Aeroelastic and Flight Dynamics Analysis of Folding Wing Systems

by

Ivan Wang

Department of Mechanical Engineering and Materials Science
Duke University

Date: ______________________

Approved:

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Earl H. Dowell, Supervisor

__________________________
Donald B. Bliss

__________________________
Kenneth C. Hall

__________________________
Thomas P. Witelski

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mechanical Engineering and Materials Science in the Graduate School of Duke University
2013
Abstract
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Abstract

This dissertation explores the aeroelastic stability of a folding wing using both theoretical and experimental methods. The theoretical model is based on the existing clamped-wing aeroelastic model that uses beam theory structural dynamics and strip theory aerodynamics. A higher-fidelity theoretical model was created by adding several improvements to the existing model, namely a structural model that uses ANSYS for individual wing segment modes and an unsteady vortex lattice aerodynamic model. The comparison with the lower-fidelity model shows that the higher-fidelity model typically provides better agreement between theory and experiment, but the predicted system behavior in general does not change, reinforcing the effectiveness of the low-fidelity model for preliminary design of folding wings. The present work also conducted more detailed aeroelastic analyses of three-segment folding wings, and in particular considers the Lockheed-type configurations to understand the existence of sudden changes in predicted aeroelastic behavior with varying fold angle for certain configurations. These phenomena were observed in carefully conducted experiments, and nonlinearities - structural and geometry - were shown to suppress the phenomena. Next, new experimental models with better manufacturing tolerances are designed to be tested in the Duke University Wind Tunnel. The testing focused on various configurations of three-segment folding wings in order to obtain higher quality data. Next, the theoretical model was further improved by adding aircraft longitudinal degrees of freedom such that the aeroelastic model may predict
the instabilities for the entire aircraft and not just a clamped wing. The theoretical results show that the flutter instabilities typically occur at a higher air speed due to greater frequency separation between modes for the aircraft system than a clamped wing system, but the divergence instabilities occur at a lower air speed. Lastly, additional experimental models were designed such that the wing segments may be rotated while the system is in the wind tunnel. The fold angles were changed during wind tunnel testing, and new test data on wing response during those transients were collected during these experiments.
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List of Abbreviations and Symbols

Symbols

\( A \quad \text{Aeroelastic matrix, coordinate transform matrix} \)
\( AR \quad \text{Aspect ratio} \)
\( a \quad \text{Elastic axis offset} \)
\( b \quad \text{Half chord} \)
\( c \quad \text{Chord} \)
\( D \quad \text{Aerodynamic drag, Generalized Theodorsen function} \)
\( E \quad \text{Young's modulus} \)
\( F \quad \text{Out-of-plane torsion generalized coordinate} \)
\( f \quad \text{Constraint function} \)
\( G \quad \text{Angular description transformation matrix} \)
\( G \quad \text{Shear modulus} \)
\( H \quad \text{Out-of-plane bending generalized coordinate} \)
\( \mathbf{h} \quad \text{Hinge direction} \)
\( h \quad \text{Out-of-plane bending displacement} \)
\( Im \quad \text{Imaginary part} \)
\( I_{xx} \quad \text{Bending area moment of inertia} \)
\( I_{yy} \quad \text{Pitching moment of inertia} \)
\( I \quad \text{Generalized inertia, equal to moment of inertia about elast axis of entire wing segment} \)
Moment of inertia about elastic axis per unit length of wing segment

Polar area moment of inertia

Stiffness matrix, vortex kernel function

Generalized stiffness

Rotational spring constant

Length of wing segment, Lagrangian, aerodynamic lift

Mass matrix

Generalized mass, total mass, aerodynamic moment

Mass per unit length of wing segment

Generalized force

Generalized coordinate

Position vector, rotation matrix

Frame translation coordinate

Real part

Mode shape in discrete form

Kinetic energy

Wing thickness, time

Air speed

Displacement

Potential energy

Velocity, spring direction

Work

Wash

Angle of attack

Virtual quantity, e.g. \( \delta W \) is virtual work
Γ  Circulation
ζ  Damping
θ  Aircraft pitch
λ  Lagrange multiplier, eigenvalue
ρ  Structural material density
σ  Flutter strength
Φ  Relative y component of mode shape
φ  Out-of-plane torsion displacement
φ_{ip}  In-plane rotation
X  Relative x component of mode shape
Ψ  Mode shape vector containing all 3 components
Ψ  Angle of wing segment relative to horizontal, relative z component of mode shape
ψ  Fold angle
ω  Natural frequency

Superscripts
-  Non-dimensional, relative frame
.  Time derivative
^  Unit vector
'  Spatial derivative
"  Amplitude in frequency domain, undeformed

Subscripts
EA  Elastic axis
e  Elastic
h  Bending
$i$ Wing segment index, horseshoe vortex index

$j$ Constraint or Lagrange multiplier index, collocation point index

$LE$ Leading edge

$m$ Mode index

$NC$ Non-circulatory

$o$ Coordinate origin

$p$ Point mass

$R$ Frame translation

$s$ Hinge spring index
The author would like to acknowledge the members of the Duke Aeroelasticity Research Group for their help throughout his graduate school career, and in particular his advisor Dr. Earl Dowell, as well as his defense committee members Dr. Donald Bliss, Dr. Kenneth Hall, and Dr. Thomas Witelski. The author would also like to gratefully acknowledge the Department of Defense and the American Society of Engineering Education for their financial support through the National Defense Science and Engineering Graduate (NDSEG) Fellowship from 2011 through 2013. The help of everyone in the Department of Mechanical Engineering and Materials Science over the past four years is also greatly appreciated. Lastly, the author would like to acknowledge the support of his friends and family. Most important of all are his father Yuan, his mother Joy, and his best friend Dinh.
In the past decade, the morphing wing concept has garnered much interest within the aerospace community. The objective of the community is to design and analyze aircraft wings that can change shape in flight such that a single aircraft can optimally perform multiple missions.\cite{1, 2, 3, 4, 5} Several types of morphing wings have been designed and analyzed. One such design implements a morphing airfoil whose shape changes at each mission leg in order to maximize the lift-to-drag ratio and delay the onset of flow separation. A typical design uses servos to control the wing’s internal structure and morph the shape of the surrounding flexible skin that forms the airfoil shape. Theoretical and experimental studies have quantified potential improvements in fuel efficiency as a result of implementing morphing airfoils.\cite{6, 7, 8} Another type of morphing aircraft has the ability to make in-plane changes to the wing planform. The implementation methods include variable wing sweep on the F-14, as well as more complicated planform shape changes demonstrated by the NextGen morphing wing concept. Past research has discussed the detailed finite element model build-up of the NextGen wing\cite{9} and aerodynamic effects of simpler in-plane morphing concepts\cite{10, 11, 12, 13}.
The present work focuses on a third type of morphing wing called the folding wing or the gull wing. Figure 1.1 shows a diagram of a three-segment folding wing. Each wing consists of at least two wing segments as well as hinges that connect each pair of adjacent wing segments such that they can rotate relative to each other. Previous studies have considered the potential performance benefits\cite{14, 15}, the actuation mechanisms\cite{16}, the flight dynamics\cite{17, 18, 19}, and the aeroelastic behavior of folding wings\cite{20, 21, 22, 23, 24}.

![Diagram of Three-Segment Folding Wing](Figure 1.1: Diagram of Three-Segment Folding Wing)

The figure shows a three-segment folding wing. One wing segment is clamped - this is the most inboard wing segment - and the other wing segments are attached one by one using hinges. The inboard direction is the direction toward the root of the clamped wing, and the outboard direction is the direction toward the wing tip.

Understanding the aeroelastic behavior of morphing wings is particularly important because the additional mechanisms and unconventional geometries that are needed to achieve the desired shape changes can dramatically alter the dynamics of the wing, as exhibited by the unusually low in-plane natural frequencies and low
torsional frequencies of the NextGen morphing wing.[9] In addition, there is growing interest in implementing flexible morphing wing technologies on micro air vehicles, which makes aeroelastic effects more important.[25, 15] However, most of the existing analyses, especially on folding wing aeroelastic behavior, consider only specific configurations and typically employ numerical tools such as MSC/NASTRAN for the aeroelastic solution. This is a high-fidelity approach, but it can be inefficient for preliminary design when it is more advantageous to use a simple yet robust model that can quickly produce relatively accurate results.

There are also some publications that discuss analyses of simpler models. Radcliffe and Cesnik[26] created a simplified model of a hinged wing and explored the aeroelastic behavior of such systems. Liska et al.[22] created a continuum model of a folding wing and conducted parameter variation studies for the aeroelastic behavior. The present thesis will extend the work done on the simplified folding wing by discussing detailed aeroelastic behavior, exploring configurations that give non-smooth flutter behavior as the fold angle varies, and describing experimental results.

To create a more robust model that can be used to quickly analyze arbitrary configurations, and can therefore be used as a preliminary design tool, the author and Duke University’s Aeroelasticity Research Group created a theoretical model based on beam theory structural dynamics and strip theory aerodynamics. This work was summarized in the author’s Master’s Degree thesis, as well as two journal articles.[27, 28] The model uses Lagrange’s equations with Lagrange multipliers as the general framework for deriving the equations of motion. The structural dynamics were analyzed using beam theory, and Theodorsen’s unsteady thin airfoil model was used for the aerodynamics. The following simplifying assumptions were applied to the structural model.

1. Each wing segment is modeled as a beam with uniform cross-sectional proper-
ties.

2. The linear analysis assumes small deflections and negligible in-plane elastic deformation.

3. The masses of the torsional springs are negligible compared to the masses of the wing segments.

4. The wing is unswept, and the elastic axes of all wing segments are collinear and perpendicular to the clamped inboard edge.

5. The folding wing hinges are parallel to the free stream direction.

6. The effect of folding motion is not considered in the present quasi-static analysis.

The structural dynamics model gives the natural frequencies of the system for different configurations. For example, the previous work studied a three-segment folding wing whose inboard fold angle is fixed at 30 degrees and the outboard angle varies from -90 to 90 degrees. Figure 1.2 shows the theoretical results for natural frequency versus the outboard fold angle for this particular case. The theoretical results are plotted as dots (·) and the experimental results are plotted as triangles (△).

The following simplifying assumptions were applied to the aerodynamic model.

1. The flow around the wing is modeled as potential flow.

2. The unsteady aerodynamic forces are modeled using strip theory and Theodorsen thin airfoil theory.

3. 3D flow effects near the folding wing hinges and wing tips are ignored.
For a fixed configuration, the aeroelastic model gives the aeroelastic eigenvalues of the system as the air speed changes. For example, the previous work studied a three-segment folding wing whose inboard fold angle is fixed at 30 degrees and the outboard angle is fixed at -75 degrees. Figure 1.3 shows the theoretical results for the real and imaginary parts of the system aeroelastic eigenvalues versus air speed. The theoretical results show that there are multiple ranges of air speeds at which the system is unstable. The flutter speed and frequency may be determined from these results.

Experiments were also conducted in the previous work, with favorable agreement between theory and experiment for the majority of cases that were considered. However, there were also some discrepancies between theory and experiment for certain cases, and the experimental set up could be improved to obtain higher quality data. The following list summarizes the discrepancies and improvements that will be addressed in the present work.

1. Larger discrepancies between predicted and experimentally measured natural frequencies.
frequencies were observed for higher modes.

2. Predicted flutter speeds and flutter frequencies agreed very well in trend to measured values, but not always in magnitude.

3. Only a limited number of configurations were tested. In particular, the Lockheed-type configuration - for which the inboard hinge and outboard hinge vary together such that the most outboard wing segment is always parallel to the clamped wing segment - was not tested, and higher fold angles were not tested.

4. The theoretical model applies only to a clamped wing and does not apply to an aircraft with rigid body motion. The two types of systems have different aeroelastic characteristics.

5. Experiments consider only fixed fold angles, and there is a lack of experimental data for folding wings that undergo folding motion during wind tunnel testing or flight.

Some preliminary experiments were then conducted on a three-segment Lockheed-type folding wing configuration. The preliminary experiments measured flutter speed
and frequency for a few different fold angles. The theory was able to predict the flutter boundary for the system for lower fold angles, but the predicted flutter behavior drastically changes near 100-degree fold angle even though the phenomenon was not observed in experiment.

The goal of the present work is to address the above issues and gain better insight into the behavior of folding wing systems. This dissertation describes the present work in the following chapters.

- Chapter 2 shows the derivation of a structural model using component modal analysis and individual wing segments that are analyzed in ANSYS, a standard commercially-available finite element software package. Using ANSYS modes for individual wing segments improves the fidelity of the structural model. The results are compared to the beam theory results to assess the accuracy of the simpler structural model.

- Chapter 3 shows the derivation of an unsteady vortex lattice aerodynamic model. The vortex lattice method allows 3D aerodynamic effects to be modeled, which increases the fidelity of the aeroelastic model. Again, the results are compared to the theoretical results.

- After creating a higher fidelity aeroelastic model in the previous chapters, Chapter 4 takes a closer look at the aeroelastic results for the three-segment folding wing. The chapter focused on two configurations: one with inboard angle at 30 degrees and varying outboard angle, and the Lockheed-type configuration whose outboard wing segment is always horizontal. The analysis considers the origin of the sudden change in predicted flutter boundary for the Lockheed-type folding wing. Several different hypotheses were explored theoretically and a few likely causes of the phenomenon are identified.
• Chapter 5 describes the design process for a new set of fixed-angle experiments. The first goal is to obtain higher quality experimental data of the system flutter boundary. The second goal is to design a Lockheed-type configuration that avoids the abrupt change in flutter behavior and see if the experimental results agree with the theory in that case. This will help narrow down the possible causes of this particular behavior. The Chapter also discusses the design process and summarizes wind tunnel test results for the new three-segment folding wing test models. Lastly, the Chapter discusses results of more carefully conducted experiments for the original three-segment Lockheed-type wing, and the fact that the sudden change in flutter speed was in fact observed.

• In order to obtain a robust aeroelastic model that may be used for preliminary design of folding wing aircraft, the aeroelastic model needs to be improved by including aircraft rigid body modes. Chapter 6 discusses the addition of aircraft longitudinal degrees of freedom in the structural model.

• Chapter 7 continues the work in Chapter 6 in describing the addition of aircraft longitudinal degrees of freedom in the aerodynamic and aeroelastic model. The theoretical model was validated by calculating the flight dynamics modes for a Cessna 172 aircraft and comparing the results to those from a traditional flight dynamics analysis.

• Chapter 8 focuses on the design of morphing wing experiments for the three-segment folding wing. Two mechanisms for controlling fold angles were designed and prototyped. Experimental results were summarized and additional experiments are in progress.
The results from the previous structural dynamics analysis showed that there are some discrepancies between the measured natural frequencies and the beam theory predictions. One explanation is that beam theory is inaccurate for low aspect ratio wing segments. Each experimental model has a total aspect ratio of 6, but the wing segment aspect ratio varies from 1 at the lowest to 5 at the highest. The cross section of each wing segment has a thickness-to-chord ratio of about 3%, so a more sophisticated structural model for the wing segments is a linear plate model.

When considering the added complexity of using a plate model, it makes sense to begin using finite element methods to obtain the natural frequencies and modes. This allows non-rectangular geometries, and the computational time is still very small. Allowing the use of ANSYS modes means that ground vibration test (GVT) results may also be used in the structural dynamics analysis, but the present work did not consider using GVT results and no further discussion is included.

Tang et al [24] has already created a structural model for a folding wing with wing segments that are modeled in ANSYS. The present work extends the results of
Tang et al in two ways. First, the present work elaborates on the use of Lagrange multipliers and lays out a systematic procedure for building the structural model for the folding wing system. Second, the present work generalizes the structural model formulation such that a folding wing with arbitrarily shaped wing segments and arbitrarily oriented hinges may be analyzed. Along the way, the derivation will also explain some numerical and kinematics considerations that must be taken into account when analyzing special cases. The discussion will focus on the previously tested case of a folding wing comprising rectangular wing segments with uniform cross sectional properties. In particular, this chapter will discuss the issues that are encountered concerning redundant constraints and the discrepancy between linear extensible mode shapes and the geometrically nonlinear system.

2.1 Kinematics for Folding Wing

Before building the structural model, it is necessary to establish the coordinate systems and kinematics of the folding wing. Figure 2.1 shows a flattened folding wing with arbitrary planform geometry. In particular, note that the hinges are not aligned with the flow direction, as was done in Tang et al [24].

![Figure 2.1: Diagram of Arbitrary Folding Wing](image)

Figure 2.2 shows the relative coordinate systems for the first wing segment and an arbitrary wing segment. For each wing segment, the local coordinate system is defined relative to the absolute coordinate system - the \( \hat{x}_i \) axis points in the flow direction and the \( \hat{y}_i \) axis is perpendicular to the clamped edge of the first wing.
segment. The origin of each wing segment’s relative coordinate system is at the midpoint of the inboard hinge. The vector direction of each hinge is denoted by $\mathbf{h}_i$, and is the direction of the hinge when the folding wing is flat. Lastly, the point $(\tilde{x}_{i,o}, \tilde{y}_{i,o})$ denotes the midpoint of the outboard hinge of the $i$th wing segment, relative to the origin of the $i$th wing segment.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure-a.png}
\caption{1st Wing Coordinates}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure-b.png}
\caption{ith Wing Coordinates}
\end{subfigure}
\caption{Relative Coordinate Systems for the 1st and ith Wing}
\end{figure}

Before building the structural model, coordinate transformations are done on the relative coordinates of the undeformed wing to obtain the absolute position of each point on the folding wing, assuming all of the fold angles $\psi_2, \ldots, \psi_N$ are known. In the present work, the fold angles are defined relative to the previous inboard wing segment. For example, the third fold angle $\psi_3$ is the angle between the second and third wing segments. One mathematical tool needed for this coordinate transformation is a 3D rotation matrix $R(h, \psi)$ that rotates a point counter-clockwise about an axis $h$ by angle $\psi$. Such a matrix may be found in literature or a standard multi-body dynamics textbook[29].

First consider the coordinate transformations for calculating the absolute coordinates of points on the second wing segment. From a flat wing segment, first it is necessary to rotate the second wing segment about the absolute direction of the second hinge $h_2$ by the second fold angle $\psi_2$, and then the origin of the wing seg-
ment must be translated from zero to the absolute coordinates of the midpoint of the outboard end of the first wing segment.

This transformation is shown in Fig. 2.3. The relative position vector \((\tilde{x}_2, \tilde{y}_2, 0)\) starts at the second wing relative origin and ends at an arbitrary point on the wing. That point has coordinates \((x_2, y_2, z_2)\) in the absolute frame. The position of the origin of the second wing local coordinate system in the absolute frame is equal to \((x_{1,o}, y_{1,o}, z_{1,o})\), which is the absolute coordinates of the midpoint of the first wing segment’s outboard hinge. The transformation is accomplished in two steps. First, the wing is rotated about the second hinge \(h_2\) by the fold angle \(\psi_2\). Since the first wing segment is always horizontal, \(h_2 = \tilde{h}_2\) is a special case. The second step is to shift the wing such that the wing relative origin is at the midpoint of the first wing’s outboard hinge.

\[
\begin{align*}
\mathbf{x}_2 &= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R(h_2, \psi_2) \begin{pmatrix} \tilde{x}_2 \\ \tilde{y}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{1,o} \\ y_{1,o} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\
\end{align*}
\]

\( (2.1) \)

This two-step transformation can be described by Eq. \((2.1)\).

The normal vector also needs to be transformed to absolute coordinates, pri-
marily for the aeroelastic model in which normal wash must be calculated. This transformation uses only the rotation matrix and is applied to the generic normal vector \((0, 0, 1)\) in the wing segment local frame.

\[
n_2 = R(h_2, \psi_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.2}
\]

To prepare for the coordinate transformation for the third wing segment, it is necessary to first compute the direction vector of the third hinge in the absolute frame. This is calculated by rotating the relative direction vector for the third hinge \(\tilde{h}_3\) using the second wing rotation matrix about the second hinge \(h_2\). Equation (2.3) describes this transformation and Fig. 2.4 depicts the transformation.

\[
h_3 = R(h_2, \psi_2) \tilde{h}_3 \tag{2.3}
\]

![Figure 2.4: Transformation for the 3rd Hinge](image)

It is also necessary to compute the absolute coordinates for second wing outboard midpoint. This can be done by applying Eq. (2.1) with \((\tilde{x}_2, \tilde{y}_2, 0) = (\tilde{x}_{2,o}, \tilde{y}_{2,o}, 0)\).

\[
x_{2,o} = R(h_2, \psi_2) \tilde{x}_{2,o} + x_{1,o} \tag{2.4}
\]

Equations (2.1)-(2.4) complete the set of coordinate transformation equations for the second wing segment. A similar procedure is followed to calculate the coordinate transformations for the third wing segment, but the rotation matrix will be a
composition of two matrices - first a rotation about the second hinge $h_2$ by angle $\psi_2$, and then a rotation about the third hinge $h_3$ by angle $\psi_3$ - and the translation step will place the third wing relative origin at the outboard midpoint of the second wing outboard hinge. The two-part rotation matrix transforms the wing similar to how a real wing might be folded. First the wing is folded about the second hinge to rotate the second wing segment to the desired fold angle, and the third wing segment rotates along with the second wing segment. Then the third wing segment rotates about the third hinge to the desired fold angle $\psi_3$ relative to the second wing segment. The two-step rotation is shown in Fig. 2.5.

![Coordinate Transformation for 3rd Wing Segment](image)

Equation (2.5) defines the rotation matrix for the third wing segment. Equation (2.6) expresses the coordinate transformation.

$$ R_3(h_3, \psi_3) = R(h_3, \psi_3)R_2(h_2, \psi_2) $$

$$ x_3 = R_3(h_3, \psi_3) \begin{pmatrix} \tilde{x}_2 \\ \tilde{y}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{2,o} \\ y_{2,o} \\ z_{2,o} \end{pmatrix} $$

The steps above for the 2nd and 3rd wing segments can be generalized to a systematic procedure for conducting the coordinate transformations from inboard to
outboard. For the $i$th wing segment, first assume that the previous inboard wing (the $i-1$th wing) has already been converted to absolute coordinates, such that the outboard midpoint $x_{i-1,o}$ and the direction for the next hinge $h_i$ are known in absolute coordinates. Also assume that the rotation matrix for the previous inboard wing $R_{i-1}$ is available. Then the rotation matrix for the $i$th wing segment may be defined by Eq. (2.7) or explicitly by Eq. (2.8).

$$R_i(h_i, \psi_i, R_{i-1}) = R(h_i, \psi_i)R_{i-1} \quad (2.7)$$

$$R_i(h_i, \psi_i, R_{i-1}) = \prod_{n=2}^{i} R(h_n, \psi_n) \quad (2.8)$$

Points on the $i$th wing can be transformed to absolute coordinates using the rotation matrix $R_i$ and a translation by vector $x_{i-1,o}$, which is assumed to be known from the calculations for the previous wing segment.

$$x_i = R_i\tilde{x}_i + x_{i-1,o} \quad (2.9)$$

The normal vector of the $i$th wing is given by Eq. (2.10). The direction of the next hinge is given by Eq. (2.11).

$$n_i = R_i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.10)$$

$$h_{i+1} = R_i\tilde{h}_{i+1} \quad (2.11)$$

The last step is to calculate the absolute coordinates for the midpoint of the outboard edge of the $i$th wing, as shown in Eq. (2.12).

$$x_{i,o} = R_i\tilde{x}_{i,o} + x_{i-1,o} \quad (2.12)$$

Equations (2.7)-(2.12) can be implemented in the specified order to compute the absolute coordinates at any point on each wing segment of an undeformed folding
wing. The rotation matrix and the midpoint of the outboard edge of the previous inboard wing must be known to start the procedure, but they are easily obtained when starting at the first (most inboard) wing segment. For the first wing segment, the rotation matrix is the identity matrix because the first wing segment is aligned with the inertial coordinates, and the midpoint of the outboard edge is the origin of the absolute coordinate system.

The elastic deformations for each wing segment must be transformed into the absolute reference frame as well for the structural dynamics model. The elastic deformation in the wing relative frame $\tilde{u}_i$ is given as a modal expansion of ANSYS modes $\tilde{\Psi}_{i,m}$.

$$\tilde{u}_i(x_i, y_i, t) = \sum_m q_{i,m}(t) \tilde{\Psi}_{i,m}(x_i, y_i)$$ (2.13)

The elastic deformation in absolute coordinates is equal to the relative deformation transformed using the appropriate rotation matrix for the particular wing segment.

$$u_i = \sum_m q_{i,m}(t) R_i \tilde{\Psi}_{i,m}$$ (2.14)

The absolute coordinates $x_i'$ of the deformed wing segment is the transformed mesh $x_1$ plus the absolute elastic deformations.

$$x_i' = x_i + \sum_m q_{i,m}(t) R_i \tilde{\Psi}_{i,m}$$ (2.15)

### 2.2 Build up of energy equations and Lagrange’s equations

The standard modal expansion approach with Lagrange’s equations is used to derive the equations of motion for the folding wing system. The relative displacement of the $i$th wing segment is assumed to be a series expansion in terms of natural modes (computed from ANSYS) and corresponding generalized coordinates, as shown in
Eq. (2.13) above. Note the dependence of the generalized coordinates on time and the dependence of the ANSYS mode shapes on the planform spatial coordinates.

In order to use Lagrange’s equations, it is necessary to obtain expressions for the total kinetic energy and potential energy of the system. The kinetic energy comes from the motion - both rigid body and elastic - of the wing segments. The potential energy comes from the strain energy stored in the elastic deformation of the wing segments, as well as the energy stored in the hinges when the wing segments deflect relative to each other.

In general, the kinetic energy is one half times a mass element times the speed squared, integrated over the entire wing. The kinetic energy expression uses the velocity in the relative coordinate system for convenience. This is correct because the wing segment coordinate systems are all inertial and energy is independent of the coordinate definition. The following equations show the algebraic manipulation from the general expression to an expression specific to the assumed modal expansion. In the general formulation, the mode shape is assumed to have a nonzero component in each coordinate direction.

\[
T_i = \frac{1}{2} \int \int \frac{d\tilde{x}_i}{dt} \cdot \frac{d\tilde{x}_i}{dt} \ dm
\]

\[
T_i = \frac{1}{2} \int \int \left( \sum_m \sum_n \dot{q}_{i,m} \dot{q}_{i,n} \Phi_{i,m} \Phi_{i,n} \right) \ dm
\]

\[
+ \frac{1}{2} \int \int \left( \sum_m \sum_n \dot{q}_{i,m} \dot{q}_{i,n} \tilde{X}_{i,m} \tilde{X}_{i,n} \right) \ dm
\]

\[
+ \frac{1}{2} \int \int \left( \sum_m \sum_n \dot{q}_{i,m} \dot{q}_{i,n} \tilde{Y}_{i,m} \tilde{Y}_{i,n} \right) \ dm
\]

\[
T_i = \frac{1}{2} \int \int \sum_m \sum_n \dot{q}_{i,m} \dot{q}_{i,n} \left( \Phi_{i,m} \Phi_{i,n} + \tilde{X}_{i,m} \tilde{X}_{i,n} + \tilde{Y}_{i,m} \tilde{Y}_{i,n} \right) \ dm
\]
The generalized mass is defined as the mass-weighted integral in the kinetic energy expression.

\[ M_{i,mn} = \int \int \left( \Phi_{i,m} \Phi_{i,n} + X_{\dot{i},m} X_{\dot{i},n} + \Psi_{\dot{i},m} \Psi_{\dot{i},n} \right) dm \]  

In general, the mode shapes are not orthogonal to each other, and the generalized mass values may be organized into a fully-populated mass matrix. However, the mode shapes are orthogonal when ANSYS natural mode (both rigid body and elastic) results are used, in which case the mass matrix is diagonal. In addition, ANSYS normalizes all mode shapes such that the generalized mass is 1.

Assuming that the natural modes are orthogonal, the kinetic energy of the wing segments can be written in the standard modal expansion form using the generalized mass and the generalized coordinates. In addition, the potential energy from the wing segments’ elastic deformation can be written in a similar form using the natural frequencies, which are also obtained from ANSYS.

\[ T_i = \frac{1}{2} \sum_m M_{i,m} \ddot{q}_{i,m}^2 \]  

\[ V_i = \frac{1}{2} \sum_m M_{i,m} \omega_{i,m}^2 q_{i,m}^2 \]

Spring potential energy is computed using the slope of the out-of-plane mode shape in the relative frame along the direction of the spring connection, which is perpendicular to the direction of the hinge. The spring direction is computed as the cross product of the hinge direction and the wing segment normal vector when the wing is flat. It is permissible (and easier) to use the relative frame because the spring elastic potential energy is a result of the relative motion of the wing segments.
Even when the wing is folded up, the spring is always perpendicular to the hinge and the derivatives will always be in the same direction in the relative frame. The sign convention is chosen such that the spring direction vector points in the outboard direction.

\[
\hat{v}_{s,i} = -\tilde{h}_i \times (0, 0, 1)
\]  

(2.19)

The spring potential energy is equal to one half, times the spring stiffness, times the square of the angular displacement. For the linearized model, the angular displacement is equal to the difference in the slope of the deformed wing segments along the spring directions. This is expressed using directional derivatives.

\[
V_{s,i} = \frac{1}{2} k_i \left( \nabla \tilde{\Psi}_i \cdot \tilde{v}_{s,i} - \nabla \tilde{\Psi}_i \cdot \tilde{v}_{s,i} \right)^2
\]  

(2.20)

Lastly, it is necessary to constrain the wing segments such that they are connected at the hinges. Because the displacements have already been computed in absolute coordinates in the kinematics analysis, the constraint equations can simply equate the absolute displacement vectors of the two wing segments at each hinge. The absolute displacement is defined in Eq. (2.14), and the general constraint equation is given in Eq. (2.21).

\[
u_i = u_{i+1}
\]  

(2.21)

This can be re-written as modal expansions in terms of generalized coordinates and mode shapes.

\[
\sum_m q_{i,m}(t) R_i \tilde{\Psi}_{i,m} = \sum_m q_{i-1,m}(t) R_{i-1} \tilde{\Psi}_{i-1,m}
\]  

(2.22)

Define the wing segment mode shape in absolute coordinates.

\[
\Psi_{i,m} = R_i \tilde{\Psi}_{i,m}
\]  

(2.23)

Then the constraint equation can be re-written in a simpler notation, as shown in Eq. (2.24). The constraint function is the constraint equation with all nonzero
terms moved to one side, as shown in Eq. (2.25). The notation \( f_{i+1,s} \) means the constraint function for one spring on the \((i+1)\)th hinge.

\[
\sum_m q_{i,m}(t) \Psi_{i,m} = \sum_m q_{i+1,m}(t) \Psi_{i+1,m} \tag{2.24}
\]

\[
f_{i+1,s} = \sum_m q_{i+1,m}(t) \Psi_{i+1,m}(x_{i+1,s}, y_{i+1,s}) - \sum_m q_{i,m}(t) \Psi_{i,m}(x_{i,s}, y_{i,s}) \tag{2.25}
\]

At this point, all of the kinetic energies, potential energies, and constraint equations have been expressed in terms of the generalized coordinates and the mode shapes. The next step is to obtain the equations of motion using Lagrange’s equations. Each \( q_{i,m} \), which is the generalized coordinate corresponding to the \(m\)th mode of the \(i\)th wing, appears in the kinetic energy of the \(i\)th wing \(T_i\), the potential energy of the \(i\)th wing \(V_i\), and the potential energy of all springs on the \(i\)th hinge and the \((i+1)\)th hinge. In addition, the coordinate \( q_{i,m} \) shows up in the constraint equations for all springs on the \(i\)th hinge and the \((i+1)\)th hinge. Then Lagrange’s equation for each generalized coordinate can be written as follows.

\[
0 = -\frac{d}{dt} \left( \frac{\partial T_i}{\partial \dot{q}_{i,m}} \right) - \frac{\partial}{\partial q_{i,m}} \left( V_i + \sum_s V_{s,i} + \sum_s V_{s,i+1} \right) + \sum_s \lambda_{i,s} \frac{\partial f_{i,s}}{\partial q_{i,m}} + \sum_s \lambda_{i+1,s} \frac{\partial f_{i+1,s}}{\partial q_{i,m}} \tag{2.26}
\]

The equation above can be algebraically manipulated into a second order ordinary
differential equation for each generalized coordinate.

\[ 0 = -M_{i,m} \ddot{q}_{i,m} - M_{i,m} \omega_{i,m}^2 q_{i,m} \]

\[ - \sum_s k_{i,s} \left( \sum_n q_{i-1,n} \nabla \Psi_{i-1,n} \cdot \vec{v}_{s,i} - \sum_n q_{i,n} \nabla \Psi_{i,n} \cdot \vec{v}_{s,i} \right) \left( -\nabla \Psi_{i,m} \cdot \vec{v}_{s,i} \right) \]

\[ - \sum_{s} k_{i+1,s} \left( \sum_n q_{i,n} \nabla \Psi_{i,n} \cdot \vec{v}_{s,i+1} - \sum_n q_{i+1,n} \nabla \Psi_{i+1,n} \cdot \vec{v}_{s,i+1} \right) \left( \nabla \Psi_{i,m} \cdot \vec{v}_{s,i+1} \right) \]

\[ + \sum_s \left[ \lambda_{i,s} \cdot \Psi_{i,m,in} - \lambda_{i+1,s} \cdot \Psi_{i,m,out} \right] \tag{2.27} \]

Finally, the set of differential equations of motion can be written as a matrix equation.

\[ M \cdot \ddot{q} + K \cdot q = 0 \tag{2.28} \]

When assuming harmonic response for each generalized coordinate, the matrix equation becomes an eigenvalue problem whose solution gives the natural frequencies and natural modes of the folding wing. The natural frequencies are the eigenvalues, and the natural modes are given as eigenvectors, which represent the relative contributions of each wing segment natural mode to the system mode shape. Each system mode shape can be computed with Eq. (2.15), in which the values for \( q_{i,m} \) are the values of the eigenvector for that mode.

2.3 Structural Dynamics Results

The structural dynamics results here were computed for the three-segment Lockheed-type folding wing.

The general structural dynamics model is used to analyze the series of simple rectangular folding wings that have been previously analyzed using beam theory structural dynamics. ANSYS is used to calculate the natural frequencies and mode shapes of each wing segment. The wing segments were modeled using SHELL63.
Table 2.1: Parameters for the Three-Segment Folding Wing

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Wing 1</th>
<th>Wing 2</th>
<th>Wing 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (GPa)</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>Density (kg/m$^3$)</td>
<td>1145</td>
<td>1145</td>
<td>1145</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>1.59</td>
<td>1.59</td>
<td>1.59</td>
</tr>
<tr>
<td># of Springs</td>
<td>-</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Spring Separation (mm)</td>
<td>-</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Span (cm)</td>
<td>14</td>
<td>10</td>
<td>6</td>
</tr>
</tbody>
</table>

elements in ANSYS. Lagrange’s Equations with constraints are then used to analyze the folding wing system.

Since the system has only rectangular wing segments, the calculation of spring potential energy is simplified significantly. The hinges all point in the $x$ direction, and the springs always connect the wing segments along the $y$ direction. The spring potential energy expression then depends on the partial derivatives of the mode shapes at the spring locations with respect to the local $y$ variable.

Figures 2.6, 2.7, and 2.8 show the first four mode shapes of the inboard, middle, and outboard wing segments of the three-segment folding wing, respectively. The results show that most of the wing segments’ mode shapes can be described by beam theory, but some mode shapes are more complicated. This result was expected going into the analysis because the aspect ratio of each wing segment is relatively low.

Figure 2.9 compares the natural frequencies calculated by the beam theory structural model and the ANSYS structural model, both of which are also compared to experiment. The three-segment folding wing has two hinges. Each hinge comprises three springs that are symmetrically placed along the hinge, with a separation of 15 mm between every two springs. The stiffness of each spring is 0.18 Nm/rad, so the total rotational stiffness of each hinge is 0.54 Nm/rad. The theoretical results are plotted as dots (·) and the experimental results are plotted as triangles (△). The comparison shows that lower modes are not significantly affected by the change in
structural model, but the natural frequencies of higher modes have more noticeable differences. Overall, both sets of results have good agreement between theory and experiment.

Figure 2.10 shows the first six elastic mode shapes of the three-segment folding wing system. The undeformed wing is shown with the black dots; the elastic mode shape is shown with the black circles. The results for the folding wing system match very well with the beam theory structural dynamics predictions. Plate-like behavior is not observed in the mode shapes until the sixth mode; at the midspan of the first wing segment the deformation varies slightly in the chord-wise direction, but the effect is still not very significant. This further shows that even though individual
wing segments cannot be described by beam theory, the entire folding wing is still adequately modeled using beam theory structural dynamics for at least the first six modes for this particular configuration. This can be explained by the fact that for each wing segment, the first two modes are beam-like, and every wing segment except the inboard one also has rigid body modes. Therefore, there are many combinations of mode shapes that can form system mode shapes without energizing the elastic modes that are more plate-like.

As another check for the general structure model, the wing that was studied by Tang and Dowell[24] was analyzed using the structural model here. Table 2.2 lists the parameters of the Tang and Dowell folding wing. Figure 2.11 shows the
Figure 2.8: The First Four Modes of the Third Wing Segment for the Three-Segment Folding Wing

Figure 2.9: Comparison of Natural Frequency Results between Beam and ANSYS Structural Models
Figure 2.10: The First Six Modes of the Three-Segment Folding Wing
corresponding wing segment geometries.

Table 2.2: Parameters for the Tang and Dowell Folding Wing

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Wing 1</th>
<th>Wing 2</th>
<th>Wing 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (GPa)</td>
<td>3.0</td>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.45</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>1145</td>
<td>2700</td>
<td>2700</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>2.38</td>
<td>0.254</td>
<td>0.254</td>
</tr>
<tr>
<td># of Springs</td>
<td>-</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Spring Separation (cm)</td>
<td>-</td>
<td>5.5</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Figure 2.11: Geometry of Each Wing Segment of the Tang and Dowell Folding Wing

Figure 2.12 shows the comparison between the frequencies predicted by the current structural model and the measured values. The test case fixes the second wing fold angle at 30 degrees and sets the outboard wing fold angle at either -30 degrees or 30 degrees. The outboard fold angles were limited to these two values because only
30-degree angle springs were found. The past work by Tang and Dowell also tested
the structure at 0-degree outboard fold angle, but those springs could no longer be
found. The natural frequencies are generally in agreement. The theory tends to uni-
formly underestimate the torsion natural frequencies, but these results were obtained
without tuning any of the material properties. In particular, poisson’s ratio for the
aluminum may be tuned to obtain higher shear modulus, which will result in better
agreement. Figure 2.13 compares the mode shapes between the current structural
model and the published results. Qualitatively, the mode shape results are also in
good agreement. Some differences may be observed in the behavior of the outboard
wing segment in the third mode. One possible explanation is that the agreement
would improve by tuning the material properties, but this was not explored further
in the present work.

![Figure 2.12: Tang and Dowell Folding Wing Natural Frequencies: Theory vs Experiment](image)

2.4 Summary of the Structural Model and Key Assumptions

To summarize, the above sections detail the derivation for the general folding wing
structural model. Very few assumptions were made in deriving the equations of
Figure 2.13: Comparison of the First 4 Modes of the Tang and Dowell Folding Wing
motion. The following key assumptions restrict the applicability of the structural model.

1. The first wing segment is assumed to be clamped at one edge and have zero displacement all along the clamped edge.

2. The model only accounts for the kinetic energy of the wing segments, and does not include the kinetic energies of springs or additional mass.

3. The wing segments have predominantly planar geometry and are modeled as SHELL63 elements in ANSYS. Only out-of-plane elastic modes are considered, and in-plane elastic deformations are not modeled.

The structural model has general applicability and may be used for preliminary design purposes. The first wing segment is clamped at one edge to model its connection to a fuselage, but the clamp may be replaced with a more general finite-stiffness connection by adding another set of springs, or more simply by modeling an additional wing segment and making the first wing segment much stiffer than the remaining wing segments (as was done in Tang et al [24]). The model does not include the kinetic energy of the springs or any extra mass, but they can be readily incorporated using Lagrange multipliers. Lastly, the assumption of predominantly planar wing segments should not be a very restrictive assumption even when the wing segments are real airfoil sections instead of theoretically idealized flat plates.

