figure 5.3: (a): Time-averaged flow field for ellipses.
Figure 5.4: Probability distribution of survival time of D=5cm, $\theta = 60^\circ$ for ellipses, fitted by an exponential function.

Figure 5.5: Survival time $\tau_c$ vs normalized opening size for both Disks and Ellipses (Hopper wall angle $\theta_w = 60^\circ$). For Disks, $l_s = d$; for Ellipses1, $l_s = d_{\text{major}}$; for Ellipse2: $l_s = d_{\text{minor}}$. 

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size? By looking at the snap shots particle-tracking video of ellipses, we see that ellipses rotate during flow. To characterize the orientation of a ellipse, we define \( \theta \) as the angle between the major axis of a ellipse to the hopper horizontal opening line. We then collect \( \theta \) of all ellipse in each snap shot and collect them into a pool. The statistics of \( \theta \) show that the most probable ellipse orientation, is around 80 to 90 degrees. This suggest that the appropriate length scale \( l_s \) of ellipses, in contrast to the average diameter of disks, falls between \( d_{\text{minor}} \) and \( d_{\text{major}} \). No matter what the exact value of \( l_s \) is, all above results show that after normalizing D with appropriate \( l_s \) ellipses flow slower or jam more easily than disks. See Reference [31] for details.

5.4 Study of ellipse flow rheology

5.4.1 The geometry: orientation preference of ellipses in force chains

In Section 5.2 and Section 5.3, we have seen that ellipses flow slower/jam more easily than disks at the same opening size compared to diskes. As we conclude from Section 4.5, the stability of strong force chains is closely related to the hopper flow rheology. Therefore, the results from discharge rate and jamming probability suggest that force chains formed for hopper flow of ellipses are more stable than force chains of disks. To explain the difference of the stability, note that an ellipse has the intrinsic geometrical anisotropy compared to disks, i.e., different lengths of major and minor axes. Therefore, how ellipses arrange their orientation relative to the force chain direction should give us some clues to the questions of force chain stability.

Using the image registration techniques introduced in Section 2.3.2, we can quantify the force chain direction by connecting centers of ellipses with strong “Gsquare” contacts. Using the elliptical hough transform introduced in Section 2.3.1, we can also quantify the orientation of a ellipse by measuring the angle between its major axis and the x axis. After obtaining these two quantities, I characterised the geometrical arrangement of ellipses in the force chains by the “Ellipse-ForceChain
angle pair\textendash -(\theta_1, \theta_2) illustrated in Fig. 5.6 (a). Some typical examples can be seen in Fig. 5.6 (b). Note that since \((\theta_1, \theta_2)\) contact and \((\theta_2, \theta_1)\) contact are essentially the same physics, we will assume \(\theta_1 < \theta_2\) and switch their order if \(\theta_1 > \theta_2\).

![Diagram](image)

Fig 5.6: (a) Definition of the “Ellipse-ForceChain angle pair” \((\theta_1, \theta_2)\). (b) Some examples. Note \((\theta_1, \theta_2)\) and \((\theta_2, \theta_1)\) are considered as the same contact in future analysis.

With the definition of “Ellipse-ForceChain angle pair” , we now have a measure to quantify the geometrical arrangement of ellipses relative to their corresponding force chains. By collecting all such angle pairs from image sequences of hopper flow, I plot their probability density in Fig. 5.7(a), as a 2D histogram. Fig. 5.7(b) is a
plot of the value of the probability density of equal $\theta_1$ and $\theta_2$. The results suggest a slight orientation preference: Ellipses forming the force chains tend to align parallelly with their neighbors, and vertically with respect to the direction of the force chains connecting themselves. We call it “Parallel Preference”.

This “Parallel Preference” is much more obvious if we apply the same analysis only to the force chains that jam the hopper flow. We separate the particles that form the chains jamming the hopper by (to be finished). The corresponding probability density plot is shown in Fig. 5.8(a) and (b). As we can see, the probability of ellipses with the “Parallel Preference” arrangement is about 0.07, while the probability of ellipses with $(\theta_1, \theta_2) = (0, 0)$ is almost zero. To conclude, for hopper flow, the “stressed” elliptical particles, which form the strong force chains, tend to align parallel to their contacting neighbors, and to align vertically to the direction of the force chains they form. The next subsection is devoted to explaining why this specific geometrical arrangement can enhance the mechanical stability of the ellipse force chain and therefore increase the jamming probability.
Figure 5.7: (a): 2D histogram of the pdf of “Elp-ForceChain angle pair” for hopper flow. (b): Quantitative plot along the diagonal line of (a).
Figure 5.8: (a): An example showing we can find ellipses that form the bottom arch. (b): 2D histogram of the pdf of “Elp-ForceChain angle pair” for the jammed arch. (c): Quantitative plot along the diagonal line of (b).
Section 5.4.1 has shown that the specific geometrical arrangement- “Parallel Preference”- can enhance the mechanical stability of the ellipse force chains. The remaining question is what is the physical reason for the increased stability, and whether we can quantify the dependence of the increased stability on the aspect ratio of the ellipse.

