Development of Analog Nonlinear Materials Using Varactor Loaded Split-ring Resonator Metamaterials
by
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Maiken Mikkelsen

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Electrical and Computer Engineering in the Graduate School of Duke University 2013
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Abstract

As research in electromagnetics has expanded, it has given rise to the examination of metamaterials, which possess nontrivial electromagnetic material properties such as engineered permittivity and permeability. Aside from their application in the microwave industry, metamaterials have been associated with novel phenomena since their invention, including sub-wavelength focusing in negative refractive index slabs, evanescent wave amplification in negative index media, and invisibility cloaking and its demonstration at microwave frequency with controlled material properties in space.

Effective medium theory plays a key role in the development and application of metamaterials, simplifying the electromagnetic analysis of complex engineered metamaterial composites. Any metamaterial composite can be treated as a homogeneous or inhomogeneous medium, while every unit structure in the composite is represented by its permittivity and permeability tensor. Hence, studying an electromagnetic wave’s interaction with complex composites is equivalent to studying the interaction between the wave and an artificial material.

This dissertation first examines the application of a magnetic metamaterial lens in wireless power transfer (WPT) technology, which is proposed to enhance the mutual coupling between two magnetic dipoles in the system. I examine and investigate the boundary effect in the finite sized magnetic metamaterial lens using a numerical simulator. I propose to implement an anisotropic and indefinite lens in a WPT
system to simplify the lens design and relax the lens dimension requirements. The numerical results agree with the analytical model proposed by Smith et al. in 2011, where lenses are assumed to be infinitely large.

By manipulating the microwave properties of a magnetic metamaterial, the nonlinear properties come into the scope of this research. I chose split-ring resonators (SRR) loaded with varactors to develop nonlinear metamaterials. Analogous to linear metamaterials, I developed a nonlinear effective medium model to characterize nonlinear processes in microwave nonlinear metamaterials. I proposed both experimental and numerical methods here for the first time to quantitate nonlinear metamaterials’ effective properties. I experimentally studied three nonlinear processes: power-dependent frequency tuning, second harmonic generation, and three-wave mixing. Analytical results based on the effective medium model agree with the experimental results under the low power excitation assumption and non-depleted pump approximation. To overcome the low power assumption in the effective medium model for nonlinear metamaterials, I introduced general circuit oscillation models for varactor/diode-loaded microwave metamaterial structures, which provides a qualitative prediction of microwave nonlinear metamaterials’ responses at relatively high power levels when the effective medium model no longer fits.

In addition to 1D nonlinear processes, this dissertation also introduces the first 2D microwave nonlinear field mapping apparatus, which is capable of simultaneously capturing both the magnitude and phase of generated harmonic signals from nonlinear metamaterial mediums. I designed a C-band varactor loaded SRR that is matched to the frequency and space limitation of the 2D mapper. The nonlinear field generation and scattering properties from both a single nonlinear element and a nonlinear metamaterial medium composite are experimentally captured in this 2D mapper, and the results qualitatively agree with numerical results based on the effective medium model.
To my parents
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# List of Abbreviations and Symbols

## Abbreviations

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<th>Description</th>
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<tr>
<td>SRR</td>
<td>Split ring resonator</td>
</tr>
<tr>
<td>VLSRR</td>
<td>Varactor loaded split ring resonator</td>
</tr>
<tr>
<td>MM</td>
<td>Metamaterial</td>
</tr>
<tr>
<td>NMM</td>
<td>Nonlinear metamaterial</td>
</tr>
<tr>
<td>EMM</td>
<td>Effective medium model</td>
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Introduction

Materials science is a historical, sophisticated and interdisciplinary field which applies the properties of matters to various areas of science and engineering. Materials science covers the interactions between matter and its surroundings (e.g. mechanical, chemical, thermal and electromagnetic interactions). To better represent those interactions, macroscopic properties of materials describe those interaction processes in analytical models. In the context of this dissertation, the main focus is the macroscopic electromagnetic properties of matters which describes the interaction between matters and electromagnetic(EM) waves. And for both scientific and industrial interests, developing and engineering materials with desired EM properties have generated tremendous interests in the past decades. Recently, the study of metamaterials discovered a novel way to engineer material properties, different from the conventional chemical method which usually constructs new materials at the atomic or molecular level.
1.1 Materials’ electromagnetic properties

Electromagnetic media holds a constitutive relation specific to a material or substance that approximates the response of the material to external stimuli (e.g. the applied electric and magnetic field).

In atoms or molecules, electrons may flow freely, may move only along certain paths, or may be bound and restricted from movement. Their motion will be affected by external interference, such as electric and magnetic fields. Then the 'infected' electrons move symmetrically and generate electric or magnetic dipole momentum. This non-zero electric/magnetic dipole moment does not exist if there is no external interference; the motion of electrons is randomized in natural material, and the net electric dipole momentum or magnetic dipole momentum is zero. By averaging the external field and local field strength, we obtain the local response of the polarization and magnetization, as shown in the constitutive relations of Eq. 1.1.

\[
\vec{D} = \varepsilon_0 < \vec{E}> + \vec{P} \\
\vec{B} = \mu_0(< \vec{H}> + \vec{M})
\] (1.1)

Here, \(< \vec{E} \rangle \) and \(< \vec{H} \rangle \) are the statistical average of the local electric and magnetic field in media. \(D\) and \(B\) are electric and magnetic flux. \(P\) and \(M\) are the polarization and magnetization in materials which are induced by external fields. In linear electromagnetic materials, \(P\) is linearly proportional to electric field \(< \vec{E} \rangle\), and the magnetization \(M\) also linearly depends on magnetic field \(< \vec{H} \rangle\) everywhere in the medium. The first order electric susceptibility \(\chi_e\) and magnetic susceptibility \(\chi_m\), therefore, are defined by \(P = \varepsilon_0 \chi_e < \vec{E} >\) and \(M = \mu_0 \chi_m < \vec{H} >\). Combining susceptibilities with external fields in Eq.1.1, we introduce the concept of permittivity \(\varepsilon_r = 1 + \chi_e^{(1)}\) and permeability \(\mu_r = 1 + \chi_m^{(1)}\) for electromagnetic materials[1]. Hence, the general form of constitutive relations are shown below to describe materials’

---

2
macroscopic EM properties:

\[ D = \varepsilon_0 \varepsilon_r E \]
\[ B = \mu_0 \mu_r H \]  

(1.2)

Notice that the macroscopic homogeneous medium model is built up through a field averaging method from the atomic/molecular level. By assuming polarization and magnetization are symmetric all over the space, the macroscopic properties for materials are simply obtained by taking a volume integration of the microscopic property. Mimicking the atomic/molecular system by periodically positioning unit structures with subwavelength dimensions, metamaterials are one practical example of engineered artificial material on a larger scale, formed by designed inclusions on host medium and realized by the effective material’s EM properties.

1.2 Magnetic materials and their applications

Materials may also be categorized by their EM properties, according to whether they are electric or magnetic materials. Conductor, semiconductor and dielectrics are subcategories of electric materials, determined by the moving pattern of free electrons. Magnetic materials may be diamagnetic, paramagnetic, antiferromagnetic, ferromagnetic or ferrimagnetic, according to the local magnetization’s contribution to the external magnetic field, whether it is subtracted from or added to the flux generated from external field.

All substances are magnetic to some extent. However, some materials have such small magnetic effects that they are generally called ”nonmagnetic” by the engineer and scientist. For instance, diamagnetic materials have \( \chi_m \); paramagnetic materials have but very small value, around \( 10^{-4} \); ferromagnetic materials have \( \chi_m > 0 \) with a larger relative value. On the other hand, the magnetic susceptibility in ferromagnetic and ferrimagnetic material are usually much higher, some ranging from \( 10^3 \) to \( 10^4 \).
One of the oldest and most well-known applications of magnetic materials is their use in compass. In the current electronic era, magnetic materials play a large role in our daily lives, and can be found in products from voice and video recording tapes and devices, to computer hard disk drives. Still other devices – such as electromagnets, transformer, electric motors, inductors and magnetic assemblies– have a magnetic core with high permeability to confine and guide magnetic fields.

Over the recent few years, some researches have also proposed that magnetic materials may be used in wireless power transfer systems to charge electric vehicles. Results show that introducing a magnetic core into resonant coupling wireless charging stations can enhance energy transfer efficiency. However, the performance for such system is limited by at least three known factors: magnetic materials are not machine friendly, which limits the viable shape configuration of a magnetic core; natural magnetic materials are usually heavy; and the hysteresis property in magnetic materials causes energy loss in such high power system. In order to overcome these issues, I propose the application of artificial magnetic metamaterials, which provides similar enhancement, but easy to shape, light-weight and linear materials' properties.

1.3 Magnetic metamaterials

In contrast to natural materials, metamaterials are engineered artificial materials with designed material EM properties on demands, which are frequently composed of planar, conducting metals periodically patterned on substrates at microwave frequency. The elements in a metamaterial composite are usually identical, and described by an equivalent oscillation model based on its effective circuit[2]. By adjusting the circuit properties (e.g. inductance, capacitance or resistance), the sub-wavelength elements can be designed to exhibit a wide range of effective EM properties. The surface current flowing on the metallic surface in some metamaterials' designs gener-
ates either electric dipole or magnetic dipole, depending on the shape of the current flowing path. Similar to the averaging method in natural materials, macroscopic properties for metamaterials are presented by their effective polarization and magnetization properties[3, 2], which result in the effective permittivity and permeability for metamaterials. Thus the electric or magnetic response in metamaterials can be controlled by modifying their unit cell structure or by exciting that structure at different modes and frequencies.

The split-ring resonator (SRR) is one of the most important unit-cell structures. Pendry(1999)[2] predicted that SRR could possess negative effective permeability; this was later confirmed in experiments by Dr. Smith (2000)[4]. Given the negative effective permeability from an SRR structure; and the negative effective permittivity from wire structures, scientists were able to verify artificial composites that exhibit negative refraction phenomena, which have been theoretically predicted by Veselago in 1968[5] and experimentally verified by Dr. Smith in 2001[6].

SRR structures provides an easy and quick unit-cell design for artificial magnetic material development. Their effective permeability follows the well-known Drude-Lorentz formula, which is predicted by its equivalent oscillation circuits. The Drude-Lorentz formula for effective permeability of SRR structures exhibits an frequency dependent artificial medium with controllable permeability from a negative value to a positive value. Moreover, engineers can design artificial material with a continuous value of permeability by slightly tuning SRR unit structures. Further, scientists can design artificial materials with isotropic, anisotropic, homogeneous or inhomogeneous material properties.

The application of magnetic metamaterials covers a wide range. Scientists have applied magnetic metamaterials to conventional microwave devices, such as antennas and microstrip devices, and have investigated their use in microwave applications—such as perfect lenses that break the diffraction limits and transformation optics.
designed for multi-function antenna random or microwave lenses. Magnetic metamaterials could also replace natural magnetic materials in some applications, such as wireless transfer technology. Magnetic metamaterials observe the advantages originated from general metamaterials: light-weight, easy to shape, and their material properties are linear. Magnetic metamaterials overcome these three defects in natural magnetic materials in the WPT application. In chapter 2, I will discuss the application of magnetic metamaterials in WPT system in details. Encouraging results show the promising of applying artificial magnetic materials in such system.

1.4 Nonlinear metamaterial

In the high-power application of magnetic metamaterials, nonlinear effects have negative effects, and can bring about issues similar to the hysteresis effect in natural magnetic materials, which causes a loss in transformer design. However it is still meaningful to study nonlinear metamaterials for a number of reasons. First, in extremely high power cases, even natural materials used to construct metamaterial structures display nonlinearity. Second, some semiconductor integrated metamaterial designs involve nonlinear materials and display nonlinear properties. Third, nonlinear metamaterials extend the control of materials’ properties to a higher level, which can include the control of higher order susceptibilities in linear metamaterials.

Nonlinear metamaterials possess the capability to manipulate the linear response of materials, and can excite stronger nonlinear responses or nonlinear polarization/magnization in nonlinear metamaterials than natural materials. This enhanced nonlinear responses result from the resonant intrinsic properties of metamaterials’ elements, their unit structures accumulate strong localized fields in capacitive or inductive region where nonlinear material or components are embedded[7, 8].

Like conventional linear metamaterials, nonlinear metamaterials exhibit the potential for nonlinear material properties desired by target applications. As long as the
element of the nonlinear metamaterial can be presented by the oscillation model[9] and the effective susceptibilities of the composite can be extracted by a perturbative method. Taking the artificial material composed by VLSRR element for example, a nonlinear medium model is formed by effective permittivity, permeability and the nonlinear magnetic susceptibilities. The frequency dependent and enhanced nonlinear response is represented in the effective susceptibilities. Extraordinary nonlinear susceptibilities, comparable to that of natural nonlinear materials, have been discovered near the resonant frequency of the VLSRR elements.

In addition to theoretical studies, some experimental verification of the nonlinear phenomena in nonlinear metamaterials has been conducted, both in single element[10, 11] and composites with arrays of elements[12, 13]. Several nonlinear phenomena (including hysteresis in the frequency response[8, 11], power dependent frequency tuning[9, 8, 11], and harmonic generation[13, 14])—have been discovered and discussed by many in the field. A portion of this dissertation seeks to analyze, experimental studies of power-dependent frequency tuning and harmonic generation, and to compare the results of those studies with results based on the effective medium model.

The phenomena studied in nonlinear metamaterials are the same as those in natural materials, most of which began in 1961 with Franken et. al’s discovery of second-harmonic generation[15]. By pumping a strong fundamental signal into nonlinear material, signals two or three times the pump signal’s frequency have been discovered in transmitted or reflected signal spectrum. These generated signals are called harmonic signals.

This dissertation also examines more complicated nonlinear processes such as sum/difference frequency generation in three-wave mixing, and four-wave mixing when two or more fundamental signals are imposed on nonlinear materials simultaneously.
The nonlinear process has been used to bring about several tremendous applications, including optical amplification, tunable laser, optical sensing and detection. A later chapter of this dissertation discusses my experiments and analyses with second harmonic generation and three-wave mixing in nonlinear metamaterial.

In addition to harmonic generation, I also examined other interesting phenomena, such as hysteresis, bi-stabilities and power-dependent frequency tuning, which is the product of a self-phase modulation process from the third order nonlinear response resulting in a power-dependent refractive index. I interpret and analyze this phenomenon in a subsequent chapter.

1.5 Overview

The first portion of this work examines the application of linear magnetic metamaterials in high power wireless power transfer systems. Triggered by the high power application, nonlinear properties in metamaterials are subsequently considered. This is followed by an introduction to the design and characterization process of nonlinear magnetic materials. Next is an examination of the methodology to obtain effective nonlinear medium properties from experimental data and numerical data, as well as a discussion of the validation and limitations of the proposed effective medium model for nonlinear metamaterials. To overcome the lower power limitation of the proposed effective medium model, a general model for diode/varactor loaded SRR unit cell is then proposed and discussed.

The next portion of this study provides an in-depth analysis of several nonlinear phenomena (resonant frequency-tuning, second harmonic generation and wave mixing in nonlinear metamaterials) that have been studied experimentally and precisely explained by the effective nonlinear medium model.

The third section of this dissertation presents a 2D experimental setup that can enable scientists to study wave generation and propagation in nonlinear materials at
microwave frequency. This includes a detail of technique setup and system requirements for using a C-band VLSRR nonlinear metamaterial composite. Results of the experiment qualitatively agree with numerical result based on the effective medium model.

The final section offers concluding statements regarding the current work offers suggestions for future study and examination.
Magnetic metamaterials enhance the mutual coupling between magnetic dipole...

Since the earliest suggestions of transferring electric power wirelessly (WPT) by Tesla in 1891[16], the concept of conveying electromagnetic energy from one point to another without wires has captured enormous interest. Numerous efforts are now underway, investigating new configurations capable of transporting energy, and generally seeking to increase the efficiency of all WPT schemes as much as possible. WPT techniques have crossed over from research to product, with several commercial units being offered for charging low-power home and medical electronics, also RFID transponders[17, 18]. Most current WPT schemes employ a near-field based, inductive coupling mechanism to achieve high transfer efficiency over extremely short distances; for acceptable efficiency in these schemes, the distance between the source and receiver must be considerably smaller than the operating wavelength. This near-field requirement makes the integration of WPT protocols into many other electronic systems difficult or unfeasible, since the transfer efficiency decreases exponentially with the increasing transfer distance. (Note that transfer efficiency is defined here...
as the fraction of power delivered to the resistive load relative to the total consumed power.)

Recently, a WPT system with a relatively high transfer efficiency over a moderate propagation range has been developed, based on a pair of coupled, resonant coils with large quality factors (Q-factors) [19]. In subsequent experiments, the efficiency in this resonant WPT system was shown to be as high as 75% over a distance of 0.5m between the source and receiver coils [Intel][20]. Two key factors in the resonant WPT system determine the transfer efficiency: the Q-factors of the resonators and the mutual coupling strength. High Q resonators confine energy locally at the operation frequency. Strong coupling between the two resonators increases the energy exchange rate, thus increasing the power transfer efficiency. Though high Q resonant WPT systems can have excellent power transfer efficiency, they are usually not preferred in practical applications since the coils tend to be more sensitive to the environment, and dynamic control is difficult to implement [21]. Nevertheless, given the demand for wireless powering and charging of devices, the efficiency associated with resonant WPT systems motivates the continued exploration and improvement of such systems.

An obvious means of increasing the efficiency of an inductive, resonant WPT system is to increase the mutual inductive coupling between the source and the receive resonators. Because the distance between the source and receiver is so much smaller than the wavelength, the relevant field distribution is quasi-static and the inductive coupling relates predominantly to the amount of magnetic flux from one coil captured by the second coil. Enhancing coupling efficiency equates to modifying this field distribution, which in turn means focusing or otherwise controlling the near-fields.

The traditional means of controlling fields is through the use of a material. High permeability magnetic materials, for example, can guide magnetic flux and are used
in motors and transformers. Inherently magnetic materials, however, tend to be heavy and cumbersome, and have limited ranges of applicability. Over the past decade, however, the concept of artificial magnetism—using inductive elements to mimic magnetic media—has been introduced and achieved widespread interest[22, 23, 24]. In particular, the use of artificially structured metamaterials with negative effective permeability and permittivity has led to new opportunities for managing near-fields, as exemplified by the "perfect" lens. The perfect lens, introduced by Pendry in 2000[25], is a planar slab whose electric permittivity and magnetic permeability both assume the values of -1 at a given frequency; a source placed on one side of the perfect lens is reproduced at an image on the other side of the slab, with both far- and near-fields refocused to the image.

In the WPT context, the perfect lens geometry seems a likely starting point for improving the efficiency of coupling between two coils. Viewing the first coil as a source, any loss in transfer efficiency to the receiver can be understood as magnetic flux that is too far from the receiver to be captured. A perfect lens, then, that images the near-fields should, in principle, concentrate the flux at the receiver, enhancing coupling. The use of a perfect lens to improve WPT efficiency was considered by Mitsubishi Electric Research Lab[26], who, in recent experiments[27], demonstrated efficiency gains in a resonant WPT system. A rigorous analysis of the coupling enhancement offered by a perfect lens situated between two magnetic dipoles was subsequently performed by Urzhumov et.al[28]. One of the key conclusions of this work was that the efficiency increase through enhanced mutual coupling could predominate over the reduction of efficiency caused by material losses, assuming practical loading of the receiver coil. This conclusion is critical for consideration of metamaterials in WPT schemes, since metamaterials are frequently formed using conducting circuits that can exhibit significant Ohmic losses.

The analysis pursued by Urzhumov et al. employed a simplified geometry-such
as an infinitely large slab and point dipoles for source and receiver-to achieve closed-form, analytical expressions that would provide guidance and basic trends. Here, we make use of full-wave, finite-element based simulations to analyze the mutual coupling between finite-diameter loops situated on either side of a finite-sized planar lens. We make use of the prior analytical results to compare with numerical simulations of loops approximating point dipoles and placed next to an infinitely large slab. After confirming the agreement between simulation and theory for several test cases, we next consider more realistic geometries, investigating the effects of finite loop diameter; finite slab width; and material loss and anisotropy.

2.1 Analytical model

We analyze the mutual inductance between the coils numerically by evaluating the magnetic flux in the receiver coil generated from the source coil in COMSOL Multiphysics. Here we represent the resonant coil by a finite size loop with constant current flowing through, which forms a magnetic dipole. This magnetic dipole approximation is valid since the mode for the high Q resonators, such as spiral and solenoid, is dipole mode like.

In general terms, the efficiency $\eta$ of a WPT system is defined as

$$\eta = \frac{P_2^0}{P_1 + P_2} = \frac{R_2}{R_2^{\text{eff}}} \frac{\chi}{1 + \chi}$$

(2.1)

where

$$\chi = \frac{P_2}{P_1} = \frac{R_2^{\text{eff}} \omega^2 |L_{21}|^2}{R_1^{\text{eff}} |Z_2|^2}$$

(2.2)

Here $P_2^0$ is the power extracted out the WPT system at the receiver end; $P_1$ and $P_2$ represent the power dissipated at the source and receiver coil, respectively the
radiation loss; \( R_2 \) is the resistive load at the receiver coil; \( R_{2}^{\text{eff}} \) and \( R_{1}^{\text{eff}} \) is the effective resistive loss at receiver and source coil, which includes the loss from the self- and mutual inductances correspondingly; \( \omega \) is the operating frequency of the system; \( Z_2 \) is the effective impedance of the receiver coil, including the self-inductance and capacitance. Once having optimized all other factors, consideration of Eqs. 2.1 and 2.2 shows that the transfer efficiency increases as the mutual inductance \( L_{21} \) between the loops is increased.

The mutual coupling between the two loops can be evaluated by numerically calculating the mutual inductance between them, which is the ratio of the induced voltage \( V_{\text{induced}} \) on the receiver coil and the excitation current \( I_s \) on the source coil. The induced voltage is evaluated by the time derivative of the magnetic flux at the receiver coil, which is the area integration of the normal component of magnetic flux density or the closed loop integration of the electrical field on the loop enclosing the surface area \( S_2 \), as shown below:

\[
L_{21} = \frac{V_{\text{induced}}}{I_s} = \frac{-j\omega \int_{S_2} \vec{B} \vec{n} dA}{I_s} = \frac{-j\omega \oint_{S_2} \vec{E} d\vec{l}}{I_s}
\]

(2.3)

where \( \vec{B} \) is the magnetic flux density which can be computed either analytically or in full wave simulations; \( \vec{n} \) is the unit vector normal to the surface of \( S_2 \); \( \vec{E} \) is the electric field; \( \vec{l} \) is the loop encloses the area \( S_2 \); and \( S_2 \) is the surface enclosed by the receiver coil. The two coil system diagram is shown in Fig.2.1 (a).

In analytical study[28], mutual inductance in Eq.2.3 is calculated analytically under two assumptions. First of all, the size of coil is negligible comparing to the discussion wavelength. Thus we could assume the radiation field pattern is the same as dipole radiation. And near field description for small size coil is set to dipole radiation formula, which provide an analytical formula for coil radiation field. Secondly, dimension of metamaterial slab is chosen to be infinitely large, so no finite
boundary effect needs to be taken into consideration. Under this assumption, a one dimensional transmission and reflection coefficient could be calculated to evaluate the field intensity from source coil after penetrating metamaterial slab. The final analytical formula could be found in Ref.[28] Eqn. 32.

In numerical modeling, the presence of a metamaterial lens is simulated by including a slab of material with finite diameter $W$ and thickness $L$ placed between the two loops, as in Fig.2.1 (b). The dominate field components are magnetic, since $D << \lambda_0$, where $\lambda_0$ is the free space wavelength and hence in the quasi-magnetostatic regime. For this reason, it is expected that only the magnetic response of the slab will impact the near-field refocusing, and thus only a magnetic component is assumed in this study. The magnetic slab is also desirable since it is easier to fabricate a metamaterial where fewer elements of the constitutive tensor need to be controlled. We consider
then magnetic slabs with diagonal material properties, where the material properties can be represented as $\mu_{\text{eff}} = [\mu_z, \mu_y, \mu_x]$, and $\epsilon_{\text{eff}}$ is irrelevant. For simplicity, we assume $\epsilon_{\text{eff}} = 1$.

2.2 Numerical validation

To perform the preliminary comparisons between the numerical and analytical models, an axisymmetric two-dimensional geometry is simulated, in which the loop current coil is represented by an out of plane line current, as shown in Fig. 2.1(c). The axisymmetric geometry equates to an infinite slab and takes advantage of the rotational symmetry of the configuration. Please note this rotational symmetry only exists in the z-oriented dipole shown in Fig. 2.1(a), which is the second case studied in Ref. [28]. For the other dipole orientations, such as x-oriented dipole, this rotational symmetry does not exist, thus the numerical model is no longer valid. However, the system shown in Fig. 2.1(a) is the optimal configuration since the coupling is twice stronger than other dipole orientation configuration[28]. The magnetic flux density $B$ can be precisely determined over the surface enclosed by the receiver coil through a full wave simulation. Thus, we can numerically evaluate the mutual inductance between the two coils since all the terms in Eq. 2.3 can be determined in the simulator.

In Ref. [28], the mutual coupling between two magnetic point dipoles is studied analytically in the presence of metamaterial lens. The mutual inductance is calculated and presented in a closed form expression. The resonant coil is approximated as a single loop, finite-sized coil with a constant current and infinitesimal coil cross section area, and then further approximated by a magnetic point dipole with equivalent magnetic dipole moment $M = I_s A = \pi R^2 I$. In the numerical study, the resonant coils are also approximated by finite size single loop coils, however, which are not further approximated by point dipoles. The analytical expression for the mutual inductance in the presence of the slab is:
\[ L_{21} = -\frac{\mu_0 \pi R^4}{2} \frac{4\alpha/\mu_x}{a(2\alpha L)^3} \Phi_L \left( -\frac{b}{a}, 3, \frac{\alpha L + D - L}{2\alpha L} \right) \]  

where \( \mu_0 \) is the free space permeability; \( \mu_x, \mu_y, \mu_z \) is the effective permeability of the MM lens; \( \alpha = \sqrt{\mu_x/\mu_z} \); \( a = -(\alpha/\mu_x + 1)^2 \); \( b = (\alpha/\mu_x - 1)^2 \); and \( \Phi_L \) is the standard Lerch transcendent function.

In the absence of the slab when \([\mu_x, \mu_y, \mu_z] = [1, 1, 1]\), the mutual inductance between the two loops reduces to \( L_{21}^{\text{eff}} = -\frac{\mu_0 \pi R^4}{2D^3} \), when the two coils has the same radius \( R \). In the following, we will use the enhancement factor defined as the ratio between the calculated mutual inductance and theoretically predicted mutual inductance in vacuum as the criteria to evaluate the enhancement of mutual coupling by implementing the slab.

\[ \rho = \frac{L_{21}}{L_{21}^{\text{vac}}} \]  

Before the analysis of the MM lens enhancement effect, we validate the numerical model by investigating the retardation effect of the mutual inductance between two coils in free space numerically. The analyzed system has two coils with radius \( R = 0.01 \text{m} \) and the distance between two coils is \( D = 0.5 \text{m} \). The frequency of the system sweeps from 5MHz to 50MHz, with the wavelength between 6m to 60m. The radius of the coil is less than 0.2% of wavelength, and the transfer distance is less than 10% of the wavelength, thus we call this kind of WPT system as sub-wavelength WPT system. As shown in Fig. 2.1(d), the mutual coupling calculated from the numerical model is the same as predicted by the analytical model. Only less than 2% of deviation presents in the analyzed broad frequency range. Therefore, the numerical model is validated by the excellent agreement of the mutual inductance between the numerical model and analytical analysis, which will be further proved.
in the system with metamaterial lens. Any discrepancy between the numerical and analytical models is attributable to the finite radius of the wires, finite diameter of the coils and also the numerical error from the applied FEM simulator.

In the practical situation, the actual system deviates from the ideal case. Taking the system MIT group studied for example, the high Q resonator–solenoid or spiral–has a finite dimension, varies from the infinitesimal point dipole approximation in the analysis. Also the practical MM lens will have a finite size in the transverse direction, instead of the infinite size without any boundary effect. In this numerical study, we investigate those finite dimension effects along with the application of various type of MM lens. If it’s not specified, the numerical model in this study operates at 10MHz, which is in the frequency range widely captured by research and companies[19, 20, 27]; the radius of the loop size is $R = 0.01\text{m}$; the transfer distance is $D = 0.5\text{m}$; the diameter of the MM lens is $W = 2\text{m}$ and the thickness $L = D/2 = 0.25\text{m}$.

2.3 Finite size isotropic MM slab application

First of all, we investigate the finite coil size effect on the mutual inductance in the system without MM lens. In analytical study, resonators or coils are approximated to an infinitesimal point dipole. In the numerical model shown here, the coils have finite radius $R$. Apparently, the deviation in the mutual inductance comparison between numerical and analytical models increases as the coil size increases, since the dipole approximation is no longer valid when coils are too large. The result is shown in Fig. 2.2(a).

The comparison shows that the deviation remains within 10% between the numerical and analytical result when the radius of the coil is approximately 0.4% of the free space wavelength or quarter of the transfer distance. Further increasing the coil radius, the mutual inductance decrease as increase radius and Eq. 2.4 no longer
provides a correct evaluation for the mutual inductance. At the frequency of 10MHz, the free space wavelength is 30m; thus the equation is valid when the coil radius is less than 12cm. This is obeyed in the following analysis where coil radius is 1cm unless specified. This study shows us when the coil radius is less than 0.4% of the wavelength, the theory[28] provides an adequate model to analyze the system.

Secondly, a finite size of MM lens is applied in the numerical model, comparing to the infinite 2D MM lens in the analysis. In Ref.[28], MM lens with material property $[\mu_x, \mu_y, \mu_z] = [-1, -1, -1]$ could enhance the mutual coupling which further enhance the transfer efficiency in the WPT system. And the MM lens has a thickness $L = D/2$, in which case the two coils have equal distance $D/4$ to the surface of the MM lens. This is an optimal configuration since the source and receiver coils are at the source and focal plan respectively from the super lens point of view[25].

Here we apply the same MM lens property and configurations in the numerical model. A small imaginary part $0.001j$ is added to all three diagonal components of permeability to ensure the numerical computation converges, thus $\mu = -1 - 0.001j$. Strong mutual coupling enhancement has been predicted in the analysis, which is also shown in the numerical result here.

As shown in Fig.2.2 (b), the enhancement ratio is a function of $W$. $\rho$ saturates when $W \approx D$. In this sub-wavelength regime, the field excited along the MM lens surface is called magnetostatic surface resonance (MSR) [28]. The field reconstruction at the receiver coil is hard to predict when the lens radius changes as the effective permeability $\mu_{\text{eff}} = -1$. When the lens radius is too small, enhancement ratio is small since the field deconstruct at the receiver coil due to the boundary effect; enhancement ratio turns to larger than normal at specific lens radius since the field reconstruct there due to the boundary effect; enhancement ratio flatten when the lens radius is large enough, in which the boundary effect no longer affects the field reconstruction at the receiver end, since the dominant MSR modes are the first few
mode close to the center, as shown in Fig. 2.3(c). A similar study is presented later when we discuss the application of indefinite medium lens.

Thirdly, the field intensity in the receiver coil increases as the thickness of the MM lens increases, when the distance between the two coils is fixed, $D = 0.5$ m; and the MM lens material properties keep the same, $\mu = -1 - 0.001j, \epsilon = 1$. The effective distance between the two coils is $D - 2L$ with isotropic MM lens comparing to $D$ in the absence of the MM lens. Thus the mutual inductance increases with the increase of the MM lens thickness while the overall distance $D$ remains the same, result shown in Fig. 2.2(c).

![Figure 2.2](image)

**Figure 2.2:** (color online) (a) the ratio of simulated $L_{21}$ to the theoretical $L_{21}$ calculated in the point-dipole limit; (b) the mutual inductance enhancement dependency on the MM lens width; (c) the mutual inductance enhancement dependency on the lens thickness; (d) Material loss dampen mutual coupling enhancement.

From the other point of view, when the thickness of the MM lens increases, the
receiver coil is closer to the surface of the MM lens since the total distance $D$ is fixed. In this case, the field intensity at the receiver coil is higher than further away since the field intensity decays exponentially when the position moves away from the surface in the MSR mode.

Last, material loss always presents when we discuss negative index medium properties, the same to the negative permeability MM lens we implemented here. Material loss is an inevitable problem when MM is used in the WPT system, especially for high power transmission systems. In the previous text, we applied an idealized MM lens with a small magnetic loss tangent $\sigma = 0.001$. Here, we study the system performance when the MM medium has a reasonable loss and show the mutual coupling still been enhanced.

The numerical modeling still applies an isotropic MM medium while the permeability of the MM lens is $\mu_{\text{eff}} = -(1 + j\sigma)$ and $\epsilon = 1$, in which $\sigma$ is the magnetic loss tangent of the MM lens. The dipole radius is $R = 0.01m$; and the MM lens radius is $W/2 = 1m$; both dimensions are enough to ignore all finite size effects discussed earlier. The thickness of the MM lens is optimum, half of the separation distance $D = 0.5m$. Results from analytical theory and numerical model are shown in Fig. 2.2(d).

From both the analytical and numerical models, the results show that even with a unit loss tangent, the mutual inductance still twice as large as in the case without MM lens. The mutual inductance enhancement ratio $\rho$ has a magnitude inverse to that of the magnetic loss tangent $\sigma$. In practical, magnetic response MM can reach a low loss tangent at the level of 0.1 or less when $\mu \approx -1$. In this case, the mutual inductance enhancement ratio $\rho$ is at two orders of magnitude.
2.4 Anisotropic MM slab application

Until now, we studied the mutual inductance enhancement with isotropic MM lens, where the optimal system configuration occurs when the MM lens is $D/2$ thick and the source and receiver coil are placed at the source and imagine plane of the lens. This configuration results in the MM lens with a bulk thickness and occupied half of the space between the source and receiver. In the case of long propagation distance, this bulky MM lens is not practical. Transformation optics[29] can shrink the lens thickness while remain the same functionality–known as compression lens–by using anisotropic materials, comparing to the isotropic material in conventional lens. Here we study the WPT system with anisotropic lens.

In the configuration analyzed both numerically and analytically, the distance between the two coils $D$ remains constant as 0.5m. And the MM lens is placed in the middle of the two coils. We introduce the anisotropic factor $a$ to represent the compression factor. Based on the transformation theory, it requires the MM material property to be $[\mu_x, \mu_y, \mu_z] = [-a, -a, -1/a]$ in order to reduce the lens thickness from $L$ to $L/a$[29]. We performed a parametric study, where $a$ sweeps from 0.6 to 2 and the lens thickness varies from 1.6$L$ to 2$L$. And a constant imaginary part of permeability $\text{Im}(\mu) = 0.001$ has been added to all three component of the MM permeability. The parametric study result is shown in Fig.2.3(a).

As shown, the enhancement ratio is larger when the anisotropic factor $a$ is small, in which case the receiver coil is closer to the MM lens surface. In this situation, stronger local field exists at the receiver coil due to the MSR mode. The MSR mode when $a = 0.6, 1, 2$ are shown in Fig.2.3(b, c, d). As the anisotropic factor $a$ increases, the lens thickness decreases, and both the receiver coil and source coil moves away from the MM lens surface. Less magnetic flux flows through the receiver coil, thus the enhancement ratio decreases. Even the enhancement ratio becomes small when
\( a = 2 \). The overall enhancement is still large, \( \rho \approx 220 \), comparing to the case without MM lens.

\[
\begin{align*}
\text{Numerical Result} & \quad \text{Analytical Result} \\
\text{Anisotropy Factor: } a & \quad \rho
\end{align*}
\]

**Figure 2.3:** (color online) (a) Mutual coupling enhancement factor depends on the compression factor; (b) System configuration and SP mode when \( a = 0.6 \); (c) System configuration and SP mode when \( a = 1 \); (d) System configuration and SP mode when \( a = 2 \).

From the lens compression technique, the compressed MM lens with anisotropic medium would perform as well as isotropic lens. The theory predicts that the enhancement factor \( \rho \) shall remain constant for any value of the anisotropic factor \( a \), which differs from the result shown in Fig.2.3(a). The reason for the inconstant mutual coupling enhancement results from the loss in the anisotropic medium. In order to preserve the same performance as isotropic lens, the loss in the anisotropic medium needs to keep in certain level. We investigated several different type of anisotropic medium with different loss tangent, where \( \sigma = 0.1, 0.1/a, 0.1/a^2 \). The result is shown in Fig.2.4(a). As shown, the anisotropic medium with \( \sigma = 0.1/a^2 \) stabilize the mutual inductance.

The loss in the anisotropic MM lens affects the performance of the system. In the following, all components in the permeability tensor share the same loss tangent

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σ. For σ = 0.1/a^2, σ = 1/a^2 the enhancement factor ρ reduces from a value around 50 to a unit value. Both the analytical and numerical result is shown in Fig.2.4(b). In practical case where , the enhancement ρ still achieves around 50 even when the MM lens is only quarter of the transfer distance.

The results in the above study indicate that material loss of MM lens needs to be well controlled if the system implement anisotropic lens, especially the magnetic loss of the permeability component in the transverse direction, μx, μy. It requires MM lens to increase the anisotropy but reduce the loss in order to maintain the performance.

2.5 Indefinite MM slab application

In previous section, we analyzed the performance of anisotropic lens, which compresses MM lens’s thickness in the propagation direction. Here we study the system when the MM lens is formed by an indefinite medium, where the three components of permeability have different sign[30]. The motivation to apply indefinite medium is to simplify the MM lens configuration in reality. By eliminating one negative index
component in the permeability tensor, one resonant unit can be removed from each array element in the periodic artificial composite. Thus a MM lens with periodicity in all three directions in Cartesian coordinate can be simplified to lens with periodicity in two or even one direction. Equivalent performance of WPT system with anisotropic MM lens has been verified experimentally comparing to the WPT system with isotropic MM lens[26].

In our specific study, the anisotropic material properties for the MM lens is: \( \mu_z = -a \mu_x = \mu_y = 1 \) and \( \epsilon = 1 \). And the lens has a thickness \( L = D/2 \), with finite width in the numerical model, and infinite width in analytical modeling. The analytical result shows the enhancement ratio is less than 10, the solid blue curve in Fig.2.5. Two indefinite MM lens with different lens width \( W = 3.6m \) and \( W = 4m \) are studied numerically. The fitting curve (dashed green curve in Fig.2.5) based on the numerical result matches well with the analytical prediction. The short oscillations in the numerical result are due to the Fabry-Perot resonances, which results from the finite diameter of the MM lens. The waves in MSR modes on the lens surface will be reflected back on the surface, which will further help to reconstruct/deconstruct the field at receiver coil. This process depends heavily on the width of the MM lens, which result in the oscillation in \( \rho \).

In the end, we compare the performance of isotropic, anisotropic and indefinite MM lens. The result is shown in Fig.2.6. In this comparison, both isotropic and indefinite medium lens share the same loss tangent \( \sigma = 0.1 \) and lens thickness \( L = D/2 = 0.25m \). The indefinite medium lens has the loss tangent \( \sigma = 0.1/a^2 \) and lens thickness \( L = D/4 = 0.125m \) since the anisotropic factor \( a = 2 \). In the indefinite medium lens, the oscillation in \( \rho \) results from the Fabry-Perot resonance modes associate with the finite dimension of the MM lens.

By better control the material loss in the anisotropic medium lens, the mutual coupling enhancement performs as well or even better than isotropic negative per-
Figure 2.5: (color online) The comparison of enhancement ratio between analytical and numerical result. (b) The comparison of enhancement ratio between isotropic MM lens, anisotropic MM lens and indefinite lens when the lens radius changes. Isotropic lens has \( \mu = -1 - 0.1i \) and \( L = D/2 \); Anisotropic lens has \( [\mu_x, \mu_x, \mu_x] = [-2-0.2i, -2-0.2i, -0.5-0.05i] \) and \( L = D/4 \); Indefinite lens \( a = 1 \) has \( [\mu_x, \mu_x, \mu_x] = [-1-0.1i, 1, 1] \) and \( L = D/2 \); Indefinite lens \( a = 1 \) has \( [\mu_x, \mu_x, \mu_x] = [-2-0.2i, 1, 1] \) and \( L = D/2 \).

meability MM lens. And the thickness for anisotropic lens is only half of that in the isotropic lens. This result shows a potential application of compressed MM lens in the WPT system to form an more compact product design.

In indefinite medium lens, optimum mutual coupling enhancement can be achieved by constructively rebuilding the field in the receiver coil and manipulating the MSR mode on the lens surface through the boundary effect. In particular, when the lens has width equal to 0.15m, and the lens with \( \mu_z = -2 - 0.2i \), the mutual coupling
enhancement of the indefinite lens outperforms that of the isotropic and anisotropic MM lens when all three lenses share the same width. The results shown here predict that the indefinite medium can equally perform or outperform isotropic lens if it is designed and optimized including all the finite size and boundary effect.

Figure 2.6: (color online) The comparison of enhancement ratio between isotropic MM lens, anisotropic MM lens and indefinite lens when the lens radius changes. Isotropic lens has \( \mu = -1 - 0.1i \) and \( L = D/2 \); Anisotropic lens has \([\mu_x, \mu_x, \mu_x] = [-2 - 0.05i, -2 - 0.05i, -0.5 - 0.0125i]\) and \( L = D/4 \); Indefinite lens \( a = 1 \) has \([\mu_x, \mu_x, \mu_x] = [-1 - 0.1i, 1, 1]\) and \( L = D/2 \); Indefinite lens \( a = 1 \) has \([\mu_x, \mu_x, \mu_x] = [-2 - 0.2i, 1, 1]\) and \( L = D/2 \).

2.6 Conclusions

In this chapter, we analyze the mutual coupling enhancement by implementing MM lenses in the resonant coupling wireless power transfer system. A few practical issues have been addressed here:

- Numerically verified the theory developed by Dr. Smith and Dr. Urzhumov in...
• Investigated the finite lens dimension in wireless power system, metamaterial lens’ performance remains when lens’ edge dimension is comparable to the transfer distance;

• Metamaterial lens boosts mutual coupling even with substantial large material loss, like loss tangent $\delta = 1$;

• Anisotropic lens helps the enhancement in WPT system, which could maintain the enhancement from metamaterial lens but shrink the thickness of lens;

• Indefinite lens helps the enhancement in WPT system with careful control of its dimensions, which provide almost equivalent enhancement while simplifies the design of metamaterial lens;

• Several patents have been applied, and one journal paper publication [detailed information listed in biography].
Effective medium model for bulk nonlinear metamaterials

The design and characterization processes presented here are targeted for the nonlinear metamaterials composed by arrays of varactor loaded split ring resonators (VLSRRs) and varactor loaded I beam resonators (VLIs). The prior one forms the most general microwave nonlinear magnetic mediums while the later one could form microwave nonlinear electric mediums, while both exhibit strong nonlinear responses. Here, both designs are mainly at microwave frequency. However, it could be extended to terahertz range by reducing the unit cell size and integrating unpackaged diode directly into MM structure or replacing the lumped components by other nonlinear materials, such as nature optical materials or nonlinear organic materials.

Both the linear and nonlinear characteristics in metamaterial composites can be precisely described by effective mediums under certain limitations, in which the linear/nonlinear properties of NMMs are presented by their effective linear and nonlinear material parameters. For linear properties, the linear material parameters are permittivity and permeability in constitutive relations, same as in many linear metamaterials. For those nonlinear properties, effective nonlinear materials' pa-
rameters are second and higher order susceptibilities. Both linear and nonlinear parameters are generally geometry and frequency dependent. In general, the permittivity/permeability are the first order component in polarizations/magnetizations, which directly relates to the first order electric/magnetic susceptibility. Thus both the materials’ linear and nonlinear properties can be described by electric/magnetic susceptibilities.

In general, all susceptibilities are in the tensor form. However, we simplify our problem here. It is assumed that nonlinear mediums analyzed here have only diagonal components; and the external excitation fields are linearly polarized. Thus we only need to consider the diagonal components in the permittivity/permeability tensor along the E/H field direction. And, the associate nonlinear properties, second or higher order susceptibilities only have component in the related directions as well. These assumptions have also been applied in Ref.[31] and later chapters.

The agreement between theoretical results and experimental results validates the effective medium model for NMMs, and its effectiveness to analyze linear and nonlinear response in NMMs. Thus the effective medium model for NMMs have been used to study the 1D/2D wave generation/propagation in NMMs, in chapter 5 and 6.

For more general applications, a retrieval method based on transmittance and reflectance calculation from one unit structure’s numerical study in commercial software is widely used to obtain a quick and initial result[32] for metamaterial composite based on that unit structure. Together with the advanced technology for linear and nonlinear modeling in computational commercial softwares, we can fast and accurately obtain the effective properties for various linear and nonlinear unit structures prior to the physical fabrication process.
3.1 Introduction

In recent work[31], an analytical theory was developed which states that a quantitative description of the higher order (second or higher) susceptibilities for the VLSRR composites can be determined by known factors. These factors include lorentz parameters describing their effective linear properties and the parameters related to the nonlinear properties of the integrated nonlinear components/materials in NMM samples.

This theory presents a direct relation between the first order susceptibility and the higher order susceptibilities. And we can predict the exact value for higher order susceptibilities as long as we know the NMM’s linear properties and the properties of nonlinear devices/materials. With the help of the advanced full-wave numerical modeling tools nowadays, we can determine the linear properties of NMMs with adequate accuracy. Along with the known nonlinear property information of the integrated nonlinear component/materials, nonlinear properties for NMMs can then be precisely predicted. Although these models are valid only when the external excitation signals’ power is low, it is still a very useful tool for scientists and engineers to study the nonlinear processes in artificial mediums.

Only under the small power excitation condition, the perturbable approach derived susceptibilities are valid. And the none-depleted pump approximation is also satisfied in low power case. Diodes, varactors and other similar components have large nonlinear response that quickly render the traditional nonlinear optical description invalid. Nevertheless, composite metamaterials based on such components currently provide extremely useful samples to test the emerging nonlinear metamaterial concepts. Our goal here is to demonstrate that the analytical model can be used to design numerous different nonlinear metamaterials.
3.2 Equivalent circuit models in conventional linear metamaterial analysis

Dating back to the first development of the effective medium models for metamaterials, the equivalent circuit models for metamaterial unit structures play a significant role in history. The polarization/magnetization for each unit structure in metamaterial composites are first studied by analyzing their equivalent quasi-static circuit models. Takeing the most famous magnetic metamaterial– split ring resonator (SRR)– for example, it is first proposed by Prof. John Pendry in 1999[2]. Prof. Pendry predicted the Drude-Lorentz shape of the SRR’s effective permeability through analyzing the local magnetization for each SRR structures. He also projected the possibility of realizing artificial material with negative permeability at certain microwave frequencies just by using SRR structures as unit structure to form concretes. One year later, Prof. Smith experimentally demonstrated the negative refraction phenomenon by using both wire medium and SRR medium, which proved Prof. Pendry’s proposal. Hence, we understand the importance of the equivalent circuit model in analyzing the properties of metamaterial unit structures. In latter section, it is shown how we analyze NMMs’ structures by using their equivalent circuit models.

Circuit models for unit structures provide an intuitive understanding of the electric/magnetic responses inside. By investigating the electric current flow in each structure, we know the origin of electric/magnetic response. Take the most two common microwave structures – the wire structure and the SRR structure – for example. Current flowing in wire structure at resonance is like bounded line current, which is identical to that in a finite size electric dipole. Because the sub-wavelength size of the wire structures, unit structures in wire composites can be treated as sub-wavelength electric dipoles. Vice versa, current flow in sub-wavelength SRR structures takes the
shape of loop current, which forms sub-wavelength magnetic dipole. In macroscopic, wire/SRR composites display the electric/magnetic response accordingly, and they exhibits abnormal effective electric/magnetic properties.

The equivalent circuit models for miniature structures provide the method to estimate the electric/magnetic dipole moments for each unit structures by relating the induced current to external excitation field. Then the macroscopic polarization/magnization for metamaterial medium are predicted by considering the volume integration of the local electric/magnetic dipole moment.

The famous Drude model for electric mediums’ effective permittivities and Drude-Lorentz model for magnetic mediums’ permeabilities can be extracted from the unit structures’ equivalent circuit representations. In the following, we will take the two common composites—wire and SRR—to show the process of how we obtain their effective medium properties by studying their representative circuits.

3.2.1 Electric response

Metallic structures with electric responses have been studied for decades. The first finding of artificial dielectrics was reported by Prof. Brown in conducting wires with artificial dielectric properties in 1960[33]. And the first realization of the negative refraction phenomenon in the artificial negative index medium in 2001[34] refocused research highlights on artificial materials. Then other artificial electric dielectrics have been found, which includes continuous wire structure, cut wire structure(I beam structure), electric L-C resonators, and so on. Here we take the cut wire medium for example to investigate how we achieve their effective medium properties from their equivalent circuit representations.

The geometry we studied here is shown in Figure 3.2.1. Consider serval layers of cut wire structures placed periodically in 3D. The distance between each layers is $d$. The external excitation field is parallel to the metallic wire direction in structures. If
Figure 3.1: (a) Periodically positioned cut wire composites; (b) The equivalent circuit model for single cut wire structure.

distance $d$ is much smaller than the wavelength $\lambda$, the space occupied by the cut wires can be modeled as a homogeneous medium, where we could apply a homogeneous method to obtain their effective properties.

Assume the current flowing through the cut wires is $I$, then the effective local polarization is

$$\frac{dp}{dt} = J = \hat{z} \frac{I}{A}$$

(3.1)

where $J$ is the total current; $A = d^2$ is the effective cross-section area which current $J$ flows through; $p$ is the effective polarization of the wire media. By solving the relation between current $I$ on wires and the external excitation field $E_0$, we get the relationship between local polarization and external field $p \sim E_0$. Integrating the local polarization in a unit volume, we obtain the macroscopic polarization $P =$
\[ \int \rho dV \]. Hence, we obtain the effective permittivity for the artificial medium in the direction of the applied field in constitutive relations.

Since the cut wires composites are metal strips patterned on dielectric substrates, they have equivalent lumped circuit components to represent resistive, inductive and capacitive effect when currents flow through. This builds up the foundation for the equivalent circuit model of the cut wire composite. In our simplified model, a serially connected RLC circuit is the equivalent circuit diagram for cut wire medium. In which, the resistor \( R \) counts for the loss from dielectric material, and radiation loss; the capacitor \( C \) counts for the capacitive effect from strip fringe and the gap between cut wire end to the periodic boundaries, and inductor \( L \) stands for the inductance in the metal strip.

The external excitation field \( E_0 \) provides an virtual voltage source for the RLC circuit, \( U = E_0 d \). Thus we have the circuit equation as:

\[
RI + V_C + L\frac{dI}{dt} = U = E_0 d \tag{3.2}
\]

where \( V_C = q/C = \) and \( I = \frac{dq}{dt} \), and \( q \) is the time dependent charge accumulated on capacitor; \( V_C \) is the voltage across the capacitor. Thus Eq. 3.2 turns to be a second order differential equation of \( q \), as below:

\[
R\frac{dq}{dt} + \frac{q}{C} + L\frac{dq^2}{dt^2} = E_0 d \tag{3.3}
\]

The time harmonic solution for Eq. 3.3 gives a direction relationship between \( q \) and \( E_0 \), as

\[
q = \frac{E_0 dC}{1 + j\omega CR - LC\omega^2} \tag{3.4}
\]

where \( \omega = 2\pi f \) is the angular frequency of the external excitation field.
Then the solution for the local polarization is:

\[
\frac{dp}{dt} = \hat{z} \frac{I}{A} = \hat{z} \frac{Cdq}{Adt} \quad (3.5)
\]

\[
p = \hat{z} \frac{C}{A} q = \hat{z} \frac{C}{A} \frac{E_0 dC}{1 + j\omega CR - LC\omega^2} \quad (3.6)
\]

And the macroscopic polarization equals to \( P = Np = \frac{1}{Ad}p \) for a unit volume of cut wire composites. Thus the effective permittivity \( \epsilon_{z, eff} = 1 + \frac{P}{\epsilon_0 E_0} = 1 + N \frac{p}{\epsilon_0 E_0} \), which gives the following result:

\[
\epsilon_{z, eff} = 1 + \frac{C^2}{\epsilon_0 A^2} \frac{1}{1 + j\omega CR - LC\omega^2} \quad (3.7)
\]

When the capacitive effect is negligible, like in the continues wire medium, Eq.3.7 returns to classical Drude model. However, in the cut wire medium, we have the general Lorentz model.

It worth to mention that the spatial dispersion effect is not considered in the current equivalent circuit model. Approximately, this spatial dispersion effect is noticeable when \( d/\lambda > 1/10 \). For the linear case, in current case, the relationship between the Lorentz shaped effective material parameters and the effective parameters with spatial information is discussed in ref.[35] and ref.[36].

### 3.2.2 Magnetic response

In the previous section, we have discussed the procedure to achieve the effective permittivity for the cut wire medium by analyzing its equivalent circuit model. Here, a similar procedure is taken to analyze another widely implemented metamaterial structure – split ring resonator, which provides magnetic responses and acts like a magnetic dipole. Instead of calculating the effective electric dipole moments from the cut wire medium, we calculate the effective magnetic dipole moments in SRR media. And we apply a volume integration method to obtain the effective macroscopic
magnetization, and have its effective material property—permeability.

The SRR geometry studied here is shown in Fig.3.2.2. SRR structures are circular metal strip with effective radius $R$, and enclosed area $A = \pi R^2$. The external time varying excitation field $H_0$ propagates through the enclosed area $A$ orthogonally, which provides the driven voltage in its equivalent circuits. The equivalent driven voltage in equivalent circuit equals to $U = \frac{\partial B}{\partial t}A = j\omega\mu_0 H_0 A$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{srr.png}
\caption{(a)SRR structure; (b) The equivalent circuit model for single SRR structure.}
\end{figure}

In the equivalent circuit model for SRR structures, the resistor $R$ stands for the substrates' resistive loss and the radiation loss; inductor $L$ stands for the inductive effect from the metal strips; and capacitor $C$ stands for the capacitance across the metal slit. Known the circuit representation and the drive source, we have the relationship between the current $I$ in circuit and the external driven field $H_0$ as:

$$V = \frac{\partial BA}{\partial t} = j\omega\mu_0 H_0 A = RI + L \frac{dI}{dt} + V_C$$

(3.8)

where $V_C = \frac{q}{C}$, and $q$ is the time dependent charge across the capacitor. The current $I = \frac{dq}{dt}$. Thus we have:
\[
R \frac{dq}{dt} + L \frac{d^2 q}{dt^2} + \frac{q}{C} = j\omega \mu_0 H_0 A
\]  

(3.9)

By solving the \( q \) in Eq.3.9, we have a direct formula to link the real time charge \( q \) to the external excitation field \( H_0 \). And the magnetic dipole moment \( m \) for SRR unit cell is \( m = IA = A \frac{dq}{dt} = j\omega qA \). The macroscopic magnetization \( M \) for the artificial medium composed by SRR structures is \( M = Nm \), where \( N \) is the volume density. And the effective permeability is

\[
\mu_{\text{eff}} = \frac{B}{H} = \frac{\mu_0(H+M)}{\mu_0 H} = 1 + \frac{NIA}{H} 
\]

(3.10)

Here we have the equivalent material properties for the artificial material composed by SRR unit structures, which simplifies the process to analyze EM wave propagating and scattering properties in and outside of the metamaterial composites. Same as in the electric mediums, the circuit representation does not count for the spatial dispersion in metamaterials.

The equivalent circuit models provide an insight view of the mechanism in metamaterials’ unit structures. Even if only two structures – cut wire and SRR, are analyzed here, the same methodology can be applied to other metamaterial composites, such as continuous wire, ELC resonator, and structures providing magnetic response. More importantly, we show how to use the equivalent circuit models in analyzing the linear metamaterial structures.

### 3.3 Effective medium model: nonlinearity

As shown above, the basic metamaterial elements with either magnetic or electric response can be presented by their effective circuit models. Hence, their effective permeability and permittivity can be obtained from their equivalent circuit models.
And the parameters are presented by the circuit components and geometric information. Next, we discuss how to obtain both their effective linear and nonlinear material properties from the equivalent circuit models of microwave nonlinear metamaterials, which includes the effective linear permittivity and permeability, nonlinear electric susceptibility and magnetic susceptibility. And we discussed the important relationship between the linear and nonlinear material properties.

The same as in the linear metamaterial discussion, the coupling effect between unit structures and the finite element effect in the composite mediums are neglected in the following analysis. Meanwhile, we assume all nonlinear processes are independent between each other, and all of them do not affect the linear process which means no power is transferred from linear pump field to generated nonlinear products. This is known as the non-depleted pump approximation or undepleted pump approximation.

3.3.1 EMM for electric medium

Recently, an analytical approach has been reported in which the nonlinear properties of microwave metamaterials are presented by their effective nonlinear properties—nonlinear susceptibilities[31]. The nonlinear susceptibilities of varactor loaded split ring resonator (VLSRR) have been described by the equivalent lumped components’ parameters and geometric parameters. Based on the equivalent medium model for nonlinear metamaterials, several nonlinear phenomena have been studied analytically and experimentally, such as power dependent material properties, second harmonic generation and wave mixing.

In addition to the analytical modeling of the nonlinear magnetic composites, we study the effective material properties for the nonlinear electric mediums here. And we take the cut wire medium as the foundation to form nonlinear electric mediums. Based on the quasi-statistic circuit model for the structure, the effective permittivity,
nonlinear electric susceptibilities are obtained and form the basis of the effective nonlinear medium model.

The infinite wire medium loaded with nonlinear components (varactors) is studied, which provides the most fundamental equivalent circuit model to investigate.

A quasi-static model for the wire medium has been reported in Ref.[37], in which the electric polarization for the wire medium and its effective permittivity have been calculated. Based on the same procedure, we study the nonlinear polarization of the varactor loaded wire, and its related nonlinear electric susceptibilities.

In varactor loaded cut wire medium, the external excitation electric field provides the driven source. And it is modeled as a virtual voltage source in the circuit model. The relationship between the voltage $U$ across the wire section with length $d$ in Fig.3.3 and the current $I$ flowing through the wire is as below:

$$U = L \frac{dI}{dt} + RI + V_D = E_Z d$$

where $L$ is the total inductance of the metal wire with length $d$, including the self-
inductance and mutual inductance; \( R \) is the effective resistor; \( V_D \) is the voltage across the varactor; \( E_Z \) is the average field along \( z \) direction.

The varactor’s voltage dependent capacitor follows the form

\[
C(V_D) = C_0(1 - V_D/V_p)^{-M}
\]

from which we have the relationship between the voltage and charge as:

\[
V_D = \frac{1}{C} \int_0^t I \, dt = V_p[1 - (1 - \frac{Q}{C_0} \frac{1 - M}{V_p})^{1-M}] \approx q + aq^2 + bq^3 + \cdots \tag{3.12}
\]

where \( q = Q/C_0 \) is the normalized voltage; \( a = -M/2V_p \), \( b = M(2M-1)/6V_p^2 \) are the second and third order Taylor expansion coefficients in Eq.3.12.

The current \( I = dQ/dt = C_0 dq/dt \). The simplified oscillation model presented by \( q \) is shown below, in which only the first three components in Eq.3.12 have been considered. These three components represent the linear process and lowest second and third order nonlinear process, which will be discussed in chapter 5.

\[
L C_0 \frac{dq^2}{dt^2} + R \frac{dq}{dt} + q + aq^2 + bq^3 = E_z d \tag{3.13}
\]

Represent Eq.3.13 in terms of symbols:

\[
\frac{dq^2}{dt^2} + \alpha_1 \frac{dq}{dt} + \omega_0^2 q + \beta q^2 + \gamma q^3 = \omega_0^2 E_z d \tag{3.14}
\]

where \( \alpha_1 = \omega_0^2 R \), \( \beta = a\omega_0^2 \) and \( \gamma = b\omega_0^2 \);

Apply the perturbation theory\[38\] to solve the normalized voltage \( q \) in Eq.3.14, we have the first (linear) and higher order components of \( q \) as below:
\[ q^{(1)}(\omega) = \frac{\omega_0^2 E_z d}{\alpha - i\alpha_1 \omega - \omega^2} \triangleq \frac{\omega_0^2 E_z d}{D(\omega)} \] (3.15)

\[ q^{(2)}(2\omega) = -\frac{\beta \omega_0^4 d^2 E_z^2}{D(2\omega) D^2(\omega)} \] (3.16)

\[ q^{(2)}(\omega_1 + \omega_2) = -2\frac{\beta \omega_0^4 d^2 E_z^2}{D(\omega_1 + \omega_2) D(\omega_1) D(\omega_2)} \] (3.17)

\[ q^{(3)}(3\omega) = -\frac{2\beta q^{(1)}(\omega) q^{(2)}(2\omega) + \gamma [q^{(1)}(\omega)]^3}{D(3\omega)} \] (3.18)

\[ q^{(3)}(\omega) = \frac{-2\beta \left[q^{(1)}(\omega) q^{(2)}(0) + q^{(1)}(-\omega) q^{(2)}(2\omega)\right] + 3\gamma [q^{(1)}(\omega)]^2 q^{(1)}(-\omega)}{D(0)} \] (3.19)

The relationship between the current through the wire medium and the effective polarization is:

\[ \frac{dP}{dt} = J = \frac{\hat{z} I}{A} = \frac{\hat{z} C_0 dq}{A dt} \] (3.20)

\[ P = \frac{\hat{z} C_0}{A} q \] (3.21)

where \( J \) is the current density; \( A \) is the effective cross-section area which the current flowing through and \( P \) is the macroscopic polarization of the wire media. Combine Eq.3.20,3.15 and the definition of electric susceptibilities, we have the analytical result for the electric susceptibilities as below:
\[ \chi_{e,z}^{(1)}(\omega) = \frac{C_0 \omega^2}{AD(\omega)} \Delta \frac{F}{D(\omega)} \] (3.22)

\[ \chi_{e,z}^{(2)}(2\omega) = -\beta C_0 d \omega^4 \Delta \frac{F \omega^2 d}{D(2\omega)D^2(\omega)} \] (3.23)

\[ \chi_{e,z}^{(2)}(\omega_1 + \omega_2) = -2\beta C_0 d \omega^4 \Delta \frac{F \omega^2 d}{D(\omega_1 + \omega_2)D(\omega_1)D(\omega_2)} \] (3.24)

\[ \chi^{(3)}(3\omega) = \left[ \frac{2\beta^2}{D(2\omega)} - \gamma \right] \frac{C_0 \omega^6 d^3}{AD(\omega)^3 D(3\omega)} \] (3.25)

\[ \chi^{(3)}(\omega) = \left[ 3\gamma - \frac{4\beta^2}{D(0)} - \frac{2\beta^2}{D(2\omega)} \right] \frac{C_0 \omega^6 d^3}{D(\omega)^3 D(-\omega)} \] (3.26)

### 3.3.2 EMM for magnetic medium

In the study of the metamaterial structure with magnetic response, the most fundamental structure—split ring resonator— is considered here. And we introduce a nonlinear component, varactor, into the gap regime to introduce nonlinear response in the geometry. We call this kind of structure—varactor loaded split ring resonator (VLSRR).

We formed a thin layer effective nonlinear medium by constructing a composite with $3 \times 15 \times 1$ split ring resonator, while each element’s gap in the array is embedded with Skyworks, SMV1231 varactor, as shown in Fig.3.4 (a, b). The analyzed geometry and its dimension information are shown in Fig.3.4 (a, e).

The varactor is the source of nonlinear response in this VLSRR model, from both its nonlinear capacitance and nonlinear resistance. This varactor’s capacitance follows the form of $C(V_D) = C_0(1 - V_D/V_p)^{-M}$, where $V_D$ is the bias voltage, $C_0 = 2.4\text{pF}$ is the unbiased capacitance, $V_p = 1.5\text{V}$ is the intrinsic potential, and $M = 0.8$ is the gradient coefficient. This nonlinear relationship between the bias voltage $V_D$ and effective capacitance $C(V_D)$ is the dominate source of the nonlinear
response in the investigated metamaterial, under the condition that the nonlinear response from nonlinear resistance of the applied varactor is negligible comparing to the response from the nonlinear capacitance. This is satisfied in our investigation in this report that the pump signal is small to excite a strong nonlinear response from nonlinear resistance.

The effective linear properties of the VLSRR metamaterial are identical to the case of bulk metamaterial medium composed by SRR element loaded with fixed value capacitance, which has been studied and modeled previously[39]. The effective permeability in its equivalent medium model follows Lorentz form as the model for the conventional split ring resonator composed artificial medium, presented in Eq.(2), where $\omega = 2\pi f$ is the angular frequency of the driven source; $F$ is the oscillation strength; $\omega_0$ is the angular frequency of the resonator and $\gamma$ is the damping coefficient of the resonant system.
\[
\mu_{\text{eff}} = 1 + \frac{F\omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \triangleq 1 + \frac{F\omega^2}{D(\omega)}
\] (3.27)

where \(D(\omega) = \omega_0^2 - \omega^2 - i\gamma\omega\). Spatial dispersion [36, 35] in the specific type of nonlinear metamaterials we investigated here is small enough to be neglected since the unit cell size is only about \(1/30\) of the free space wavelength, which is much smaller comparing to its value \((1/10)\) in traditional bulk metamaterials. As a result, the effective permittivity of the artificial medium can be approximated as constant in the interested frequency range, since the variation of effective permittivity in conventional metamaterials originates from the nontrivial spatial dispersion relationship. The non-dispersive effective permittivity \(\epsilon_{\text{eff}}\) has been verified through a numerical study of the structure later.

It is critical to determine the linear material parameters of metamaterial mediums accurately. Not only their linear responses are described by the linear property parameters, but also their nonlinear properties, nonlinear susceptibilities, are directly determined by the parameters \(F, \omega_0, \gamma\) in the Drude-Lorentz like form as shown in Eq.3.27, which describes the linear effective permeability of the investigated artificial medium. Take the second order nonlinear susceptibility at the second harmonic frequency for example, the effective nonlinear susceptibility \(\chi_{\text{eff}}^{(2)}\) of the metamaterial medium is expressed in Eq.3.28, where \(\alpha\) is the parameter determined by nonlinear components’ properties, calculated from the Taylor expansion of the varactor’s voltage dependent capacitance relationship; \(A\) is the effective area of the SRR structure[9].

\[
\chi_{\text{eff}}^{(2)}(2\omega) = -i\alpha \frac{2\omega_0^4 \omega^3 \mu_0 A F}{D(\omega)^2 D(2\omega)}
\] (3.28)
3.4 Parameter extraction

Standard retrieval method has been employed to extract the effective permeability and permittivity of nonlinear metamaterial mediums[40], under the condition that the transmittance and reflectance are received from low power pump signals. A nonlinear curve fitting for the effective permeability to Drude-Lorentz like form, shown in Eq.3.27, provides the value of $F, \omega_0, \gamma$ which are presented in higher order nonlinear susceptibilities. These parameters, $[F, \omega_0, \gamma]$, are named Drude-Lorentz parameters here.

Two different methods have been applied to extract and predict the Drude-Lorentz parameters of the nonlinear metamaterial sample. The first method is based on the transmittance result from experiments, which provides the best prediction to the linear material properties. The other method is based on the simulated transmission properties of an equivalent numerical model. Certain accuracy has been paid to obtain the design capabilities in the numerical method. The results and procedures in both methods have been discussed in the following sections.

3.4.1 based on experimental result

Given the fabricated sample, the transmission properties of the metamaterial slab could be measured in a transmission line apparatus[41], which supports TEM mode for both the fundamental frequency and harmonic frequency waves.

The standard retrieval procedure for the nonlinear metamaterial sample is not applicable here since it is hard to measure the reflectance’s phase and magnitude for the nonlinear sample in the transmission line apparatus. This difficulty results from the imperfection of the calibration processes in the apparatus. With only the transmittance, a numerical fitting method has been adopted to extract the Drude-Lorentz parameters.
Conventionally, we assume the metamaterial composite can be presented by an effective medium model with effective permittivity and permeability. With the medium model and its effective material properties, the transmittance and reflectance are evaluated through transfer matrix method. Here, we used the same method to perform the analytical study, whose result is compared with experimental data.

The detailed procedures for parameter fitting are shown in Fig. 2. First of all, we sweep each Drude-Lorentz parameters over a defined range, and then apply linear transfer matrix method to calculate the transmittance for each parameter set. In the end, we compare the calculated transmittance with experimental result till the difference between the two reaches the predefined tolerance threshold. It is worthwhile to notify that the transmittance comparison is evaluated over a defined frequency range, instead of each single frequency point. This is primarily different from the least square method used to retrieve material properties at optical frequency.
Table 3.1: Fitted Drude-Lorentz parameters by TMM method

<table>
<thead>
<tr>
<th>$\epsilon_{\text{eff}}$</th>
<th>$F$</th>
<th>$\gamma$</th>
<th>$f_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.63</td>
<td>0.214</td>
<td>$2\pi \times 25$</td>
<td>816MHz</td>
</tr>
</tbody>
</table>

Figure 3.6: The comparison between measured transmittance and fitted result from TMM method.

In the current fitting, we chose the $S_{21}$ parameter value threshold at 0.95, which limited our frequency range from 0.77GHz to 0.87GHz. Only the measured $S_{21}$ results in this frequency range have been adopted in the fitting procedure. Ranges for each parameter are: $F$ from 0.15 to 0.25, $\gamma$ from $2\pi \times 10$MHz to $2\pi \times 40$. The resonant frequency is set to $f_0 = \omega_0/2\pi = 816$MHz and the permittivity $\epsilon_{\text{eff}} = 1.63$ from numerical model in the medium model.

The best fitted result is shown in Fig.3.6. And the result for the fitted parameters is shown in Table.3.1. It has been verified from the matched result that the linear material properties of the investigated nonlinear magnetic metamaterial sample could be simply presented by its Drude-Lorentz like permeability and a constant permittivity.
3.4.2 based on numerical result

The fitting from experimental result of transmittance provides an characterization method of the linear properties of nonlinear metamaterial samples. However, it lacks the abilities to predict the material properties before fabrication. In this section, a numerical method to design nonlinear material and predict its linear properties in certain accuracy level is presented.

The effective linear properties of the nonlinear metamaterial medium, are retrieved from the numerical result of transmittance and reflectance under the assumption that the VLSRR medium acts like a linear material under low power excitation. The conventional retrieval process for linear metamaterials[40] is adopted here. Under those assumptions, both the resonant frequency shift from nonlinear process and the power depletion are neglected here. Thus, a finite-difference time-domain solver based simulator, Microwave Studio (CST), has been adopted to perform the numerical analysis of the actual SRR structure embedded with lumped varactor built in the gap of SRR structure, as shown in Fig.3.4 (a). The varactor model has been replaced by its equivalent zero biased circuit. This equivalent circuit model contains the modified lumped components: resistance, inductance and capacitance in zero bias condition which corresponds to the extra low power excitation assumption. The values of the circuit components are: \( R_{\text{var}} = 2.05\Omega \), \( C_{\text{var}} = 2.1\text{pF} \) and \( L_{\text{var}} = 2.64\text{nH} \). These zero bias component values are slightly different from the value provided in the applied varactor data sheet[42]. Here we incorporated the additional capacitance, inductance and resistance from fabrication process into model.

With the numerical model, numerical result of transmittance and reflectance are used in standard retrieval method to obtain the effective permittivity and permeability of the artificial medium composed by VLSRR elements. In the end, a curve fitting to the effective permeability extracts the value of Drude-Lorentz parameters.
Table 3.2: Fitted Drude-Lorentz parameters from numerical model

<table>
<thead>
<tr>
<th>$\epsilon_{\text{eff}}$</th>
<th>$F$</th>
<th>$\gamma$</th>
<th>$f_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.63</td>
<td>0.1305</td>
<td>$2\pi \times 20.5$</td>
<td>817.6MHz</td>
</tr>
</tbody>
</table>

Figure 3.7: The comparison between measured transmittance and numerical result.

The fitted Drude-Lorentz Parameters are shown in Table 3.2. Comparing to the similar value provided in Table 3.1, it is noticeable that the value for oscillation strength factor $F$ is much less than that fitted from experimental result. One consequence is that the numerical result of the transmittance is wider and less lossy than experimental result at the resonant frequency of the metamaterial structure, as shown in Fig. 3.7.

As we can see from the figure below, certain accuracy has been paid to obtain the design abilities. Even though the numerical model does not predict the linear properties exactly as measured, it still provides us a useful and close enough result for our study and applications for the nonlinear properties.

More important, the same model could be used in various designs as long as the varactor and the fabrication process are the same as in this design. Later, a
further verification of the numerical model with different unit cell spacing has been conducted and the result confirmed the correctness of the numerical model, which will be discussed later. For elements with different nonlinear components and fabrication processes, similar approach could be used to estimate effective material properties with several additional steps: first, presenting the nonlinear components by circuit model, and perform the perturbation analysis in the oscillation model as in Ref.[9]; measure the transmittance of one sample, and estimate the zero bias circuit model for embedded nonlinear components. Once the circuit model has been characterized through the experimental result of the initial sample, the same model can be used to design and predict the material properties in various applications.

3.5 Conclusion

In this chapter, I have developed the steps to setup an effective medium model for both electric and magnetic nonlinear metamaterials. Two key contributions are listed below:

• Developed the effective medium model for microwave nonlinear electric and magnetic metamaterials based on their equivalent circuit model;

• Developed the methodology to estimate the effective medium’s parameters from experimental and numerical results;

Later on, the effective medium model for nonlinear metamaterials have been applied to study nonlinear processes in NMMs analytically. And analytical results based on the EMM agrees with experimental results in good precision for a number of nonlinear processes.
Resistive effect in nonlinear metamaterials

4.1 Introduction

Physical scientists and engineers have been interested in metamaterials (MMs) for their unique properties and applications in construction ever since their advent in 1999[43]. New research and applications for MMs has included perfect lens[25, 44], lens deforming, and compression, etc. In most instances, the linear properties–permittivity and permeability–have been controlled or manipulated. In addition to materials’ linear properties, nonlinear polarization or magnetization in MMs have also been introduced by incorporating semiconductor components/nonlinear materials into MM structure at microwave frequency, or by utilizing optical nonlinear material in MMs construction at optical frequency. Other phenomena such as enhanced nonlinear response have also been discovered and discussed by scholars and researchers.

Many MMs are based on metal-patterned substrate that operates analogously to an effective RLC circuit, which provides a useful approach to analyze MMs linear properties. Those circuit models for linear MMs usually include effective linear cir-
cuit components such as resistors, inductors, and capacitors, which represent their distributed characteristics. For nonlineral metamaterials (NMMs), similar circuit representatives include additional nonlinear components like varactors and diodes at microwave frequency. Several popular NMM unit designs are of these types. One famous example is a varactor-loaded SRR (VLSRR). In strict conditions, it has been proven that scientists can represent VLSRR’s nonlinear properties in terms of nonlinear susceptibilities from its equivalent nonlinear circuit model, which is a continuous medium model in analog to materials properties shown in optical nonlinear materials.

Various nonlinear responses, such as the second harmonic generation and the wave mixing, the phased matched second harmonic generation, have been studied theoretically and experimentally. Enhanced nonlinear responses have been observed in these experimental studies, and their results agree with theoretical predictions where NMMs are modeled by effective medium as a continuous medium.

In the NMM studies, the effect circuit model has been shown as a very convenient and accurate predictor of NMMs’ properties and responses. The nonlinear components usually come with explicit and well-built spice models for the applied varactors or diodes. Earlier, I presented one success story for a varactor-loaded magnetic resonatorvaractor-loaded split ring resonators (VLSRRs). Good agreements have been found between the theoretical and experimental results in various nonlinear processes. In theoretical analysis, the effective nonlinear properties come from an equivalent RLC circuit with a nonlinear capacitance model that represents the varactor.

This quasi-static modeling method has proven effective when two conditions are met: the power of the pump signal at fundamental frequency must be low, under which level non-depleted pump approximation is valid; and the spatial dispersion effect in MMs is negligible, under which the oscillation circuit formulation is sufficient to describe the process in unit structure. When stronger pump signal presents, large
excitation fields bias the nonlinear components, thus the nonlinear resistive effect is no longer negligible. Hence it is meaningful to have a more general model including both the nonlinear capacitance in varactors and the nonlinear resistance effect in most microwave nonlinear components. This is the problem we are going to address in this chapter.

A varactor is also known as a variable capacitance diode or varicap. If the origin of the varactor and diode are the same the P-N junction we can treat both in a uniformed circuit model. In spice model representations, both varactor and diode share the same set of equations to describe their behaviors. In equivalent circuit representations, both varactor or diode share the same circuit model, as described in Fig.4.2(c). Thus we can use a single circuit representation to identify the behavior of the varactor or the diode in NMMs designs.

\[
\frac{I_d}{I_s}
\]

\[
\frac{C(V_d)}{C_0}
\]

\[
MA4SPS402, \text{ Diode} \quad SMV1231, \text{ Varactor}
\]

\[
-0.5 \quad 0 \quad 0.5
\]

\[
0.8 \quad 1 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \quad 2
\]

Bias Voltage (Vd) (V)

\[
3 \times 10^8
\]

\[
-0.5 \quad 0 \quad 0.5
\]

\[
0.8 \quad 1 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \quad 2
\]

Bias Voltage (Vd) (V)

(a) Relationship between the normalized current and bias voltage; (b) Voltage controlled capacitance for bias voltage between -0.5V to 0.5V.

The difference between a varactor and a diode is presented in their capacitance-voltage and current-voltage relationship. One of the most important variations is the zero biased capacitance \( C_0 \), which determines how much current flows through the nonlinear capacitance. As shown in Fig.4.1, we take the varactor \( SMV1231 \) from

54
Skyworks and diode MA4SPS402 from M/A-Com Tech to show the similarities and differences. SMV1231 and MA4SPS402 are also used later in the analysis of varactor-loaded SRR and diode-loaded SRR structures.

Here we present a general model for both varactor- and diode-loaded NMMs. Nonlinear responses in varactor- and diode-loaded SRR structures have been qualitatively explained by the proposed general circuit model.

4.2 General model

The nonlinear responses in both varactor and diode originate from the parallel connected nonlinear capacitance and nonlinear resistance in them, as shown in Fig.4.2(c). Here, we simplify our analyzed circuit by adding the package resistance, inductance, and capacitance into the total according components in the final effective circuit.

\[ R(V_d) \quad C(V_d) \]

**Figure 4.2**: (a) SRR structure loaded with lumped nonlinear component; (b) The equivalent circuit model for the component loaded structure; (c) equivalent small signal circuit model for both varactor and diode.

The basic circuit is shown in Fig.4.2, in which nonlinear components are represented by their nonlinear resistance and nonlinear capacitance. In a diode, nonlinear resistance is the dominant mechanism that results in a relationship like an exponential function between the bias voltage and current. In a varactor, nonlinear capacitance is the dominant mechanism; thus the relationship between bias voltage and current is in power function form.
Taking the circuit analysis for the circuit shown in Fig.4.2, we have:

\[ L \frac{dI}{dt} + RI + V_D = \varepsilon(t) \]  \hspace{1cm} (4.1)

The uniform circuitry for both diode and varactor is a parallel sub-circuit with nonlinear resistance on one branch and nonlinear capacitance on the other, as shown in Fig.4.2(c). The total current flowing through the overall geometry contains two parts: the current on nonlinear resistance branch and current on nonlinear capacitance branch, as below:

\[ I = g_d V_d + \frac{dQ_d}{dt} + C \frac{dV_d}{dt} \]  \hspace{1cm} (4.2)

Taking Eq.4.2 into Eq.4.1, we have the completed nonlinear equation describing varactor/diode-loaded SRR structure as:

\[ L \left( \frac{d(g_d(V_d)V_d)}{dt} + \frac{d^2Q_d}{dt^2} + C \frac{d^2V_d}{dt^2} \right) + R \left( g_d V_d + \frac{dQ_d}{dt} + C \frac{dV_d}{dt} \right) + V_d = \varepsilon(t) \]  \hspace{1cm} (4.3)

First of all, we consider the current flow through the nonlinear capacitance branch. We know the voltage-dependent capacitance is:

\[ C(V_d) = C_0 \left( 1 - \frac{V_d}{V_J} \right)^{-M} \]  \hspace{1cm} (4.4)

where \( C_0 \) is the zero-biased capacitance, \( V_J \) is the intrinsic junction voltage, and \( M \) is the gradient coefficient of the varactor.

As presented in Ref.[31], based on Eq.4.4 and \( Q(V_d) = \int C(V_d) dV_d \), we have the relationship between bias voltage \( V_d \) and total charge \( q = \frac{Q(V_d)}{C_0} \) as:
\[ V_d(q) = V_p \left[ 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right] \] (4.5)

Secondly, we have the current flowing through nonlinear resistance dependent on its bias voltage as:

\[ I_d(V_d) = I_s \left( \frac{V_d}{V_p} - 1 \right) \] (4.6)

where \( I_s \) is the forward saturation current, \( V_d \) is the bias voltage, \( V_p = NV_T \) is the intrinsic voltage, and \( V_T \) is thermal voltage. \( I_s \) and \( V_p \) are intrinsic properties of the applied diodes, which are determined by the fabrication process and usually provided in the components data sheet.

Based on Eq.4.6, we know the nonlinear admittance of the varactor:

\[ g_d(V_d) = \frac{dI_d}{dV_d} = \frac{I_s}{V_p} \left( \frac{V_d}{V_p} - 1 \right) = I_s \frac{V_p}{V_T} \exp \left\{ \frac{V_p}{V_T} \left[ 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right] \right\} \] (4.7)

As in Eq.4.5 and Eq.4.7, also \( Q(V_d) = C_0q \), the total circuit oscillation function Eq.4.3 could be represented in normalized voltage \( q \) as:

\[
\begin{align*}
\left\{ L C_0 + LC \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{M}{1-M}} \right\} \frac{d^2q}{dt^2} &= - \left\{ L C \frac{M}{V_p} \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{2M-1}{1-M}} \left( \frac{dq}{dt} \right)^2 + \\
RC_0 + RC \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{M}{1-M}} + L \frac{I_s}{V_p} \left( \frac{V_p}{V_T} \left( 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right) + 1 \right) \right\} \frac{dq}{dt} + \\
V_p \left\{ \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{M}{1-M}} \exp \left( \frac{V_p}{V_T} \left( 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right) \right) \right\} \frac{d^2q}{dt^2} + \\
\left\{ 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right\} \left( 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right) \right\} \left( 1 - \left( 1 - q \frac{1 - M}{V_p} \right)^{\frac{1}{1-M}} \right) = \varepsilon(t)
\end{align*}
\] (4.8)
The driven term $\varepsilon(t)$ in Eq.4.1,4.3,4.8 is determined by the external driven magnetic field imposed on the SRR structure, which can be expressed as:

$$\varepsilon(t) = -\frac{\partial (BA)}{\partial t} = i\mu_0\omega A H_0 \exp(-i\omega t) \quad (4.9)$$

where $A$ is the area enclosed by the SRR structure, $\omega$ is the angular frequency of the driven signals, and $H_0$ is the external magnetic field perpendicular to the cross-section of the ring.

With a stated driven field $H_0$, we can obtain the time-dependent bias voltage by solving Eq.4.8. Further, we have the current flowing through the circuit related to the bias voltage shown in Eq.4.2; thus we know the time-dependent AC current flowing through the SRR structure. With this information, the magnetic dipole moment from the single SRR structure is calculated as $\hat{m} = \mathcal{I}(t)A$.

And the magnetization of the bulk nonlinear metamaterial medium can be related to the local magnetic dipole moment as:

$$\mathcal{M}(t) = N\hat{m} = N\mathcal{I}(t) \quad (4.10)$$

Till now, the properties assessed are in the time domain with a given excitation field intensity $H_0$ and driven frequency $\omega$. Solving Eqn.4.8 with the driven source in Eqn.4.9 in the time domain. We have a time-dependent $q(t)$, and further we have time-dependent current $\mathcal{I}(t)$ and magnetization $\mathcal{M}(t)$ as well.

Taking the Fourier transformation of macroscopic magnetization $\mathcal{M}(t)$, we have:

$$\mathcal{M}(t) = M(0) + M(\omega) \exp(-j\omega t) + M(2\omega) \exp(-j2\omega t) + \cdots \quad (4.11)$$

Thus, we have the nonlinear magnetization for all frequency components with the driven field at $\omega$. Similar to Ref.[31], we have the first order susceptibility $\chi^{(1)}_{m,eff}$, higher order susceptibilities $\chi^{(2,\cdots)}_{m,eff}$, and the linear properties permeability
Table 4.1: Algorithm to obtain effective material properties through numerical method

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Solving Eq.4.8 in time domain with given driven field $\mathcal{H}(t) \exp(-j\omega t)$</td>
</tr>
<tr>
<td>2</td>
<td>Calculate $\mathcal{I}(t)$ and $\mathcal{M}(t)$ with solved $q(t)$</td>
</tr>
<tr>
<td>3</td>
<td>Fourier transform $\mathcal{M}(t)$ for frequency dependent $M(t)$</td>
</tr>
<tr>
<td>4</td>
<td>Calculate susceptibilities $\chi_{m,eff}$</td>
</tr>
</tbody>
</table>

$\mu_{r,eff} = 1 + \chi^{(1)}_{m,eff}$. Along with the effective nonlinear susceptibilities, this model provides a direct method to analyze the nonlinear response and process in NMMs.

In summary, the algorithm we applied here helps us obtain effective material properties in NMMs by solving the strong nonlinear oscillation Eq.4.8 in the time domain.

The advantage of this numerical method is that we avoid Taylor series expansion for Eq.4.6 and Eq.4.4 to obtain a standard form of nonlinear oscillation equation[31], which could be applied as a perturbation method to obtain magnetization in frequency domain as well[45, 31]. By avoiding the use of both the Taylor series expansion and perturbation method, this algorithm can obtain effective medium model properties at higher power levels, which breaks the limitation for the method in Ref.[31].

There are two conditions to apply this algorithm to obtain an effective medium model for NMMs. First of all, the spice model for circuit components should be valid at the power level and frequency of the external excitation source. At microwave frequency, it is usually true that the bias voltage does not break the threshold where Eq.4.6 and Eq.4.4, since the excitation power we applied is usually under watts, or several watts. And frequency range we are mainly focused on is below 10GHz with free space wavelength $\lambda_0 = 30mm$. The size of lumped component chip is 1mm, thus nonlinear components could be treated as lumped devices and distribution effect is negligible. At higher frequency, un-packaged P-N junction diode or transistor could be represented by its equivalent circuit model even at Terahertz. And we can still
apply the same method here to analyze similar NMM design.

Secondly, the overall unit cell is much smaller than the wavelength the structure is operating. Under this condition, spatial dispersion effect in traditional MM structure is not that obvious, and the spatial dispersion in the nonlinear process is negligible as well.

In the following, we will discuss the resistive effect in two types of nonlinear MM unit structures, one that originates nonlinearity from nonlinear resistance, and another from nonlinear capacitance. We still take SRR structure as the foundation in both analyses; however similar analysis could be applied to other geometries by changing the circuit representation and nonlinear component’s properties accordingly.

4.3 Nonlinear resistive metamaterials

In diode-loaded NMM structures, nonlinear resistance in the diode is the dominant source of nonlinearity. This also affects nonlinear phenomena, such that the transmittance is dominated by damping instead of frequency shifting by changing external excitation signal’s power.

Here, we consider an SRR structure with internal radius \( r = 2 \text{mm} \), outer radius 2.2mm, and copper thickness 17um. The copper SRR structure is patterned on a square FR4 substrate, with edge length 5mm, thickness 200um, and gap width 0.5mm. The overall unit cell dimension is \( 5 \text{mm} \times 5 \text{mm} \times 5 \text{mm} \), thus the volume density is \( N = 1/(5 \text{mm})^3 \), and the enclosed area is \( A = \pi \times r^2 \).

A low diode with low zeros bias capacitance \( MA4SPS402 \) from M/A-COM Tech.[46] was placed into the gap of SRR structures to form the diode-loaded SRR(DLSRR) structure. The junction capacitance for \( MA4SPS402 \) is \( C_0 = 0.045pF \). Together with distributive inductance \( L = 15nH \) from SRR, the DLSRR structure resonates around 5.5GHz.

The key spice model parameters for \( MA4SPS402 \) are \( I_s = 1 \times 10^{-14} \text{A} \), \( V_p = 0.7V \),
$V_T = 0.02585V, C_0 = 0.045pF, M = 0.5$. The effective inductance and capacitance of SRR structure is $L = 15nH, C = 0.7pF$. The effective linear resistance in the equivalent circuits includes two parts: the distributed resistance in the SRR structure and the package resistance from the integrated diode. Usually, package resistance is the dominant source. The overall linear resistance in the equivalent circuit model is $6\Omega$.

Together with the dimension information of SRR and the applied diode, we have the completed parameters for the DLSRR structure. Solving Eq.4.8 for each combination of $H_0$ and $\omega$, we can obtain the time-dependent $q, I$ and $M$.

Here, we investigated the following cases: $H_0$ changes from 1mT to 15mT, and frequency $\omega/2\pi$ covers the range from 5.5GHz to 6GHz. And all solved $I(t), M(t)$, has been transformed to frequency domain. We focus on those frequency-dependent properties changing with varying excitation power.

For each excitation power $H_0$ and frequency $\omega$ combination, three spectra components of $I(\omega)$ are captured here: $I(\omega), I(2\omega)$ and $I(3\omega)$, corresponding to the fundamental, second harmonic, and third harmonic components, respectively, which are used to determine nonlinear susceptibilities later.

As shown in Fig.4.3, the magnitude of current at fundamental frequency $I(\omega)$ decreases as $H_0$ increases. This is usually called the damping effect in NMMs. Damping not only exists at the fundamental frequency, but also at the second and third harmonic frequencies. Even the amplitude of the current increases with the enlarged external excitation field, though the rate of increase diminishes. The damping phenomena in higher-order nonlinear processes indicate that conversion efficiency from the pump signal to the generated harmonic frequency components decreases as the power increases.

Damping phenomena is dominant in nonlinear resistive type MMs. Unlike varactor-loaded SRR structures[31, 9], resonant frequency shifts with changing excitation pow-
er. This variation is one of the major differences between nonlinear resistive type MMs and nonlinear capacitive type MMs.

With the solved frequency component $I(\omega)$ and Eq.4.10, we have the power-dependent magnetization components at fundamental frequency, and also the first order susceptibility $\chi_{m,\text{eff}}^{(1)} = \frac{M(\omega)}{H_0}$ and effective permeability $\mu_{\text{eff}} = 1 + \chi_{m,\text{eff}}^{(1)}$. Thus we have the power-dependent effective permeability for the NMM medium composed by the proposed DLSRR unit cell here.

![Figure 4.3](image1)

**Figure 4.3:** (a) Power dependent current response at fundamental frequency; (b) Current response at second harmonic frequency; (c) Current response at third harmonic frequency.

![Figure 4.4](image2)

**Figure 4.4:** (a,b,c) The real, imaginary and magnitude of power dependent effective permeability for metamaterial medium constructed by DLSRR.

Like in VLSRR an NMM medium[9], power-dependent permeability results in a power-dependent refractive index, and usually causes noticeable changes on the transmission curve at different excitation power levels if a plane wave is imposed on
the MM composite and transmittance is calculated. The origin of power-dependent permeability is a higher-order nonlinear process in nonlinear medium generates signals at fundamental pump frequency. Like the "Kerr effect" in nonlinear optics, a self-modulated third-order process could result in a power-dependent refractive index, and a cascaded second-order process could also result in a field component at fundamental frequency. Here, field components from all the high-order nonlinear processes are considered in this semi-analytical approach, since no Taylor expansion or perturbation method is applied. In contrast to Ref.[9], only second- and third-order process are considered. The approach discussed in this paper could be extended to relative high-power cases, which continues discussion of Ref.[9].

![Figure 4.5: Transmittance shifts with increased excitation power.](image)

The higher order susceptibilities $\chi^{(2)}_{m,\text{eff}}(2\omega), \chi^{(3)}_{m,\text{eff}}(3\omega)$ are dependent on magnetization at the second and third harmonic signals, as:

$$\chi^{(2)}_{m,\text{eff}}(2\omega) = \frac{M(2\omega)}{H_0^2}$$

$$\chi^{(3)}_{m,\text{eff}}(3\omega) = \frac{M(3\omega)}{H_0^2}$$

(4.12)

In fig.4.3, second-order susceptibility is damping as excitation power increases, where the damping effect also exists. This indicates the conversion efficiency from the
pump signal at a fundamental frequency to generated wave at harmonic frequency is decreasing with increasing pump signal power at nonlinear resistive type nonlinear MMs.

4.4 Resistive effect in nonlinear capacitive metamaterials

Taking the same analytic approach for DLSRR unit structures as in the previous section, we analyze a varactor-loaded SRR structure. As reported in prior publications [8, 31], resonant frequency shifting with excitation power is the dominant phenomena in low-power cases, while $H_0 \leq 30$ mT. At higher power levels, both frequency shifting and resonance damping phenomena appear [9, 31]. Here we apply the general model to explain the shifting and damping effect in VLSRR mediums.

Taking the same VLSRR model as discussed in Ref.[9], SRR comes with interior radius $R = 4$ mm, and copper trace width 0.5 mm and gap 1 mm. It is patterned on FR4 substrate with thickness 0.2 mm. The unit cell has an edge length of $d = 10$ mm. Thus the volume density is $N = 1/d^3$. The integrated varactor is $SMV1231$. The key spice model parameters for $SMV1231$ are $I_s = 1 \times 10^{-14}$ A, $V_P = 1.58$ V, $V_T = 0.02585$ V, $C_0 = 2.4$ pF, $M = 0.8$. The effective inductance and capacitance of the SRR structure is $L = 15.7$ nH, $C = 0.01$ pF. The effective linear resistance in the equivalent circuits includes two parts: the distributed resistance in the SRR structure and the package resistance from the integrated diode. Usually, package resistance is
the dominant source. The overall linear resistance in the equivalent circuit model is 2.5Ω.

At low-power excitation, resonant frequency shift is the dominant phenomena, which has been intensively discussed in Ref.[8, 9]. When further increasing the excitation signal’s power, nonlinear resistance phenomena begin to interfere with resonant frequency shifting, Thus, we not only observe resonant frequency shift, but also resonant damping along with frequency shifting, as discussed in Ref.[31, 47]. The resonant frequency shift is explained by a power-dependent refractive index; specifically, a power-dependent effective permeability.

In this case, we analyzed the sample with an excitation field $H_0$ from 1mT to 100mT, and frequency range from 750MHz to 800MHz. The chosen frequency range is around the resonance, where higher conversion efficiency exists for multiple nonlinear processes.

With the given information and conditions, solving Eq.4.8 for the VLSRR sample provides the current flow in VLSRR circuit at the different power level, shown in Fig.4.4.

![Normalized current response at various incident power levels](image)

**Figure 4.7**: Normalized current response at various incident power levels: (a) fundamental frequency; (b) second harmonic frequency; (c) third harmonic frequency.

With solved $I(\omega)$, the macroscopic magnetization is linearly dependent on $I(\omega)$, as in Eq.4.10. Thus, we have power-dependent linear properties $\mu_{m,\text{eff}}$ and nonlinear
susceptibilities $\chi^{(2)}_{\text{m,eff}}(2\omega)$. The result is shown in Fig.4.4. As described earlier, the absolute value for nonlinear susceptibilities decreases when the excitation power is over a threshold, which is presented in the numerical results.

\[
\begin{align*}
&\text{Figure 4.8: The effective permeability and second order susceptibility change with incident power. (a) effective linear permeability;} \\
&(\text{b) effective second order susceptibility.})
\end{align*}
\]

With the effective medium model, we can predict the power-dependent linear transmittance at a relative high-power level. The result is shown in Fig.4.4. This agrees qualitatively with the experimental result discussed below.

\[
\begin{align*}
&\text{Figure 4.9: Power dependent linear transmission.}
\end{align*}
\]
4.5 Experimental result

To verify the observations in the nonlinear resistive type and nonlinear capacitive type MMs, we experimentally examined DLSRR and VLSRR samples with the same configurations as presented above.

We tested the single DLSRR or VLSRR with a small loop antenna, as shown in Fig.4.5. An Agilent PNA-X N5245A network analyzer was used to measure the reflectance and the generated SHG signal from nonlinear elements that reflects back to the testing port of the network analyzer. The power-dependent linear properties for NMMs are presented by measured reflectance from this one port network. Nonlinear processes—second harmonic generation—are presented by measured reflection SHG signal at the source port.

![Test setup with loop antenna.](image)

The testing loop antenna is a rectangular loop antenna with edge lengths equal to 4mm and patterned by a 17um copper. The loop antenna is placed on a 1.3mm-thick FR4 substrate. An SMA adapter is used to connect the loop antenna with a network analyzer. Isolated DLSRR/VLSRR unit structures are placed directly on the top of the loop antenna, as shown in Fig.4.5.

As shown in Fig.4.5, the dominant response in the DLSRR sample is resonant damping. Further increasing the power of DLSRR structure, the resonant response
in the DLSRR structure disappears. As predicted in general model, this phenomena is also shown in Fig. 4.3.

The resonant frequency is slightly shifted from the analytical result. This results from the inductive coupling between the loop antenna and DLSRR structure, which drives the frequency down from 5.7GHz to 5.1GHz.

In the same test setup, the VLSRR sample performs quite differently, due to the nontrivial capacitive effect. As shown in Fig.4.5, the resonant frequency shifts down as the excitation power level is increased, while the resonance shape maintains at the lower power range. In a high-power range, the nonlinear resistive effect plays a more important role, and resonance begins to damp.

Due to the coupling between the loop antenna and VLSRR structure, the resonant frequency shifts from 780MHz to around 1GHz. Under low-power excitation conditions, resonant frequency shifting is dominant, which is contributed by the nonlinear capacitance. At high-power excitation cases, the frequency shifts and the resonance peak damps, which is contributed by the nonlinear resistive effect. Similar
phenomena has been reported in a VLSRR medium composite as well [47, 31].

As shown in Fig.4.5(a) and (b), the resonant frequency shifts in a VLSRR medium composite is in a much narrow range comparing to the single unit testing with loop antenna. This difference results from the different two feeding and sensing mechanisms. In transmission line apparatus, field is a quasi-TEM mode like. Thus its pattern looks like a plane wave and the total power output from network analyzer is almost evenly distributed inside the cross-section. Thus the excitation field $H_0$ intensity is smaller than that in the single feded loop antenna. Also testing apparatus captured an averaged response from all metamaterial units in the composite. While in loop antenna, field interaction between loop antenna and VLSRR is much stronger, and loop antenna is more sensitive to local field from the single VLSRR unit cell.

4.6 Conclusion

In summary, we have presented a general model to analyze the nonlinear response for both varactor- and diode-loaded MM structures. A list of contributions is shown below:
• Proposed the microwave nonlinear metamaterial design by loading diode into metamaterial structure, and experimentally examined its properties;

• Achieved qualitative agreement between numerical and experimental result are observed for the high power responses from both the diode and varactor-loaded SRR structures;

• Explained the resonance damping phenomenon in relative high power excited VLSRR medium by the additional nonlinear resistor in the general circuit model for VLSRR composites;
1D wave generation and propagation in nonlinear metamaterial

Studying of nonlinear processes in natural materials dates back to 1962 when Franken et al. first experimentally observed second harmonic generation by projection an intense beam through crystalline quartz [15]. And the nonlinear material model has been developed later to explain nonlinear process in materials, where nonlinear electric (magnetic) susceptibilities are introduced to describe the nonlinear relationship between polarization (magnetization) and electric (magnetic) field. And scientists have applied this model to analyze various nonlinear phenomena at optical frequencies successfully.

In chapter 3, I have built up analog continuous media model for microwave nonlinear metamaterials. Here, I present experimental studies of nonlinear processes in proposed microwave metamaterial structures, and the analytical model to study those nonlinear processes. Generally, the experimental results have good agreement with the analytical prediction for three nonlinear processes, including second order nonlinear process as second harmonic generation, wave mixing and third order nonlinear process as power dependent refractive index.
Analytical results presented in this chapter are based on the nonlinear transfer matrix method (TMM) and the effective medium model discussed in chapter 3. And metamaterial composite studied here is composed by VLSRR unit structure for its simple Drude-Lorentz form of permeability and easily predictable higher order nonlinear susceptibilities, which have been discussed in Chapter 3.

Additionally, the power level of fundamental excitation signal is assumed to be low enough in both experimental and analytical studies. Hence both the non-depleted pump approximation and Taylor expansion in varactor’s C-V relationship are valid in order to use transfer matrix method in analyzing all those nonlinear properties and obtaining nonlinear susceptibilities from circuit oscillation model. A detailed discussion of this excitation power level effect can be found in Ref.[31].

5.1 Harmonic generation

First of all, this chapter discussed the most fundamental nonlinear process—second harmonic generation. In this example, it is presented the methodology to analyze both the wave generation and propagation in nonlinear metamaterial mediums.

5.1.1 Analytical model

A general effective medium model is introduced earlier in chapter 3 to analyze the harmonic signal generation in nonlinear metamaterial samples, which is in consistency of the effective medium model for linear metamaterials and nonlinear medium model for natural nonlinear materials. Specifically, the second order nonlinear process is studied at the frequency which is twice of the pump signal’s frequency here. This process is well known as second harmonic generation (SHG).

As presented in Ref.[31], the nonlinear susceptibility of SHG process is as Eq.3.28. We have the effective second order nonlinear susceptibility precisely, with the given Drude-Lorentz parameters—\( F, \gamma \) and \( f_0 \), the nonlinear component characteristic
information—a, and unit cell’s dimension information A. These provide an equivalent continuous medium model to study the fundamental wave propagation, second harmonic signal generation and propagation in this artificial nonlinear medium.

About a decade ago, a nonlinear transfer matrix method has been proposed to analyze the nonlinear wave generation and propagation under the non-depleted pump approximation (NDP)[48]. It assumes the power is negligible which goes to harmonic signal from fundamental pump signal, which works well at extra-low case. Recently, a revised transfer matrix method specifically designed for SHG process and second order nonlinear susceptibility retrieval has been developed[49]. Besides retrieving nonlinear susceptibilities, the same method can predict the generated harmonic signal (magnitude and phase) in both the forward and backward directions relative to the propagation direction of the pump signal, provided that the material’s linear and nonlinear properties are given. Extended works have introduced this method to other nonlinear processes as well[50], and oblique incident case.

In the following sections, the analytical data of the generated second harmonic signal in the forward direction is predicted by applying the transfer matrix method[49] on the metamaterial composite who is represented by its effective nonlinear medium model and its material properties are in Chapter 3. The nonlinear metamaterial composite has 3×13 unit cell in transverse direction and 1 unit cell in propagation direction, while unit cell edge length are 10mm. Hence, the artificial composite is modeled as a single layer nonlinear medium with a thickness of 10mm in the propagation direction and infinite dimension in transverse directions when the pump signal is at 800MHz, as shown in Fig.5.1.1(b).

5.1.2 Experimental result

To explore the harmonic generation in the nonlinear metamaterial composite, the same VLSRR medium discussed in chapter.3, and its effective medium model has
been used for analysis as well. A transmission line apparatus shown in Fig. 5.1.2 have been designed. And the presented transmission linear apparatus is designed to provide a better transmission property at the harmonic signal frequency, at a cost that only half of the power into the apparatus is confined in effective area to excite the hosted nonlinear medium. The redesigned waveguide with material inside is shown in fig. 5.1.2.

A 12dBm microwave signal with frequency swept from 700 to 900 MHz is pumped into the transmission line apparatus from an Agilent E8267D vector signal generator. Harmonic signals generated from the signal generator as noise are removed by a low

---

Figure 5.1: (a) The investigated nonlinear composite; (b) its equivalent nonlinear medium model; (c) the effective second order susceptibility when the space between elements is 10mm (solid line: real part; dash line: imaginary part).

Figure 5.2: Redesigned transmission line apparatus for both fundamental and SHG signal transmission.
pass filter, $VLF - 800+$, before it reaches the metamaterial slab. The forward propagated component of the generated SHG signal in the VLSRR metamaterial is measured by an Agilent E4440A spectrum analyzer. Meanwhile the backward propagated component of the generated SHG signal is eliminated by a fixed 10dB attenuator, as shown in Fig.5.1.2.

Please note, it is critical to use an attenuator in the experimental setup to remove the backward propagation SHG signal generated from NMM sample. Otherwise, the backward propagation SHG signals will be reflected by the low pass filter and interference with forward propagated signal. This results a interference pattern on the measured data from spectrum analyzer. This issue has been reported in Ref.[14].

An average loss around 12.34dB exists before the pump signal reaching the medium within the pump signals’ frequency range. And another 0.97dB loss exists for the SHG signal between transmission line apparatus and the spectrum analyzer in the generated harmonic signal frequency range. Magnetic field intensity in the transmission line apparatus is calculated through $H = \sqrt{2 \frac{P}{Z_0 S}}$ which is used in the transfer matrix method analysis, where $P$ is the corrected power level of either pump signal or SHG signal, $Z_0 = 377\Omega$ is the free space impedance, and $S = 18\text{cm}^2$ is the cross-section of the waveguide.

![SHG test setup](image_url)
Both results from the experiments and the analytical analysis are shown in Fig.5.1.2. A good agreement between the experimental result and analytical prediction is reached here. And as presented in Ref.[14], the retrieved second order susceptibility from experimental result also matches the analytical result.

Predicted by the oscillation system model, the second order nonlinear process reaches its maximum when the fundamental frequency is at the resonant frequency. This is indicated by the value of second order susceptibility in Eq.3.28 and Fig.3.2.2(c) as well. When each element in the composite is excited by the external pump signal, the field strength in each element is much larger on resonance than off-resonance. This local field provides a stronger driven source on the nonlinear components or material at resonance. As a result, a strong nonlinear response is detected when the pump frequency is close to the element resonant frequency.
5.1.3 Enhanced second harmonic generation

In the previous sample, the space between adjacent elements is designed to be the same as unit structure edge length, in which case the coupling is weak and the oscillation circuit model in chapter 3 could describe the quasi-static case of the VLSRR element in good accuracy. However, by reducing spacing between adjacent unit cells, the volume density \( N \) is increased. And the magnitude of nonlinear susceptibility increases as well. Thus it is predictable to excite a stronger nonlinear response by reducing the spacing between elements in the composite.

Here, one direct enhancing method has been taken by reducing the distance between the adjacent elements in the same direction as the magnetic field. It is a preferred orientation since the major field component is magnetic field when the VLSRR element is resonating. The orientation is shown in Fig.5.1.1(a). The separation distance \( d \) ranges from 6 to 10mm are investigated here.

The effective linear properties of the nonlinear metamaterial composites are retrieved with the numerical model presented in Chapter 3 for different configuration. For each value of \( d \), the retrieved linear material properties are shown in Fig.5.1.3 (a, b). Those properties are identical to the configuration when \( d = 10 \text{mm} \). The effective permittivities are almost constant and the effective permeabilities follow the classical Drude-Lorentz form. And the predicted effective second order susceptibilities is shown in Fig.5.1.3 (c). And table5.1 shows the fitted Drude-Lorentz parameters from the numerical result of permeabilities.

Frequency tuning has been observed which is due to the change of the parasitic capacitance from adjacent elements and the coupling effect between adjacent elements. Similar result has been reported elsewhere[9]. More interestingly, the resonant response increases as the separation decreasing. So does the effective second order nonlinear susceptibilities.
Figure 5.5: (a) The effective permittivities; (b) the effective permeabilities; (c) the predicted second order susceptibilities. In all three figures, solid line is the real part; dash line is the imaginary part; and color from dark, purple, light blue, dark blue and green present separation distance $d$ from 10mm to 6mm.

Table 5.1: Retrieved Lorentz parameters for different spacing configuration

<table>
<thead>
<tr>
<th>$d$(mm)</th>
<th>$f_0$(MHz)</th>
<th>$F$</th>
<th>$\gamma$</th>
<th>$\epsilon_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>780.7</td>
<td>0.1955</td>
<td>$18.1e6 \times 2\pi$</td>
<td>2.02</td>
</tr>
<tr>
<td>7</td>
<td>792.2</td>
<td>0.1805</td>
<td>$20.7e6 \times 2\pi$</td>
<td>1.88</td>
</tr>
<tr>
<td>8</td>
<td>803.8</td>
<td>0.1617</td>
<td>$20.7e6 \times 2\pi$</td>
<td>1.77</td>
</tr>
<tr>
<td>9</td>
<td>811.8</td>
<td>0.1468</td>
<td>$21.1e6 \times 2\pi$</td>
<td>1.69</td>
</tr>
<tr>
<td>10</td>
<td>817.6</td>
<td>0.1305</td>
<td>$20.5e6 \times 2\pi$</td>
<td>1.63</td>
</tr>
</tbody>
</table>

With the effective medium model for the various samples and the test setup in previous section, both the analytical and experimental studied have been conducted. The result is shown in Fig.5.1.3. The magnitude of the transmitted SHG signal in experimental result matches the analytical result coming from transfer matrix method with the given linear and nonlinear material properties. And the nonlinear response becomes stronger as the excitation frequency get closer to the resonant frequency for each medium, as indicated by the properties of second order susceptibility.

However, this characterization process is only valid when the nonlinear resistance in the varactor is negligible—comparing to nonlinear capacitance—as in the case $d = 7,...,10$mm. A strong local field exists when $d = 6$mm, which drives the varactor
to exhibit higher resistance than that used in the numerical model. This increased resistance affects the evaluation of the damping factor $\gamma$ in Eq.3.27 and Eq.3.28. Smaller $\gamma$ is predicted in numerical model when $d = 6\text{mm}$ than in reality. As a result, the analytical prediction of SHG signal exceeds the experimental result, especially around the resonant frequency where stronger effect from the increased resistance exhibits. Hence, the system error increases. And a modification of the linear resistance $R_{\text{var}}$ to $2.5\Omega$ from $2.05\Omega$ provides a better prediction in analytical model.

In both the analytical result of $\chi^{(2)}_{\text{eff}}$ and the analytical/experimental result of the generated SHG signal, the nonlinear response increases with increasing the volume

---

**Figure 5.6**: (a) The magnitude of effective second order nonlinear susceptibility; (b) The analytical and experimental result of the generated second harmonic magnetic field intensity (in mT) with different configurations of artificial medium. Dashed line is the analytical result, and solid line is the measurement result. Colors from green to dark represent the artificial medium with space factor $d=7, 8, 9, 10\text{mm}$.
density which is physical realized by reducing the space $d$ between adjacent elements.

5.1.4 Conclusion

The agreement between the analytical and experimental results for second harmonic process in VLSRR nonlinear metamaterials demonstrates that the nonlinear response of VLSRR metamaterials can be effectively described by their effective medium model. And the effective linear and nonlinear material properties can be determined precisely in linear numerical study or in fitting an experimental result. The main properties are permittivity, permeability as in Eq.3.27 and nonlinear susceptibilities as in Eq.3.28 in the effective medium model. In addition, the linear and nonlinear properties of VLSRR medium can be predicted through a numerical study of the VLSRR elements in numerical simulator, like COMSOL multiphysics, which provides a convenient method to design and optimize nonlinear metamaterial medium for more complicated applications. In addition, the nonlinear response in the nonlinear medium can be enhanced by increasing the volume density, which provides a mechanical method to adjust the effective properties of nonlinear mediums.

5.2 Three wave mixing

Wave mixing, as the subject of the many advanced device technologies in optical equipment, is widely used in spectroscope[51], remote sensing[52], optical radar[53] and various other optical applications. In these applications, strong optical pump signals are usually present as sources since the interaction between light and material is usually weak in natural materials.

Here, the experiment study of wave mixing process in nonlinear magnetic metamaterial has been presented. Both three-wave mixing and four-wave mixing phenomena have been observed in our experiments.
5.2.1 Analytical model

The analysis of the SFG process in the nonlinear metamaterial composite is conducted on the same VLSRR metamaterial slab we studied in the previous chapters. We use a nonlinear effective medium model with the effective linear properties remains the same as discussed in the previous chapters. And the effective nonlinear material property $\chi^{(2)}_{\text{SFG}}$ at the sum frequency $f_n + f_m$ is applied here instead of the second order susceptibilities at the harmonic frequency in previous section, which is shown in Eq.5.1[31].

$$\chi^{(2)}_{\text{SFG}}(f_n + f_m) = i a f_0^4 \mu_0 A F \frac{f_n}{D(f_n)} \frac{f_m}{D(f_m)} \frac{(f_n + f_m)}{D(f_n + f_m)}$$ (5.1)

where $a, A$ and the Drude-Lorentz parameters $F, f_0, \gamma$ in $D(f)$ are the same as previously discussed. $f_n$ and $f_m$ are the frequencies of the two pump signals. The overall medium model is displayed as below:

A transfer matrix method specifically designed for nonlinear wave mixing processes, previously published in Ref.[50], has been used to estimate the generated SFG

Figure 5.7: The metamaterial element (a), composite (b) and its effective nonlinear model (c) for the analysis of wave mixing process.
signal strength with the given effective medium model.

5.2.2 Experimental result

The same material composite discussed for SHG process is studied here. And the separation between elements is 10mm.

The experiment setup is shown in Fig.5.2.2, which is to investigate the cross modulation process of the artificial nonlinear medium. The nonlinear metamaterial medium is hosted in the same transmission line apparatus as used for SHG experiments, which supports TEM mode over a broad frequency range covering both the fundamental and SFG frequencies. Two RF pump signals are combined with a Mini-circuits ZAPD-1+(RF combiner) to form a single RF signal to excite the nonlinear metamaterial simultaneously. They are generated separately from one Agilent PSG8267D and one Agilent N9310A. One low pass filter, LPF-800+ from Mini-circuits, has been placed between the combiner and the TL apparatus to block the harmonic signals generated from the two RF signal generators, which are noise in the system. The signal spectrum of the TL apparatus’ output is analyzed by an Agilent E4440A spectrum analyzer. Three-wave mixing and four-wave mixing signal generation have been detected in this experiment.

Two different RF signals, with one’s frequency at 800MHz and another’s frequen-
Figure 5.9: (a) the transmission line output spectrum without nonlinear metamaterial slab inside. (b) The transmission line structure output spectrum with nonlinear metamaterial slab inside.

Multi-wave mixing phenomena have been observed in this experiment. The measured output signal spectrum covers the frequency range from 700MHz to 1.9GHz. Second harmonic signals $2f_n$ and $2f_m$ with power level at $-27$dBm and $-40$dBm...
have been captured in the output spectrum. The difference in the two harmonic signals’ power results from the frequency dispersive nonlinear susceptibilities which determines the conversion efficiency, as described in previous section and also in Ref.[14]. Four-wave mixing signals \( 2f_n - f_m \) and \( 2f_m - f_n \) have also observed in the output signals’ spectrum. Even higher order wave mixing processes, five-wave mixing signal at \( 3f_n - f_m \), also present in the output signal spectrum.

It should be mentioned that the higher order nonlinear processes may be a result of the cascaded lower order nonlinear processes. Such as the three wave-mixing products \( 2f_n - f_m \) can be the result of cascaded second order nonlinear responses, like \( 2f_n - f_m = f_n + (f_n - f_m) \), which is the pump signal \( f_n \) mixed with the differential signal product \( f_n - f_m \) in a second order sum frequency generation process. Thus, this cascaded effect needs to be considered when we study the high order nonlinear processes analytically. In the following study of SFG, there is no such consideration since it is the lowest nonlinear process presented in the nonlinear composite. And we ignore the cascaded effect from processes which requires three or more orders of cascading process, since their effects are small enough at the power level in this experiment. Related discussions have been published in Ref.[31].

Take the SFG for example, this process is investigated by exciting the nonlinear metamaterial medium with two RF pump signals \( f_n \) and \( f_m \) from generators. The same experimental test setup as the one for the wave mixing experiments has been used here with an extra 10dB attenuator placed between the low pass filter and the TL waveguide, as shown in Fig.5.2.2. This attenuator plays the same role as the one used in SHG experiment, which eliminates the generated backward propagated SFG signal in the test environment. Two signals’ frequencies sweep from 740MHz to 890MHz, with 2MHz step for pump signal at frequency \( f_n \) and 10MHz step for pump signal at frequency \( f_m \). Both of the two pump signals’ powers are at 12dBm which are measured out from the signal generators. With an average loss about 15.78dB,
Figure 5.10: SFG signal power with pump wave frequencies sweeping. (a) SFG signal when one pump signal frequency fixed at 820MHz, another pump signal sweeps; (b) SFG experimental result when two pump signals are sweeping; (c) Analytical result of SFG signal when two pump signals both are sweeping.

The actual pump signals’ power reaching the nonlinear metamaterial is $-3.78\text{dBm}$. These pump signals correspond to magnetic field intensities of 19.6mA/m through the conversion formula we used in the previous section.

Identical to SHG process in this nonlinear metamaterial[14], strong nonlinear response has been detected in SFG process near the resonant frequency $f_0 = 816\text{MHz}$, as shown in Fig. 5.2.2(a, b). The SFG signals reach their maximum when the pump signal frequency reaches $f_0$ as in Fig.5.2.2(a). A global maximum of the SFG signal is achieved when both two pump signals are close to the resonant frequency, which corresponds to the central red spot in Fig.5.2.2(b). A maximum of 0.36mA/m SFG signal is detected in the nonlinear metamaterial sample.

5.2.3 Conclusion

In conclusion, multi-wave mixing processes in the nonlinear metamaterial medium have been verified experimentally. And an analytical analysis of the sum frequency
generation process is presented based on its EMM and the TMM for SFG process. The wave mixing nonlinear phenomenon can be analytically studied with their EMM with both linear and nonlinear properties of the composite medium. This provides a useful tool to further investigate other wave mixing phenomena in artificial nonlinear mediums.

5.3 Frequency tuning

With the effective medium model of the nonlinear metamaterial composites, it is first investigated the resonant frequency shifting in the artificial medium when the pump signals’s power varies. This frequency tuning mechanism shares the same physical model as the third order self-phases modulation in optical nonlinear materials, which is also known as ”Kerr effect” when the nonlinear modulation results from nonlinear electrical polarization. With the information of the third order nonlinear susceptibility at the fundamental frequency, the analytical prediction of the resonant frequency shifting matches the experiment result when the excitation power is low. Good agreement have been found both in the single varactor loaded VLSRR medium and the double varactor loaded VLSRR medium. Later, relatively high power excitation experimental results demonstrate broad range of resonant frequency tuning could be achieved in these VLSRR composites.

5.3.1 Analytical model

We consider the self-phase modulation in the third order nonlinear processes as our interests here, which leads to the effective linear material properties of nonlinear metamaterials depending on the external excitation signal’s power. Two different VLSRR metamaterial composites are considered here, which are formed by single varactor or double varactor loaded VLSRR element, as shown in Fig.5.3.1.

For single varactor VLSRR element composed metamaterial, the effective third
order nonlinear susceptibility at the pump frequency $\omega$ can be expressed as [31]:

$$\chi_{\text{single}}^{(3)}(\omega; \omega, \omega, -\omega) = \frac{F_0 \omega^4 \omega^2 A^2 \mu_0^2}{D(\omega)^2 D(-\omega)} \left( \frac{4a^2 \omega^2}{D(0)} + \frac{2a^2 \omega^2}{D(2\omega)} - b \right)$$

(5.2)

where $N$ is unit volume density of magnetic dipole moment, and $D(\omega), a, A$ are the same as in Eq.3.28. Thus the third order susceptibility for single varactor loaded nonlinear metamaterial composite can be solely determined by the same Drude-Lorentz parameters.

For the double varactor loaded VLSRR composite, the signs of the bias voltage are opposite from their back-to-back connection (Note: small variation due to the distributed resistance is ignored in our analysis here.) As a result, the voltage dependent capacitance has changed in this case. Thus, the nonlinear susceptibility at the pump signal frequency is as:

$$\chi_{\text{Dual}}^{(3)}(\omega; \omega, \omega, -\omega) = \frac{F' \omega_0^4 \omega^4 A^2 \mu_0^2}{D'(\omega) D'(-\omega)}$$

(5.3)

where $D'(\omega) = \omega_r^2 - \omega^2 - i\gamma \omega$, and $\omega_r = 1.215GHz \times 2\pi$ is the resonant frequency of the double varactor VLSRR element. $F' = 0.1362$. $\omega_0$ is the resonant frequency of the single varactor loaded element. The opposite sign of $\chi_{\text{eff}}^{(3)}$ between Eq.5.2 and
Eq. 5.3 implies that the frequency shifts to different direction in these two samples.

Thus we have the intensity dependent effective permeability considering the third order nonlinear product as below, which is in analog to the intensity dependent refractive index at optical frequencies.

$$\mu_{eff}^{NL} = 1 + \frac{F\omega^2}{D(\omega)} + 3\chi_{\text{single/double}}^{(3)}(\omega)|H|^2$$

(5.4)

where $H$ is magnetic field intensity of the pump signal exciting the sample.

Combining the power dependent effective permeability with the constant linear permittivity, the transmittance and reflectance for wave propagated through the NMM slab are predicted by the linear transfer matrix method when we assume a plane wave propagating through the effective nonlinear medium. As a result, we can control and tune the resonant frequency of the medium by varying the pump signal power, which affects the effective permeability as in Eq. 5.4. In low power case, such as $H = 0$, Eq. 5.4 turns back to the classical Lorentz model as shown in Eq.3.27[32].

Due to the limitation of the varactor model and perturbative method used in the derivation process, the current effective medium and the associate formula to describe the material properties are only applicable when the sample is excited by a relative small power, up to $H = 20\text{mA/m}$ for composite with single varactor loaded SRR and $100\text{mA/m}$ for composite with dual varactor loaded SRR from our investigation[31]. In the following discussion, the presented experimental results are collected when the pump signal powers are under these limitations.

5.3.2 Experimental result

In the composed nonlinear metamaterial sample, VISRR elements have been periodically placed in the $E$ and $H$ direction, and one layer is placed in the wave propagation direction ($k$). And the total unit cell is $3 \times 15 \times 1$. The fabricated metamaterial mediums are presented in Fig.5.3.2 (a). In the first sample, one varactor is integrated
Figure 5.12: Nonlinear medium sample and waveguide testing setup, (a) Original SRR medium and two varactor VISRR medium; (b) Sample in waveguide testing assemble

on the gap in the SRR structure. In the second sample, an additional gap (double split SRR) is created to integrate the second varactor, and two varactors’ cathodes and anodes are connected by copper strips. The dimension information of both the VLSRR element is the same as shown in Fig.5.3.1.

Transmission line apparatus which supports TEM wave propagation below 2GHz is used to measure the transmittance when wave propagates through the nonlinear magnetic metamaterial samples, as in Fig.5.3.2(b). Transmittance is measured by an Agilent network analyzer N5230. In this test setup, the frequency dispersion and system loss and phase variation from the utilized TL apparatus, amplifier and the connecting cable have been calibrated in the network analyzer.

Transmittance is measured at different network analyzer’s output power levels.
The output power increases from −10dBm to +15dBm. And the total loss from the TL apparatus, the connection cables and adapters was measured to be 4.6 dB from 500MHz to 1GHz. So the actual power exciting the sample from −14 to 11 dBm, which falls in the valid power range for the effective medium model. The resonant frequency of the VLSRR composites shifts down in single varactor nonlinear medium when increasing the incident power, which is identical to the single unit cell case presented in[8], as in Fig.5.3.2(a). Oppositely, the magnetic resonant frequency shifts up in double varactor VISRR medium with incident power increase, as in Fig.5.3.2(b).

![Increasing Power](image)

**Figure 5.13**: Experimental result of the power dependent frequency tuning. (a) Single varactor VLSRR sample; (b) double varactor VLSRR sample.

### 5.3.3 Analytical Result

Based on the medium model proposed in chapter 3, this power dependent frequency shifting can be explained by the third order self-phase modulation in the nonlinear medium.

With the given information for both the effective linear and nonlinear material properties, the transmission properties for both single and double varactor loaded metamaterial have been analyzed. The results shown in Fig.5.3.3 cover several chosen
pump signal power levels, which correspond to the comparison case in experiments. As predicted in the previous text and also shown in the published experimental result[8], the frequency shifting in the two analyzed sample are in different directions.

A low power excitation experiment has been performed to compare the analytical prediction and the experimental data. Pump signal power ranges from $-16\text{dBm}$ to $-1\text{dBm}$ excluding the system loss. Divergence between the analytical result and experiment result appears when further increases the pump signal’s power, which has been explained in our previous statements that the current model is only valid in relative low power case.

The comparison between the analytical and experimental transmission data demonstrates that the varactor loaded nonlinear metamaterials can be described by their effective second and third order nonlinear susceptibilities, for values of the excitation magnetic field magnitude up to $27\text{mA/m}$ and $21\text{mA/m}$ for the single-and double-varactor VLSRR nonlinear metamaterial, respectively.

**Figure 5.14**: Analytical result of the frequency shifting in single (a) and double (b) varactor loaded nonlinear metamaterial composites.
Figure 5.15: Comparison between experimental and analytical result of the intensity dependent resonant frequency shift. (a) Small power experimental result for single varactor loaded sample; (b) small power experimental result for double varactor loaded sample; (c) comparison for single varactor loaded sample; (d) comparison for double varactor loaded sample.

5.3.4 Conclusion

In this chapter, we applied the effective medium model presented in chapter 3 to explain three nonlinear processes in a thin layer VLSRR magnetic metamaterial sample. The key contributions are listed below:

- Experimentally investigated the second harmonic generation nonlinear process in VLSRR composites, and applied the effective medium model to analyze this process by analog VLSRR composite as a continuous medium;

- Experimentally studied the three wave mixing nonlinear process in VLSRR composites, and applied the same effective medium model to analyze this pro-
cess analytically, good agreement between experimental and analytical result is reported.

- Experimentally studied the frequency tuning in VLSRR nonlinear composites with different power level of excitation wave, and analytically explained this process by a third order nonlinear process at the fundamental frequency, in which power dependent effective permeability is applied.

In sum, the good agreement between the experimental and analytical results for the three nonlinear processes validates the effective medium model of microwave nonlinear metamaterials.
6.1 Introduction

Metamaterials, as artificial composites, represent an important extension to conventional natural materials. Novel and unusual phenomena have been demonstrated in metamaterials, which include but are not limited to negative refraction, evanescent wave amplification, perfect lensing, electromagnetic cloaking, etc. While some recent studies involve wave propagation and refraction in/out of metamaterial composites in one dimension (1D) only, such as evanescent wave amplification and tunneling effects in zero index materials, other phenomena are fundamentally two- or three-dimensional (2D, 3D), such as negative refraction, gradient index lensing and cloaking. Thus, there is a need for test systems operating not only in a 1D environment, but also in 2D or even 3D environments, such as the 3D Luneburg lens.

In 1D, there are a number of methods for studying wave propagation, including the transmission line apparatus which supports a quasi-TEM mode over a broad frequency range, or metallic waveguides for selected modes and frequency combina-
tions. In the 2D case, the 2D field mapping apparatus, in which the sample and interacting waves are confined between two mobile conducting plates, has proven highly successful in visualizing a number of novel metamaterial-related phenomena, such as the first visualization of negative refraction, and the first validation and visualization of the metamaterial-based electromagnetic cloak. The 2D field mapping system is an ideal setup to explore EM wave propagation in complex and inhomogeneous materials and geometries. In the 3D case, anechoic chambers are often used to visualize scattering from objects and antenna systems, such as the Luneburg lens, but with a correspondingly high increase in the complexity of the setup and sample requirements.

Metamaterials are artificial material composites, typically periodic metallic patterns, arranged in such a way as to interact with electromagnetic radiation in a desired and often unconventional manner. In the exploration of metamaterials, effective medium models represent a wide-spread and convenient set of methods for studying EM wave interaction with the artificial composite. In the context of effective properties, metamaterials open previously unavailable parameter space, for the control or guiding of EM waves, and have enabled such advancements as transformation optics and negative refraction.

In addition to this flexibility in the linear response, metamaterials can be hybridized with nonlinear and active components, vastly extending their achievable parameter space. The key element of nonlinear metamaterials is that their artificial structure offers a means to enhance the weak optical nonlinearities of natural materials. This enhancement originates from the inhomogeneous field distribution within metamaterial structures: extremely strong local fields can accumulate in critical volumes, which offers increased sensitivity of the overall metamaterial to the local properties, as well as enhancement of the local nonlinear response.

A number of nonlinear process have been studied experimentally and analytical-
ly in nonlinear metamaterials, especially at microwave frequencies, including second harmonic generation, wave mixing, power dependent refractive index, etc. Good agreement between experimental results and nonlinear effective medium models has been reported, demonstrating the effectiveness and convenience of effective medium models in analyzing, optimizing, and projecting nonlinear processes in metamaterials. In the mentioned experiments, most of the nonlinear measurements took place in effectively 1D testing setups, such as transmission line apparatuses or metallic waveguides. For example, second harmonic generation and wave mixing in thin nonlinear metamaterial slabs has been performed in a transmission line apparatus which supported a quasi-TEM like mode at both the fundamental and harmonic frequencies. Similarly, phased matched second harmonic generation over macroscopic distances has been measured in a custom designed metallic waveguide. In both cases, the experimental setups restricted measurements to just nonlinear wave generation in the transmitted and reflected directions.

Recently, however, several novel predictions in nonlinear metamaterials demand 2D studies of wave generation, propagation and scattering, including time reversal enabled negative refraction, and loss-compensated negative refractive super lensing. Another interesting finding is the refocusing of a second harmonic signal in a medium possessing a negative index at the harmonic frequency but a positive index at the fundamental frequency. In order to experimentally study these predictions, a new testing system is required, capable of performing non-invasive field measurements over a 2D domain.

Here, we propose a new field mapping system capable of measuring the phase and amplitude of the electric field at all mix and harmonic frequencies in a particular nonlinear process over a macroscopic 2D domain. In what follows, we detail the system requirements and capabilities, validating our procedure by measuring the dipole-like harmonic field radiation pattern from a single nonlinear metamaterial
unit structure. Furthermore, we present field mappings of the harmonic generation inside a bulk nonlinear metamaterial, as well as the resulting scattered field into free space, finding excellent agreement with numerical results.

6.2 Nonlinear Field Mapping System

The field mapping apparatus that we utilize is similar in concept to the system employed in Ref. [54]. By confining the sample and electromagnetic waves between two conducting plates, the scattering is reduced to an effectively two dimensional domain, in which only the fundamental TEM mode is excited, provided the exciting and mapping frequencies are kept below the cut-off frequencies of all higher-order modes. Coaxial feeds in both plates serve as the source and measurement ports, and are sufficiently mismatched to the fundamental mode to avoid significant perturbation of the local fields. By manipulating one plate with respect to the other, relative motion between the source and measurement ports is achieved, giving access to the local fields over a relatively large, discrete 2D domain.

When unloaded, the fundamental TEM mode is invariant in the z-axis of the mapping chamber, such that the field just above either conducting plate is equivalent to the field in the structure. Thus it is convenient to measure the field at the top of the structure through a series of coaxial feeds. Although adding an inhomogeneous material breaks this z-axis invariance, the boundary condition enforced by the conducting plates ensures that the local fields are indicative of the desired macroscopic scattering. In the linear 2D field mapping chamber[54], the distance between two conducting plates is 11mm (distance along z axis). Often it is the case that structures fill 10mm of the height, and 1mm is intentionally reserved for the measurement probes to ensure there is no contact between the probes and the structure as the plates are moved relative to each other.

In the 2D field mapping chamber, two common excitation methods are used. One
of them involves excitation from a waveguide adapter, which produces a planar beam (plane wave like) eventually after correction. This excitation method is widely used in the experiments for negative refraction, cloaking and lensing demonstrations. The frequency range of this method, however, is limited by the waveguide adapter. The other method of exciting waves in the 2D chamber involves a center coaxial feed, which produces an effectively cylindrical wave. This method provides a standard radial wave across a wide frequency range, limited by the frequency range of the chamber itself. Due to its broad operational frequency range, we employ the central coaxial feed method for the nonlinear field mapping apparatus. If a planar beam is required in some measurements, however, there are several other methods for converting a cylindrical wave into a planar wave, such as reflection lenses or focusing lenses.

In our proposed setup, the bottom plate is used to support the source feeds and the sample, and is itself supported by a computer controlled linear stage. This stage performs automated scanning of the bottom plate, while the top plate, containing the measurement ports, is fixed and unmovable. In the linear field mapping apparatus, a vector network analyzer (VNA) provides the source microwave signal and phase sensitive detection of the measured signals. A customized program controls the stages and stores data (in the form of S-parameters) from the VNA.

In the nonlinear field mapping chamber setup, we utilized the same mechanical setup as in the linear field chamber, including the two conduction plate which supports TEM mode; automated linear stage which provides automatically scan; and the center feeding method which provides identical transmission and refection characteristic at both fundamental and harmonic frequencies. However, we employed the PNA-X with model number N5245A, whose receiver design not only measures the signal at the same frequency at incident signal and source port, but can also measure absolute phase and amplitude at the harmonic and mix-frequencies.
The PNA-X, has been applied to a series of nonlinear metamaterial studies, such as harmonic field measurement, difference frequency generation, etc. It is well-suited for nonlinear field mapping in the current case, since it provides the capability of measuring nonlinear field components at the source and receiver ports. With an optimized port receiver design, the harmonic noise is -10 to -20 dB lower than that in traditional VNA. This avoids filters or attenuators previously required for sensitive nonlinear field measurements[47].

6.3 Design of Cband Nonlinear Metamaterial – VLSRR

Our implementation of the 2D mapping chamber has a bottom plate with dimensions 91.5cm × 91.5cm, a scan area of 54cm × 36cm, and the frequency limited feeding method. As an initial demonstration, we studied the varactor loaded split ring resonator(VLSRR) with resonant frequency in C band. The VLSRR represents the canonical microwave nonlinear metamaterial, in which the inductively driven LC resonance of the split copper ring is coupled to the nonlinear properties of a varactor-diode, whose capacitive properties are dependent on applied voltage. For example, this VLSRR sample has been used previously to demonstrate second-harmonic generation. For excitation near the VLSRRs fundamental resonance of 4.5GHz, we expect to measure and map the second harmonic frequency around 9GHz, verifying our experimental field mappings with those obtained from numerical effective models.

The proposed VLSRR unit cell is composed of a patterned copper SRR on Fr4 substrate and a flip chip varactor. The SRR structure consists of a 17um thick copper ring with inner diameter 3mm, trace width 0.25mm, and a 0.25mm gap. These are arranged periodically with a lattice constant of 5mm in all directions. The flip-chip varactor MA46H146 is integrated into the gap of each SRR structure. MA46H146 is chosen for its compact size, low zero bias capacitance and high working frequency. This type of varactor’s internal resonance is as high as 20GHz, and
so does not significantly interfere with the resonance from VLSRR unit structure. Commercialized bonding service ensures symmetric connections between flip chip varactors and the SRR copper traces.

As in other low frequency VLSRR structures, the nonlinear properties originate from the loaded varactors. Thus it is important to know the nonlinear parameters for MA46H146 flip chip varactor, including the zero bias capacitance $C_0 = 0.0421 \text{pF}$, gradient coefficient $M = 0.5$, and intrinsic voltage potential $V_P = 1.2 \text{V}$, and the package capacitance $C_P = 0.0209 \text{pF}$.

A transmission line apparatus (TLA), as shown in Fig.6.1.(a), is specially designed to fulfill testing linear and nonlinear response for this VLSRR medium. The TLA supports quasi-TEM like mode between 4 – 12GHz, thus it fits for both linear transmittance and second harmonic signal testing.

![Transmission Line Apparatus](image)

**Figure 6.1:** (a) Transmission line apparatus for transmittance and second harmonic generation measurement; (b) Transmittance comparison between the experimental result and transfer matrix result.

The characterization method for microwave nonlinear metamaterials has been well established[55]. By parametric fitting of the experimental linear transmission curve obtained at extremely low power, Lorentz parameters are extracted for modeling the linear permeability (see Fig.6.1(b)). Using the nonlinear properties of the varactor, the nonlinear susceptibilities are determined as well. However, nonlinear processes in these nonlinear metamaterials are known to be very complicated, of-
ten containing higher order and cascaded nonlinear processes usually contribute to low order processes. Second harmonic generation, for example, can be the result of a second-order process, a cascaded second order process, as well as a fourth order process. However, for sufficiently low power excitation, effects from higher order or cascaded processes are usually negligible compared to the lowest order. In what follows, it is assumed the fundamental nonlinear process is the only source of the desired nonlinear process, simplifying the subsequent discussion.

The linear properties from the extraction procedure are shown in Fig.6.2. A standard Lorentz shape presents in the effective permeability, since the dominant resonance is inductively driven. An anti-resonance effect presents in its effective permittivity property in this VLSRR medium since its unit cell size is relatively large comparing to the wavelength at its resonant frequency, beyond the $1/10$th threshold.

![Figure 6.2: Effective medium properties for Cband VLSRR composite.](image)

**Figure 6.2**: Effective medium properties for Cband VLSRR composite, (a) effective permittivity, (b) effective permeability, (c) effective second order susceptibility

Similar to other microwave nonlinear metamaterials, stronger nonlinear process is predicted to exist around the resonance frequency. Thus we anticipate strong harmonic generation when excited near the resonance frequency, which is confirmed later in experimental result.
6.4 Harmonic field mapping of a point source

To assess the performance of the nonlinear field mapping apparatus, a nonlinear field generated from a point source is tested in the 2D chamber. A similar testing procedure presented in Ref.[54] for a point source is launched here.

The test setup is shown in Fig.6.3. A single C-band VLSRR unit cell is placed close to the center feeding probe at the bottom plate. Thus cylindrical waves are launched from the feeding probe, exciting the C-band VLSRR unit, which in turn generates harmonic waves which propagates into the chamber. The sensor probe at the top plate is used to measure both fundamental signal and harmonic signals while the bottom plate is moving spatially. Absorbers have been placed around the outer boundary of bottom plate, to avoid any boundary reflection from the edges of bottom and top plates.

PNAX N5245A’s port 1 and 2 are connected to the center feeding probe on bottom plate and sensor probe on top plate accordingly. For fundamental field mapping, network analyzer measures frequency dependent $S_{21}$, which is spatial dependent. Excellent agreement has been reported in previous article[54]. We simultaneously measure the power at the second-harmonic frequency from the same ports. To obtain meaningful phase information at the harmonic frequency, we also measure the ratio of the received signal at port 2 to the received signal at port 1. While the bottom plate is moving, the closed space within mapping chamber is thought to be a stational

![Figure 6.3: 2D mapper setup for the point source.](image-url)
space, which means the received signal at port 1 is invariant with the movement. Thus we obtain at each point in the measurement space both magnitude and phase information of the generated second harmonic signal.

We model the system through the previously determined effective properties in COMSOL, a finite element method based commercial full wave simulator. By coupling frequency domain simulations at both the fundamental and harmonic frequencies, we solve both the linear and nonlinear scattering simultaneously for precisely the same geometries employed in experiment.

For a frequency sufficiently below the resonant frequency of the VLSRR element, such as 4.2GHz, the VLSRR element exhibits positive index properties, while the second harmonic process is still strong enough to generate a detectable signal. The result is shown in fig.6.4. As shown in fig.6.4, the numerical result agrees well with experimental mapping result obtained from the nonlinear field mapping chamber. Even though there is only one frequency’s field pattern is shown here. We checked field mapping result at other frequencies, which also agrees with the numerical result qualitatively.

Due to the small feature size of the applied VLSRR sample, the interference between point feeding probe and VLSRR structure is small, thus the scattering pattern for the fundamental field is almost cylindrical field, and agrees with the Hankle function. For the measured second harmonic signal, the scattering pattern is 2D dipole like, because the applied VLSRR unit cell could be treated a point source at second harmonic frequency as well.

6.5 Harmonic field mapping of a VLSRR medium

With the successful demonstration of the nonlinear field mapping system for a single metamaterial element, we turn now to a larger medium composed of arrays of VLSRRs. This bulk nonlinear metamaterial is composed of the proposed C-band VL-
Figure 6.4: (a) Measured field at fundamental frequency; (b) Numerical field result at fundamental frequency; (c) Measured field at second harmonic frequency; (d) numerical field result at second harmonic frequency.

SRR unit cell discussed earlier. The whole medium composite is formed by $18 \times 6 \times 2$ unit cells. A picture of the constructed sample is shown in fig.6.5. Since the 2D mapper ideally is mimicking a 2D environment, the dimension of the proposed nonlinear metamaterials composite roughly approaches $3\lambda_{SH} \times \lambda_{SH}$ where $\lambda_{SH}$ is the wavelength at the second harmonic frequency.

The nonlinear metamaterial medium is closely placed to the center feeding probe on the bottom plate, in order to have sufficient power excitation, as shown in fig.6.6. The distance from nonlinear medium edge to center feeding probe is 2cm. Thus there is a cylindrical-patterned wave, as shown in fig.6.4.(a), incident on the rectangular
nonlinear medium which excites an effective nonlinear magnetization. With the capability of the nonlinear 2D field mapper, we capture both the field propagating inside of the medium and the scattered field pattern outside of the medium, at both fundamental and harmonic frequencies.

Again, we measure the second harmonic component of the field at each spatial position while the bottom plate of the 2D mapper moves. The measured fundamental field and second harmonic field are shown in fig.6.7(a,c). The result is shown at 4.3GHz, before the resonant frequency point. At this frequency, the VLSRR medium has an effective index larger than 1, thus resulting in waves with shorter wavelength inside the medium compared to free space. The result is shown in fig.6.7(a). Again, a cylindrical field pattern exists in the regime which is away from the area occupied by the nonlinear metamaterial medium. The field pattern of the generated second
As before, we employ the effective properties of the metamaterial in COMSOL to validate the experimental measurements. The numerical result is shown in fig.6.7(b,d). As expected, the experimental result agrees qualitatively with numerical result from effective medium model, even within the metamaterial itself, showcasing the ability of the nonlinear 2D field mapper to visualize complex nonlinear processes in inhomogeneous materials and geometries.
6.6 Conclusion

Here, we reported a field mapping system designed to map the nonlinear field pattern spatially, along with a design of C-band VLSRR nonlinear metamaterials.

- By studying the proposed VLSRR metamaterial in the field mapping chamber in both single element and bulk configurations, it is demonstrated the capability and performance of the nonlinear field mapping chamber.

- Remarkably, it is found that the homogeneous description of the nonlinear metamaterial can be directly confirmed by the presence of harmonic field patterns with both magnitude and phase information that coincide extremely well with effective medium simulations.

- With the capability to measure harmonic and mix-wave magnitude and phase information, the presented nonlinear field mapping chamber may prove a valuable tool to study complex field scattering and wave generation problems in nonlinear metamaterials, especially in spatially varying metamaterial samples.

And a number of other experiments, including phase matched harmonic generation, third harmonic generation, power dependent refractive index, could be verified experimentally in this mapping chamber, yielding unprecedented visualization of exciting and even novel nonlinear processes in metamaterials.
In this dissertation, I have developed a useful methodology to study microwave nonlinear metamaterials. This includes physical design of various nonlinear metamaterials, effective medium model extraction, and experimental examination of 1D and 2D nonlinear processes. Here, it covered most analytical and experimental aspects in research. The analytical studying based on the effective nonlinear medium model and transfer matrix method provides a convenient means to study the nonlinear wave generation and propagation in nonlinear metamaterials.

To construct nonlinear metamaterials, either diode or varactor could be integrated and treated as nonlinear sources. While varactor is preferred to be more efficient in harmonic signal generation and frequency tuning, while diode is preferred to have controllable transition and higher frequency (above 10GHz). Varactor is hard to find which works above 10GHz, and are usually more expensive than diode. Varactor and diode are easy to find lumped microwave nonlinear components. Nonlinear metamaterial design are not only limited to those. We could implement other natural nonlinear materials into metamaterial structures to introduce nonlinearity as well, such as liquid crystal, semiconductor, organic materials as well.
With known chosen nonlinear components/materials, both experimental characterization method or numerical modeling method could be applied to extract the effective medium model properties for the constructed nonlinear metamaterials at microwave frequency. The extracted effective medium model could be validated experimentally in both linear and nonlinear properties. Like the transmission line apparatus I applied in this report, both linear transmission and nonlinear wave generation process have been tested. And results are used to validate associate effective medium model.

With both the design of nonlinear metamaterials and its validated effective medium model, many interesting experiments could been proceeded, such as nonlinear wave generation, phase matched harmonic generation, electromagnetic phenomena, and so on. All those experimental study, analytical study or numerical studies could be either 1D or 2D case, since we have both the 1D and 2D experiment capability. Moreover, other applications based on nonlinear metamaterials could be realized by the methodology proposed in this report as well, such as metamaterial proposed optical parametric amplifier, metamaterial proposed power dependent refractive index material, etc.

As a continuation of current nonlinear metamaterial study, a few further research topics are projected here:

1. Phase matched second harmonic generation

In the 2D field mapping experimental result, I presented the study of a nonlinear field mapping result for a block of C-band VLSRR metamaterials. This block of metamaterial unit structures composes a effective medium slab with magnetic resonant only in 1 direction, and non phased matched nonlinear process is studied here. Thus, it is meaningful to actually observe phased matched nonlinear wave generation and propagation phenomena experimentally. With 2D mapper, it is doable.

One way to visualize phased matched nonlinear wave generation and propagation
is to design a metamaterial medium with negative refractive index at fundamental frequency, and positive index at harmonic frequency. In this case, perfect phase match is automatically satisfied for backward propagating second harmonic wave, if material properties are not frequency dispersive. In metamaterials, backward generated second harmonic process should exist in certain frequency point since both linear material properties and nonlinear material properties are frequency dependent.

2. Active nonlinear metamaterial

In chapter 3 and chapter 4, I have shown that the effective medium model could be directly derived from the effective circuit model for nonlinear metamaterials. A further analysis could be extended to more complicated circuit geometry, such as lumped nonlinear device FET/BJT transistor. With a well-developed spice model for transistor, it is possible to combine it with either SRR or electric medium’s equivalent circuit model. This could provide a basis to exam physical realizable complex nonlinear metamaterial. Even it is possible to analytical study active metamaterials with this equivalent circuit model.

3. Terahertz nonlinear metamaterial

In this report, I mostly focus on the microwave metamaterial design and characterization, also observing nonlinear phenomena from them. However, there is very limited application for microwave nonlinear metamaterials. One of the main reason is that most of traditional nonlinear devices at microwave frequency, such as amplifier, mixer, frequency multiplexer e.t.c have already been well built at low cost. However, there is a huger and better market at higher frequency than microwave, like terahertz, as the demanding of terahertz communication. Nonlinear terahertz metamaterial device, if it is well developed, could play a significant role in the next generation of terahertz communication infrastructure.
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