The wing segments also do not have to be quadrilateral, as was assumed in Fig. 2.1, or even polygonal. No assumptions are made about the sides of the wing segments. In terms of computational time, the cost is almost exclusively in setting up the ANSYS modal analysis because the eigenvalue problem setup and solution process takes a negligible amount of time in comparison.
The results of the general structural model supports the original assumption that beam theory is sufficient for capturing the general behavior of the rectangular folding wing systems that are analyzed in the present work. The mode shapes predicted by the general structural model, using ANSYS modes for individual wing segments, are nearly identical to those predicted by beam theory. The primary difference is in the prediction of natural frequencies because ANSYS provides more accurate estimates of natural frequencies of individual wing segments, especially those with low aspect ratio and low thickness-to-chord ratio. The comparison to the Tang and Dowell folding wing shows that the general structural model works for structures that are well outside of the beam theory regime. Therefore, the similarity between the new results and the beam theory results for the rectangular folding wings shows that despite plate-like behavior for some of the wing segments, the entire folding wing system is predominately beam-like for the first few modes.

The aeroelastic code does not distinguish between the type of theory used to compute the structural dynamics results. Only the natural frequencies and mode shapes enter into the aeroelastic model. The similarity between the beam theory results and the general structural dynamics results suggest that using one model or the other will not significantly affect the aeroelastic predictions for these particular configurations.
Three dimensional aerodynamic effects may become significant when folding wing hinges, especially inboard hinges, are at large angles. In order to model the 3D aerodynamic effects, a vortex lattice unsteady aerodynamic model is implemented in the aeroelastic code. This allows the theory to take into account several factors, in particular vortex shedding at hinges and the wing tip, as well as the aerodynamic influence of the wing segments on each other.

The approach used in the present theoretical model follows largely from the work done by Hall[30] and Tang et al[31, 32]. Therefore, the following sections only outline the general concept, and the detailed derivation of the vortex lattice equations of motion is contained in Appendix A. However, some modifications are made for the folding wing case, and the following sections will also describe the features that are specific to the present work in more detail.

3.1 Theory and Implementation of Vortex Lattice

The vortex lattice method is a numerical method for solving the potential flow model, in which the velocity field can be described by a potential function $\Phi$ that must satisfy
Laplace’s equation.

\[ \nabla^2 \Phi = 0 \] (3.1)

One solution that satisfies the differential equation is a horseshoe vortex, which is composed of one finite-length vortex filament connecting two semi-infinite vortex filaments that are parallel to each other and go off to infinity in the same direction. The shape of the horseshoe vortex is shown in Fig. 3.1. Because the potential flow equation is linear, any arbitrary number of horseshoe vortices may be used to build up more complicated solutions to the potential flow equation. Furthermore, when in the presence of a free stream potential, the horseshoe vortex has a pressure difference across its plane. Therefore, the horseshoe vortex may be used to model lifting surfaces such as wings.

![Diagram of a Single Horseshoe Vortex](image)

**Figure 3.1: Diagram of a Single Horseshoe Vortex**

In general, the vortex lattice method discretizes the surface of the wing into a mesh, each surface area element has a single horseshoe vortex with constant circulation, and the flow-tangency boundary condition is applied at a single collocation point on each surface area element. For the unsteady vortex lattice method, the wake after the object is also modeled downstream to a large but finite distance compared to the chord of the wing.

More specifically, each segment of the folding wing is discretized into uniformly distributed rectangular panels, and a horseshoe vortex is placed on each panel. The
bound portion of the horseshoe vortex is at the 1/4 chord of its panel, and the trailing vortices are coincident with the side edges of their panel. The trailing vortices of the horseshoes continue into the wake, which ends at some fixed distance after the trailing edge of the wing. Finally, each panel has a single collocation point located at the 3/4 chord and directly downstream of the midpoint of the bound vortex. Figure 3.2 shows a diagram of the vortex lattice mesh on the wing and in the wake, with some representative horseshoe vortices. Note that the figure shows the air flow from left to right, and the wing would be the right-side wing of an aircraft. The figure is shown only to demonstrate the concept of vortex lattice method. In the present analysis, the model is for a left-side wing.

The behavior of the circulation on the wing and in the wake is governed by potential flow theory. The vortex lattice mesh is categorized into 4 distinct regions, as indicated in Fig. 3.2. Each region has a set of governing equations.

1. The horseshoe vortices in the first region are bound to the wing. The governing equations in this region are the boundary conditions for the wing, that is, the velocity induced by all horseshoe vortices on the wing and in the wake must
be tangent to the wing. This condition is imposed at the collocation point of each mesh panel on the wing. The collocation point is at the 3/4 chord and midspan of each panel.

2. In the second region, the governing equations specify that the circulation at each horseshoe vortex immediately after the wing trailing edge must be equal to the change in the total circulation of all upstream horseshoe vortices on the wing. This is a consequence of Kelvin’s circulation theorem.

3. In the wake, the vorticity is not bound to any structure and is therefore convected downstream. The governing equations in this region specify that the circulation at a mesh location at some point in time must be equal to the circulation of the immediate upstream mesh location at a previous time point, provided that the time step is chosen such that the fluid particles travel exactly the distance between the two panels.

4. The final region is the end of the vortex lattice mesh. At the end of the wake mesh, the vorticity in reality convects past the endpoint, and its effect on other vortices decrease. This can be modeled by accumulating the vorticity at the end of the wake mesh, but reducing the total accumulated value by a “relaxation” factor to simulate the decreasing effect that is actually due to the vorticity convecting farther downstream at each time step.

The general equations of motion are discussed in Appendix A. The following sections will describe more specific details that pertain to the implementation of the vortex lattice method for folding wing systems.

3.1.1 Horseshoe Vortex Influence Function

To apply boundary conditions on the wing, it is necessary to determine the induced velocity field of each horseshoe vortex. Consider a horseshoe vortex aligned with the
\( \tilde{x} - \tilde{y} - \tilde{z} \) coordinate system, as shown in Fig. 3.1. This horseshoe vortex induces a velocity field that can be described by a vector-valued function that is equal to the strength of the horseshoe vortex \( \Gamma_i \) multiplied by a vector-valued kernel function that depends only on the geometry of the horseshoe vortex - \( \tilde{y}_b \) is half the length of the bound vortex filament, and \( \tilde{x}_c \) is the collocation point in the relative frame.

\[
\tilde{v}_i = \Gamma_i \tilde{K}(\tilde{y}_b, \tilde{x}_c)
\]  

(3.2)

The vector-valued kernel function contains the \( \tilde{x} \), \( \tilde{y} \), and \( \tilde{z} \) components of the induced velocity, equal to \( \tilde{K}_u \), \( \tilde{K}_v \), and \( \tilde{K}_w \), respectively.

\[
\begin{align*}
\tilde{K}_u &= \frac{\tilde{z}_c}{4\pi} \left\{ \frac{-1}{\tilde{x}_c^2 + \tilde{z}_c^2} \left[ \frac{\tilde{y}_c - \tilde{y}_b}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2}} - \frac{\tilde{y}_c + \tilde{y}_b}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2}} \right] \right\} \\
\tilde{K}_v &= \frac{\tilde{z}_c}{4\pi} \left\{ \frac{1}{(\tilde{y}_c + \tilde{y}_b)^2 + \tilde{z}_c^2} \left[ 1 + \frac{\tilde{x}_c}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c + \tilde{y}_b)^2 + \tilde{z}_c^2}} \right] \right\} \\
\tilde{K}_w &= \frac{1}{4\pi} \left\{ \frac{x_c}{\tilde{x}_c^2 + \tilde{z}_c^2} \left[ \frac{\tilde{y}_c - \tilde{y}_b}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2}} - \frac{\tilde{y}_c + \tilde{y}_b}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2}} \right] \right\} \\
&\quad + \frac{\tilde{y}_c - \tilde{y}_b}{(\tilde{y}_c + \tilde{y}_b)^2 + \tilde{z}_c^2} \left[ 1 + \frac{\tilde{x}_c}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c + \tilde{y}_b)^2 + \tilde{z}_c^2}} \right] \\
&\quad - \frac{\tilde{y}_c + \tilde{y}_b}{(\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2} \left[ 1 + \frac{\tilde{x}_c}{\sqrt{\tilde{x}_c^2 + (\tilde{y}_c - \tilde{y}_b)^2 + \tilde{z}_c^2}} \right] \right\}
\end{align*}
\]  

(3.3-3.5)

In any vortex lattice method, the horseshoe vortices are not necessarily centered at the origin of the absolute coordinate system since a number of them are distributed over the wing planform. For the folding wing problem, the vortices also do not lie on the x-y plane if the fold angles are nonzero. Consider the general horseshoe vortex for the rectangular folding wing problem, as shown in Fig. 3.3. The two vertices of
the horseshoe vortex are located at \((x_a, y_a, z_a)\) and \((x_a, y_b, z_b)\). Define the relative coordinate system \(\tilde{x} - \tilde{y} - \tilde{z}\) such that the horseshoe vortex lies in the \(\tilde{x} - \tilde{y}\) plane and is centered at the relative system origin.

\[
\begin{align*}
(x_a, y_a, z_a) \\
(x_a, y_b, z_b)
\end{align*}
\]

\[\text{Figure 3.3: Diagram of a General Horseshoe Vortex on a Folding Wing}\]

The general kernel function gives the induced velocity at a collocation point \((x_c, y_c, z_c)\) in absolute coordinates. The kernel may be derived by first using coordinate transformations from \(x - y - z\) to \(\tilde{x} - \tilde{y} - \tilde{z}\), calculating the relative induced velocities, and then transforming the relative induced velocity into the absolute coordinate system.

The coordinate transformation is done by a rotation about the \(x\) axis followed by a translation. The translation vector \(x_T\) is the midpoint between the two vertices of the horseshoe vortex. The rotation angle \(\theta_T\) is the angle of the horseshoe plane relative to the \(x - y\) plane.

\[
x_T = \left( x_a, \frac{1}{2}(y_a + y_b), \frac{1}{2}(z_a + z_b) \right)^T
\]

\[
\theta_T = \tan^{-1}\left( \frac{z_b - z_a}{y_b - y_a} \right)
\]

In the relative coordinate system, the collocation point has the following coordinates.

\[
\tilde{x}_c = R(-\theta_T) \cdot (x_c - x_T)
\]

In the relative coordinate system, the horseshoe vortex end points have the fol-
lowing coordinates.

\[ \tilde{x}_a = 0 \]  
(3.9)

\[ \tilde{y}_b = \frac{1}{2} (y_b - y_a) \]  
(3.10)

The induced velocity at the collocation point in relative coordinates is given by the kernel function in Eq. (3.2). This vector is then rotated back by the angle \( \theta_T \) to obtain the induced velocity of the horseshoe vortex in absolute coordinates.

\[ \mathbf{v}_i = \Gamma_i \mathbf{R}(\theta_T) \cdot \tilde{\mathbf{K}}(\tilde{y}_b, \tilde{x}_c) \]  
(3.11)

\[ \mathbf{v}_i = \Gamma_i \mathbf{R}(\theta_T) \cdot \tilde{\mathbf{K}} \left( \frac{1}{2} (y_b - y_a), \mathbf{R}(-\theta_T) \cdot (x_c - x_T) \right) \]  
(3.12)

The following equation, combined with Eqs. (3.6) and (3.7), define the kernel function for an arbitrary horseshoe vortex with the limitations that the short vortex filament is parallel to the absolute \( y \) axis and the tails extend along the absolute \( x \) axis.

\[ \mathbf{K}(x_a, x_b, x_c) = \mathbf{R}(\theta_T) \cdot \tilde{\mathbf{K}} \left( \frac{1}{2} (y_b - y_a), \mathbf{R}(-\theta_T) \cdot (x_c - x_T) \right) \]  
(3.13)

### 3.1.2 Time and Spatial Discretization

The unsteady aerodynamic force depends on the time rate of change of circulation in the flow. Therefore, it is necessary to keep track of the state variables at two times - one at time \( t \) and another at time \( t + \Delta t \). However, the time step \( \Delta t \) cannot be arbitrarily chosen. The time step should be large enough to capture the behaviors of the important vibration modes. This is determined by the period of the highest frequency mode shape that is used in the analysis, with frequency \( \omega_{\text{max}} \). In this analysis, Eq. (3.14) is used to determine the appropriate time step for the vortex lattice analysis. The parameter \( k_{dt} \) is a constant that determines how small the time step is. A value of 1 for \( k_{dt} \) means that the time step is exactly equal to the period
of the highest frequency mode shape. A large value of $k_{dt}$ means a smaller time step. The results in the present work were computed using $k_{dt} = 20$.

$$\Delta t = \frac{2\pi}{\omega_{max} k_{dt}}$$  \hspace{1cm} (3.14)

Setting the time step also determines the spatial resolution of the horseshoe vortices in the wake. The vorticity convects through the wake after it is shed from the wing, and the stream-wise distance between horseshoe vortices in the wake, $\Delta x_w$, must be specified such that after each time step the wake vorticity convects downstream to the next horseshoe. The time step and wake mesh resolution are related by the free stream air speed in Eq. (3.15).

$$\Delta x_w = U \Delta t$$  \hspace{1cm} (3.15)

Typically the mesh resolution on the wing is the same as that in the wake. Therefore, specifying $\Delta x_w$ also specifies the stream-wise distance between horseshoe vortices on the wing.

$$\Delta x = \Delta x_w$$  \hspace{1cm} (3.16)

The number of panels in the span-wise direction is not as critical. It should be enough to capture the behavior of the mode shapes and the variation in circulation along the wing segments. In the present analysis, the span-wise direction is partitioned into 30 intervals.

Lastly, the length of the wake is set to 10 times the chord of the wing for all of the vortex lattice analyses in the present work.

3.1.3 Boundary Condition

In the first region, the governing equations enforce the boundary condition that the total induced flow must not go through the wing. In the numerical code, this boundary condition is applied at each collocation point on the wing. The velocity
induced by each horseshoe vortex can be computed using the kernel function $K_{ij}$ of Eq. (3.13), which computes the induced velocity vector due to the $i$th horseshoe at the $j$th collocation point. This is shown in Eq. (3.17).

$$v_j = \sum_i \Gamma_i K_{ij}(x_i, x_j)$$  \hspace{1cm} (3.17)

The boundary condition is that the component of the total induced velocity that is normal to the wing must be equal to the normal wash on the wing, which is related to the motion and/or deflection of the wing. For a typical wing, the normal direction is the $z$ direction, so the normal wash is typically referred to as downwash. For the folding wing system, the normal wash is equal to the component of the fluid particle velocity that is in the direction normal to the wing segment.

$$v_j \cdot \hat{n}_j = D x_j \cdot \hat{n}_j$$  \hspace{1cm} (3.18)

$$v_j \cdot \hat{n}_j = \left( \dot{x}_j - U_\infty \frac{\partial x_j}{\partial x} \right) \cdot \hat{n}_j$$  \hspace{1cm} (3.19)

The material derivative in the equations above is equal to a local time derivative plus a convective term, that is, $\frac{D}{Dt} = \frac{\partial}{\partial t} + (U \cdot \nabla)$. For a system in which the $x$ axis points in the direction of aircraft forward flight and therefore against the direction of the flow, the flow vector $U$ is approximately equal to $(-U, 0, 0)$.

In the above equations, the vector $\hat{n}_j$ is the unit normal vector of the wing segment at the $j$th collocation point, and the vector $x_j$ is the absolute position of the $j$th collocation point. The wash on the wing is the material derivative of the fluid as it travels over the wing at the collocation point, and has contributions from the instantaneous displacement of the collocation point as well as from the motion of the fluid particle convecting downstream along the wing surface. The boundary condition is applied at the collocation point of each panel on the wing.
3.1.4 Solution of the Vortex Lattice Aeroelastic Equations

The final vortex lattice aeroelastic equation is expressed in matrix form in the following equation. The equation uses the notation from Hall[30]. The aeroelastic state vector consists of both aerodynamic coordinates $\Gamma$ and structural coordinates $q_s$

$$\begin{bmatrix} A & -E \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} \Gamma \\ q_s \end{bmatrix}^{n+1} + \begin{bmatrix} B & 0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} \Gamma \\ q_s \end{bmatrix}^n = 0 \quad (3.20)$$

The above matrix equation may be interpreted as two smaller matrix equations. The equation in the first row governs the aerodynamics: the matrices $A$ and $B$ multiply the vector of circulation strengths $\Gamma$ to compute the induced normal wash on the wing, the matrix $E$ multiplies the vector of wing generalized coordinates $q_s$ to compute the corresponding normal wash from the motion of the structure, and the equation specifies that the two must be equal to each other. The equation in the second row governs the structural dynamics: the matrices $C_1$ and $C_2$ compute the generalized forces on the wing due to aerodynamics from the circulation vector $\Gamma$, the matrices $D_1$ and $D_2$ compute the modal mass, damping, and stiffness properties of the folding wing system, and the equation enforces Newton's second law.

The vortex lattice aeroelastic equations may be solved in two ways. In the time domain, the equations can be time-marched given initial conditions to obtain the behavior of the system over time. This is a straight-forward process and can be used to find system stability by imposing a perturbation in the initial condition, time-marching the solution, and estimating the damping from the time series results. However, this is not efficient for this case, in which only system stability is of concern, because stability can be calculated in the frequency domain by posing the equation of motion as an eigenvalue problem.

In the frequency domain, the state variables are assumed to be exponential in time with eigenvalue $\lambda$. Then the state variables at time $n+1$ differ from the
state variables at time $t$ by a factor of $e^{\lambda \Delta t}$. Equation (3.20) then simplifies to the following eigenvalue problem, with eigenvalue equal to $e^{\lambda \Delta t}$, the vector $q$ equal to the combination of $\Gamma$ and $q_s$, and the matrices $M_1$ and $M_2$ equal to the matrices in Eq. (3.20).

$$\left(e^{\lambda \Delta t}M_1 + M_2\right) \cdot \vec{q} = 0$$

(3.21)

The aeroelastic eigenvalues $\lambda$ of the continuous time system can be computed from the solution of the above eigenvalue problem. In the present work, MATLAB’s `eigs` function was used to solve the vortex lattice aeroelastic eigenvalue problem with a specified search for the largest magnitude eigenvalues. This tended to return aeroelastic modes, which are the desired results, instead of the purely aerodynamic modes. However, purely aerodynamic modes sometimes do appear in the eigenvalue results along side the aeroelastic modes.

### 3.2 Steady Vortex Lattice Results

A steady version of the vortex lattice aerodynamic model was used to first study the lift over the folding wing while accounting for three-dimensional flow effects. The steady vortex lattice aerodynamic model differs from the unsteady vortex lattice model in the following ways.

1. The steady model ignores the wake mesh completely because no vorticity is shed into the wake during steady flow. Only the horseshoe vortices on the wing are modeled.

2. The only governing equations are the no-flow-through boundary conditions at the wing collocation points.

3. Since there is no unsteady flow, the lift on the wing is only from the Kutta-Joukowsi lift, equal to air density times the flight speed times the total circulation strength.
Figure 3.4 shows the steady state circulation over the three-segment Lockheed-type folding wing for four different sets of fold angles. The wing root is at the right side of each plot, and the wing tip is at the left side. The inboard angle is 0, 60, 90, or 120 degrees, and the outboard wing segment is kept parallel to the first wing segment. The results are calculated with a vortex lattice mesh that has only one chord-wise panel, and is equivalent to a discretized version of the lifting line model. The figures show the effect of the fold angles changing the three-dimensional flow field and affecting the circulation over the wing segments. The circulation is plotted along the span of each wing segment in order to show the three dimensional flow effects on the circulation over each wing segment, regardless of their orientation. The results show a clear drop in circulation over the second segment, as well as reduced circulation over the first and third wing segments. However, when looking at the change in circulation from 0-degree to 60-degree configuration, and comparing it to the change in circulation from 60-degree to 90-degree configuration, the comparison shows that the change in circulation occurs more drastically closer to 90 degrees. Consequently, this suggests that the 2D aerodynamic theory should still give sufficiently accurate results at high fold angles, but the accuracy will deteriorate more as the fold angles get closer to 90 degrees.

Another way to show the circulation is to plot the values over the actual wing segment geometry, as shown in Fig. 3.5. The circulation results are plotted to scale and calculated for the same normal wash. Again, the results show a general reduction in circulation as the fold angles increase.

The plots show that even at 60-degree fold angles, the circulation distribution over the wing segments is not vastly different from the circulation distribution over a flat wing. This suggests that a 2D aerodynamic model may still be effective in that range of fold angles. This agrees with experimental results as well - results obtained up to 60-degree fold angles for the Lockheed-type folding wing flutter experiments.
Figure 3.4: Steady-State Circulation over the Three-Segment Folding Wing for Four Different Sets of Fold Angles

were in reasonable agreement with the predictions from a beam theory/strip theory aeroelastic model.

Another effect that was included in the vortex lattice aerodynamic model is the existence of gaps between wing segments. The gaps force the wing to shed vorticity such that the circulation drops to zero at the gaps, very much like how the circulation drops to zero at the edge of a wing. In a discrete aerodynamic model such as vortex lattice, the circulation will decrease noticeably near the gaps. Figure 3.6 shows plots of circulation for the 4 different configurations when the gaps between wing segments are modeled. Despite the difference in circulation distribution, however, the presence of the gaps does not significantly change the behavior of the system. It simply
Figure 3.5: Circulation Plotted Normal to Wing Segments for the Three-Segment Folding Wing

increases the flutter speed uniformly for the different fold angles. Physically the higher flutter speed predictions make sense because the gaps reduce the circulation over the wing. Some results will be discussed in more detail in the proceeding sections.

In summary, the steady vortex lattice aerodynamic model provides a tool for estimating the reliability of the 2D strip theory aerodynamic model for higher fold angle configurations. The results show that the circulation distribution is not drastically affected by fold angles up to 60 degrees, and explains why reasonable agreement between strip theory aeroelastic results and experiment was obtained even at those angles.
Figure 3.6: Circulation Surface for 4 Configurations of the 3-Segment Folding Wing with Gaps

3.3 Unsteady Vortex Lattice Aeroelastic Results

The unsteady vortex lattice aeroelastic model provides a tool for quantifying the three-dimensional flow effects and for obtaining a more accurate flutter prediction. The solution to the aeroelastic eigenvalue problem gives the aeroelastic eigenvalues of the folding wing system. Figure 3.7 shows two views of the aeroelastic eigenvalue plot for the three-segment folding wing with inboard fold angle of 30 degrees and outboard fold angle of -75 degrees.

For the vortex lattice results, the plot on the left shows that the flutter speed is near 27 m/s for the hump mode, the flutter speed is near 41 m/s for the coalescence
mode, and the divergence speed is near 46 m/s. The plot on the right shows the real part and imaginary part of the eigenvalues versus the air speed plotted in a 3D plot. Figure 3.8 shows the eigenvalue results from the strip theory simulation. Both model are able to predict three types of instabilities, but the eigenvalue results show some differences between the Theodorsen strip theory aeroelastic predictions and the vortex lattice aeroelastic predictions. The first difference is that the flutter speed predicted by the vortex lattice model is higher than the result predicted by the strip theory model. This makes sense because the strip theory model does not take 3D
aerodynamic effects into account, and the 3D aerodynamic effects generally reduces the aerodynamic force on the wing. The second difference is that in the vortex lattice results, the first bending mode becomes overdamped before diverging. On the other hand, the Theodorsen strip theory model predicts that the first bending mode continues to increase in frequency, while the divergence mode suddenly appears. This is due to a fundamental difference between the two types of aerodynamic models, namely the Theodorsen model does not allow an eigenvalue on the negative real axis because of a branch cut in the Theodorsen function on the negative real axis. In general, however, the difference between the aerodynamic models does not affect the flutter behavior since the general behavior of the flutter mode does not change. Another difference between the two sets of results is that the vortex lattice results may show aerodynamic modes in addition to aeroelastic modes.

The unsteady vortex lattice model is next used to analyze the folding wings that were tested in experiment.

3.3.1 Two-Segment Folding Wing

The first case considers the two-segment folding wing. Figure 3.9 shows the flutter speeds and frequencies of the two-segment folding wing as the fold angle varies from 0 to 90 degrees. The figures also compare the vortex lattice predictions to the strip theory predictions as well as the experimental results.

Even for a simple system, there is a difference in flutter behavior between the vortex lattice and strip theory results. In the strip theory results, the flutter speed simply increases monotonically as the fold angle increases. In the vortex lattice theoretical results, however, a hump mode was observed at fold angles near 40 degrees. The disappearance of the hump mode causes a sudden change in flutter speed and frequency at 55 degrees. However, there was not enough previous experimental data for the two-segment folding wing to verify these theoretical results. At the fold an-
gages that were measured in experiment (which were 0, 30, and 60), the experimental results were in reasonable agreement with both sets of predictions. More precise experiments may be conducted to obtain more data points and better validate the theoretical results.

3.3.2 Three-Segment Folding Wing with Fixed Inboard Angle

The second case considers the three-segment folding wing with the inboard fold angle fixed at 30 degrees, and the outboard fold angle varying from -90 to 90 degrees. Figure 3.10 shows the flutter speeds and frequencies of this particular 3-segment folding wing as the outboard fold angle varies. The figures also compare the vortex lattice predictions, strip theory predictions, and the experimental results.

The plots show several sets of theoretical results. The solid lines are the vortex lattice results, and the dashed lines are the strip theory results. The thick lines show the coalescence flutter boundary, and the thin lines show the hump flutter boundary.

For this particular configuration, both a hump flutter mode and a coalescence flutter mode were predicted by the strip theory aeroelastic model. The results show that there is no difference in general trend between the strip theory results and the vortex lattice results. The vortex lattice results also predicted both the hump flutter
mode as well as the coalescence flutter mode. The flutter speed predictions, however, are higher for the vortex lattice results as expected. The flutter frequency predictions do not vary significantly between the two aerodynamic models.

The figures show the coalescence flutter boundary as well as the hump flutter boundary. The hump flutter boundary is defined as the air speed at which the hump flutter mode first becomes unstable. No significant changes were observed in the flutter frequencies between the 2D strip theory results and the vortex lattice results. The vortex lattice flutter speed predictions were greater than the strip theory flutter speed predictions for all fold angles between -90 and 90 degrees. The experimental data fall between the coalescence flutter boundary curves of the vortex lattice and strip theory results.

3.3.3 Three-Segment Folding Wing with Lockheed-Type Configurations

The final case to consider is the three-segment folding wing with Lockheed-type geometry, that is, configurations where the inboard fold angle varies but the outboard fold angle is always equal to the negative of the inboard fold angle such that the outboard wing segment is horizontal. Figure 3.11 shows the flutter speeds and fre-
quencies of this particular three-segment folding wing as the inboard fold angle varies while the outboard wing segment stays horizontal. The figures compare the vortex lattice predictions, strip theory predictions, and the experimental results.

![Flutter Speed](image1)

![Flutter Frequency](image2)

Figure 3.11: Vortex Lattice Flutter Results for the Three-Segment Folding Wing in Lockheed-Type Configuration

As shown in the theoretical results, using a vortex lattice aeroelastic model as opposed to a strip theory aeroelastic model does not qualitatively affect the flutter speed and frequency predictions. The flutter frequency results showed very little change in both trend and value. In Fig. 3.11a there is first a jump in flutter speed near 30 degrees. The second place at which the flutter behavior drastically changes is near 100 degrees. The flutter mode disappears and the flutter speed jumps up to a value that is outside the search range of the algorithm. It is possible to search higher air speeds to get the flutter speed, but the results will still be vastly different from the experimental data, which suggest a smooth increase in flutter speed versus fold angle even around 100 degrees. Additionally, the predicted flutter speeds at fold angles greater than 120 degrees are also very different from the trend suggested by the experimental data. The predicted flutter frequencies, however, agree very well with the measured values.

A closer look at the structural dynamics and aeroelastic results show that the
sudden changes in flutter behavior occur near locations where the first torsion and second bending natural frequencies cross each other. The natural frequencies of the three-segment folding wing with Lockheed-type fold angles are shown in Fig. 3.12, and there are two locations at which the natural frequencies cross: at 30-degree fold angle and at 100-degree fold angle. The figure shows theoretical predictions (●) as well as experimental data (△).

Figure 3.12: Natural Frequencies of the Three-Segment Lockheed-Type Folding Wing

Aeroelastic eigenvalue plots of the vortex lattice results at fold angles of 25 degrees and 105 degrees show how the sudden changes in flutter speed and frequency occur. The plots are shown in Fig. 3.13.

The eigenvalue plots show that the interaction between the first torsion and second bending mode resulted in behavior that is not typical of coalescence flutter. The interaction tended to make one of the modes unstable - this is similar to the presence of a hump flutter mode in the three-segment folding wing system whose inboard fold angle is 30 degrees and the outboard fold angle is -75 degrees, shown in Fig. 3.7 and Fig. 3.8. This suggested that the second bending mode had a strong effect on the flutter behavior.

This motivated another vortex lattice calculation in which second bending mode was manually excluded from the equations of motion. This was achieved by first
Figure 3.13: Vortex Lattice Flutter Results for the Three-Segment Wing in Lockheed-Type Configuration near Frequency Crossings

(a) 25-deg Imaginary Part
(b) 100-deg Imaginary Part
(c) 25-deg Real Part
(d) 100-deg Real Part
(e) 25-deg 3D Plot
(f) 100-deg 3D Plot
building the structural dynamics and aerodynamics equations of motion, diagonalizing the structural dynamics matrices using the structural eigenvectors such that the system was expressed in terms of system elastic modes that represent the motion of the entire folding wing, and finally removing the equation that corresponds to the second bending mode. Note that this was done purely out of interest because there is no physical basis for removing this particular mode. The resulting flutter speed and frequency are shown in Fig. 3.14 along with the experimental data.

![Flutter Speed](image1)

![Frequency](image2)

Figure 3.14: Vortex Lattice Flutter Results for the Three-Segment Folding Wing in Lockheed-Type Configuration Neglecting the 2nd Bending Mode

Surprisingly, taking out the second bending mode results in excellent agreement between theory and experiment for both flutter speed and frequency. Not only do the sudden changes in flutter behavior disappear altogether at both 30-degree fold angle and 100-degree fold angle, the values of flutter speed and frequency are very close to the values measured in experiment. However, there is no physical basis for ignoring the second bending mode in the aeroelastic analysis. In fact, the existing results show that this mode is very important because it is very close to the first torsion and first bending modes.

The results suggest that a frequency crossing has a significant affect on the aeroelastic simulation results, and seems to cause drastic changes in flutter behavior. Re-
moving the second bending mode resulted in excellent agreement between theory and experiment, but there is no physical basis for doing so. Several hypotheses were considered as potential reasons for why the drastic changes in flutter behavior exist in the theory, or why the phenomenon was not observed in experiment. It is possible for the system to be sensitive to certain physical parameters, such as static imbalance. The set of experimental data came from a physical test model that connected wing segments by three pieces of spring steel, each individually bent to the desired fold angle. Since the three pieces of spring steel were not bent exactly to the same angle, the slight misalignment between wing segments may cause static imbalance in the system. The misalignment could also cause static deflection under steady free stream velocity, which was in fact observed in experiment for some of the configurations. In those cases, the static deflection may affect the aeroelastic stability. Additional theoretical analyses and experiments are needed to validate these hypotheses.

3.3.4 Summary of Results

The steady vortex lattice aeroelastic results showed the extent of the three dimensional flow effects on each wing segment by calculating the circulation over the wing due to a constant system angle of attack, i.e. the aircraft angle of attack. For a Lockheed-type three-segment folding wing, the results showed that even up to a 60-degree inboard fold angle, the strip theory aerodynamic model may still be sufficiently accurate. However, the circulation distributions are strongly affected by the three dimensional flow effects at higher fold angles, especially for inboard fold angles higher than 90 degrees.

The unsteady vortex lattice aeroelastic results showed that using the vortex lattice aerodynamic model does not qualitatively change the flutter behavior compared to using the strip theory aerodynamic model. In general the predicted flutter speeds increased, which is expected because the strip theory model overestimates the aero-
dynamic forces near hinges and near the wing tip.

In general, the vortex lattice aeroelastic model appears to be more sensitive to interactions between the first torsion and second bending mode. For the two-segment folding wing, the vortex lattice results had a significant change in flutter speed near 55-degree fold angle, but the Theodorsen strip theory results showed a smooth change in flutter speed throughout the range of fold angles.

Lastly, the vortex lattice aeroelastic model should give more accurate answers because the three dimensional effects are modeled properly, but the result also shows that the strip theory aeroelastic model does a good job at predicting aeroelastic behavior. The two sets of results usually do not differ qualitatively, and the vortex lattice results do not necessarily improve all-around agreement between theory and experiment, so the analysis shows that the combination of beam theory structural model and Theodorsen strip theory aerodynamic model is still a useful model, at least for the particular configurations studied in the present work, for preliminary design purposes.
The aeroelastic results for the three-segment folding wing warrants additional attention. As discussed in Chapter 3, the three-segment folding wing exhibits very sudden changes in flutter behavior as the fold angles change in a way that mimics the Lockheed folding wing concept, that is, the inboard hinge rotates through a range of angles but the outboard hinge always rotates in a way that keeps the outboard wing segment in the horizontal orientation, parallel to the clamped wing segment. In particular, the natural frequencies of the first torsion and second bending modes cross each other twice as the inboard fold angle is increased, as shown in Fig. 4.1a, and a sudden change in flutter behavior occurs near each of the two crossings, as shown in Fig. 4.1b. Figure 4.1b shows the vortex lattice theoretical results, the strip theory theoretical results, and the experimental results.

One main difference between theory and experiment is the absence of sudden changes in flutter behavior, in the experimental results, near either location on the natural frequency plot where the structural frequencies cross each other. There are a few hypotheses for why the discrepancy occurs.
1. The aeroelastic solution has not converged. Refining the aeroelastic solution may improve results, especially for cases with significant three-dimensional flow geometry, such as configurations with high fold angles.

2. The physical experimental model was not built with perfect alignment. In particular, three spring steel pieces formed at a particular angle were used to connect each pair of adjacent wing segments. Because the spring steel pieces were handmade, there may have been sufficient variation in the angle of each spring to cause slight misalignment between wing segments.

3. The system is sensitive to the accuracy of the aerodynamics model, especially in configurations in which the real part of the eigenvalue changes slowly with air speed as the eigenvalue goes unstable.

This chapter takes a closer look at the aeroelastic results, especially for the three-segment folding wing that undergoes Lockheed-type fold angle variation. The set of results for the three-segment folding wing with fixed inboard hinge at 30 degrees and varying outboard fold angle is also discussed.
4.1 Beam Theory and Strip Theory Aeroelastic Results

In flutter analysis, typically the metrics of interest are the flutter speed and flutter frequency. In classical flutter analysis using an approximation method such as the V-g method, the eigenvalue results are only correct when the real part of the eigenvalue is zero, i.e. at either flutter or divergence points, but nowhere else.

In the present study, the exact eigenvalues of the system are solved using a brute force method coupled with a search algorithm. The brute force method partitions the complex number space into a large number of points on a grid, and each point is tested as a potential eigenvalue by substituting it for the eigenvalue in the aeroelastic matrix and then calculating the determinant of the resulting matrix. If the point is indeed the eigenvalue of the system, the real and imaginary parts of the determinant should both be equal to zero. However, a zero determinant cannot be obtained numerically at discrete test points. Instead, the determinant calculation is done over a large number of points in the complex number space, and the contour function in MATLAB is used to calculate the locations at which either the real or imaginary part of the determinant is zero. Lastly, a curve intersection code - obtained from MATLAB Central File Exchange and written by S. Hölz - is used to calculate the intersections of the contour lines, which are the aeroelastic eigenvalues of the system. Figure 4.2 shows an example of contour plots for a particular configuration at a particular air speed. The two types of contour lines mark points at which either the real or imaginary part of the aeroelastic matrix determinant is zero.

The efficiency of the brute force method may be improved by assuming that the aeroelastic eigenvalues must start at the structural natural frequencies at zero air speed, and then smoothly vary as the air speed increases. This means that instead of searching a large region of the complex number space, it is only necessary to search a small neighborhood around the aeroelastic eigenvalues of the previously tested air...
speed. The area of the search neighborhood depends on how fast the analysis steps up the air speed, but that is a detail that depends on the particular configuration and will not be discussed further in the present thesis.

By implementing a brute force algorithm, the results show the evolution of the exact aeroelastic eigenvalues as the air speed varies, which is typically not done for flutter analyses. Even though this is more time consuming than the V-g method, whose results are only accurate at the flutter and divergence points, the computational time is still not significant given today’s computing power. For example, the brute force method was used to analyze the three-segment wing configurations in the present work. The air speed was increased by 1 m/s each time from 0 to 50. The code was able to find the aeroelastic eigenvalues at each air speed in about half a second, and the analysis was completed in less than half a minute. More importantly, the analysis gives additional information on the behavior of the system away from the flutter point or divergence point. A new metric, called the flutter strength $\sigma_f$, is

![Figure 4.2: Example of Determinant Contour Plot](image)
defined as the change in the system aeroelastic damping coefficient versus air speed.

\[
\sigma_f \equiv \frac{\partial \zeta}{\partial U} \quad (4.1)
\]

\[
\sigma_f \approx \frac{\text{Re}(\lambda(U + \Delta U))/\text{Im}(\lambda(U + \Delta U)) - \text{Re}(\lambda(U))/\text{Im}(\lambda(U))}{\Delta U} \quad (4.2)
\]

The damping \( \zeta \) includes the effects of structural damping as well as aerodynamic damping, and is equal to the real part divided by the imaginary part of the aeroelastic eigenvalue. A high flutter strength means that the real part of eigenvalue changes quickly versus air speed, which is typical of coalescence flutter. A low flutter strength, on the other hand, may occur for hump flutter modes. This is also a useful measure when comparing the theoretical and experimental flutter speed results. A low flutter strength means that the theoretical flutter speed is sensitive to the accuracy of the aeroelastic model, and the physical system is sensitive to conditions that may cause imperfect structural or flow conditions.

A detailed representation of the flutter boundary is plotted for one three-segment folding wing configuration in Fig. 4.3. The strip theory aeroelastic results assumed zero structural damping. The plots are results for the three-segment folding wing with inboard angle fixed at 30 degrees and outboard angle varying from -90 to 90 degrees. The figure shows three plots for each folding wing: flutter speed, flutter frequency, and flutter strength.

In addition to plotting the flutter strength, there are several new features to the flutter speed and frequency results compared to the previous work. These include marking multiple points at which an eigenvalue becomes unstable, and also marking the points at which an unstable eigenvalue becomes stable. In particular, the information shown in the plots make it easier to determine when a flutter mode is either hump flutter or coalescence flutter.

The flutter speed plot in Fig. 4.3a shows that for outboard fold angles greater
Figure 4.3: Flutter Results for the Three-Segment Folding Wing with 30 Degrees Inboard Angle

than -35 degrees, the flutter speed smoothly varies between 20 m/s to 33 m/s. For fold angles lower than -35 degrees, the flutter speed is near 20 m/s, the eigenvalue then becomes stable at a higher speed, and another eigenvalue becomes unstable at a yet higher speed. This is indicative of a hump flutter mode occurring at a lower air speed than the coalescence flutter mode. Figure 4.4 shows the real parts and imaginary parts of aeroelastic eigenvalues versus air speed for the case where the outboard angle is -75 degrees. In particular, the hump mode and coalescence mode are clearly seen in the plots of the real parts of eigenvalues.

The experimental results for flutter speed agreed well with the strip theory aeroelastic model results for the configurations that only had coalescence flutter. For the
configurations that had hump flutter, the hump flutter mode was observed for the configuration with outboard angle of -75 degrees, but not for the configuration with outboard angle of -45 degrees. The fact that the hump flutter mode did not appear in experiment for the latter was most likely due to the real part of the eigenvalue being too small. This caused the flutter behavior to be sensitive to two factors: 1) the inherent structural damping in the system, and 2) the accuracy of the aerodynamics model. Figure 4.5 shows the aeroelastic eigenvalues for the -45-degree case. When compared to the results for the -75-degree case in Fig. 4.4, it is clear that both the range of air speeds and the real part of the eigenvalue for the hump mode were smaller. Nevertheless, the fact that the hump mode was found for the -75-degree configuration provided good validation for the aeroelastic model.