An intuitive interpretation is that aligning ellipse chains vertically with the “Parallel Preference” prevent ellipses from rotating. Rotations of particles always change the friction between particles by a small amount, therefore moving the particles’ relative positions. For the disks case, this small mobilization will destabilize the chain, as shown in Fig. 5.9. For the ellipse chain, rotating the ellipse require much bigger deformation of particles due to geometrical constraints of neighboring particles. Although this interpretation is easy to understand, it is difficult to quantify its effect on flow rate and jamming.

From another point of view, we can use knowledge learned from structural mechanics. There, the stability of a beam is a major concern of civil engineers. Consider the force chain (the chain of stressed particles) as a beam that supports the weight of the materials as well as the momentum impulse from particle flow. If the beam is not stable, it will only stay momentarily and then collapse by impulses of particles momentum, while a stable beam can withstand and jam the flow eventually. Using the conclusion about “Parallel Preference” from Section 5.4.1, we can think of a chain of ellipses has larger cross section area than a chain of disks and therefore can be think of as a thicker beam. Indeed, this idea is supported by calculating the mean contact area of all contacts of ellipses and disks in a random image, as shown in Fig. 5.10. For current aspect ratio $A_p=2$, the mean contact area (13 pixels) of ellipses is around two times the mean contact area (7 pixels) of disks.

It is intuitive that a thicker beam is stronger and more stable than a thin beam.
Figure 5.9: (a): A stable chain of disks. (b): Rotation of the central particle of (a) lead to an unstable chain of disks. (c): A stable chain of ellipses. Due to geometrical constraints, the central elliptical particle is very difficult to move downwards, thus destabilizing the chain.

Figure 5.10: Compare probability distribution of number of pixel in contact for neighboring ellipses and neighboring disks. Rectangular Dash box of the inset plot is an example showing how we define pixels in contact.
Civil engineers have quantify the stability of a beam based on its geometrical shape. Considering bending a beam with its ends fixed. Fig. 5.11(a) shows the x-y cross section of a small segment of the beam before bending. After bending, the top surface is shortened by compression and the bottom surface is elongated by tension. This create a relative rotational degree of y-z cross section 1-1 to y-z cross section 2-2. See Fig. 5.11(b). Somewhere between top surface and bottom surface there is a surface whose length does not change. The interaction of this surface with the x-y cross section, indicated by the $O_1, O_2$ line, is called the neutral axis. Denote the curvature of neutral axis as $\rho_n$, then the deformation of a line segment a-b at y to $a' - b'$ (Fig. 5.11(b)) can be described as:

$$a'b' - ab = (\rho + y)d\theta - dx = (\rho + y)d\theta - \rho d\theta = y d\theta$$

(5.2)

Then the normal strain and the corresponding normal stress inside the beam are

$$\epsilon = \frac{yd\theta}{dx} = \frac{yd\theta}{\rho d\theta} = \frac{y}{\rho}$$

(5.3)

$$\sigma = E \epsilon = \frac{Ey}{\rho}$$

(5.4)

To connect the stress $\sigma$ with applied torque $T_q$, note we have:

$$\int_A y \sigma dA = T_q$$

(5.5)

dA is an area segment on y-z cross section. Now plug equation (5.4) into equation (5.5), we have

$$\frac{E}{\rho} \int_A y^2 dA = T_q$$

(5.6)

Note $I_z = \int_A y^2 dA$ is the moment of inertia of the beam around the neutral axis. Finally we have for the bending stress:

$$\sigma = \frac{T_q y}{I_z}$$

(5.7)
Figure 5.11: (a): The x-y cross section of a small segment of the beam before bending. $O_1 O_2$ is the neutral axis. (b): The same segment of (a) after bending.

It follows that the bending stress reaches maximum at the boundary of the beam, that is

$$\sigma_{\text{max}} = \frac{T_q y_0}{I_z} \quad (5.8)$$

Here, $y_0$ is equal to one half of the beam thickness (one of the beam width for our 2D system). For a quasi 2D rectangular beam, we have:

$$I_z \sim y_0^3 \quad (5.9)$$

Also note that $y_0$ is proportional to the cross section of the beam, which is essentially proportional to the contact area of the neighboring particles in the force chain, as indicated in Fig. 5.10. Therefore, we suggest:

$$y_0 = A_R y_d \quad (5.10)$$

$$\sigma_{\text{max}} = \frac{T_q y_0}{y_0^3} \sim \frac{T_q}{A_R^2} \quad (5.11)$$

Note The yield stress is a constant depending on materials properties and room temperature, etc. Therefore, for beams made of the same materials, we propose
the stability of the beam can be considered as inversely proportional to its maximum bending stress.