The predicted flutter frequencies were also in good agreement with experimental results. The flutter strength plot shows that the flutter strength for the hump modes was significantly weaker than the flutter strength for the coalescence modes. In particular, the results suggest that the hump flutter mode may be more sensitive to variations in structural and aerodynamic damping, and greater discrepancy is expected between theory and experiment.
Figure 4.5: Aeroelastic Results for Three-Segment Folding Wing with $\psi_2 = 30^\circ$ and $\psi_3 = -45^\circ$

Figure 4.6 shows the flutter analysis results for the Lockheed-type configuration. The aeroelastic analyses were carried out to 100 m/s for this configuration. The figures show that the results may be divided into three regions.

The first region contains the results from fold angles between 0 and 30 degrees. In this region, the flutter speed and the flutter frequency both gradually decrease. The theoretical results were within about 15% of the experimental results in this region.

The second region contains the results from fold angles between 30 degrees and 90 degrees. In this region, the flutter speed first decreases and then increases, and the flutter frequency continues to decrease. The significant feature in the aeroelastic results in this region is that the flutter mode stabilizes at a higher air speed, as shown by the circles in Fig. 4.6a. For the flutter frequency, the agreement between theory and experiment was generally good. For the flutter speed, however, the agreement between experimental results and theoretical predictions became worse in this region, both in terms of magnitude and in terms of the trend. In the experiment, the flutter speeds continued to increase smoothly. The initial decrease in the theory created greater separation with the measured results, and the anticipated sharp increase in
flutter speed near 90-degree fold angle was also not observed in experiment.

Figure 4.6c shows that the flutter strength drops drastically inside the first region, and remains low in the second region. This means that the real part of the eigenvalue changes slowly with increasing air speed. This is a possible explanation for why the predicted flutter speed is lower than the measured flutter speed. It is important to note that the strip theory aeroelastic model results presented here were calculated without any structural damping. For cases where the flutter strength is low, the amount of structural damping would have a greater effect on the flutter speed prediction.

The last region contains the results from fold angles above 90 degrees. In this
region, the flutter speed first jump up drastically to near 90 m/s, and then drops drastically to near 10 m/s. The flutter frequency jumps up to near 37 Hz, and then drops down to near 15 Hz before smoothly increasing for higher fold angles. The experimental results are very different from the theoretical results, since the measured flutter speed still increased relatively smoothly up to 120-degree fold angle, which was the last fold angle that was tested.

First, it is important to discuss what is happening in the theoretical results. In the flutter speed results for fold angles just below 90 degrees, Fig. 4.6a shows that the first flutter mode stabilizes again at a later air speed. As the fold angle approaches 90 degrees, the air speed at which the flutter mode stabilizes also approaches the flutter speed. After 90 degrees, the mode does not become unstable anymore. Instead, the next mode that becomes unstable is a higher mode with frequency near 37 Hz, and the speed at which that occurs is near 90 m/s. Figure 4.7 shows the disappearance of the hump flutter mode by plotting the real parts of the eigenvalue versus air speed for two configurations, one with an 85-degree fold angle and one with a 95-degree fold angle.

![Graph showing real parts of eigenvalues for fold angles 85 and 95 degrees](image)

**Figure 4.7**: Aeroelastic Results for the Three-Segment Lockheed-Type Folding Wing for Two Different Fold Angles Near 90 Degrees

After the hump mode disappears and the flutter speed jumps up, another hump
mode with very low flutter velocity appears after the fold angle increases further, which is why the flutter speed suddenly drops down. Figure 4.8 shows the aeroelastic results for a fold angle of 120 degrees. Note that in the preliminary wind tunnel experiments, the measured flutter speed was near 45 m/s.

Figure 4.8: Aeroelastic Results for the Three-Segment Lockheed-Type Folding Wing with 120-Degree Fold Angle

When including structural damping in the aeroelastic model, the flutter speed prediction would increase due to the low flutter strength of the hump mode. On top of that, uncertainties in the aerodynamics would also affect the agreement between theory and experiment for flutter speed. However, this does not explain the discrepancy between theory and experiment for every data point. In particular, for a fold angle of 105 degrees, the measured flutter speed was approximately 37 m/s. However, the theoretical results predicted no unstable air speeds around 37 m/s. Even if there was a ±5 uncertainly in the fold angle from manufacturing tolerance, the theory does not provide a prediction that would agree with the measured flutter speed. Either the predicted flutter speed is near 100 m/s, or below 30 m/s.

The last analysis is to include some static imbalance in the system. The wing segments are connected by individually manufactured spring steel pieces bent into a certain angle. Three spring steel pieces were used between each pair of wing segments.
Given the manufacturing tolerances, it is possible for the three springs to have slightly
different angles, which may result in some misalignment between wing segments.
Manufacturing tolerance may also cause the mounting holes for the spring steel
pieces to not line up perfectly, which may result in the elastic axes of wing segments
being not perfectly collinear with each other. Static imbalance may be introduced
into the physical model due to these reasons. Therefore, aeroelastic analyses were
conducted on the Lockheed-type three-segment folding wing with varying amounts
of static imbalance. The static imbalance was achieved by adding a small point mass
at the tip of each wing segment off the elastic axis. The point mass was calculated
to give either 1%, 5%, or 10% static imbalance. The percentage $\tilde{s}$ is defined by the
following formula.

$$\tilde{s} \equiv \frac{3m_p(1/2)b/L_i}{m_hb} \quad (4.3)$$

The point mass $m_p$ is located at the outboard edge of each wing segment at
quarter chord downstream of the elastic axis. This is equivalent to approximately
three times the point mass distributed along the span at the same chord-wise position,
as shown in the numerator. The denominator is the static imbalance if the entire
wing mass is located at half chord from the elastic axis.

Figure 4.9 shows the flutter speed and flutter frequency predictions for the 3
cases with static imbalance. The abrupt changes in flutter behavior remained in all
3 cases. Therefore, it may be concluded that sensitivity to static imbalance was not
the cause of these abrupt changes in flutter behavior.

In general, the results show that these unexpected behaviors occur when the first
torsion and second bending natural frequencies cross each other. A more detailed
analysis using the vortex lattice aeroelastic model will be conducted next to study
the effect of the three-dimensional flow field, which was not analyzed in the strip
theory aeroelastic model.
Figure 4.9: Aeroelastic Results for Three-Segment Lockheed-Type Folding Wing with Varying Static Imbalance
4.2 Beam Theory and Vortex Lattice Aeroelastic Results

For higher fold angles, the folding wing becomes less like a regular wing, and three dimensional flow effects may become significant such that the strip theory aerodynamic model is no longer accurate. This was discussed in the derivation of the vortex lattice method. The vortex lattice aeroelastic model was then used to predict the flutter behavior of the three-segment folding wings.

As previously discussed, the flutter speeds from the vortex lattice model were generally higher than the flutter speeds for the strip theory model. This makes sense because the three dimensional flow effects reduce the circulation at several locations along the span of the wing, including the wing tip and in the gaps between each pair of adjacent wing segments. For the three-segment folding wing with inboard fold angle fixed at 30 degrees, the result was essentially a uniform increase in the predicted flutter speed for both the hump mode and the coalescence mode, as shown in Fig. 3.10a, versus strip theory. In addition, the fact that both aeroelastic models predicted the hump flutter mode suggests that the hump mode is not a three dimensional aerodynamics effect, but is rather due to the interactions between the first torsion mode and the second bending mode.

The more interesting case is the Lockheed-type three-segment folding wing. As shown in Fig. 4.10, the vortex lattice aeroelastic model predicts higher flutter speeds and about the same flutter frequencies. However, the model still has the abrupt changes in flutter behavior near the two frequency crossings. The vortex lattice aeroelastic results computed in this section used a structural damping of 1%.

The folding wing experimental models were connected by spring steel pieces with gaps between each pair of adjacent wing segments, and the gaps would affect the circulation profile over the wing, as discussed in the steady vortex lattice results. The gaps were then modeled in the unsteady vortex lattice aeroelastic analysis. Figure
4.11 compares the vortex lattice code flutter speed predictions for the Lockheed-type three-segment folding wing with and without modeling the gaps between wing segments. In general, the result is an increase in flutter speed by modeling the gaps.

Since adding the gaps did not affect the general flutter behavior, a careful convergence analysis is done for this case to explore the possibility that increasing the mesh density would remove the abrupt changes in flutter behavior. The mesh refinement
analysis was done on top of the aeroelastic model with gaps between wing segments. In one analysis, the number of spanwise elements was doubled in order to better model the regions near the gaps and near the wing tip. The results are shown in Fig. 4.12. In another separate analysis, the time step was halved to better capture the dynamics of higher modes. Decreasing the time step also decreases the chordwise mesh size by the same amount, since they are proportional to each other by the air speed. It is important to note that even for the nominal case, the time step was chosen to be 20 times smaller than the period of the highest frequency mode used in the analysis. The results are shown in Fig. 4.13. In a third separate analysis, the wake length was doubled to reduce the effect of the finite wake. The results are shown in Fig. 4.14.

The results showed that increasing the span-wise mesh density slightly increased the flutter speed prediction, while decreasing the time step and doubling the wake length had no significant effect on the flutter speed. In all three cases, the abrupt changes in flutter behavior remained. Therefore, the convergence analysis suggests that the abrupt changes in flutter behavior were not due to insufficient mesh density or insufficient precision in modeling the three dimensional flow effects.
4.3 Three-Degree-of-Freedom Airfoil System

The vortex lattice results showed that the three-dimensional flow effects did not affect the sudden jump in flutter speed and frequency near the high fold angle. There is increasing evidence that this phenomenon is due to the system sensitivity when the natural frequencies cross each other. An aeroelastic analysis of a simpler system with similar natural frequency crossings was drawn up in order to better understand the theoretical predictions for the folding wing case. The simpler system is a three-
degree-of-freedom (3DOF) system that contains an airfoil with mass and moment of inertia, and an attached rigid object with mass but no moment of inertia. A diagram of the system is shown in Fig. 4.15. A detailed derivation of the equations of motion and a table of parameters are given in Appendix B.

![Diagram of the 3 DOF Airfoil System](image)

Figure 4.15: Diagram of the 3 DOF Airfoil System

By varying the system parameters, the two plunge natural frequencies of the 3DOF airfoil system can be tuned to equal the first two bending natural frequencies of the folding wing. By setting static imbalance in the 3DOF airfoil system to zero, the torsion frequency may be varied without affecting either plunge frequency. Figure 4.16 shows the flutter speed and flutter frequency of the simple system versus the torsion natural frequency, expressed in non-dimensional terms. The torsion natural frequency crosses the second bending natural frequency near 3.7, which is close to the location at which a sudden change in flutter behavior is predicted.

The analysis of the simple 3DOF airfoil system reinforces the notion that sudden changes in flutter behavior may occur when the first torsion and second bending natural frequencies cross each other. A worthy project for future work is to conduct wind tunnel experiments on this airfoil system. A 3DOF airfoil model would have several advantages over the folding wing experimental model for the purposes of
understanding this type of behavior.

1. The aerodynamics of an airfoil is simpler to analyze, and the air flow will be more streamlined around the airfoil than the folding wing models.

2. The airfoil model may be designed with better control over the three degrees of freedom. In particular, the system may be more easily adjusted to minimize static deflection.

3. The airfoil model would have simpler motions than the folding wing model due to its limited degrees of freedom. This should result in cleaner experimental data.

4.4 Summary of Detailed Analyses for the Three-Segment Folding Wing

The above detailed analysis for the three-segment folding wing showed that the hypothesized effects were not significant in causing the phenomenon of abrupt changes
in flutter behavior with varying fold angle. The results of the detailed aeroelastic analysis are summarized as follows.

1. The phenomenon is not due to convergence since both structural convergence and aerodynamic mesh convergence were examined, and neither had an effect on the general flutter behavior of the system.

2. The phenomenon is not due to potential static imbalance from misalignment between wing segments, since a study of the system with varying degrees of static imbalance yielded no significant changes in flutter behavior for fold angles near 100 degrees.

3. The sensitivity of the flutter speed for certain cases may explain the discrepancy between theory and experiment for certain combinations of fold angles. However, this reason alone does not explain why these abrupt changes in flutter speeds were not observed in experiment. In particular, the data point for the 105-degree case could not be predicted by the theory even with low flutter strength taken into account, and the data point for the 120-degree case is difficult to explain only using low flutter strength as the argument.

4. The phenomenon appears to be a consequence of the first torsion mode and second bending mode having frequencies that are close together and crossing, and not a consequence of three dimensional aerodynamic effects, since the vortex lattice aerodynamic model was able to predict this phenomenon.

5. The phenomenon does not occur for all cases involving the first torsion and second bending mode having close together frequencies. In particular, no such behavior was observed for the configuration where the inboard fold angle was fixed at 30 degrees and the outboard angle varied.
Modeling the nonlinearities in the system, which have not been explored in the present work, may resolve this discrepancy. In particular, static deflections near the flutter speed would alter the circulation profile and the aerodynamic forces. In addition, large deflections would affect the structural dynamics by introducing both geometric and structural nonlinearities. For the folding wing system that is very sensitive to the relative magnitudes of natural frequencies, it is possible for nonlinear structural dynamics effects to affect the flutter behavior. Nevertheless, there are still some studies that can be done to better understand this folding wing system. One is to design improved experiments with better manufacturing in order to obtain higher quality flutter test data. This effort is discussed in the next chapter.
Improved Fixed-Angle Folding Wing Experiments

The fixed-fold-angle experiments from the previous work showed that the theory provided reasonable estimates of the system structural dynamics and the aeroelastic behavior. However, a few problems were encountered during actual testing. First, it was noted that for every two adjacent wing segments, which were connected by three spring steel pieces bent to the desired fold angle, the spring steel pieces were not all at the same angle. This is a result of making each piece individually, and bending each piece by hand. The tolerance on the fold angle was ±5 degrees. Since the spring steel pieces were screwed on to the wing segments, differences in angle resulted in the wing segments being forced to align with the springs, and created an uneven stress distribution in the wing segment, which may have affected the system dynamics. A related second problem was that when the spring steel pieces were not all at the same angle, the wing segments may have been misaligned with the flow direction as well. In other words, when one wing segment is at zero angle of attack with the flow, another wing segment may not be at zero angle of attack. This resulted in static deflection during wind tunnel testing since there were configurations that had a non-zero average angle of attack with the flow. The last problem was that the
spring steel pieces must be bent to an angle during manufacturing, and cannot be bent to a different angle later in order to avoid fatigue, so only a limited number of fold angles may be tested.

The previous experiments had some results, in particular the results for the three-segment Lockheed-type folding wing configurations, that did not agree well with the theory. However, it was uncertain whether the manufacturing quality caused any of the discrepancies between theory and experiment. In addition, the flutter prediction was sensitive to the system natural frequency spacing, as discussed in the previous Chapter. Therefore, a new set of experiments was designed to minimize the aforementioned problems. In particular, the new experiments were designed to avoid the following three problems.

1. Avoid misalignment between wing segments.

2. Avoid static deflection during flutter testing.

3. Avoid natural frequency crossings as the fold angle varies.

5.1 Experimental Design

The two primary modifications from the previous work were using friction hinges to connect wing segments and clamped the wing onto a mounting structure with adjustable angle of attack. A friction hinge is a simple but effective way to improve alignment between wing segments, and it also allows for a continuous range of fold angles. In terms of structural dynamics, each friction hinge adds mass that must be modeled in the theory, unlike the previous spring steel pieces whose weights were not significant compared to the wing segments. In addition, a friction hinge theoretically has infinite torsion stiffness in the model since the hinge maintains zero angular displacement up to some torque limit, which is not exceeded in the linear regime.
These two effects must be modeled in the structural dynamics model since they have significant effects on the system behavior.

5.1.1 Theoretical Model of Friction Hinge

For the three-segment folding wing, there are two friction hinges. The inboard hinge is modeled as two point masses located on either side of the elastic axis at the outboard end of the first wing segment, and the outboard hinge is modeled the same way on the outboard end of the second wing segment. Each hinge is modeled with two point masses to capture added rotational inertia as well as added mass. Each point mass is located at a distance from the elastic axis equal to the radius of gyration.

The friction hinge stiffness was modeled by setting a very large value for the rotational stiffness of the hinge. In the non-dimensional structural dynamics equations from the previous work, the non-dimensional rotational spring stiffness is the ratio of rotational spring stiffness to the beam bending stiffness, as shown in Eq. (5.1).

\[ \tilde{k} = \frac{kL}{EI_{xx}} \]  

(5.1)

The numerical simulation encountered some round-off errors when using very high values of dimensional spring stiffness. Specifically, the system natural frequencies were obtained in two ways and compared to each other. The eigenvalue solver gives the system natural frequencies directly, and the natural frequencies may also be calculated by diagonalizing the stiffness matrix using the eigenvectors as shown in the following equation, in which the matrix \( \Omega^2 \) is a diagonal matrix containing the squares of the natural frequencies and the matrix \( V \) contains the system eigenvectors.

\[ M\Omega^2 = V^T K V \]  

(5.2)

The two sets of natural frequency results were compared to each other. At high values of spring stiffness, the difference between the two sets may exceed 0.001%.
This difference is negligible in terms of structural dynamics results, but serves as a convenient boundary for setting the spring stiffness when approximating an infinitely stiff hinge. For the new experimental configurations, the value of spring stiffness was set to the largest possible value that did not produce more than 0.001% difference between the two methods of calculating the natural frequencies.

Table 5.1 summarizes the parameters for the hinge. The hinge mass includes both the hinge and the associated fasteners. The hinge radius of gyration was assumed to be 1/4 of the hinge width.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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</thead>
<tbody>
<tr>
<td>Mass of hinge</td>
<td>12.1</td>
<td>g</td>
</tr>
<tr>
<td>Mass of fasteners</td>
<td>5.2</td>
<td>g</td>
</tr>
<tr>
<td>Radius of gyration</td>
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<td>mm</td>
</tr>
<tr>
<td>Stiffness</td>
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<td>Nm/rad</td>
</tr>
<tr>
<td>Nondimensional Stiffness</td>
<td>$10^5$</td>
<td>-</td>
</tr>
</tbody>
</table>

5.1.2 Design of Flutter Experiment

The experimental configurations must be designed such that the expected flutter speed is near 30 m/s, since based on experience that is the best air speed for operating the wind tunnel. At higher speeds, the tunnel vibrations become noticeable on the instrumentation. The original three-segment folding wing configuration from the previous work was used as a starting point. The known modifications were included in the theoretical model first: the high stiffness of the friction hinges increased the bending natural frequencies, and the point masses of the friction hinges decreased the bending and torsion frequencies. Then the span of each wing segment and the chord of the wing were tuned to give the desired aeroelastic behavior. Adding more point masses was also explored as a method of tuning the structural dynamics and aeroelastic behavior, but the idea was not used in the final configurations since it
required additional design work and would have also affected the flow field. Tuning the geometry alone was enough to obtain the desired results. Table 5.2 summarizes the parameters for the Three-Segment Folding Wing with Friction Hinges.

Table 5.2: Parameters for the Newly-Designed Three-Segment Folding Wings

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Config 1</th>
<th>Config 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord (cm)</td>
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<td>6.5</td>
</tr>
<tr>
<td>Span 1 (cm)</td>
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<td>14</td>
</tr>
<tr>
<td>Span 2 (cm)</td>
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<td>10</td>
</tr>
<tr>
<td>Span 3 (cm)</td>
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<td>6</td>
</tr>
<tr>
<td>Young’s Modulus (GPa)</td>
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<td>3.0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
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<td>0.45</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
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<td>1145</td>
</tr>
<tr>
<td>Thickness (in)</td>
<td>1/16</td>
<td>1/16</td>
</tr>
</tbody>
</table>

The goal of this set of experiments is to obtain flutter data on the more carefully constructed experimental models, which will also help better understand the discrepancy between theory and experiment in the previous work. To that end, two configurations were designed for this set of experiments. For the first configuration, the crossings between first torsion and second bending were designed to be slightly closer together such that there is at least one point at which the flutter behavior changes. In the second configuration, the crossings were designed to be as separated as possible, resulting in a wider range of fold angles over which there is no expected abrupt change in flutter behavior.

The initial design modeled each friction hinge as only one mass located at the elastic axis, did not take the mass of fasteners into account, and also used the nominal value of 1/16” for the thickness of the wing segment. Figure 5.1 shows the natural frequencies for the two configurations.

Figures 5.2 and 5.3 show the predicted flutter speeds and flutter frequencies of the two experimental configurations over a wide range of fold angles from 0 to 150. The theoretical results come from the vortex lattice aeroelastic model. The experiments
are for the Lockheed-type configuration so the outboard wing segment is always horizontal. The figures show that the first configuration has an abrupt change in flutter behavior near 10 degrees fold angle, and the second configuration does not have any abrupt change in flutter behavior until past 140 degrees fold angle.

The following list summarizes the predicted results.

1. By designing both configurations with smoothly-varying aeroelastic behavior for nearly full range of fold angles, good agreement between theory and experiment will mean that the aerodynamic model captures the physics of the flow
field even for a case with strong three-dimensional aerodynamic effects.

2. Assuming good agreement between theory and experiment, as stated above, one may infer that in the previous work, the drastic change in flutter behavior near 90 degrees fold angle was primarily a result of modal interactions and the frequency crossings, and not due to inaccurate modeling of the three-dimensional aerodynamics.

3. For the first configuration, the drastic change in flutter frequency near 10 degrees fold angle will be easier to observe than the very small change in flutter speed. If this change could be observed in experiment, then it would show that these types of sudden changes in flutter behavior do indeed occur, which is significant from a design point of view.

5.2 Ground Vibration Test Data

Ground vibration tests were conducted on the two configurations. The wing is assembled and clamped between two L-brackets. The L-brackets are then secured onto the test table using C-clamps. One accelerometer was placed on the wing to measure vibration response. Figure 5.4 shows the experimental set up for the ground
vibration tests, and specifically points out 1) the wing model, 2) the base of the clamped wing, 3) the impact hammer, 4) the data acquisition hardware, and 5) the data acquisition software.

To measure the natural frequencies, impact tests were conducted using an impact hammer. The structure was hit with the hammer, and the transfer function of the accelerometer response versus the hammer force input was calculated using the PULSE software. The natural frequencies were the frequencies corresponding to the peaks of the transfer functions. The hammer was used the hit the structure at 27 different locations, and transfer function results were collected after a hammer impact at each location. Figure 5.5 shows the impact locations on the folding wing.

Two sets of data were collected with the accelerometer at one of two locations on the wing: off the elastic axis either near the mid-span of the first wing segment or on the outboard hinge with accelerometer direction parallel to the hinge. Figure 5.6
Figure 5.5: Impact Locations on the Three-Segment Folding Wings for Impact Test

Figure 5.6: Uni-Axial Accelerometer Positions and Direction of Measured Motion on the Three-Segment Folding Wings for Ground Vibration Testing

shows the accelerometer positions and the direction of motion that could be measured by the accelerometer. The first location was good for measuring bending frequencies, but it was noted that torsion frequencies were more difficult to excite using the impact hammer. The second accelerometer location was very good at picking up torsion motion because for non-zero fold angles, the torsion motion resulting in rigid body translation of the outboard hinge in the direction of the hinge axis. Figure 5.7 shows two transfer functions. Both were a result of impacting the wing at position 0, but each one had the accelerometer at a different location. The figures show that different modes were captured by the accelerometer because some peaks are located at different frequencies. The ones captured using accelerometer position 1 were bending modes, and the ones captured using accelerometer position 2 were torsion modes. There are times when the accelerometer at position 1 may also pick up a torsion mode, and vice versa, but using data at two different accelerometer positions ensure that no modes were missed.
Figure 5.8 compares two transfer functions similar to in Fig. 5.7, but the hammer impacted the structure at position 1, which is on the elastic axis. For accelerometer position 1, the transfer function was approximately the same as that of results from impact position 0. For accelerometer position 2, the first torsion mode was less apparent and the response of both torsion modes were reduced by about a factor of 3, which shows up as 0.5 on the plot of log-base-10 of the transfer function. This is expected because the transfer function values at the natural frequencies give the mode shapes. For accelerometer position 1, which primarily captures bending modes that have mode shapes that are independent of chord-wise position, changing the impact position to a different chord-wise position but the same span-wise position should yield the same result. For accelerometer position 2, which primarily captures torsion modes that have approximately zero displacement at the elastic axis, impacting the structure at the elastic axis should result in very low values of the transfer function at torsion modes. Because the experimentalist cannot perfectly impact the structure at the elastic axis, and because there is a small amount of coupling between bending and torsion due to manufacturing tolerances as well as accelerometer placement, some torsion motion was still be measured. Nevertheless, the behavior of measured
transfer functions agreed with expectations.

Figure 5.8: Transfer Functions of Config 1 at 90° Fold Angle, Impact at Position 1

(a) Accel Position 1
(b) Accel Position 2

The natural frequency results were categorized into two groups: one set of results measured from accelerometer position 1, and a second set of results measured from accelerometer position 2. The natural frequencies are plotted over the theoretical natural frequencies for both configurations in Fig. 5.9. The data from accelerometer position 1 are indicated with a triangle (△), and the data from accelerometer position 2 are indicated with a star symbol (*). As mentioned before, the two groups tend to separate bending and torsion frequencies, though some overlaps may also occur.

Figure 5.9: Natural Frequencies of Configs 1 and 2 Measured from Impact Tests

(a) Config 1
(b) Config 2
Some data points for the first bending modes were missing because the PULSE system was not able to measure very low frequencies. Additional impact tests were done with the spectrum analyzer and the natural frequencies were obtained by reading the locations of frequency peaks in the Fourier Transform of the accelerometer response instead of the transfer function. Figure 5.10 shows theory versus experiment for the two wing configurations using natural frequency data obtained from the spectrum analyzer.

![Natural Frequencies for Config fixed3_test1](image1)

![Natural Frequencies for Config fixed3_test2](image2)

(a) Config 1  
(b) Config 2

**Figure 5.10:** Natural Frequencies of Configurations 1 and 2 Measured from Impact Tests and Spectrum Analyzer

At this point, it is important to first note that the general agreement between theory and experiment is already very good for the first three modes, but also that the folding wing parameters had not been tuned in the theory yet. In particular, corrections needed to be made to some of the parameters.

1. The measured thickness of the wing segments is about 2% less than the nominal value.

2. The mass of the friction hinge should also include fasteners, which increase the mass by a little less than 50%. Similarly, the friction hinge should really be modeled as two point masses to account for inertia of the hinge.
After making these corrections, the parameters were then tuned to obtain a best overall fit of the experimental data for the lowest three modes since those modes participate the most in flutter. The geometric parameters are easily measurable and only the wing segment thickness needed a small correction. Then the only parameters that can be reasonably changed are the material properties: density, Young’s modulus, and shear modulus. Density is also easily measurable. The approach consisted of varying the two modulus values in the appropriate direction given the placement of bending or torsion modes. The end result was a 13% increase in Young’s modulus and a 10% decrease in shear modulus. Figure 5.11 shows the natural frequency data plotted over theoretical results obtained from the tuned sets of wing parameters. Figure 5.12 shows the same data but zoomed in to the first three natural frequencies.

![Figure 5.11: Natural Frequencies of Configurations 1 and 2 and Theoretical Results of Tuned Parameters](image)

The results show that the theoretical model does a good job of predicting the system natural frequencies. The large variations in natural frequencies were predicted by the theoretical model and observed in the experiment. The agreement is worse for the higher modes, which is expected because higher modes involve higher natural frequency modes of individual wing segments, and beam theory is only accurate
for calculating the natural frequencies of the first one or two modes for each wing segment. Chapter 2 in the present work showed that the agreement may be improved by implementing a more accurate structural model for individual wing segment. However, the results as shown here are sufficient for verifying the structural model, and no further tuning of wing parameters is deemed necessary.

5.3 Flutter Tests

Flutter tests were conducted for the two configurations in the Duke University Wind Tunnel. The recorded data include the FFT of the accelerometer response at various air speeds, and the particular value of the FFT at the frequency that is the most unstable, which is typically near the flutter frequency and between the first bending and first torsion natural frequencies. Two methods of calculating the FFT were used. The first method was to use the FFT function in LabVIEW, which allowed for an easy way to save the FFT data for later analysis. The second method was to read the FFT off the spectrum analyzer, which did not have a way to output data for future analysis. The spectrum analyzer is specifically designed for vibration measurements, and consequently had better filtering and precision in the computed FFT compared
to the LabVIEW built-in function. Therefore, the LabVIEW program was used to compute and save the FFT, but the value of the FFT at the unstable frequency was read from the spectrum analyzer and recorded manually.

Figure 5.13 shows a typical plot of the FFT amplitude at the most unstable frequency versus air speed. The amplitude data were manually read and recorded from the spectrum analyzer. The amplitude is converted from dB to a linear unit by taking 10 to the power of one-tenth the decibel value. The actual value does not matter as long as all data points are converted to the same linear unit. The result shows a jump in response at 40 m/s and an even sharper jump in response at 42 m/s. The flutter speed may be inferred from the plot to be between 40 m/s and 42 m/s.

![Figure 5.13: Example Plot of FFT Amplitude versus Air Speed for the Least Stable Mode](image)

Another way to look at the data is to create a waterfall plot of the FFT at the various air speeds. Figure 5.14 shows a waterfall plot of the system response versus air speed: at each air speed, the figure plots the FFT of the system response as acquired by LabVIEW. The data have a slightly higher noise floor than data from
the spectrum analyzer, but near the flutter speed the peak in the FFT near the flutter frequency is still very noticeable. In particular, Fig. 5.14 shows flutter mode near 7 Hz becoming sharper and increasing in magnitude as the air speed increases.

![Example Waterfall Plot of FFT versus Air Speed](image)

**Figure 5.14**: Example Waterfall Plot of FFT versus Air Speed

The waterfall plot gives a general idea of where flutter is occurring and how the frequency response evolves as the air speed changes. A more detailed look at individual FFTs at each air speed is also helpful. Figure 5.15 shows six FFTs from six different air speeds in increasing order, with the last one being the last air speed at which data were taken.

Figure 5.16 shows the flutter speed and flutter frequency for the first friction hinge configuration. Each plot includes two sets of theoretical results. The first set uses an aeroelastic damping ratio of zero as the boundary for stability, and the second set uses an aeroelastic damping ratio of 0.02 as the boundary for stability. Plotting the two sets of results is a method of seeing how sensitive the flutter behavior of the system is to structural damping and accuracy of the aerodynamic model.
Figure 5.15: FFT of System at 6 Air Speeds Leading up to and Including the Highest Test Speed

Figure 5.17 shows the flutter speed and flutter frequency for the second friction hinge configuration. The plot contains the same types of content as the plot for the first friction hinge configuration: two sets of theoretical results and one set of experimental results.

In general, the results are in agreement with the theoretical predictions. The measured flutter frequencies are in excellent agreement with the theoretical results. The measured flutter speeds generally follow the trend predicted by the theory, al-
though the actual values tend to be off by about 15%. Two interesting results from the plots are that the agreement looks better for low fold angles, and that there was an abrupt change in flutter behavior for Configuration 2 near 120 degree fold angle; the higher fold angles were not tested because the flutter speed was approaching the physical limits of the wind tunnel motor. The better agreement in flutter speeds at low angles may be due to better accuracy of the 3D aerodynamic model for a relative flat wing. The abrupt change in flutter behavior was not observed in experiment. There is not enough data available at this time to make a definite conclusion about whether the phenomenon of sudden change in flutter speed with varying fold angle
actually exists. When looking at the results overall, however, the data suggest that the aeroelastic model sufficiently captures the physics of the friction hinge folding wing system.

5.4 Redo Experiments for Original Three-Segment Lockheed-Type Wings

The newly-designed experimental models showed that by avoiding natural frequency crossings and misalignment of the wing with the flow, good agreement between theory and experiment were obtained. The new wind tunnel mounting system allows adjustment of the system alignment with the flow, thereby minimizing the steady state deflection of the test model during wind tunnel testing. Because this feature was not available in the preliminary flutter experiments for the original three-segment Lockheed-type folding wing, additional experiments were then conducted on the original configuration with the new wind tunnel mounting system.

For these experiments, the data acquisition procedure was simplified to only taking time series data of the system response and air speed. The time series data was then analyzed using MATLAB’s spectrogram function to calculate the short time Fourier Transform (STFT) of the system to analyze the frequency content of the response as time varies. Figure 5.18 shows example results of a spectrogram analysis. The first subplot shows a surface plot of the spectrogram, which shows the STFT of the system response over time. The second subplot shows air speed and system response over time. At each time step, the system response was the largest value of the STFT at that point in time.

The first test was to look at the flutter speed of the system as the fold angle changes from 75 degrees to 120 degrees. The predicted behavior is that the flutter speed would first increase as the fold angle increases, but then the flutter speed would increase up to near 90 m/s before dropping down to near 20 m/s within a small range
Figure 5.18: Example Results of Spectrogram Analysis on Flutter Test Data
of fold angles. This was indeed observed in the experiment, as shown in Fig. 5.19. The scales are the same on all of the plots, and the air speed curve shows steadily increasing air speed until the system response increases drastically, indicating that flutter has occurred. The plots show a sudden drop in flutter speed when the fold angle changes from 105 degrees to 120 degrees.

Figure 5.19: Flutter Test Results of Original Three-Segment Lockheed-Type Folding Wing from 75-Degree Fold Angle to 120-Degree Fold Angle

The second test was to look at the effects of steady state deflection during wind tunnel testing on the flutter behavior. The preliminary experimental results presented in previous Chapters of the present dissertation were obtained without possibility of controlling the alignment of the test model with the air flow. Any misalignment between wing segments and the flow would cause steady state aerodynamic
forcing and deflection during testing. Three experiments were conducted for the 120-degree fold angle case: one was done with minimized steady state deflection, another was done with an imposed positive deflection by increasing the angle of attack, and the last experiment was done with an imposed negative steady state deflection by setting the angle of attack in the opposite direction. Figure 5.20 qualitatively demonstrates the definition of positive deflection and negative deflection. In the actual experiments, the deflection at the outboard end of the first wing segment was approximately equal to half the chord length.

![Figure 5.20: Deformed Folding Wing Geometry due to Misalignment of Wing with Flow](image)

The flutter results for the case with minimized deflection was already discussed in the above comparison. Figure 5.21 shows the flutter results for both the case with positive deflection and the case with negative deflection. The case with positive deflection had a slightly lower flutter speed compared to the case with minimized deflection. The change was very small and the overall behavior was not significantly different. However, the case with negative deflection had a significantly higher flutter speed: near 47 m/s instead of near 26 m/s. This shows that static deflection can have a profound effect on the flutter behavior of this system. The results also show that the previous test model, used during the preliminary flutter experiments, must have
had some deflection during flutter testing because a high flutter speed was observed for the 120-degree fold angle case.

![Flutter Results for the 120-Degree Fold Angle Case with Imposed Static Deflection](image)

(a) Positive Deflection
(b) Negative Deflection

**Figure 5.21:** Flutter Results for the 120-Degree Fold Angle Case with Imposed Static Deflection

Figure 5.22 shows comparisons between theory and experiment for the flutter speed and flutter frequency. Two sets of theoretical results are plotted: a flutter boundary predicted by the strip theory aeroelastic model and another one predicted by the vortex lattice aeroelastic model. The experimental results were plotted without averaging to show the different results of different trials when varying the static deflection of the test model. There was no quantitative measurement of the static deflection, and implementing a method for taking such a measurement would be an important task for future work.

In summary, the new experimental results showed that the sudden change in flutter speed as the fold angle varies does in fact occur. This is caused by the interactions of the second bending and first torsion modes as the air speed increases. One caveat is that the results is sensitive to the amount of static deflection in the system; the results for the 120-degree fold angle case showed that the system with static deflection may have a significantly different flutter speed versus the same system without static deflection. This effect was exacerbated by the fact that the aeroelastic experimental
Figure 5.22: Flutter Speed and Frequency Results for the Recent Experiments for the Original Three-Segment Lockheed-Type Folding Wing
model was designed to be very flexible such that the flutter speed would be low. A nonlinear analysis is necessary to predict the sensitivity of the system aeroelastic behavior to static deflection. This is outside the scope of the present dissertation, but is an important question for future work.
The previous chapters have shown that the clamped-wing aeroelastic model may be used for the design of folding wings. However, an actual folding wing aircraft is very different from the theorized clamped wing in the existing work. The wing is only a part of the aircraft, and there are several other components affect the system dynamics. For an elastic system with low natural frequencies, which may be the case for folding wing systems and especially for micro air vehicles, the flight dynamics modes may couple with the elastic modes. In order to create a theoretical model that can be used for preliminary design of folding wing aircraft systems, it is necessary to account for at least the aircraft fuselage and tail, which affect the system stability. This chapter extends the clamped wing aeroelastic model to apply to an aircraft system by including aircraft rigid body motion, tail and fuselage inertia, and tail aerodynamic forces. The resulting theoretical model can predict both aeroelastic instabilities of the wings as well as flight dynamics instabilities of the aircraft. As a starting point, the present work will focus only on longitudinal dynamics during level flight, and limit the aircraft degrees of freedom to plunge, roll, and motion in flight direction.
To create an aeroelastic model that includes longitudinal flight dynamics, it is necessary to look again at the basic kinematics framework. The clamped wing model from the previous work neglects two important issues associated with aircraft system kinematics: the existence of longitudinal rigid body motion of the entire wing, and the need to carefully define component coordinate systems. For the clamped wing, there is no rigid body motion, and the coordinate system for the first wing segment is fixed and therefore is an inertial system. When considering the entire aircraft, only part of which is the wing, it now makes sense to have multiple coordinate systems for different parts of the aircraft: wing, fuselage, and tail. Each coordinate system must now allow rigid body motion since the aircraft, and therefore each component of the aircraft, must have at least the 3 longitudinal rigid body degrees of freedom. In addition, there is a choice of using either a rotating or non-rotating coordinate system, whereas this choice was absent for the clamped wing since all coordinate systems were non-rotating. The added complexity may be worked out using the existing model as a starting point, but the effort is significant enough that it is worthwhile to consider a different kinematics framework that is more easily understood and more conducive to deriving the equations of motion.

A natural choice for deriving the aircraft aeroelastic model is to use a general multi-body dynamics framework, which can handle not only multiple wing segments, but also multiple aircraft components. The derivation uses the method described in Shabana[29], with some simplifications that are specific to the present problem. The objective of this chapter is to derive a general multibody dynamics framework, and then define the wing coordinate system in a way that is most conducive to later incorporating the existing clamped-wing aeroelastic model. Specifically, this chapter gives an overview of the derivation, and discusses the structural dynamics. The derivation of the aeroelastic model will be discussed in the following chapter.
6.1 Overview

6.1.1 Wing Dynamics

The derivation starts with the wing dynamics. Each wing segment has its own coordinate system centered at the inboard edge of the wing segment on its elastic axis, the x axis is in the flow direction, and the y axis is aligned with the elastic axis. The coordinate systems are non-rotating, which is not typical of a flight dynamics analysis in which the wing and aircraft coordinate systems rotate with reference plane on the aircraft. However, the choice to use a non-rotating coordinate system is based on the fact that it is much easier to build the equations of motion. The transformation from wing frame to inertial frame depends only on the wing fold angle, which is assumed to be constant. Figure 6.1 shows the relative coordinate systems of each wing segment from a head-on point of view. The x axis in the figure points out of the page.

![Figure 6.1: Relative Coordinate Systems of Each Wing Segment](image)

For each wing segment, the generalized coordinates are grouped into ”frame translation modes” and ”wing modes”. The frame translation modes are the three rigid body translations in the x, y, and z directions; the y translation mode is the same as the in-plane translation mode from Wang et al[27]. The wing modes are all other modes: the 3 rigid body rotations and the elastic modes. The z-rotation mode is similar to the in-plane rotation mode as described in Wang et al[27], the x-rotation
mode can be considered one of the zero-frequency bending modes, and the y-rotation mode can be considered a zero-frequency torsion mode. For the first wing, the y-translation mode, x-rotation mode, and the z-rotation mode are not included since the present work only considers longitudinal motion.

Constraint equations simply state that the absolute positions of adjacent wing segments at each spring connection must be the same. For a beam theory model where only bending and torsion modes exist, it is sufficient to constrain only two locations at each hinge. For the present work, one constraint was enforced at the elastic axis and the other was enforced at the leading edge of the wing.

6.1.2 Wing Aerodynamics

Unlike traditional flight dynamics convention, the free stream velocity components are defined in the inertial frame directions. Again, this simplifies the equations of motion since no coordinate transformations due to frame motion are necessary when calculating the aerodynamic loads. However, it is still necessary to transform the free stream into the local wing reference frame using the fold angle.

The aerodynamic force and moment are based on the downwash at the 3/4 chord for the circulatory terms, and the rigid body plunge and rotation for the non-circulatory terms. In general, the aerodynamic force acts in all three coordinate directions after rotating the relative frame lift vector, first by the angle of attack about the y axis and then by the fold angle about the x axis, into the inertial frame. The angle of attack is the plunge velocity divided by the nominal free stream velocity.

The virtual work at each point on the wing is the dot product between the aerodynamic force vector and the virtual displacement vector at that point. The generalized force is the partial derivative of the virtual work with respect to a particular virtual displacement. The virtual work terms due to lift and moment are calculated in the wing relative frame.
6.1.3 Aircraft Dynamics and Aerodynamics

To model an entire aircraft, not just the wings, additional coordinates need to be introduced to keep track of aircraft components, namely the tail and the fuselage in the simple model here. These coordinates are defined to be body-fixed rotating coordinates, such that the coordinates rotate with the aircraft pitch. Figure 6.2 summarizes the coordinate systems of each component of the aircraft. The relative coordinate systems of the tail and fuselage are body-fixed, and the wing relative coordinate system has the $\tilde{x}$ axis always pointing in the inertial $x$ direction.

![Figure 6.2: Relative Coordinate Systems of Aircraft Components](image)

The kinetic energies of the tail and fuselage are described by translational and rotational kinetic energy components. There is no coupling between translation and rotation because the coordinate system is defined at the center of gravity of each component. There is no potential energy associated with the aircraft components. The aerodynamic forces due to the tail can be calculated from the downwash at the tail 3/4 chord. Either a quasi-steady or the Theodorsen unsteady aerodynamic model may be used to calculate the aerodynamic forces. Lastly, it is necessary to constrain the tail and fuselage relative to the wing to form a complete aircraft.
6.2 Structural Dynamics

The general structural dynamics derivation follows the multi-body dynamics framework established in Shabana.[29] The notation from Shabana will be used throughout the text such that it will be easier to make direct comparisons to the literature.

6.2.1 Coordinate System and Kinematics

The absolute position of any point on an elastic structure is given by the following equation.

\[ R_i = R_{io} + A_{ii} (\tilde{u}_i + \bar{u}_{ie}) \]  

(6.1)

The above equation states that the absolute position of a point on the deformed structure is equal to the absolute position of the coordinate system origin \( R_{io} \) plus the absolute position of the deformed point relative to the coordinate system origin, with the latter equal to the position of the undeformed structure \( \tilde{u}_i \) plus any elastic deformations \( \bar{u}_{ie} \). The \( \tilde{\cdot} \) symbol signifies that the vector is expressed in terms of the relative coordinate basis vectors, which is transformed into the inertial coordinate system using the matrix \( A_{ii} \). Figure 6.3 shows the definitions of the above terms.

![Figure 6.3: Schematic of Vectors in the Multibody Dynamics Framework](image)

Specifically, the coordinate transformation matrix \( A_{ii} \) for the \( i \)th wing contains the angle \( \Psi_i \) of the \( i \)th wing segment relative to the horizontal. The matrix rotates the vector counterclockwise by the angle \( \Psi_i \) about the \( \tilde{x}_i \) axis to obtain the vector
in terms of inertial coordinate basis vectors.

\[
A_{ii} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Psi_i & -\sin \Psi_i \\ 0 & \sin \Psi_i & \cos \Psi_i \end{pmatrix}
\] (6.2)

The elastic deformation term can be written as the product of generalized coordinates and mode shapes. Here it is expressed as a matrix inner product following Shabana’s notation, but it can be equivalently expressed as a modal summation as well. In the equation below the matrix \( \tilde{S}_i \) is a 3 by \( N \) matrix, where \( N \) is the number of modal coordinates, that contains the \((x, y, z)\) displacements of the mode shapes at some point on the structure.

\[
\tilde{u}_{ie} = \tilde{S}_i q_{ie} \quad (6.3)
\]

\[
R_i = R_{io} + A_{ii} \left( \tilde{u}_i + \tilde{S}_i q_{ie} \right) \quad (6.4)
\]

Next, the absolute velocity is the time derivative of the absolute position. The equation below uses the notation in Shabana.

\[
\dot{R}_i = \begin{bmatrix} I_3 & -A_{ii} \hat{u}_i \tilde{G}_i & A_{ii} \tilde{S}_i \end{bmatrix} \begin{bmatrix} \dot{R}_{io} \\ \dot{\theta}_i \\ \dot{q}_{ie} \end{bmatrix} \quad (6.5)
\]

The equation is a vector dot product expression for calculating the absolute velocity based on time derivatives of the system generalized coordinates, including the reference frame position \( R_{io} \), the reference frame angular description vector \( \theta_i \), and the elastic coordinates \( q_{ie} \). In the row vector, the matrix \( \tilde{G}_i \) multiplies the vector \( \theta_i \) to give the angular velocity vector, and the matrix \( \hat{u}_i \) performs a cross product of the deformed position with the angular velocity to obtain the additional term in the velocity due to having a rotating coordinate system. The third element of the row vector converts changes in the elastic generalized coordinate to changes in position.

Up to this point, the expressions are general and apply to any multi-body elastic system.
6.2.2 Kinetic Energy

The kinetic energy is equal to the general expression of one half times the velocity squared, integrated over the mass of the structure.

\[ T_i = \int \frac{1}{2} dM_i \dot{\mathbf{R}}_i^T \dot{\mathbf{R}}_i \]  

(6.6)

Since Eq(6.5) expresses the absolute velocity in terms of system generalized co-ordinates, the equation can be used here to express kinetic energy using the same coordinates.

\[ T_i = \frac{1}{2} \dot{q}_i^T M_i \dot{q}_i \]  

(6.7)

The mass matrix \( M_i \) is given by the following equations.

\[ M_i = \int \begin{bmatrix} I_3 & \left( -A_{ii} \hat{u}_i \hat{G}_i \right)^T \\ \left( A_{ii} \hat{S}_i \right) \end{bmatrix} \begin{bmatrix} I_3 & -A_{ii} \hat{u}_i \hat{G}_i \\ A_{ii} \hat{S}_i \end{bmatrix} dM_i \]  

(6.8)

\[ = \int \begin{bmatrix} I_3 I_3 & -I_3 A_{ii} \hat{u}_i \hat{G}_i \\ \hat{G}_i^T \hat{u}_i \hat{G}_i & I_3 A_{ii} \hat{S}_i \end{bmatrix} dM_i \]  

(6.9)

\[ = \begin{bmatrix} M_{i,RR} & M_{i,R\theta} & M_{i,Re} \\ M_{i,\theta R} & M_{i,\theta \theta} & M_{i,\theta e} \\ \text{sym} & \text{sym} & M_{i,ee} \end{bmatrix} \]  

(6.10)

In the equations above, the lower triangular parts are not shown because the mass matrix is symmetric. In the end, the mass matrix is divided into six distinct components. The diagonal components are associated with uncoupled rigid translation, rigid rotation, and elastic deformation. The off-diagonal components are associated with coupling between any two of the three types of motions. The matrices associated with rigid translation and elastic deformation are relatively easy to compute.
The component of the mass matrix associated with uncoupled rigid body translation is simply a diagonal matrix with entries equal to the mass of the structure. The component of the mass matrix associated with uncoupled elastic deformation is the generalized mass matrix of the elastic modes; the size of this matrix is equal to the number of elastic modes chosen to describe the system, and the matrix is diagonal if the modes are orthogonal. Orthogonality of structural modes is assumed, so the off-diagonal terms are zero and are not shown.

\[
M_{i,RR} = \begin{bmatrix}
M_i & 0 & 0 \\
0 & M_i & 0 \\
0 & 0 & M_i \\
\end{bmatrix}
\] (6.11)

\[
M_{i,ee} = \int \begin{bmatrix}
\tilde{\Psi}_1 \cdot \tilde{\Psi}_1 \\
\tilde{\Psi}_2 \cdot \tilde{\Psi}_2 \\
\vdots \\
\end{bmatrix} dM_i
\] (6.12)

The mass matrices associated with the frame rotation are more difficult to write down because they depend on the type of description used for the frame rotation, such as Euler angles or Rodriguez parameters. They are discussed in the next section.

6.2.3 Wing Kinetic Energy

The wing itself is an elastic multi-body structure. For each wing segment, a coordinate system is defined at the elastic axis at the inboard edge. Furthermore, the coordinate system is defined to be non-rotating, and the x-axis is aligned with the inertial x-axis. This simplifies the equations because now the angular coordinates \( \theta \) are equal to zero and the corresponding terms in the mass matrix vanish.

The following equations result from the definition of non-rotating coordinate systems. First, the angular coordinates are equal to zero. Second, the generalized coordinates for the wing segment contain frame rigid body translations and structural modes. Third, the absolute velocity matrix expression no longer contains the angular coordinates. Lastly, the mass matrix contains only two uncoupled contribu-
tions and one coupling component between structural deformations and frame rigid body translations.

\[ \theta_i = 0 \]  
\[ q_i = [ R_{io} \quad q_{ie} ]^T \]  
\[ \dot{R}_i = [ I_3 \quad A_{ii} \tilde{S}_i ] \left[ \begin{array}{c} \dot{R}_{io} \\ \dot{q}_{ie} \end{array} \right] \]  
\[ M_i = \begin{bmatrix} M_{i,RR} & M_{i,Re} \\ M_{i,Re}^T & M_{i,ee} \end{bmatrix} \]

However, by not using the angular coordinates, it is necessary to now use other coordinates to describe rigid body rotations. This is done by including rigid body rotation modes in the structural dynamics analysis. All three rigid body rotational degrees of freedom are needed for each wing segment except for the first (i.e. most-inboard) wing segment. The first wing segment has only rotation about the y axis, which corresponds to pitching motion.

The generalized coordinates for these rigid body modes are most easily included within the vector \( q_{ie} \). Note that the discussion has been careful not to use the word "elastic" to describe the structural modes. The distinction is necessary because the rigid body modes are not elastic, but are structural modes that need to be kept in the analysis. Alternatively, one may think of these rigid body modes as direct replacements for the frame angular coordinates. Defining coordinates this way is more convenient in this case because by treating rigid body rotations as structural modes, much of the structure from the clamped wing aeroelastic code can be reused in this general multi-body dynamics code.

With the above changes, the kinetic energy can be written more explicitly by
carrying out the matrix multiplication.

\[
T_i = \frac{1}{2} \left( \dot{R}_{i,o}^T M_{i,RR} \dot{R}_{i,o} + \dot{R}_{i,o}^T M_{i,Re} \dot{q}_{i,e} + \dot{q}_{i,e}^T M_{i,Re}^T \dot{R}_{i,o} + \dot{q}_{i,e}^T M_{i,ee} \dot{q}_{i,e} \right) \quad (6.17)
\]

Lastly, the kinetic energy can be expanded in terms of the individual generalized coordinates. The first term represents kinetic energy from only translational motion of the reference frame. The second term represents kinetic energy from additional motion of the structure, due to either rigid body motion within the reference frame or elastic motion. The last three terms represent additional kinetic energy due to coupling between the frame motion and any additional structural motion relative to the frame.

\[
T_i = \frac{1}{2} M_i \left( \dot{R}_{i,x}^2 + \dot{R}_{i,y}^2 + \dot{R}_{i,z}^2 \right) + \frac{1}{2} \sum_m \sum_n M_{i,ee,mn} \dot{q}_{i,m} \dot{q}_{i,n} \\
+ \dot{R}_{i,x} \sum_m \dot{q}_{i,m} \int X_{i,m} dM_i + \dot{R}_{i,y} \sum_m \dot{q}_{i,m} \int \Phi_{i,m} dM_i \\
+ \dot{R}_{i,z} \sum_m \dot{q}_{i,m} \int \Psi_{i,m} dM_i \quad (6.18)
\]

6.2.4 Constraints

To form a folding wing system, constraint equations must be defined to enforce a connection at each hinge between adjacent wing segments. For a simple hinge model, the absolute positions of all points on the two adjacent wing segments must be equal all along the hinge, since the adjacent edges of the wing segments are held firmly by the hinge all along the edges. For the beam theory model, it is only necessary to match the absolute positions at two locations along the hinge. This is because beam theory only allows two degrees of freedom - bending and torsion.

The general constraint equation is that the absolute positions of the two wing segments are equal at the hinge.

\[
0 = R_{i-1} - R_i \quad (6.19)
\]
The equation may be expanded by breaking down the absolute position vector into three components: coordinate system origin, nominal position, and elastic deformation.

\[
0 = \left[ R_{i-1,o} + A_{i-1,i-1} \left( \tilde{u}_{i-1} + \tilde{S}_{i-1}q_{i-1,e} \right) \right] - \left[ R_{i,o} + A_{ii} \left( \tilde{u}_i + \tilde{S}_iq_{ie} \right) \right]
\] (6.20)

Define the constraint function as the right hand side of the constraint equation. The vector function \( f_i \) is the constraint function on the \( i \)th hinge, which connects the \((i-1)\)th wing segment and the \( i \)th wing segment.

\[
f_i \equiv \left[ R_{i-1,o} + A_{i-1,i-1} \left( \tilde{u}_{i-1} + \tilde{S}_{i-1}q_{i-1,e} \right) \right] - \left[ R_{i,o} + A_{ii} \left( \tilde{u}_i + \tilde{S}_iq_{ie} \right) \right]
\] (6.21)

The terms for the undeformed position \( \tilde{u}_{i-1} \) and \( \tilde{u}_i \), and the terms for the mode shapes \( \tilde{S}_{i-1} \) and \( \tilde{S}_i \), are evaluated in the wing segments’ respective local coordinates at the point where the constraint is being enforced. In this work, the constraints are applied at the elastic axis and the leading edge. These locations are chosen for convenience, but constraining any two locations along the chord-wise direction will work. Table 6.1 summarize the locations at which the terms \( \tilde{u} \) and \( \tilde{S} \) are evaluated in this beam structural model.

<table>
<thead>
<tr>
<th>Constraint at Elastic Axis</th>
<th>Constraint at Leading Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i - 1 )th wing</td>
<td>((0, L_{i-1}))</td>
</tr>
<tr>
<td>( i )th wing</td>
<td>((b_i(1 + a_i), L_{i-1}))</td>
</tr>
<tr>
<td></td>
<td>((0, 0))</td>
</tr>
<tr>
<td></td>
<td>((b_i(1 + a_i), 0))</td>
</tr>
</tbody>
</table>

Each hinge has two vector constraint functions: one for the constraint at the elastic axis and one for the constraint at the leading edge. The two functions are
shown in the equations below.
\[ f_{i,EA} = \left[ R_{i-1,o} + A_{i-1,i-1} \left( \tilde{u}_{i-1}(0, L_{i-1}) + \tilde{S}_{i-1}(0, L_{i-1})q_{i-1,e} \right) \right] 
- \left[ R_{i,o} + A_{ii} \left( \tilde{u}_i(0,0) + \tilde{S}_i(0,0)q_{ie} \right) \right] \]  \hspace{1cm} (6.22)

\[ f_{i,LE} = \left[ R_{i-1,o} + A_{i-1,i-1} \left( \tilde{u}_{i-1}(\frac{c}{2}, L_{i-1}) + \tilde{S}_{i-1}(\frac{c}{2}, L_{i-1})q_{i-1,e} \right) \right] 
- \left[ R_{i,o} + A_{ii} \left( \tilde{u}_i(\frac{c}{2},0) + \tilde{S}_i(\frac{c}{2},0)q_{ie} \right) \right] \]  \hspace{1cm} (6.23)

Equations (6.22) and (6.23) are vector constraint functions that equate the x, y, and z displacements between two adjacent wing segments at the elastic axis and leading edge, respectively. Each vector constraint function can be expanded into three scalar functions. For example, Eq. (6.22) can be written as the following three scalar constraint functions.

\[ f_{i,EA,x} = \left[ X_{i-1,o} + [1 \ 0 \ 0] \cdot A_{i-1,i-1} \left( \tilde{u}_{i-1}(0, L_{i-1}) + \tilde{S}_{i-1}(0, L_{i-1})q_{i-1,e} \right) \right] 
- \left[ X_{i,o} + [1 \ 0 \ 0] \cdot A_{ii} \left( \tilde{u}_i(0,0) + \tilde{S}_i(0,0)q_{ie} \right) \right] \]  \hspace{1cm} (6.24)

\[ f_{i,EA,y} = \left[ Y_{i-1,o} + [0 \ 1 \ 0] \cdot A_{i-1,i-1} \left( \tilde{u}_{i-1}(0, L_{i-1}) + \tilde{S}_{i-1}(0, L_{i-1})q_{i-1,e} \right) \right] 
- \left[ Y_{i,o} + [0 \ 1 \ 0] \cdot A_{ii} \left( \tilde{u}_i(0,0) + \tilde{S}_i(0,0)q_{ie} \right) \right] \]  \hspace{1cm} (6.25)

\[ f_{i,EA,z} = \left[ Z_{i-1,o} + [0 \ 0 \ 1] \cdot A_{i-1,i-1} \left( \tilde{u}_{i-1}(0, L_{i-1}) + \tilde{S}_{i-1}(0, L_{i-1})q_{i-1,e} \right) \right] 
- \left[ Z_{i,o} + [0 \ 0 \ 1] \cdot A_{ii} \left( \tilde{u}_i(0,0) + \tilde{S}_i(0,0)q_{ie} \right) \right] \]  \hspace{1cm} (6.26)

In summary, there are two vector constraint equations for each hinge, one at the elastic axis and one at the leading edge. There are \( N - 1 \) hinges for an \( N \)-segment folding wing, so there are a total of \( 2(N - 1) \) vector constraint equations in the folding wing structural dynamics model.

There is one vector Lagrange multiplier \( \lambda_j \) for each vector constraint function \( f_j \). The vector Lagrange multiplier is the force vector necessary to enforce the constraint.
Equivalently, there are three scalar Lagrange multipliers for each vector constraint function, corresponding to the components of the constraint force.

### 6.2.5 Potential Energy

The potential energy comes from the elastic energies of wing segments undergoing deformation and the elastic springs undergoing angular displacement. The potential energy depends only on the elastic modes in local coordinates, and is unaffected by the general definitions of the coordinate system, the frame rigid body translation coordinates, and the removal of frame angular coordinates.

The potential energy of each wing segment is given by the following equation, which is common to modal analysis. The summation is over beam natural modes and it is simple to calculate the natural frequencies of a uniform beam.

$$V_i = \frac{1}{2} \sum_m M_i \omega_{i,m}^2 q_{i,m}^2$$  \hspace{1cm} (6.27)

The potential energy of each hinge is given by Hooke’s law since each hinge is modeled as a linear torsional spring. This is the same result as the derivation in Chapter 2. The spring potential energy is equal to one half times the spring stiffness times the square of the angular displacement. For the linearized model, the angular displacement is equal to the difference in the slope of the deformed wing segments along the spring directions. The angular displacement of the spring is computed using the slope of the out-of-plane mode shape in the relative frame along the spring direction. It is permissible (and easier) to use the relative frame because the spring elastic potential energy is created from the relative motion of the wing segments. The spring direction is first computed using cross product of the hinge direction and the wing segment normal vector when the wing is flat. This works even when the wing is folded up because the spring is always perpendicular to the hinge and the
derivatives will be in the same direction.

\[ \tilde{v}_{s,i} = -\tilde{h}_i \times (0, 0, 1) \]  

(6.28)

The angular displacement, linearized to the slope of the wing segment, can be expressed using directional derivatives. The equation below expresses the potential energy of each spring for an arbitrarily-shaped wing segment.

\[ V_{s,i} = \frac{1}{2} k_i \left( \nabla \tilde{\Psi}_i \cdot \tilde{v}_{s,i} - \nabla \tilde{\Psi}_i \cdot \tilde{v}_{s,i} \right)^2 \]  

(6.29)

For rectangular wing segments in which the hinge is always in the x-direction, the potential energy depends on the slopes of the mode shapes in the y-direction. The expression is then simplified.

\[ V_{s,i} = \frac{1}{2} k_i \left( \sum_n q_{i-1,m} \frac{\partial \tilde{\Psi}_{i-1,n}}{\partial y_{i-1}} (L_{i-1}) - \sum_n q_{i,m} \frac{\partial \tilde{\Psi}_{i,n}}{\partial y_i} (0) \right)^2 \]  

(6.30)

6.2.6 Lagrange’s Equation

The kinetic energy, potential energy, and constraint equations are substituted into Lagrange’s Equation for each generalized coordinate. The generalized coordinates include the frame translation coordinates, the rigid body z-rotation coordinate, and the beam mode shape coordinates for each wing segment.

\[- \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial V}{\partial q} + \sum_j \lambda_j \frac{\partial f_j}{\partial q} = 0 \]  

(6.31)

An assumption is made in the above equation that the kinetic energy depends on only the time derivative of generalized coordinates \( \dot{q} \), and the potential energy depends only on the generalized coordinates \( q \). Also the constraints are geometric constraints that only depend on the generalized coordinates. These assumptions are true for the structural dynamics models of many systems, including the folding wing
system in the present work. The variable \( q \) is used here to represent any generalized coordinate in the system.

Consider the \( i \)th wing segment. The following analysis computes the terms in Lagrange’s equations for an arbitrary generalized coordinate of the \( i \)th wing segment.

**Frame Translation Coordinates**

The frame translation coordinates are \( R_{i,x}, R_{i,y}, \) and \( R_{i,z} \). The terms that depend on those coordinates are the wing segment kinetic energy, the constraint equations for the \( i \)th hinge, and the constraint equations for the \((i + 1)\)th hinge.

The wing segment kinetic energy terms that depend on the three frame translation coordinates are shown below, where \(*\) can be \( x, y, \) or \( z \). The terms come from Eq. (6.18).

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial R_{i,*}} \right) = \frac{d}{dt} \left\{ \frac{\partial}{\partial R_{i,*}} \left[ \frac{1}{2} M_i \left( \dot{R}_{i,x}^2 + \dot{R}_{i,y}^2 + \dot{R}_{i,z}^2 \right) + \dot{R}_{i,x} \sum_m \ddot{q}_{i,m} \int X_{i,m} dM_i \right. \right. \\
+ \left. \dot{R}_{i,y} \sum_m \ddot{q}_{i,m} \int \Phi_{i,m} dM_i \right. \right. \\
+ \left. \dot{R}_{i,z} \sum_m \ddot{q}_{i,m} \int \Psi_{i,m} dM_i \right] \right\} \tag{6.32}
\]

The contribution to Lagrange’s equation of each frame translation coordinate from the kinetic energy is shown below.

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial R_{i,x}} \right) = -M_i \ddot{R}_{i,x} - \sum_m \ddot{q}_{i,m} \int X_{i,m} dM_i \tag{6.33}
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial R_{i,y}} \right) = -M_i \ddot{R}_{i,y} - \sum_m \ddot{q}_{i,m} \int \Phi_{i,m} dM_i \tag{6.34}
\]
\[-\frac{d}{dt} \left( \frac{\partial T}{\partial \ddot{R}_{i,z}} \right) = -M_i \ddot{R}_{i,z} - \sum_m \ddot{q}_{i,m} \int \psi_{i,m} dM_i \quad (6.35)\]

The above three equations can also be expressed in matrix form. Alternatively, the matrix form can be derived from the matrix form of the wing kinetic energy in Eq. (6.17).

\[-\frac{d}{dt} \left( \frac{\partial T}{\partial \ddot{R}_{i,o}} \right) = -M_{RR} \ddot{R}_{i,o} - M_{Re} \ddot{q}_{i,e} \quad (6.36)\]

The next step is to consider the potential energy. The frame translation coordinates represent rigid body motions and therefore do not contribute to elastic potential energy. The gravitational potential energy term results in a constant force in the negative \( z \) direction, which will not have any effect after the system is linearized, so that term is not retained.

\[-\frac{\partial V}{\partial R_{i,x}} = 0 \quad (6.37)\]
\[-\frac{\partial V}{\partial R_{i,y}} = 0 \quad (6.38)\]
\[-\frac{\partial V}{\partial R_{i,z}} = 0 \quad (6.39)\]

The last step is to consider the constraint equations. The frame translation coordinates appear in four sets of constraint equations: the constraint equations at the elastic axis and leading edge for the \( i \)th hinge, and the constraint equations at the elastic axis and leading edge for the \((i + 1)\)th hinge. The four terms that contribute to the equations of motion are shown in Eq. (6.40), in which the placeholder \( * \) can
be \( x, y, \) or \( z \).

\[
\sum_j \lambda_j \frac{\partial f_j}{\partial R_i} = \lambda_{i,EA,x} \frac{\partial f_{i,EA,*}}{\partial R_i} + \lambda_{i,LE,x} \frac{\partial f_{i,LE,*}}{\partial R_i} + \lambda_{i+1,EA,x} \frac{\partial f_{i+1,EA,*}}{\partial R_i} + \lambda_{i+1,LE,x} \frac{\partial f_{i+1,LE,*}}{\partial R_i} \tag{6.40}
\]

The contribution to Lagrange’s equation from the constraints is shown for each frame translation coordinate in Eqs. (6.41)-(6.43).

\[
\sum_j \lambda_j \frac{\partial f_j}{\partial R_i,x} = -\lambda_{i,EA,x} - \lambda_{i,LE,x} + \lambda_{i+1,EA,x} + \lambda_{i+1,LE,x} \tag{6.41}
\]

\[
\sum_j \lambda_j \frac{\partial f_j}{\partial R_i,y} = -\lambda_{i,EA,y} - \lambda_{i,LE,y} + \lambda_{i+1,EA,y} + \lambda_{i+1,LE,y} \tag{6.42}
\]

\[
\sum_j \lambda_j \frac{\partial f_j}{\partial R_i,z} = -\lambda_{i,EA,z} - \lambda_{i,LE,z} + \lambda_{i+1,EA,z} + \lambda_{i+1,LE,z} \tag{6.43}
\]

Alternatively, the contributions from the constraint terms can be written in matrix form, as shown in Eq. (6.44). The Lagrange multipliers are grouped into vectors. The matrix \( I_3 \) is the 3 by 3 identity matrix.

\[
\sum_j \lambda_j \frac{\partial f_j}{\partial R_i,o} = -I_3 \lambda_{i,EA} - I_3 \lambda_{i,LE} + I_3 \lambda_{i+1,EA} + I_3 \lambda_{i+1,LE} \tag{6.44}
\]

The equations of motion for the frame translation coordinates are obtained by combining the above terms into a single equation.

\[
0 = -M_i \ddot{R}_{i,x} - \sum_m \ddot{q}_{i,m} \int X_{i,m} dM_i - \lambda_{i,EA,x} - \lambda_{i,LE,x} + \lambda_{i+1,EA,x} + \lambda_{i+1,LE,x} \tag{6.45}
\]

\[
0 = -M_i \ddot{R}_{i,y} - \sum_m \ddot{q}_{i,m} \int \Phi_{i,m} dM_i - \lambda_{i,EA,y} - \lambda_{i,LE,y} + \lambda_{i+1,EA,y} + \lambda_{i+1,LE,y} \tag{6.46}
\]

\[
0 = -M_i \ddot{R}_{i,z} - \sum_m \ddot{q}_{i,m} \int \Psi_{i,m} dM_i - \lambda_{i,EA,z} - \lambda_{i,LE,z} + \lambda_{i+1,EA,z} + \lambda_{i+1,LE,z} \tag{6.47}
\]
Alternatively, the equations of motion for the frame translation coordinates can be written more compactly in matrix form.

$$0 = -M_{RR} \ddot{R}_{i,o} - M_{Re} \dot{q}_{i,e} - I_3 \lambda_{i,EA} - I_3 \lambda_{i,LE} + I_3 \lambda_{i+1,EA} + I_3 \lambda_{i+1,LE}$$  \hspace{1cm} (6.48)

**Wing Segment Coordinates**

The wing segment coordinates are $q_{i,m}$ where $i$ denotes the $i$th wing segment and $m$ denotes the $m$th mode. The terms that depend on those coordinates are the wing segment kinetic energy, the wing segment potential energy, the potential energies for the $i$th hinge and the $(i+1)$th hinge, and the constraint equations for the $i$th hinge and the $(i+1)$th hinge.

The kinetic energy terms that depend on the wing segment coordinates from Eq. (6.18) are substituted into Lagrange’s Equation. The result is shown in Eq. (6.49).

$$-\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_{i,m}} \right) = -\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{R}_{i,*}} \left[ \frac{1}{2} \sum_m \sum_n M_{i,ee,mn} q_{i,m} \dot{q}_{i,n} \right. \right.$$  

$$+ \dot{R}_{i,x} \sum_m \dot{q}_{i,m} \int X_{i,m} dM_i \right.$$  

$$+ \dot{R}_{i,y} \sum_m \dot{q}_{i,m} \int \Phi_{i,m} dM_i \right.$$  

$$+ \dot{R}_{i,z} \sum_m \dot{q}_{i,m} \int \Psi_{i,m} dM_i \left. \right\}$$  \hspace{1cm} (6.49)

Because the wing segment generalized coordinates represent normal modes, the modes are orthogonal to each other and the double sum in the above equation is simplified to a single sum. Then for each wing segment generalized coordinate $q_{i,m}$, the contribution to the equation of motion from the kinetic energy is expressed in
the following equation.

\[- \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_{i,m}} \right) = - M_{i,ee,m} \ddot{q}_{i,m} - \bar{R}_{i,x} \sum_{m} X_{i,m} dM_{i} \]

\[- \bar{R}_{i,y} \sum_{m} \int \Phi_{i,m} dM_{i} - \bar{R}_{i,z} \sum_{m} \int \Psi_{i,m} dM_{i} \quad (6.50)\]

Each wing segment generalized coordinate \(q_{i,m}\) appears in the potential energy of the \(i\)th wing segment, the potential energy of the hinge between the \((i-1)\) and \(i\)th wing segments, and the potential energy of the hinge between the \((i+1)\) and \(i\)th wing segments. Those potential energy terms from Eqs. (6.27) and (6.30) are substituted into Lagrange’s Equation. The result is shown in Eq. (6.51).

\[- \frac{\partial V}{\partial q_{i,m}} = - M_{i,ee,m} \omega_{i,m}^{2} q_{i,m} \]

\[- k_{i} \left( \sum_{n} q_{i-1,n} \frac{\partial \Psi_{i-1,n}}{\partial y_{i-1}} (L_{i-1}) - \sum_{n} q_{i,n} \frac{\partial \Psi_{i,n}}{\partial y_{i}} (0) \right) \left( - \frac{\partial \Psi_{i,m}}{\partial y_{i}} (0) \right) \]

\[- k_{i+1} \left( \sum_{n} q_{i,n} \frac{\partial \Psi_{i,n}}{\partial y_{i}} (L_{i}) - \sum_{n} q_{i+1,n} \frac{\partial \Psi_{i+1,n}}{\partial y_{i+1}} (0) \right) \left( \frac{\partial \Psi_{i,m}}{\partial y_{i}} (L_{i}) \right) \quad (6.51)\]

The last step is to consider the constraint equations. The wing segment generalized coordinates appear in four sets of constraint equations: the constraint equations at the elastic axis and leading edge for the \(i\)th hinge, and the constraint equations at the elastic axis and leading edge for the \((i+1)\)th hinge. The four terms that
contribute to the equations of motion are shown in Eq. (6.52).

$$\sum_j \lambda_j \frac{\partial f_j}{\partial q_{i,m}} =$$

$$+ \lambda_{i,EA,x} \frac{\partial f_{i,EA,x}}{\partial q_{i,m}} + \lambda_{i,LE,x} \frac{\partial f_{i,LE,x}}{\partial q_{i,m}} + \lambda_{i+1,EA,x} \frac{\partial f_{i+1,EA,x}}{\partial q_{i,m}} + \lambda_{i+1,LE,x} \frac{\partial f_{i+1,LE,x}}{\partial q_{i,m}}$$

$$+ \lambda_{i,EA,y} \frac{\partial f_{i,EA,y}}{\partial q_{i,m}} + \lambda_{i,LE,y} \frac{\partial f_{i,LE,y}}{\partial q_{i,m}} + \lambda_{i+1,EA,y} \frac{\partial f_{i+1,EA,y}}{\partial q_{i,m}} + \lambda_{i+1,LE,y} \frac{\partial f_{i+1,LE,y}}{\partial q_{i,m}}$$

$$+ \lambda_{i,EA,z} \frac{\partial f_{i,EA,z}}{\partial q_{i,m}} + \lambda_{i,LE,z} \frac{\partial f_{i,LE,z}}{\partial q_{i,m}} + \lambda_{i+1,EA,z} \frac{\partial f_{i+1,EA,z}}{\partial q_{i,m}} + \lambda_{i+1,LE,z} \frac{\partial f_{i+1,LE,z}}{\partial q_{i,m}}$$

$$= (6.52)$$

The partial derivative of the constraint with respect to a wing segment generalized coordinate is, to a factor of \(\pm 1\), the value of the corresponding mode shape at the hinge expressed in the inertial reference frame. The contributions to the equation of motion from the constraint terms are expressed in Eq. (6.53).

$$\sum_j \lambda_j \frac{\partial f_j}{\partial q_{i,m}} =$$

$$- \lambda_{i,EA,x} X_{i,m}(0, 0) - \lambda_{i,EA,y} \Phi_{i,m}(0, 0) - \lambda_{i,EA,z} \Psi_{i,m}(0, 0)$$

$$- \lambda_{i,LE,x} X_{i,m}(c/2, 0) - \lambda_{i,LE,y} \Phi_{i,m}(c/2, 0) - \lambda_{i,LE,z} \Psi_{i,m}(c/2, 0)$$

$$+ \lambda_{i+1,EA,x} X_{i,m}(0, L_i) + \lambda_{i+1,EA,y} \Phi_{i,m}(0, L_i) + \lambda_{i+1,EA,z} \Psi_{i,m}(0, L_i)$$

$$+ \lambda_{i+1,LE,x} X_{i,m}(c/2, L_i) + \lambda_{i+1,LE,y} \Phi_{i,m}(c/2, L_i) + \lambda_{i+1,LE,z} \Psi_{i,m}(c/2, L_i)$$

$$= (6.53)$$

The equation of motion for each wing segment generalized coordinate is obtained by combining the contributions of kinetic energy, potential energy, and constraints from Lagrange’s Equation. Equation (6.54) is the equation of motion for a wing

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segment generalized coordinate.

\[ 0 = -M_{i,ee,m} \ddot{q}_{i,m} - M_{i,ee,m} \omega_{i,m}^2 q_{i,m} \]

\[ - \ddot{R}_{i,x} \sum_m \int X_i,m dM_i - \ddot{R}_{i,y} \sum_m \int \Phi_i,m dM_i - \ddot{R}_{i,z} \sum_m \int \Psi_i,m dM_i \]

\[ - k_i \left( \sum_n q_{i-1,n} \frac{\partial \Psi_{i-1,n}}{\partial y_{i-1}} (L_{i-1}) - \sum_n q_{i,n} \frac{\partial \Psi_{i,n}}{\partial y_i} (0) \right) \left( - \frac{\partial \Psi_{i,m}}{\partial y_i} (0) \right) \]

\[ - k_{i+1} \left( \sum_n q_{i,n} \frac{\partial \Psi_{i,n}}{\partial y_i} (L_i) - \sum_n q_{i+1,n} \frac{\partial \Psi_{i+1,n}}{\partial y_{i+1}} (0) \right) \left( \frac{\partial \Psi_{i,m}}{\partial y_i} (L_i) \right) \]

\[ - \lambda_{i,EA,x} X_{i,m} (0, 0) - \lambda_{i,EA,y} \Phi_{i,m} (0, 0) - \lambda_{i,EA,z} \Psi_{i,m} (0, 0) \]

\[ - \lambda_{i,LE,x} X_{i,m} (b_i (1 + a_i), 0) - \lambda_{i,LE,y} \Phi_{i,m} (b_i (1 + a_i), 0) - \lambda_{i,LE,z} \Psi_{i,m} (b_i (1 + a_i), 0) \]

\[ + \lambda_{i+1,EA,x} X_{i,m} (0, L_i) + \lambda_{i+1,EA,y} \Phi_{i,m} (0, L_i) + \lambda_{i+1,EA,z} \Psi_{i,m} (0, L_i) \]

\[ + \lambda_{i+1,LE,x} X_{i,m} (b_i (1 + a_i), L_i) + \lambda_{i+1,LE,y} \Phi_{i,m} (b_i (1 + a_i), L_i) \]

\[ + \lambda_{i+1,LE,z} \Psi_{i,m} (b_i (1 + a_i), L_i) \] (6.54)

6.2.7 Post-Processing the Equations of Motion

Equations (6.45)-(6.47), (6.54), and (6.22)-(6.23) form the complete system of equations for the folding wing structural dynamics model. Specifically, the complete set of equations of motion includes Lagrange’s Equations for each generalized coordinate (frame rigid body translation and wing segment modes) plus all constraint equations.

The set of equations can be expressed in a very compact matrix form as shown in Eq. (6.55), in which the coefficients of terms containing the second time derivatives of generalized coordinates are combined into a system mass matrix \( M \), and the coefficients of terms containing the generalized coordinates themselves are combined into a system stiffness matrix \( K \). Specifically, the system mass matrix contains the kinetic energy terms, and the system stiffness matrix contains the potential energy and constraint terms. The vector \( q \) is the vector of all generalized coordinates and Lagrange
multipliers. By assuming harmonic motion such that the generalized coordinates are proportional to $e^{\lambda t}$, the equation of motion becomes an eigenvalue problem, as shown in Eq. (6.56).

$$0 = M \cdot \ddot{q} + K \cdot q$$  \hspace{1cm} (6.55)
$$0 = (\lambda^2 M + K) \cdot q$$  \hspace{1cm} (6.56)

The eigenvalues are the natural frequencies of the folding wing system and the eigenvectors describe the mode shapes of the folding wing system. MATLAB is used for all computations for this project. Before using MATLAB functions to solve the eigenvalue problem, the equations of motion must first be processed. This is done for two reasons. First, depending on whether the wing is clamped or allowed to have longitudinal degrees of freedom, different rigid body modes for the first wing segment must be removed. Second, depending on the fold angles that are being analyzed, some of the constraint equations are redundant and must be removed. These issues are addressed as follows.

**Removing First Wing Segment Rigid Body Modes**

For the first wing segment, the first four generalized coordinates represent rigid body modes: x-translation, y-translation, z-translation, and z-rotation (yaw). The other two rigid body modes are grouped into the wing segment beam modes: the x-rotation mode (roll) is considered a part of the beam bending mode, and the y-rotation mode (pitch) is considered a part of the beam torsion mode. If the folding wing is clamped, then the first wing segment should have no rigid body modes and all six modes should be removed from the system. The roll and pitch modes are kept from being introduced into the system by using clamped-free beam modes instead of free-free beam modes. The other four rigid body modes are removed from the system by removing the first four rows and columns of the system mass and stiffness matrices.
Removing rows and columns from the system mass and stiffness matrices is equivalent to setting those coordinates to zero, but former is easier to carry out in a MATLAB code than the latter. It is also easier, in terms of code organization, to form the system equations of motion with those rigid body modes in place and then removing them later depending on the configuration.

If the folding wing is allowed to have longitudinal degrees of freedom, then only the y-translation and z-rotation modes are removed from the system. In addition, the pitch mode is introduced by using free-free beam torsion modes. Note that both the clamped wing model and the aircraft model use clamped-free beam bending modes since the x-rotation (roll) coordinate is never needed in the longitudinal model. The z-rotation (yaw) coordinate is removed in both cases for the same reason.

Removing Redundant Constraints

This model uses general constraint equations that enforce the same $x$, $y$, and $z$ displacements at two locations at each hinge. However, this may result in redundant constraints in the system equations of motion. The simplest example is when the folding wing is flattened (i.e. all fold angles are equal to zero). The two $x$ constraints for each hinge specify that the $x$ displacements must be equal to zero at the leading edge and the elastic axis. But within the linear beam theory framework, the mode shapes all have zero $x$ displacement, so it is not possible for the $x$ displacement to be nonzero anywhere as long as the $x$ displacement is specified to be zero at one place. Therefore, all other $x$ constraints are redundant.

Mathematically, redundant constraints show up as linearly dependent rows in the system stiffness matrix, and the resulting stiffness matrix does not have full rank. Numerically, this prevents MATLAB eigenvalue solvers, including the commonly used `eigs` function, from working correctly. Therefore, it is necessary to remove those redundant constraints before using MATLAB’s eigenvalues solvers. A general method
for finding linearly dependent equations is to row-reduce the system of constraint
equations, using MATLAB’s `rref` function, and then look for rows that are linearly
dependent on the other rows. After identifying those linearly dependent rows, the
rows and the corresponding columns of the same index are removed from the system
mass and stiffness matrices.

6.2.8 Solving the Structural Dynamics Equations

After processing the equations of motion, they can be solved as an eigenvalue problem
to obtain the folding wing natural frequencies and mode shapes. For a clamped wing,
this is accomplished very simply using MATLAB’s `eigs` function. For a wing with
longitudinal degrees of freedom, however, `eigs` cannot be used due to the stiffness
matrix being singular from the system rigid body modes. Instead, a custom brute-
force eigenvalue solver was written in MATLAB and used to solve the structural
dynamics problem for a folding wing with longitudinal motion. By the working
principles of this eigenvalue solver, however, the code is not limited to longitudinal
degrees of freedom and applies to any physical system with rigid body degrees of
freedom. The code solves the eigenvalue problem in two steps: first the eigenvalues
are computed, and then the eigenvectors are computed.

Calculating the Eigenvalues

The characteristic equation of the structural dynamics eigenvalue problem is that
the determinant of the system matrix must be equal to zero.

\[
\text{det} (\lambda^2 M + K) = 0
\] (6.57)

To solve this using a brute force method, the above determinant may be calculated
for a large number of values for \( \lambda \), starting at zero and stopping at some upper limit
when the desired number of eigenvalues are found. Mathematically, the eigenvalues
are values of \( \lambda \) at which the determinant is zero. Practically, the eigenvalues can be
found in the brute force method by looking for two values of $\lambda$ between which the determinant changes sign. Even though this is a brute force method, the computational cost is very small because the eigenvalues are limited to the positive imaginary axis for a structural dynamics system, i.e. the eigenvalues physically correspond to purely oscillatory motion.

Calculating the Eigenvectors

After finding the eigenvalues, the corresponding eigenvectors can be computed. For a system with rigid body modes, zero is always an eigenvalue. The solver first finds the rigid body modes by computing the nullspace of the stiffness matrix. This is accomplished simply in MATLAB by invoking the `null` function. Mathematically, this is appropriate because substituting $\lambda = 0$ into Eq. (6.55) to find rigid body modes results in the following equation, whose solution is by definition the nullspace of the stiffness matrix $K$.

$$0 = K \cdot q$$

After calculating the rigid body modes, the elastic modes may be calculated using a brute force method. For each eigenvalue, the eigenvalue is substituted into the matrix equation of motion. The eigenvector of that matrix satisfies the following matrix equation.

$$0 = (\lambda_m^2 M + K) \cdot q = A \cdot q$$

Then one of the generalized coordinates is assumed to be equal to 1. The row corresponding to that generalized coordinate is removed, since that coordinate is forced to be a specified value. Doing this results in a non-homogeneous matrix equation that can be solved for the values of the remaining generalized coordinates. The solution is then the eigenvector for that particular eigenvalue.

In theory, it does not matter which generalized coordinate is set equal to 1. However, numerical errors may arise in certain situations, in which case some choices
are better than others. This is certainly true when the folding wing has no static imbalance and the bending and torsion modes are uncoupled. This means that the eigenvectors theoretically should have either all zeros for bending generalized coordinates or all zeros for torsion generalized coordinates. For this case, better results can be obtained by doing two trials, one where a bending coordinate is set equal to 1 and one where a torsion coordinate is set equal to 1. The eigenvector with a lower residual, defined as the resulting vector when multiplying the matrix by the eigenvector, is taken as the eigenvector of that eigenvalue. The brute force eigenvalue solver has shown to work very well for the analyses in the present work.

6.3 Structural Dynamics Results

6.3.1 Compare to Old Clamped Wing Results

In the absence of easily obtainable experimental data, a few theoretical validation studies were conducted on the new structural dynamics code. The first study compares the new structural dynamics results with the old results for several clamped wing configurations to check that the more general multi-body dynamics code can reproduce the old results for the more specific cases of clamped wings. The two-segment folding wing, the three-segment folding wing with inboard angle at 30 degrees, and the three-segment folding wing with Lockheed-type folding were analyzed using the new code and compared to the old results. Figure 6.4 shows the natural frequencies for the Lockheed configuration. The results are nearly identical.

Figure 6.5 shows the first four mode shapes of the three-segment folding wing with inboard angle of 30 degrees and horizontal outboard wing segment. The mode shapes also look nearly the same as the mode shapes from Chapter 2. In addition, the new code calculates and plots the system center of mass. The nominal center of mass position is shown by the unfilled circle, and the center of mass of the deformed structure is shown by the filled circle.
Figure 6.4: Natural Frequencies for the Three-Segment Lockheed Configuration: Results of Old vs. New Multi-Body Dynamics Code

Figure 6.5: Natural Modes of the Three-Segment Folding Wing with $\psi_2 = 30$ and $\psi_3 = -30$ Obtained from Multi-Body Dynamics Code
After confirming that the new multi-body dynamics code returns the same results for the clamped wing, the next analysis considers aircraft systems that include longitudinal rigid body motion. Consider again the three-segment Lockheed-type folding wing, but with a fuselage and tail included in the structural dynamics as well. The system should have three zero frequency modes due to three longitudinal system degrees of freedom, as well as additional elastic modes. The zero frequency mode shapes should be rigid body motion of the entire aircraft, and the elastic modes should be a combination of rigid body motion and elastic deformation.

Figure 6.6 shows the natural frequencies of the aircraft system over a range of fold angles. In particular, the figure shows rigid body modes as well as elastic modes. The number of rigid body modes cannot be seen from the figure, but the analysis did return three rigid body modes as expected. The elastic modes are now at different natural frequencies compared to the clamped wing results. This is expected when a system changes from fixed to free.

![Figure 6.6: Natural Frequencies of Aircraft System with a Three-Segment Lockheed-Type Folding Wing](image)

Figure 6.7 shows the natural modes of the aircraft system. The aircraft system has the same wing as the clamped three-segment Lockheed-type folding wing, the a tail and a fuselage have been added. The tail mass is a quarter of the wing mass, the tail inertia is a quarter of the wing inertia, the fuselage mass is three times the
wing mass, and the fuselage inertia is four times the wing inertia. These values were arbitrarily chosen but considered reasonable for a real system, such as a micro air vehicle.

The plots show the left side wing of the aircraft, with the free stream coming from the upper right corner of each plot. In addition to the mode shape of the wing, the plots also show the centers of mass for the wing, tail, and fuselage. The nominal center of mass positions are shown by the unfilled circles, and the deformed positions are shown by the filled circles.

The first three modes are rigid body modes. Specifically, the first mode is aircraft pitch, the second mode is aircraft translation primarily in the flow direction, and the third mode is aircraft translation primarily in the plunge direction. Note that the eigenvectors of the rigid body system are arbitrarily chosen by MATLAB's eigenvalue solver. The most physically intuitive choices for rigid body natural modes would be three modes describing x and z rigid body translation of and pitch about the aircraft center of mass. However, an infinite number of mutually orthogonal motions exist, and MATLAB does not usually choose eigenvectors that describe motion referencing the aircraft center of mass.

The elastic modes are modes 4, 5, and 6 in Fig. 6.7. The mode shapes look very similar to the clamped wing elastic mode shapes. It is difficult to see the root of the wing in the figures, but the figures do show that the root of the wing has some motion in each mode.

The last validation study is to increase the fuselage mass and inertia, while maintaining the wing and tail properties, and observe what happens to the natural frequencies of the elastic modes. As the mass and inertia of the fuselage approaches infinity, the system should behave more like a clamped wing because the inertia of the wing will not have a great effect on the motion of the fuselage. Figure 6.8 shows two plots. The plot on the left shows the natural frequencies of the aircraft system.
Figure 6.7: Natural Modes of the Three-Segment Folding Wing Aircraft System with $\psi_2 = 30$ and $\psi_3 = -30$ Obtained from Multi-Body Dynamics Code
with the same wing and tail but ten times the fuselage mass and stiffness as the previous case. The plot on the right shows the natural frequencies of a clamped wing. The side by side comparison shows that the natural frequencies do indeed approach those of a clamped wing when the fuselage mass and inertia are increased while keeping all other parameters the same. It is interesting to note that adding rigid body degrees of freedom affected the torsion modes more than the bending modes, and the torsion frequencies also converge back to the clamped wing results more slowly than the bending frequencies.

Figure 6.8: Natural Frequencies of Aircraft System with Increasing Fuselage Mass and Inertia Compared to Natural Frequencies of Clamped Wing
6.4 Summary of Multi-Body Structural Dynamics Model

This chapter discusses the derivation of a general multi-body structural dynamics model for an aircraft. The model includes the wing, the tail, and the fuselage. The model can predict the natural frequencies of a clamped folding wing, as well as the natural frequencies of a folding wing aircraft with longitudinal degrees of freedom.

Several studies were conducted to verify the accuracy of the theoretical model.

1. First, the clamped-wing natural frequencies generated by the multi-body dynamics model are compared to the results generated by the previous clamped-wing structural dynamics model. The results were nearly identical as expected.

2. The natural frequencies for an example folding wing aircraft were computed using the present model. The natural frequencies of the aircraft elastic modes were higher than the corresponding clamped wing elastic modes. This is also an expected trend.

3. As the fuselage mass and inertia approach infinity, the natural frequencies of the aircraft’s elastic modes approach those of the clamped wing elastic modes. This is also expected.

The above validation studies above show that the aircraft structural dynamics results are physically reasonable and obey typical trends of structures with rigid body degrees of freedom. The results show that the model should give reliable estimates of system structural dynamics and may be used with high confidence in an aeroelastic analysis.
The aerodynamic model calculates the forces and moment on the wing segments due to the surrounding flow. Previous studies of aerodynamic models for the folding wing show that the unsteady Theodorsen strip theory model is accurate in predicting the system behavior, and the existing numerical method for solving the eigenvalue problem calculates system eigenvalues in a reasonable amount of time. If only the flutter speed and frequency are of interest, MATLAB can calculate them very quickly using the V-g method.

For the aircraft aeroelastic and flight dynamics model, both the wing and tail aerodynamic forces and moments must be modeled. The analysis still uses Theodorsen unsteady thin airfoil theory as the fundamental theory for the aerodynamic model. The main difference between the present model and the past aerodynamic model for the clamped wing is that the inclusion of rigid body plunge and pitch modes also affect the aerodynamics. This chapter describes the derivation of the aerodynamic forces and moments, as well as the resulting aeroelastic equations of motion.
7.1 Theodorsen Unsteady Aerodynamics

Bisplinghoff et al[33] gives the lift and moment per unit span from Theodorsen’s unsteady thin airfoil theory. The lift and moment expressions above are grouped by the square brackets into circulatory and non-circulatory terms. Note that for the aerodynamic calculations, it is simpler to compute everything in the wing-relative frame.

\[ \bar{L} = \pi \rho \frac{b^2}{2} \left[ \dot{h} + U \dot{\alpha} - ba\ddot{\alpha} \right] + 2\pi \rho UbD(k) \left[ \dot{h} + U\alpha + b \left( \frac{1}{2} - a \right) \dot{\alpha} \right] \]  

(7.1)

\[ \bar{M} = \pi \rho \frac{b^2}{2} \left[ ba\ddot{h} - Ub \left( \frac{1}{2} - a \right) \dot{\alpha} - b^2 \left( \frac{1}{8} + a^2 \right) \ddot{\alpha} \right] \]

\[ + 2\pi \rho Ub^2 \left( a + \frac{1}{2} \right) D(k) \left[ \dot{h} + U\alpha + b \left( \frac{1}{2} - a \right) \dot{\alpha} \right] \]  

(7.2)

The circulatory terms depend on the generalized Theodorsen function \( D(k) \) and depend on the downwash at the 3/4 chord of the airfoil. The reduced frequency \( k \) is defined by \( k \equiv \lambda b/U \), where \( \lambda \) is the system eigenvalue, \( b \) is the half chord length, and \( U \) is the air speed. The downwash in the wing-relative frame is given by the following equation, and has contributions from airfoil plunge velocity, twist velocity, and steady angle of attack.

\[ \tilde{w}_i \equiv \dot{h} \left( \frac{1}{2} - a \right) \dot{\alpha} + U\alpha \]  

(7.3)

The non-circulatory terms contain \( \ddot{h} \), \( \ddot{\alpha} \), and \( \dot{\alpha} \). The \( \ddot{h} \) term corresponds to a uniform plunge acceleration over the airfoil, and does not include any torsion motion. Likewise, the \( \ddot{\alpha} \) and \( \dot{\alpha} \) terms correspond to a uniform angular acceleration and a uniform angular velocity over the airfoil, and does not include any bending motion. In particular, the term \( \alpha \) does not mean angle of attack. To reinforce this distinction, the terms will be renamed with a subscript \((\ )_{NC}\) and a \( (\ )\) to specify
that (1) they correspond to the plunge and twisting motion of the airfoil, (2) they are calculated in the wing-relative frame, and (3) they are used to calculate the non-circulatory lift and moment on the airfoil.

In general, the lift and moment per unit span are given by the following equations, which now contain the downwash and the airfoil motions that contribute to non-circulatory forcing. These equations will be applied to both the wings and the tails of the folding wing aircraft. The downwash and non-circulatory terms for the wings and tails will be expressed in terms of the appropriate generalized coordinates.

\[
\bar{L} = \pi \rho b^2 \left[ \ddot{h}_{NC} + U \dot{\alpha}_{NC} - ba \ddot{\alpha}_{NC} \right] + 2\pi \rho UbD(k)\tilde{w}_i \tag{7.4}
\]

\[
\bar{M} = \pi \rho b^2 \left[ ba \ddot{h}_{NC} - Ub \left( \frac{1}{2} - a \right) \dot{\alpha}_{NC} - b^2 \left( \frac{1}{8} + a^2 \right) \ddot{\alpha}_{NC} \right] \\
+ 2\pi \rho Ub^2 \left( a + \frac{1}{2} \right) D(k)\tilde{w}_i \tag{7.5}
\]

7.2 Wing Aerodynamic Model

First it is necessary to set up the problem and note the convention. Consider the aircraft in level flight. The air speed is equal to the speed of the fuselage, clamped wing, or tail coordinate system, since the three components are rigidly attached. Consider the clamped wing coordinate system origin as the reference point. The free stream velocity in inertial coordinates is given by the vector \( \mathbf{V}_\infty = (\dot{R}_{1,x}, 0, \dot{R}_{1,z}) \), where the components \( \dot{R}_{1,x} \) and \( \dot{R}_{1,z} \) are the \( x \) and \( z \) velocity components of the origin of the clamped wing coordinate system.

The aerodynamic forcing is divided into two types: circulatory and non-circulatory. The circulatory terms are terms that multiply the Theodorsen function, and the non-circulatory terms are all other terms, which include apparent mass and some additional damping. To calculate the circulatory aerodynamic forcing, it is necessary to determine the normal wash \( \tilde{w}_i \) at the 3/4 chord of each wing segment. The
expression for the normal wash can be derived from the more general expression of Eq. (7.6) for the negative of the material derivative of the fluid particle at the wing 3/4 chord. The negative is added such that downward velocity is positive.

\[ \tilde{w}_i = - \left[ \frac{\partial}{\partial t} \left( A_{i,i}^T R_i \right) - U \frac{\partial}{\partial \tilde{x}_i} \left( A_{i,i}^T R_i \right) \right] \quad (7.6) \]

The position vector \( R_i \) is the same as the one defined in Eq. (2.15), and accounts for wing rigid body translation and elastic deformation. The following step assumes that the coordinate rotation matrix \( A_{i,i} \), defined in Eq. (6.2), does not change with time. This means that the fold angles will be treated as quasi-steady. This allows the rotation matrix to be taken out of the time derivative. The downwash vector in the wing-relative frame is equal to the downward velocity of the structure at the 3/4 chord. First, the full velocity vector of the wing structure at the 3/4 chord is equal to the material derivative of the position of the 3/4 chord point on the wing cross section, and includes contributions from the frame translation and the wing deformation. The following equations expand the material derivative expression of Eq. (7.6).

\[ \tilde{w}_i = - A_{i,i}^T \left[ \dot{R}_{i,o} + A_{i,i} \left( \dot{\tilde{u}}_i + \dot{\tilde{u}}_{i,e} \right) \right] + U A_{i,i}^T \frac{\partial}{\partial \tilde{x}_i} \left[ R_{i,o} + A_{i,i} \left( \tilde{u}_i + \tilde{u}_{i,e} \right) \right] \quad (7.7) \]

\[ \tilde{w}_i = - A_{i,i}^T \dot{R}_{i,o} - \dot{\tilde{u}}_{i,e} + U \frac{\partial}{\partial \tilde{x}_i} \tilde{u}_{i,e} \quad (7.8) \]

\[ \tilde{w}_i = - A_{i,i}^T \dot{R}_{i,o} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m}(3c/4) + U \sum_m q_{i,m} \frac{\partial}{\partial \tilde{x}_i} \tilde{\Psi}_{i,m}(3c/4) \quad (7.9) \]

Equation 7.10 gives the normal wash, which is equal to the \( \tilde{z}_i \) component of the wash vector.

\[ \tilde{w}_i = \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m}(3c/4) + U \sum_m q_{i,m} \frac{\partial}{\partial \tilde{x}_i} \tilde{\Psi}_{i,m}(3c/4) \quad (7.10) \]
The non-circulatory aerodynamic forcing depends on plunge and twist motion of the wing. The apparent mass contributes to non-circulatory forcing, and depends on the acceleration of the plunge and twist motion, \( \ddot{h}_{NC} \) and \( \ddot{\alpha}_{NC} \). In addition, the angular velocity of the twist motion \( \dot{\alpha}_{NC} \) also contributes to the non-circulatory forcing. As discussed previously, the non-circulatory terms depend on the motion of the wing itself in the flow field, so even though \( \alpha \) is used here, the terms do not mean angle of attack. In addition, the terms are evaluated at the wing elastic axis. The following equations express the non-circulatory terms using wing generalized coordinates. Again, the derivation assumes quasi-steady fold angles such that the folding motion does not contribute significantly to the unsteady aerodynamic forcing.

\[
\ddot{h}_{NC} = \sin \Psi \dot{R}_{i,y} - \cos \Psi \dot{R}_{i,z} - \sum_{m} \ddot{q}_{i,m} \ddot{\Psi}_{i,m}(EA) \tag{7.11}
\]

\[
\ddot{\alpha}_{NC} = \sum_{m} \ddot{q}_{i,m} \frac{\partial \ddot{\Psi}_{i,m}}{\partial x_i}(EA) \tag{7.12}
\]

\[
\dot{\alpha}_{NC} = \sum_{m} \dot{q}_{i,m} \frac{\partial \Psi_{i,m}}{\partial x_i}(EA) \tag{7.13}
\]

The total lift force on each wing segment is the integral of the lift per unit span over the span of the wing segment.

\[
L_i = \int \pi \rho b^2 \left[ \ddot{h}_{NC} + U \dot{\alpha}_{NC} - b a \ddot{\alpha}_{NC} \right] + 2\pi \rho U b D(k) \tilde{w}_i \, d\tilde{y}_i \tag{7.14}
\]

Equations (7.10)-(7.13) are substituted into the equation for unsteady lift per unit span, Eq. (7.4), to obtain an expression for the lift force per unit span on a
wing segment in terms of the wing segment generalized coordinates.

\[ \bar{L}_i = \pi \rho b^2 \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \ddot{q}_{i,m} \tilde{\Psi}_{i,m} (EA) + U \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial x_i} (EA) - ab \sum_m \ddot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial x_i} (EA) \right] \\
+ 2\pi \rho U b D(k) \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m} (3c/4) + U \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial x_i} (3c/4) \right] \tag{7.15} \]

In order to obtain flight dynamics results, it is not necessary to include profile drag or induced drag in the aerodynamics model. The simple flight dynamics models will return the expected phugoid and short period modes even when drag is not modeled. Therefore, this analysis will not include any drag calculations for simplicity. Only the lift force will be considered. The aerodynamic lift vector must be normal to the flow direction. From the structural dynamics derivation, the matrix \( A_{i,i} \) transforms a vector in the wing segment coordinate system to the inertial coordinate system. Then the aircraft velocity can be transformed to the wing segment coordinate system using the transpose of \( A_{i,i} \).

\[ \tilde{V}_\infty = A_{i,i}^T V_\infty \tag{7.16} \]

\[ \tilde{V}_\infty = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Psi_i & \sin \Psi_i \\ 0 & -\sin \Psi_i & \cos \Psi_i \end{pmatrix} \begin{pmatrix} \dot{R}_{i,x} \\ \dot{R}_{i,y} \\ \dot{R}_{i,z} \end{pmatrix} \tag{7.17} \]

\[ \tilde{V}_\infty = \begin{pmatrix} \dot{R}_{1,x} \\ \cos \Psi_i \dot{R}_{i,y} + \sin \Psi_i \dot{R}_{i,z} \\ -\sin \Psi_i \dot{R}_{i,y} + \cos \Psi_i \dot{R}_{i,z} \end{pmatrix} \tag{7.18} \]

Note that even though the aircraft has zero \( y \) velocity in a longitudinal dynamics analysis, individual wing segments can have non-zero \( y \) velocity when the fold angles
are non-zero. In the inertial frame, for each wing segment the free stream flow angle \( \alpha_f \) relative to horizontal is approximately equal to the negative \( z \) velocity component of the free stream velocity vector divided by the full air speed.

\[
\sin \alpha_f = \frac{-\dot{R}_{i,z}}{\sqrt{\dot{R}_{i,x}^2 + \dot{R}_{i,z}^2}} \quad (7.19)
\]

\[
\alpha_f \approx -\frac{\dot{R}_{i,z}}{U} \quad (7.20)
\]

In the relative frame, the free stream flow angle \( \tilde{\alpha}_f \) is computed using the same formula, but using the \( \tilde{z}_i \) component of the free stream velocity vector in the wing relative frame.

\[
\tilde{\alpha}_f \approx \frac{\sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z}}{U} \quad (7.21)
\]

The lift force vector per unit length in the wing relative frame is then given by the following equations. If the relative frame flow angle is zero, the lift vector points exactly in the \( \tilde{z} \) direction. With non-zero relative frame flow angle \( \tilde{\alpha}_f \), the lift vector must be rotated counterclockwise about the \( \tilde{y} \) axis from vertical by the angle \( \tilde{\alpha}_f \). A small angle approximation is made in the last step.

\[
\tilde{F}_i = \begin{pmatrix}
\cos \tilde{\alpha}_f & 0 & \sin \tilde{\alpha}_f \\
0 & 1 & 0 \\
-\sin \tilde{\alpha}_f & 0 & \cos \tilde{\alpha}_f
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\bar{L}_i
\end{pmatrix} \quad (7.22)
\]

\[
\tilde{F}_i = \begin{pmatrix}
\sin \tilde{\alpha}_f \bar{L}_i \\
0 \\
\cos \tilde{\alpha}_f \bar{L}_i
\end{pmatrix} \quad (7.23)
\]

\[
\tilde{F}_i \approx \begin{pmatrix}
\left( \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} \right) \bar{L}_i/U_\infty \\
0 \\
\bar{L}_i
\end{pmatrix} \quad (7.24)
\]

The virtual work due to lift on the wing segment is equal to the lift force vector per unit length dotted with the virtual change in displacement at each point along
the wing, and then integrated over the wing span.

$$\delta W_{Li} = \int \tilde{F}_i \cdot \delta \tilde{R}_i \, d\tilde{y}_i \quad (7.25)$$

The virtual displacement in the relative frame can be expressed in terms of system generalized coordinates as follows.

$$\delta \tilde{R}_i = A_{ii}^T \begin{pmatrix} \delta R_{i,x} \\ \delta R_{i,y} \\ \delta R_{i,z} \end{pmatrix} + \sum_m \delta q_{i,m} \tilde{\Psi}_{i,m}(EA) \quad (7.26)$$

$$\delta \tilde{R}_i = \begin{pmatrix} \delta R_{i,x} \\ \cos \Psi_i \delta R_{i,y} + \sin \Psi_i \delta R_{i,z} \\ -\sin \Psi_i \delta R_{i,y} + \cos \Psi_i \delta R_{i,z} \end{pmatrix} + \sum_m \delta q_{i,m} \tilde{\Psi}_{i,m}(EA) \quad (7.27)$$

The expanded form of the virtual work is obtained by substituting unsteady lift in Eq. (7.15) into the lift force vector.

$$\delta W_{Li} = \int \tilde{\alpha}_f \tilde{L}_i \delta R_{i,x} + \tilde{\alpha}_f \tilde{L}_i \sum_m \delta q_{i,m} \tilde{X}_{i,m}(EA) \, d\tilde{y}_i$$

$$+ \int \tilde{L}_i (-\sin \Psi_i \delta R_{i,z} + \cos \Psi_i \delta R_{i,y}) + \tilde{L}_i \sum_m \delta q_{i,m} \tilde{\Psi}_{i,m}(EA) \, d\tilde{y}_i \quad (7.28)$$

$$\delta W_{Li} = \delta R_{i,x} \int \tilde{\alpha}_f \tilde{L}_i \, d\tilde{y}_i + \sum_m \delta q_{i,m} \int \tilde{\alpha}_f \tilde{L}_i \tilde{X}_{i,m}(EA) \, d\tilde{y}_i - \delta R_{i,y} \sin \Psi_i \int \tilde{L}_i \, d\tilde{y}_i$$

$$+ \delta R_{i,z} \cos \Psi_i \int \tilde{L}_i \, d\tilde{y}_i + \sum_m \delta q_{i,m} \int \tilde{L}_i \tilde{\Psi}_{i,m}(EA) \, d\tilde{y}_i \quad (7.29)$$

### 7.2.1 Linearizing the Lift Expression

To obtain the phugoid mode, it is necessary to consider perturbations in the flight speed $U$. The free stream velocity vector is $(\dot{R}_{i,x}, \dot{R}_{i,z})$. The square of the free stream velocity is obtained exactly by the Pythagorean theorem, but can be linearized using the binomial expansion. The $x$ frame translation velocity $\dot{R}_{i,x}$ perturbs about a steady state value of $U_\infty$, and the $z$ frame translation velocity $\dot{R}_{i,z}$ perturbs about a
steady state value of zero for this level flight analysis. Then the free stream velocity and the square of the free stream velocity may be linearized as follows.

\[
U \approx \dot{R}_{i,x} = U_\infty + \dot{r}_{i,x} \\
U^2 = \dot{R}_{i,x}^2 + \dot{R}_{i,z}^2 \approx U_\infty^2 + 2U_\infty \dot{r}_{i,x}
\] (7.30)

The terms in the lift expression that do not contain the flight speed \(U\), which include only the apparent mass terms, are linear in terms of the generalized coordinates. The terms that contain \(U\) are nonlinear in the generalized coordinates. Lastly, the term \(\tilde{\alpha}_f \tilde{L}_i\) is nonlinear in the generalized coordinates because the flow angle \(\tilde{\alpha}_f\) contains the frame \(y\) and \(z\) translation coordinates. The nonlinear terms in the lift and the virtual work must be linearized.

First, consider the terms that are proportional to \(U\) but not \(U^2\). Substitute in Eq. (7.30) to obtain the following terms.

\[
\bar{L}_{i,1} = \pi \rho b^2 U \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i}(EA)
\]

\[
+ 2\pi \rho U bD(k) \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m}(3c/4) \right]
\]

\[
\bar{L}_{i,1} = \pi \rho b^2 U_\infty \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i}(EA) + \pi \rho b^2 \dot{r}_{i,x} \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i}(EA)
\]

\[
+ 2\pi \rho U_\infty bD(k) \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m}(3c/4) \right]
\]

\[
+ 2\pi \rho \dot{r}_{i,x} bD(k) \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m}(3c/4) \right]
\]

(7.32)

For the terms above that multiply \(\dot{r}_{i,x}\), each term is a nonlinear term that contains the product of time derivatives of two generalized coordinates. Since the time derivatives of the generalized coordinates \(\dot{R}_{i,y}, \dot{R}_{i,z}\), and \(\dot{q}_{i,m}\) all perturb about a steady state value of zero for this particular case of level flight, the terms that multiply \(\dot{r}_{i,x}\)
are neglected when the mathematical model is linearized. Therefore, the terms that multiply $U_\infty$ are the only terms that remain.

\[ \bar{L}_{i,1} = \pi \rho b^2 U_\infty \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (EA) \]

\[ + 2\pi \rho U_\infty b D(k) \left[ \sin \Psi \dot{R}_{i,y} - \cos \Psi \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m} (3c/4) \right] \quad (7.34) \]

Next, consider the terms that are proportional to $U^2$, and substitute in Eq. (7.31).

\[ \bar{L}_{i,2} = 2\pi \rho U^2 b D(k) \sum_m q_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (3c/4) \quad (7.35) \]

\[ \bar{L}_{i,2} = 2\pi \rho U^2_\infty b D(k) \sum_m q_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (3c/4) + 2\pi \rho (2\dot{r}_{i,x}) b D(k) \sum_m q_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (3c/4) \quad (7.36) \]

The first set of terms multiply the constant $U^2_\infty$, so they are linear in the generalized coordinates and remain as they are. The second set of terms are nonlinear terms containing the product of $\dot{r}_{i,x}$ and $q_{i,m}$. The modal expansion containing $q_{i,m}$ physically represents the angle of attack. Each generalized coordinate $q_{i,m}$ perturbs around a steady state value such that the modal expansion perturbs about the trim angle of attack. Therefore, the product of $\dot{r}_{i,x}$ and the modal expansion containing $q_{i,m}$ linearizes to $\dot{r}_{i,x}$ times the trim angle of attack.

\[ \bar{L}_{i,2} = 2\pi \rho U^2_\infty b D(k) \sum_m q_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (3c/4) + 2\pi \rho (2\dot{r}_{i,x}) b D(k) \alpha_{i,ss} \quad (7.37) \]

The linearized expression for lift is then given by combining the apparent mass
terms and the terms from the above two linearizations.

\[
\bar{L}_i = \pi \rho b^2 \left[ \sin \Psi_i \bar{R}_{i,y} - \cos \Psi_i \bar{R}_{i,z} - \sum_m \bar{q}_{i,m} \bar{\Psi}_{i,m} (EA) \right. \\
+ U_\infty \sum_m \bar{q}_{i,m} \frac{\partial \bar{\Psi}_{i,m}}{\partial \bar{x}_i} (EA) - ab \sum_m \bar{q}_{i,m} \frac{\partial \bar{\Psi}_{i,m}}{\partial \bar{x}_i} (EA) \\
\left. + 2\pi \rho U_\infty b D(k) \left[ \sin \Psi_i \bar{R}_{i,y} - \cos \Psi_i \bar{R}_{i,z} - \sum_m \bar{q}_{i,m} \bar{\Psi}_{i,m} (3c/4) \right. \\
+ U_\infty \sum_m \bar{q}_{i,m} \frac{\partial \bar{\Psi}_{i,m}}{\partial \bar{x}_i} (3c/4) \right] \\
+ 2\pi \rho (2 \dot{r}_{i,x}) b D(k) \alpha_{i,ss} \quad (7.38)
\]

Lastly, consider the terms in \( \bar{\alpha}_f \bar{L}_i \).

\[
\bar{\alpha}_f \bar{L}_i = \frac{\sin \Psi_i \bar{R}_{i,y} - \cos \Psi_i \bar{R}_{i,z}}{U_\infty} \bar{L}_i \quad (7.39)
\]

\[
\bar{\alpha}_f \bar{L}_i \approx \text{steady state} + \frac{\sin \Psi_i \bar{R}_{i,y} - \cos \Psi_i \bar{R}_{i,z}}{U_\infty} \bar{L}_{i,ss} + \frac{\sin \Psi_i \bar{R}_{i,y,ss} - \cos \Psi_i \bar{R}_{i,z,ss}}{U_\infty} \bar{L}_i \quad (7.40)
\]

\[
\bar{\alpha}_f \bar{L}_i \approx \text{steady state} + \frac{\sin \Psi_i \bar{R}_{i,y} - \cos \Psi_i \bar{R}_{i,z}}{U_\infty} \bar{L}_{i,ss} \quad (7.41)
\]

In the equation above, the term \( \bar{L}_{i,ss} \) is the steady state lift per unit span on the wing in the trim state. For level flight and longitudinal dynamics, the steady state frame translation velocities \( \bar{R}_{i,y,ss} \) and \( \bar{R}_{i,z,ss} \) are zero, so the term that contains them vanishes. The linearized result contains one term, which is proportional to the \( y \) and \( z \) translation velocities and the steady state lift force on the wing. This result means that the steady state lift on each wing segment at the trim condition must be computed for the set of fold angles being studied. The steady state lift will vary depending on the fold angle of each wing segment. This is clear when considering that a wing segment at 90-degree angle with the horizontal cannot provide any lift during
steady level flight. The dependence on steady state lift also exists in the conventional flight dynamics model for rigid aircraft. For the folding wing problem, the steady state lift on each wing segment can be calculated easily before the aeroelastic analysis.

The generalized force for each generalized coordinate is equal to the partial derivative of the virtual work with respect to a virtual change in that coordinate. The virtual work due to lift depends on virtual changes in \( x \) frame translation, \( y \) frame translation, \( z \) frame translation, and the beam modes \( q_{i,m} \), as shown in Eq. (7.29).

Therefore, there are four sets of generalized forces:

\[
Q_{i,x} = \frac{\sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z}}{U_\infty} \int \bar{L}_{i,ss} \, d\bar{y}_i \tag{7.42}
\]

\[
Q_{i,y} = -\sin \Psi_i \int \bar{L}_i \, d\bar{y}_i \tag{7.43}
\]

\[
Q_{i,z} = \cos \Psi_i \int \bar{L}_i \, d\bar{y}_i \tag{7.44}
\]

\[
Q_{q,im} = \frac{\sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z}}{U_\infty} \int \bar{L}_{i,ss} \bar{X}_{i,m}(EA) \, d\bar{y}_i + \int \bar{L} \bar{\Psi}_{i,m}(EA) \, d\bar{y}_i \tag{7.45}
\]

The generalized force due to moments are computed in a similar way, but using the expression for moment per unit span along the wing. Equations (7.10)-(7.13) are substituted in Eq. (7.5) to obtain an expression for the total moment about the wing elastic axis in terms of the wing segment generalized coordinates.

\[
\bar{M}_i = \pi \rho b^3 \left[ a \left( \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \bar{q}_{i,m} \bar{\Psi}_{i,m}(EA) \right) \right.
\]

\[
\left. -U \left( \frac{1}{2} - a \right) \sum_m \bar{q}_{i,m} \frac{\partial \bar{\Psi}_{i,m}(EA)}{\partial \bar{x}_i} - b \left( \frac{1}{8} + a^2 \right) \sum_m \bar{q}_{i,m} \frac{\partial \bar{\Psi}_{i,m}(EA)}{\partial \bar{x}_i} \right]
\]

\[
+ 2\pi \rho U b^2 \left( \frac{1}{2} + a \right) D(k) \left[ \left( \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} \right) - \sum_m \bar{q}_{i,m} \bar{\Psi}_{i,m}(3c/4) \right.
\]

\[
\left. + U \sum_m q_{i,m} \frac{\partial \bar{\Psi}_{i,m}(3c/4)}{\partial \bar{x}_i} \right] \tag{7.46}
\]
The moment expression may be linearized in a similar manner as the linearization of the lift expression. From the above linearization of the lift expression, we know that the terms that multiply $U$ remain unchanged in the linearized version and multiply $U_\infty$ instead, and the terms that multiply $U^2$ linearize to two sets of terms: one that is proportional to $q_{i,m}$ times $U_\infty$, and another that is proportional to $\dot{r}_{i,x}$ times the trim angle of attack.

The linearized aerodynamic moment is shown in the equation below.

$$\bar{M}_i = \pi \rho b^3 \left[ a \left( \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m q_{i,m} \tilde{\Psi}_{i,m} (EA) \right) ight.$$

$$- U_\infty \left( \frac{1}{2} - a \right) \sum_m \ddot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (EA) - b \left( \frac{1}{8} + a^2 \right) \sum_m \dot{q}_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (EA) \right]$$

$$+ 2\pi \rho U_\infty b^2 \left( \frac{1}{2} + a \right) D(k) \left[ \sin \Psi_i \dot{R}_{i,y} - \cos \Psi_i \dot{R}_{i,z} - \sum_m \dot{q}_{i,m} \tilde{\Psi}_{i,m} (3c/4) \right]$$

$$+ 2\pi \rho U_\infty b^2 \left( \frac{1}{2} + a \right) D(k) \sum_m q_{i,m} \frac{\partial \tilde{\Psi}_{i,m}}{\partial \tilde{x}_i} (3c/4)$$

$$+ 2\pi \rho (2\dot{r}_{i,x}) b^2 \left( \frac{1}{2} + a \right) D(k) \alpha_{i,ss} \tag{7.47}$$

The virtual work due to moment is equal to the aerodynamic moment multiplied by virtual displacements in torsion coordinates. This includes both rigid body torsion modes as well as elastic torsion modes. The term $F_{i,m}$ in the equations below represent the generalized coordinates of all torison modes, and is a subset of the system generalized coordinates $q_{i,m}$. The term $\tilde{\Phi}_{i,m}$ in the equations below represent
the torsion mode shapes, and are equal to $\partial \tilde{\Psi}_{i,m}/\partial \tilde{x}_i$ for torsion modes.

$$\delta W_{Mi} = \int \tilde{M}_i \delta \phi \ d\tilde{y}_i \quad (7.48)$$

$$\delta \phi = \sum_m \delta F_{i,m} \tilde{\Phi}_{i,m} \quad (7.49)$$

$$\delta W_{Mi} = \sum_m \delta F_{i,m} \int \tilde{M}_i \tilde{\Phi}_{i,m} \ d\tilde{y}_i \quad (7.50)$$

The generalized force due to moments on the wing with respect to the torsion coordinates is given by the following expression.

$$Q_{F,im} = \int \tilde{M}_i \tilde{\Phi}_{i,m} \ d\tilde{y}_i \quad (7.51)$$

The generalized forces due to the wing aerodynamic lift and moment can be substituted into the equations of motion for the wing generalized coordinates to complete the aeroelastic model for the wing.

### 7.3 Tail Aerodynamic Model

The tail coordinate system is defined to rotate with the aircraft coordinate system. The downwash at the tail 3/4 chord is equal to the following equation. The first term represents uniform plunge of the tail. The second term represents the downwash induced by the tail pitching about its elastic axis. The third term represents a steady angle of attack between the tail and the horizontal.

$$w_i = -\dot{z}_t + \dot{\theta} \left( \frac{1}{2} - a \right) b_t + U\theta \quad (7.52)$$

This is in exactly the same form as the downwash expression for the wing and the expression in Theodorsen’s lift and moment formulas, as expected. The downwash expression can be substituted into Theodorsen’s expressions for lift and moment per
unit span, shown in Eqs. (7.53) and (7.54).

\[
\bar{L}_t = \pi \rho b^2 \left[ -\dot{z}_{to} + U \dot{\theta} - ba\ddot{\theta} \right] + 2\pi \rho UbD(k) \left[ -\dot{z}_{to} + \dot{\theta} \left( \frac{1}{2} - a \right) b + U\theta \right] \tag{7.53}
\]

\[
\bar{M}_t = \pi \rho b^2 \left[ ba\ddot{z}_{to} - Ub \left( \frac{1}{2} - a \right) \dot{\theta} - b^2 \left( \frac{1}{8} + a^2 \right) \dot{\theta} \right]
+ 2\pi \rho Ub^2 \left( a + \frac{1}{2} \right) D \left[ \dot{z}_{to} + U\theta + b \left( \frac{1}{2} - a \right) \dot{\theta} \right] \tag{7.54}
\]

As was done for the wing lift and moment, the tail lift and moment expression can also be linearized in the same way.

\[
\bar{L}_t = \pi \rho b^2 \left[ -\dot{z}_{to} + U_\infty \dot{\theta} - ba\ddot{\theta} \right] + 2\pi \rho U_\infty bD(k) \left[ -\dot{z}_{to} + \dot{\theta} \left( \frac{1}{2} - a \right) b + U_\infty \theta \right]
+ 2\pi \rho (2\dot{r}_{i,x})bD(k)\alpha_{t,ss} \tag{7.55}
\]

\[
\bar{M}_t = \pi \rho b^2 \left[ ba\ddot{z}_{to} - U_\infty \dot{\theta} - b^2 \left( \frac{1}{8} + a^2 \right) \dot{\theta} \right]
+ 2\pi \rho U_\infty b^2 \left( a + \frac{1}{2} \right) D(k) \left[ \dot{z}_{to} + U_\theta + b \left( \frac{1}{2} - a \right) \dot{\theta} \right]
+ 2\pi \rho (2\dot{r}_{i,x})b^2 \left( \frac{1}{2} + a \right) D(k)\alpha_{t,ss} \tag{7.56}
\]

This model will further assume that the lift and moment per unit span are constant along the span of the tail, so the total lift and moment are simply equal to the per-span expressions times the span of the tail. For the tail, the flow angle \( \alpha_f \) is equal to the negative of the \( z \) velocity \( \dot{z}_t \) divided by the free stream air speed \( U_\infty \). If the flow angle \( \alpha_f = 0 \), then the aerodynamic force vector contains only the lift \( L_t \) in the \( z \) component. For non-zero flow angles, the purely vertical lift force vector must be rotated counterclockwise about the \( y \) axis by the flow angle such that the
lift force vector is perpendicular to the flow direction.

\[
F_t = \begin{pmatrix}
\cos \alpha_f & 0 & \sin \alpha_f \\
0 & 1 & 0 \\
-\sin \alpha_f & 0 & \cos \alpha_f
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
L_t
\end{pmatrix}
\] (7.57)

\[
F_t \approx \begin{pmatrix}
\alpha_f L_t \\
0 \\
L_t
\end{pmatrix}
\] (7.58)

The virtual work due to lift is equal to the dot product between the aerodynamic force vector and the virtual displacement of the tail. The virtual displacement should be taken at the tail elastic axis because Theodorsen’s expression for lift acts through the elastic axis. However, the displacement at the coordinate system origin \((\delta x_{to}, 0, \delta z_{to})\) is used for simplicity. This case is correct when the elastic axis is coincident with the center of mass, which is the case being studied here.

\[
\delta W_{Lt} = \alpha_f L_t \delta x_{to} + L_t \delta z_{to}
\] (7.59)

The generalized forces are partial derivatives of the virtual work with respect to the virtual displacement coordinates.

\[
Q_{xt} = -L_t \frac{\dot{z}_{to}}{U_\infty}
\] (7.60)

\[
Q_{zt} = L_t
\] (7.61)

The virtual work due to moment is equal to the product of the tail moment and the virtual displacement in tail angle \(\delta \theta\). Then there is only one generalized force corresponding to the equation of motion for the coordinate \(\theta\).

\[
Q_\theta = M_t
\] (7.62)

The generalized forces due to the tail aerodynamic lift and moment can be substituted into the equations of motion for the tail generalized coordinates to complete the aeroelastic model for the folding wing aircraft with horizontal tails.
7.3.1 Different Aerodynamic Models

The new multi-body aeroelastic code is designed to be able to use multiple types of aerodynamic models. For the clamped wing, it has been shown before that the Theodorsen unsteady aerodynamic model is best for predicting aeroelastic instabilities. However, traditional flight dynamics analyses do not use the Theodorsen model, and instead use a quasi-steady aerodynamic model in which the lift and moment has only circulatory components that are proportional to the downwash. In addition, some flight dynamics analyses\cite{34} use an unsteady aerodynamic model for the wing and a steady aerodynamic model on the tail. By having the option to mix and match aerodynamic models for the wing and the tail in the present work, more direct comparisons can be made with flight dynamics analyses in literature for validating the theory.

For the present work, the available strip theory aerodynamic models are the quasi-steady aerodynamic model and the Theodorsen unsteady aerodynamic model. The quasi-steady model can be obtained from the unsteady model by modifying two key parts of the unsteady model.

1. The quasi-steady model has no non-circulatory terms, unlike the Theodorsen model, which has apparent mass and non-circulatory damping effects.

2. The quasi-steady model assumes low reduced frequency, so the Theodorsen function is assumed to be equal to 1.

The equations of motion may be written as a matrix equation containing the mass matrix, stiffness matrix, and several aerodynamic matrices. The same aerodynamic matrices are constructed regardless of the choice of aerodynamic model. For unsteady flow, all of the aerodynamic matrices are used in the final equation of motion, and the circulatory aerodynamic matrices are multiplied by the Theodorsen function.
For quasi-steady flow, only the circulatory aerodynamic matrices are used in the final equation of motion, and the matrices do not get multiplied by the Theodorsen function.

The derivation ignores effects of wing motion and wing wake on the tail aerodynamic forces, and vice versa.

7.4 Verification of Aeroelastic Results

The first step is to check that the new and more general multi-body dynamics code can reproduce the aeroelastic results of the old code for the clamped wing. As a test case, the 3-segment Lockheed-type folding wing was analyzed at two different fold angles using the old clamped wing code and the new multi-body dynamics code. The system was analyzed at 30-degree fold angle and at 60-degree fold angle. The aeroelastic eigenvalues are plotted side by side in Fig. 7.1. The eigenvalue plots are nearly identical, showing that the multi-body dynamics aeroelastic code can reproduce the results of the older code for a clamped wing.

The next step is to test the ability of the code to generate flight dynamics modes. For this purpose, a Cessna 172 was modeled in terms of folding wing parameters. The parameters for the Cessna 172 are listed in Table 7.1. The top half of the table lists the wing and tail parameters, and the bottom half of the table lists parameters for the aircraft as a whole. Only the rigid body modes are used in this validation analysis, so the individual values of wing mass, tail mass, and fuselage mass do not matter as long as the total aircraft mass and total aircraft inertia about the aircraft center of mass match the specifications. Therefore, the wing mass, tail mass, and fuselage mass were arbitrarily selected. Additionally, the bending and torsion stiffness of the wing do not matter since no elastic modes are used in the analysis.

The rigid body flight dynamics analysis was done by first forming the aeroelastic equations of motion, and then removing all equations associated with elastic modes.
The eigenvalues were found using a brute force search algorithm in which the values of the determinant of the equation of motion matrix was calculated at many different test eigenvalues uniformly distributed over a rectangular partition of the complex plane, and the locations at which the determinant was equal to zero were found using numerical tools in MATLAB. Figure 7.2 shows the result of the brute force eigenvalue solver. The lines represent points at which either the real part or the imaginary part of the determinant is equal to zero, and the locations at which the lines cross are the eigenvalues. The plot on the left shows both phugoid and short period modes. The plot on the right zooms into the area near zero to better distinguish the phugoid mode from the zero eigenvalue.
Table 7.1: Equivalent Parameters for Cessna 172 in the Folding Wing Convention

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord</td>
<td>5 ft</td>
</tr>
<tr>
<td>Span</td>
<td>18 ft</td>
</tr>
<tr>
<td>Wing mass per span</td>
<td>1 kg/m</td>
</tr>
<tr>
<td>Wing radius of gyration</td>
<td>25% of Chord</td>
</tr>
<tr>
<td>Wing elastic axis offset</td>
<td>0</td>
</tr>
<tr>
<td>Wing bending stiffness</td>
<td>$10^6$ Nm$^2$/ (kg/m)</td>
</tr>
<tr>
<td>Wing torsion stiffness</td>
<td>$10^8$ Nm$^2$/ (kg-m)</td>
</tr>
<tr>
<td>Tail span</td>
<td>5 ft</td>
</tr>
<tr>
<td>Tail Chord</td>
<td>3 ft</td>
</tr>
<tr>
<td>Tail elastic axis offset</td>
<td>0</td>
</tr>
<tr>
<td>Total aircraft mass</td>
<td>510 kg</td>
</tr>
<tr>
<td>Total aircraft inertia about c.g.</td>
<td>420 kg-m$^2$</td>
</tr>
<tr>
<td>Aircraft c.g. location</td>
<td>Wing 1/4 Chord</td>
</tr>
<tr>
<td>Tail 1/4 chord to wing 1/4 chord</td>
<td>15 ft</td>
</tr>
</tbody>
</table>

Figure 7.2: Determinant Contours of the Cessna 172 Configuration at Air Speed of 25 m/s
Figure 7.3 shows the aeroelastic eigenvalues of the Cessna 172 configuration versus air speed. The first plot shows the imaginary part of the aeroelastic eigenvalues versus air speed, and the second plot shows a three-dimensional view of how the eigenvalues change. The plots show that both the short period mode and the phugoid mode were captured by the aeroelastic model. The short period mode increases in frequency and damping as the air speed increases. The phugoid mode decreases in frequency as the air speed increases, and shows the typical behavior that frequency is approximately inversely proportional to the air speed. In addition, the eigenvalue is slightly unstable, which is also typical of the phugoid mode.

Table 7.2 summarizes the eigenvalues of the phugoid and short period modes that were produced by the present multi-body dynamics model versus the eigenvalues from a simple flight dynamics model of a rigid aircraft. The simple model considers four state variables: x velocity, z velocity, pitch rate, and pitch angle. The derivation of the model is shown in the Appendix, and follows largely from information taught in an introductory course in aircraft flight dynamics. Table 7.2 shows that the two different models agree very well in terms of the eigenvalues for both modes. This validation analysis here show that the flight dynamics modes are captured by the
7.5 Aircraft Aeroelastic and Flight Dynamics Results

After checking the new aeroelastic models against some known results, the model is then applied to examples of aircraft systems consisting of a folding wing, a horizontal tail, and a fuselage. The tail and fuselage masses and inertias are arbitrarily specified, but are in the right order of magnitude for a real aircraft.

Two sets of configurations were studied using the new multi-body aeroelastic model. Both studies analyzed the 3-segment folding wing that was considered in the previous clamped-wing analyses. The first study considered configurations with inboard fold angle at 30 degrees and outboard fold angle varying between -90 and 90 degrees. The second study considered the Lockheed-type configurations with inboard fold angle varying between 0 and 120 degrees, and the outboard wing segment always horizontal. For both configurations, the tail is assumed to have a quarter of the wing mass and inertia, and the fuselage is assumed to have three times the wing mass and four times the wing inertia. The aircraft center of mass is assumed to be at the wing half chord. The distance from the wing quarter-chord to the tail quarter-chord is assumed to be five times the wing chord. These aircraft parameters are summarized in Table 7.3. In addition, the tail chord is assumed to be equal to the wing chord, the tail span is equal to a quarter of the total wing span, the tail life curve slope is assumed to be 0.7 times $2\pi$ to account for the lower aspect ratio. All other wing parameters remain the same.

<table>
<thead>
<tr>
<th>Mode</th>
<th>4-State Model</th>
<th>Multi-Body Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phugoid</td>
<td>0.0027 + 0.2283i</td>
<td>0.0004 + 0.2265i</td>
</tr>
<tr>
<td>Short Period</td>
<td>-6.5697 + 9.2304i</td>
<td>-6.6602 + 9.3619i</td>
</tr>
</tbody>
</table>
Table 7.3: Aircraft Parameters for Three-Segment Folding Wing

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tail Mass</td>
<td>$M_t$</td>
<td>$6.8E-3$ kg</td>
</tr>
<tr>
<td>Tail Inertia</td>
<td>$I_{yy,t}$</td>
<td>$1.42E-6$ kg-m$^2$</td>
</tr>
<tr>
<td>Fuselage Mass</td>
<td>$M_f$</td>
<td>$81.8E-3$ kg</td>
</tr>
<tr>
<td>Fuselage Inertia</td>
<td>$I_{yy,f}$</td>
<td>$22.7E-6$ kg-m$^2$</td>
</tr>
<tr>
<td>Center of Mass Location</td>
<td>$X_{cg}$</td>
<td>$0.0125$ cm</td>
</tr>
<tr>
<td>Wing to Tail Distance</td>
<td>$L_t$</td>
<td>$25$ cm</td>
</tr>
<tr>
<td>Tail Chord</td>
<td>$c_t$</td>
<td>$5$ cm</td>
</tr>
<tr>
<td>Tail Span</td>
<td>$L_t$</td>
<td>$7.5$ cm</td>
</tr>
<tr>
<td>Tail Lift Curve Slope</td>
<td>$C_{Lt}$</td>
<td>$4.4$</td>
</tr>
</tbody>
</table>

7.5.1 Results for 30-Degree Inboard Fold Angle

Figure 7.4 shows several plots of the aeroelastic eigenvalues for the first configuration. Two sets of results are shown. The figures on the left show results for outboard angle of -60 degrees, and the figures on the right show results for outboard angle of 60 degrees. For each set of results, the first plot shows the imaginary part of the eigenvalues versus air speed, the second plot shows the real part of the eigenvalues versus air speed, and the third plot zooms in near zero eigenvalue to show the evolution of the phugoid mode. There are several notable features of the eigenvalue plots.

1. The configuration with negative 60-degree fold angle still has a hump flutter mode followed by a coalescence flutter mode.

2. The configuration with positive 60-degree fold angle still has a coalescence flutter mode.

3. The phugoid modes are very weakly unstable, and is inversely proportional to air speed as expected, but eventually couples with the divergence mode.

4. There is a neutrally stable mode whose mode shape is motion of the aircraft exclusively in the x direction.
5. Divergence occurs at a lower air speed. Compared to the clamped wing, the divergence speed is about 10% lower.

6. The short period mode couples with the first bending mode but quickly becomes very damped. The mode disappears from the plots of the imaginary parts of eigenvalues because it leaves the brute force eigenvalue solver’s search range due to its high damping.

The flutter modes in this result are very similar to the flutter modes of the clamped-wing results. Figure 7.5 shows a side-by-side comparison of the aircraft and clamped-wing eigenvalues for both configurations. In general, the flutter boundaries for the case with -60 degrees outboard angle and the coalescence flutter boundary for the case with 60 degrees outboard angle occur at higher air speeds. This agrees with the observation that torsion natural frequencies increase faster than bending natural frequencies when aircraft rigid body modes are modeled in the system, resulting in greater frequency separation. However, the general flutter behavior does not change with the addition of the aircraft rigid body degrees of freedom. The divergence boundary is not shifted by a significant amount (estimated to be 10% from the results), but since the flutter boundary is at a higher speed for the aircraft system, divergence now occurs at a lower air speed than flutter. Therefore, the overall aeroelastic behavior of the system is expected to change from a flutter instability to a divergence instability.

7.5.2 Results for Lockheed-Type Configuration

Figure 7.6 shows plots of the aeroelastic eigenvalues for the second configuration with Lockheed-type fold angles. Two sets of results are shown. The figures on the left show results for inboard angle of 30 degrees, and the figures on the right show results for inboard angle of 90 degrees. For each set of results, the first plot shows
Figure 7.4: Aeroelastic Eigenvalues for the Three-Segment Folding Wing, $\psi_2 = 30$, Two Different Outboard Angles
the imaginary part of the eigenvalues versus air speed, the second plot shows the real part of the eigenvalues versus air speed, and the third plot zooms in near zero eigenvalue to show the evolution of the phugoid mode. There are several notable features of the eigenvalue plots, and many of the features are the same as the ones observed for the first set of configurations.

1. The configuration with 30-degree fold angle still has a coalescence flutter mode.

2. The configuration with 90-degree fold angle does not flutter. The coalescence mode disappears as was the case for the clamped wing.

3. The phugoid modes are again very weakly unstable, and is inversely propor-
tional to air speed as expected, but eventually couples with the divergence mode.

4. There is round-off error near zero eigenvalue at high speeds because there is one zero eigenvalue and one unstable eigenvalue that is very close to zero.

5. Divergence occurs at a lower air speed. Compared to the clamped wing, the divergence speed is about 10% lower.

6. The short period mode couples with the first bending mode but quickly becomes very damped.

As was observed with the first set of theoretical results, the flutter modes in this set of results are very similar to the flutter modes of the corresponding clamped-wing results. Figure 7.7 shows a side-by-side comparison of the aircraft and clamped-wing eigenvalues for both configurations. In general, the flutter speeds increased and the divergence speed decreased. The overall result is that the aircraft system is more likely to enter a divergence instability than the clamped wing.
Figure 7.6: Aeroelastic Eigenvalues for the Lockheed-Type Three-Segment Folding Wing, Two Different Inboard Angles, Outboard Wing Horizontal
Figure 7.7: Comparison of Aeroelastic Eigenvalues for an Aircraft System and a Clamped Wing
In addition to designing the new set of fixed-angle experiments using friction hinges, two more sets of experiments were designed with hinges between wing segments that may rotate. Due to the lack of transient experimental data, the present work aimed to build and test experimental models that can undergo folding motion during wind tunnel testing.

The first series of experiments was designed for a three-segment folding wing with a controllable outboard hinge. This is an improvement over reports of existing folding wing experiments in literature that generally consider fold angles that do not change during each round of testing. The outboard hinge is a loose hinge with very little static friction, and the inboard hinge is a friction hinge. A servo is attached to the outboard end of the second wing segment, and a simple linkage mechanism is used to connect the servo to the outboard wing segment, such that the servo can be actuated to rotate the outboard wing segment. This set of experiments was then conducted to study transients of folding motion as well as the aeroelastic behavior of the system as the fold angle changes and moves the wing across the flutter boundary, as defined by the flutter speeds and frequencies measured during fixed-angle experiments.
The second series of experiments was designed for a three-segment folding wing with controllable inboard and outboard hinges. The outboard hinge is actuated using the same servo and linkage mechanism as the mechanism from the second set of experiments. The inboard hinge is also a loose hinge, and the fold angle change is actuated using two strings that are attached to the second wing segment, one from each face of the wing segment. The wing segment moves in one direction or the other depending on which string is being pulled. These experiments were designed along with the second set of experiments to study the transients and aeroelastic behavior of the system across the fixed-fold-angle flutter boundary. The added control allowed the Lockheed configuration to be studied.

This chapter will discuss the design of each series of experiments, the construction of each model, and both ground vibration and wind tunnel test results.

8.1 Design of Moving Wing Experiments

The fixed-fold-angle experiments validated the theoretical aeroelastic model. When the configurations are fixed and the fold angles do not change, the theoretical model is able to predict the system stability with good accuracy. However, the stability of the system as fold angles change has not been addressed, neither in the present model nor in the prior literature. In general, there is a lack of experimental aeroelastic results for folding wing systems where the wing morphs during testing. The present work addresses that need by designing a set of experiments in which a three-segment folding wing has movable inboard and outboard hinges. The fold angles are actively and independently controlled during wind tunnel testing, and time series data of structural displacement are used to analyze system stability.
8.1.1 Design of Hinge Control Mechanisms

The first step in designing the experiments was designing the mechanism that would control the fold angles. Several different ideas were considered, but the focus was on creating a mechanism that could control the fold angles without drastically changing the wing structural dynamics properties or affecting the air flow over the wing. The type of hinge, either a friction hinge or a loose hinge, was also a design choice that has yet to be determined.

Concept Generation

The first idea was to control the wing segment position by attaching thin strings to each wing segment, and pulling on the strings to move the wing segments relative to each other. The strings can be lengthened or shortened to move the wing segments, and the lengths are controlled by servo motors. Two sets of servos and strings are required for each wing segment since it is necessary to pull either way to increase or decrease the fold angle. The strings are perpendicular to the wing segment planes and the servos are mounted to the wind tunnel walls. Figure 8.1 shows a diagram of the concept.

Figure 8.1: Concept 1 for Moving Wing Experiment: String-Controlled Fold Angles
A variation of the string-controlled fold angle idea is to route the strings close to the wing segments and attach them through a structure on the wing, and then attach them to a point on the wing segments. This is a more realistic design because the servos can all be mounted on one side of the wing, which is like in the fuselage. Also the strings hug the shape of the wing, resulting in a more compact package. The hinge may be either friction hinge or loose hinge, but a friction hinge will require greater force from the servos, and result in greater reaction force on the wing. Figure 8.2 shows a diagram of the concept.

The third idea comes from the micro air vehicles lab at University of Florida [25, 17]. Dr. Rick Lind’s research group built a model aircraft with morphing wings. Their wing achieved morphing using a linkage system. By providing a short moment arm at the hinge, a large range of fold angles may be achieved with little displacement of the linkage. Using linkages will remove the need for two sets of controls per hinge because a linkage can push and pull, whereas a string can only pull. This idea may be used without modification to control the inboard hinge. A variation of this design that includes two linkages is considered as a potential mechanism for controlling the outboard hinge. Figure 8.3 shows a diagram of the concept.

The fourth method is to have one servo at each hinge to rotate each hinge independently. In this case, it would be easier to use a loose hinge to reduce torque
requirements. This will increase the weight of the wing and negatively impact the aerodynamic shape of the wing. However, this method has very little unwanted reaction force on the wing because the servo will apply torque very close to the hinge. This design is also the simplest to control because there is a direct correlation between a single control input to the servo and the resulting fold angle. For the string-controlled concepts, at least two servo inputs are required to control each hinge, which necessitates careful coordination between the multiple inputs. Figure 8.4 shows a diagram of the concept.

Concept Evaluation

The concepts were evaluated based on potential disturbances to air flow, realistic design, effects on structural dynamics, and ease of set up and control.
The effect on air flow is important because the added mechanism should not disturb the surrounding air flow too much. For the first concept, the effect on aerodynamic shape is small because thin strings are used. Thin strings, such as fishing line, should not significantly affect the flow field when compared to other features such as hinges or fasteners. For the second concept, the effect on aerodynamic shape is worse because a standoff structure is needed to route the string. However, the effect should be limited close to the hinges, especially if loose hinges are used, since there is no need for a large moment arm in that case. For the third concept, the effect is similar to, but slightly worse than the second design. A small standoff structure is still needed to attach the linkage, and the linkage itself will be less aerodynamic because a stiff linkage will have greater flow obstruction than a thin string. A linkage that is flat with the flow direction is ideal. The moment arm required for the linkage also adds slight flow obstruction. Lastly in the case of the direct-mount servo concept, the effect on aerodynamics is generally worse than the other concepts because each servo will require a smooth fairing for improved flow. The effect again is limited to the area near the hinges, but the servo will take up more of the wing segment span than the other ideas, resulting in greater effects on the lift and moment on the wing. This also depends on the size of the servo. The effect for the outboard hinge, which requires the least torque, will be less than the effect for the inboard hinge.

Whether the design is realistic for an actual aircraft is an important criterion because part of the goal is to show the validity of the proposed theoretical model for a realistic system. The first concept is unrealistic because a real aircraft would not have the servos so far from the actual aircraft. On the other hand, the second concept is a more realistic design because it is possible for all of the servos to be inside the fuselage. The third concept is certainly a realistic design since it has already been done by Dr. Rick Linds research group. Lastly, the direct-mount servo design is a realistic design because the servos can be mounted on the wing inside a fairing,
similar to how trailing edge flap actuators are mounted in the wing store, and the control cables can be routed along the wing and into the fuselage.

The effect on structural dynamics is important because too much of a change to the structural dynamics will invalidate the existing theory as an appropriate mathematical model for the moving wing experimental system. The added mass is not as important because additional mass is easily modeled in the theory. However, adding stiffness to the system should be avoided as much as possible. For the first concept, the design does not add any mass to the existing wing structural dynamics, but it adds stiffness because the servos exert a force on the wing via the strings to maintain the fold angles. Because the strings are predominately normal to the wing segments, most of the force is out-of-plane, which adds to the bending stiffness of the wing. The second concept is the same as the first in terms of added stiffness. The third concept will add significant stiffness to the system because stiff linkages are used to connect hinges to servos. This means that the wing segments cannot bend as easily as when the linkages are not present. This design will also add some mass from the moment arm connections and the linkage itself. Lastly, the fourth concept will not add stiffness to the system because the servo control is localized near the hinge. This design will add some mass depending on the weight of the servos, but the servos will range from 5 grams to 15 grams, which is about the same as a hinge.

The last evaluation criterion is ease of set up and control. For the first concept, the design requires careful control of pairs of servos. For each hinge, one servo must apply tension and one servo must give slack at the same time. However, this is relatively easy to set up because no other parts are required. The second concept has the same amount of difficulty as the first in terms of control. In addition, the set up for this concept requires building the stand-offs so that the wires can be routed through them. However, the parts can be easily made on the rapid prototyping machine. The third concept has easier control because there is only one servo per
hinge. The set up for this concept requires more work, including designing and making the linkages and making the moment arms that can connect to the hinges. The last concept also has easier control because there is only one servo per hinge. The set up for this design requires more work, including mounting parts for the servo and moment arms that connect to the hinges.

**Summary of Concept Evaluations**

The designs are compared in Table 8.1. The default string-controlled fold angle design has a score of zero for each category to serve as a point of reference. The other concepts are given point scores, with positive score being better than the reference design.

<table>
<thead>
<tr>
<th>Candidate Design</th>
<th>Aero Shape</th>
<th>Realistic Design</th>
<th>Add Mass or Stiffness</th>
<th>Ease of Setup</th>
<th>Ease of Control</th>
<th>Total Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>String 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>String 2</td>
<td>-1</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Linkage control</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
<td>-2</td>
<td>+2</td>
<td>-1</td>
</tr>
<tr>
<td>Servo control</td>
<td>-2</td>
<td>+1</td>
<td>+2</td>
<td>-2</td>
<td>+2</td>
<td>+1</td>
</tr>
</tbody>
</table>

The concept evaluation study shows that the servo control is the best design, while the two string-control concepts are tied as the second-best design. This result comes from un-weighted scores for each category. For the aeroelastic experiments, the most important criterion is still minimizing the effects on system structural dynamics. The compact string-control design is better than the default string design except for aerodynamic shape and ease of setup, both of which may be solved relatively easily. Therefore, both the servo control and the compact string control designs are considered in more detail.
8.1.2 Calculations for Servo Torque Requirements

Required Torque

The servos must resist torque about each folding wing hinge. The calculations consider gravity and aerodynamic forcing as sources of torque. When the wing segment center of gravity is not vertical to the hinge axis, the weight of the wing segment exerts a torque about the hinge. When the wing segment is folding, the induced wash causes aerodynamic lift over the wing segment, with exerts a torque about the hinge as well.

![Diagram of Folding Wing for Calculating Required Torque due to Gravity](image)

**Figure 8.5**: Diagram of Folding Wing for Calculating Required Torque due to Gravity

Consider a three-segment folding wing mounted vertically from the wind tunnel top wall, as shown in Fig. 8.5. Gravity acts through the mid-span of each wing segment, and exerts torques about each hinge. The outboard hinge must resist the torque from gravity acting on the outboard wing segment, which in general depends on the fold angle.

\[
T_{3g} = m_3 L_3 g \left( \frac{1}{2} L_3 \sin \psi_3 \right)
\]  

(8.1)

The inboard hinge must resist the torque from gravity acting on the outboard wing segment, the inboard wing segment, and any additional mass such as the out-
board hinge and the outboard servo motor. The following equation shows how each effect contributes to the total required torque.

\[
T_{2g} = m_3g \left( \frac{1}{2}L_3 \sin \Psi_3 + L_2 \sin \Psi_2 \right) + m_2g \frac{1}{2}L_2 \sin \Psi_2 + M_s g L_2 \sin \Psi_2 \quad (8.2)
\]

The term \(M_s\) includes both the hinge and the servo, if a servo is attached there, and assumed that the masses are concentrated at the hinge. The required torque is maximized when the wing segments are horizontal, in which case the moment arms for the gravitational forces are the greatest.

The second source of required torque is from the aerodynamic forces. When the wing is undergoing folding motion, there is a resisting aerodynamic force at each point of the wing that is proportional to the translational velocity and translational acceleration of the wing. The following equation is the aerodynamic force taken from Theodorsen’s expression for lift on an airfoil with only terms that depend on plunge and without the Theodorsen function.

\[
\vec{L}_i = -\pi \rho b^2 \ddot{z}_i - 2\pi \rho U b \dot{z}_i \quad (8.3)
\]

The coordinate \(z\) is the displacement perpendicular to the wing cross section, and is equal to the angular velocity \(\dot{\psi}_i\) of the rotating wing segment times the span-wise distance from a point on the wing segment to the hinge.

\[
\dot{z}_i = \dot{\psi}_i \tilde{y}_i \quad (8.4)
\]

The terms in Theodorsen’s lift expression that depend on angle of attack are ignored because the folding motion contributes only to plunge motion in the wing segment relative frame. The Theodorsen function was also ignored to simplify the calculations, but is not necessary for the design calculation since the Theodorsen function has a maximum magnitude of 1. In fact, the Wagner function is more appropriate because at each cross section of the wing, the motion of the folding wing
looks like an impulsively-started airfoil. However, it is not necessary to include it for
the design calculation since the Wagner function also has a maximum magnitude of
1.

The total torque exerted about the hinge due to aerodynamics is equal to the
integral of the per-span lift force in Eq. (8.3), multiplied by the distance from the
hinge, and then integrated along the wing segment.

\[ M_{ia} = \int \bar{L}_i \ddot{y}_i \, d\bar{y}_i \quad (8.5) \]

Substituting Eq. (8.4) into the above integral results in the following expression
for the torque due to aerodynamics.

\[ M_{ia} = \frac{1}{2} L_i^2 \left( \pi \rho b^2 \ddot{\psi}_i + 2 \pi \rho U \dot{b} \dot{\psi}_i \right) \quad (8.6) \]

The equation above can be used directly to calculate the required torque for the
outboard hinge. For the inboard hinge, the required torque includes the motion of
both the second wing segment and the outboard wing segment. For a worst case
calculation, assume that the outboard hinge is at zero degrees such that the second
wing segment is parallel to the outboard wing segment, resulting in the greatest
translational velocity and also the greatest induced torque. Then the effective wing
segment span is \( L_2 + L_3 \). The equations below show the required torque for the
outboard and inboard hinges, respectively.

\[ M_{3a} = \frac{1}{2} L_3^2 \left( \pi \rho b^2 \ddot{\psi}_3 + 2 \pi \rho U \dot{b} \dot{\psi}_3 \right) \quad (8.7) \]

\[ M_{2a} = \frac{1}{2} (L_2 + L_3)^2 \left( \pi \rho b^2 \ddot{\psi}_2 + 2 \pi \rho U \dot{b} \dot{\psi}_2 \right) \quad (8.8) \]

The calculations can ignore the apparent mass (second derivative) term because
the folding motion typically will be of constant angular velocity. The above equations
for the gravity-induced and aerodynamically-induced torques are used to compute the
required torques for the inboard and outboard hinges for the two test configurations. The calculations assume an angular velocity of 1 rad/s, which is approximately 60 degrees per second. Table 8.2 summarizes the numerical results.

### Table 8.2: Required Torque for Direct-Mount Servo

<table>
<thead>
<tr>
<th>Source of Torque</th>
<th>Config 1 Inboard (oz-in)</th>
<th>Config 1 Outboard (oz-in)</th>
<th>Config 2 Inboard (oz-in)</th>
<th>Config 2 Outboard (oz-in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity</td>
<td>7.81</td>
<td>0.23</td>
<td>10.31</td>
<td>0.30</td>
</tr>
<tr>
<td>Aero</td>
<td>7.85</td>
<td>1.44</td>
<td>13.33</td>
<td>1.88</td>
</tr>
<tr>
<td>Total</td>
<td>15.66</td>
<td>1.67</td>
<td>23.65</td>
<td>2.17</td>
</tr>
</tbody>
</table>

For the mechanism in which the servo is mounted directly on the wing segment and right next to the hinge, the values in Table 8.2 are approximately the torques that the servo must be able to resist. For both configurations, the outboard hinge does not have to resist a lot of torque. This means that a small servo will work well for that hinge. For the inboard hinge, the required torque is approximately 25 oz-in. This means that if a servo is attached to the wing segment directly, it should resist about 30 oz-in of torque. The induced torque is linearly proportional to the angular velocity, so stronger and larger servos are necessary to achieve greater angular velocities, for example, to study the wing during transient folding motions.

**Required Torque for String-Controlled Hinge**

For the case where the folding wing positions are controlled by strings, hand-calculations and rough prototypes showed that with the loose hinges, any internal friction can be overcome relatively easily as long as there is some distance between the string and the hinge axis to provide sufficient moment arm. This is accomplished by adding some stand-off structures near the hinges such that the strings can route through those structures first, providing some fixed distance away from the hinge axis. To move the wing segments, the strings are tensioned to supply a torque equal to the tension force times the moment arm from the string to the hinge axis. The moment
arm geometry differs depending on whether the string is on the bottom or top of the folding wing, since the bottom string hugs closer to the hinge axis, resulting in a smaller moment arm and therefore smaller torque.

The servo torque is not directly related to the required torque about the hinge in this case. The required torque is the string tension times the moment arm. Suppose that the string moment arm about the hinge axis a minimum value of about 1 cm, assuming good design of stand-off structures. Also assume that the string length must change by 5 cm to achieve the desired range of motion. The servo arm must be as long as the change in string length to achieve the same angular velocity as the case where the servo is directly attached to the wing segment. Therefore, the required torque is equal to the required tension in the string times the required servo arm length.

Table 8.3 summarizes the required string tension and servo torque for the servo that controls the inboard hinge. Preliminary search of servo motors showed that there are small servo motors that can meet the needs of the outboard hinge. Therefore, the calculations will not consider string-controlled outboard hinge because it will be much more complicated than directly attaching a servo to the wing segment.

Table 8.3: Required String Tension and Torque for String-Controlled Hinge

<table>
<thead>
<tr>
<th>Config</th>
<th>Tension (oz)</th>
<th>Torque (oz-in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Config 1</td>
<td>39.8</td>
<td>78.3</td>
</tr>
<tr>
<td>Config 2</td>
<td>60.1</td>
<td>118.2</td>
</tr>
</tbody>
</table>

These results suggest that a servo with a torque rating of over 100 oz-in is required for the bottom string. Note that gravity accounts for about half of the required torque, but gravity acts in favor of the bottom string, so the actual torque requirement is less. Also note that these calculations are very rough estimates, but still suggest an order of magnitude for the required servo size.
Final Designs for Hinge Mechanisms

The above calculations show that both concepts are feasible provided that the servos can supply the required torque. For the outboard hinge, the servo should be small enough to minimize its effect on the air flow, but also have enough torque to move the wing segment. The smallest commercially available servo motor is the HS-5035HD from Hitec. The servo has dimensions of 18.6 mm x 7.6 mm x 15.5 mm, weighs 4.5 grams, and has a torque output of 11 oz-in. This fits the requirements for the outboard hinge very well. The torque is also large enough to allow a wide range of angular velocities.

The inboard hinge requires about 30 oz-in of torque. A commercially available servo that meets the requirement is the Hitec HS-5065MG, which can provide 30.5 oz-in of torque. The dimensions are 23 mm x 25 mm x 12 mm, and the servo weighs 12 grams. This is certainly a feasible option, but it will add significant weight to the experimental model and require a larger fairing to enclose the servo. Additionally, the servo will most likely not be strong enough to rotate the wing segment much faster than 60 degrees per second. In this case, the better choice is the string-control mechanism because the servos are located outside the wind tunnel test section, so any large size servo may be used. The servo that is chosen for the string-control mechanism is the Hitech HS-5645MG, which can supply 168 oz-in of torque.

In summary, three servos were chosen for the experimental model. The nominal design uses two of the servos, with the small servo directly mounted to the wing segment to control the outboard hinge, and two large servos mounted outside the wind tunnel test section and connected to strings that attach to the wing to control the inboard segment. An alternate method for controlling the inboard hinge is to attach the medium sized servo directly to the wing. Table 8.4 summarizes the chosen servos.
Table 8.4: Final Choices for Servo Motors

<table>
<thead>
<tr>
<th>Servo</th>
<th>Torque (oz-in)</th>
<th>Weight (oz)</th>
<th>Length (in)</th>
<th>Width (in)</th>
<th>Height (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS-5035HD</td>
<td>11.1</td>
<td>0.16</td>
<td>0.73</td>
<td>0.61</td>
<td>0.30</td>
</tr>
<tr>
<td>HS-5065MG</td>
<td>30.6</td>
<td>0.42</td>
<td>1.03</td>
<td>0.94</td>
<td>0.46</td>
</tr>
<tr>
<td>HS-5645MG</td>
<td>168.0</td>
<td>2.10</td>
<td>1.59</td>
<td>1.48</td>
<td>0.77</td>
</tr>
</tbody>
</table>

8.1.3 Quasi-Steady Limit of Folding Motion

When the fold angle changes very slowly, the system should behave as if the fold angle is not changing at all, and the aeroelastic behavior should be the same as was measured in the fixed-angle folding wing experiments. When the fold angle changes very quickly, the transient effects will be significant and the system can no longer be treated as if the fold angle is quasi-steady. A scaling analysis is carried out to determine the maximum angular velocity for the folding motion at which the fold angle may still be considered quasi-steady.

The physical effects under consideration are the changes in structural dynamics and aerodynamics. For the structural dynamics, the mass distribution of the folding wing changes as the fold angles vary, resulting in changes in natural frequencies. The time scale of the folding motion is compared to the time scale of the natural system vibrations. For the aerodynamics, downwash is added to wing segments that undergo folding motion and it affects the circulatory lift and moment over the wing segments. The folding-induced downwash is compared to the downwash due to the free vibrations of the folding wing. The details of the analysis are described in Appendix D. The final result is a relation between the angular velocity of the folding motion and the folding wing flutter frequency, shown in Eq.(8.9), that determines whether the effect of wing folding is negligible.

\[
\frac{\omega_f b}{U} \ll \frac{\omega_f b}{U} \left( \frac{t}{b} \right) \tag{8.9}
\]
The left hand side in the above equation is the angular velocity of the folding motion normalized by half chord \( b \) and air speed \( U \). The right side of the above equation is the reduced flutter frequency multiplied by the wing segment thickness-to-half-chord ratio. For the particular folding wing experimental model that was constructed for wind tunnel testing, the right hand side of the above equation gives a value of 180 deg/s. Therefore, the angular velocity of the folding motion should be much less than 180 deg/s, say around 20 deg/s, for the folding motion to have negligible effect on the aeroelastic behavior of the system.

8.1.4 Design of Supporting Components

The hinge control mechanisms are finalized as discussed above. The next tasks are to design the additional components for each concept. These tasks are itemized in the list below.

1. For the direct-mount servo, additional components are needed to transmit the rotation of the servo to rotation of the hinge.

2. For the direct-mount servo, a fairing needs to be designed and manufactured to house the servo and reduce the effect of the added mechanism on air flow.

3. For the string-control mechanism, stand-off structures need to be designed to maintain some moment arm between the string and the hinge axis.

For the direct-mount servo, the simplest system that will transmit the servo torque to torque about the hinge is a two-linkage system, as shown in Fig. 8.6. The servo arm that comes with the servo motor provides one linkage, and a second set of linkages connects the servo arm to a mounting point on the outboard wing segment. The servo motor is positioned along the span of the second wing segment to allow the greatest range of motion for a certain linkage length. This design allows the fold
angle to vary from 0 to approximately 120 degrees, which is sufficient for the current test goals.

Figure 8.6 also shows the fairing design. The fairing is made in two parts that are glued together. Figure 8.7 shows the two components in more detail.

Each half has several important design features.

1. A large slot to allow the servo arm to move without interference.

2. A small slot to allow the servo control cable to pass through the fairing.
3. The half that is in contact with the hinge (the top half) has a bolt hole for securing the fairing to the hinge.

4. The top half has four locating protrusions to help locate the servo motor within the fairing during assembly.

5. The bottom half has three protrusions to push against the servo and secure it when the system is fully assembled.

For the string-control mechanism, the stand-off structures were designed to be bolted to the hinge. The only purpose is to provide a location through which the string can be routed. To minimize the aerodynamic effects, the structure was designed such that each cross section along the flow direction was shaped like an airfoil. Figure 8.8 shows detailed renderings of the stand-off structures. A mirror image of the version shown in the rendering was also designed.

The string that routes along the bottom of the wing segment will have a smaller moment arm to the hinge axis. To maintain sufficient moment arm, the stand-off structure should be placed as close to the hinge as possible. For the top string, this is not necessary because the moment arm is always sufficient. In fact, placing the stand-off structure close to the hinge will cause interference with the wing segment at higher fold angles, so the structure should be moved away from the hinge. These
contradicting requirements resulted in the two designs that are mirror images of each other.

Furthermore, the final design uses two stand-off structures on the bottom side of the wing to maintain distance between the string and the hinge axis. This is shown in Fig. 8.9.

![Figure 8.9: Renderings of the String-Control Mechanism](image)

Figure 8.10 shows a rendering of the overall folding wing system, which employs a directly-mounted servo at the outboard hinge and the string-control mechanism at the inboard hinge.

![Figure 8.10: Rendering of Moving Wing Experimental Model](image)
8.1.5 Manufacturing and Control

The components were all very small and have contoured shapes that would be difficult to manufacture by conventional means such as milling. Therefore, the rapid prototyping machine at Duke University was used to manufacture a majority of the necessary components. Figure 8.11 shows two photographs of the completed direct-mount servo assembly, showing the fairing and linkage system. The mounting point structure on the outboard wing segment was bonded to the hinge using an epoxy for plastic materials. The same epoxy was used to secure the second wing segment to the outboard hinge because the presence of the servo fairing blocked access to the hinge bolt hole at the trailing edge side. Super glue was used to bond the two halves of the fairing. Lastly, the fairing was very flexible at the trailing edge because it was bolted to the hinge at only one point near the leading edge of the fairing. In terms of structural dynamics, this resulted in noisy measurements of the wing during impact testing, and also altered the natural frequencies. Therefore, small amounts of epoxy were used to secure the fairing to the hinge at the trailing edge.

![Figure 8.11: Photos of the Completed Direct Mount Servo Mechanism](image)

(a) (b)

**Figure 8.11**: Photos of the Completed Direct Mount Servo Mechanism
Both LabVIEW and Arduino are used to control the servo. The Arduino is used because it is a very simple programmable micro-controller with an existing library for controlling servo motors. However, it is difficult to program in active control as well as data acquisition. Therefore, LabVIEW is used for both data acquisition and user control interface. The user specifies a wing angle, the conversion to the corresponding servo angle is done in LabVIEW, and LabVIEW subsequently sends an analog output signal to the Arduino. The Arduino then interprets the analog output signal as an angle for the servo motor, and sets the servo motor angle accordingly. Figure 8.12 shows a flowchart of how the control is implemented. A more detailed description of the control set up is discussed in the appendix.

![Servo Control Flowchart](image)

**Figure 8.12: Servo Control Flowchart**

8.2 Experiment with Moving Outboard Hinge

For the first step in testing a moving wing experimental model, a folding wing was constructed with a friction hinge for the inboard hinge and a movable hinge for the outboard hinge. This configuration is the same as the second configuration of the
friction hinge experiments, except that the outboard hinge has been replaced with a loose hinge with associated servo mechanisms. Figure 8.13 shows two photographs of the folding wing experimental model mounted in the wind tunnel. The figure emphasizes 1) the clamped root of the wing mounted to the wind tunnel top wall, 2) the servo on the outboard hinge, and 3) the accelerometer on the inboard wing segment. The accelerometer may also be placed on the second wing segment to obtain larger response.

Figure 8.13: Photos of the Three-Segment Folding Wing with Movable Outboard Hinge

The outboard hinge is controlled with a servo that is mounted on the wing segment. The servo fairing is mounted to the outboard hinge on the side of the wing that is away from the camera in the second photo, but is difficult to see in the first photo. For wind tunnel testing, the servo cables and the accelerometer cable are
routed along the trailing edge of the wing, and secured by tape, to minimize flow obstruction. The accelerometer was placed at the trailing edge of the outboard end of the first wing segment. This is because that position is optimal for picking up the responses of the first three natural modes, which are the modes that participate the most in the expected flutter instability.

8.2.1 Verifying Structural Dynamics

The first step was to verify the structural dynamics of the system by measuring the natural frequencies and comparing them to the predicted values after correcting for the added mass. The parameters for the second friction hinge configuration were kept the same, and an additional point mass was added to the system with mass equaling the total mass of the servo mechanism and the position equal to the approximate center of gravity for the total system, as measured in SolidWorks. The mass of the outboard hinge remained approximately the same because the loose hinge is slightly heavier than the friction hinge, but half of the fasteners were removed in favor of using epoxy to attach the second wing segment.

To test the system, the wing was clamped at the root, and the servo was powered on and set at specified fold angles. Figure 8.14 shows the predicted natural frequencies (·) and measured natural frequencies (△) versus fold angle for the Lockheed-type configurations. The left plot shows the frequencies up to 80 Hz, and the right plot zooms into the first three frequencies.

The results show excellent agreement for the natural frequencies of the first three modes. Data for the next two higher modes were of lower quality due to the addition of the servo mechanism, which was not as rigid as the friction hinge. Therefore, it was more difficult to pick out the transfer function peaks. Nevertheless, some data points were obtained and the results are in approximately the same range as the predicted values. The results show that other than added mass, the addition of
the servo mechanism did not significantly affect the structural dynamics for at least the first three modes. The fact that excellent agreement was obtained without any additional tuning of other parameters reinforces this conclusion.

This test was done to verify that the servo did not have a significant effect on structural dynamics, and in particular the hinge stiffness. However, this particular model cannot actually achieve Lockheed-type configurations in the wind tunnel since the inboard hinge is fixed. Instead, the experiments tested this configuration with the inboard angle fixed at 30 degrees and the outboard angle varying from 0 to 120 degrees. A series of ground vibration tests were also done for that configuration, again with excellent agreement between theory and experiment. Only the first three natural frequencies were tracked during the experiment.

8.2.2 Verifying Fixed-Fold-Angle Aeroelastic Behavior

The next step before conducting any morphing wing experiments is to verify that the fixed-fold-angle aeroelastic behavior matches the predictions from the aeroelastic model. For this set of experiments, the outboard fold angle was set by the servo control when the wind tunnel was off, and not changed again when the wind tunnel...
was on. The flutter speed was determined by tracking the amplitude of the fluttering mode in the FFT of the system response as the air speed was varied. For this particular configuration, the flutter speed was easy to determine because a sharp increase in amplitude was observed for each case, therefore narrowing the error bar on the flutter speed.

Figure 8.16 shows the flutter speed and flutter frequency for this particular system with fixed fold angles. All theoretical results were computed assuming a structural damping of 1%. Two sets of theoretical results are plotted. The first set defines zero aeroelastic damping ratio, equal to the real part divided by the imaginary part of the aeroelastic eigenvalue, as the boundary across which the system becomes unstable. The second set defines an aeroelastic damping ratio of 0.02 as the boundary across which the system becomes unstable. The two results are shown together to show the sensitivity of the predicted flutter boundary on structural damping and accuracy of the aerodynamic model. In this particular case, the results were not very sensitive to those factors.

The agreement between theory and experiment for flutter frequency is excellent, as typically seen for the cases that are studied in the present work. The trend in flutter speed is accurately captured by the theoretical model, as expected. The theory
underestimates the flutter speed, which has also been a typical result in the present work. The discrepancy between theory and experiment for this particular configuration is approximately 15%. Overall, the theory is able to predict the aeroelastic behavior of the system with fixed fold angles.

8.2.3 Aeroelastic Results with Transient Folding Motion

The goal of these experiments is to measure the folding wing system behavior as the wing is taken across the flutter boundary formed by the fixed-angle experimental results, herein referred to as the quasi-steady flutter boundary. For each experiment, a fold angle sweep is done where the system starts at a stable state, and then the fold angle changes such that the system enters the quasi-steady flutter region, and then returns to the starting stable state. Different tests were conducted in which the wing folding motion took the system either up to the quasi-steady flutter boundary or past the boundary and into the flutter region. Figure 8.17 shows the quasi-steady flutter boundary result for the three-segment folding wing under consideration, and shows an example operating path for the moving wing experiment. The wind tunnel is first brought up to an air speed that just exceeds the flutter speed of the negative
45-degree configuration, and the fold angle is initially set at -75 degrees such that the system is stable. This path is labeled as 0 in the figure. Path 1 varies the fold angle first toward -45 degrees and then back to -75 degrees, so path 1 represents an experiment in which the folding motion took the system up to the quasi-steady flutter boundary. Path 2 starts at the same place but takes the fold angle to -15 degrees, so path 2 represents an experiment in which the folding motion took the system deeper into the quasi-steady flutter region.

Figure 8.17: Example Operating Path for the Fold Angle Sweep Experiments

If the folding motion is slow enough, the system should flutter as it crosses the quasi-steady flutter boundary. The scaling analysis estimated that the angular velocity of the folding motion should be greater than 20 deg/s for the folding motion to have an effect on system aeroelastic behavior. The angular rate was varied from 5 deg/s at the minimum to 90 deg/s at the maximum. The response of the accelerometer over time was measured and recorded using LabVIEW. The frequency content of the signal over time was reproduced from the time-series data in MATLAB using the spectrogram function, which computes the short time Fourier Transform (STFT) of the signal.

Figure 8.18 shows the spectrogram results for an experiment in which the fold
angle started at -60 degrees and moved toward -15 degrees. The wind tunnel was brought up to 32 m/s, and the fold angle was first set to -60 degrees. Then the angle was folded to -15 degrees and then back to -60 degrees at the following rates: 5, 10, 15, 20, 30, 60, and 90 deg/s. The bottom plot shows the spectrogram of the signal, zoomed into the frequencies near the flutter frequency. The maximum value of the STFT at each time step was then plotted over time in the top plot. Superimposed over the plot of max STFT is a curve that indicates the specified fold angle. There are no units because the angle linearly varied between two values: the system started at -60 degrees and moved toward and away from -15 degrees. Each triangle in the dashed curve is a single fold angle sweep, and the width of the triangle is inversely proportional to the angular velocity of the folding motion.

The results show that large responses were observed for low wing folding angular velocities, i.e. wider triangles in the angle setting curve. At higher rates the response was at signal noise level, and no visible oscillations were observed. The plot may be summarized into a more concise figure by plotting the response level versus the angular velocity of the folding motion. The expected trend is that faster folding motion would correlate with lower system response, and that trend was in fact observed in the experiments. Figure 8.19 shows the response amplitude versus angular velocity of folding motion when the air speed is fixed at the flutter speed for the -45-degree configuration and the fold angle starts at -75 degrees. Figure 8.20 shows the response amplitude versus angular velocity of folding motion when the air speed is fixed at the flutter speed for the -75-degree configuration and the fold angle starts at -115 degrees.

8.2.4 Conclusions

In summary, these experiments provided new data on the aeroelastic response of a folding wing system during wing morphing. The experimental results show that if
the wing folding occurs at fast enough angular velocity, the system may enter and leave the quasi-steady flutter zone without going into a limit cycle oscillation. When looking at the experimental data, the cut off is between 30 deg/s and 60 deg/s, which agrees well with the results of the theoretical scaling analysis. So far there have been no observed cases in which the folding motion destabilizes the system, but the possibility is certainly not ruled out. Furthermore, these tests were conducted for a clamped wing at zero angle of attack. In a real application in which the wing
Figure 8.19: Response Amplitude vs. Rate of Folding Motion for Air Speed Above Flutter Speed of -45-Degree Fold Angle Configuration

Figure 8.20: Response Amplitude vs. Rate of Folding Motion for Air Speed Above Flutter Speed of -75-Degree Fold Angle Configuration
must support some nominal lift force, the resulting static deflection may again alter the aeroelastic behavior if the wing is very flexible.

There is significant opportunities for future work, including corroborating theoretical results using the experimental setup and conducting more experiments to see whether there may be cases in which folding motion has a de-stabilizing effect.
Conclusions

The present work conducted more detailed aeroelastic analyses of folding wings and folding wing aircraft. A higher-fidelity model, which consists of a finite element structural model and a vortex lattice aerodynamic model, was derived and tested against the existing beam theory/strip theory aeroelastic model. The analyses showed that while the higher-fidelity model resolved the discrepancies between predicted and measured natural frequencies, there were still differences between the predicted and measured flutter boundaries. In particular, the aeroelastic model tended to underestimate the flutter speed, but the flutter frequencies were very accurately predicted. The complicated construction of the folding wing experimental models, including the presence of hinges and gaps between wing segments, may have adversely affected the flow field and caused the disagreement between theory and experiment. In addition, the flat plate experimental model itself may not be conducive to smooth flow, and a more sophisticated experiment with a more streamlined wing may yield better results.

The aeroelastic behavior for the three-segment folding wing was analyzed more thoroughly using the higher-fidelity model, with the goal of understanding the sudden
changes in flutter speed and frequency at specific fold angles for certain configurations. Several different hypotheses, including mesh convergence and accounting for low flutter strength, were tested but did not yield any definite conclusions. However, the analyses did show that for many folding wing configurations, the strength of the flutter mode, as defined by how fast the real part of the eigenvalue changes with respect to air speed, varies significantly over the full range of fold angles. Agreement between theory and experiment typically was worse for the weaker flutter modes.

Nevertheless, the correct trends are modeled by both the low-fidelity theory and the high-fidelity theory for most of the configurations that were studied, and the predicted values are in the range of measured values. The theoretical results confirm that the aeroelastic model may be used as an efficient preliminary design tool. This was certainly true when designing improved aeroelastic experiments for the fixed-angle wind tunnel tests. The aeroelastic code was used to conduct parameter variation studies very quickly, and the results were used to design the geometry and additional components of two folding wing test models that had the desired flutter behavior. Two significant conclusions were reached from the improved fixed-angle experiments.

The first conclusion was that the aeroelastic model was able to predict the correct trends for the flutter boundary of both of the Lockheed-type configurations that were tested, and the flutter speed was uniformly underestimated by the theory. This suggests that the vortex lattice aerodynamic model captures the overall physics of the flow field. This also added a large collection of fixed-fold-angle folding wing aeroelastic test data.

The second conclusion was that the static deflection was a significant factor in changing the flutter speed, and sometimes flutter frequency as well. The predicted sudden change in flutter speed with varying fold angle for the three-segment Lockheed-type folding wing test model was indeed observed in experiment, but the
flutter behavior was very sensitive to the amount of steady state structural deflection in the system. In general for all of the experiments that were conducted for the present dissertation, structural deflection in one direction tended to make the system more unstable and reduce the flutter speed, while structural deflection in the other direction tended to increase the flutter speed but also resulted in a more violent flutter response when the flutter point is finally reached. The source of the behavior still comes down to the proximity of the first torsion and second bending modes; geometric and structural nonlinear effects can both affect the spacing of the two modes and cause drastically different flutter behaviors during testing. Further work should consider the nonlinear structural dynamics in order to better understand this phenomenon.

In parallel with the effort to better understand the aeroelastic behavior of clamped wings, the aeroelastic theory was also improved by the addition of aircraft longitudinal degrees of freedom. As a part of the multi-body dynamics derivation, a fuselage and a tail were added to the aircraft system along with the wing. Additional aerodynamic forcing terms were derived and some of the terms were important for obtaining the basic aircraft flight dynamics modes: the phugoid mode and the short period mode. The theoretical results showed that when comparing a clamped wing to the same wing with an attached fuselage and tail, the aircraft system has higher flutter speeds but lower divergence speeds. In some cases, the system may reach the divergence instability first. This is also an important result when considering the design of such an aircraft.

In another parallel effort, several series of experiments with controllable hinges were designed such that the fold angles may be changed while the wind tunnel is in operation. Ground vibration tests were done with the fold angle fixed to ensure that the natural frequencies were approximately equal to those of the fixed-angle experimental models. Fixed-angle aeroelastic tests were then conducted to measure
the flutter speed and flutter frequency at each particular fold angle, and the resulting data formed a quasi-steady flutter boundary. Lastly, the wing was moved across the quasi-steady flutter boundary at different rates by changing the fold angles at different angular velocities. The results show that the system may avoid going into flutter when the fold angles are changed quickly enough. There was no case in which the wing motion became unstable due to changes in fold angle. The present aeroelastic model has not been developed enough to answer this question theoretically, but the multi-body aeroelastic model may be readily extended to have time-varying fold angles, and the equations of motion may then be solved to obtain time series results of wing response due to prescribed folding motion.

In conclusion, the present work showed that the multi-body aeroelastic model may be used either as a preliminary design tool to obtain relatively accurate predictions of system stability very quickly, or the higher-fidelity enhancements to the model (ANSYS beam modes and vortex lattice) may be used to obtain more accurate results. The present work also discussed configurations with interesting aeroelastic behavior. For many configurations, the variation of fold angles causes large shifts in natural frequencies that tend to change the system sensitivity to structural damping, aerodynamic damping, and steady state structural deflections. The inherent flexibility of the wind tunnel test models exacerbated the sensitivity, and this finding highlights the importance of both nonlinear analyses and experimental studies in understanding these sensitivities for micro air vehicles that employ flexible folding wings.

The present work also produced a wealth of experimental data, both for fixed-angle wind tunnel tests and morphing wing wind tunnel tests. The experimental results of wing motion during transients of changing fold angles are particularly valuable because there is a lack of both theoretical and experimental studies of folding wing transient behavior in the prior literature.
Appendix A

Vortex Lattice Method Governing Equations

A.1 Vortex Lattice Aerodynamic Model

A vortex lattice mesh may be divided into four regions, each of which is governed by a physical principle. To implement the vortex lattice theory in a numerical code, the governing behaviors of the horseshoe vortex circulations in each of the four regions are written as equations. The circulation of each horseshoe vortex on the wing and in the wake is a state variable in the equations of motion.

In the first region, the governing equations enforce the boundary condition that the total induced flow must not go through the wing. In the numerical code, this boundary condition is applied at each collocation point on the wing. The total velocity induced by all of the horseshoe vortices can be computed using a kernel function $K_{ij}$ that computes the induced velocity vector due to the $i$th horseshoe at the $j$th collocation point, as shown in Eq. (A.1). The boundary condition is that the component of the total induced velocity that is normal to the wing must be equal to
the wash on the wing, which is related to the motion and/or deflection of the wing.

\[ \mathbf{v}_j = \sum_i \Gamma_i \mathbf{K}_{ij}(x_i, x_j) \quad (A.1) \]

\[ \mathbf{v}_j \cdot \hat{n}_j = \frac{Dx_j}{Dt} \cdot \hat{n}_j \quad (A.2) \]

\[ \mathbf{v}_j \cdot \hat{n}_j = \left( \dot{x}_j + U_\infty \frac{\partial x_j}{\partial x} \right) \cdot \hat{n}_j \quad (A.3) \]

Note that the wash on the wing is the material derivative of the fluid as it travels over the wing at the collocation point, and has contributions from the instantaneous displacement of the collocation point as well as from the motion of the fluid particle convecting downstream along the wing surface. The boundary condition of Eq. (A.2) is applied at the collocation point of each panel on the wing. The collocation point is located at the 3/4 chord and midspan of the panel. For a wing that is divided into \( N_c \) panels and a vortex lattice system that has a total of \( N_v \) vortices on the wing and in the wake, there are \( N_c \) equations that enforce the no-flow-through boundary condition on the wing.

\[ \sum_{i=1}^{N_v} \Gamma_i \mathbf{K}_{ij}(x_i, x_j) \cdot \hat{n}_j = \frac{Dx_j}{Dt} \cdot \hat{n}_j \quad \text{for } j = 1, 2, \ldots, N_c \quad (A.4) \]

In the second region, which encompasses the single row of panels immediately downstream of the wing along the span-wise direction, the circulation of each horseshoe vortex in that region must be equal to the change in total circulation of all upstream horseshoe vortices on the wing at each time step. This is a consequence of Kelvin’s circulation theorem, which states that the circulation integrated over a material domain must remain in the same in irrotational flow. Equation (A.5) describes the vortex shedding phenomenon. In the equation below, \( \Gamma_i \) is the circulation of a horseshoe vortex in the second region, and \( \Gamma_{us} \) represents a horseshoe vortex that is upstream of \( \Gamma_i \). The equation sums over all horseshoe vortices upstream of \( \Gamma_i \). There
are $N_y$ equations in this region, where $N_y$ is the number of panels in the span-wise direction.

$$\Gamma^{n+1}_i = \sum_{us} \Gamma^n_{us} - \Gamma^{n+1}_{us} \text{ for } i = 1, 2, \ldots, N_y$$ (A.5)

In the third region, the circulation is convected downstream in the wake. The circulation of any horseshoe vortex at any time step is equal to the circulation of the immediately upstream horseshoe vortex at the previous time step, provided that the spatial and time resolutions of the vortex lattice mesh are appropriately related by the air speed, as specified in Eq. (3.15). The governing equation in this region is shown in Eq. (A.6), in which $\Gamma_i$ is the circulation of an arbitrary horseshoe vortex, and $\Gamma_{i,us1}$ is the circulation of the horseshoe vortex that is immediately upstream. For a wake mesh with $N_{xw}$ rows of vortices, there are a total of $N_y$ times $N_{xw} - 2$ equations.

$$\Gamma^{n+1}_i = \Gamma^n_{i,us1} \text{ for } i = 1, 2, \ldots, N_y(N_{xw} - 2)$$ (A.6)

In the fourth region, the circulation is accumulated at the last row of wake vortices, with a relaxation factor that gradually reduces the effect of previously-accumulated circulation. This simulates the presence of circulation past the end of the numerically-defined finite wake. The relaxation factor ($\alpha < 1$) accounts for the fact that the circulation is convected past the last row of wake vortices, and the decreased influence due to larger distance is numerically equivalent to decreased influence by reduced circulation. The governing equation in this region is shown in Eq. (A.7), in which $\Gamma_i$ is a horseshoe vortex in the last row in the wake, and $\Gamma_{i,us1}$ is the horseshoe vortex that is immediately upstream. At each time step, the equation first relaxes the circulation that has built up at the end of the wake by the previous time step, and then adds the circulation of the immediately upstream horseshoe vortex due to convection. There are a total of $N_y$ equations in this region.

$$\Gamma^{n+1}_i = \alpha \Gamma^n_i + \Gamma^n_{i,us1} \text{ for } i = 1, 2, \ldots, N_y$$ (A.7)
The circulation distribution over the wing can be used to compute the force on the wing by using the unsteady Bernoulli’s equation.

\[
L = \int \rho \frac{D\Gamma}{Dt} \, dA \tag{A.8}
\]

\[
L = \int \rho \left[ \frac{\partial \Gamma}{\partial t} + U_\infty \frac{\partial \Gamma}{\partial x} \right] \, dA \tag{A.9}
\]

\[
L = \int \rho \left[ U_\infty \gamma(x) + \frac{\partial}{\partial t} \int_{LE}^x \gamma(\xi) \, d\xi \right] \, dA \tag{A.10}
\]

\[
L = \rho U_\infty \int \gamma(x) \, dA + \rho \int \frac{\partial}{\partial t} \left[ \int_{LE}^x \gamma(\xi) \, d\xi \right] \, dA \tag{A.11}
\]

The equation can also be written for the \(i\)th panel at time step \(n + 1\), as shown in Eq. (A.12). In the equation below, \(A_i\) is the area of the \(i\)th panel.

\[
L_{n+1}^i = \rho U_\infty \int \gamma(x) \, dA_i + \rho \int \frac{\partial}{\partial t} \left[ \int_{LE}^x \gamma(\xi) \, d\xi \right] \, dA_i \tag{A.12}
\]

The first term in Eq. (A.12) is the quasi-steady portion of the lift force, and is equal to the Kutta-Joukowski lift. The integral of vorticity over the panel is equal to the circulation of the horseshoe vortex on that panel multiplied by the span-wise size of the panel \(\Delta y\).

\[
\int \gamma(x) \, dA_i = \Gamma_{n+1}^i \Delta y \tag{A.13}
\]

The second term in Eq. (A.12) is the unsteady portion of the lift force. The inner integral is the integral of vorticity from the leading edge of the wing to a particular point at location \(x\) downstream on the wing. In the vortex lattice method, that integral is equal to the sum of the circulation of horseshoe vortices upstream of the point in question.

\[
\int_{LE}^x \gamma(\xi) \, d\xi = \sum_{us} \Gamma_{us} \tag{A.14}
\]
The time derivative is discretized by the difference between the two time steps.

$$\frac{\partial}{\partial t} \int_{LE}^{x} \gamma(\xi) \, d\xi = \sum_{us} \Gamma_{us}^{n+1} - \Gamma_{us}^{n} \quad (A.15)$$

The second term in the lift equation is then equal to the air density times the integral of the above expression over the panel. For the quarter of the $i$th panel that is in front of the $i$th horseshoe vortex, the vorticity integral only includes the horseshoe vortices that are upstream of the $i$th panel. For the remaining 3/4 of the $i$th panel, the $i$th horseshoe vortex is also included in the vorticity integral along with the upstream horseshoe vortices.

$$\rho \int \frac{\partial}{\partial t} \left[ \int_{LE}^{x} \gamma(\xi) \, d\xi \right] \, dA_i = \rho \Delta y_i \Delta x_i \left[ \sum_{us} (\Gamma_{us}^{n+1} - \Gamma_{us}^{n}) + \frac{3}{4} (\Gamma_{i}^{n+1} - \Gamma_{i}^{n}) \right] \quad (A.16)$$

A.2 Vortex Lattice Method Aeroelastic Analysis

The vortex lattice method equations govern the strengths of horseshoe vortices in the aerodynamic mesh. If the normal wash on the wing is known, then the circulation of each horseshoe vortex, and consequently the lift on the wing, can be computed. This is the way that a vortex lattice aerodynamic analysis is done. For the aeroelastic analysis, the normal wash is also unknown because it depends on the structural deformations. Therefore, the aerodynamic model must be coupled to a structural model.

The structural model that is employed in this analysis is a linear structural model that describes the system by its natural modes. The mass and stiffness matrices are diagonalized such that the stiffness is equal to the mass times the square of the natural frequency. For the folding wing, this means that the structural model is in terms of generalized coordinates that represent folding wing modes and not wing segment modes. The structural dynamics equations can then be written in the general form
of Eq. \((A.17)\), in which the vector \(q\) contains the generalized coordinates of the system.

\[
M \ddot{q} + Kq + Q = 0 \tag{A.17}
\]

To combine the structural and aerodynamic models, the structural model needs to be cast in state-space form, and also be discretized in time, in order to match the vortex lattice equations. To cast the equations in state-space form, the velocities of the generalized coordinates is added to the vector \(q\) to form the structural dynamics state-space vector \(q_s\).

\[
q_s = \begin{bmatrix} q_1, q_2, \cdots, q_m, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_m \end{bmatrix}^T = \begin{bmatrix} q \quad \dot{q} \end{bmatrix} \tag{A.18}
\]

To discretize the structural dynamics equations in time, the second time derivative is approximated as the difference between the values of the state variables at the two times divided by the time step, and the values of the generalized coordinates are approximated as the arithmetic average of the values at the two times.

\[
M \left( \frac{q_n^{n+1} - q^n}{\Delta t} \right) + K \left( \frac{q_n^{n+1} + q^n}{2} \right) = 0 \tag{A.19}
\]

In adding \(m\) additional state variables, an additional \(m\) equations need to be added to define the new states \(\dot{q}_i\) as the time derivatives of the existing states \(q_i\). This is discretized in time by taking the difference between the values of the generalized coordinates at the two times and dividing by the time step.

\[
\dot{q} = \frac{q_n^{n+1} - q^n}{\Delta t} \tag{A.20}
\]

Equation \((A.19)\) contains the time-discretized structural dynamics equations, and Eq. \((A.20)\) contains the equations that define the first derivatives of the generalized coordinate as state-space variables. Together they form a stand-alone structural
dynamics model and can be written in matrix form.

\[ D_2 q^{n+1} + D_1 q^n + Q = 0 \] (A.21)

The structural dynamics model has one unknown - the generalized force vector \( Q \). At this point, the two models can be coupled together. The generalized force must be obtained from the unsteady aerodynamic model, and the downwash must be obtained from the structural model.

First, the generalized force is equal to the partial derivative of the virtual work with respect to a virtual change in one of the generalized coordinates. The virtual work is equal to the integral of the unsteady lift times the virtual displacement of the wing over the wing area. The integral is discretized as a sum over the panels on the wing, and the wing displacement is expressed as a modal summation using the system natural modes. In this equation, the mode shape \( \Psi_m \) is assumed to be the displacement in the direction of lift. Note that for the folding wing, this is the displacement in the wing-relative \( \tilde{z} \) direction.

\[
\delta W = \sum_i L_i^{n+1} \left( \sum_m \delta q_m \Psi_m \right) 
\] (A.22)

The generalized force for each generalized coordinate is then given by the following equation.

\[ Q_m = \sum_i L_i^{n+1} \Psi_m \] (A.23)

The lift itself depends linearly on the horseshoe circulations on the wing and in the wake, so the generalized force vector can be written as a matrix multiplied by the vector of circulations.

\[ Q = C_2 \Gamma^{n+1} + C_1 \Gamma^n \] (A.24)
The structural dynamics equation then becomes the following.

\[
\begin{bmatrix}
C_2 & D_2
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^{n+1}
+
\begin{bmatrix}
C_1 & D_1
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^n = 0
\]  
(A.25)

The second step is to calculate the normal wash on the wing. The normal wash is given by the following equation.

\[
\frac{Dw^{n+1}}{Dt} = \frac{\partial w}{\partial t} + U_\infty \frac{\partial w}{\partial x}
\]  
(A.26)

\[
\frac{Dw^{n+1}}{Dt} = \sum_m q_m \dot{\Psi}_m + U_\infty \sum_m q_m \frac{\partial \Psi_m}{\partial x}
\]  
(A.27)

The above equation can be expressed in matrix form as shown in the equation below, in which the vector \( w^{n+1} \) is the normal wash for the first \( N_c \) equations, and equal to zero for the remaining equations that govern the horseshoe vortex strengths in regions 2 through 4.

\[
w^{n+1} = E \cdot q_s
\]  
(A.28)

The above equation can then be substituted into the vortex lattice aerodynamic equations.

\[
\begin{bmatrix}
A & -E
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^{n+1}
+
\begin{bmatrix}
B & 0
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^n = 0
\]  
(A.29)

Lastly, Eqs. (A.25) and (A.29) can be combined together as a single matrix equation to complete the vortex lattice aeroelastic model.

\[
\begin{bmatrix}
A & -E \\
C_2 & D_2
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^{n+1}
+
\begin{bmatrix}
B & 0 \\
C_1 & D_1
\end{bmatrix}
\begin{pmatrix}
\Gamma \\
q_s
\end{pmatrix}^n = 0
\]  
(A.30)
Appendix B

Three Degree of Freedom Airfoil Problem

As discussed in the previous sections, the wing natural frequencies can cross each other as the fold angles of the folding wing system change because the mass distribution of the system changes. This caused abrupt changes in predicted flutter behavior that are sometimes not noticeable, but are other times drastic in magnitude. In particular, the Lockheed-type folding wing configuration is sensitive to such circumstances. Typically, a hump mode or the first coalescence flutter mode either appears or disappears, resulting in a sudden change in flutter speed and flutter frequency. However, this change in behavior was observed only in one case in wind tunnel testing - the three-segment folding wing with 30-degree inboard fold angle and negative 75-degree outboard fold angle - but was not observed consistently for all configurations that theory predicts should have this behavior.

For the three-segment folding wing undergoing Lockheed-type motion in particular, the change in flutter behavior was not observed at all in experiment. A more interesting result is that the vortex lattice model predictions for flutter speed and frequency agree very well with the experimental results when the second bending
mode of the system was forcibly removed from the equations of motion. This is done by first writing the matrix equations of motion using generalized coordinates that represent folding wing system modes, and then removing the rows and columns corresponding to the second bending mode from all matrices. Figure B.1 shows theory versus experiment for flutter speed and frequency with and without the second bending mode. This suggests that there may be some physical characteristic of the system that was not modeled, or the flutter behavior was strongly sensitive to that physical characteristic.

**Figure B.1:** Flutter Results for Lockheed-Type 3-Segment Folding Wing with and without 2nd Bending

(a) Flutter Speed with 2nd Bending  
(b) Flutter Frequency with 2nd Bending  
(c) Flutter Speed without 2nd Bending  
(d) Flutter Frequency without 2nd Bending

In order to better understand this phenomenon from a theoretical perspective, a toy problem was created to simulate the structural dynamics and aerodynamic
characteristics of the folding wing system. The aeroelastic analyses of the folding wing system suggests that the important modes are the first two bending modes and the first torsion mode, and that the changes in flutter behavior occurs when the torsion mode crosses the second bending mode. Therefore, the toy problem was set up as a three degree-of-freedom (DOF) airfoil system. This section discusses the derivation of the equations of motion and the aeroelastic analysis results of the three DOF airfoil system.

B.1 System Definition and Equations of Motion

Figure B.2 shows a diagram of the 3 DOF airfoil system.

![Diagram of the 3 DOF Airfoil System](image)

**Figure B.2: Diagram of the 3 DOF Airfoil System**

The airfoil has mass $m_1$ and moment of inertia $i$. The attached second structure has mass $m_2$. The airfoil has pitch, $\alpha$, and plunge, $h_1$, degrees of freedom. The sign convention is the same as that used in the aerodynamic lift and moment equations in Theodorsen’s unsteady thin airfoil theory[33]. The linear springs have stiffnesses $k_1$ and $k_2$, and the torsion spring has stiffness $k_\alpha$. The parameter $b$ is half the chord length. The elastic axis is downstream of the mid-chord by distance $ab$, and the center of mass for the airfoil is downstream of the elastic axis by distance $eb$. 

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The parameters \( a \) and \( e \) are non-dimensional. Lastly, the free-stream direction is perpendicular to the plunge direction, and has air speed \( U \).

The equations of motion were derived using Lagrange’s equations. Equations (B.1)-(B.3) are the three equations of motion for the three DOF airfoil system. The aerodynamic model, from Theodorsen, expresses the lift \( L \) and moment \( M \) in terms of the state variables and the air speed. Specifically, the lift and moment expressions depend linearly on the groups of terms shown in Eqs. (B.1) and (B.3).

\[
0 = -m_1 \ddot{h}_1 - m_1 eb \ddot{\alpha} - (k_1 + k_2)h_1 + k_2 h_2 - L(\dot{h}_1, U\dot{h}_1, \ddot{\alpha}, U\dot{\alpha}, U^2\alpha) \quad \text{(B.1)}
\]

\[
0 = -m_2 \ddot{h}_2 - m_2 eb \ddot{\alpha} - k_2 h_2 + k_2 h_1 \quad \text{(B.2)}
\]

\[
0 = -i \ddot{\alpha} - m_1 eb \dot{h}_1 - m_2 eb \dot{h}_2 - k_\alpha \alpha + M(\ddot{h}_1, U\dot{h}_1, \ddot{\alpha}, U\dot{\alpha}, U^2\alpha) \quad \text{(B.3)}
\]

The next step is to non-dimensionalize the equations of motion. The variables that need to be non-dimensionalized are the plunge displacements \( h_1 \) and \( h_2 \) and the time \( t \). The displacement variables are non-dimensionalized by the half chord length, and time is non-dimensionalized by a time constant \( T \), which has yet to be determined.

\[
\tilde{h}_1 = h_1/b \quad \text{(B.4)}
\]

\[
\tilde{h}_2 = h_2/b \quad \text{(B.5)}
\]

\[
\tau = t/T \quad \text{(B.6)}
\]

The equations of motion can then be expressed in terms of non-dimensional parameters and variables.

\[
0 = -\ddot{\tilde{h}}_1 - \epsilon \ddot{\alpha} - \frac{k_1 + k_2}{m_1} T^2 \tilde{h}_1 + \frac{k_2}{m_1} T^2 \tilde{h}_2 - \frac{T^2}{m_1 b} L(\ddot{h}_1, U\dot{h}_1, \ddot{\alpha}, U\dot{\alpha}, U^2\alpha) \quad \text{(B.7)}
\]

\[
0 = -\ddot{\tilde{h}}_2 - \epsilon \ddot{\alpha} - \frac{k_2}{m_2} T^2 \tilde{h}_2 + \frac{k_2}{m_2} T^2 \tilde{h}_1 \quad \text{(B.8)}
\]

\[
0 = -\ddot{\alpha} - \frac{m_1 eb}{i} \dddot{\tilde{h}}_1 - \frac{m_2 eb}{i} \dddot{\tilde{h}}_2 - \frac{k_\alpha T^2}{i} \alpha + \frac{T^2}{i} M(\ddot{h}_1, U\dot{h}_1, \ddot{\alpha}, U\dot{\alpha}, U^2\alpha) \quad \text{(B.9)}
\]
The expressions for lift and moment are non-dimensionalized as follows. The term $D(k)$ is the generalized Theodorsen function, which depends on the reduced frequency $k$.

\[
\tilde{L} \equiv T^2 \frac{\rho b^2}{m_1} \left( \tilde{L}'' + \frac{UT}{b} \alpha' - a\alpha'' \right) \\
+ \frac{2\pi \rho b^2 D(k)}{m_1} \left[ \frac{UT}{b} \tilde{h}'_1 + \left( \frac{UT}{b} \right)^2 \alpha + \frac{UT}{b} \left( \frac{1}{2} - a \right) \alpha' \right]
\]  (B.10)

\[
\tilde{M} \equiv \frac{T^2}{i} M = \pi \rho b^2 \frac{m_1 b^2}{i} \left[ a\tilde{h}''_1 - \left( \frac{1}{2} - a \right) \frac{UT}{b} \alpha' - \left( \frac{1}{8} + a^2 \right) \alpha'' \right]
\]

\[
+ \frac{2\pi \rho b^2 m_1 b^2}{m_1} \left( \frac{1}{2} + a \right) D(k) \left[ \frac{UT}{b} \tilde{h}'_1 + \left( \frac{UT}{b} \right)^2 \alpha + \frac{UT}{b} \left( \frac{1}{2} - a \right) \alpha' \right]
\]  (B.11)

The equations above suggest defining the following non-dimensional groups. Define the air mass ratio $\mu_a$ as the ratio of mass of air surrounding the airfoil versus the mass of the airfoil. Define the inertia ratio $\mu_i$ as the ratio of inertia if the airfoil mass was concentrated at the leading or trailing edge versus the actual inertia of the airfoil. Define the attached mass ratio as the ratio of attached mass to airfoil mass. Finally, the non-dimensional lift and moment expressions contain groupings of $UT/b$, and does not explicitly depend on either air speed or the time constant. Therefore, define reduced velocity $\tilde{U}$ as the velocity it takes for an air particle to travel the half
Since the time constant $T$ is absorbed into the reduced velocity and does not show up explicitly in the aerodynamic expressions anymore, it makes sense to scale the time by a natural frequency. First define the uncoupled natural frequencies of the plunge and twist motions as the square root of stiffness over mass or inertia, then let the time constant $T$ be equal to the inverse of the uncoupled natural frequency of the airfoil plunge motion.

$$\hat{\omega}_1 \equiv \sqrt{k_1/m_1} \quad (B.16)$$
$$\hat{\omega}_2 \equiv \sqrt{k_2/m_2} \quad (B.17)$$
$$\hat{\omega}_\alpha \equiv \sqrt{k_\alpha/i} \quad (B.18)$$
$$T \equiv 1/\hat{\omega}_1 \quad (B.19)$$

At this point, the equations of motion can be written as the following three
equations in terms of non-dimensional variables and parameters.

\[ 0 = - \tilde{h}_1'' - \alpha'' - \left(1 + \mu_2 \frac{\tilde{\omega}_2^2}{\tilde{\omega}_1^2}\right) \tilde{h}_1 + \mu_2 \frac{\tilde{\omega}_2^2}{\tilde{\omega}_1^2} \tilde{h}_2 - \mu_a \left(\tilde{h}_1'' + \tilde{U} \alpha' - a \alpha''\right) \]

\[ - 2 \mu_a D(k) \left[ \tilde{U} \tilde{h}_1' + \tilde{U}^2 \alpha + \tilde{U} \left(\frac{1}{2} - a\right) \alpha'\right] \quad (B.20) \]

\[ 0 = - \tilde{h}_2'' - \alpha'' - \frac{\tilde{\omega}_2^2}{\tilde{\omega}_1^2} \tilde{h}_2 + \frac{\tilde{\omega}_2^2}{\tilde{\omega}_1^2} \tilde{h}_1 \quad (B.21) \]

\[ 0 = - \alpha'' - \mu_i e \tilde{h}_1'' - \mu_2 \mu_i e \tilde{h}_2'' - \frac{\tilde{\omega}_2^2}{\tilde{\omega}_1^2} \alpha + \mu_a \mu_i \left[a \tilde{h}_1'' - \left(\frac{1}{2} - a\right) \tilde{U} \alpha' - \left(\frac{1}{8} + a^2\right) \alpha''\right] \]

\[ + 2 \mu_a \mu_i (\frac{1}{2} + a) D(k) \left[ \tilde{U} \tilde{h}_1' + \tilde{U}^2 \alpha + \tilde{U} \left(\frac{1}{2} - a\right) \alpha'\right] \quad (B.22) \]

Next, define non-dimensional parameters for the ratios of uncoupled natural frequencies that show up in the above equations.

\[ \tilde{\omega}_2 \equiv \frac{\tilde{\omega}_2}{\tilde{\omega}_1} \quad (B.23) \]

\[ \tilde{\omega}_\alpha \equiv \frac{\tilde{\omega}_\alpha}{\tilde{\omega}_1} \quad (B.24) \]

The last term that needs to be expressed in terms of the non-dimensional parameters in the present model is the reduced frequency that is used in the Theodorsen function. The Theodorsen function \( C(k) \) depends on the reduced frequency \( k \), which is equal to \( \lambda b/U \) where \( \lambda \) is the system dimensional eigenvalue in radians per second. The eigenvalue problem solution first assumes an exponential time dependence with eigenvalue \( \lambda \). In the non-dimensional system, there is a corresponding non-dimensional eigenvalue \( \tilde{\lambda} \), defined using the following equation.

\[ e^{\lambda t} = e^{\lambda T \tau} \equiv e^{\tilde{\lambda} \tau} \quad (B.25) \]

Then the reduced frequency \( k \) can be expressed in terms of the non-dimensional parameters.

\[ \frac{\lambda b}{U} = \frac{\lambda T}{T U} = \tilde{\lambda} \frac{b}{U T} = \frac{\tilde{\lambda}}{\tilde{U}} = k \quad (B.26) \]
Finally, the equations of motion are fully non-dimensionalized as follows.

\[ 0 = -\ddot{h}_1 - e\alpha'' - (1 + \mu_2\ddot{\omega}_2) \dot{h}_1 + \mu_2\ddot{\omega}_2\dot{h}_2 - \mu_a (\ddot{h}_1 + \ddot{U}\alpha' - a\alpha'') \]

\[ \text{B.27} \]

\[ 0 = -\ddot{h}_2 - e\alpha'' - \ddot{\omega}_2\dot{h}_2 + \ddot{\omega}_2\dot{h}_1 \]

\[ \text{B.28} \]

\[ 0 = -\dddot{\alpha} - \mu_1\dddot{h}_1 - \mu_2\mu_1\dddot{h}_2 - \dddot{\omega}_\alpha\alpha + \mu_\lambda\mu_i \left[ a\dddot{h}_1 - \left( \frac{1}{2} - a \right)\dddot{U}\alpha' - \left( \frac{1}{8} + a^2 \right)\dddot{\alpha} \right] \]

\[ + 2\mu_\lambda\mu_i (\frac{1}{2} + a)D(\dddot{\lambda}/\dddot{U}) \left[ \dddot{U}\dddot{h}_1 + \dddot{U}^2 \alpha + \dddot{U} \left( \frac{1}{2} - a \right)\alpha' \right] \]

\[ \text{B.29} \]

The equations of motion can be expressed in matrix form. The generic aeroelastic matrix equation of motion is given by the following equation.

\[ \dddot{\lambda}^2 \mathbf{M} + \dddot{\lambda} \dddot{\mathbf{K}} + \dddot{\lambda} \dddot{\mathbf{A}}_{dd} + \dddot{\lambda} \dddot{U} \mathbf{A}_d + \dddot{\lambda} \dddot{U} D(k) \mathbf{A}_{ddD} + \dddot{U}^2 D(k) \mathbf{A}_D = 0 \]

\[ \text{B.30} \]

The equation contains six matrices: the mass matrix, stiffness matrix, and four aerodynamic matrices. The matrices are given in the following equations.

\[ \mathbf{M} = \begin{bmatrix} -1 & 0 & -e \\ 0 & -1 & -e \\ -\mu_1 e & -\mu_2 \mu_1 e & -1 \end{bmatrix} \]

\[ \text{B.31} \]

\[ \mathbf{K} = \begin{bmatrix} -(1 + \mu_2\dddot{\omega}_2) & \mu_2\dddot{\omega}_2 & 0 \\ \dddot{\omega}_2 & -\dddot{\omega}_2 & 0 \\ 0 & 0 & -\dddot{\omega}_\alpha \end{bmatrix} \]

\[ \text{B.32} \]

\[ \mathbf{A}_{dd} = \mu_\lambda \begin{bmatrix} -1 & 0 & -a \\ 0 & 0 & 0 \\ a\mu_i & 0 & -\left( \frac{1}{4} + a^2 \right)\mu_i \end{bmatrix} \]

\[ \text{B.33} \]

\[ \mathbf{A}_d = \mu_\lambda \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -\left( \frac{1}{2} - a \right)\mu_i \end{bmatrix} \]

\[ \text{B.34} \]

\[ \mathbf{A}_{ddD} = 2\mu_\lambda \begin{bmatrix} -1 & 0 & -\left( \frac{1}{2} - a \right) \\ 0 & 0 & 0 \\ \left( \frac{1}{2} + a \right)\mu_i & 0 & -\left( \frac{1}{4} - a^2 \right)\mu_i \end{bmatrix} \]

\[ \text{B.35} \]
Aeroelastic analyses were conducted on the 3 DOF airfoil system without any structural coupling, that is, the center of mass of the wing is at the midchord and \( e = 0 \) in the aeroelastic equations of motion. The elastic axis is also assumed to be at the midchord, so \( a = 0 \) as well. Table B.1 summarizes the parameters used in the aeroelastic analyses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eccentricity</td>
<td>( e )</td>
<td>0</td>
</tr>
<tr>
<td>Elastic axis offset</td>
<td>( a )</td>
<td>0</td>
</tr>
<tr>
<td>Uncoupled plunge freq ratio</td>
<td>( \tilde{\omega}_2 )</td>
<td>2.5</td>
</tr>
<tr>
<td>Air mass ratio</td>
<td>( \mu_a )</td>
<td>0.1</td>
</tr>
<tr>
<td>Inertia ratio</td>
<td>( \mu_i )</td>
<td>1</td>
</tr>
<tr>
<td>Attached mass ratio</td>
<td>( \mu_2 )</td>
<td>1</td>
</tr>
</tbody>
</table>

In the aeroelastic analyses, the torsion frequency was varied from a value right above the first system plunge frequency to a value above the second system plunge frequency. Because the plunge and twist motions are uncoupled, the plunge natural frequencies are constant throughout the analysis. For the configuration in Table B.1 in particular, the system plunge frequencies are 0.677 and 3.52.

Two sets of analyses were conducted. In the first set, the second plunge mode was removed from the system equations of motion, and the system becomes a typical section airfoil model. In the second set, all three degrees of freedom were included in the analyses.
B.2.1 2 DOF Airfoil System

Figure B.3 shows the flutter speed and frequency of the typical section airfoil system as the torsion frequency varies. Both values increase as the torsion frequency increases. This trend is expected of typical coalescence flutter as the separation between the bending and torsion frequencies increase.

![Figure B.3: Theoretical Flutter Results for the Two DOF Airfoil System](image)

Figure B.4 shows the real and imaginary parts of the aeroelastic eigenvalues versus air speed when the torsion natural frequency is 4. The behavior of the aeroelastic eigenvalues is representative of the behavior at other values of torsion natural frequency, and is indicative of coalescence flutter.

B.2.2 Three DOF Airfoil System

The general case considers three degrees of freedom: two plunge DOFs and one twist DOF. The torsion frequency at first is between the two bending frequencies, but increases past the second bending frequency. The interesting cases are 1) the typical case when torsion frequency is between the bending frequencies, 2) the case when the
torsion frequency is just past the second bending frequency but the flutter behavior has not changed, 3) the case when the flutter behavior just starts to change, and 4) the case when the flutter behavior is typical of when the torsion frequency is above second bending.

The overall flutter results of the torsion frequency sweep are shown in Fig. B.5.

Four cases are chosen to be examined in more detail. The four cases are marked in
Fig. B.5. The first case is when the torsion frequency is 1.5. The torsion frequency is between the two bending frequencies, and the behavior is typical coalescence flutter, as shown in Fig. B.6.

The second case is when the torsion frequency is 3.7. At this point, the torsion natural frequency is just greater than the second bending frequency. The flutter behavior is still that of coalescence flutter, with the second bending mode becoming torsion-like very quickly. However, the mode that does not participate in flutter, originally the torsion mode that gained bending characteristics, is now very close to the imaginary axis. The aeroelastic eigenvalues are shown in Fig. B.7.

The third case is when the torsion frequency is 3.8. At this point, the flutter behavior changes and the third mode that was not unstable before now goes unstable at a higher frequency and a lower air speed when compared to the coalescence flutter point. This result is essentially the same as the previous case where the torsion frequency is 3.7, but now the third mode is slightly unstable. The aeroelastic eigenvalues are shown in Fig. B.8.

The last case is when the torsion frequency is 7. Now the system looks like coalescence flutter again, except that it is more accurately described as a stronger version
Figure B.7: Aeroelastic Eigenvalues of the Three DOF Airfoil System at $\tilde{\omega}_\alpha = 3.7$

Figure B.8: Aeroelastic Eigenvalues of the Three DOF Airfoil System at $\tilde{\omega}_\alpha = 3.8$

of the $\tilde{\omega}_\alpha = 3.8$ case. The higher frequency flutter mode decreases in frequency first, and then goes unstable. The lower mode is still going into flutter, but the eigenvalue is changing very rapidly and the in-house algorithm used to automatically track the eigenvalues was not able to find the next eigenvalue with a nearby search. The aeroelastic eigenvalues are shown in Fig. B.9.
B.3 Conclusions of the Three DOF Airfoil Analysis

The three DOF airfoil problem was studied because the problem retains only the simplest and most essential features that still produces flutter behavior that was observed in the folding wing system. In particular, the aerodynamic model for the airfoil was used with higher confidence than the aerodynamic model for the folding wing system. The structural dynamics was also simplified to essentially two lumped masses with moment of inertia for only one of the masses.

The analysis shows that even with the simplified system, abrupt changes in flutter behavior may be observed as the structural dynamics parameters are varied. Without the second plunge mode, the system was essentially the typical airfoil model and the aeroelastic analysis resulted in coalescence flutter. With the second plunge mode, interactions between the torsion and second plunge mode caused an additional unstable mode to appear near the point at which the natural frequencies crossed.

This analysis does not fully explain why sudden changes in theoretical flutter behavior occur at natural frequency crossings, but they could not be observed in experiment. It is possible that while the sudden change in behavior occurs, the magnitudes of the changes in flutter speed and flutter frequency were not very large, and
could be smoothed out in experiment by imperfect flow condition and inaccuracies in the fold angles (and therefore frequency separation). Nevertheless, this model problem can serve as the basis of a more detailed exploration of these types of sudden changes in flutter behavior. A more detailed consideration of the three DOF problem, including parameter variation studies in static imbalance and elastic axis offset, as well as experimental studies, is a worthy subject for future work.
A simple flight dynamics model is derived in this section. The results of this model is used to compare to the more robust aircraft aeroelastic model. The model assumes quasi-steady aerodynamics, as explained in more detail throughout this section.

C.1 Simple Phugoid Model

A simple model for predicting the phugoid mode of an aircraft is derived using Newton’s second law in the flight path tangential and normal directions. In the tangential direction, the thrust, drag, and a component of weight must balance the aircraft linear acceleration. In the normal direction, the lift and weight must balance the aircraft centripetal acceleration. The aircraft states are flight path velocity $V$ and flight path angle $\Theta$. The equations of motion are shown below, and Fig. C.1 shows how the flight paths and forces are defined.

\[
\dot{V} = \frac{1}{m} [T - D - W \sin \Theta] \quad (C.1)
\]

\[
\dot{\Theta} = \frac{1}{mV} [L - W \cos \Theta] \quad (C.2)
\]
To solve for the flight dynamics, the next step is to linearize the system about nominal values of $V$ and $\Theta$. For longitudinal dynamics, the analysis assumes that $V$ perturbs about the cruise velocity $V_0$, and $\Theta$ perturbs about 0.

$$V = V_0 + v$$  \hspace{1cm} (C.3)

$$\Theta = 0 + \theta$$  \hspace{1cm} (C.4)

Linearizing the equations of motion gives a matrix eigenvalue problem.

$$\begin{pmatrix} \dot{v} \\
\dot{\theta} \end{pmatrix} = g \begin{bmatrix} -2 \frac{C_D}{C_L} \frac{1}{V_0} & -1 \\
\frac{2}{V_0^2} & 0 \end{bmatrix} \begin{pmatrix} v \\
\theta \end{pmatrix}$$  \hspace{1cm} (C.5)

The characteristic equation for this eigenvalue problem is a quadratic equation in terms of the eigenvalue $\lambda$.

$$\lambda^2 + 2 \frac{C_D g}{C_L V_0} \lambda + 2 \frac{g^2}{V_0} = 0$$  \hspace{1cm} (C.6)

The eigenvalue are solved using the quadratic formula.

$$\lambda = \frac{g}{V_0} \left[ - \frac{1}{L/D} \pm \sqrt{\left( \frac{1}{L/D} \right)^2 - 2} \right]$$  \hspace{1cm} (C.7)

The equations assume that the aircraft pitch is aligned with the direction of travel. Since the lift-to-drag ratio of aircraft is typically in the range of 7 to 9,
the frequency of the phugoid motion is largely independent of the lift-to-drag ratio, and therefore independent of any aircraft-specific parameter except the cruise speed. Typical phugoid motion has a frequency of about 0.25 rad/s, and the frequency in general is proportional to the inverse of the flight speed for relatively high values of lift-to-drag ratio.

This simple result can be used to check the validity of the aircraft aeroelastic model.

C.2 Static Stability

Static stability is also a simple analysis that can be used to validate the more complicated aircraft aeroelastic model. In particular, the zero-frequency root can be compared directly to the static stability of the aircraft. The analysis requires a moment balance about the aircraft center of gravity, as shown in Eq. (C.8). The aircraft geometry - centers of gravity of aircraft components and locations of wing and tail quarter-chord lines - is shown in Fig. C.2.

\[
\Delta M = \frac{1}{2} \rho U^2 \left[ x_{cg} \left( S_w \frac{\partial C_{Lw}}{\partial \alpha} + S_t \frac{\partial C_{Lt}}{\partial \alpha} \right) - l_t S_t \frac{\partial C_{Lt}}{\partial \alpha} \right] \Delta \Theta \quad (C.8)
\]

![Figure C.2: Diagram of Aircraft Geometry for Static Stability](image)
For stability, the coefficient of $\Delta \Theta$ should be negative, which physically corresponds to the aircraft providing a restoring moment when the pitch angle changes. The result is an inequality for the aircraft center of gravity position such that the aircraft is statically stable.

$$x_{cg} \leq l_t S_t \left[ \frac{\partial C_{Lt}}{S} \frac{\partial C_{Lw}}{\partial \alpha} + \frac{S_t \partial C_{Lt}}{S} \right]$$ (C.9)

The aircraft aeroelastic model does not have coordinates for the aircraft center of gravity. Instead, the aircraft center of gravity is implicit in the structural dynamics model through constraints between the components of the aircraft: wing, tail, and fuselage. To compare the aircraft aeroelastic model with the simple flight dynamics model results, there needs to be a way to convert from one system to another. Specifically, there needs to be an expression that computes the locations of the wing and tail relative to the fuselage given the desired aircraft center of gravity location. This is done by considering a moment balance of the weights of each aircraft component about the aircraft center of gravity. The following result assumes that the wing and tail centers of gravity are located at the half chord.

$$\begin{pmatrix} M_f + M_t - M_t \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} x_{wf} \\ x_{tf} \end{pmatrix} = \begin{pmatrix} (M_w + M_f + M_t) (x_{cg} - \bar{x}_w) \\ l_t - \bar{x}_w + \bar{x}_t \end{pmatrix}$$ (C.10)

C.3 Phugoid and Short-Period Model

The traditional rigid body longitudinal flight dynamics model uses a rotating coordinate system that is attached to the aircraft and centered at the aircraft center of gravity. The aircraft has velocity components $U$, $V = 0$, and $W$ in the aircraft relative reference frame. The velocity vector in the inertial frame is the vector $(U, 0, W)$ rotated counter-clockwise by the aircraft pitch angle. The aircraft also have pitch rate $Q$, and pitch angle $\Theta$. 226
To model both the phugoid mode and the short-period mode, it is necessary to have 4 coordinates. There are two coordinates for each mode since each mode has two eigenvalues that form a complex conjugate pair. The 4 coordinates are the velocity components $U$ and $W$, the pitch rate $Q$, and the pitch angle $\Theta$. The variables are defined in Fig. C.3.

\[ \begin{bmatrix}
  X_u & X_w & 0 & -mg \\
  Z_u & Z_w & Z_q + mU_0 & 0 \\
  M_u & M_w & M_q & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  w \\
  q \\
  \dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
  m & 0 & 0 & 0 \\
  0 & m & 0 & 0 \\
  0 & -M_{w\dot{w}} & I_{yy} & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \dot{u} \\
  \dot{w} \\
  \dot{q} \\
  \dot{\theta}
\end{bmatrix}
\] (C.11)

It is assumed that the flight speed $(U, 0, W)$ perturbs about a steady state flight speed $(U_0, 0, 0)$ with perturbation $(u, 0, w)$. The task in any flight dynamics calculation is to calculate the stability derivatives, which are terms in the square matrix on the right hand side of the above equation. The forces in the $x$ and $z$ directions, as well as the moment about the pitch axis, are considered.

The forces in the $x$ direction come from the drag, as well as components of lift and weight when the aircraft angle of attack is nonzero.

\[ X = \frac{1}{2} \rho \left( (U_0 + u)^2 + w^2 \right) S \left( C_D \cos \alpha - C_L \sin \alpha \right) - mg \sin(\theta + \alpha) \] (C.12)

**Figure C.3:** Diagram of Aircraft for the 4-State Model
The angle of attack $\alpha$ is approximately equal to the $z$ velocity divided by the $x$ velocity for small angles, noting again that the positive $z$ direction is defined downward. Therefore, the angle of attack depends on the perturbation in $z$ velocity. Consequently, the chain rule must be used when taking partial derivatives of the forces or moment with respect to $w$.

$$\alpha \approx \frac{w}{U_0 + u} \quad \text{(C.13)}$$

$$\frac{\partial \alpha}{\partial w} = \frac{1}{U_0} \quad \text{(C.14)}$$

The stability derivatives for the forces in the $x$ direction with respect to the state variables are given by the following equations.

$$\frac{\partial X}{\partial u} = -\rho U_0 S C_D \quad \text{(C.15)}$$

$$\frac{\partial X}{\partial w} = -\frac{1}{2} \rho U_0^2 S \left( \frac{\partial C_D}{\partial \alpha} - C_L \right) \frac{1}{U_0} - \frac{mg}{U_0} = \frac{C_L \cos \alpha + C_D \sin \alpha}{U_0}$$ 

The forces in the $z$ direction come from the lift and weight, as well as the vertical component of drag when the aircraft angle of attack is nonzero.

$$Z = -\frac{1}{2} \rho \left[ (U_0 + u)^2 + w^2 \right] S \left( C_L \cos \alpha + C_D \sin \alpha \right) + mg \cos(\theta + \alpha) \quad \text{(C.17)}$$

The stability derivatives for the forces in the $z$ direction with respect to the state variables are given by the following equations.

$$\frac{\partial Z}{\partial u} = -\rho U_0 S C_L \quad \text{(C.18)}$$

$$\frac{\partial Z}{\partial w} = -\frac{1}{2} \rho U_0^2 S \left( \frac{\partial C_L}{\partial \alpha} + C_D \right) \frac{1}{U_0} \quad \text{(C.19)}$$

The partial derivatives of the lift and drag coefficients with respect to $\alpha$ are:

$$\frac{\partial C_L}{\partial \alpha} = \frac{S_w}{S} \frac{\partial C_{Lw}}{\partial \alpha} + \frac{S_t}{S} \frac{\partial C_{Lt}}{\partial \alpha} \quad \text{(C.20)}$$

$$\frac{\partial C_D}{\partial \alpha} = \frac{S_w}{S} 2C_{Lw} \frac{\partial C_{Lw}}{\partial \alpha} + \frac{S_t}{S} 2C_{Lt} \frac{\partial C_{Lt}}{\partial \alpha} \quad \text{(C.21)}$$
To calculate the stability derivatives for the pitching moment, the moment coefficient must first be expressed in terms of the lift coefficients. For a linear flight dynamics analysis, it is not necessary to consider the moment generated by the airfoil itself, since it is a constant value. Therefore, consider only the moment generated about the aircraft center of gravity due to lift forces at the wing and tail.

\[
M = \frac{1}{2} \rho U_0^2 \left[ S_w x_{cg} C_{Lw} - S_t (l_t - x_{cg}) C_{Lt} \right] \quad (C.22)
\]

\[
C_M = \frac{S_w x_{cg}}{S_C} C_{Lw} - \frac{S_t (l_t - x_{cg})}{S_C} C_{Lt} \quad (C.23)
\]

The stability derivatives for the pitching moment are:

\[
\frac{\partial M}{\partial u} = 0 \quad (C.24)
\]

\[
\frac{\partial M}{\partial w} = \frac{1}{2} \rho U_0^2 \left[ S_w x_{cg} \frac{\partial C_{Lw}}{\partial \alpha} - S_t (l_t - x_{cg}) \frac{\partial C_{Lt}}{\partial \alpha} \right] \quad (C.25)
\]

Lastly, the \( z \) component of the forces and the pitching moment also depend on the pitch rate through the angle of attack. In particular, a nonzero pitch rate is equivalent to a change in the velocity in the \( z \) component. The following equations show the dependence of the angle of attack on the pitch rate, without the effect of \( w \). Note that pitching affects the wing and tail angle of attack in different amounts because the wing and tail are at different distances from the aircraft center of gravity.

\[
\alpha_w = \frac{q c_w}{U_0} \left( \frac{1}{2} - \frac{x_{cg}}{c_w} \right) \quad (C.26)
\]

\[
\alpha_t = \frac{q c_w}{U_0} \left( \frac{l_t}{c_w} - \frac{x_{cg}}{c_w} + \frac{1}{2} \frac{c_t}{c_w} \right) \quad (C.27)
\]

Then the stability derivatives of \( Z \) and \( M \) with respect to \( q \) are:

\[
\frac{\partial Z}{\partial q} = -\frac{1}{2} \rho U_0^2 S \left( \frac{\partial C_L}{\partial \alpha} + C_D \right) \frac{\partial \alpha}{\partial q} \quad (C.28)
\]

\[
\frac{\partial M}{\partial q} = \frac{1}{2} \rho U_0^2 \left[ S_w x_{cg} \frac{\partial C_{Lw}}{\partial \alpha} \frac{\partial \alpha_w}{\partial q} - S_t (l_t - x_{cg}) \frac{\partial C_{Lt}}{\partial \alpha} \frac{\partial \alpha_t}{\partial q} \right] \quad (C.29)
\]
This completes the calculations for the stability derivatives. The stability derivatives are then substituted into the state-space matrix equation for the aircraft flight dynamics, and the eigenvalue problem can be used numerically using MATLAB’s eigenvalue solver \texttt{eig}.

Lastly, the equations may be non-dimensionalized using the following scaling definitions for time, velocity perturbations, aircraft moment of inertia (described by non-dimensional radius of gyration), and aircraft mass (described by mass ratio between aircraft and surrounding air).

\[
\tau \equiv \frac{tU_0}{c} \quad (C.30)
\]

\[
\tilde{u} \equiv \frac{u}{U_0} \quad (C.31)
\]

\[
\tilde{w} \equiv \frac{w}{U_0} \quad (C.32)
\]

\[
\tilde{q} \equiv \frac{qc}{2U_0} \quad (C.33)
\]

\[
\tilde{r}_{yy}^2 \equiv \frac{I_{yy}}{mc^2} \quad (C.34)
\]

\[
\mu \equiv \frac{m}{\rho S_w c} \quad (C.35)
\]

The resulting non-dimensional equations of motion are given by the following matrix equation.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2\tilde{r}_{yy}^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{\tilde{u}}' \\
\dot{\tilde{w}}' \\
\dot{\tilde{q}}' \\
\dot{\theta}'
\end{pmatrix}
=

\begin{pmatrix}
-C_D & -\frac{1}{2}(C_{DA} - C_L) & 0 & -\frac{\mu g c}{U_0^2} \\
-C_L & -\frac{1}{2}(C_{LA} + C_D) & -\frac{1}{2}\frac{\partial C_L}{\partial \tilde{q}} + 2\mu & 0 \\
0 & \frac{1}{2}C_{Ma} & \frac{1}{2}\frac{\partial C_M}{\partial \tilde{q}} & 0 \\
0 & 0 & \frac{2}{\mu} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{u}' \\
\tilde{w}' \\
\tilde{q}' \\
\theta'
\end{pmatrix}
\quad (C.36)
\]
C.4 Typical Results

Before comparing the aircraft aeroelastic model to this rigid body longitudinal flight dynamics model, the rigid body model is applied to some test cases for benchmarking. First, the model is applied to a Cessna 172 aircraft using the following parameters. Table C.1 summarizes the parameters that were used in the flight dynamics test case.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wing Span</td>
<td>36 ft</td>
<td>Tail Span</td>
<td>10 ft</td>
</tr>
<tr>
<td>Wing Chord</td>
<td>5 ft</td>
<td>Tail Chord</td>
<td>3 ft</td>
</tr>
<tr>
<td>Wing Span Efficiency</td>
<td>-</td>
<td>Tail Span Efficiency</td>
<td>-</td>
</tr>
<tr>
<td>Wing Drag Coefficient</td>
<td>-</td>
<td>Tail Drag Coefficient</td>
<td>-</td>
</tr>
<tr>
<td>Aircraft Mass</td>
<td>1020 kg</td>
<td>Aircraft Inertia</td>
<td>838 kg-m²</td>
</tr>
<tr>
<td>Trim Speed</td>
<td>54 m/s</td>
<td>Air Density</td>
<td>1.2 kg/s</td>
</tr>
<tr>
<td>C.G. Distance</td>
<td>0</td>
<td>Tail Distance</td>
<td>15 ft</td>
</tr>
</tbody>
</table>

The simple phugoid model predicts that the eigenvalue of the phugoid mode is equal to \(-0.0216 + 0.2540i\) rad/s. This corresponds to a frequency of approximately 25 seconds, which is typical of aircraft phugoid modes. The light damping is also a reasonable result. Figure C.4 shows the predicted flight path of the aircraft, with data markers at equal time intervals and a dashed line showing the unperturbed flight path.

Next, the 4-state model is applied to the test case to calculate both phugoid and short period modes. The phugoid mode eigenvalue is equal to \(-0.0177+0.2315i\) rad/s, which is very close to the eigenvalue predicted by the simple phugoid model. The short period mode eigenvalue is equal to \(-6.2410+9.2282i\) rad/s. The short period mode result is also reasonable, with high frequency and high damping. Figure C.5
Figure C.4: Aircraft Flight Path Predicted by Simple Phugoid Model

(a) Absolute Position
(b) Relative Position

shows the absolute flight paths of the aircraft in phugoid mode and short period mode with arrows indicating the aircraft pitch, which was not independently modeled in the simple phugoid model.

Figure C.5: Aircraft Flight Path Predicted by 4-State Longitudinal Dynamics Model

(a) Phugoid Mode
(b) Short Period Mode
Appendix D

Scaling Analysis of Folding Wing Motion

The structural dynamics and aeroelastic analyses so far have assumed that the fold angle is fixed. This is a common assumption in the literature as well. The present work focuses on an experimental study of the folding wing transients during the folding motion. The new designs for moving hinges allow each hinge to be controlled from outside the wind tunnel during flutter testing.

Before starting the tests, it is necessary to have an order of magnitude estimate of how slow the folding motion should be if the fold angles are to be considered quasi-steady, and how fast the folding motion should be before the transients of the structural motion and flow field become important. For the structural dynamics, the order of magnitude estimate compares the time scale of the folding motion versus the time scale of the structural vibrations. For the aerodynamics, the order of magnitude estimate compares the induced downwash from the folding motion versus the downwash due to motion of the structure.
D.1 Structural Dynamics

The folding wing system consists of an arbitrary number of wing segments that are oriented at arbitrary angles. The structural dynamics equations of motion may be separated into at least two components. The mass matrix contains the kinetic energies of the system, and the stiffness matrix contains the potential energies as well as any constraints in the system. After solving the structural dynamics eigenvalue problem, however, the system equations may then be re-written in terms of the system normal modes, and the modified equations of motion may be expressed as the following matrix equation, in which the mass matrix $M$ and stiffness matrix $K$ are diagonal matrices.

\[ M \ddot{q} + K \cdot q = 0 \]  \hspace{1cm} (D.1)

When considering a particular folding wing system oriented at two different fold angles, the mass matrix in the original equation of motion is the same for the two cases because individual wing segments have not changed. The stiffness matrix contains terms that are different between the two cases because the constraint terms are now different: one set of constraints is used to specify a certain combination of fold angles, and a different set of constraints are needed to specify a different combination of fold angles. Therefore, the resulting natural frequencies are also different.

Now consider the first torsion mode in particular because torsion modes undergo more significant changes in natural frequencies than bending modes. The physical explanation for the drastic changes in torsion natural frequencies as the fold angles change is that the system moment of inertia also changes. Varying the fold angles varies the mass distribution of the folding wing system, and the moment of inertia increases significantly as the system moves away from a flat wing. This phenomenon is discussed in detail in Wang et al.[27] This is different from how the equations of
motion change because the mass matrix remains constant while the stiffness matrix varies with fold angle.

This suggests that the best way to think about changes in natural frequencies as the fold angles change is that the mass distribution of the system changes, or that the modal mass changes. When considering a folding wing system that undergoes folding motion, it is equivalent to a spring-mass system whose mass varies with time. This provides a simple way to begin a scaling analysis.

Consider a single degree of freedom spring mass system with the following equation of motion. The first term accounts for the variation of the system mass with time.

\[-\frac{1}{m} \frac{dm}{dt} \dot{x} - \ddot{x} - \omega^2 x = 0 \quad (D.2)\]

Define time scale \( T \) to be equal to the inverse of a reference natural frequency \( \omega_1 \), and define non-dimensional time \( \tau = t/T \) to be equal to the dimensional time divided by the time scale. The definition of non-dimensional time is then substituted into the equation above to obtain the non-dimensional equation of motion.

\[-\frac{1}{m} \frac{dm}{d\tau} \dot{x}' - x'' - \left( \frac{\omega}{\omega_1} \right)^2 = 0 \quad (D.3)\]

Next assume that the response is exponential in time \( t \) with dimensional eigenvalue \( \lambda \), or equivalently exponential in non-dimensional time \( \tau \) with non-dimensional eigenvalue \( \tilde{\lambda} \). The resulting eigenvalue problem is then given in the following equation.

\[-\frac{1}{m} \frac{dm}{d\tau} \tilde{\lambda} - \tilde{\lambda}^2 - \left( \frac{\omega}{\omega_1} \right)^2 \bar{x} = 0 \quad (D.4)\]

The first term is the time varying mass term and the second term is the quasi-steady mass term. For the system to behave as if the mass is not changing, the first term must be much smaller than the second term. The equation may then be
rearranged to compare time scales instead of frequencies.

\[
\frac{1}{m} \frac{dm}{d\tau} \tilde{\lambda} \ll \tilde{\lambda}^2 \quad (D.5)
\]

\[
\frac{1}{\tilde{\lambda}} \ll \left( \frac{1}{m} \frac{dm}{d\tau} \right)^{-1} \quad (D.6)
\]

The left hand side of the preceding equation is the period of a single system oscillation, in units of reference time \( \tau \). The right hand side is the mass divided by the rate of change in mass, so it is the time it takes for the mass to reach zero or double at the specified rate. But in general it is the time scale of how quickly the mass changes. The physical meaning of the preceding equation is that for the system to be considered quasi-steady, the time scaling for varying the system mass must be much larger than the time scale for one system oscillation.

The terms may be converted to dimensional quantities.

\[
\frac{1}{\tilde{\lambda}} = \frac{1}{\lambda T} \quad (D.7)
\]

\[
\frac{1}{m} \frac{dm}{d\tau} = T \frac{1}{m} \frac{dm}{dt} \quad (D.8)
\]

The quasi-steady limit can then be expressed in terms of dimensional quantities.

\[
\frac{1}{m} \frac{dm}{dt} \ll \lambda \quad (D.9)
\]

For a single degree of freedom system, the above equation delineates the quasi-steady limit, in which changes in mass may be neglected from the system structural dynamics. The final step is to compute the equivalent mass rate of change for the folding wing system. Note that the original physical explanation for why natural frequencies vary with fold angles is that fold angles affect the mass distribution. Then a simplified analogy with the single degree of freedom system may be formed by assuming that the modal stiffness of each folding wing mode is constant, and each
modal mass is inversely related to the square of the corresponding natural frequency.

\[ m_i = \frac{\bar{k}_i}{\omega_i^2} \]  

(D.10)

In the equation above, the \( \bar{\cdot} \) symbol is used to stress that these are modal mass and modal stiffness values. The mass rate of change is then obtained by taking the time derivative of the above equation. The natural frequency \( \omega_i \) varies with time, so the chain rule applies when taking the derivative.

\[ \frac{1}{m} \frac{dm}{dt} = 2 \frac{d\omega_i}{\omega_i} \frac{d\dot{\psi}}{dt} \]  

(D.11)

Lastly, the change in natural frequency with respect to time is equal to the angular velocity of the folding motion times the derivative of the natural frequency with respect to the fold angle, which may be obtained from the theory or experimental data.

\[ \frac{d\omega_i}{dt} = \frac{d\omega_i}{d\psi} \dot{\psi} \]  

(D.12)

\[ \frac{1}{m} \frac{dm}{dt} = 2 \frac{d\omega_i}{\omega_i} \frac{d\dot{\psi}}{d\psi} \]  

(D.13)

The final result is an equation for the angular velocity of the folding motion that determines whether the motion may be considered quasi-steady.

\[ \frac{2}{\omega_i^2} \frac{d\omega_i}{d\psi} \dot{\psi} \ll 1 \]  

(D.14)

When determining the largest angular velocity of folding motion for which the system may still be considered quasi-steady, Equation (D.14) limits that value in two ways.

1. For large changes in natural frequency versus fold angle, the wing must move slower to minimize transient motion.
2. For folding wings with low natural frequencies, the wing must move slower to minimize transient motion.

D.2 Aerodynamics

The folding motion affects the aerodynamics by contributing additional downwash. In particular, the folding motion affects the circulatory aerodynamic forces and moments, but do not contribute to the non-circulatory effects such as apparent mass. Consider an airfoil with Theodorsen aerodynamics, the downwash expression is given by the following equation. The last term on the right hand side is the additional contribution to the downwash from the folding motion. The variable $r$ is the distance from a point on the wing to the hinge axis.

$$w = \dot{h} + U\alpha + b \left( \frac{1}{2} - a \right) \dot{\alpha} + r\dot{\psi} \quad (D.15)$$

For the wing folding motion to be negligible to the aeroelastic behavior of the system, the additional downwash due to the folding motion must be small compared to the existing downwash due to structural deformations. The downwash may be separated into two groups: local deformation term and the convective term. The local deformation is equal to $\dot{h} + b(1/2 - a)\dot{\alpha}$, which is the actual downward motion of the wing at the 3/4 chord. The convective term is equal to $U\alpha$.

First consider the local deformation. The downwash due to folding motion must be much smaller than the downwash due to local deformation at the 3/4 chord. In the equation below, the term $1/2 - a$ is neglected since it is typically between $1/2$ and $1$, and is not important for this order of magnitude analysis.

$$r\dot{\psi} \ll \dot{h} + b\dot{\alpha} \quad (D.16)$$

The bending displacement $h$ may be non-dimensionalized by the half chord $b$. The frequency of the wing motion is equal to the flutter frequency. The parameter
$r$ varies between zero and the wing segment span, so the analysis will use the wing segment span $L$. The final result shows that the angular velocity of the folding motion is proportional to the flutter frequency, as expected.

$$\frac{r\dot{\psi}}{b\lambda \bar{h} + \bar{\alpha}} \ll 1 \quad (D.17)$$

$$\frac{\dot{\psi}}{\omega_f \bar{h} + \bar{\alpha}} \ll 1 \quad (D.18)$$

Next, consider the convective term in the downwash. The final result is given in Eq. (D.19).

$$\frac{\dot{\psi}b \ AR}{U \ \bar{\alpha}} \ll 1 \quad (D.19)$$

The two scaling analysis results in this section may be expressed in terms of non-dimensional groupings. The aspect ratio has been ignored since it will typically be between 1 and 5. In the following two equations, the first equation is the effect of local deformation and the second equation is the convective effect.

$$\frac{\dot{\psi}b}{U} \ll \frac{\omega_f b}{U} (\bar{h} + \bar{\alpha}) \quad (D.20)$$

$$\frac{\dot{\psi}b}{U} \ll \bar{\alpha} \quad (D.21)$$

The right hand side of the first equation contains the reduced flutter frequency, which is usually less than 1, so the first equation is a more limiting equation on the angular velocity of the folding motion. Lastly, the bending displacement is on the order of the wing thickness in the linear regime, and the angular displacement also causes a leading edge displacement on the order of the wing thickness. Therefore, the term $\bar{h} + \bar{\alpha}$ is approximately equal to the wing thickness divided by the half chord. The final equation is given below. The parameter $t$ represents the wing thickness.

$$\frac{\dot{\psi}b}{U} \ll \frac{\omega_f b}{U} \left( \frac{t}{\bar{b}} \right) \quad (D.22)$$
D.3 Scaling Analysis for Three-Segment Folding Wing

The following example calculations use the parameters for the three-segment folding wing with controllable outboard hinge. First consider the structural dynamics. The parameters are substituted into Eq. (D.14), and the first three natural frequencies are analyzed because they are the ones that participate in the flutter mode.

\[
\begin{align*}
\frac{d\omega_1}{d\psi} &= 0.48 \text{ (rad/s)/s} \quad \omega_1 = 18.85 \text{ rad/s} \quad \dot{\psi} \ll 370 \text{ rad/s} \\
\frac{d\omega_2}{d\psi} &= 18.9 \text{ (rad/s)/s} \quad \omega_2 = 87.78 \text{ rad/s} \quad \dot{\psi} \ll 204 \text{ rad/s} \\
\frac{d\omega_3}{d\psi} &= 12.8 \text{ (rad/s)/s} \quad \omega_3 = 123.0 \text{ rad/s} \quad \dot{\psi} \ll 591 \text{ rad/s}
\end{align*}
\]

These values are very high because for this particular configuration, the natural frequencies do not change very much over the range of fold angles that will be tested in the experiment. Therefore, the aerodynamics will limit the rate of folding motion.

For this particular configuration, the flutter frequency is approximately 10 Hz. The thickness to half chord ratio is approximately 0.05. Substituting these values into Eq. (D.22) gives a limit of 172 deg/s.

\[
\dot{\psi} \ll 172 \text{ deg/s}
\]  

(D.23)
Bibliography


Biography

Ivan Wang was born on July, 22, 1987, in Shanghai, China. He first came to the United States in 1998 and lived in Vernon Hills, IL, about 30 minutes outside of Chicago. He graduated from West Oak Middle School in May of 2001, and graduated from Adlai E. Stevenson High School in May 2005. He attended Duke University and was a part of the Department of Mechanical Engineering and Materials Science for all three of his higher degrees. He graduated from Duke with a Bachelors of Science in Engineering in May of 2009. His research over the first two years of graduate school culminated in a thesis titled "Component Modal Analysis of Folding Wings," and he obtained Masters degree in Mechanical Engineering from Duke in May of 2011. His work over the final two years of graduate school resulted in this dissertation, and he obtained his Doctor of Philosophy degree in Mechanical Engineering in May of 2013.

Ivan was supported by various funding sources throughout his graduate school career at Duke University. He was awarded the James B. Duke Fellowship upon entering graduate school at Duke. He also received an honorary mention from the National Science Foundation’s Graduate Research Fellowship Program. Lastly, he won the Department of Defense National Defense Science and Engineering Graduate (NDSEG) Fellowship in 2011.