In order to derive \( D_{\text{eff}}(D, A_R) \), we can assume the jamming probability of hopper flow \( J(D, A_R) \), which depends on opening size \( D \) and aspect ratio \( A_R \) of the particles, is directly proportional to the stability of the beam formed by the force chain, as suggested by Section 4.5. Then we have

\[
J(D, A_R) \sim \frac{1}{\sigma_{\text{max}}} \implies J(D, A_R) \sim \frac{A_R^2}{T_q} \tag{5.12}
\]

The connection between \( J(D, A_R) \) and the flow rate \( V_F(D, A_R) \) at the opening rate exists in \( T_q \): we assume the bending torque \( T_q \) is proportional to the kinetic energy impulse \( V_F^2 \) due to collisions. So we have

\[
J(D, A_R) = k A_R^2 V_F^2 \tag{5.13}
\]

We can argue that the proportional constant \( k \) in equation (5.13) depends solely on \( D \), since the geometrical effect brought by \( A_R \) is all reflected by the effect of \( I_z \). To solve \( k(D) \), we can make use of our knowledge of disks in Chapter 4:

\[
J_{\text{Disk}}(D) = J(D, A_R = 1) = \frac{k(D)}{V_F(D, A_R = 1)^2} = \frac{k(D)}{C(D - kd)} \tag{5.14}
\]

\[
k(D) = J_{\text{Disk}}(D) C(D - kd) \tag{5.15}
\]

Here, \( V_F(D, A_R = 1) = \sqrt{C(D - kd)} \) follows from Beverloo Correlation introduced in Section 3.2.3. Finally, for \( D_{\text{eff}} \), we have:

\[
J_{\text{Disk}}(D_{\text{eff}}) = J(D, A_R) = \frac{k(D)A_R^2}{V_F(D, A_R)^2} = \frac{k(D)A_R^2}{V_F(D_{\text{eff}}, A_R = 1)^2} = \frac{k(D)A_R^2}{C(D_{\text{eff}} - kd)} \tag{5.16}
\]

Plug equation (5.15) into equation (5.16):

\[
J_{\text{Disk}}(D_{\text{eff}}) = J_{\text{Disk}}(D)(D - kd) \frac{A_R^2}{D_{\text{eff}} - kd} \tag{5.17}
\]
If $J_{\text{Disk}}(D) \sim D^{-3}$. Plug this relation into equation (5.17) and assume $D \gg d$, we obtain

$$D_{\text{eff}} \sim \frac{D}{A_R} \tag{5.18}$$

This matches with what we have seen in Section 5.2 and Section 5.3. However, from Section 4.3, we know $J_{\text{Disk}}(D)$ depends exponentially on $D$: $J_{\text{Disk}}(D) \sim e^{-\alpha D}$, so the above model needs further careful examinations.
6.1 Summary of results

In this thesis, I have extensively studied flow and jamming of granular materials in a 2D hopper flow. Through synchronized data of particle tracking and photoelastic data, the results of this thesis provides novel insights of particle-level dynamics of hopper flow and jamming. I divide the study into 3 stories: the time-averaged flow and stress profile of hopper flow of disks, the physical mechanism of jamming of hopper flow, and the impact of shape of particles on hopper flow and jamming. Below I briefly summarize the results of these three stories.

For the first story, I did a comprehensive review of previous experimental and theoretical studies of time-averaged flow profile of hopper flow, such as theories based on continuum mechanics, the Janssen picture and the mesoscopic models. I then compared my data to those theoretical predictions, such as the radial solution and the spot model. The comparison showed that the experimental flow and stress profile are more complex then what the radial solution would predict.

For the second story, I proposed and experimentally proved that the hopper
jamming events can be modeled as a Poission stochastic process with no memory of history based on the “Free-fall” arch model. I then used the detailed results of packing fraction, stress and contact number to demonstrate the connection of hopper jamming to the recent proposed “shear jamming” concept [2] and its difference with previously studied “isotropic jamming”. I suggested a heuristic but novel phase diagram of hopper unjamming-jamming phase transition, based on my analysis of the correlation between “arch-shaped” force chains and the velocity of the flow. The phase diagram suggests that hopper jamming needs both an “initially” shear jammed state and a mechanically stable arch of particles near the hopper opening to withstand the particles’ momentum impulse and to jam the flow (Fig. 4.7).

For the third story, I varied the shape of particles from disks to ellipse with aspect ratio $A_R = 2$. By comparing the flux rate and the jamming probability of ellipses to disks, I observed that elliptical particles experienced more resistance than disks during flow for the same confined hopper geometry (same opening size $D$ and hopper wall angle $\theta_w$). Analyses of the stress network geometry shows that “stressed” elliptical particles, which form the strong force chains, tend to align parallel to their contacting neighbors, and to align vertically to the direction of the force chains they form. I call this orientation preference as the “Parallel Preference”. To quantify how the “Parallel Preference” can affect the hopper flow, I make use of the concept of beam mechanics from structural engineering. If we consider a “force chain” that stressed particles form as a beam, changing the shape of particles is then similar to changing the cross section of the beam. I then built a tentative mathematical model based on beam mechanics to quantitatively predict how the aspect ratio of elliptical particles can change the flux rate and the jamming probability of hopper flow.
6.2 Outlook

Results of this thesis have provided pioneer insights into particle-scale dynamics for hopper flow and jamming. However, there is still a lot of room for future work. Below I will suggest some possible directions for future study on in my opinion.

One of the key differences between hopper flow and ordinary fluid is the effect of fluctuations, which are caused by the discrete and random nature of granular materials. For instance, are there any connections among particles that eventually form the arches? Can we predict in which part of the hopper they come from? Also, a lot of recent studies have observed “cage breaking” and formations of “particle clusters” of granular flow. Can we quantify these phenomenon by making use of our photoelastic stress analysis?

Another direction is to explore the impact of particle shape. In this thesis, I have only changed the shape of particles to ellipses and have only tested one value of aspect ratio, $A_R = 2$. It is certainly interesting to systematically measure the relation between aspect ratio and the flow rate or the jamming probability. It is also interesting to check the effect of other shapes, such as pentagons or concave particle shapes. The current mathematical model proposed at the end of Chapter 5 does not correctly match with the experimental observations. Can we improve the model based on force chain analyses, with more experimental evidence provided?
A sample bash script to run the ffmpeg software in order to fastly make a video of image sequences.

```bash
#!/bin/bash
set -x
COUNTER=1
MYDIR=$( mktemp --directory )
PATTERN=$1
EXTENSION=$2
for FILENAME in /${PWD}/${PATTERN}*.${EXTENSION}
do
  echo "$FILENAME " $( printf %04d ${COUNTER} )
  ln -s $FILENAME /$MYDIR/img-$( printf %04d $COUNTER ).${EXTENSION}
  COUNTER=$(($COUNTER+1))
  # if [[ $COUNTER -ge 100 ]] then
```

99
# break

# fi
done
echo $MYDIR

MOVIENAME=$(basename $PWD)-${PATTERN}30fps.mpg

/usr/bin/ffmpeg -f image2 -b 5000k -s 1024x768 -i /$MYDIR/img
-
\%04d.${EXTENSION} -r 25 -//HopperVideo/${MOVIENAME}

rm -rf $MYDIR
A sample condor submission script to run the Parallel computing software, vsfPT.submit:

```plaintext
executable=/usr/bin/matlabR2010b
arguments= -nodisplay
input=vsfPT_Condor.m
environment= ID=$(Process)
output=CondorOutput/vsfPT-out$(Process).txt
error=CondorOutput/vsfPT-error$(Process).txt
log=CondorOutput/vsfPT-log$(Process).txt

notification=error
universe=vanilla
+Department = Physics
should_transfer_files=YES
when_to_transfer_output=ON_EXIT
```
requirements = OpSys == "LINUX" && Arch =="X86_64" &&
UidDomain == "phy.duke.edu" && TARGET.HAS_MATLAB==True

transfer_input_files=vsfPT.m,otsu.m

queue 2900

The matlab file vsfPTCondor.m. Note that how we use getenv to dynamically
get picture number:

information={'4','15','','2','2'};
cmpixel=43.64;

pnum1=str2double(getenv('ID'))+1;%since condor starts at 0
pnumstr1=num2str(pnum1,'%04d');
pnumstr2=num2str(pnum1+1,'%04d');

display(['Processing', pnumstr1]);

vsfPT(information,pnumstr1,pnumstr2,cmpixel);
Bibliography


Biography

Junyao Tang was born on July 26th, 1984, in Wuhu, Anhui Province, China. He obtained his B.S. degree in Applied Physics from University of Science and Technology of China in 2006. He then came to United States to pursue his PhD degree in Department of Physics at Duke University. Since 2007, he joined Professor Bob Behringer’s lab and started studying granular physics. His PhD study has been focusing on 2D hopper flow and jamming of disks and ellipses. Meanwhile, Junyao has developed a strong interest in image processing. He obtained a concurrent MS degree in Electrical Engineering focused on image/signal processing at Duke University during the period from 2009 to 2011.

Current Publications:


