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Loop di Wilson supersimmetrici nelle teorie di gauge e dualità gauge/gravità

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Prefazione

La struttura di questa tesi può sembrare strana, e in effetti, a mio parere, lo è.

Essa consta di tre capitoli in lingua italiana all'interno dei quali viene introdotto l'argomento di studio di questa tesi di ricerca, si enunciano i risultati ottenuti e le principali considerazioni. La seconda parte, inclusa in questo documento sotto forma di allegato e scritta in lingua inglese, è la tesi vera è propria.

Chiaramente, questa suddivizione linguistica comporta inevitabili ridonzanze e ripetizioni. Mi accingo dunque a illuminarvi sul motivo di tale, apparentemente, infelice scelta stilistica. Nel mondo scientifico, la lingua inglese è oggigiorno la lingua ufficiale. Redarre una tesi in inglese è un grosso vantaggio perchè è direttamente spendibile nel "mercato" scientifico, vale a dire pubblicarla se il lavoro ne è degno o semplicemente presentarla in una università non italiana. Tale discorso assume ancora maggior valore quando, come nel presente caso, la materia in questione è molto tecnica e rivolta ad un ristretto pubblico di specialisti. D'altra parte, le leggi dell'Università di Parma non permettono al candidato di scrivere un documento di tesi ufficiale in una lingua straniera. Il risultato del compromesso al quale si è giunti è rappresentato dalle seguenti pagine.

L'invito a coloro che non avessero particolari problemi con la lingua inglese e che fossero interessati all'argomento trattato, è di iniziare a leggere direttamente a partire dall'allegato.

Capitolo A

La corrispondenza AdS/CFT

Nella teoria quantistica dei campi gli oggetti di studio fondamentali sono le particelle elementari presenti in natura, che vengono rappresentate come punti matematici nello spazio-tempo. La teoria di stringa è una radicale generalizzazione della teoria quantistica dei campi in cui gli oggetti fondamentali sono invece linee o loop 1-dimensionali, in cui le varie particelle elementari osservate corrispondono a differenti modi vibrazionali della stringa. Purtroppo non è ancora possibile isolare e osservare direttamente una stringa in natura, ma ponendosi molto lontano da essa, studiandola quindi a basse energie, è possibile osservare le sue oscillazioni puntiformi e misurare le particelle elementari associate. Il notevole vantaggio di questa descrizione è che mentre ci sono molte particelle elementari, esisterebbe una solo ente, la stringa, in grado di spiegarne le diverse proprietá. Questo argomento è una prima indicazione che la teoria di stringa possa rappresentare una teoria di unificata per le interazioni fondamentali.

Tra gli stati a massa nulla nello spettro della stringa è presente inoltre una particella a spin 2 che interagisce come un gravitone. In realtà, le uniche interazioni massless consistenti a spin 2 sono le interazioni gravitazionali. La teoria di stringa, dunque, include in maniera naturale la relatività generale, e questa caratteristica la candida come una teoria di gravitá, unificata con tutte le forze fondamentali. La teoria di stringa è quindi una teoria gravitá quantistica consistente, libera da divergenze ultraviolette, la quale necessita del gravitone per la sua consistenza globale.

In questo ultimo decennio, grande interesse è stato dedicato alla connessione tra la teoria di stringa e la teoria dei campi, in particolare studiando la dualità tra una teoria di stringa in uno spazio-tempo di Anti-de Sitter e una teoria di campo conforme. Questa equivalenza, nota come corrispondenza AdS/CFT, è stata originariamente congetturata da Maldacena [1], sebbene indicazioni della equivalenza erano già presenti in lavori precendenti [2, 3, 4]. Da un lato della corrispondenza abbiamo la teoria di stringa di tipo IIB 10-dimensionale sullo spazio prodotto $AdS_5 \times S^5$, dove la 5-forma di flusso di tipo IIB attraverso S^5 è un intero N e entrambi gli AdS_5 e S^5 hanno lo stesso raggio L dato da $L^4 = 4\pi g_s N \alpha'^2$, dove g_s è la costante di accoppiamento di stringa. Dall'altro lato della corrispondenza abbiamo invece la teoria di Yang Mills 4-dimensionale, con massima supersimmetria $\mathcal{N} = 4$, gruppo di gauge $SU(N)$ e costante di accoppiamento $g^2_{YM} = g_s$, nella fase conforme. La congettura di Maldacena afferma che queste due teorie, compresi operatori osservabili, stati, funzioni di correlazione e dinamiche complete, sono equivalenti una all'altra. Nella forma più forte della congettura, la corrispondenza vale per tutti i valori di N e per tutti i regimi della costante di accoppiamento $g_s = g^2_{YM}$. Finora si è introdotta l'idea principale alla base della corrispondenza: vale la pena, a questo punto, di discutere l'argomento più in dettaglio.

La teoria $\mathcal{N} = 4$ SYM è classicamente conforme, la sua funzione beta è nulla ed è quindi una teoria di campo quantistica conforme. I gradi di libertà sono un campo di gauge A_μ, sei scalari Φ_i e quattro spinori di Majorana Ψ, i quali tutti trasformano nella rappresentazione aggiunta del gruppo di gauge. La Lagrangiana, nella spazio euclideo, è data da

$$\mathcal{L} = \frac{1}{g^2_{YM}} \mathrm{Tr} \left\{ \frac{1}{2} F^2_{\mu\nu} + (D_\mu \Phi_i)^2 - \sum_{i<j} [\Phi_i, \Phi_j]^2 + i\bar{\Psi}\Gamma^\mu D_\mu \Psi + i\bar{\Psi}\Gamma^i [\Phi_i, \Psi] \right\} .$$

Questa azione può essere ottenuta come una riduzione dimensionale della teoria 10-dimensionale $\mathcal{N} = 1$ SYM a 4 dimensioni. Questa caratteristica si riflette nella notazione usata, nella quale i fermioni sono inseriti in un singolo spinore di Majorana-Weyl 10-dimensionale a 16 componenti dove (Γ^μ, Γ^i) rappresentano le matrici di Dirac 10-dimensionali nella rappresentazione di Majorana-Weyl.

La soluzione duale di supergravità è data dalla geometria *near-horizon* di una *black* D3-brana con N unità di flusso-RR. Questo soluzione corrisponde in teoria di stringa ad uno stato di N D3-brane coincidenti. La metrica rilevante può essere scritta nella forma

$$ds^2 = L^2 \, \frac{dx^\mu dx^\mu + dy^i dy^i}{y^2}, \qquad \mu = 1, ..., 4, \; i = 1, ..., 6 \; . \tag{A.0.1}$$

Il vettore di modulo unitario y^i/y^2 parametrizza S^5 mentre x^μ, y sono le coordinate di AdS_5. Dalla forma esplicita dell'equazione (A.0.1) è evidente che AdS_5 e S^5 hanno lo stesso raggio di curvatura L. Il bordo dello spazio è situato a $y = 0$, mentre l'orizzonte di AdS_5 è locato a $y = \infty$. La metrica scritta esplicitamente nella forma di prodotto dei due spazi è

$$ds^2 = L^2 \frac{dx^2_\mu + dy^2}{y^2} + L^2 d\Omega^2_{S^5} \; . \tag{A.0.2}$$

Nella corrispondenza AdS/CFT, il raggio L è legato alla costante di accoppiamento di Yang Mills dalla relazione

$$L = \sqrt{\alpha'} \left(g_{YM}^2 N\right)^{1/4} \ . \tag{A.0.3}$$

L'elemento di linea (A.0.1) è invariante sotto le trasformazioni di coordinate che formano il gruppo di isometria di AdS_5, $SO(2,4)$. Il rimanente gruppo di isometria di (A.0.1) è il gruppo di simmetria di S^5, vale a dire $SO(6) \sim SU(4)$. Unendo queste simmetrie *geometriche* con la supersimmetria, si ottiene il super-gruppo di Lie $SU(2,2|4)$.

Dal punto di vista della teoria di gauge, le simmetrie bosoniche sono date dal gruppo $SO(2,4)$ che rappresenta la simmetria conforme e dal gruppo $SU(4)$ che rappresenta la R-simmetria della teoria SYM. Consistemente le trasformazioni del gruppo $SO(2,4)$ che lasciano invariante la metrica di AdS_5 (A.0.1) si riducono a trasformazioni conformi sul bordo, dove le osservabili della teoria di campo sono definite.

La teoria di stringa sulla metrica di background (A.0.1) è un σ-model con costante di accoppiamento data dall'inverso della tensione di stringa effettiva adimensionale

$$T = L^2/2\pi\alpha' = \sqrt{g_{YM}^2 N}/2\pi \ . \tag{A.0.4}$$

Inoltre, la constante di accoppiamento di stringa g_s e la corrispondente di Yang Mills g_{YM} sono legate dalla relazione

$$g_s = 4\pi g_{\rm YM}^2 \ . \tag{A.0.5}$$

Questa relazione può essere capita dal fatto che l'azione della teoria di gauge, a fronte della quale la costante di accoppiamento di gauge appare come un fattore del tipo $1/g_{YM}^2$, è ottenuta perturbativamente in teoria di stringa all' ordine $1/g_s$.

Con queste identificazioni, la teoria di stringa e la teoria SYM sono congetturate essere equivalenti. Purtroppo è difficile riuscire a trovare conferme esplicite generali essendo entrambi i lati della corrispondenza rappresentati da sistemi fortemente interagenti. Tuttavia, in alcuni limiti la congettura offre possibilitá tutt'altro che banali. Primo fra tutti, il limite di 't Hooft [5] nella teoria super Yang Mills (SYM), in cui la costante di 't Hooft $\lambda \equiv g_{YM}^2 N$ è fissata mentre $N \to \infty$, corrisponde alla teoria di stringa classica su $AdS_5 \times S^5$, vale a dire senza loop di stringa. In altre parole, la teoria di stringa classica su $AdS_5 \times S^5$fornisce una formulazione Lagrangiana classica della dinamica a grande N della teoria $\mathcal{N} = 4$ SYM.

La teoria ha ancora una parametro libero λ. Una forma ancora più debole della corrispondenza consiste nel prendere il limite $\lambda \to \infty$, il quale riduce la teoria classica di stringa alla supergravità classica di tipo IIB su $AdS_5 \times S^5$. In questo modo, la dinamica ad accoppiamento forte nella teoria $\mathcal{N} = 4$ SYM è mappata nella dinamica classica a basse energie in supergravità.

Le implicazioni della congettura sono notevoli, siccome la corrispondenza è tra una teoria della gravità 10-dimensionale e una teoria 4-dimensionale senza gravità. Il fatto che tutti i gradi di libertà 10-dimensionali siano in qualche modo codificati in una teoria in 4 dimensioni che vive nel bordo di AdS_5, suggerisce che la dinamica di gravità di volume è il risultato di una *immagine olografica* generata dalla dinamica della teoria del bordo. Per questo motivo, spesso la corrispondenza è nota come *olografica*.

La più limpida evidenza in supporto alla congettura di Maldacena è costituita dalla simmetria. Le simmetrie globali in entrambi i lati della corrispondenza si combinano nel super-gruppo di Lie $SU(2,2|4)$, come già accennato sopra. In aggiunta, non solo le simmetrie globali coincidono, ma anche le proprietá di quegli oggetti che portano le rappresentazioni del gruppo di simmetria (per esempio lo spettro degli operatori chirali in teoria di campo e i campi in supergravità sono correlati)[6]. Entrambe le teorie inoltre posseggono una dualità $SL(2,\mathbb{Z})$ nota anche come dualità di Montonen-Olive.

Il fatto che la teoria $\mathcal{N} = 4$ SYM è superconforme restringe notevolmente la forma delle funzioni di correlazione e, in diversi casi, le protegge da correzioni radiative, in modo tale che la dipendenza dalla costante di accoppiamento risulti banale. Un certo numero di tali quantità sono state calcolate usando la corrispondenza AdS/CFT e si è trovato accordo con il limite di campi liberi. Questo può essere visto come una simultanea conferma dei teoremi di non rinormalizzazione supersimmetrici e delle predizioni della corrispondenza AdS/CFT. Esempi di questo tipo sono le funzioni a due e tre punti degli operatori chirali primari[7].

Nell'ambito della corrispondenza AdS/CFT, quantità calcolate a costante di accoppiamento debole usando la descrizione delle teoria di gauge sono legate a quantità calcolate a costante di accoppiamento forte usando le tecniche proprie della teoria di stringa. Purtroppo, il range di validità dei due approcci non si sovrappongono e, di conseguenza, è molto difficile confrontare i risultati ottenuti perturbativamente con quelli di teoria di stringa. Per fortuna qualche eccezione esiste. Per esempio, usando il Bethe-ansatz per calcolare le dimensioni anomale di operatori locali è possibile interpolare alcuni risultati da accoppiamento debole a forte. Un altro esempio è il loop di Wilson circolare, il cui valore di aspettazione sul vuoto calcolato dal punto di vista della teoria di gauge sembra essere descritto da un modello di matrici [8, 9]. Questi risultati sono in accordo con i calcoli di teoria di stringa comprendendo sia una serie infinita di correzioni in $1/N$, sia alcune proposte di calcoli di stringa validi per tutti gli ordini in $1/\sqrt{g_{YM}^2 N}$ [10].

La corrispondenza originale è tra la teoria $\mathcal{N} = 4$ SYM nella sua fase conforme e la teoria di stringa sullo spazio $AdS_5 \times S^5$. Tale corrispondenza assume ancora più valore considerando che la congettura può essere estesa a situazioni senza l'invarianza conforme e con meno, o addirittura nessuna, supersimmetria. In tali casi, lo spazio-tempo $AdS_5 \times S^5$è rimpiazzato da altre varietà o soluzioni di orbifold della teoria di stringa di tipo IIB.

Uno degli scopi di questo lavoro è cercare di estendere alcuni risultati noti per la teoria $\mathcal{N}=4$ SYM a teorie con meno supersimmetria, nell'ambito della corrispondenza AdS/CFT. Si considera una teoria di gauge $\mathcal{N}=1$ supersimmetrica ottenuta mediante una deformazione marginale, rappresentata da un parametro reale β, della teoria $\mathcal{N}=4$ SYM. La teoria è descritta dal superpotenziale deformato

$$ih\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)\ , \qquad \text{(A.0.6)}$$

dove Φ_i sono i tre supercampi chirali $\mathcal{N}=1$. La teoria è conforme quantisticamente a patto che una certa condizione sui parametri h,β e la costante di accoppiamento di gauge τ sia soddisfatta. La teoria risultante preserva una supersimmetria superconforme $\mathcal{N}=1$ e ha una simmetria globale $U(1)\times U(1)$

$$\begin{aligned} U(1)_1: &\qquad (\Phi_1,\Phi_2,\Phi_3)\to(\Phi_1, e^{i\partial_1}\Phi_2, e^{-i\partial_1}\Phi_3) \\ U(1)_2: &\qquad (\Phi_1,\Phi_2,\Phi_3)\to(e^{-i\partial_2}\Phi_1, e^{i\partial_2}\Phi_2, \Phi_3). \end{aligned} \qquad \text{(A.0.7)}$$

La rimanente R-simmetria $U(1)$ agisce come

$$U(1)_R: \qquad (\Phi_1,\Phi_2,\Phi_3)\to e^{i\partial}(\Phi_1,\Phi_2,\Phi_3). \qquad \text{(A.0.8)}$$

Mettendo tutto insieme, la teoria $\mathcal{N}=1$ SYM β-deformata è invariante sotto una simmetria $U(1)^3$.

La descrizione duale in supergravità della teoria β-deformata è stata ricavata da Lunin e Maldacena in [11]. Il background di Lunin-Maldacena può essere ottenuto dallo spazio prodotto $AdS_5\times S^5$attraverso una trasformazione di T-dualità, uno shift sulle coordinate e infine un'altra T-dualità. Tutte queste trasformazioni agiscono sulla 5-sfera dello spazio originale, mentre la parte di AdS_5 non viene intaccata dalla deformazione. In tutto il lavoro ci concentreremo sul caso in cui β sia reale, in modo che la teoria così ottenuta sia una deformazione esatta della teoria originale $\mathcal{N}=4$ SYM. La descrizione di supergravità è valida nel limite di curvatura piccola

$$L=(4\pi g_s N)^{1/4}\gg 1 \qquad \text{(A.0.9)}$$

e piccole deformazioni

$$L\beta \ll 1 \qquad \text{(A.0.10)}$$

mentre

$$L^2\beta := \hat{\gamma} \quad \text{fissata.} \qquad \text{(A.0.11)}$$

In generale, supponiamo di conoscere il duale gravitazionale della teoria originale, cioè non deformata, e che questa geometria abbia due isometrie associate alle due simmetrie globali $U(1)$. La geometria, dunque, contiene al suo interno un toro bidimensionale. La descrizione gravitazionale della deformazione (A.0.6) consiste solo nell'effettuare la seguente sostituzione

$$\tau \equiv B + i\sqrt{g} \longrightarrow \tau_\gamma = \frac{\tau}{1+\gamma\tau} \tag{A.0.12}$$

nella soluzione originale, dove $\sqrt{g}$ è il volume del toro bidimensionale e B e' il flusso del campo $B_{\mu\nu}$. È possibile vedere (A.0.12) come una trasformazione che genera soluzioni. In altre parole, si riduce la teoria 10-dimensionale a 8 dimensioni sul toro bidimensionale. La teoria gravitazionale 8-dimensionale è invariante sotto la trasformazione $SL(2,\mathbb{R})$ che agisce sul parametro τ. Si nota che la deformazione (A.0.12) è un particolare elemento del gruppo $SL(2,\mathbb{R})$, il quale ha la proprietà fondamentale di produrre una metrica non singolare se la metrica originale è non singolare. Un'altra caratteristica importante della trasformazione (A.0.6) è che non cambia la topologia della soluzione.

Capitolo B

Loop di Wilson e superfici minimali

Una delle osservabili più interessanti nelle teorie di gauge è l'operatore loop di Wilson, il $\mathcal{P}$-esponenziale del campo di gauge

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left(i \oint A_\mu dx^\mu \right) \tag{B.0.1}$$

dove la traccia è presa nella rappresentazione fondamentale. Il loop di Wilson può essere definito per ogni percorso chiuso nello spazio, fornendo così una grande classe di osservabili gauge invarianti. In effetti, questi operatori, e i prodotti degli stessi, formano una base completa di operatori gauge invarianti nella teoria di puro Yang Mills.

Una delle applicazioni fisiche del loop di Wilson deriva dal fatto che ad un quark infinitamente massivo nella rappresentazione fondamentale del gruppo di gauge, che si muove lungo il percorso identificato dal loop, può essere associato il fattore di fase (B.0.1). In questo modo gli effetti dinamici delle forze di gauge su sorgenti di quark esterne vengono misurate dal loop di Wilson. In particolare, per una coppia quark anti-quark statici, il loop di Wilson è l'esponenziale del potenziale effettivo tra i quark e costituisce un parametro d'ordine per il confinamento.

La corrispondenza AdS/CFT afferma che la teoria di stringa di tipo IIB nello spazio $AdS_5 \times S^5$è duale alla teoria $\mathcal{N} = 4$ super Yang Mills in 4 dimensioni. Questa teoria di gauge, però, non contiene quark nella rappresentazione fondamenttale del gruppo di gauge. Per costruire il loop di Wilson, si ricordi la riduzione dimensionale della teoria $\mathcal{N} = 1$ SYM in 10 dimensioni. Si considerino $N + 1$ D3-brane coincidenti, e si separi una D-brana dalle rimanenti N portandola molto lontana dalle altre. In questo modo abbiamo rotto la simmetria $SU(N + 1) \to SU(N) \times U(1)$. Per grande N, è possibile ignorare i campi sulla brana lontana, eccetto la stringa aperta che si allunga tra essa e le altre N brane. Lo stato fondamentale di tale stringa aperta corrisponde ai bosoni W e i loro superpartner del gruppo di gauge rotto $SU(N)$, le cui traiettorie producono lo stesso effetto di una particella infinitivamente massiva nella rappresentazione fondamentale.

In questa procedura, oltre all'accoppiamento del loop con il campo di gauge, appare un accoppiamento aggiuntivo con gli scalari Φ^i, che si ottengono mediante la riduzione dimensionale dalla teoria 10-dimensionale. Quando lo spazio 4-dimensionale ha segnatura euclidea, il fattore di fase associato al loop è dato dal valore di aspettazione sul vuoto dell'operatore

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left(\oint (i A_\mu dx^\mu + |\dot{x}| \theta^i \Phi^i) ds \right). \tag{B.0.2}$$

Si noti che il *fattore di fase* nella teoria euclidea non è una vera fase, ma contiene una parte reale. Le variabili θ^i sono coordinate angolari che parametrizzano S^5.

Questo operatore gioca un ruolo più importante dell'usuale loop di Wilson (B.0.1) nell'ambito della corrispondenza AdS/CFT. Una ragione in favore di questa affermazione è la supersimmetria. Le trasformazioni di supersimmetria dei campi di gauge e scalari sono date da

$$\delta_\epsilon A_\mu(x) = \bar{\Psi} \Gamma_\mu \epsilon, \qquad \delta_\epsilon \Phi_i(x) = \bar{\Psi}(x) \Gamma_i \epsilon. \tag{B.0.3}$$

Sotto queste trasformazioni di supersimmetria, l'esponente del loop di Wilson varia della quantità

$$\bar{\Psi} \left(i \Gamma_\mu \dot{x}^\mu(s) - \Gamma_i \theta^i |\dot{x}(s)| \right) \epsilon. \tag{B.0.4}$$

La combinazione lineare delle matrici di Dirac $\left(i \Gamma_\mu \dot{x}^\mu(s) - \Gamma_i \theta^i |\dot{x}(s)| \right)$ ha otto autofunzioni relative all'autovalore nulla, come si può facilmente vedere quadrando la matrice. Se queste autofunzioni non dipendono dal parametro del loop s, l'operatore (B.0.2) mantiene quindi metà della supersimmetria. In realtà questo accade solo quando il percorso C è una linea retta. In questo caso $W[C]$ è un operatore BPS che commuta con metà delle supercariche e, proprio per questo, sembra sia protetto da correzioni radiative. Effettivamente, sia all'ordine dominante in teoria delle perturbazioni e sia nel limite a strong coupling, calcolato dalla corrispondenza AdS/CFT, risulta essere indipendente dalla costante di accoppiamento e in particolare

$$\langle W(\text{straight line}) \rangle = 1. \tag{B.0.5}$$

Nel caso invece di un loop di Wilson che in generale è un curva liscia ancora si ha supersimmetria locale, e si hanno proprietà ultraviolette migliori rispetto a quelle del loop di Wilson convenzionale (B.0.1).

Come si è già affermato, la corrispondenza AdS/CFT può essere utilizzata per calcolare il valore di aspettazione di un loop di Wilson nel limite $N \to \infty$, $\lambda \to \infty$. Nel primo limite, il valore di aspettazione del loop di Wilson è dato dall'azione di una stringa aperta le cui estremità sono forzate lungo la curva che definisce il loop al bordo dello spazio

$$\langle W[C]\rangle = \int_{\partial X=C} \mathcal{D}X \, \exp(-\sqrt{\lambda}S[X]), \tag{B.0.6}$$

dove X rappresenta tutte le coordinate della stringa, sia bosoniche che fermioniche, e $S[X]$ è un'appropriata azione di stringa. Nel limite di accoppiamento forte, la tensione di stringa $T = \sqrt{\lambda}/2\pi$ diventa grande e sopprime le fluttuazioni di stringa. L'integrale (B.0.6) è dominato dall'azione bosonica al suo punto sella, che corrisponde ad una superficie minimale in $AdS_5 \times S^5$

$$\langle W\rangle \simeq K \exp(-\sqrt{\lambda}A). \tag{B.0.7}$$

dove K è una costante di normalizzazione ed A l'area della superficie.

L'azione di stringa al punto sella è ottenuta minimizzando l'azione di Nambu-Goto o, equivalentemente, l'azione di Polyakov

$$\text{Area}(C) = \int d^2\sigma \, \frac{1}{Y^2} \sqrt{\det_{ab}(\partial_a X^\mu \partial_b X^\mu + \partial_a Y \partial_b Y)}. \tag{B.0.8}$$

Per determinare il fattore di normalizzazione in (B.0.7) si dovrebbe calcolare la funzione a due punti. In realtà, l'azione corretta da inserire in (B.0.7) non è l'area A della superficie descritta dalla stringa, ma la sua trasformata di Legendre rispetto alle sei coordinate ortogonali alla D3-brane. Questa modifica non cambia le equazioni del moto, e le soluzioni sono ancora superfici minimali. Effettivamente l'area della superficie il cui bordo è il loop C è infinita. Questa parte dell'area divergente deve essere cancellata dal termine di massa del bosone W, che si dimostra essere proprio la trasformata di Legendre dell'area stessa.

Diversi aspetti della soluzione duale in supergravità del loop di Wilson nella teoria $\mathcal{N} = 1$ β-deformata SYM sono già stati studiati [12, 13]. Ancora, però, non si è identificata la forma precisa dell'operatore di teoria di campo duale alle configurazioni di supergravità. In questo lavoro si è mostrato che l'operatore loop di Wilson definito in (B.0.2) è non BPS, cioè non è supersimmetrico, siccome i bosoni di gauge e gli scalari della teoria sono in multipletti di supersimmetria $\mathcal{N} = 1$ diversi, e di conseguenza, le loro rispettive variazioni di supersimmetria non possono cancellarsi a vicenda. Inoltre, neanche la supersimmetria conforme mescola questi multipletti. Si dimostra che anche permettendo un accoppiamento fermionico generico, non è possibile costruire un loop di Wilson supersimmetrico. Appare dunque impossibile construire un operatore loop di Wilson che rispetti alcune delle simmetrie superconformi $\mathcal{N} = 1$ della teoria β-deformata SYM.

Benchè il loop di Wilson (B.0.2) non sia BPS, in questo lavoro si forniscono diverse evidenze che, se il coupling con gli scalari soddisfa la solita condizione di BPS località $\dot{x}^2 = \dot{y}^2$, esso possegga una proprietà fondamentale del loop di Wilson BPS nella teoria $\mathcal{N} = 4$ SYM, cioè che abbia valore di aspettazione sul vuoto finito. Seguendo la nomenclatura in [14], chiamiamo questo tipo di operatori *loop di Wilson quasi BPS*. La caratteristica che sembra emergere dagli esempi si considerano in seguito, è che non solo questi operatori hanno un valore di aspettazioni finito, ma coincide esattamente con il valore che si trova nella teoria non deformata, almeno a strong coupling.

I loop di Wilson per i quali si è calcolato il valore di aspettazione sul vuoto nella teoria $\mathcal{N} = 1$ β-deformata SYM sono i seguenti (per i calcoli si fa riferimento all'Allegato):

- Due longitudini su una due-sfera connesse da un angolo δ arbitrario. L'accoppiamento scalare corrispondente è costituito da due punti sull'equatore di $\tilde{S}^2 \subset \tilde{S}^5$ separati da un angolo pari a $\pi - \delta$, dove con $\tilde{S}^n$ si indica una sfera n-dimensionale β-deformata. Nella teoria $\mathcal{N} = 4$ SYM tale loop identifica un operatore $1/4$ BPS. Per ricavare la soluzione di stringa duale si è notato che tale operatore è legato da una proiezione stereografica ad un loop di Wilson definito da due raggi semi-infiniti sul piano con angolo d'apertura pari a δ.

- Loop toroidali, costruiti considerando S^3 come una fibrazione di Hopf, vale a dire come una fibra S^1 sopra la varietà S^2. Si sono considerati dapprima loop associati a latitudini sulla base di Hopf, e successivamente più generali loop doppiamente periodici su qualunque toro in S^3. In entrambi i casi l'accoppiamento con gli scalari è tutt'altro che banale, in quanto si verifica una periodicità su $\tilde{S}^5$.

In tutti i casi considerati, il valore di aspettazione per il loop di Wilson coincide con quello calcolato nella teoria non deformata in [15]. Si noti che ciò avviene anche per loop che non sono lisci, come nel caso delle due longitudini, nel quale sono presenti due cuspidi.

Altre osservabili molto interessanti, oltre ai valori di aspettazione sul vuoto di loop di Wilson, sono le funzioni di correlazione di due loop. Si è affrontato lo studio del correlatore di due loop circolari che descrivono paralleli su S^2 nella teoria β-deformata. Questa quantità è nota in letteratura nella teoria $\mathcal{N} = 4$ SYM [16, 17], e presenta un interessante fenomeno di transizione di fase detta di Gross-Ooguri [18]. La motivazione per la transizione di fase di Gross-Ooguri [19] è che l'azione di stringa, vale a dire l'area di una superficie minimale delimitata dal loop, genericamente ha due punti sella in competizione. La superficie minimale può avere la topologia di un anello (si considera qui il caso in cui i due cerchi hanno orientazione opposta) o può essere composta da due superfici sconnesse, ognuna delle quali è la superficie relativa ad un singolo loop. L'anello chiaramente spanna un'area minore quando i loop sono sufficientemente vicini. Ma l'area dell'anello cresce quando la separazione tra i cerchi aumenta e, eventualmente, la superficie disconnessa diventa energeticamente favorita. A grandi distanze, la stringa si comporta classicamente solo in vicinanza dei loop e il correlatore connesso è saturato da scambi perturbativi dei modi di supergravità più leggeri tra le parti disconnesse del worldsheet classico. Il salto tra i due punti sella dovrebbe condurre alla transizione di fase nel correlatore dei loop di Wilson come funzione della separazione tra i due loop e i loro raggi.

L'ansatz che viene considerato studia due cerchi concentrici nello spazio AdS_5, ma le soluzioni trovate non differiscono da quelle in studiate in [16, 17], dove si considerano due cerchi concentrici di raggi uguali prima e diversi poi, giacenti su piani paralleli. Infatti, due cerchi concentrici su piani paralleli definiscono una due-sfera o un piano in $\mathbb{R}^4$. Siccome è possibile legare una due-sfera a un piano mediante una trasformazione confome, i due sistemi considerati sono equivalenti.

Per quanto riguarda l'accoppiamento con gli scalari, ogni loop è accoppiato ad un cerchio sulla sfera deformata, così che si ha una doppia rotazione, altamente non banale, su $\tilde{S}^5$. Il risultato che si trova per il correlatore è ancora una volta uguale al risultato per la quantità calcolata nella teoria non deformata. Si è svolto il calcolo sia per il caso in cui i cerchi hanno la stessa orientazione sia nel caso in cui abbiano orientazione opposta. La principale conseguenza di questo risultato è che il sistema presenta la stessa transizione di fase di Gross-Ooguri osservata in [18] per il caso non deformato.

È importante notare che nonostante i valori di aspettazione dei loop di Wilson e dei correlatori sono uguali nelle due teorie considerate, le soluzioni di stringa duali a tali loop sono diverse. Infatti, nella teoria $\mathcal{N} = 1$ β-deformata SYM è presente un angolo aggiuntivo sulla sfera deformata $\tilde{S}^5$, necessario per la consistenza della soluzione con le condizioni al contorno, a causa del fatto che nella teoria β-deformata il campo B è acceso. Tale angolo è proporzionale al parametro della deformazione $\hat{\gamma} = L^2\beta$ e si annulla nel limite di deformazione nulla $\beta = 0$, ritrovando così la soluzione non deformata. La deformazione è parametrizzata in qualche modo da tale angolo, ma non influenza l'azione di stringa che così risulta indipendente dalla deformazione. Si noti che la tale cancellazione avviene prima della procedura di regolarizzazione dell'azione mediante la trasformata di Legendre.

Per affrontare esplicitamente i calcoli di cui si sono riportati i risultati risulta notevole notare che l'azione di Polyakov di una stringa in $AdS_5 \times S^5$è espressa in termini dell'usuale modello σ

$$\frac{1}{4\pi\alpha'} \int d\tau\, d\sigma\, \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN} \tag{B.0.9}$$

dove G_{MN} è la metrica di target per il nostro spazio prodotto. L'anstaz che si considera permette di fattorizzare il modello σ in una parte di puro AdS_5 e una parte di puro S^5, permettendo di maneggiare equazioni del moto indipendenti per le rispettive variabili. Queste due parti dell'anstaz sono connesse solo in due modi. Il primo è il range di validità delle coordinate di worldsheet, che chiaramente deve coincidere su entrambi gli spazi. Il secondo sono le condizioni di Virasoro, che sono date dall'annullamento del tensore energia-momento. Quindi, risulta molto conveniente considerare separatamente anstaz per AdS_5 e S^5. Questa infatti è la strategia seguita in Appendice.

Capitolo C

Loop di Wilson supersimmetrici e pure spinors

La totalità dei loop di Wilson supersimmetrici studiati finora nell'ambito della teoria $\mathcal{N} = 4$ SYM appartengono a due classi: i loop che descrivono un percorso arbitrario su $\mathbb{R}^4$, scoperti da Zarembo in [20] e i loop di forma arbitraria su una tre-sfera $S^3 \subset \mathbb{R}^4$, scoperti da Drukker, Giombi, Ricci e Trancanelli in [21], anche conosciuti come loop di DGRT. I loop di Zarembo hanno valore di aspettazione sul vuoto banale su $\mathbb{R}^4$ e sono gli stessi loop di Wilson che appaiono nel *Langlands twist* topologico della teoria $\mathcal{N} = 4$ SYM [22]. I loop studiati in questo lavoro nella teoria $\mathcal{N} = 1$ β deformata SYM appartengono alla classe dei loop di DGRT. Non è stato chiaro se queste due classi contengono tutti i possibili loop di Wilson supersimmetrici fino a che Pestun e Dymarsky hanno risposto in modo sistematico a questa domanda: in [23] i due autori hanno trovato tutti i possibili operatori loop di Wilson W che sono invarianti almeno sotto una simmetria superconforme Q. In aggiunta, hanno classificato le sottoclassi di coppie (Q, W) modulo equivalenza sotto l'azione del gruppo superconforme di $\mathcal{N} = 4$ SYM. Il risultato principale è che emergono loop di Wilson supersimmetrici che non erano stati identificati prima. In diversi casi questi nuovi operatori coinvolgono accoppiamenti complessi ai campi scalari che li distinguono nettamente dai casi studiati precedentemente. In alcuni casi i nuovi operatori possono essere legati ai loop precedenti tramite una trasformazione conforme complessificata. Comunque, a meno di definire la teoria su uno spazio-tempo complessificato, rimanendo nell'ambito della teoria convenzionale formulata nello spazio Euclideo reale, i nuovi operatori non sono equivalenti a quelli precedentemente conosciuti.

L'ingrediente cruciale nella costruzione di tali operatori sono i cosiddetti *pure spinors* 10-dimensionali. Per definizione, uno spinore ϵ nel gruppo $Spin(\mathbb{R}^{2n})$ è detto *puro* se è annichilato da metà delle matrici gamma. In altre parole, esiste un sottospazio di dimensione n, $L \subset \mathbb{R}^{2n} \otimes \mathbb{C}$ tale che

$$v^I \Gamma_I \epsilon = 0 \Leftrightarrow v \in L \ . \tag{C.0.1}$$

Di conseguenza, un *pure spinor* definisce una struttura quasi complessa sullo spazio vettoriale $\mathbb{R}^{2n} \otimes \mathbb{C}$ dicendo che L è lo spazio dei vettori anti-olomorfi $L = V^{(0,1)}$.

La loro importanza non deve essere così sorprendente siccome la teoria $\mathcal{N} = 4$ SYM in 4 dimensioni si ottiene come riduzione dimensionale della teoria $\mathcal{N} = 1$ SYM, nella quale i *pure spinors* emergono naturalmente [24]. Infatti, lo spinore $\epsilon(x)$ (che in generale dipende dalle variabili di spazio-tempo) che parametrizza le trasformazioni superconformi della teoria $\mathcal{N} = 4$ SYM, può essere visto come la riduzione di uno spinore chirale 10-dimensionale.

Localmente ad un punto x dello spazio-tempo, il loop di Wilson può essere descritto dal vettore tangente alla curva e all'accoppiamento scalare in x. È possibile combinare questi dati in un unico vettore $v(x)$ 10-dimensionale. La condizione che il loop di Wilson rispetti la supersimmetria generata dallo spinore $\epsilon(x)$ si traduce in un sistema di equazioni su $v(x)$. Si osservano due differenti situazioni:

- Se $\epsilon(x)$ non è un *pure spinor*, il sistema di equazioni ha una soluzione unica, e il vettore tangente alla curva e l'accoppiamento con gli scalari in x sono completamente fissati. In altre parole, il vettore tangente alla curva e l'accoppiamento scalare possono essere combinati in un unico vettore $v(x)$ 10-dimensionale. Le curve, generate in questo modo a partire da un parametro di supersimmetria $\epsilon(x)$ generico, non sono altro che le orbite della trasformazione conforme generata da Q_ϵ^2, dove si indicano con Q_α le supercariche di Poincaré della teoria. Se, come è naturale, si chiede che le orbite siano compatte, allora, modulo equivalenza conforme, le sole curve compatte sono le figure di Lissajous (p, q), dove $\frac{p}{q} \in \mathbb{Q}$ è il rapporto dei due autovalori della matrice che rappresenta l'azione di Q^2 sullo spazio-tempo $\mathbb{R}^4$.

- Se $\epsilon(x)$ è un *pure spinor* allora ci sono più soluzioni per il vettore $v(x)$. Più precisamente, un *pure spinor* $\epsilon(x)$ definisce una struttura quasi complessa 10-dimensionale $J(x)$, e la condizione che il loop di Wilson sia supersimmetrico si traduce nella condizione che $v(x)$ sia un vettore anti-olomorfo rispetto a $J(x)$. All'interno del sottospazio Σ dello spazio dove $\epsilon(x)$ è un *pure spinor* c'è uno spazio ricco di soluzioni per loop di Wilson supersimmetrici. In generale, per qualunque curva che giace in Σ si possono trovare accoppiamenti scalari in modo che il loop di Wilson risulti supersimmetrico.

Lo spinore di supersimmetria $\epsilon(x)$ della teoria $\mathcal{N} = 4$ SYM può essere naturalmente esteso nello spazio $AdS_5 \times S^5$dove assume il ruolo di spinore di supersimmetria della teoria di stringa di tipo IIB.

In questo ambito, lo scopo di questo lavoro è stato quello di trovare la soluzione di stringa duale a loop di Wilson generati dalle figure di Lissajous (p, q), e calcolarne poi il valore di aspettazione sul vuoto, per la teoria $\mathcal{N} = 4$ SYM. Il risultato ottenuto a strong coupling dalla teoria di stringa è consistente con quello calcolato a weak coupling attraverso la teoria delle perturbazioni. La caratteristica comune di queste soluzioni è che si verifica sempre una cancellazione esatta tra l'azione di S^5 e quella di AdS_5. Di conseguenza, il valore di aspettazione sul vuoto dell'operatore ha una dipendenza molto semplice in termini dei parametri che definiscono il loop, vale a dire $\frac{p}{q}$. Si è verificato che anche per questo nuovo tipo di loop si ritrovano gli stessi risultati nella teoria β-deformata.

Inoltre, il correlatore di due cerchi di raggi generici è stato calcolato. Il risultato del caso limite in cui i due cerchi sono coincidenti suggerisce che la soluzione connessa potrebbe non esistere. Questo perchè il valore dell'azione di stringa coincide esattamente con il valore che darebbe la soluzione disconnessa, cioè la somma delle aree dei singoli cerchi. È naturale, dunque, porsi la domanda se la soluzione connessa esiste o, in altre parole, se si verifica transizione di fase di Gross-Ooguri. I diversi scenari che si possono presentare sono i seguenti:

- Non è presente transizione di fase, non esiste la soluzione connessa e dunque la soluzione è descritta dalle superfici dei due cerchi separatamente.
- Non è presente transizione di fase, però esiste la soluzione connessa, che è sottodominante rispetto a quella disconnessa.
- Il sistema presenta transizione di fase, e per certi valori dei parametri la soluzione connessa è dominante rispetto a quella disconnessa.

Per rispondere a tale questito, la strategia è stata la seguente: si è fissato uno dei due cerchi all'equatore di S^2, e si è tenuto come parametro libero il raggio dell'altro cerchio. In virtù dei nuovi accoppiamenti scalari di Pestun, tutte le altre quantità sono fissate dalla geometria, una volta fissata la distanza tra i due cerchi. Si noti come questo non accade per i loop di Wilson di DGRT. Attraverso opportuni cambi di variabili, è possibile arrivare a scrivere una singola equazione che, al variare del raggio del cerchio "libero", descrive una curva nel diagramma di fase di Gross-Ooguri. A seconda del fatto che questa curva intersechi o meno la curva di transizione di fase, oppure in quale regione del diagramma giaccia piuttosto che in un'altra, sarà verificata una delle ipotesi presentate sopra.

Allegato

Introduction

In conventional quantum field theory, the fundamental objects are mathematical points in spacetime, modeling the elementary point particles of nature. String theory is a rather radical generalization of quantum field theory whereby the fundamental objects are extended, one-dimensional lines or loops. The various elementary particles observed in nature correspond to different vibrational modes of the string. While we cannot see a string (yet) in nature, if we are very far away from it we will be able to see its point-like oscillations, and hence measure the elementary particles that it produces. The main advantage of this description is that while there are many particles, there is only one string. This indicates that strings could serve as a good starting point for a unified field theory of the fundamental interactions.

Moreover, among the massless string states, there is a spin 2 particle that interacts like a graviton. In fact, the only consistent interactions of massless spin 2 particles are gravitational interactions. Thus string theory naturally includes general relativity, and it was thereby proposed as a unified theory of the fundamental forces of nature, including gravity, rather than a theory of hadrons. Indeed, string theory is a consistent quantum theory, free from ultraviolet divergences, which necessarily requires gravitation for its overall consistency.

In the last decade, great interest has been given to the connection between string and field theory, namely the duality between a string theory in a space Anti-de-Sitter and a conformal field theory. This equivalence, known as AdS/CFT correspondence, has been originally conjecturated by Maldacena [1], even though indications of the equivalence had been already given in previous works [2, 3, 4]. On one side of the correspondence, we have 10-dimensional Type IIB string theory on the product space $AdS_5 \times S^5$, where the Type IIB 5-form flux through S^5 is an integer N and the equal radii L of AdS_5 and S^5 are given by $L^4 = 4\pi g_s N \alpha'^2$, where g_s is the string coupling. On the other side of the correspondence, we have 4-dimensional super-Yang-Mills (SYM) theory with maximal $\mathcal{N} = 4$ supersymmetry, gauge group $SU(N)$, Yang-Mills coupling $g_{YM}^2 = g_s$ in the conformal phase. The AdS/CFT conjecture states that these two theories, including operator observables, states, correlation functions and full dynamics, are equivalent to one another.

In the strongest form of the conjecture, the correspondence is to hold for all values of N and all regimes of coupling $g_s = g_{YM}^2$. Certain limits of the conjecture are, however, also highly non-trivial. The 't Hooft limit on the SYM-side, in which $\lambda \equiv g_{YM}^2 N$ is fixed as $N \to \infty$ corresponds to *classical string theory on $AdS_5 \times S^5$* (no string loops) on the AdS-side. In this sense, classical string theory on $AdS_5 \times S^5$ provides with a classical Lagrangian formulation of the large N dynamics of $\mathcal{N} = 4$ SYM theory. A further limit $\lambda \to \infty$ reduces classical string theory to classical Type IIB supergravity on $AdS_5 \times S^5$. Thus, strong coupling dynamics in SYM theory (at least in the large N limit) is mapped onto classical low energy dynamics in supergravity and string theory, a problem that offers a reasonable chance for solution.

The correspondence is a strong-weak duality: it relates the nonperturbative strong coupling regime of one theory to the weak coupling perturbative regime of the other. On one hand, supergravity limit of superstring theory should describe the strong coupling regime of gauge theory. On the other hand, perturbative regime $\lambda = N g_{YM}^2$ of gauge theory should describe the strongly coupled string theory.

The conjecture is remarkable because its correspondence is between a 10-dimensional theory of gravity and a 4-dimensional theory without gravity at all, in fact, with spin ≤ 1 particles only. The fact that all the 10-dimensional dynamical degrees of freedom can somehow be encoded in a 4-dimensional theory living at the boundary of AdS_5 suggests that the gravity bulk dynamics results from a *holographic image* generated by the dynamics of the boundary theory. Therefore, the correspondence is also often referred to as *holographic*.

The original correspondence is between a $\mathcal{N} = 4$ SYM theory in its conformal phase and string theory on $AdS_5 \times S^5$. The power of the correspondence is further evidenced by the fact that the conjecture may be adapted to situations without conformal invariance and with less or no supersymmetry on the SYM side. The $AdS_5 \times S^5$ space-time is then replaced by other manifold or orbifold solutions to Type IIB theory, whose study is usually more involved than was the case for $AdS_5 \times S^5$ but still reveals useful information on SYM theory.

One of the goals of this work is to try to extend some of these results to theories with less supersymmetries. We will consider an $\mathcal{N} = 1$ supersymmetric gauge theory obtained by a marginal β-deformation of the $\mathcal{N} = 4$ SYM [25, 26, 27].

The supergravity dual of the β-deformed SYM was found by Lunin and Maldacena in [11]. The Lunin-Maldacena background can be obtained from the $AdS_5 \times S^5$ via a series of T-duality transformation, shift and T-duality transformation acting on the five-sphere, whereas the AdS space is untouched by the deformation. We will look at the real β case throughout the work, so that the theory is an exact deformation of the original theory.

Wilson loops are extended object which are related to the propagation of

$SU(N)$ bosons in $\mathcal{N}=4$ SYM theory and supersymmetry implies here that scalar fields must also couple to the contour. In particular, the expectation value of the Wilson operator is the phase factor associated to the propagation of the boson.

In AdS/CFT correspondence, the computation of the exact expectation value of the Wilson loop, in the limit of large 't Hooft coupling, turns into a problem of finding the minimal area swept by the string worldsheet whose boundary is the Wilson loop itself.

In this work, we would like to analyze aspects of the supergravity dual of Wilson loops in the β-deformed SYM theory. Within this framework of the Lunin-Maldacena background, we prove that the vacuum expectation value of Wilson loops coincide with the value found in the undeformed theory. Moreover, we show that this results does not hold for 1/2 or 1/4 BPS Wilson loop only, the cases treated in literature so far, but also for operators that preserve a less amount of supersymmetry (for example, the toroidal loop which are 1/8 BPS operators), or even non-BPS operators. This last case corresponds to correlators of Wilson loops. In this contest, the same Gross-Ooguri phase transition, namely the transition through the solution described by a connected minimal surface to a disconnected one, is obtained in the two theories. This result reinforces the conjecture that the expectation value of the Wilson loop in $\mathcal{N}=1$ β-deformed SYM might be described by the same matrix model as in the non-deformed theory.

We then turn to consider toroidal loops with a different scalar coupling on S^5, proposed recently by Dymarsky and Pestun [23]: they proved that the two classes of couplings contain all possible supersymmetric Wilson loop with this type of geometry. In $\mathcal{N}=4$ SYM theory, the result obtained at strong coupling matches with the result calculated at weak coupling using perturbation theory. The common feature of these solution is an exact cancellation between the S^5 part of the action and a piece of the AdS_5 part of the action. As a consequence, the expectation value of such operators posses a trivial dependence on the parameters that define the loops. Moreover, we show that even for this new type of Wilson loop the same results are obtained in the $\mathcal{N}=1$ β-deformed SYM.

Furthermore, we evaluate the correlator of two circles of generic radii. The result found for the action when the two circles are coincident suggests us that the connected solution might not exist. This because the value of the string action is equal to the value of the disconnected solution, that is the value of the circles separately. It is worth to ask if the connected solution effectively exists, namely if the Gross-Ooguri phase transition take places. We derived an equation that describes a curve in the Gross-Ooguri phase diagram. Our next step will be to determine if this curve intersects the phase transition line or if it lies in only one single region of the diagram, that is no phase transition takes place.

The work is organized as follow. In the first chapter we give a brief introduction to the AdS/CFT correspondence. We describe the main features of the $AdS_5 \times S^5$ space where the string theory lives and the conformal group, which is the bosonic sector of the supersymmetry group of the two theories. We then give a precise formulation of the correspondence, underlying its three different limits of validity. Finally, we provide some basic tests which have confirmed the soundness of the conjecture, namely the symmetry matching and the holographic principle, which permits to relate the observables of the two theories.

In the second chapter, we introduce the notion of Wilson loop as known from field theory and extend it to be suitable for our $\mathcal{N} = 4$ SYM theory, including the coupling to the scalar fields. We then describe how to associate a regularized area to the loops by defining a modified string action by a Legendre transform of the Nambu-Goto action.

In the third chapter, we introduce the $\mathcal{N} = 1$ β-deformed SYM theory. We show that it is effectively a marginal deformation of $\mathcal{N} = 4$ SYM, and we recall how the supergravity solution, leading to the Lunin-Maldacena background, is obtained. Then we describe how extended objects like D-brane are modified between the two geometries. Finally, we introduce the Wilson loop in the β-deformed theory and we show that it cannot be a BPS operator since the gauge bosons and scalars are in different $\mathcal{N} = 1$ supersymmetry multiplets.

In the forth chapter, we begin the computation of expectation value of Wilson loop. We start with a well-known example in order to familiarize with the analysis of such problems and to prove explicitly the equivalence of the Nambu-Goto and the Polyakov formulations. Then, we evaluate the dual string solution of certain class of Wilson loop in the β-deformed theory, and we conclude that, although the solutions are different from the undeformed ones, the results for the vev of the Wilson loops are exactly the same.

In the fifth chapter, we turn to the computation of correlators of circular Wilson loops in the $\mathcal{N} = 1$ β-deformed SYM theory. Again, the results we find agree completely with the values found in the undeformed theory.

In the sixth chapter, we introduce another class of supersymmetric Wilson loops, with a different scalar coupling on S^5. In $\mathcal{N} = 4$ SYM theory, the result obtained at strong coupling matches with the result calculated at weak coupling using perturbation theory. We show that the expectation value of such operators posses a trivial dependence on the parameters that define the loops, due to a exact cancelation within the string action. We show that even for this new type of Wilson loop the same results are obtained in the $\mathcal{N} = 1$ β-deformed SYM. Furthermore, we evaluate the correlator of two circles of generic radii, and the result seems to suggests that the connected solution might not exists, that is no Gross-Ooguri phase transition takes place.

Chapter 1

AdS/CFT correspondence

The AdS/CFT correspondence [1] is a conjecture about the connection between string theory on certain curved background and field theory. The conjecture asserts that string theory on the space $AdS_5 \times S^5$ is dual to a conformal quantum field theory, which lives on the boundary of AdS_5.

In this chapter we shall recall some basic properties of anti de Sitter spaces, the fact that they may be associated to a boundary and that the isometry group of anti de Sitter space can be viewed as the conformal group on the boundary. Furthermore, we shall presents the correspondence in more detail, and explain its three relevant limits of validity. Finally, we shall present the most important evidences that seem to confirm the conjecture.

1.1 AdS geometry

We consider here Euclidean metric. D-dimensional Anti de Sitter spaces are solutions of the empty space Einstein equation with positive cosmological constant[1]

$$\left.\begin{array}{rcl} R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R & = & \frac{1}{2}\,\Lambda g_{\mu\nu} \\ R & = & \frac{D}{2-D}\,\Lambda \end{array}\right\} \Rightarrow R_{\mu\nu} = \frac{\Lambda}{2-D} g_{\mu\nu}\ . \tag{1.1.1}$$

AdS spaces satisfy an additional symmetry

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma})\ . \tag{1.1.2}$$

Let us proceed to the construction of such spaces. We start by considering an $(n+1)$-dimensional AdS_{n+1} as imbedded in an Euclidean $(n+2)$-dimensional space with coordinates $(x^a) = (x^0, x^1, ..., x^n, x^{n+1})$, on which

[1] De Sitter spaces are solution of empty space Einstein equation with negative cosmological constant

we take the metric

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n+1})^2 \tag{1.1.3}$$

and the scalar product

$$x_1 \cdot x_2 = \eta_{ab} x_1^a x_2^b \tag{1.1.4}$$

where $\eta_{ab} = \mathrm{diag}(+, -, -, ..., -)$.

The quantity

$$x^2 \equiv (x^0)^2 - \sum_{i=1}^{n+1} (x^i)^2 \tag{1.1.5}$$

is preserved by the action of the group $SO(1, n+1)$

$$x^a \to x'^a = \Lambda^a{}_b x^b, \quad \Lambda^a{}_b \in SO(1, n+1) \ . \tag{1.1.6}$$

We define AdS_{n+1} by the condition

$$x^2 = L^2 \tag{1.1.7}$$

where L is a constant and it is called the "radius" of AdS.

We want to show, as (1.1.6) suggests, that the isometry group of AdS_{n+1} is $SO(1, n+1)$. Let us take three points x_0^a, $x_0^a + dx_{(1)}^a$ and $x_0^a + dx_{(2)}^a$ lying in AdS_{n+1}, and their images x'^a_0, $x'^a_0 + dx'^a_{(1)}$ and $x'^a_0 + dx'^a_{(2)}$ under the $SO(1, n+1)$ transformation defined in (1.1.6). Obviously, these points are included in AdS_{n+1} as well and, in addition the following identity holds

$$dx_{(1)} \cdot dx_{(2)} = dx'_{(1)} \cdot dx'_{(2)} \ . \tag{1.1.8}$$

Hence, the embedding metric on AdS_{n+1} is $SO(1, n+1)$ invariant.

Now we want to find out explicit expressions of the coordinates in AdS_{n+1} spaces. Let us take a different set of coordinates (ρ, z^μ) on the $(n+2)$-dimensional embedding space, where $(z^\mu) = (z^1, ..., z^{n+1})$ are the coordinates on AdS_{n+1}, related to the previous ones by the following relations

$$\begin{aligned} x^0 &= \rho \frac{1+z^2}{1-z^2} \\ x^\mu &= \rho \frac{2z^\mu}{1-z^2}, \quad \mu = 1, ..., n+1 \end{aligned} \tag{1.1.9}$$

where we have defined

$$z^2 \equiv (z^1)^2 + ... + (z^n)^2 + (z^{n+1})^2 \ . \tag{1.1.10}$$

A simple calculation shows that $x^2 = \rho^2$, and consequently the AdS condition (1.1.7) is fulfilled when $\rho = L$. From (1.1.9) we work out the metric in the new coordinates

$$ds^2 = d\rho^2 - \frac{4\rho^2}{(1-z^2)^2} dz^2 \ . \tag{1.1.11}$$

Hence the metric factorizes into a trivial radial part and a AdS part

$$g_{\mu\nu} = \frac{4L^2}{(1-z^2)^2}\eta_{\mu\nu} \ . \tag{1.1.12}$$

Let us prove that the metric (1.1.12) satisfies the Einstein equation (1.1.1) and (1.1.2). In other words, we want to show that $R_{\mu\nu} \propto g_{\mu\nu}$ and find the proportionality constant in terms of the dimension D and the AdS radius L.

Let us take a conformally flat metric

$$g_{\mu\nu}(z) = e^{\phi(z)}\eta_{\mu\nu} \ . \tag{1.1.13}$$

We recover (1.1.12) by setting

$$\phi(z) = \ln 4L^2 - 2\ln(1-z^2) \ . \tag{1.1.14}$$

First, we recall the definitions of the Christoffel symbols and the Riemann tensor

$$\begin{aligned} \Gamma^\mu_{\nu\rho} &= \frac{1}{2}g^{\mu\lambda}(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}) & (1.1.15)\\ R^\mu{}_{\nu\rho\sigma} &= \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho} \ . & (1.1.16) \end{aligned}$$

Then, the calculus is quite simple

$$\begin{aligned} \Gamma^\mu_{\nu\rho} &= \frac{1}{2}(\partial_\nu\phi\delta^\mu_\rho + \partial_\rho\phi\delta^\mu_\nu - \partial^\mu\phi\eta_{\nu\rho}) \\ \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} &= \frac{1}{2}\left(\delta^\mu_\sigma\partial_\rho\partial_\nu\phi + \delta^\mu_\nu\partial_\rho\partial_\sigma\phi - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi\right) \\ &\quad -\frac{1}{2}\left(\delta^\mu_\rho\partial_\sigma\partial_\nu\phi + \delta^\mu_\nu\partial_\sigma\partial_\rho\phi - \eta_{\nu\rho}\partial_\sigma\partial^\mu\phi\right) \\ &= \frac{1}{2}\left(\delta^\mu_\sigma\partial_\rho\partial_\nu\phi - \delta^\mu_\rho\partial_\sigma\partial_\nu\phi - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi + \eta_{\nu\rho}\partial_\sigma\partial^\mu\phi\right) \\ \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho} &= \frac{1}{4}\left(\delta^\mu_\sigma\partial_\nu\phi\partial_\rho\phi + \eta_{\rho\nu}\partial^\mu\phi\partial_\sigma\phi + \eta_{\nu\sigma}\delta^\mu_\rho(\partial\phi)^2\right) \\ &\quad -\frac{1}{4}\left(\delta^\mu_\rho\partial_\nu\phi\partial_\sigma\phi + \eta_{\sigma\nu}\partial^\mu\phi\partial_\rho\phi + \eta_{\nu\rho}\delta^\mu_\sigma(\partial\phi)^2\right) \end{aligned} \tag{1.1.17}$$

when the derivatives of the conformal scaling factor are

$$\begin{aligned} \partial_\mu\phi\partial_\nu\phi &= \frac{16z_\mu z_\nu}{(1-z^2)^2} \\ \partial_\mu\partial_\nu\phi &= \frac{4}{1-z^2}\eta_{\mu\nu} + \frac{8z_\mu z_\nu}{(1-z^2)^2} \ . \end{aligned} \tag{1.1.18}$$

The equation of maximal symmetry (1.1.2) is satisfied since

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} &= -\frac{4}{(1-z^2)^2}\left(\eta_{\nu\sigma}\delta^\mu_\rho - \eta_{\nu\rho}\delta^\mu_\sigma\right) \\ &= -\frac{1}{L^2}(-g_{\nu\rho}\delta^\mu_\sigma + g_{\nu\sigma}\delta^\mu_\rho) \ . \end{aligned} \tag{1.1.19}$$

Contracting (1.1.16) we work out the expression for the Ricci tensor,

$$R_{\nu\sigma} = -\frac{D-1}{L^2} g_{\nu\sigma} \tag{1.1.20}$$

when in our case $D = n + 1$, and (1.1.1) is indeed satisfied with the cosmological constant

$$\Lambda = \frac{n(n-1)}{L^2} = \frac{(D-1)(D-2)}{L^2} \ . \tag{1.1.21}$$

Hence, our definition of AdS (1.1.7) as a submanifold of a Euclidean embedding space is indeed correct.

It turns useful to describe Euclidean AdS_{n+1} by introducing the so called "light cone" coordinates

$$u = x^0 + x^{n+1}, \quad v = x^0 - x^{n+1} \ . \tag{1.1.22}$$

The condition (1.1.7) then becomes

$$x^2 = uv - \vec{x}^2 = L^2 \tag{1.1.23}$$

where $\vec{x} = (x^1, \dots, x^n)$.

We now turn to consider various set of coordinate on AdS_{n+1} and the corresponding metrics.

Maldacena coordinates

We define

$$\begin{aligned} \xi^\alpha &\equiv \frac{x^\alpha}{u}, \qquad \alpha = 1, ..., n \\ \vec{\xi}^2 &\equiv \sum_{\alpha=1}^{n} (\xi^\alpha)^2 \ . \end{aligned} \tag{1.1.24}$$

From (1.1.23) we have

$$\begin{aligned} x^2 &= uv - \vec{x}^2 = uv - u^2\vec{\xi}^2 = L^2 \\ \Rightarrow v &= \xi^2 u + \frac{L^2}{u} \ . \end{aligned} \tag{1.1.25}$$

We use the set (u, ξ^α) on AdS_{n+1}, and it is simple to work out the metric

$$(ds^2)_{AdS_{n+1}} = \frac{L^2 du^2}{u^2} + u^2 d\vec{\xi}^2 \ . \tag{1.1.26}$$

This set of coordinates is the one used by Maldacena in [1].

Poincaré coordinates

We take the set $(\xi^0, \vec{\xi}) \equiv (u^{-1}, \vec{\xi})$ on AdS_{n+1}, where we have put the radius of the space L equal to one. The metric is then given by

$$(ds^2)_{AdS_{n+1}} = \frac{1}{(\xi^0)^2}\left((d\xi^0)^2 + d\vec{\xi}^2\right) . \tag{1.1.27}$$

This set is identified with Poincaré coordinates on the projective plane.

1.1.1 The boundary of AdS

Now we turn to the problem of how a sort of *projective* boundary arise in anti de Sitter space. We consider the points $(x^0, x^\mu) \in AdS_{n+1}$ in embedding space and take the limit in which x is large. Let us redefine the coordinates

$$x^a = R\tilde{x}^a, \qquad u = R\tilde{u}, \qquad v = R\tilde{v} . \tag{1.1.28}$$

Now we consider the limit $R \to \infty$. Hence

$$x^2 = L^2 \Rightarrow \tilde{u}\tilde{v} - \vec{\tilde{x}}^2 = L^2/R^2 \to 0 . \tag{1.1.29}$$

Therefore, the condition that identifies the boundary is

$$\tilde{u}\tilde{v} - \vec{\tilde{x}}^2 = 0 . \tag{1.1.30}$$

We note that in our limit, R can be replaced without loss of generality by tR with t real. Then, the boundary is identified with the projective equivalence classes

$$\begin{aligned} uv - \vec{y}^2 &= 0 \\ (u, v, \vec{y}) &\sim t(u, v, \vec{y}) \end{aligned}$$

from which we see that the boundary is n dimensional, as we expected for it to be.

Next, we can use the equivalence scaling to probe the topology of the boundary. We note that the boundary condition (1.1.31) may be written as

$$(x^0)^2 - (x^{n+1})^2 = 1 = \vec{x}^2 \tag{1.1.31}$$

with Euclidean signature. It is straightforward to conclude that the boundary has the topology of $S^1 \times S^{n-1}$.

The equivalence scaling may also be used to find appropriate coordinates on the boundary. Indeed

$$v \neq 0 \xrightarrow{\text{scaling}} v = 1 \;\Rightarrow\; u = \vec{x}^2 \tag{1.1.32}$$

$$u \neq 0 \xrightarrow{\text{scaling}} u = 1 \;\Rightarrow\; v = \vec{\tilde{x}}^2 \tag{1.1.33}$$

and the connection between the two sets is

$$\vec{\tilde{x}} = \frac{\vec{x}}{x^2}\,. \tag{1.1.34}$$

If one of the two conditions holds, only one of the two sets may be used

$$v = 0 \quad \Rightarrow \quad \vec{\tilde{x}} = \vec{0} \tag{1.1.35}$$

$$u = 0 \quad \Rightarrow \quad \vec{x} = \vec{0}\,. \tag{1.1.36}$$

In fact, we may think of the one point (1.1.35) as the point at infinity in the $\vec{x}$ coordinates, and analogously for (1.1.36). This argument shows that the boundary (1.1.31) is already compactified.

1.2 The conformal group

In this section, we want to demonstrate that the isometry group $SO(1, n+1)$ acts on the boundary as the conformal group acting on Euclidean space.

We begin our analysis by checking that the dimension of the conformal group of a n-dimensional Euclidean space is the same as the dimension of $SO(1, n+1)$, which is $\frac{1}{2}(n+2)(n+1)$, *i.e.*, the number of linearly independent antisymmetric $(n+2)\times(n+2)$ matrices. By the other hand, the conformal group is generated by translations P^μ, Lorentz transformation $L_{\mu\nu}$, dilations D and special conformal transformations K^μ. P^μ and $L_{\mu\nu}$ together form the Poincaré group, which has dimension $\frac{1}{2}n(n+1)$. The dilations D have unit dimension since they act as

$$\vec{x} \to \lambda\vec{x}, \quad \lambda \in \mathbb{R} \tag{1.2.1}$$

while the special conformal transformations are given by

$$x'^\mu = \frac{x^\mu + \alpha^\mu x^2}{1 + 2\vec{\alpha}\cdot\vec{x} + \alpha^2 x^2} \tag{1.2.2}$$

which provide additional n generators. Thus, the number of generators of the conformal group is exactly $\frac{1}{2}(n+2)(n+1)$, as we claimed, so this concludes our dimensional analysis.

Now we dive into the main point of this section, that is to demonstrate that $SO(1, n+1)$ acts on the boundary points exactly as the conformal group. Let us take a point $(u, v, \vec{x})$ in AdS_{n+1}. Clearly, it satisfies the AdS relation $uv - \vec{x}^2 = L^2$, and we consider a transformation through $SO(1, n+1)$

$$\Lambda \begin{pmatrix} u \\ v \\ \vec{x} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \vec{x}' \end{pmatrix}, \qquad \Lambda \in SO(1, n+1) \tag{1.2.3}$$

where Λ preserves the norm of the vector, so that $u'v' - \vec{x}'^2 = L^2$.

As we have seen in the previous section, a point $(u, v, \vec{x})$ on the boundary of AdS_{n+1} satisfies the following relations

$$(i) \qquad uv - \vec{x}^2 = 0 \tag{1.2.4}$$
$$(ii) \qquad (u, v, \vec{x}) \sim \lambda(u, v, \vec{x}) \tag{1.2.5}$$

and Λ acts on it as before.

Now we take Λ to be infinitesimal, $\Lambda = \mathbf{1}_{n+2} + \omega$, where the $(n+2) \times (n+2)$ dimensional matrix ω is an infinitesimal transformation. The condition $\Lambda \in SO(1, n+1)$, that is, the norm is preserved by the transformation, turns into a condition on the form of ω

$$\omega = \begin{pmatrix} a & 0 & \vec{\alpha}^T \\ 0 & -a & \vec{\beta}^T \\ \frac{1}{2}\vec{\beta} & \frac{1}{2}\vec{\alpha} & \omega_n \end{pmatrix} \tag{1.2.6}$$

where $\vec{\alpha}, \vec{\beta}$ are n-vectors columns and ω_n is an $n \times n$ antisymmetric matrix.

It is straightforward to show that $u'v' - \vec{x'}^2 = uv - \vec{x}^2$ to the first order in the infinitesimal parameters $a, \vec{\alpha}, \vec{\beta}$ and ω_n, by simply writing the explicit mapping

$$(\mathbf{1}_{n+2} + \omega) \begin{pmatrix} u \\ v \\ \vec{x} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \vec{x}' \end{pmatrix} = \begin{pmatrix} u(1+a) + \vec{\alpha} \cdot \vec{x} \\ v(1-a) + \vec{\beta} \cdot \vec{x} \\ \left(\vec{x} + \frac{u}{2}\vec{\beta} + \frac{v}{2}\vec{\alpha}\right) + \omega_n \vec{x} \end{pmatrix} . \tag{1.2.7}$$

Let us take $v = 1, u = \vec{x}^2$ on the boundary[2]. Within this choice, a point on the boundary is represented by the coordinate $\vec{x}$. Now we map the point according to (1.2.7) and we apply (1.2.5) in order to rescale the transformed point in the same convention $v' = 1$. The explicit mapping is

$$\vec{x} \to \vec{x}'/v' = \vec{y}(1 + a - \vec{\beta} \cdot \vec{x}) + \frac{x^2}{2}\vec{\beta} + \frac{1}{2}\vec{\alpha} + \omega_n \vec{x} . \tag{1.2.8}$$

We want to show that (1.2.8) is actually a general conformal transformation. If we put all parameters except $\vec{\alpha}$ equal to zero we obtain a translation

$$\vec{x} \to \vec{x} + \frac{1}{2}\vec{\alpha} . \tag{1.2.9}$$

If we allow only ω_n to be non-zero we obtain a rotation

$$\vec{x} \to \vec{x} + \omega_n \vec{x} . \tag{1.2.10}$$

If only $a \neq 0$ we pick a dilation

$$\vec{x} \to \vec{x}(1 + a) . \tag{1.2.11}$$

[2]This is rigorous for all points but one for which $v = 0$ corresponding to a point at infinity on the boundary.

At last, if we let only $\vec{\beta} \neq 0$ we select a special conformal transformation.

$$\vec{y} \to \vec{y}(1 - \vec{\beta} \cdot \vec{y}) + \frac{1}{2} y^2 \vec{\beta} \,. \tag{1.2.12}$$

Indeed, by taking the first order expansion of (1.2.2)

$$\vec{x} \to \frac{\vec{x} + \frac{1}{2}\vec{\beta}x^2}{1 + \vec{\beta} \cdot \vec{x} + \frac{1}{4}\beta^2 x^2} = \vec{x}(1 - \vec{\beta} \cdot \vec{x}) + \frac{1}{2} x^2 \vec{\beta} + O(\beta^2) \tag{1.2.13}$$

we find precisely (1.2.12).

Therefore, we have finally shown that the isometry group of AdS_{n+1}, $SO(1, n+1)$, acts as the conformal group on the boundary of AdS_{n+1}.

1.3 Solitonic p-branes in low energy supergravity

The goal of this section is to briefly review the classical supergravity solutions in presence of branes, considering both the so called extremal and non-extremal solutions.

We begin by considering the generic Euclidean action in the Einstein frame[3] in D dimension

$$S = -\frac{1}{2\kappa_D^2} \int d^D x \sqrt{g} \{ R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \sum_n \frac{1}{n!} e^{a_n \phi} F_n^2 + .. \} \,. \tag{1.3.1}$$

Here ϕ is the dilaton, F_n's are the n-form field strengths belonging to the RR sector and the dots represent fermionic terms. For IIA strings we only have even values of n while for IIB strings we only have odd values of n.

We shall search for classical solutions of the type of flat translationally invariant p-branes, which are isotropic in transverse directions. We decompose the dimension of the theory as

$$D = p + 1 + d \tag{1.3.2}$$

where d is the dimension of the transverse space to the p-brane.

To fix our notation, we choose coordinates appropriate for our ansatz

$$z^\mu = (t, x^i, y^a), \quad \mu = 0, ..., D-1; \quad i = 1, 2, ..., p; \quad a = 1, 2, ..., d \tag{1.3.3}$$

and a metric suitable for the symmetries of the theory

$$ds^2 = g_{\mu\nu} dz^\mu dz^\nu = sB^2 dt^2 + C^2 \sum_{i=1}^{p} (dx^i)^2 + F^2 dr^2 + G^2 r^2 d\Omega_{d-1}^2 \tag{1.3.4}$$

[3]We switch from the Einstein frame to the string frame by a certain Weyl rescaling $g_{\mu\nu}^{\text{Einst}} = e^{-\frac{1}{2}\phi} g_{\mu\nu}^{\text{string}}$

where all components only depend on the transverse "length square" coordinate $r^2 = \sum_{a=1}^{d}(y^a)^2$, and $d\Omega_{d-1}^2$ is the metric on the unit sphere S^{d-1} in the transverse space.

Now we are ready to write the equations of motion for the action (1.3.1)

$$\begin{aligned} R^\mu{}_\nu &= \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \frac{1}{2n!}e^{a\phi}\left(nF^{\mu\xi_2\ldots\xi_n}F_{\nu\xi_2\ldots\xi_n} - \frac{n-1}{D-2}\delta^\mu_\nu F_n^2\right) && (1.3.5)\\ \nabla^2\phi &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial_\nu\phi g^{\mu\nu}) = \frac{a_n}{2n!}F_n^2 && (1.3.6)\\ 0 &= \partial_\mu(\sqrt{g}e^{a\phi}F^{\mu\nu_2\ldots\nu_n})\ . && (1.3.7)\end{aligned}$$

As we will see, $F_n \neq 0$ only for one value of n, thus we consider only this case from now on. In addition F_n must also satisfy the Bianchi identity

$$\partial_{[\mu_1}F_{\mu_2\ldots\mu_{n+1}]} = 0\ . \quad (1.3.8)$$

Moreover, our ansatz requires that the metric ought to tend to a flat value for $r \to \infty$, that is, the coefficients B, C, F, G ought to tend to 1 in this limit.

We will see that our problem possesses a electric/magnetic duality. Indeed if we define

$$\tilde{F}_{D-n} = e^{a\phi} * F_n \quad (1.3.9)$$

where $*F$ is the Hodge dual to n-form F, it is possible to show that the equations (1.3.5)-(1.3.7) are invariant under the transformations

$$a\phi \to -a\phi, \quad n \to D-n, \quad F_n \to \tilde{F}_{D-n}\ . \quad (1.3.10)$$

We explore first the electric case, in which the ansatz for the field strength is

$$F_{ti_1\ldots i_p r}(r) = \epsilon_{i_1\ldots i_p}k(r)\ . \quad (1.3.11)$$

If $\gamma_{\alpha\beta}$ is the metric on S^{d-1} we can write

$$\sqrt{g} = BC^pF(Gr)^{d-1}\sqrt{\gamma_{d-1}} \quad (1.3.12)$$

and using (1.3.6) we obtain

$$F^{ti_1\ldots i_p r} = \frac{1}{B^2C^{2p}F^2}\epsilon_{i_1\ldots i_p}k(r)\ . \quad (1.3.13)$$

Plugging this into (1.3.7) we find

$$\left(\frac{1}{BC^pF}(Gr)^{d-1}e^{a\phi}k(r)\right)' = 0 \quad (1.3.14)$$

and the result is given by

$$\begin{aligned} k(r) &= e^{-a\phi}BC^pF\frac{Q}{(Gr)^{d-1}} \\ F_{ti_1\ldots i_p r} &= \mathrm{e}_{\,i_1\ldots i_p}e^{-a\phi}BC^pF\frac{Q}{(Gr)^{d-1}}\ . \end{aligned} \quad (1.3.15)$$

Here Q is an integration constant that arise from

$$\begin{aligned} \tilde{F}_{\alpha_1\ldots\alpha_{d-1}} &= \sqrt{\gamma_{d-1}}\,\mathrm{e}_{\,\alpha_1\ldots\alpha_{d-1}}Q \\ \mu_p &= \frac{1}{\sqrt{16\pi G_D}}\int_{S^{d-1}}\tilde{F}_{d-1} = \frac{\Omega_{d-1}Q}{\sqrt{16\pi G_D}} \end{aligned} \tag{1.3.16}$$

where μ_p is the density of electric charge on the p-brane and Ω_{d-1} is the volume of S^{d-1}.

Now we turn to the magnetic ansatz: from (1.3.9) we see that $n = D-(p+2) = d-1$ and the field strength tensor

$$F_{\alpha_1\ldots\alpha_{d-1}} = \sqrt{\gamma_{d-1}}\,\mathrm{e}_{\,\alpha_1\ldots\alpha_{d-1}}Q(r)\ . \tag{1.3.17}$$

This is simple to work out: the equation of motion for F_n (1.3.7) is trivially satisfied, while in order for the Bianchi identity (1.3.8) to be fulfilled, Q must be a constant. In an analogous way to (1.3.16) we write the density of magnetic charge

$$g_p = \frac{1}{\sqrt{16\pi G_D}}\int_{S^{d-1}}F_{d-1} = \frac{\Omega_{d-1}}{\sqrt{16\pi G_D}}Q\ . \tag{1.3.18}$$

For the magnetic case, since

$$F^{\alpha_1\ldots\alpha_{d-1}} = \frac{1}{\sqrt{\gamma_{d-1}}}\,\mathrm{e}_{\,\alpha_1\ldots\alpha_{d-1}}\frac{Q}{(Gr)^{2(d-1)}} \tag{1.3.19}$$

we easily find that

$$\begin{aligned} \frac{1}{n!}F_n^2 &= \frac{Q^2}{(Gr)^{2(d-1)}} \\ \frac{1}{(n-1)!}F^{\mu\xi_2\ldots\xi_n}F_{\nu\xi_2\ldots\xi_n} &= \delta^\mu_\nu\frac{Q^2}{(Gr)^{2(d-1)}} \end{aligned} \tag{1.3.20}$$

while the corresponding expression for the electric case are the same but multiplied by a factor $e^{-2a\phi}$.

The next step is to find the form of the equations of motion for our ansatz. To accomplish that, we must find out the Riemann tensor for the metric. First, since the metric (1.3.4) is clearly diagonal, we rewrite it as

$$ds^2 = (A_0)^2(dz^0)^2 + \sum_{\mu=1}^{D-1}(A_\mu)^2(dz^\mu)^2\ . \tag{1.3.21}$$

Then, it is simply a matter of calculus to find out the only non-vanishing components of the Ricci tensor when the metric depends only on r [28]. It will be useful to define the function $f(r)$ by the relation

$$f(r)r^{d-1} \equiv BC^pF^{-1}(Gr)^{d-1}\ . \tag{1.3.22}$$

Now we are ready to write down the equations of motion for the metric and the dilaton in the electric ansatz

$$\begin{aligned} R^{\bar t}{}_{\bar t} &= -\frac{(d-2)e^{-a_n\phi}n!}{2(D-2)}F_n^2 \equiv -(d-2)\frac{K^2}{F^2} \\ R^{\bar i}{}_{\bar i} &= -(d-2)\frac{K^2}{F^2} \\ R^{\bar r}{}_{\bar r} &= -(d-2)\frac{K^2}{F^2} + \frac{1}{2F^2}(\phi')^2 \\ R^{\bar\alpha}{}_{\bar\alpha} &= (p+1)\frac{K^2}{F^2} \\ \phi'' + \phi'(\log(fr^{d-1}))' &= a_n(D-2)K^2 \\ K^2 &\equiv \frac{1}{2(D-2)}e^{-a\phi}F^2\frac{Q^2}{(Gr)^{2(d-1)}}\ . \end{aligned} \tag{1.3.23}$$

We have shown in the previous section that in order for the metric to assume a form of the type of $AdS_q \times S^{D-q}$, the Riemann tensor must be proportional to the metric tensor, that is, it has to satisfy the empty space Einstein equation. To realize that, we see from (1.3.23) that the dilaton has to decouple and becomes a constant, which we are free to set to zero. Therefore the possible spaces are $AdS_4 \times S^7$(2-branes) or $AdS_7 \times S^4$(5-branes) which both correspond to 11-dimensional supergravity, or $AdS_5 \times S^5$(D3-branes) corresponding to IIB string theory. We will concentrate our discussion on this last case.

A little more effort [28] leads us to the final solution

$$ds^2 = H^{-2\frac{d-2}{\Delta}}\left(sfdt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H^{2\frac{p+1}{\Delta}}\left(f^{-1}dr^2 + r^2(d\Omega_{d-1})^2\right) \tag{1.3.24}$$

where

$$\begin{aligned} B &= f^{\frac{1}{2}}H^{-\frac{d-2}{\Delta}}, \quad C = H^{-\frac{d-2}{\Delta}}, \quad F = f^{-\frac{1}{2}}H^{\frac{p+1}{\Delta}} \\ G &= H^{\frac{p+1}{\Delta}}, \quad e^{\phi} = H^{a\frac{D-2}{\Delta}} \\ H &= 1 + \left(\frac{h}{r}\right)^{d-2}, \quad f = 1 - \left(\frac{r_0}{r}\right)^{d-2} \\ \Delta &= (p+1)(d-2) + \frac{1}{2}a^2(D-2) \\ h^{2(d-2)} + r_0^{d-2}h^{d-2} &= \frac{\Delta Q^2}{2(d-2)(D-2)}\ . \end{aligned} \tag{1.3.25}$$

It is simple to check that indeed the diagonal metric tensor components tend to 1 for $r \to \infty$. The 5-form field strength, in the electric case, is given by

$$F_{ti_1\ldots i_p r} = \mathrm{e}_{\,i_1\ldots i_p}H^{-2}\frac{Q}{r^{d-1}}\ . \tag{1.3.26}$$

We point out that the solutions (1.3.24) and (1.3.25) are a 2-parameter (r_0, Q) sub-family of solutions. We have already seen in (1.3.16) that Q is related to the mass and the density of charge of the D-brane. While for $r_0 \neq 0$ a horizon develops at $r = r_0$, by putting $r_0 = 0$ we obtain the very interesting so called extremal solution, which we will describe closely in the next section.

1.3.1 The extremal solution

As we have already pointed out, the extremal solution is realized when we require r_0 to vanish. In a quantum description, this solution is consistent with the brane in the ground state, whereas the non-extremal solution describes excitations of the branes. Here we consider no dilaton coupling, that is, we require it to vanish as already mentioned above, and we concentrate on the $n = 5$ case.

We can simplify the general solution by noticing that in this case the following identity holds

$$h^4 = \frac{Q}{4} \tag{1.3.27}$$

that comes from the fact that $\Delta = 2(D-2) = 16$. Then

$$\begin{aligned} f(r) &\equiv 1 \\ H &= 1 + \frac{Q}{4r^4} \\ ds^2 &= H^{-\frac{1}{2}}(sdt^2 + \sum_{i=1}^{3}(dx^i)^2) + H^{\frac{1}{2}}\sum_{a=1}^{6}(dy^a)^2 \\ \sum_{a=1}^{6}(dy^a)^2 &\equiv dr^2 + r^2(d\Omega_5)^2 \, . \end{aligned} \tag{1.3.28}$$

Following [29], we write the electric flux defined in (1.3.16) for a single D3-brane

$$\mu_3 = T_3\sqrt{16\pi G_{10}} \tag{1.3.29}$$

where T_3 is the D3-brane tension and G_{10} is the ten-dimensional Newton's constant

$$T_3 = \frac{2\pi}{(2\pi\ell_s)^4 g_s} \tag{1.3.30}$$

$$G_{10} = \frac{(2\pi\ell_s)^8}{32\pi^2} g_s^2 \, . \tag{1.3.31}$$

We now take N coincident Dp-branes, therefore we set a sort of flux normalization

$$\frac{\mu_3}{T_3\sqrt{16\pi G_{10}}} = N \tag{1.3.32}$$

which, by (1.3.16), turns into a constraint for Q

$$\frac{Q\Omega_5}{15\pi G_{10}} \frac{(2\pi\ell)^4 g_s}{2\pi} = N \ . \tag{1.3.33}$$

According to (1.3.28) we obtain

$$H = 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \ . \tag{1.3.34}$$

We now introduce the scaled variable

$$U = r/\ell_s^2 \tag{1.3.35}$$

and we consider $\alpha' = \ell_s^2 \to 0$ as well as $r \to 0$. The result is

$$\begin{aligned} H &\simeq \frac{4\pi g_s N}{U^4 \ell_s^4} \\ ds^2 &= \ell_s^2 \left\{ \frac{U^2}{\sqrt{4\pi g_s N}} dx_4^2 + \sqrt{4\pi g_s N} \left(\frac{dU^2}{U^2} + d\Omega_5^2 \right) \right\} \\ &= \frac{U^2}{L^2} d\tilde{x}_4^2 + L^2 \frac{dU^2}{U^2} + L^2 d\Omega_5^2 \end{aligned} \tag{1.3.36}$$

where $\tilde{x}$ is related to x by an appropriate scaling. (1.3.36) is precisely the product metric of $AdS_5 \times S^5$, where the radii of both spaces are the same and it is given by L

$$L^4 = 4\pi g_s N \ell_s^4 \ . \tag{1.3.37}$$

We shall review these results in a deeper way in the next section.

1.4 The Maldacena limit

The Dp-brane is a $(p+1)$-dimensional hyperplane in spacetime where an open string can end. D-branes carry mass and charge, and therefore the spacetime around them is curved. On the worldsheet of a type II string, the left-moving degrees of freedom and the right-moving degrees of freedom carry separate spacetime supercharges. Since the open string boundary condition identifies the left and right movers, the D-brane breaks at least one half of the spacetime supercharges. In type IIA (IIB) string theory, precisely one half of the supersymmetry is preserved if p is even (odd), that is, it is 1/2 BPS.

Now we want to give a more physical interpretation of the general construction that we have presented in the previous section. It was shown in [4] that, if we put N Dp-branes on top of each other, the resulting $(p+1)$-dimensional hyperplane carries exactly N units of the $(p+1)$-form charge. Let us consider N parallel separated D3-branes, then the end points of an open string may or may not be attached to the same brane. If it is so, these strings can have arbitrarily small length and will be massless. In the low energy limit these excitation modes induce a massless $U(1)^N$ gauge theory with $\mathcal{N} = 4$ supersimmetry. An open string can, however, have its ends attached to two different branes, then the mass of such a string cannot be arbitrarily small. Indeed, the length of the string cannot be shorter than the separation distance between the branes. In the limit where the N branes tend to be coincident, all string states would be massless and the gauge symmetry is increased to a $U(N)$ gauge symmetry. Hence, in the low energy limit, N coincident branes hold up an $\mathcal{N} = 4$ Super Yang-Mills theory in 4-dimensions with gauge group $SU(N)$[4].

Now we want to motivate the AdS/CFT correspondence by considering excitations around the ground state and taking a low energy or decoupling limit.

Let us take the metric of N coincident D3-branes

$$ds^2 = \left(1 + \frac{R^4}{y^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{R^4}{y^4}\right)^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2) \tag{1.4.1}$$

where $\eta_{ij} = \mathrm{diag}(- + ++)$ and R is the radius of the D3-brane

$$R^4 = 4\pi g_s N {\alpha'}^2 \ . \tag{1.4.2}$$

In order to study such a geometry, we will consider to different regimes.

If we take $y \gg R$ we simply recover the flat space-time $\mathbb{R}^{10}$. Otherwise, when $y < R$ we obtain the geometry of the throat, that it will be singular as $y \ll R$. In order to understand it better, we introduce the coordinate

$$u \equiv \frac{R^2}{y} \ . \tag{1.4.3}$$

The asymptotic form of the metric (1.4.1) in the large u limit is given by

$$ds^2 = R^2 \left[\frac{1}{u^2}\, \eta_{ij} dx^i dx^j + \frac{1}{u}\, du^2 + d\Omega_5^2\right] \ . \tag{1.4.4}$$

This geometry is described by the product of the five-sphere S^5 and the hyperbolic space AdS_5, where both spaces have identical radius R. We refer to this geometry close to the brane as $AdS_5 \times S^5$.

The Maldacena limit corresponds to keeping fixed g_s and N as well as all physical length scales, while letting $\alpha' \to 0$. In this limit, only the $AdS_5 \times S^5$ region of the D3-brane geometry survives and contributes to the string dynamics of physical processes, while the dynamics in the asymptotically flat region decouples from the theory.

[4]We ignore the $U(1)$ factor corresponding to the overall position of the branes.

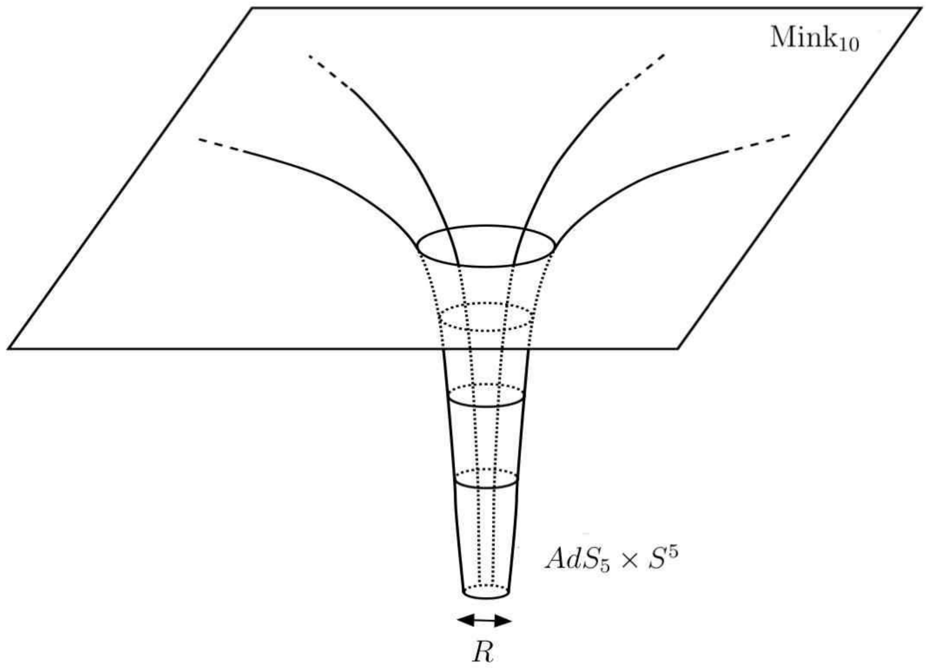

Figure 1.1: Space-time around D3-branes.

1.5 The Maldacena conjecture

The AdS/CFT correspondence [1] states the equivalence between the following theories

- Type IIB superstring theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius R, where the 5-form F_5^+ has integer flux $N = \int_{s^5} F_5^+$ and where the string coupling is g_s;
- $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions, with gauge group $SU(N)$ and coupling g_{YM} in its superconformal phase [5];

The identification between the parameters of the two theories is given by

$$g_s = g_{YM}^2, \qquad R^4 = 4\pi g_s N \alpha'^2 \; . \tag{1.5.1}$$

This statement of the conjecture is referred to as the *strong form*, as it is to hold for all values of N and of $g_s = g_{YM}^2$, and it also implies that the $AdS_5 \times S^5$ background is an exact solution of type IIB superstring theory. In addition to this, there are other two level of the conjecture.

[5]The physical states and operators are gauge invariant and trasform under unitary representations of $SU(2,2|4)$.

The 't Hooft limit consists in keeping the 't Hooft constant $\lambda \equiv g^2_{YM} N$ fixed and letting $N \to \infty$. In Yang-Mills theory, this limit corresponds to a topological expansion of the field theory's Feynman diagrams while, on the AdS side, the 't Hooft limit corresponds to a weak coupling string perturbation theory.

This form of the conjecture is clearly weaker than the previous version, but it is very interesting since it states a correspondence between classical string theory and the large N limit of gauge theories.

The weakest form of the conjecture is realized taking the 't Hooft parameter $\lambda \gg 1$ while $N \to \infty$. In this case the conjecture relates the strong coupling limit, or the $\lambda^{-1/2}$ expansion, of the Yang-Mills theory with the low energy, supergravity limit of type IIB superstring theory on $AdS_5 \times S^5$.

Even this last version of the correspondence has profound consequences, since it enables one to compute correlation functions in the large N, large $g^2_{YM} N$ limit. In fact, this limit contains the highly nontrivial sum of all planar Feynman diagrams, and emphasizes those diagrams which have infinitely many vertices.

However, the predictions of the AdS/CFT correspondence are very difficult to check in a direct way, and the main evidence which supports the correspondence comes from symmetry arguments.

1.6 Global symmetries

The natural first step is to compare global symmetries. In this section we switch from the Euclidean to the Minkowski formulation of the theory, since it is more natural in order to compare the symmetris. Therefore, the relevant group of AdS_5 it will be $SO(2,4)$ instead of $SO(1,5)$. The IIB string theory has an isometry group $SO(2,4) \times SO(6)$, the first being the isometry group of AdS_5, as we have already pointed out, while the last $SO(6)$ being the isometry group of S^5. However, due to the presence of spinors, we must take as the relevant group the product of the covering groups $SU(4)$ of $SO(6)$ and $SU(2,2)$ of $SO(2,4)$, that is $SU(2,2) \times SU(4)$. But a better looking at the way the 32 Majorana spinor supercharges trasform under this group show us the in fact the global symmetry is given by the Lie-supergroup $SU(2,2|4)$.

Now we ought to show that the global continuous symmetry group of $\mathcal{N} = 4$ SYM is given by the same supergroup. First of all, the theory being conformal, it is invariant under the conformal group $SO(2,4) \sim SU(2,2)$. The group $SO(6) \sim SU(4)$ comes from the R-symmetry of the theory, since the different supercharges may be rotated into one another. To set this more rigorous, let us consider the field content of the $\mathcal{N} = 1$ ten dimensional pure SYM. This theory contains the gauge potential a_μ, $\mu = 0, \ldots, 9$ (8 bosonic physical degrees of freedom, the gluons) and the 8-dimensional Majorana-Weyl gluinos λ_α, $\alpha = 1, \ldots, 8$, all in the adjoint representation. Moreover, the theory has 16 Majorana supercharges. Now, $\mathcal{N} = 4$ SYM is obtained as the dimensional reduction of such a theory: the gluon fields reduce to one gauge field (2 degrees of freedom) producing 6 scalar fields, transforming under the fundamental representation of $SO(6)$, and the gluino fields turn into 4 Weyl spinors in 4 dimension (8 degrees of freedom). Finally, the 16 supercharges turn into 4 sets of complex Majoranas charges, transforming under the fundamental and the antifundamental of the group $SU(4)$. So, the conformal group together with the R-symmetry give the supergroup $SU(2,2|4)$ as the global symmetry of the field theory.

In $\mathcal{N} = 4$ Super Yang-Mills theory there is also a Montonen-Olive or S-duality, realized on the complex coupling constant τ by Möbius transformations in $SL(2,\mathbb{Z})$

$$\tau = \frac{\theta_I}{2\pi} + i\frac{4\pi}{g^2_{YM}} \tag{1.6.1}$$

where θ_I is the instanton angle. But also the IIB theory contains a $SL(2,\mathbb{Z})$ invariance, arising from compactification of M-theory on a 2-torus with modular parameter $\tau = \chi + ie^{-\phi}$ with χ the axion of IIB [30].

1.7 AdS/CFT fields, operators and correlators

The Maldacena conjecture establishes a duality between two theories: string theory on a manifold of the form of $AdS_n \times \mathcal{M}$, where $\mathcal{M}$ is a compactification manifold, and an appropriate conformal field theory on the boundary of AdS_n. However, this conjecture does not state the precise way in which these two theories are related. So far we have proved that the global symmetries on both sides of the correspondence agree. In this section we want to demonstrate that also the representations, *i.e.* the states of the theories, of the supergroup $SU(2,2|4)$ match on both sides.

We recall that the single color trace operators are the fundamental bricks of our construction, since all higher trace operators may be compose using the OPE. Therefore it is natural for single trace operators on the SYM side to be correlated to single particle states on the AdS side, while multiple trace states are regarded as bound states of these particle states.

Let us pick out the irreducible representations of the supergroup $SU(2,2|4)$ on the string theory side. In order to do this, we identify all type IIB string degrees of freedom as fields φ in $AdS_5 \times S^5$. We split the metric

$$ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^S_{uv} dy^u dy^v \tag{1.7.1}$$

where z^μ, $\mu = 0, 1, \cdots, 4$ are coordinates on AdS_5 and y^u, $u = 1, \cdots, 5$ are the corresponding on S^5. It is useful to expand the fields $\varphi(z, y)$ in a basis

Type IIB string theory	$\mathcal{N}=4$ conformal super Yang Mills
Supergravity Excitations 1/2 BPS, spin ≤ 2	Chiral primary + descendants $\mathcal{O}_2 = \mathrm{tr}\, X^{\{i}X^{j\}}$ + desc.
Supergravity Kaluza-Klein 1/2 BPS, spin ≤ 2	Chiral primary + Descendants $\mathcal{O}_\Delta = \mathrm{tr}\, X^{\{i_1} \cdots X^{i_\Delta\}}$ + desc.
Type IIB massive string modes non-chiral, long multiplets	Non-Chiral operators, dimensions $\sim \lambda^{1/4}$ e.g. Konishi $\mathrm{tr}\, X^i X^i$
Multiparticle states	products of operators at distinct points $\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)$
Bound states	product of operators at same point $\mathcal{O}_{\Delta_1}(x)\cdots\mathcal{O}_{\Delta_n}(x)$

Table 1.1: Mapping between the representations of $SU(2,2|4)$ on both sides of the correspondence.

$Y_\Delta(y)$ of spherical harmonics on S^5

$$\varphi(z,y) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(z) Y_\Delta(y) \ . \tag{1.7.2}$$

We expect that fields compactified on S^5 receive a contribution to the mass. Indeed, from the eigenvalues of the Laplacian on S^5, the following relations between mass and scaling dimension hold

$$\begin{array}{rl} \text{scalars} & m^2 = \Delta(\Delta-4) \\ \text{spin } 1/2,\ 3/2 & |m| = \Delta - 2 \\ p-\text{form} & m^2 = (\Delta-p)(\Delta+p-4) \\ \text{spin2} & m^2 = \Delta(\Delta-4)\ . \end{array} \tag{1.7.3}$$

We summarize the complete mapping between string and sugra states onto super Yang Mills operators in Table 1.1.

We emphasize the result that the single trace half BPS operators in the SYM theory correspond in a one-to-one way with the canonical fields of supergravity, compactified on $AdS_5 \times S^5$. Now we further discuss the AdS/CFT correspondence by describing how correlators on both sides are related.

Taking the Poincaré metric on Euclidean AdS_5

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2) \tag{1.7.4}$$

the boundary of the space $x_0 = 0$ is identified to be $\mathbb{R}^4$, where the metric diverges. A Weyl rescaling is required to remove the overall factor and regularize the metric, but, in general, such rescaling is not unique. Thus, in order to obtain a well-defined limit to the boundary we must require for the boundary theory to be scale invariant. Indeed, $\mathcal{N}=4$ SYM theory is scale invariant and it lives at the boundary of AdS_5.

We suppose that the bulk fields $\varphi_\Delta(z)$ are asymptotically free, and the two independent free solutions have the following asymptotic behaviors

$$\varphi^0_\Delta(z_0,\vec{z}) = \begin{cases} z_0^\Delta & \text{if normalizable} \\ & \\ z_0^{4-\Delta} & \text{if non-normalizable} \end{cases} . \tag{1.7.5}$$

While the normalizable solutions are related to the vacuum expectation values of operators of dimensions Δ and corresponding quantum numbers, the non-normalizable solutions do not correspond to bulk excitations. In [6] it was argued that they epitomize the coupling of external sources to the supergravity or string theory. The boundary fields related to the non normalizable solution φ_Δ are defined by

$$\bar{\varphi}_\Delta(\vec{z}) \equiv \lim_{z_0\to 0} \varphi_\Delta(z_0,\vec{z}) z_0^{4-\Delta} . \tag{1.7.6}$$

In order to realize the mapping between the correlators in the SYM theory and the dynamics of string theory, we construct a standard generating functional for all the correlators of single trace operators $\mathcal{O}_\Delta$ on the boundary

$$\begin{aligned} Z[\{\bar{\varphi}_\Delta\}] &= \sum_q \frac{1}{q!} \int \prod_{k=1}^q d^n z_k \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_q(z_q)\rangle \bar{\varphi}_1(z_1)\cdots\bar{\varphi}_q(z_q) \\ &= \left\langle \exp\left\{ \int d^n z \sum_i \bar{\varphi}_i(z)\mathcal{O}_i(z) \right\} \right\rangle . \end{aligned} \tag{1.7.7}$$

If the operators $\mathcal{O}_i$ on the field theory living on the boundary have conformal dimension Δ_i then the currents $\bar{\varphi}_i$ should have conformal dimension $n-\Delta_i$. Moreover, the relation (1.7.7) is supposed to hold order by order in a perturbative expansion. The dynamics of Type IIB string theory on $AdS_5\times S^5$ is described by an action $S[\varphi_\Delta]$, which we can relate to the generating functional (1.7.7) by

$$Z[\{\bar{\varphi}_\Delta\}] = \text{extrem } S[\varphi_\Delta] \tag{1.7.8}$$

where the extremum is taken over all fields which assume the asymptotic behavior (1.7.6).

However, we note that $\mathcal{N}=4$ super-Yang-Mills theory is certainly not a free quantum field theory, and generic correlators will receive quantum corrections from their free field values, and therefore they will acquire non-trivial coupling $g_s = g^2_{YM}$ dependence, but we will not discuss this matter here.

Chapter 2

Wilson loops in $\mathcal{N} = 4$ SYM

2.1 Definition

In gauge theories the Wilson loop operator W is defined as the path-ordered exponential of the gauge field

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left(i \oint_C A_\mu dx^\mu \right) \tag{2.1.1}$$

with the trace in the fundamental representation and the integral is taken over any closed path in space-time. The most famous physical application of such an operators comes from the fact that they are the phase factor associated with the trajectory of a heavy quark in the fundamental representation of the gauge group. Hence, the Wilson loop measures the effects of the gauge dynamics on external quark sources. In particular, in the well known case of a parallel quark anti-quark pair, the Wilson loop serves as an order parameter for confinement, since it is the exponent of the effective potential between the quarks.

However, we easily see that the definition (2.1.1) does not fill in the case of $\mathcal{N} = 4$ SYM, since this theory does not contain quarks in the fundamental representation. From the string point of view, we can construct the Wilson loop by considering the usual system of $N + 1$ coincident D-branes and separate one of them taking it far away from the others. In such a way, we break the group $SU(N+1) \to SU(N) \otimes U(1)$ and, for large N, we can ignore the dynamics of the distant brane. The new features arise from the string stretching between the stack of coincident branes and the separated one. The ground state of this open string are the W-bosons and their superpartners of the broken $SU(N+1)$ gauge group. From the dimensional reduction from the 10-dimensional pure gauge theory we get the correct form of the Wilson loop in the Euclidean theory

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left(\oint_C (i A_\mu dx^\mu + |\dot{x}| \theta^i \Phi^i) ds \right) \tag{2.1.2}$$

where Φ^i are the extra six component of the gauge field and the θ^i coupling to the scalar fields are angular coordinates of unit magnitude and can be thought as coordinates on S^5. Although the presence of the scalar coupling in (2.1.2) might be seem surprisingly, we will show that it will be crucial to preserve supersymmetry.

We note here that in the euclidean formulation the Wilson loop is no longer a pure phase factor, because of the real term arising from the coupling to the scalars. The operator in the trace is no longer an unitary operator, hence the inequality $\langle W \rangle \leq 1$ does not hold anymore.

Let us analyze more rigorously the supersymmetry of the Wilson loop. Let us start from the general operator

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left(\oint_C (iA_\mu dx^\mu + \Phi^i \dot{y}^i) ds \right) . \tag{2.1.3}$$

We can study the perturbation theory of the expectation value of the loop $\langle W \rangle$ to the first order in $g_{YM}^2 N$ and we find that [31]

$$W = 1 + \frac{g_{YM}^2 N}{4\pi^2 \epsilon} \oint_C ds\, |\dot{x}| \left(1 - \frac{\dot{y}^2}{\dot{x}^2} \right) + \text{finite terms} \tag{2.1.4}$$

where ϵ is a cutoff[1]. We easily note that the linear divergence cancels only when the constraint $\dot{x}^2 = \dot{y}^2$ is satisfied. It is possible to show [31] that also at high order in the 't Hooft constant the divergence cancels when the constraint is satisfied.

2.2 Minimal surface in AdS space

In section 2.1 we have pointed out that the W-boson in string theory is described by an open string stretching between a single separated D-brane and the other N coincident D-branes. In the Maldacena limit, as we have seen in section 1.4, the flat ten-dimensional space decouple and the N D-branes are replaced by the $AdS_5 \times S^5$ geometry, and the open string is stretched from the boundary to the interior of AdS_5. This reasoning makes us aware that the Wilson loop operator and the string in AdS_5 geometry are related to each other.

The Maldacena conjecture tells us how effectively the expectation value of the Wilson Loop is given in terms of the action of a string bounded by the loop itself at the boundary of space

$$\langle W[C] \rangle = \int_{\partial X = C} \mathcal{D}X \, \exp(-\sqrt{\lambda} S[X]) \tag{2.2.1}$$

where X represents both the bosonic and the fermionic coordinates of the string and $S[X]$ is a suitable string action. In the strong 't Hooft limit,

[1] We have regularized the propagator replacing $1/x^2$ with $1/(x^2 + \epsilon^2)$.

that is, for large λ, we can approximate the integral to be the area A of the minimal surface bounded by the contour C

$$\langle W \rangle \simeq K \exp(-\sqrt{\lambda} A) \tag{2.2.2}$$

where K is a normalization factor that may depend on the loop variables (x^μ, y^i).

In fact, we will show below that the correct choice of the action is not the area of the surface as in (2.2.2), but instead the Legendre transform of it. While the value of the classical action for a surface is different than the area, the equation of motion are unchanged and the solution are still minimal surfaces.

2.2.1 Boundary conditions

In this section we will give a precise treatment of boundary conditions on the string on $AdS_5 \times S^5$ and how these are tied to another explanation of the constraint $\dot{x}^2 = \dot{y}^2$, that was originally obtained by making use of the coupling of the string to the gauge fields and to the scalars.

In order to do that, we start as usually by considering the ten-dimensional $\mathcal{N} = 1$ pure Super Yang-Mills, and analyzing the boundary conditions on the bosonic variables. We recall that this theory is fulfilled by stocking the space with D9-branes, and the Wilson loop is equivalent to an open string constrained to end on the loop, therefore we assume to impose full Dirichlet boundary conditions on the string worldsheet. Indeed, these conditions are complementary to the full Neumann boundary conditions imposed along the D9-brane.

We now reduce the theory to 4 dimensions by performing a T-duality along 6 directions and, as a result, a string bounded along a D3-brane is forced to obey 6 Dirichlet and 4 Neumann boundary conditions. Accordingly, the Wilson loop operator in 4 dimension imposes complementary boundary conditions, that is, 4 Dirichlet and 6 Neumann boundary conditions. We parameterize the loop by the variables $(x^\mu(s), y^i(s))$, where the 4 loop variables $\dot{x}^\mu$ are to be recognized as the 4 Dirichlet boundary conditions, while the 6 loop variables $\dot{y}^i$ as the 6 Neumann boundary conditions.

We select (τ, σ) to be the string wordsheet coordinates, in such a way that the boundary is situated at $\sigma = 0$, and we take the following $AdS_5 \times S^5$ metric

$$ds^2 = \sqrt{\lambda} Y^{-2} \left(\sum_{\mu=0}^{3} dX^\mu dX^\mu + \sum_{i=1}^{6} dY^i dY^i \right) \tag{2.2.3}$$

where we put $\alpha' = 1$. Following the identification made above, we impose Dirichlet conditions on X^μ

$$X^\mu(\tau, 0) = x^\mu(\tau) \tag{2.2.4}$$

and Neumann condition on Y^i, which we write as

$$J_\tau{}^\alpha \partial_\alpha Y^i(\tau, 0) = \dot{y}^i(\tau), \tag{2.2.5}$$

up to a normalization constant[2]. $J_\alpha{}^\beta$ is the complex structure defined on the string worldsheet by means of the induced metric $g_{\alpha\beta}$

$$J_\alpha{}^\beta = \frac{1}{\sqrt{g}} g_{\alpha\gamma} \epsilon^{\gamma\beta} \tag{2.2.6}$$

which enters in the condition (2.2.5) because of the reparametrization invariance on the worldsheet.

The minimal surface obeying the boundary conditions (2.2.4) and (2.2.5) may or may not end at the boundary of AdS_5, therefore the condition $Y^i(\tau, 0) = 0$, which identifies the boundary, is a supplemental Dirichlet conditions. It can be shown that this condition is consistent with (2.2.5) if and only if the loop variables satisfy the constraint $\dot{x}^2 = \dot{y}^2$. Let us take into consideration the Hamilton-Jacobi equation for the area A of a minimal surface that end along a loop parameterized by $(X^\mu(s), Y^i(s))$ in $AdS_5 \times S^5$

$$\left(\frac{\delta A}{\delta X^\mu}\right)^2 + \left(\frac{\delta A}{\delta Y^i}\right)^2 = \frac{1}{(2\pi)^2 Y^4}\left((\partial_\tau X^\mu)^2 + (\partial_\tau Y^i)^2\right) . \tag{2.2.7}$$

It is useful rewrite this equation in terms of the momenta conjugate to the X^μ's and the Y^i's

$$\frac{\delta A}{\delta X^\mu} = \frac{1}{2\pi Y^2} J_\tau{}^\alpha \partial_\alpha X^\mu, \qquad \frac{\delta A}{\delta Y^i} = \frac{1}{2\pi Y^2} J_\tau{}^\alpha \partial_\alpha Y^i . \tag{2.2.8}$$

Substituting these expression into (2.2.7) we obtain

$$(J_\tau{}^\alpha \partial_\alpha X^\mu)^2 + (J_\tau{}^\alpha \partial_\alpha Y^i)^2 = (\partial_\tau X^\mu)^2 + (\partial_\tau Y^i)^2 . \tag{2.2.9}$$

Implementing the boundary conditions (2.2.4) and (2.2.5) we get

$$\dot{x}^2 - \dot{y}^2 = (J_\tau{}^\alpha \partial_\alpha X^\mu)^2 - (\partial_\tau Y^i)^2 . \tag{2.2.10}$$

The last step is to impose the condition for the string worldsheet to end at the boundary of AdS_5. From $Y^i(\tau, 0)$ we get $\partial_\tau Y^i(\tau, 0) = 0$, so that the left hand side of (2.2.10) cannot be negative. Furthermore, due to the factor Y^{-2} in (2.2.3), keeping $J_\tau{}^\alpha \partial_\alpha X^\mu \neq 0$ near the boundary costs a large area, so we have to put it equal to zero at the boundary. Thus, we have demonstrated that the condition $\dot{x}^2 = \dot{y}^2$ is required for a minimal surface ending at the boundary of AdS_5.

[2] However the condition $\dot{x}^2 = \dot{y}^2$ suggests that the normalization constant should be equal to 1.

We can show that, once the constraint $\dot{x}^2 = \dot{y}^2$ is satisfied, the 6 Neumann boundary conditions (2.2.5) can be thought as Dirichlet conditions on S^5. Let us reparameterize the coordinates Y^i as

$$Y^i = Y\theta^i \tag{2.2.11}$$

where $(\theta^i)^2 = 1$. At the boundary of AdS_5 the classical solution satisfies $\partial_\alpha Y^i = (\partial_\alpha Y)\theta^i$, therefore the conditions (2.2.5) become

$$\theta^i(\tau, 0) = \frac{\dot{y}^i}{|\dot{y}|} \tag{2.2.12}$$

which can be viewed as Dirichlet conditions on S^5.

2.2.2 Legendre transform

As we have already pointed out above, the Wilson loop is associated to a string which end points lying along the loop on the boundary of space, and we expect that the string solution is described in terms of minimal surfaces. However, there are many actions, which differ by boundary terms, whose equations of motion are solved by minimal surfaces. These terms may be important, since the surfaces we consider have boundaries. In (2.2.2) we argued that $\langle W \rangle$ was given in terms of the area A of the minimal surface.

The goal of this section is to show that the Wilson loop is actually described by the Legendre transform of such an area. Since we are dealing with the area of the minimal surface the most ordinary choice for the action is the Nambu-Goto action[3], which is a natural functional of $X^\mu(s)$ and $Y^i(s)$. From (2.2.12) we see that this action is more suitable for the full Dirichlet boundary conditions. However, we have seen in the previous section that the loop variables $\dot{y}^i$ force 6 Neumann conditions (2.2.5) on the coordinates Y^i. For this reason we consider the Legendre trasform

$$\tilde{\mathcal{L}} = \mathcal{L} - \partial_\sigma \left(P_i Y^i\right) \tag{2.2.13}$$

$$\tilde{A} = A - \oint d\tau P_i Y^i \tag{2.2.14}$$

where P_i are the conjugate momenta to Y^i

$$P_i = \frac{\delta A}{\delta \partial_\sigma Y^i} = \frac{1}{2\pi\sqrt{\lambda}} \sqrt{g} g^{\sigma\alpha} \partial_\alpha Y^j G_{ij} \; . \tag{2.2.15}$$

It is simple to show that $\tilde{A}$ is actually a functional of (X^μ, P_i). Let us consider a variation of the area A under a general variation of the Y^i's

$$\begin{aligned} \delta A &= \int d\tau d\sigma \left(\frac{\delta A}{\delta Y^i} - \partial_\alpha \frac{\delta A}{\delta \partial_\alpha Y^i} \right) \delta Y^i(\tau, \sigma) + \oint d\tau \frac{\delta A}{\delta \partial_\sigma Y^i} \delta Y^i(\tau, 0) \\ &= \oint d\tau P_i(\tau, 0) \delta Y^i(\tau, 0) \; . \end{aligned} \tag{2.2.16}$$

[3] In the following chapters we will use the Polyakov action, which is classically equivalent to the Nambu-Goto action[32].

We now implement the Legendre transform, obtaining

$$\delta\tilde{A} = -\oint d\tau Y^i(\tau,0)\delta P_i(\tau,0)\ . \qquad (2.2.17)$$

Hence $\tilde{\mathcal{A}}$ and, as a consequence, $\langle W\rangle$ are functionals of the momenta P_i at the boundary.

The Neumann conditions (2.2.5) turn into conditions on the momenta P_i

$$\frac{\dot{y}^i}{2\pi} = P^i = Y^2 P_i\ . \qquad (2.2.18)$$

We assume here the loop variables $\dot{y}^i(s)$ to be continuous. As a consequence, from (2.2.12) and (2.2.11) we see that the Y^i's and the P_i's are parallel to each others. Within this assumption, we evaluate the Legendre transform

$$\tilde{A} = A - \frac{1}{2\pi}\oint d\tau\frac{\dot{y}^i}{Y^2}Y^i = A - \frac{1}{2\pi}\oint d\tau\frac{|\dot{y}|}{Y} = A - \frac{1}{2\pi\epsilon}\oint ds|\dot{y}|\ . \qquad (2.2.19)$$

In the last equality we have set $Y=\epsilon$, where ϵ is a regulator. In such a way, we have obtained the regularize area for $Y\geq\epsilon$.

It is straightforward to show that the divergent piece of the area of the minimal surface is proportional to the circumference of the loop L, that is, the boundary of the surface. We consider the curve $\mathcal{C}$ parameterized by $(x^\mu(\tau), y^i(\tau))$ to run in the bulk of $AdS_5\times S^5$ and then we take the projection into the boundary by letting $y^i\to 0$. In fact, we parameterize the coordinates as follow

$$y^i(\tau) = \theta^i\epsilon,\quad Y^i(\tau,y) = y\theta^i,\quad X^\mu(\tau,y) = x^\mu(\tau) + \mathcal{O}(y^2) \qquad (2.2.20)$$

where ϵ is the same regulator as in (2.1.4).

Now we are able to extract the divergent piece from the Nambu-Goto action

$$\begin{aligned}
\mathrm{A}(C) &= \frac{1}{2\pi}\int d\tau d\sigma\,\frac{1}{Y^2}\sqrt{\det_{ab}\langle\partial_a X^\mu\partial_b X^\mu + \partial_a Y^i\partial_b Y^i} = \\
&= \frac{1}{2\pi}\int d\tau\int_\epsilon dy\,\frac{1}{y^2}\sqrt{\dot{X}^2 + \dot{X}^2 X'^2 - (\dot{X}\cdot X')^2} \\
&= \frac{1}{2\pi}\int d\tau\int_\epsilon\frac{dy}{y^2}\langle\sqrt{\dot{x}^2} + O(y^2) = \\
&= \frac{1}{2\pi\epsilon}L(C) + \text{finite terms}\ . \qquad (2.2.21)
\end{aligned}$$

If we merge (2.2.21) and (2.2.19), we obtain

$$\tilde{A} = \frac{1}{2\pi\epsilon}\oint ds(|\dot{x}| - |\dot{y}|) + \text{finite terms}\ . \qquad (2.2.22)$$

Hence, whenever the constraint $\dot{x}^2 = \dot{y}^2$ is fulfilled, the linear divergence vanishes, and such loops, if smooth, do not require regularization. However, if the loop has a cusp or an intersection, the linear divergence still cancels, but it arise a logarithmic divergence

$$\tilde{A} \sim -\frac{1}{2\pi} \log \frac{L}{\epsilon} + \text{finite terms} \; . \qquad (2.2.23)$$

In fact, the constraint $\dot{x}^2 = \dot{y}^2$ is not satisfied either at a cusp or at an intersection point [31].

2.3 Supersymmetric Wilson loops

The Euclidean action for the $\mathcal{N} = 4$ Super Yang-Mills theory is unique and given by

$$S = \frac{1}{g^2} \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i{}^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 + \bar{\Psi} \Gamma^\mu D_\mu \Psi + i \bar{\Psi} \Gamma^i [\Phi_i, \Psi] \right\} \qquad (2.3.1)$$

where $\Gamma^M = (\Gamma^\mu, \Gamma^i)$ are ten-dimensional Dirac matrices. The gauge fields A_μ, the six scalars Φ_i ($i = 4 \dots 9$) and the four Majorana fermions Ψ^A are all in the adjoint representation of $SU(N)$, and we put fermions into a single Majorana-Weyl spinor of $Spin(9,1)$.

The supersymmetry transformations of the bosonic field are

$$\delta_\epsilon A_\mu = \bar{\Psi} \Gamma^\mu \epsilon \qquad (2.3.2)$$

$$\delta_\epsilon \Phi_i = \bar{\Psi} \Gamma^i \epsilon \qquad (2.3.3)$$

where the parameter of transformation ϵ is a ten-dimensional Majorana-Weyl spinor.

The supersymmetry variation of the Wilson loop dual to the string in AdS space is

$$\delta_\epsilon W = \frac{1}{N} \mathrm{Tr} \mathcal{P} \int ds \, \bar{\Psi} (i \Gamma^\mu \dot{x}^\mu + \Gamma^i \theta^i |\dot{x}|) \epsilon \, \exp \int ds' \, (i A_\mu \dot{x}^\mu + \Phi_i \theta^i |\dot{x}|) \; . \qquad (2.3.4)$$

In order to preserve some part of the supersymmetry we require that

$$(i \Gamma^\mu \dot{x}^\mu + \Gamma^i \theta^i |\dot{x}|) \epsilon = 0 \; . \qquad (2.3.5)$$

This equation has eight independent solution for every given s, due to the fact that the combinations of Dirac matrices between brackets squares to zero. Since these solutions, in general, will depend on s, the Wilson loop is only locally supersymmetric, but the action is not locally symmetric. Then we must require ϵ to be s-independent, and this will turn into a constraint on $x^\mu(s)$ and $\theta^i(s)$. In this picture, the number of conserved supercharges will be the same as the number of linearly independent supersymmetry parameters ϵ that satisfy (2.3.5).

Let us begin with the simplest case, namely when we take θ^i to be a constant. Then it is simple to see that the only solution to eq. (2.3.5) is when the contour in $\mathbb{R}^4$ is a straight line. To show that, we choose the parameterization of the contour such that $|\dot{x}| = 1$ and we differentiate (2.3.5)

$$i\Gamma^\mu \ddot{x}^\mu \,\epsilon = 0 \tag{2.3.6}$$

which implies $\ddot{x} = 0$ identically.

Although for a general loop with a curved contour in $\mathbb{R}^4$ and non constant θ^i, (2.3.5) translates in an high redundant number of equation for 16 unknown quantities, it is possible to find non trivial solutions implementing a simple ansatz. We require that the loop on S^5 is given by the tangent vector $\dot{x}^\mu$, which actually lives on S^3. Therefore, we need a map from S^3 to S^5, which we define through an immersion of $\mathbb{R}^4$ in $\mathbb{R}^6$

$$x^\mu \longmapsto x^\mu M^i_\mu \tag{2.3.7}$$

where the matrix M^i_μ are a sort of projection operator[4]

$$W_s(C) = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\oint_C dx^\mu\,(iA_\mu + M^i_\mu \Phi_i)\;. \tag{2.3.8}$$

Then, our ansatz reads

$$\theta^i = M^i_\mu \frac{\dot{x}^\mu}{|\dot{x}|} \tag{2.3.9}$$

and the Wilson loop is given by

$$W = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\oint_C dx^\mu\,(iA_\mu + M^i_\mu \Phi_i)\;. \tag{2.3.10}$$

The supersymmetry equation (2.3.5) for this operator is

$$i\dot{x}^\mu(\Gamma^\mu - iM^i_\mu\Gamma^i)\epsilon = 0\;. \tag{2.3.11}$$

It is straightforward to note that all the s-dependence disappears, and we have to deal with four algebraic equations

$$(\Gamma^\mu - iM^i_\mu\Gamma^i)\epsilon = 0\;. \tag{2.3.12}$$

[4]An explicit form of M^i_μ is not required because of $SO(4)\times SO(6)$ global symmetry of the theory.

In order to solve these equations, let us define five pair of creation and annihilation operators

$$a^\mu = \frac{1}{2}(\Gamma^\mu - iM^i_\mu \Gamma^i) \tag{2.3.13}$$

$${a_\mu}^\dagger = \frac{1}{2}(\Gamma^\mu + iM^i_\mu \Gamma^i) \tag{2.3.14}$$

$$a^4 = \frac{1}{2}(v^i_1 \Gamma^i - iv^i_2 \Gamma^i) \tag{2.3.15}$$

$${a_4}^\dagger = \frac{1}{2}(v^i_1 \Gamma^i + iv^i_2 \Gamma^i) \tag{2.3.16}$$

where the vectors $v_{1,2}$ are taken as

$$M^i_\mu v^i_{1,2} = 0, \qquad v^2_{1,2} = 1 \ . \tag{2.3.17}$$

These operators satisfy anti-commutation relations

$$\{a^M, {a_N}^\dagger\} = \delta^M_N \ . \tag{2.3.18}$$

Using this Fock space representation we can rewrite (2.3.12) as

$$a^\mu |\epsilon\rangle = 0, \qquad \mu = 0 \ldots 3 \ . \tag{2.3.19}$$

There are two states, with opposite chirality, that are solutions of the above equation

$$|\epsilon_+\rangle = a^\dagger_0 \ldots a^\dagger_3 |0\rangle \tag{2.3.20}$$

$$|\epsilon_-\rangle = a^\dagger_0 \ldots a^\dagger_3 a^\dagger_4 |0\rangle \ . \tag{2.3.21}$$

In general, there exists only one Weyl spinor that satisfy the supersymmetry equation (2.3.12) and the Wilson loop commutes with only one of the sixteen supercharges. As a consequence, the operator defined in (2.3.10) is supersymmetric and it will be 1/16 BPS.

We can also obtain Wilson loop more supersymmetric if we put some constraint on the shape of the contour C. For example, a three dimensional slice can be defined by the condition $x^0 = 0$. It follows that only three of (2.3.12) must be imposed in this case, and the state $|\epsilon\rangle$ solution of (2.3.19) will be annihilated only by three oscillators. In this way, we get additional solutions to (2.3.20), given by

$$|\epsilon^{(1)}_+\rangle = a^\dagger_0 \ldots a^\dagger_3 |0\rangle \tag{2.3.22}$$

$$|\epsilon^{(1)}_-\rangle = a^\dagger_0 \ldots a^\dagger_3 a^\dagger_4 |0\rangle \tag{2.3.23}$$

$$|\epsilon^{(2)}_+\rangle = a^\dagger_1 \ldots a^\dagger_3 |0\rangle \tag{2.3.24}$$

$$|\epsilon^{(2)}_-\rangle = a^\dagger_1 \ldots a^\dagger_3 a^\dagger_4 |0\rangle \ . \tag{2.3.25}$$

In this three dimensional case the Wilson loop commutes with two supercharges, therefore it will be 1/8 BPS. Following the same reasoning, we find that if the contour C lies in a two-dimensional slice the loop will commutes with four supercharges and will preserve 1/4 of the supersymmetry. Finally, if we constraint the contour C to be unidimensional we fall back in the famous case of the Wilson line with constant scalar coupling, which is known to be 1/2 BPS.

2.4 More BPS Wilson loops

The naive BPS Wilson loops described in the last section have trivial expectation values [20], and they are annihilated by combinations of Poincaré supercharges Q_α alone.

Nevertheless, there exist other types of Wilson loop not captured in this class. For example, an interesting operator is the circle with a coupling to a single scalar. Its corresponding Wilson loop has a non-trivial expectation value, but it preserves 1/2 of the supersymmetries. Therefore it cannot be obtained by the above construction. Moreover, it has been found an entire class of BPS Wilson loops which have non-unit expectation value in [33, 21]. The supersymmetries preserved by this type of loops with non-trivial expectation value always include combinations of superconformal generators S_α with the usual Poincaré supercharges Q_α. Explicitly, in this case, the charges consist of a combination of the type

$$\bar{\mathcal{Q}}^a = \varepsilon^{\dot{\alpha}\dot{a}} \left(\bar{Q}^a_{\dot{\alpha}\dot{a}} - \bar{S}^a_{\dot{\alpha}\dot{a}} \right) \ . \tag{2.4.1}$$

The specific way in which they are constructed is by the twisting of the supersymmetry group in order to couple the Wilson loop to the scalars in a defined scheme.

It is believed that when the shape of these loops is contained in two-sphere, the expectation value can be computed exactly in field theory by a gaussian matrix model. This result has recently been proved for a particular loop in this class, the circular Wilson loop[34]. We only recall here that the basic ingredients in this construction are the invariant one-forms on the group manifold $SU(2) = S^3$, in fact these loops generically lie on a S^3. We will see that there are different interesting subclasses of such a construction, and we will use them to test the $\mathcal{N} = 1$ β-deformed SYM theory in chapter 4.

Let us review in more detail one of the most studied examples.

2.4.1 Circular loop

It is worth to propose here the circular Wilson loop because it provided historically a direct test of AdS/CFT correspondence. In this case Erickson, Semenoff and Zarembo in [8] showed that all Feynman diagrams containing interaction vertices cancel up two loops in the perturbative computation of the expectation value: they conjectured that just planar diagrams with no internal vertices contribute at any order of perturbation theory and they were able to sum them. They observed that each planar diagram without internal vertices gives an identical contribution to the Wilson loop expectation value. Since each propagator just yields a constant [9] the sum is equivalent to a calculation in a 0-dimensional field theory, namely a matrix model

$$\langle W_{circle} \rangle = \left\langle \frac{1}{N} \mathrm{Tr} \exp M \right\rangle = \frac{1}{Z} \int \mathrm{D}M \frac{1}{N} \mathrm{Tr} \exp(M) \exp\left(-\frac{2N}{\lambda} \mathrm{Tr} M^2 \right) \tag{2.4.2}$$

that can be solved exactly in an expansion in powers of $1/N^2$. The result is

$$\begin{aligned} \langle W_{circle} \rangle &= \frac{1}{N} L^1_{N-1} \left(-\frac{\lambda}{4N} \right) \exp\left(\frac{\lambda}{8N} \right) \\ &= \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} I_4(\sqrt{\lambda}) + \ldots \end{aligned} \tag{2.4.3}$$

where

$$L^m_n(x) = \frac{1}{N!} L^1_{N-1} \left(-\frac{\lambda}{4N} \right) \exp\left(\frac{\lambda}{8N} \right)$$

are the Laguerre polynomials. Taking the large $\lambda = g^2 N$ limit we have

$$\langle W_{circle} \rangle = \frac{e^{\sqrt{\lambda}}}{(\pi/2)^{1/2} \lambda^{3/4}} \, . \tag{2.4.4}$$

The result of the gauge field theory precisely agrees with the string computation in the same limit, which we will treat explicitly in section 4.2

$$\langle W_{circle} \rangle_{AdS/CFT} \simeq e^{\sqrt{\lambda}} \, . \tag{2.4.5}$$

This is a strong evidence for the AdS/CFT correspondence, since the circular Wilson loop has a non-trivial dependence on the coupling constant λ.

Chapter 3

$\mathcal{N} = 1$ β-deformed SYM theory

Let us start by considering the $\mathcal{N} = 4$ supersymmetric gauge theory. Using the $\mathcal{N} = 1$ superfield formulation, the superpotential of the theory is given by

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \tag{3.0.1}$$

where the Φ's are chiral superfields in the adjoint representation of the gauge group. In the course of this section, it will be useful work in components, so we write the $\mathcal{N} = 4$ SYM Lagrangian

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f \tag{3.0.2}$$

where $\mathcal{L}_b$ stands for the bosonic part of the Lagrangian

$$\begin{aligned} \mathcal{L}_b \;=\; \mathrm{Tr}\Big(&\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\bar{\Phi}^i)(D_\mu\Phi_i) \\ &-\frac{g^2}{2}[\Phi_i,\Phi_j][\bar{\Phi}^i,\bar{\Phi}^j] + \frac{g^2}{4}[\Phi_i,\bar{\Phi}^i][\Phi_j,\bar{\Phi}^j]\Big) \end{aligned} \tag{3.0.3}$$

and $\mathcal{L}_f$ for the fermionic one

$$\begin{aligned} \mathcal{L}_f \;=\; \mathrm{Tr}\Big(&\lambda_A\sigma^\mu D_\mu\bar{\lambda}^A - ig([\lambda_4,\lambda_i]\bar{\Phi}^i + [\bar{\lambda}^4,\bar{\lambda}^i]\Phi_i) \\ &+\frac{ig}{2}(\epsilon^{ijk}[\lambda_i,\lambda_j]\Phi_k + \epsilon_{ijk}[\bar{\lambda}^i,\bar{\lambda}^j]\bar{\Phi}^k)\Big)\,. \end{aligned} \tag{3.0.4}$$

In our convenction Φ_i $i = 1, \ldots, 3$ are complex scalar field components of the chiral superfield defined above, while $\lambda_A = (\lambda_i, \lambda_4)$ where λ_i are the fermionic superpartner of the Φ's and λ_4 is the fermionic superpartner of the gluon. The gauge group here and throughout this section is taken to be $SU(N)$.

In the next section we describe marginal deformations of this theory.

3.1 Marginal deformations

It is possible to generalize $\mathcal{N}=4$ SYM and obtain a larger class of finite 4-dimensional field theories which preserve only $\mathcal{N}=1$ supersymmetry, by implementing a marginal deformation of the superpotential

$$\begin{aligned} \mathcal{W}_{\mathcal{N}=4} &= g\mathrm{Tr}(\Phi^1[\Phi^2,\Phi^3]) \\ \longrightarrow \mathcal{W}_{LS} &= \kappa\mathrm{Tr}\left(\Phi^1[\Phi^2,\Phi^3]_q + \frac{h}{3}\left((\Phi^1)^3+(\Phi^2)^3+(\Phi^3)^3\right)\right) \end{aligned} \tag{3.1.1}$$

where the q deformed commutator is defined as $[\Phi^i,\Phi^j]_q = \Phi^i\Phi^j - q\Phi^j\Phi^i$. Let us take a look at the symmetries: $\mathcal{W}_{\mathcal{N}=4}$ is invariant under $SU(3)\times U(1)_R$, but $\mathcal{W}_{LS}$ generically breaks the $SU(3)$ component to a discrete subgroup. Therefore the replacement (3.1.1) breaks the supersymmetry of the theory from $\mathcal{N}=4$ to $\mathcal{N}=1$. We notice that we are not considering the most general marginal $\mathcal{N}=1$ deformation, but this is a 2-parameter subgroup describing all exactly marginal theories. The proof of the finiteness of the marginally deformed theories has been given by Leigh and Strassler[25]. Indeed, they demonstrated that the condition for finiteness can be encoded in a single function of the constant g,κ,q and h. Although this function is not known in general, at one loop order is given by

$$2g^2 = \kappa\bar{\kappa}\left[\frac{2}{N^2}(1+q)(1+\bar{q}) + \left(1-\frac{4}{N^2}\right)\left(1+q\bar{q}+h\bar{h}\right)\right]. \tag{3.1.2}$$

Generally, this expression is modified, even in the planar limit, at four loop order and higher. A special case is the so called real β deformed theory, relative at $h=0$ and $q=e^{i\beta}$ with β real, which we will describe in more detail in the next section. Indeed, in this case the finiteness condition (3.1.2) is exact to all orders in the planar perturbation theory. Moreover, since (3.1.2) reads $g^2=\kappa\bar{\kappa}$ and it does not depend on q, it is the same for the $\mathcal{N}=4$ SYM.

Let us discuss briefly the symmetries of the full $(q-h)$-deformed theory. In addition to the $U(1)_R$ symmetry preserved by the residual $\mathcal{N}=1$ symmetry, the theory has a remaining $\mathbb{Z}_3\times\mathbb{Z}_3$ symmetry. The action of such symmetries are given by

$$\mathbb{Z}_3^A : \Phi^1\to\Phi^2 \;,\; \Phi^2\to\Phi^3 \;,\; \Phi^3\to\Phi^1 \tag{3.1.3}$$
$$\mathbb{Z}_3^B : \Phi^1\to\Phi^1 \;,\; \Phi^2\to\omega\Phi^2 \;,\; \Phi^3\to\omega^2\Phi^3 \tag{3.1.4}$$
$$U(1)_R : \Phi^1\to\omega\Phi^1 \;,\; \Phi^2\to\omega\Phi^2 \;,\; \Phi^3\to\omega^2\Phi^3 \tag{3.1.5}$$

where $\omega^3=1$. These three symmetries does not commute with each other and combined together they form a trihedral group. We end this short analysis pointing out that the real β-deformed theory actually preserves a larger $U(1)^3$ subgroup of $SU(3)\times U(1)_R$.

3.2 Real β-deformation

Here and throughout the rest of the paper, we will consider only real β-deformations, corresponding to $\bar{q} = 1/q, h = 0$ and $\beta \in \mathbb{R}$. In this setting, the superpotential of the $\mathcal{N} = 4$ SYM (3.0.1) is deformed into a simple expression

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \to ig\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)\ . \quad (3.2.1)$$

In fact, the result of the deformation is the addition of certain phases in the Lagrangian. We have already seen in the previous section that this theory maintains $\mathcal{N} = 1$ supersymmetry and preserves a global $U(1) \times U(1)$ non-R-symmetry. Lunin and Maldacena showed [11] that the deformation (3.2.1) can be seen as arising from a new definition of the product $*$ between fields in the $\mathcal{N} = 4$ SYM Lagrangian

$$f * g \equiv e^{i\pi\beta(Q_1^f Q_2^g - Q_2^f Q_1^g)} fg\ . \quad (3.2.2)$$

From the symmetry transformations (3.1.3) and (3.1.4) it is simple to find the values of the charges for all superfields of the theory

$$\begin{array}{lll} \Phi_1 \ :\ (Q_1\,,Q_2) = (0\,,-1) & & \bar{\Phi}_1\ :\ (Q_1\,,Q_2) = (0\,,1) \\ \Phi_2\ :\ (Q_1\,,Q_2) = (1\,,1) & & \bar{\Phi}_2\ :\ (Q_1\,,Q_2) = (-1\,,-1) \\ \Phi_3\ :\ (Q_1\,,Q_2) = (-1\,,0) & & \bar{\Phi}_3\ :\ (Q_1\,,Q_2) = (1\,,0) \\ & V\ :\ (Q_1\,,Q_2) = (0\,,0) & \end{array}$$

From this it follows that

$$\mathrm{Tr}(\Phi_1 * \Phi_2 * \Phi_3 - \Phi_1 * \Phi_3 * \Phi_2) = \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) \quad (3.2.3)$$

which is exactly the same expression as (3.2.1). Hence, we have showed that indeed the real β-deformed superpotential is caused by the product (3.2.2).

Now we are able to write the component Lagrangian of the real β-deformed theory

$$\begin{aligned} \mathcal{L} = \mathrm{Tr}\Bigg(& \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\bar{\Phi}^i)(D_\mu\Phi_i) - \frac{g^2}{2}[\Phi_i,\Phi_j]_{\beta_{ij}}[\bar{\Phi}^i,\bar{\Phi}^j]_{\beta_{ij}} \\ & + \frac{g^2}{4}[\Phi_i,\bar{\Phi}^i][\Phi_j,\bar{\Phi}^j] + \lambda_A\sigma^\mu D_\mu\bar{\lambda}^A - ig([\lambda_4,\lambda_i]\bar{\Phi}^i + [\bar{\lambda}^4,\bar{\lambda}^i]\Phi_i) \\ & + \frac{ig}{2}(\epsilon^{ijk}[\lambda_i,\lambda_j]_{\beta_{ij}}\Phi_k + \epsilon_{ijk}[\bar{\lambda}^i,\bar{\lambda}^j]_{\beta_{ij}}\bar{\Phi}^k)\Bigg) \end{aligned} \quad (3.2.4)$$

where the β-deformed commutator is defined in the same way as in (3.1.1)

$$[f_i, g_j]_{\beta_{ij}} := e^{i\pi\beta_{ij}}\, f_i g_j - e^{-i\pi\beta_{ij}}\, g_j f_i \quad (3.2.5)$$

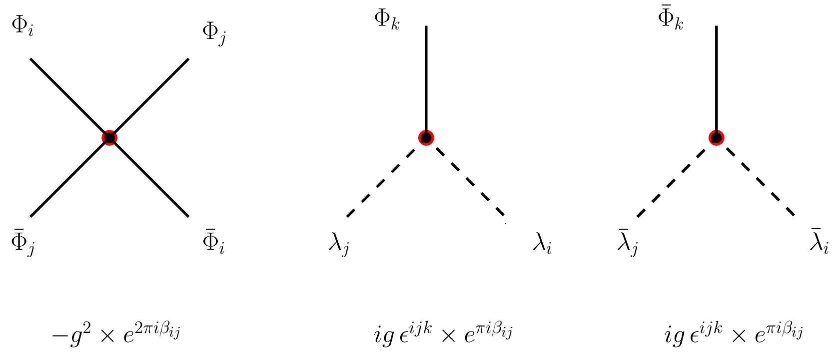

Figure 3.1: *All β-dependent Feynman vertices in perturbation theory.*

and the matrix β_{ij} is defined as

$$\beta_{ij} = -\beta_{ji}\,, \quad \beta_{12} = -\beta_{13} = \beta_{23} := \beta\ . \tag{3.2.6}$$

In Fig. 3.2 we have presented all the β-dependent Feynman vertices. The first is a ϕ^4 interaction and it is given by the third term of (3.2.4) while the other two, coming from the last term of the Lagrangian, are Yukawa interactions.

3.3 Supergravity solution

We have pointed out in the previous section that the β-deformed theory has two $U(1)$ symmetries, therefore let us consider a string theory with such a symmetry realized geometrically. This means that we can take two coordinates φ_1, φ_2 on which the two $U(1)$ symmetries act as shifts. As a result, we obtain a two torus parameterized by the φ's, which will be fibered over an eight dimensional manifold.

Let us consider a closed string theory compactified on a two torus. As a consequence, the remaining eight dimensional theory is symmetric under a $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ transformation, which acts on the complex structure of the torus[1] and on the complex parameter

$$\tau = B_{12} + i\sqrt{g} \tag{3.3.1}$$

where $\sqrt{g}$ represent the volume of the two torus in string metric. If we put ourselves at the supergravity level, the symmetry is enhanced to a full $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, which obviously is not a symmetry of the full string theory. The interesting feature of such a supergravity symmetry is that it can be used as a solution generating transformation [35]. In our case, the

[1]However, this symmetry does not play an important role in our present discussion, therefore we neglect it in the following.

interesting $SL(2,\mathbb{R})$ symmetry is the one which acts as on the parameter defined in (3.3.1) as

$$\tau \to \tau' = \frac{\tau}{1+\gamma\tau} \,. \tag{3.3.2}$$

The transformation (3.3.2) is also called a TsT transformation, since we can think of it as a result of doing a T-duality on one circle, then a shift of the coordinates, and finally another T-duality. We referred to it as a solution generation transformation because after implementing (3.3.2) we get a new solution.

Now we want to convince ourselves that it we start with a non-singular geometry, after applying (3.3.2) we obtain again a non-singular metric. Let us start with a non-singular ten dimensional geometry and suppose that there the B field vanishes when $\tau_2 \to 0$. The original geometry is non-singular if τ_1 tend to an integer when $\tau_2 \to 0$, but we must avoid the case in which τ_1 goes to different integers if there are different regions where $\tau_2 \to 0$. These are the assumptions we have to require in order for the new geometry to be non-singular. Indeed, it is possible to insert a singularity by means of an $SL(2,\mathbb{R})$ only in the points where the torus shrinks to zero size, when both τ_2 and τ_1 tend to zero. Therefore, by (3.3.2), $\tau' = \tau$ for small τ, and, as a consequence, the region near the possible singularity becomes equal to what it was before the transformation. Hence, the new geometry is still non-singular, and it shows that the topology of the solution remains the same.

Let us consider a D-brane, invariant under the $U(1) \times U(1)$ symmetry in the original geometry. After applying (3.3.2) we obtain a corresponding brane in the new geometry. Now we want to find out what field theory lives on this transformed brane. The conjecture pointed out in [11] states that the open string field theory on the brane living in the new geometry arise by redefining the star product

$$f *_\gamma g \equiv e^{i\pi\gamma(Q^1_f Q^2_g - Q^2_f Q^1_g)} f *_0 g \tag{3.3.3}$$

when $*_0$ is the product between fields in the open string field theory in the original background. Comparing this to (3.2.2), $Q^i_{f,g}$ are the $U(1)$ charges of the fields f and g respectively. If we consider a open string field theory with a B field turned on, the effective metric is the sum of the open string metric and a non-commutativity parameter

$$G^{ij}_{open} + \Theta^{ij} = \left(\frac{1}{g+B}\right)^{ij} \sim \frac{1}{\tau} \,. \tag{3.3.4}$$

If we perform the transformation (3.3.2), we have

$$\frac{1}{\tau} \to \frac{1}{\tau'} = \frac{1}{\tau} + \gamma \,. \tag{3.3.5}$$

Hence, the effect of the deformation is the appearance of a non-commatutavity parameter $\Theta^{12} = \gamma$, namely, the B field is turned on in the new background, while the open string metric remains the same.

Let us now examine branes located at the origin of the space, that is the point where both circles reduce to zero size. Since the $U(1)$ directions are global symmetries of the field theory, for such a brane the star product (3.3.3) does not lead to a non-commutative field theory at low energies. All that happens applying (3.3.3) to the low energy conventional field theory living on the brane is that we obtain another conventional field theory with some phases in the Lagrangian. However, we might be concerned by the fact that we are deriving the theory on the brane located at a point where the volume of the torus shrinks to zero, $\sqrt{g} = 0$. A simple argument in favor of this procedure is realized considering a $D(p+2)$ brane anti-brane system, wrapped along the two torus, with a magnetic flux of their worldvolume on the two torus in order to have net Dp brane charge. These branes can annihilate via tachyon condensation to form the Dp brane at the origin and the brane anti-brane system can be located far from the origin. This process of tachyon condensation is insensitive to the Θ parameter in (3.3.4), so it proceeds in the same way in the theory after the $SL(2,\mathbb{R})$ transformation as in the theory before the transformation. The net result is that we obtain the same field theory on the Dp brane at the origin, but with the extra phases (3.3.3).

A fundamental issue is to establish whether of not the deformation (3.3.3) preserves supersymmetry. Let us start with the original ten dimensional background possessing a supersymmetry invariant under $U(1) \times U(1)$. As a consequence, the deformed background will also be invariant under the same supersymmetry. In our case, in which we have a D3-brane at the origin, the theory is $\mathcal{N} = 1$ supersymmetric.

So far we have analyzed the theory on the brane and how it is modified after performing the deformation (3.3.2). Now we want to discuss the gravity dual of these theories. The main concept is very simple: if we know the gravity dual of the field theory living an a D-brane in the original background, then the gravity dual of the deformed field theory living on the D-brane on the new background is given by performing the $SL(2,\mathbb{R})$ transformation (3.3.2) on the original solution.

Now we will show explicitly how to get the exact gravity dual solution for β-deformation, so we set $\gamma = \beta$ and we keep it real.

We start by writing the metric[2] of S^5

$$\frac{ds^2}{R^2} = \sum_{i=1}^{3} d\mu_i^2 + \mu_i^2 d\phi_i^2 \ , \qquad \text{with} \quad \sum_i \mu_i^2 = 1 \tag{3.3.6}$$

[2]It is clear form the above discussion that the AdS_5 part of the space will be unmodified by the deformation.

$$
\begin{aligned}
\frac{ds^2}{R^2} &= d\alpha^2 + s_\alpha^2 d\theta^2 + c_\alpha^2 (d\psi - d\varphi_2)^2 \\
&\qquad + s_\alpha^2 c_\theta^2 (d\psi + d\varphi_1 + d\varphi_2)^2 + s_\alpha^2 s_\theta^2 (d\psi - d\varphi_1)^2 \\
&= d\alpha^2 + s_\alpha^2 d\theta^2 + \frac{9 c_\alpha^2 s_\alpha^2 s_{2\theta}^2}{4 c_\alpha^2 + s_\alpha^2 s_{2\theta}^2} d\psi^2 + s_\alpha^2 \left[d\varphi_1 + c_\theta^2 d\varphi_2 + c_{2\theta} d\psi \right]^2 \\
&\qquad + (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2) \left[d\varphi_2 + \frac{(-c_\alpha^2 + 2 s_\alpha^2 s_\theta^2 c_\theta^2)}{c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2} d\psi \right]^2 .
\end{aligned} \tag{3.3.7}
$$

The coordinate φ_1 and φ_2 are shifted by the action of the two $U(1)$ symmetries (3.1.3), (3.1.4). Therefore the metric of the two torus is given by the last line of (3.3.7), and we can evaluate the τ parameter of this two torus

$$
\tau = i\sqrt{g_0} = i[R^2 s_\alpha^2 (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2)]^{1/2} = iR(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)^{1/2} \tag{3.3.8}
$$

where $R = (4\pi g_s N)^{1/4}$. At this point we apply the transformation (3.3.2) and we obtain the solution corresponding to the gravity dual of the β-deformed theory in the string frame

$$
ds_{str}^2 = R^2 \left[ds_{AdS_5}^2 + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G \mu_1^2\mu_2^2\mu_3^2 (\sum_i d\phi_i)^2 \right] \tag{3.3.9}
$$

$$
\begin{aligned}
G^{-1} &= 1 + \hat{\gamma}^2 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2) && (3.3.10) \\
e^{2\phi} &= e^{2\phi_0} G, \qquad \hat{\gamma} = R^2\gamma, \qquad R^4 \equiv 4\pi e^{\phi_0} N && \\
B^{NS} &= \hat{\gamma} R^2 G (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) && (3.3.11) \\
C_2 &= -3\gamma(16\pi N) w_1 d\psi \ , \qquad \text{with} \quad dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha d\theta && \\
C_4 &= (16\pi N)(w_4 + G w_1 d\phi_1 d\phi_2 d\phi_3) && \\
F_5 &= (16\pi N)(\omega_{AdS_5} + G\omega_{S^5}) && (3.3.12) \\
\omega_{S^5} &= dw_1 d\phi_1 d\phi_2 d\phi_3 \ , \qquad \omega_{AdS_5} = dw_4 &&
\end{aligned}
$$

where ω_{S^5} is the volume element of a unit radius S^5.

The regime of validity for this solution are given by

$$
R\gamma \ll 1 \ , \qquad R \gg 1 \ . \tag{3.3.13}
$$

The first inequality represents the condition that the size of the two torus does not become smaller than the string scale after the transformation.

3.4 TsT of D-branes

In this section our study focuses more precisely on D-brane in the Lunin-Maldacena background. The idea is to consider a specific D-brane in the undeformed background and to map it via (3.3.2) to the corresponding configuration in the deformed geometry. We note that this can be made explicitly because the transformation (3.3.2) that led to the deformed solution (3.3.9) is equivalent to a TsT transformation of the undeformed $AdS_5 \times S^5$ solution (3.3.6).

Let us review how the strategy operates from the TsT point of view. To avoid confusion, we use the tilde to indicate coordinates in the undeformed background. We start by redefining the coordinates which parameterize the next-to-be deformed part of the sphere in the original $AdS_5 \times S^5$ by the relations

$$\tilde{\phi}_1 = \tilde{\varphi}_3 - \tilde{\varphi}_2\,, \qquad \tilde{\phi}_2 = \tilde{\varphi}_3 + \tilde{\varphi}_1 + \tilde{\varphi}_2\,, \qquad \tilde{\phi}_3 = \tilde{\varphi}_3 - \tilde{\varphi}_1\,. \tag{3.4.1}$$

Performing a T-duality along φ_1 direction takes to a type IIA solution, whose coordinates we indicate as $\tilde{\tilde{\varphi}}_i$. In this background, we perform a shift $\tilde{\tilde{\varphi}}_2 \to \tilde{\tilde{\varphi}}_2 + \hat{\gamma}\tilde{\tilde{\varphi}}_1$ and, finally, we return to a type IIB background by acting with another T-duality along $\tilde{\tilde{\varphi}}_1$.

Now we concentrate on the main goal of this section, that is, how boundary conditions for open strings modify under the TsT transformation outlined just above. The first step is write down the boundary condition obtained by performing a T-duality along the X^1 direction of a background with metric $G_{\mu\nu}$ and NSNS two-form $B_{\mu\nu}$ in terms of the dual coordinate $\tilde{X}^1 = 1/X^1$

$$\epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 = \eta^{\alpha\beta} G_{1a}\partial_\beta X^a - \epsilon^{\alpha\beta} B_{1a}\partial_\beta X^a \tag{3.4.2}$$

where the worldsheet coordinates are $\alpha, \beta = (\tau, \sigma)$, $\eta_{\alpha\beta}$ is the worldsheet metric and we have defined $\epsilon^{\tau\sigma} = +1$. In order to find a relation between the coordinates of the undeformed and the deformed background we use (3.4.2) twice and take into account the effect of the shift

$$\tilde{G}_{\tilde{\phi}_i\tilde{\phi}_j}\partial_\alpha \tilde{\phi}_j = G_{\phi_i\phi_j}\partial_\alpha \phi_j - \eta_{\alpha\beta}\epsilon^{\beta\kappa} B_{\phi_i\phi_j}\partial_\kappa \phi_j \tag{3.4.3}$$

where the label $i, j = 1, 2, 3$.

In the original undeformed background the $\tilde{B}_{\mu\nu}$ vanishes and we consider D-branes without any world-volume flux turned on along the $\tilde{\phi}_i$ directions. The boundary conditions for open strings ending on the brane along a $\tilde{\phi}_i$ direction are

$$\begin{aligned} \partial_\sigma \tilde{\phi}_i &= 0 \qquad \text{(Neumann)} \\ \partial_\tau \tilde{\phi}_i &= 0 \qquad \text{(Dirichlet)}\,. \end{aligned} \tag{3.4.4}$$

In our analysis, we will consider only the direction ϕ_i of the five-sphere, since the other coordinate are not modified by the TsT transformation. As we can see in (3.3.9), in the undeformed background the NSNS two-form $B_{\mu\nu}$ is no more vanishing and, in addition, we will consider a non-zero world-volume flux F_{ab}. Then, the boundary conditions are given by

$$\begin{aligned} G_{\phi_i\phi_j}\partial_\sigma\phi_j + \left(B_{\phi_i\phi_j} - 2\pi F_{\phi_i\phi_j}\right)\partial_\tau\phi_j &= 0 \qquad \text{(mixed Neumann)} \\ \partial_\tau\phi_i &= 0 \qquad \text{(Dirichlet)}\ . \end{aligned} \tag{3.4.5}$$

So far we have established that D-branes in the undeformed background are characterized by boundary conditions (3.4.4), while D-branes in the deformed background are characterized by boundary conditions (3.4.5). Now we want to map the branes in the two geometries by using of the transformation (3.4.3). Let us take into account all the four possible cases.

NNN conditions

Let us take Neumann boundary conditions along all three coordinates $\tilde{\phi}_i$ in the undeformed background. Applying the transformation (3.4.3) to the first of (3.4.4) we find

$$G_{\phi_i\phi_j}\partial_\sigma\phi_j + B_{\phi_i\phi_j}\partial_\tau\phi_j = 0\,. \tag{3.4.6}$$

This relation matches with the first of the (3.4.5) if we set $F_{\phi_i\phi_j} = 0$, i.e., no world-volume flux is turned on. As a result, if we start with Neumann boundary conditions along all of the $\tilde{\phi}_i$ directions in the undeformed background, we end up with all Neumann boundary conditions along the ϕ_i directions in the Lunin-Maldacena background with no world-volume flux.

DNN conditions

Now we consider a Dirichlet boundary condition along $\tilde{\phi}_1$ and Neumann boundary condition along the remaining. In the same way, we apply the transformation (3.4.3) to (3.4.4) and we obtain

$$B_{\phi_1\phi_j}\partial_\sigma\phi_j + G_{\phi_1\phi_j}\partial_\tau\phi_j = 0\,. \tag{3.4.7}$$

This relation is not explicitly of the same form as the boundary condition in (3.4.5). But, if we use (3.4.6) twice (for ϕ_2 and ϕ_3), (3.4.7) turns into a pure Dirichlet boundary condition $\partial_\tau\phi_1 = 0$. Therefore, as the same way as in the previous case, the boundary conditions are mapped without changes from the undeformed background to the deformed one, with no world-volume flux turned on.

NDD conditions

Let us take in account the more interesting case in which we have two Dirichlet and one Neumann boundary conditions in the undeformed background. This leads to the equations

$$\begin{cases} G_{\phi_1\phi_j}\partial_\sigma\phi_j + B_{\phi_1\phi_j}\partial_\tau\phi_j = 0 \\ B_{\phi_2\phi_j}\partial_\sigma\phi_j + G_{\phi_2\phi_j}\partial_\tau\phi_j = 0 \\ B_{\phi_3\phi_j}\partial_\sigma\phi_j + G_{\phi_3\phi_j}\partial_\tau\phi_j = 0 \end{cases} \quad . \tag{3.4.8}$$

Using (3.3.11) and (3.3.11), the system reduces to

$$\begin{cases} G_{\phi_1\phi_j}\partial_\sigma\phi_j + B_{\phi_1\phi_j}\partial_\tau\phi_j = 0 \\ G_{\phi_2\phi_j}\partial_\sigma\phi_j + (B_{\phi_2\phi_j} - \frac{R^2}{\hat{\gamma}})\,\partial_\tau\phi_j = 0 \\ G_{\phi_3\phi_j}\partial_\sigma\phi_j + (B_{\phi_3\phi_j} + \frac{R^2}{\hat{\gamma}})\,\partial_\tau\phi_j = 0 \end{cases} . \tag{3.4.9}$$

It is clear that for the resulting D-brane these represents all Neumann boundary conditions in the deformed background. Moreover, the world-volume has increased its dimension by two and now wraps a two-torus spanned by the coordinates ϕ_2 and ϕ_3. Furthermore, there is a world-volume flux along the torus directions:

$$F_{\phi_2\phi_3} = \frac{1}{2\pi}\frac{R^2}{\hat{\gamma}} = \frac{1}{2\pi}\frac{1}{\gamma}\,. \tag{3.4.10}$$

We notice that since the world-volume is turned on a compact two-torus, it must obey a quantization condition, which is satisfied when γ is rational, $\gamma = \frac{m}{n}$ where m, n are integers[3].

DDD conditions

The last configuration is the case of all Dirichlet boundary conditions. Let us switch to coordinates φ_i defined in (3.4.1). The resulting system of equation reads

$$\begin{cases} G_{\varphi_1\varphi_j}\ \partial_\sigma\varphi_j + (B_{\varphi_1\varphi_j} - \frac{R^2}{\hat{\gamma}})\,\partial_\tau\varphi_j = 0 \\ G_{\varphi_2\varphi_j}\ \partial_\sigma\varphi_j + (B_{\varphi_2\varphi_j} + \frac{R^2}{\hat{\gamma}})\,\partial_\tau\varphi_j = 0 \\ \partial_\tau\varphi_3 = 0 \end{cases} . \tag{3.4.11}$$

We see that the two initial Dirichlet conditions for φ_1 and φ_2 become Neumann after the transformation, while φ_3 simply remains Dirichlet. As a consequence, the brane wraps again a two-torus, but this time it is parameterized by φ_1 and φ_2, and, similarly, the world-volume flux along the torus is

$$F_{\varphi_1\varphi_2} = \frac{1}{2\pi}\frac{R^2}{\hat{\gamma}} = \frac{1}{2\pi}\frac{1}{\gamma}\,. \tag{3.4.12}$$

Still, the reasoning about the quantization of the world-volume field strength is the same as in the previous case, so the requirement is that γ has to be rational.

3.5 β-deformed Wilson loops

We have already pointed out that Wilson loops in certain representations of the gauge group are dual to D-brane configurations in the bulk geometry.

[3]For more generic value of γ, i.e. if γ is not rational, the brane must sit at special points where the would-be Neumann directions shrink and the brane world-volume effectively loses two directions.

Our aim in this section is to study such objects in the Lunin-Maldacena background.

For $\mathcal{N} = 4$ SYM, n [36, 37, 38] it has been shown that for type IIB string theory on $AdS_5 \times S^5$, there are three objects preserving the same supersymmetry and the same global symmetry $SU(1,1) \times SU(2) \times SO(5)$ as the straight line Wilson loop:

- Fundamental string with AdS_2 world-sheet,
- D5-brane with $AdS_2 \times S^4$ world-volume (where $S^4 \subset S^5$),
- D3-brane with $AdS_2 \times S^2$ world-volume (where $S^2 \subset AdS_5$).

The first corresponds to the original prescription by Maldacena [36], while the study of D-brane configurations in the context of Wilson loops has been initiated in [39].

The analysis of [39, 37, 38] shows that an $AdS_2 \times S^2$ D3-brane with k units of world-volume flux along the AdS_2 directions represents a Wilson loop in the k-index symmetric representation, while an $AdS_2 \times S^4$ D5-brane with the same flux represents a Wilson loop in the k-index antisymmetric representation. The proposal of [38] also includes Wilson loops in arbitrary representations, which can be described as collections of either D3 or D5-branes. From this perspective, the expectation value of the loop is computed by evaluating the action of a probe D-brane in the $AdS_5 \times S^5$background.

Our goal is then to perform an analogous study of D-brane configurations which are dual to Wilson loops in (anti)symmetric representations of the gauge group in the $\mathcal{N} = 1$ β-deformed SYM theory. The strategy we follow will be to follow the relevant D-branes of the undeformed $AdS_5 \times S^5$background through the TsT transformation which leads to the LM background, making use of the results we presented in section 3.4.

The main results are that a Wilson loop in the gravity dual of the β-deformed theory with k units of fundamental string charge can be related by the following D-branes:

- D5-brane with $AdS_2 \times \tilde{S}^4$ world-volume (where $\tilde{S}^4$ is a deformed four-sphere inside the deformed S^5),
- D5-brane with $AdS_2 \times S^2 \times T^2$ world-volume (where $S^2 \subset AdS_5$ and there is a world-volume flux $F = \frac{1}{\gamma}$ along the $T^2 \subset S^5$. These configurations only exist for rational values of γ),
- D3-brane with $AdS_2 \times S^2$ world-volume (where $S^2 \subset AdS_5$).

As in the undeformed $AdS_5 \times S^5$case, all of the above D-brane configurations have a world-volume flux along the AdS_2 part of the world-volume that gives the appropriate fundamental string charge.

These D-brane configurations are found to preserve two $U(1)$ symmetries, given by appropriate combinations of the $U(1)_R \times U(1)_1 \times U(1)_2$ symmetry of the theory. The main result we find is that the D-brane probes reproduce the same results one finds in $AdS_5 \times S^5$for Wilson loops in $\mathcal{N} = 4$ SYM.

3.6 Supersymmetry

In this section we want to prove explicitly that the β-deformed SYM theory is effectively $\mathcal{N} = 1$ supersymmetric.

Let us start with the off-shell Lagrangian for the $N = 1$ β-deformed theory

$$\begin{aligned}\mathcal{L}_\beta &= \mathrm{Tr}\Bigg(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \frac{1}{2}D^2 \\ &\quad + \sum_{i=1}^{3}\Bigg(\frac{1}{2}\nabla_\mu A_i \nabla^\mu A_i + \frac{1}{2}\nabla_\mu B_i\nabla^\mu B_i + \frac{i}{2}\bar{\psi}_i\gamma^\mu\nabla_\mu\psi_i \\ &\quad + \frac{1}{2}F_i^2 + \frac{1}{2}G_i^2 - i[A_i.B_i]D - i\bar{\psi}_i[\lambda, A_i] - i\bar{\psi}_i\gamma_5[\lambda, B_i]\Bigg) \\ &\quad -\frac{i}{2}\sum_{ijk}\epsilon_{ijk}\left(\bar{\psi}_i[\psi_j, A_k]_{\beta_{jk}} - \bar{\psi}_i\gamma_5[\psi_j, B_k]_{\beta_{jk}} + [A_i, A_j]_{\beta_{ij}}F_k\right. \\ &\qquad \left. -[B_i, B_j]_{\beta_{ij}}F_k + 2[A_i, B_j]_{\beta_{ij}}G_k\right)\Bigg)\end{aligned} \tag{3.6.1}$$

where the β-deformed commutator is defined as in (3.2.5).

The equations of motion for the auxiliary fields are

$$D = i\sum_{i=1}^{3}[A_i, B_j] \tag{3.6.2}$$

$$F_k = \frac{i}{2}\sum_{ij}\epsilon_{ijk}\left([A_i, A_j]_{\beta_{ij}} - [B_i, B_j]_{\beta_{ij}}\right) \tag{3.6.3}$$

$$G_k = i\sum_{ij}\epsilon_{ijk}[A_i, B_j]_{\beta_{ij}} \ . \tag{3.6.4}$$

We notice that the equation for the D field is exactly the same as in the undeformed case.

We define four Majorana spinors by

$$\begin{aligned}&\lambda_i = \psi_i \qquad \text{for } i = 1, 2, 3 \\ &\lambda_4 = \lambda\end{aligned} \tag{3.6.5}$$

and antisymmetric 4×4 matrices of scalars and pseudoscalars by

$$\begin{aligned}A_{ij} &= -\epsilon A_k; \qquad & B_{ij} &= -\epsilon_{ijk}B_k \\ A_{4i} &= -A_{i4} = A_i; \qquad & B_{4i} &= -B_{i4} = B_i \ .\end{aligned} \tag{3.6.6}$$

Our goal is to see if it is possible to rewrite the deformed Lagrangian defined in (3.6.1) with enhanced supersymmetry. In order to do that we should eliminate the auxiliary fields from (3.6.1) and try to find out an expression for the Lagrangian that involves only $O(4)$ invariant quantities, as the self-dual and anti-self-dual tensors of $O(4)$, respectively A_{ij} and B_{ij}. But a more detailed analysis will show that the Lagrangian is not $O(4)$ invariant. Indeed, in order to express all the terms in the Lagrangian in an explicitly $O(4)$ invariant form we should be dealing with expressions similar to

$$\sum_{ij}\sum_{rsmn}\sum_{tupq}\epsilon_{rsi}\epsilon_{mni}\epsilon_{tuj}\epsilon_{pqj}[A_{rs},A_{mn}]_{\beta_{ij}}[A_{tu},A_{pq}]_{\beta_{ij}}\ . \tag{3.6.7}$$

This expression is easily summed in the undeformed case, where $\beta_{ij}=0\ \forall i,j$, and it yields to

$$4\sum_{rsmn}[A_{rs},A_{mn}][A_{rs},A_{mn}]\ . \tag{3.6.8}$$

However, it is not possible to resolve the sum over i,j in the deformed case, due to the fact that such indices are involved in the definition of the β-deformed commutator.

Now we take a Majorana parameter ξ and we have the following supersymmetry transformations

$$\begin{aligned}
\delta A_\mu &= i\bar{\xi}\gamma_\mu\lambda \\
\delta\lambda &= -\frac{1}{2}i\sigma^{\mu\nu}\xi F_{\mu\nu}-\gamma_5\xi D \\
\delta A_i &= \bar{\xi}\psi_i \\
\delta B_i &= \bar{\xi}\gamma_5\psi_i \\
\delta\psi_i &= -(F_i+\gamma_5 G_i)\xi - i\partial\!\!\!/(A_i+\gamma_5 B_i)\xi
\end{aligned} \tag{3.6.9}$$

where the expression for the auxiliary fields D, F and G are obtained by their equations of motion (3.6.2)-(3.6.4). It is clear that it is not possible to rewrite the supersymmetry transformation (3.6.9) in terms of the $O(4)$ invariant quantities defined above. Thus, no supersymmetric extension is possible for generic real β parameter and the theory is simple $\mathcal{N}=1$ supersymmetric.

3.6.1 Supersymmetric β-deformed Wilson loops

In section (2.3) we have discussed the supersymmetry property of Wilson loops in $\mathcal{N}=4$ super Yang Mills. In particular, we have derived how the shape of the contour influences the BPS property of the operator. Now we want perform a similar analysis for Wilson loops in the β-deformed theory.

Let us consider small β deformation, so that we can write the supersymmetry transformation (3.6.9) at the first order in β parameter expansion

$$
\begin{aligned}
\delta A_\mu &= i\bar{\xi}\gamma_\mu\lambda \\
\delta\lambda &= -\frac{1}{2}i\sigma^{\mu\nu}\xi F_{\mu\nu} - i\gamma_5\xi\sum_{i=1}^{3}[A_i, B_j] \\
\delta A_i &= \bar{\xi}\psi_i \\
\delta B_i &= \bar{\xi}\gamma_5\psi_i \\
\delta\psi_k &= -\frac{i}{2}\sum_{ij}\epsilon_{ijk}([A_i, A_j] - [B_i, B_j] + 2\gamma_5[A_i, B_j])\xi - i\partial\!\!\!/(A_k + \gamma_5 B_k)\xi \\
&\quad +\frac{\pi}{2}\sum_{ij}\epsilon_{ijk}\beta_{ij}\left(\{A_i, A_j\} - \{B_i, B_j\} + 2\gamma_5\{A_i, B_j\}\right)\xi\ .
\end{aligned}
\tag{3.6.10}
$$

Moreover, if one digs a little bit inside the properties of the tensors involved in the last expressions, it should be easy to see that $\epsilon_{ijk}\beta_{ij} = \beta$. The last equation then becomes

$$
\begin{aligned}
\delta\psi_k &= -\frac{i}{2}\sum_{ij}\epsilon_{ijk}([A_i, A_j] - [B_i, B_j] + 2\gamma_5[A_i, B_j])\xi - i\partial\!\!\!/(A_k + \gamma_5 B_k)\xi \\
&\quad +\beta\frac{\pi}{2}\sum_{ij}\left(\{A_i, A_j\} - \{B_i, B_j\} + 2\gamma_5\{A_i, B_j\}\right)\xi\ .
\end{aligned}
\tag{3.6.11}
$$

First, we notice that the supersymmetry transformation of the scalars and boson are unaffected by β terms.

Now, let us recall the definition of the Wilson loop operator

$$
W(C) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left(\oint_C (iA_\mu dx^\mu + |\dot{x}|\theta^i\Phi^i)ds\right) \tag{3.6.12}
$$

where $\Phi^i = (A_j, B_k)$, $i = 1, \ldots, 6$ are the six scalars of the theory. In this expression C is the contour parameterized in $\mathbb{R}^4$ by $x^\mu(s)$ and θ^i is a unit six-vector, assumed to be constant (no dependence on s). We can consider the supersymmetry variation of the Wilson loops, which is given by

$$
\begin{aligned}
\delta W(C) &= \frac{1}{N}\mathrm{Tr}\mathcal{P}\left[\int ds(i\delta A_\mu\dot{x}^\mu + \delta\Phi^i\theta^i|\dot{x}|)\, e^{\int ds(iA_\mu\dot{x}^\mu + \Phi^i\theta^i|\dot{x}|)}\right] \\
&= \frac{1}{N}\mathrm{Tr}\mathcal{P}\left[\int ds\bar{\xi}(-\gamma_\mu\lambda\dot{x}^\mu + \Psi_i\theta^i|\dot{x}|)\, e^{\int ds(iA_\mu(x)\dot{x}^\mu + \Phi_i(x)\theta^i|\dot{x}|)}\right]
\end{aligned}
\tag{3.6.13}
$$

where

$$
\Psi_i = \begin{cases} \psi_i, & \text{if } i = 1, 2, 3 \\ \gamma_5\psi_{i-3}, & \text{if } i = 4, 5, 6 \end{cases}\ . \tag{3.6.14}
$$

The result we find here is quite different from that we obtained in section (2.3). Indeed, there we put fermions into a single 10 dimensional Majorana-Weyl spinor of $Spin(9,1)$ and then we introduced ten dimensional Dirac matrices (Γ^μ, Γ^i). As a result, the $\mathcal{N}=4$ action took a simple form as well as the supersymmetry transformations for the bosonic fields, which were given by

$$\delta A_\mu = \bar{\Psi}\Gamma^\mu \epsilon \tag{3.6.15}$$

$$\delta \Phi_i = \bar{\Psi}\Gamma_i \epsilon \tag{3.6.16}$$

where ϵ is again a ten dimensional Majorana-Weyl spinor. We recall that when we performed the variation on the loop in the undeformed theory and then imposed it equal to zero the fermions factorize out and the condition for the loop to be supersymmetric was simply

$$(i\Gamma^\mu \dot{x} + \Gamma^i \theta^i |\dot{x}|)\epsilon = 0 \tag{3.6.17}$$

according to (2.3.11).

In the deformed case, even though the the supersymmetry transformations of the bosonic fields do not depend on β parameter, we ought to apply the $\mathcal{N}=1$ transformations:

$$\delta A_\mu = i\bar{\xi}\gamma_\mu \lambda \tag{3.6.18}$$

$$\delta A_i = \bar{\xi}\psi_i \tag{3.6.19}$$

$$\delta B_i = \bar{\xi}\gamma_5 \psi_i\ . \tag{3.6.20}$$

This prevent us to find for the loop in the deformed theory a supersymmetry condition as simple as (2.3.11) in the undeformed theory.

The relevant issue it to note that the β-deformed SYM theory contains two multiplets $\mathcal{N}=1$ invariant that do not mix themselves. While it is possible to write the β-deformed Lagrangian as the undeformed $\mathcal{N}=4$ SYM Lagrangian plus terms linear in β which broke the supersymmetry down to $\mathcal{N}=1$, the same procedure cannot be done with the supersymmetry transformations, which remain simply those of $\mathcal{N}=1$ SYM theory.

Chapter 4

D-brane configurations dual to Wilson loops

4.1 The σ model in $AdS_5 \times S^5$

The Polyakov action of a string in $AdS_5 \times S^5$ is given in terms of a standard σ model

$$\frac{1}{4\pi\alpha'} \int d\tau\, d\sigma\, \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}. \tag{4.1.1}$$

At this stage, G_{MN} is the target space metric for our product space, where both AdS_5 and S^5 have curvature radius equal to L. We have already pointed out that the AdS/CFT correspondence relates it to the 't Hooft coupling $\lambda = g_{YM}^2 N$ of the dual gauge theory and the string scale by $L^4 = \lambda \alpha'^2$.

The ansatz we shall consider factorizes the σ model in (4.1.1) into an AdS_5 part and an S^5 part, providing independent equations of motion for the respective variables. These two parts of the ansatz are connected only in two ways. One is the range of the world-sheet coordinates, which clearly has to agree on both spaces, and the other are the Virasoro constraints. Hence, it turns very worthwhile to consider the S^5 and AdS_5 parts of the ansatz separately.

The Virasoro constraints are given by the vanishing of the stress-energy tensor which in the conformal gauge is given by

$$\begin{aligned} T_{\sigma\sigma} = -T_{\tau\tau} &= \frac{1}{8\pi\alpha'} \left[\partial_\sigma X^M \partial_\sigma X^N - \partial_\tau X^M \partial_\tau X^N\right] G_{MN} = 0\,, \\ T_{\sigma\tau} = T_{\tau\sigma} &= \frac{1}{4\pi\alpha'} \partial_\sigma X^M \partial_\tau X^N G_{MN} = 0\,. \end{aligned} \tag{4.1.2}$$

Since our space has a product structure we can decompose the stress-energy tensor into independent contributions from AdS_5 and from S^5

$$T_{\alpha\beta} = T^{AdS_5}_{\alpha\beta} + T^{S^5}_{\alpha\beta}. \tag{4.1.3}$$

The Virasoro constraints are then

$$T^{AdS_5}_{\alpha\beta} + T^{S^5}_{\alpha\beta} = 0. \tag{4.1.4}$$

For notational simplicity we label $a^2 \equiv 8\pi\alpha' T^{S^5}_{\sigma\sigma}/L^2$, and in AdS_5 this parameter serves a role similar to a mass term coming from the Kaluza-Klein reduction on the sphere.

Since the stress-energy tensors of both σ-models are separately conserved, we have

$$\begin{aligned} &\partial_\sigma T^{S^5}_{\sigma\sigma} + \partial_\tau T^{S^5}_{\tau\sigma} = 0 \\ &\partial_\sigma T^{AdS_5}_{\sigma\sigma} + \partial_\tau T^{AdS_5}_{\tau\sigma} = 0. \end{aligned} \tag{4.1.5}$$

In the ansatze we will use below, $T^{S^5}_{\sigma\tau}$ is always constant, thus it follows that a^2 is constant, that may be either positive or negative. Using (4.1.4), the second of (4.1.5) reads

$$T^{AdS_5}_{\sigma\sigma} + \frac{L^2}{8\pi\alpha'}a^2 = 0. \tag{4.1.6}$$

We will now focalize on the AdS_5 part of the σ-model. As in the case of the sphere, a simple description of the system is by taking Euclidean AdS_5 as a hypersurface in flat six-dimensional Minkowski space. It is given by the hyperboloid

$$-Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 = -L^2. \tag{4.1.7}$$

Now let us define the coordinates r_0, r_1, r_2, v, ϕ_1 and ϕ_2 by

$$\begin{aligned} Y_0 &= Lr_0 \cosh v\,, & Y_5 &= Lr_0 \sinh v\,, \\ Y_1 &= Lr_1 \cos\phi_1\,, & Y_2 &= Lr_1 \sin\phi_1\,, \\ Y_3 &= Lr_2 \cos\phi_2\,, & Y_4 &= Lr_2 \sin\phi_2\,. \end{aligned} \tag{4.1.8}$$

Those coordinates satisfy the constraint $-r_0^2 + r_1^2 + r_2^2 = -1$, and the metric of the embedding flat Minkowski space is

$$ds^2 = L^2\left(-dr_0^2 + r_0^2 dv^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2\right). \tag{4.1.9}$$

In some of the specific examples we study below we will employ Poincaré coordinates. We replace r_0, r_1, r_2 and v with $\hat{y}$, $\hat{r}_1$ and $\hat{r}_2$ by the relations

$$r_0 = \frac{\sqrt{\hat{y}^2 + \hat{r}_1^2 + \hat{r}_2^2}}{\hat{y}}, \qquad r_1 = \frac{\hat{r}_1}{\hat{y}}, \qquad r_2 = \frac{\hat{r}_2}{\hat{y}}, \qquad v = \ln\sqrt{\hat{y}^2 + \hat{r}_1^2 + \hat{r}_2^2}. \tag{4.1.10}$$

In the new coordinates the metric reads

$$ds^2 = \frac{L^2}{\hat{y}^2}\left(d\hat{y}^2 + d\hat{r}_1^2 + \hat{r}_1^2 d\phi_1^2 + d\hat{r}_2^2 + \hat{r}_2^2 d\phi_2^2\right). \tag{4.1.11}$$

4.2 Maximal latitude in $\mathcal{N} = 4$ SYM

We start our analysis with a famous example, the maximal latitude on $S^2 \subset AdS_5$, that is the circular Wilson loop in $\mathcal{N} = 4$ super Yang Mills theory. Our aim is to become familiar with the methods of analysis of such problems as well as to see explicitly the equivalence of the classical solutions of the Nambu-Goto and the Polyakov action.

We choose the scalar charge of the Wilson loop to be constant, so that the string worldsheet lives at a single point on the 5-sphere.

4.2.1 Nambu-Goto action

In this example we use Poincaré coordinates, and the boundary is described by flat $\mathbb{R}^4$. In this patch we describe a circle on the boundary by constant $\hat{r}_1 = r$ and $\hat{r}_2 = 0$. This coordinates parametrize an AdS_2 subspace of AdS_5 with the metric

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr^2 + r^2 d\phi^2). \tag{4.2.1}$$

We take r and ϕ as worldsheet coordinates and we expect, due to the symmetry of the loop, that the surface described by y does not depend on ϕ, so we have

$$\partial_a x^\mu \partial_b x^\nu g_{\mu\nu} = \begin{pmatrix} \frac{1}{y^2}(y'^2 + 1) & 0 \\ 0 & \frac{r^2}{y^2} \end{pmatrix}. \tag{4.2.2}$$

The Nambu-Goto action is given by

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dr d\phi \frac{r}{y^2}\sqrt{1 + y'^2}. \tag{4.2.3}$$

In order to find the explicit form of the minimal surface we notice that the energy is a conserved quantity

$$E = -\frac{1}{y^2\sqrt{1 + y'^2}} \tag{4.2.4}$$

hence

$$y'^2 = \frac{1}{y^2 E^2} - 1. \tag{4.2.5}$$

The solution of this differential equation gives us the right expression for the minimal surface

$$y(r) = \sqrt{R^2 - r^2} \tag{4.2.6}$$

where we have set $E = 1/R$ and the integration constant equal to zero in order to recover the circular loop in the boundary of AdS_5. We notice that y has its maximum at $r = 0$, that is, at the center of the loop. Plugging this expression into the action we find

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi} \int dr d\phi \, \frac{Rr}{(R^2 - r^2)^{3/2}} \\ &= R\sqrt{\lambda} \int dr \, \frac{r}{(R^2 - r^2)^{3/2}} \\ &= R\sqrt{\lambda} \int_0^{\sqrt{R^2-\epsilon^2}} dr \, \frac{r}{(R^2 - r^2)^{3/2}} \\ &= \sqrt{\lambda} \, (\frac{R}{\epsilon} - 1). \end{aligned} \tag{4.2.7}$$

where ϵ is a cutoff and the linear divergent term is cancelled by the Legendre transform. Therefore, the vacuum expectation value of the Wilson loop is given by

$$\langle W \rangle = e^{-\sqrt{\lambda}} \tag{4.2.8}$$

as it should be [39].

4.2.2 Polyakov action

We take the AdS_5 metric

$$ds^2 = du^2 + \cosh^2 u(d\rho^2 + \sinh^2 \rho d\psi^2) + \sinh^2 u(d\chi^2 + \sin^2 \chi d\phi^2). \tag{4.2.9}$$

In this framework we describe a circle on the boundary by setting $r_2 = 0$ and the radius of the circle will be given by the value of v, $R = \exp v$. We identify (σ, τ) to be the worldsheet coordinates and it is natural to take the ansatz

$$\rho = \rho(\sigma), \quad \psi = \tau, \quad u = \chi = \phi = 0 \tag{4.2.10}$$

that leads to the Polyakov action in the conformal gauge

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau ({\rho'}^2 + \sinh^2 \rho). \tag{4.2.11}$$

The equation of motion is

$$\rho'' = \sinh \rho \cosh \rho \tag{4.2.12}$$

while the Virasoro constraint reads

$${\rho'}^2 = \sinh^2 \rho. \tag{4.2.13}$$

The solution is then given by

$$\sinh \rho(\sigma) = \frac{1}{\sinh \sigma}. \tag{4.2.14}$$

An integration constant in this equation that shifts σ was set to zero so the boundary of the world-sheet at $\sigma = 0$ is at the boundary of AdS_5. Then, the bulk part of the action is given by

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \ \sinh^2 \rho \\ &= \sqrt{\lambda} \int_{\sigma_{min}}^{\infty} d\sigma \ \frac{1}{\sinh^2 \sigma} \\ &= \sqrt{\lambda} \left(\coth \sigma_{min} - 1\right), \end{aligned} \tag{4.2.15}$$

where σ_{min} is the cutoff. Again the divergent term will cancel against the boundary term coming from the Legendre transform, and we obtain the same result for the expectation value of the Wilson loop as in the previous case

$$\langle W \rangle = e^{-\sqrt{\lambda}}. \tag{4.2.16}$$

4.3 Two longitudes in $\mathcal{N} = 4$ SYM

In this section we will resolve the string dual of an 1/4 BPS Wilson loops in $\mathcal{N} = 4$ SYM theory, which is also a special case of the loops on S^2 conjectured to be computed by a gaussian matrix model. We will find a suitable ansatz for the geometry of the loop and we will use the Polyakov formulation instead of the Nambu-Goto one. We will see that this will be very useful when, in the next section, we will solve the same loop in the $\mathcal{N} = 1$ β-deformed SYM theory.

We consider a loop made of two arcs of length π connected at an arbitrary angle δ, *i.e.* two longitudes on the two-sphere. We parameterize the loop as follows

$$\begin{aligned} x^\mu &= (\sin t, \, 0, \, \cos t, \, 0) \, , & 0 \le t \le \pi \, , \\ x^\mu &= (-\cos\delta \sin t, \, -\sin\delta \sin t, \, \cos t, \, 0) \, , & \pi \le t \le 2\pi \, . \end{aligned} \tag{4.3.1}$$

We take the metric of AdS_5 to be

$$ds^2 = L^2(-dr_0^2 + r_0^2 dv^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2) \tag{4.3.2}$$

and these coordinates satisfy the constraint

$$-r_0^2 + r_1^2 + r_2^2 = -1. \tag{4.3.3}$$

In our coordinate system the boundary of AdS_5, which is a four-sphere, is given by $r_0 \to \infty$ or, equivalently, $r_1 \to \infty$. As in the previous case we

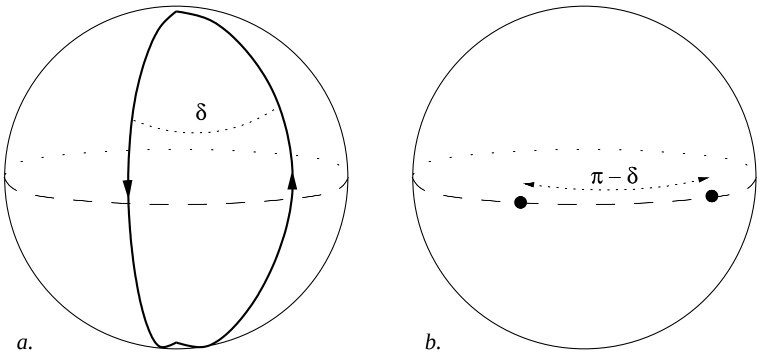

Figure 4.1: 1/4-BPS Wilson loop made of two longitudes. In a. we show the loop on $S^2 \subset \mathbb{R}^4$ obtained by taking two longitudes, connected at an angle δ. The corresponding scalar couplings in b. turn out to be two points on the equator of $S^2 \subset S^5$ separated by an angle $\pi - \delta$.

set $r_2 = \phi_2 = 0$ and, after implementing the constraint (4.3.3), the metric (4.3.2) becomes

$$ds^2 = L^2 \left((1 + r_1^2) dv^2 + \frac{1}{1 + r_1^2} dr_1^2 + r_1^2 d\phi_1^2 \right). \tag{4.3.4}$$

If we perform the limit $r_1 \to \infty$, the boundary is then described be a two-sphere

$$ds^2 = L^2 r_1^2 (dv^2 + d\phi_1^2) \tag{4.3.5}$$

spanned by the angles v and ϕ_1. In this limit we recover the loop if we set $v = \tau : [-\infty, +\infty]$ and $\phi_1 = (0, \delta)$ for the first and second branch, respectively. Hence, the most natural AdS_5 ansatz turns out to be

$$r_0 = r_0(\sigma), \quad r_1 = r_1(\sigma), \quad v = \tau + \alpha(\sigma), \quad \phi_1 = \phi_1(\sigma), \quad r_2 = \phi_2 = 0 \tag{4.3.6}$$

together with the S^1 ansatz

$$\varphi = m\tau + \beta(\sigma). \tag{4.3.7}$$

The Polyakov action is given by

$$S = \frac{L^2}{4\pi\alpha'} \int d\sigma d\tau \quad [-{r_0'}^2 + {r_1'}^2 + r_0^2 + r_0^2 \alpha'^2 + r_1^2 {\phi_1'}^2 + m^2 + \beta'^2 + \Lambda(-r_0^2 + r_1^2 + 1)], \tag{4.3.8}$$

where we have imposed the constraint (4.3.3) using the Lagrange multiplier Λ. We notice that α, ϕ_1 and β are cyclic, so we can express them in terms of the conserved momenta

$$\phi_1' = \frac{p_1}{r_1^2}, \quad \alpha' = \frac{p_0}{r_0^2}, \quad \beta' = \pi. \tag{4.3.9}$$

The equations of motion for r_0 and r_1 are

$$r_0'' = (\Lambda - 1) r_0 - \frac{p_0^2}{r_0^3} \tag{4.3.10}$$

$$r_1'' = \Lambda r_1 + \frac{p_1^2}{r_1^3}. \tag{4.3.11}$$

The diagonal component of the stress-energy tensor gives us the equation

$$-r_0'^2 + r_1'^2 + \frac{p_0^2}{r_0^2} + \frac{p_1^2}{r_1^2} - r_0^2 + \pi^2 - m^2 = 0. \tag{4.3.12}$$

Using this constraint we can rewrite the Polyakov action in a very simple form

$$S = \frac{L^2}{4\pi\alpha'} \int d\sigma d\tau [2r_0^2 + 2m^2]. \tag{4.3.13}$$

Moreover, the non-diagonal component of the stress-energy tensor gives us the following condition

$$p_0 + m\pi = 0. \tag{4.3.14}$$

Using the constraint (4.3.3), we can then eliminate r_0 from (4.3.12) and the Virasoro constraint (4.3.12) turns into an equation for r_1

$$r_1'^2 = 1 - \pi^2 + m^2 - p_0^2 - p_1^2 + (2 - \pi^2 + m^2) r_1^2 + r_1^4 - \frac{p_1^2}{r_1^2}. \tag{4.3.15}$$

It turns to be useful to switch to the coordinate $z = 1/r_1$, which reduces to zero at the boundary. The equation (4.3.15) becomes

$$z'^2 = 1 + (2 - \pi^2 + m^2) z^2 - (\pi^2 - m^2 - 1 + p_0^2 + p_1^2) z^4 - p_1^2 z^6. \tag{4.3.16}$$

Let us indicate z_1^2, z_2^2 and z_3^2 as the solutions of the right-hand side of (4.3.16) equal to zero. Then we can rewrite the above equation as

$$z'^2 = p_1^2 (z_1^2 - z^2)(z_2^2 - z^2)(z_3^2 - z^2). \tag{4.3.17}$$

The solution can be written in terms of elliptic integrals

$$\sigma = \frac{1}{p_1 z_2 \sqrt{z_1^2 - z_3^2}} F\left(\arcsin \sqrt{\frac{z^2 (z_3^2 - z_1^2)}{z_3^2 (z^2 - z_1^2)}} \middle| \frac{(z_1^2 - z_2^2) z_3^2}{z_2^2 (z_1^2 - z_3^2)} \right). \tag{4.3.18}$$

Inverting this equation gives z, or r_1 and r_0, as a function of σ. From (4.3.18) the full range of the worldsheet coordinate σ is given by the complete elliptic integral

$$\delta\sigma = \frac{2}{p_1 z_2 \sqrt{z_1^2 - z_3^2}} K\left(\frac{(z_1^2 - z_2^2) z_3^2}{z_2^2 (z_1^2 - z_3^2)} \right). \tag{4.3.19}$$

For simplicity, we introduce the quantities

$$t=\sqrt{\frac{z^2(z_3^2-z_1^2)}{z_3^2(z^2-z_1^2)}},\qquad k=\frac{(z_1^2-z_2^2)z_3^2}{z_2^2(z_1^2-z_3^2)}. \tag{4.3.20}$$

Next we can integrate the angle $\phi_1(\sigma)$:

$$\begin{aligned}\phi_1-\phi_1^i &= \int d\sigma\frac{1}{r_1^2}=p_1\int\frac{dz}{z'}z^2=p_1\int\frac{dt}{t'}z(t)^2\\ &= \frac{z_1^2}{z_2\sqrt{z_1^2-z_3^2}}\left[F\left(\arcsin t\middle|k\right)-\Pi\left(\frac{z_3^2}{z_3^2-z_1^2},\arcsin t\middle|k\right)\right]\end{aligned} \tag{4.3.21}$$

where $\phi_1^i=0$ is the initial boundary value at $z=0$ and t' is the derivative of the variable defined in (4.3.20) with respect to the worldsheet variable σ. This expression covers only half the worldsheet, while the other branch is given by a similar expression shifted by the complete elliptic integrals

$$\begin{aligned}\phi_1 &= \frac{z_1^2}{z_2\sqrt{z_1^2-z_3^2}}\left[2K\left(k\right)-2\Pi\left(\frac{z_3^2}{z_3^2-z_1^2}\middle|k\right)\right.\\ &\left.\quad -F\left(\arcsin t\middle|k\right)+\Pi\left(\frac{z_3^2}{z_3^2-z_1^2},\arcsin t\middle|k\right)\right].\end{aligned} \tag{4.3.22}$$

On the second branch, ϕ_1 reaches the final value ϕ_1^f, so the total change is

$$\delta=\delta\phi_1=\phi_1^f-\phi_1^i=\frac{2z_1^2}{z_2\sqrt{z_1^2-z_3^2}}\left[K\left(k\right)-\Pi\left(\frac{z_3^2}{z_3^2-z_1^2}\middle|k\right)\right]. \tag{4.3.23}$$

In a very similar way we integrate $\alpha(\sigma)$

$$\begin{aligned}\alpha &= p_0\int d\sigma\frac{1}{r_0^2}=p_0\int\frac{dz}{z'}\frac{z^2}{z^2+1}=p_0\int\frac{dt}{t'}\frac{z(t)^2}{z(t)^2+1}\\ &= \frac{p_0z_1^2}{p_1z_2(z_1^2+1)\sqrt{z_1^2-z_3^2}}\left[F\left(\arcsin t\middle|k\right)-\Pi\left(\frac{z_3^2(z_1^2+1)}{z_3^2-z_1^2},\arcsin t\middle|k\right)\right].\end{aligned} \tag{4.3.24}$$

Again, the correct expression on the other half of the worldsheet is given by a shift

$$\begin{aligned}\alpha &= \frac{p_0z_1^2}{p_1z_2(z_1^2+1)\sqrt{z_1^2-z_3^2}}\left[2K\left(k\right)-2\Pi\left(\frac{z_3^2(z_1^2+1)}{z_3^2-z_1^2}\middle|k\right)\right.\\ &\left.\quad -F\left(\arcsin t\middle|k\right)+\Pi\left(\frac{z_3^2(z_1^2+1)}{z_3^2-z_1^2},\arcsin t\middle|k\right)\right].\end{aligned} \tag{4.3.25}$$

The last step is to evaluate the action

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau\,(r_0^2+m^2) = \frac{T\sqrt{\lambda}}{2\pi}\int\frac{dz}{z'}\left(1+\frac{1}{z^2}+m^2\right) \\ &= \frac{1+z_1^2(1+m^2)}{p_1 z_1^2 z_2\sqrt{z_1^2-z_3^2}}\left[F\left(\arcsin t|k\right)\right]_0^{t_+} \\ &\quad -\frac{\sqrt{z_1^2-z_3^2}}{p_1 z_1^2 z_3^2 z_2}\left[\frac{1}{t}\sqrt{(1-k^2t^2)(1-t^2)}-F\left(\arcsin t|k\right)+E\left(\arcsin t|k\right)\right]_{t_0}^{t_+} \\ &= 2\frac{1+z_1^2(1+m^2)}{p_1 z_1^2 z_2\sqrt{z_1^2-z_3^2}}K\left(k\right)+2\frac{\sqrt{z_1^2-z_3^2}}{p_1 z_1^2 z_3^2 z_2}\left[\frac{1}{t_0}+K\left(k\right)-E\left(k\right)\right]. \end{aligned} \quad (4.3.26)$$

The $1/t_0$ term, where t_0 is a cutoff at small t, is the standard linear divergency and it will be regularized as usual by a boundary term.

Now it is worth to notice that the loop we are interested in is a special case of the general ansatz (4.3.6) and (4.3.7) we made above. In fact, the scalar coupling is constant, thus $m=0$. Using the non diagonal Virasoro constraint, this implies $p_0=0\Rightarrow\alpha=0$. Therefore, we can express the solutions z_1^2, z_2^2 and z_3^2 in terms of the quantities defined in [21]

$$z_1^2=-1,\quad z_2^2=-\frac{p^2}{b^2},\quad z_3^2=b^2. \quad (4.3.27)$$

Then we have

$$t=\sqrt{\frac{z^2(1+b^2)}{b^2(1+z^2)}},\quad k=\frac{b^2(p^2-b^2)}{p^2(1+b^2)}. \quad (4.3.28)$$

The string solutions are given by

$$\sigma=\frac{b}{p_1p\sqrt{1+b^2}}F\left(\arcsin t\middle|k\right) \quad (4.3.29)$$

$$\delta\sigma=\frac{2b}{p_1p\sqrt{1+b^2}}K\left(k\right) \quad (4.3.30)$$

and

$$\begin{aligned} \phi_1 &= \frac{b}{p\sqrt{1+b^2}}\left[F\left(\arcsin t\middle|k\right)-\Pi\left(\frac{b^2}{1+b^2},\arcsin t\middle|k\right)\right] \\ \phi_1 &= \frac{b}{p\sqrt{1+b^2}}\left[2K\left(k\right)-2\Pi\left(\frac{b^2}{1+b^2}\middle|k\right)-F\left(\arcsin t\middle|k\right)\right. \\ &\qquad \left.+\,\Pi\left(\frac{b^2}{1+b^2},\arcsin t\middle|k\right)\right] \end{aligned} \quad (4.3.31)$$

on the first and second half of the worldsheet respectively. The separation angle δ is then given by

$$\delta=\frac{2b}{p\sqrt{1+b^2}}\left[2K\left(k\right)-2\Pi\left(\frac{b^2}{1+b^2}\middle|k\right)\right]. \quad (4.3.32)$$

Finally, the action

$$S = 2\frac{\sqrt{1+b^2}}{p_1 p b}\left[\frac{1}{t_0} + K(k) - E(k)\right]. \tag{4.3.33}$$

So far, we have treated the general non-BPS case, that is, the non-supersymmetric cusp in the plane with opening angle δ and arbitrary jump in the scalar coupling.

The BPS loop is obtained by implementing the condition $p_1 = 1$, and for the other variables, $b^2 = p^2 \Rightarrow k = 0$, thus

$$t = \sqrt{\frac{z^2(1+b^2)}{b^2(1+z^2)}}, \quad k = \frac{b^2(p^2-b^2)}{p^2(1+b^2)}. \tag{4.3.34}$$

The string solutions are given by

$$\sigma = \frac{1}{\sqrt{1+p^2}} \arcsin\sqrt{\frac{z^2(1+p^2)}{p^2(1+z^2)}} \tag{4.3.35}$$

$$\delta\sigma = \frac{\pi}{\sqrt{1+p^2}}. \tag{4.3.36}$$

Since $\arctan(\frac{z^2}{\sqrt{p^2-z^2}}) = \arcsin(\frac{z}{p})$, we have

$$\begin{aligned} \phi_1 &= \arcsin\frac{z}{p} - \frac{1}{\sqrt{1+p^2}}\arcsin\sqrt{\frac{z^2(1+p^2)}{p^2(1+z^2)}} \\ \phi_1 &= \pi - \arcsin\frac{z}{p} - \frac{1}{\sqrt{1+p^2}}\left(\pi - \arcsin\sqrt{\frac{z^2(1+p^2)}{p^2(1+z^2)}}\right) \end{aligned} \tag{4.3.37}$$

on the two half of the worldsheet. The equation (4.3.32) for the angle δ becomes very simple

$$\delta = \pi - \frac{\pi}{\sqrt{1+p^2}} \tag{4.3.38}$$

as well as the expression for the action

$$S = 2\frac{T\sqrt{\lambda}}{2\pi}\frac{\sqrt{1+p^2}}{p^2}\left[\frac{1}{t_0} + K(k) - E(k)\right] = 2\frac{T\sqrt{\lambda}}{2\pi}\frac{1}{pz_0}. \tag{4.3.39}$$

Now we want to show explicitly that the divergence $1/z_0$ cancels against the Legendre transform of the action

$$S_{LT} = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma z'\frac{\delta L}{\delta z'} = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau \frac{dz}{z'}\frac{z^2+1}{z^2} = -S. \tag{4.3.40}$$

so the full Lagrangian locally vanishes.

Now we conformally transform to global AdS_5 with metric

$$ds^2 = L^2[d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta d\phi_1^2) + d\phi^2] \tag{4.3.41}$$

where the relations between the new and the old coordinates (4.3.2) are

$$\sinh\rho\sin\theta = \frac{1}{z}, \quad \cosh\rho = \frac{1+r^2+r^2v^2}{2rv} \tag{4.3.42}$$

where $v = \ln(r\sqrt{1+z^2})$ and $v = \ln r$ on the boundary. The equation of motion for z in the BPS case simply reads

$$z'^2 = (1+z^2)^2(p^2 - z^2). \tag{4.3.43}$$

Thus the action becomes

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\int\frac{dr}{r}\int\frac{dz}{z'}\frac{1+z^2}{z^2} = \frac{\sqrt{\lambda}}{2\pi}\int\frac{dr}{r}\int dz\frac{1}{z^2\sqrt{p^2-z^2}} \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\rho\, d\theta\frac{p\sinh^2\rho\sin\theta}{\sqrt{p^2\sinh^2\rho\sinh^2\theta - 1}}. \end{aligned} \tag{4.3.44}$$

It is not too complicated to integrate this expression. First, we notice that for a fixed ρ the angle θ varies between the two roots of the equation

$$\sin\theta\sinh\rho = 1/p \tag{4.3.45}$$

and then it comes back. Therefore, the integration over θ contributes with $2\pi\sinh\rho$, while the remaining integral to perform over ρ is between the minimal value, where $\sinh\rho = 1/p$ and a cutoff ρ_0 at large ρ

$$S = \sqrt{\lambda}\int d\rho\,\sinh\rho = \sqrt{\lambda}\left(\cosh\rho_0 - \frac{\sqrt{p^2+q}}{p}\right). \tag{4.3.46}$$

As we will see, we cannot simply discard the divergent $\cosh\rho_0$ term, because there is the possibility of some finite corrections left over from the divergent piece. In global AdS the divergent Legendre transform which must be subtracted to the bulk Lagrangian reads

$$L_{\text{bound}} = -\coth\rho_0\, p_\rho = -\coth\rho_0\,\rho'\,\frac{\delta\mathcal{L}_{NG}}{\delta\rho'}. \tag{4.3.47}$$

Following [21], we evaluate the corresponding boundary action

$$\begin{aligned} S_{\text{bound}} &= -\frac{\sqrt{\lambda}}{2\pi}\int d\theta\,\sin\theta\,\frac{p^2\sinh^2\rho_0(\sinh^2\rho_0\sin^2\theta+1) - \cosh^2\rho_0}{p(\sinh^2\rho_0\sin^2\theta+1)\sqrt{p^2\sinh^2\rho_0\sin^2\theta - 1}} \\ &\simeq -\sqrt{\lambda}\left(\sinh\rho_0 - \frac{\coth\rho_0}{p\sqrt{1+p^2}}\right). \end{aligned} \tag{4.3.48}$$

Combining this with the bulk action (4.3.46), the divergences indeed cancel and we get the final answer for the action of the string dual to the two-longitudes Wilson loop

$$S = -\frac{\sqrt{\lambda\,\delta(2\pi - \delta)}}{\pi}. \tag{4.3.49}$$

This result is consistent with the conjectured relation with the gaussian matrix model describing the zero-instanton sector of two-dimensional Yang-Mills theory on the sphere [15].

4.4 Two longitudes in $\mathcal{N} = 1$ β-deformed SYM

Now we turn to analyze the same loop made of two longitudes on the two-sphere connected at an arbitrary angle δ in the $\mathcal{N} = 1$ β-deformed SYM theory[1].

We recall that the Lunin-Maldacena metric is given by

$$ds^2 = L^2\left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\varphi_i^2) + \hat{\gamma}^2 G\mu_1\mu_2\mu_3(\sum_i d\varphi_i)^2\right] \tag{4.4.1}$$

where

$$G^{-1} = 1 + \hat{\gamma}^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2) \tag{4.4.2}$$

and the Euclidean AdS_5 metric is the same as in (4.3.2).

We parametrize the loop as ($x^4 = 0$ defines the two-sphere)

$$\begin{aligned} x^1 &= \sin\tau, \;\; x^2 = 0, \;\; x^3 = \cos\tau, && 0 \le \tau \le \pi \\ x^1 &= -\cos\delta\sin\tau, \;\; x^2 = -\sin\delta\sin\tau, \;\; x^3 = \cos\tau, && \pi \le \tau \le 2\pi \end{aligned} \tag{4.4.3}$$

and the corresponding Wilson loop will couple to the scalars according to

$$\begin{aligned} \theta^5 &= 0, \quad \theta^2 = 1, && 0 \le \tau \le \pi \\ \theta^5 &= \sin\delta, \quad \theta^2 = -\cos\delta, && \pi \le \tau \le 2\pi. \end{aligned} \tag{4.4.4}$$

Unlike the latitude loop, here we expected no time dependance in the scalar coupling, since locally the loop is a maximal circle.

For the deformed $\tilde{S}^5$, in the first branch, we parametrize the μ_i coordinates via

$$\mu_1 = \cos\theta, \quad \mu_2 = \sin\theta\cos\alpha, \quad \mu_3 = \sin\theta\sin\alpha \tag{4.4.5}$$

and consequently $\sum_i d\mu_i^2 = d\theta^2 + \sin^2\theta d\alpha^2$.

[1] We notice that when $\delta = \pi$ the loop falls back to the 1/2-BPS circe in the undeformed theory.

In order to obtain the dual string configuration for the first branch we note that

$$\begin{aligned}\theta^1 + i\theta^4 &= \mu_1 e^{i\varphi_1} = 0\\ \theta^2 + i\theta^5 &= \mu_2 e^{i\varphi_2} = 1\\ \theta^3 + i\theta^6 &= \mu_3 e^{i\varphi_3} = 0.\end{aligned} \tag{4.4.6}$$

Therefore, at the boundary the string solution must satisfy $\theta = \pi/2$ and $\alpha = \varphi_1 = \varphi_2 = \varphi_3 = 0$.

Similarly, for the second branch

$$\begin{aligned}\theta^1 + i\theta^4 &= \mu_1 e^{i\varphi_1} = 0\\ \theta^2 + i\theta^5 &= \mu_2 e^{i\varphi_2} = -\cos\delta + i\sin\delta\\ \theta^3 + i\theta^6 &= \mu_3 e^{i\varphi_3} = 0.\end{aligned} \tag{4.4.7}$$

Hence the string solution must satisfy $\theta = \pi/2$, $\varphi_2 = \pi - \delta$ and $\alpha = \varphi_1 = \varphi_3 = 0$ at the boundary. As a consequence, it is easy to see that the scalar coupling turns out to be two points on the equator of the deformed sphere $\tilde{S}^2 \subset \tilde{S}^5$ separated by an angle $\pi - \delta$, similarly to the undeformed case.

However, due to the presence of the B-field

$$B = R^2 \hat{\gamma} G(\mu_1^2\mu_2^2 d\varphi_1 \wedge d\varphi_2 + \mu_2^2\mu_3^2 d\varphi_2 \wedge d\varphi_3 + \mu_3^2\mu_1^2 d\varphi_3 \wedge d\varphi_1) \tag{4.4.8}$$

we must consider an ansatz involving at least two additional angles, let us say φ_1 and φ_2. Thus, let us consider the subspace $AdS_3 \times \tilde{S}^3$ of $AdS_5 \times \tilde{S}^5$, where AdS_3 is parametrized by r_0, r_1, v and ϕ_1, and the deformed three-sphere $\tilde{S}^3$ is parametrized by the angles θ, φ_1 and φ_2. For these coordinates we consider the following AdS_5 ansatz

$$r_0 = r_0(\sigma), \quad r_1 = r_1(\sigma), \quad v = \tau, \quad \phi_1 = \phi_1(\sigma), \quad r_2 = \phi_2 = 0 \tag{4.4.9}$$

together with the $\tilde{S}^3$ ansatz

$$\theta = \theta(\sigma), \quad \varphi_1 = \varphi_1(\sigma), \quad \varphi_2 = \varphi_2(\sigma). \tag{4.4.10}$$

The euclidean Polyakov action with the B coupling is given by

$$\begin{aligned}S =& \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[\partial_a x^\mu \partial_a x^\nu g_{\mu\nu} + \epsilon^{ij}\partial_i x^\mu \partial_j x^\nu B_{\mu\nu})\Big]\\ =& \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[-{r_0'}^2 + r_0^2 + {r_1'}^2 + r_1^2{\phi_1'}^2 + \theta'^2 + G\cos^2\theta{\varphi_1'}^2 + G\sin^2\theta{\varphi_2'}^2\\ &\qquad - 2i\hat{\gamma}G\sin^2\theta\cos^2\theta(\dot{\varphi}_1\varphi_2' - \varphi_1'\dot{\varphi}_2) + \Lambda(-r_0^2 + r_1^2 + 1)\Big]\\ =& \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[-{r_0'}^2 + r_0^2 + {r_1'}^2 + r_1^2{\phi_1'}^2 + \theta'^2 + G\cos^2\theta{\varphi_1'}^2 + G\sin^2\theta{\varphi_2'}^2\\ &\qquad + \Lambda(-r_0^2 + r_1^2 + 1)\Big].\end{aligned} \tag{4.4.11}$$

The B coupling *mixing* term vanishes since the $\tilde{S}^3$ angles do not depend on the temporal coordinate τ in our ansatz (4.4.9). Since the coordinates ϕ_1, φ_1 and φ_2 are cyclic we can introduce the conserved momenta

$$\phi_1' = \frac{\beta}{r_1^2}, \quad \varphi_1' = \frac{\pi_1}{\cos^2\theta}, \quad \varphi_2' = \frac{\pi_2}{\sin^2\theta}. \tag{4.4.12}$$

At the boundary the solution must satisfy $\theta = \pi/2$ and $\varphi_1 = 0$, so we must set $\pi_1 = 0$. As a result, the string solution does not involve the φ_1 direction. The same reasoning about φ_2 lead us to set $\pi_2 = \dfrac{\pi - \delta}{\delta}$.

The equations of motion for the other variables are given by

$$r_0'' = (\Lambda - 1) r_0 \tag{4.4.13}$$

$$r_1'' = \Lambda r_1 - \frac{\beta}{r_1^3} \tag{4.4.14}$$

$$\theta'' = \partial_\theta [G \sin^2\theta] {\varphi_2'}^2. \tag{4.4.15}$$

The resolution equation (4.4.15) of motion in terms of elliptic integrals is quite complicated, but it is still possible. However, that solution is not compatible with the boundary conditions, so we are forced to set $\theta = \pi/2$ and, as a consequence, $G = 1$. The $\tilde{S}$ part of the metric then becomes simply the metric of a undeformed S^1 parametrized by the angle φ_2, that is, we are fallen back to the case of the two longitudes in the undeformed background.

Therefore, since the AdS_5 part of the geometry is unmodified by the β-deformation, we obtain for the Polyakov action the same result as in $\mathcal{N} = 4$ SYM

$$S = -\frac{\sqrt{\lambda\, \delta (2\pi - \delta)}}{\pi}\,. \tag{4.4.16}$$

4.5 Toroidal loops in $\mathcal{N} = 1$ β-deformed SYM

In this section we will find the dual string solution to a new interesting system obtained by using the description of S^3 as an Hopf fibration, namely as a S^1 bundle over S^2. Explicitly, one can write the S^3 metric as

$$ds^2 = \frac{1}{4}\left(d\vartheta^2 + \sin\vartheta^2 d\phi^2 + (d\psi + \cos\vartheta\, d\phi)^2\right), \tag{4.5.1}$$

where the range of the Euler angles is $0 \le \vartheta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \psi \le 4\pi$. The S^1 fiber is parameterized by ψ, while the base S^2 by (ϑ, ϕ).

Consider now a Wilson loop along a generic fiber. This loop will sit at constant (ϑ, ϕ), while ψ varies along the curve. The fibers are non-intersecting great circles of the S^3, so they will each couple to a single scalar, but the interesting fact is that all the circles in the same fibration will couple to the same scalar, in this case Φ^3.

We first notice that such operators are 1/8 BPS in the undeformed theory, so this may be an important check if we shall indeed find that the vev of such type of loops in the β-deformed theory is the same as in the $\mathcal{N}=4$ SYM case. This because, as far as we know, no results are known about β-deformed near 1/8 BPS Wilson loops.

Before going into the calculus ,we want to point out that the AdS_5 and the $\tilde{S}^5$ parts of the σ-model completely decouple again. In principle the two system may be coupled because of the Virasoro constraint, that is, the vanishing of the stress-energy tensor, which should be satisfied by the entire system. In this case, since we are dealing with only single loop, and not, for example, correlator of two or more loops, the Virasoro constraint is indeed satisfied independently on the two system separately [18]. As a consequence, the action for the string will be the sum of AdS_5 part and of the S^5 part, which we shall treat separately.

4.5.1 Two scalars

We take a multiply wrapped latitude curve with winding k on the Hopf base

$$\varphi = k\tau, \quad \vartheta = \vartheta_0, \quad 0 \leq \tau \leq 2\pi \tag{4.5.2}$$

and along the fibers, in order to obtain loops with enhanced supersymmetry, we choose

$$\psi = -\int_0^\tau d\tau' \dot{\varphi}(\tau') \cos\vartheta(\tau') = -k\tau\cos\vartheta_0. \tag{4.5.3}$$

The condition for the loop to be closed leads the equation (4.5.3) to an integral condition that it is equivalent to the fact that the area bound by the curve should be quantized in units of 4π

$$\int d\varphi d\vartheta \sin\vartheta = \int_0^{2\pi} d\tau \dot{\varphi}(\tau)(1-\cos\vartheta(\tau)) = \varphi(2\pi) + \psi(2\pi). \tag{4.5.4}$$

From this we notice that $k\cos\vartheta_0$ should be an integer and we set $k = k_1 + k_2$ and $k\cos\vartheta_0 = k_1 - k_2$ where k_1 and k_2 are integer. We can write the curve in terms of the Cartesian coordinates

$$x^1 = \sqrt{\frac{k_2}{k}}\sin k_1\tau,\ x^2 = \sqrt{\frac{k_2}{k}}\cos k_1\tau,\ x^3 = \sqrt{\frac{k_1}{k}}\sin k_2\tau,\ x^4 = \sqrt{\frac{k_1}{k}}\cos k_2\tau. \tag{4.5.5}$$

This is a motion on a torus inside S^3 where the curve wraps the two cycles k_1 and k_2 times.

The scalar couplings for these loops turn out to be

$$\theta^2 = \cos(k_2-k_1)\tau, \quad \theta^5 = \sin(k_2-k_1)\tau. \tag{4.5.6}$$

Thus, the coupling involves only two scalars.

$\tilde{S}^5$ part

In order to find the dual string configuration we set

$$\begin{aligned} \theta^1 + i\theta^4 &= \mu_1 e^{i\varphi_1} = 0 \\ \theta^2 + i\theta^5 &= \mu_2 e^{i\varphi_2} = \cos(k_2-k_1)\tau + i\sin(k_2-k_1)\tau \\ \theta^3 + i\theta^6 &= \mu_3 e^{i\varphi_3} = 0 \end{aligned} \tag{4.5.7}$$

where the μ_i are given as usual by the (6.1.3). Then at the boundary we have $\theta = \pi/2$, $\varphi_2 = (k_2-k_1)\tau$ and $\alpha = \varphi_1 = \varphi_3 = 0$. This is described by a maximal latitude on $\tilde{S}^2$ but, due to the B-field, we must consider an ansatz involving an additional angle, say φ_1, so we look for solutions of the form

$$\theta = \theta(\sigma), \quad \varphi_1 = \varphi_1(\sigma), \quad \varphi_2 = (k_2-k_1)\tau. \tag{4.5.8}$$

The $\tilde{S}^5$ part of the action is[2]

$$\begin{aligned} S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \Big[\theta'^2 + G\cos^2\theta {\varphi_1'}^2 + G\sin^2\theta\dot{\varphi}_2^2 \\ &\qquad -2i\hat{\gamma}G\sin^2\theta\cos^2\theta(\dot{\varphi}_1\varphi_2' - \varphi_1'\dot{\varphi}_2)\Big] \\ &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \Big[\theta'^2 - G\cos^2\theta {\varphi_1'}^2 + G(k_2-k_1)^2\sin^2\theta \\ &\qquad -2(k_2-k_1)\hat{\gamma}G\sin^2\theta\cos^2\theta\varphi_1'\Big] \end{aligned} \tag{4.5.9}$$

where in the last equality we have performed a Wick rotation $\varphi_1 \to i\varphi_1$ in order to obtain a real configuration, and the G factor is given by

$$G^{-1} = 1 + \hat{\gamma}^2\sin^2\theta\cos^2\theta. \tag{4.5.10}$$

First, since φ_1 is cyclic, we can express it in term of his conserved momentum

$$-G\cos^2\theta\varphi_1' - (k_2-k_1)\hat{\gamma}G\sin^2\theta\cos^2\theta = \pi_1 \tag{4.5.11}$$

but, for the surface to be closed, we must set $\pi_1 = 0$ and we have

$$\varphi_1' = (k_1-k_2)\hat{\gamma}\sin^2\theta. \tag{4.5.12}$$

Substituting this equation in the S^5 part of the Virasoro constraint we get an equation for θ

$$\begin{aligned} &\theta'^2 - G\cos^2\theta{\varphi_1'}^2 - (k_2-k_1)^2 G\sin^2\theta = 0 \\ &\qquad \Rightarrow \theta'^2 = (k_2-k_1)^2\sin^2\theta. \end{aligned} \tag{4.5.13}$$

[2]For Euclidean space, the worldsheet coupling to the B-field get an extra $-i$ factor

The solutions of both equation are simply to find out

$$\sin\theta = \frac{1}{\cosh[(k_2-k_1)\sigma]} \tag{4.5.14}$$

$$\varphi_1 = \hat{\gamma}\tanh[(k_1-k_2)\sigma]. \tag{4.5.15}$$

We check that our solution behaves correctly in the boundary limit

$$\sigma\to 0 \Rightarrow \begin{cases}\sin\theta\to 1 \Rightarrow \theta\to\frac{\pi}{2} \\ \varphi_1\to 0\end{cases} . \tag{4.5.16}$$

Next we evaluate the $\tilde{S}^5$ action for this configuration

$$\begin{aligned} S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[(k_2-k_1)^2\sin^2\theta-(k_2-k_1)^2\hat{\gamma}^2G\sin^2\theta\cos^2\theta \\ &\qquad +G(k_2-k_1)^2\sin^2\theta+2(k_2-k_1)^2\hat{\gamma}^2G\sin^4\theta\cos^2\theta\Big] \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau\Big[(k_2-k_1)^2\sin^2\theta\Big] \\ &= \sqrt{\lambda}\,(k_2-k_1)^2\int_0^\infty d\sigma\,\frac{1}{\cosh^2[(k_2-k_1)\sigma]} \\ &= \sqrt{\lambda}\,(k_2-k_1)\Big[\tanh[(k_2-k_1)\sigma]\Big]_0^\infty \\ &= \sqrt{\lambda}\,(k_2-k_1). \end{aligned} \tag{4.5.17}$$

AdS_5 part

For the AdS_5 part of our space we take as the metric of the embedding flat Minkowski space

$$ds^2 = L^2(-dr_0^2+r_0^2dv^2+dr_1^2+r_1^2d\phi_1^2+dr_2^2+r_2^2d\phi_2^2) \tag{4.5.18}$$

with the constraint $-r_0^2+r_1^2+r_2^2=-1$.

The ansatz for our system of periodic motion on T^2 is

$$r_i=r_i(\sigma),\quad \phi_1=k_1\tau,\quad \phi_2=k_2\tau \tag{4.5.19}$$

and we set v to be constant. The AdS_5 part of the action is

$$S_{AdS_5}=\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau[-{r_0'}^2+{r_1'}^2+{r_2'}^2+r_1^2k_1^2+r_2^2k_2^2+\Lambda(-r_0^2+r_1^2+r_2^2)] \tag{4.5.20}$$

where Λ is the usual Lagrange multiplier.

The equations of motion of the r_i variables are

$$r_0''=\Lambda r_0,\quad r_1''=(k_1^2+\Lambda)r_1,\quad r_2''=(k_2^2+\Lambda)r_2. \tag{4.5.21}$$

The AdS_5 part of the Virasoro constraint give us the first integral of motion

$$-{r_0'}^2+{r_1'}^2+{r_2'}^2-r_1^2k_1^2-r_2^2k_2^2=0. \tag{4.5.22}$$

Using this equation we can rewrite the action as twice the kinetic piece

$$S_{AdS_5}=\frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau[r_1^2k_1^2+r_2^2k_2^2]. \tag{4.5.23}$$

The other integral of motion are

$$\begin{aligned} I_0 &= r_0^2-\frac{1}{k_1^2}(r_0r_1'-r_1r_0')^2-\frac{1}{k_2^2}(r_0r_2'-r_2r_0')^2 && (4.5.24)\\ I_1 &= r_1^2-\frac{1}{k_1^2}(r_0r_1'-r_1r_0')^2+\frac{1}{k_1^2-k_2^2}(r_1r_2'-r_2r_1')^2 && (4.5.25)\\ I_2 &= r_2^2-\frac{1}{k_2^2}(r_0r_2'-r_2r_0')^2+\frac{1}{k_2^2-k_1^2}(r_2r_1'-r_1r_2')^2 && (4.5.26) \end{aligned}$$

but they are not independent, in fact the satisfy the condition $-I_0+I_1+I_2=-1$. Studying such equations in the limit in which σ is infinite (in such limit r_1,$r_2\to 0$ while $r_0\to 1$), we find that the integration constant are $I_0=1$ and $I_1=I_2=0$.

In order to solve these equations we introduce the coordinates ζ_1 and ζ_2 as the roots of the equation

$$\frac{r_0^2}{\zeta^2}-\frac{r_1^2}{\zeta^2-k_1^2}-\frac{r_2^2}{\zeta^2-k_2^2}=0. \tag{4.5.27}$$

We find that

$$r_0=\frac{\zeta_1\zeta_2}{k_1k_2},\quad r_1=\sqrt{\frac{(\zeta_1^2-k_1^2)(\zeta_2^2-k_1^2)}{k_1^2(k_2^2-k_1^2)}},\quad r_2=\sqrt{\frac{(\zeta_1^2-k_2^2)(\zeta_2^2-k_2^2)}{k_2^2(k_1^2-k_2^2)}}. \tag{4.5.28}$$

Substituting these back into the (4.5.24) and (4.5.25) we obtain

$$\zeta_1'=\pm\frac{(\zeta_1^2-k_1^2)(\zeta_1^2-k_2^{"})}{\zeta_1^2-\zeta_2^2},\quad \zeta_2'=\pm\frac{(\zeta_2^2-k_1^2)(\zeta_2^2-k_2^{"})}{\zeta_1^2-\zeta_2^2}. \tag{4.5.29}$$

Let us now assume, without loss of generality, that $k_1<k_2$, then we can take $k_1\le\zeta_1\le k_2\le\zeta_2$ and the (4.5.29) becomes

$$\zeta_1'=-\frac{(\zeta_1^2-k_1^2)(\zeta_1^2-k_2^{"})}{\zeta_1^2-\zeta_2^2},\quad \zeta_2'=\frac{(\zeta_2^2-k_1^2)(\zeta_2^2-k_2^{"})}{\zeta_1^2-\zeta_2^2}. \tag{4.5.30}$$

We can now integrate the ratio of these two equations, and the solution is given by

$$k_1\operatorname{arctanh}\frac{\zeta_1}{k_2}-k_2\operatorname{arctanh}\frac{\zeta_1}{k_1}+k_1\operatorname{arctanh}\frac{\zeta_2}{k_2}-k_2\operatorname{arctanh}\frac{\zeta_2}{k_1}=c \tag{4.5.31}$$

where c is a constant, or

$$\left(\frac{(\zeta_1-k_1)(\zeta_2+k_1)}{(\zeta_1+k_1)(\zeta_2-k_1)}\right)^{k_2}\left(\frac{(k_2+\zeta_1)(\zeta_2-k_2)}{(k_2-\zeta_1)(\zeta_2+k_2)}\right)^{k_1}=C \tag{4.5.32}$$

where the constant C is

$$C=\left(\frac{k_2-\sqrt{k_1k_2}}{k_2+\sqrt{k_1k_2}}\right)^{k_2}\left(\frac{\sqrt{k_1k_2}+k_1}{\sqrt{k_1k_2}-k_1}\right)^{k_1}. \tag{4.5.33}$$

Although it is not easy to invert this equation in order to find the ζ's in terms of the worldsheet variableσ, that is not necessary, because it turns out that the action can be evaluated without that. In fact

$$\begin{aligned} S_{AdS_5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau[r_1^2k_1^2+r_2^2k_2^2] \\ &= \sqrt{\lambda}\int d\sigma\left(\frac{{\zeta_1'}^2(\zeta_2^2-\zeta_1^2)}{(\zeta_1^2-k_1^2)(k_2^2-\zeta_1^2)}\frac{{\zeta_2'}^2(\zeta_2^2-\zeta_1^2)}{(\zeta_2^2-k_1^2)(\zeta_2^2-k_2^2)}\right) \\ &= -\sqrt{\lambda}\left(\int_{\sqrt{k_1k_2}}^{k_1}d\zeta_1+\int_{\infty}^{k_2}d\zeta_2\right) \\ &\simeq -\sqrt{\lambda}\left(k_1+k_2-\sqrt{k_1k_2}\right) \end{aligned} \tag{4.5.34}$$

where in the last expression we have removed the divergence. If we combine this result with the $\tilde{S}^5$ part of the action (4.5.17) we finally get the total action

$$S=-\sqrt{\lambda}\left(2k_1-\sqrt{k_1k_2}\right) \tag{4.5.35}$$

which is exactly the same as in the undeformed case.

4.5.2 Three scalars

Now we want to extend this calculation to a general doubly-periodic loops on any torus in S^3. Again (θ,ϕ) will be the Hopf base, but now we shall not fix the value of θ. In fact, we take the curve to be of the form

$$x^1=\sin\frac{\vartheta}{2}\sin k_1\tau,\ x^2=\sin\frac{\vartheta}{2}\cos k_1\tau,\ x^3=\cos\frac{\vartheta}{2}\sin k_2\tau,\ x^4=\cos\frac{\vartheta}{2}\cos k_2\tau. \tag{4.5.36}$$

The scalar coupling now involve three scalars

$$\begin{aligned} \theta^1 &= \frac{1}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\left(k_2\cos^2\frac{\vartheta}{2}-k_1\sin^2\frac{\vartheta}{2}\right) \\ \theta^2 &= \frac{k_1+k_2}{2\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\sin\vartheta\cos(k_2-k_1)\tau \\ \theta^5 &= \frac{k_1+k_2}{2\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\sin\vartheta\sin(k_2-k_1)\tau. \end{aligned} \tag{4.5.37}$$

$\tilde{S}^5$ part

In order to find the dual string configuration we set

$$\begin{aligned}
\theta^1 + i\theta^4 &= \mu_1 e^{i\varphi_1} = \frac{1}{\sqrt{k_1^2 \sin^2\frac{\vartheta}{2} + k_2^2\cos^2\frac{\vartheta}{2}}}\Big(k_2\cos^2\frac{\vartheta}{2} - k_1\sin^2\frac{\vartheta}{2}\Big)\\
\theta^2 + i\theta^5 &= \mu_2 e^{i\varphi_2} = \frac{(k_1+k_2)\sin\vartheta}{2\sqrt{k_1^2 \sin^2\frac{\vartheta}{2} + k_2^2\cos^2\frac{\vartheta}{2}}}\Big(\cos(k_2-k_1)\tau + i\sin(k_2-k_1)\tau\Big)\\
\theta^3 + i\theta^6 &= \mu_3 e^{i\varphi_3} = 0.
\end{aligned} \tag{4.5.38}$$

The boundary values are $\varphi_2 = (k_2 - k_1)\tau$ and $\alpha = \varphi_1 = \varphi_3 = 0$ as before, while

$$\sin\theta_0 = \frac{(k_1+k_2)\sin\vartheta}{2\sqrt{k_1^2 \sin^2\frac{\vartheta}{2} + k_2^2\cos^2\frac{\vartheta}{2}}}. \tag{4.5.39}$$

We see that we do not have to introduce another additional angle to resolve the problem, so the ansatz is the same as in (4.5.8).

We write down the $\tilde{S}^5$ action

$$\begin{aligned}
S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[\theta'^2 - G\cos^2\theta{\varphi_1'}^2 + G(k_2-k_1)^2\sin^2\theta\\
&\qquad -2(k_2-k_1)\hat{\gamma}G\sin^2\theta\cos^2\theta\varphi_1'\Big]
\end{aligned} \tag{4.5.40}$$

where we have already performed a Wick rotation on φ_1. From this it follows that the action and, as a consequence, the equations of motion are the same as in the previous case, a part from the fact that the boundary condition (4.5.39) forces the integration constant σ_0 to be non-zero. In fact, it is straightforward to find out that the solution is given by

$$\sin\theta = \frac{1}{\cosh[(k_2-k_1)(\sigma_0 \pm \sigma)]} \tag{4.5.41}$$

$$\varphi_1 = \hat{\gamma}\big(\tanh(\sigma\pm\sigma_0) \mp \tanh(\sigma_0)\big). \tag{4.5.42}$$

The sign choice corresponds to a surface wrapping the northern or the southern hemisphere and the integration constant σ_0 is chosen so that at $\sigma = 0$ it reaches the boundary value

$$\frac{1}{\cosh[(k_2-k_1)\sigma_0]} = \frac{(k_1+k_2)\sin\vartheta}{2\sqrt{k_1^2 \sin^2\frac{\vartheta}{2} + k_2^2\cos^2\frac{\vartheta}{2}}}. \tag{4.5.43}$$

The last step is to integrate the action

$$\begin{aligned} S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau\Big[(k_2-k_1)^2\sin^2\theta\Big] \\ &= \sqrt{\lambda}\,(k_2-k_1)^2\int_0^\infty d\sigma\;\frac{1}{\cosh^2[(k_2-k_1)(\sigma_0\pm\sigma)]} \\ &= \sqrt{\lambda}\,(k_2-k_1)\Big[\pm\tanh[(k_2-k_1)(\sigma_0\pm\sigma)]\Big]_0^\infty \\ &= \sqrt{\lambda}\,(k_2-k_1)\Big(1\pm\tanh[(k_2-k_1)\sigma_0]\Big) \\ &= \sqrt{\lambda}\,(k_2-k_1)\left(1\pm\frac{k_2\cos^2\frac{\vartheta}{2}-k_1^2\sin^2\frac{\vartheta}{2}}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\right). \end{aligned} \tag{4.5.44}$$

AdS_5 part

The AdS_5 solution for the more general case in which we consider any torus inside S^3 and we couple with three scalars, instead of two, is nearly the same as the case of the latitude on the Hopf base. Indeed, if we follow the same path we find the same solution for the ζ's as in (4.5.32) but in that case the constant has a different expression from (4.5.33), depending instead on the radii $\sin\vartheta/2$ and $\cos\vartheta/2$, which are encoded in the asymptotic values of r_1 and r_2

$$C=\left(\frac{k_2-\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}{k_2+\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\right)^{k_2}\left(\frac{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}+k_1}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}-k_1}\right)^{k_1}. \tag{4.5.45}$$

In the same way we evaluate the AdS_5 part of the action

$$\begin{aligned} S_{AdS_5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau[r_1^2k_1^2+r_2^2k_2^2] \\ &= \sqrt{\lambda}\int d\sigma\left(\frac{{\zeta_1'}^2(\zeta_2^2-\zeta_1^2)}{(\zeta_1^2-k_1^2)(k_2^2-\zeta_1^2)}\frac{{\zeta_2'}^2(\zeta_2^2-\zeta_1^2)}{(\zeta_2^2-k_1^2)(\zeta_2^2-k_2^2)}\right) \\ &= -\sqrt{\lambda}\left(\int_{\frac{k_1k_2}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}}^{k_1}d\zeta_1+\int_\infty^{k_2}d\zeta_2\right) \\ &\simeq -\sqrt{\lambda}\left(k_1+k_2-\frac{k_1k_2}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\right) \end{aligned} \tag{4.5.46}$$

where again in the last equality the divergence has been removed.

Combining together the two parts of the action we find

$$S=\sqrt{\lambda}\left(-2k_1\pm\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}+(1\mp 1)\frac{k_1k_2}{\sqrt{k_1^2\sin^2\frac{\vartheta}{2}+k_2^2\cos^2\frac{\vartheta}{2}}}\right) \tag{4.5.47}$$

which again is the same as in the undeformed case.

Chapter 5

Correlators of Wilson loops in $\mathcal{N} = 1$ β-deformed SYM

In this section we consider the correlator between two concentric circles in $\mathcal{N} = 1$ β-deformed theory, where the Wilson loop couples to scalars in a periodic fashion. In fact, the same correlator with constant scalar coupling, treated in the undeformed theory by [16, 17], receives no correction from the deformed sphere, in a similar way we have seen for the two longitudes.

In our discussion, we study circles in the same plane, instead of taking them lying on parallel planes. But two concentric circles on parallel planes define a 2-sphere and, since we can relate any 2-sphere to the plane by a conformal transformation, those system are equivalent.

Again the σ model decouples, so we consider the $\tilde{S}^5$ part and the AdS_5 part of the action separately.

5.0.3 $\tilde{S}^3$ part

The ansatz for the coordinates in this case takes the form

$$\theta = \theta(\sigma), \quad \varphi_1 = \varphi_1(\sigma), \quad \varphi_2 = m\tau. \tag{5.0.1}$$

The boundary conditions on the first circle are

$$\theta = \theta_i, \quad \varphi_1 = 0, \quad \varphi_2 = m\tau \tag{5.0.2}$$

while for the second circle

$$\theta = \theta_f, \quad \varphi_1 = 0, \quad \varphi_2 = m\tau. \tag{5.0.3}$$

The $\tilde{S}^5$ part of the action is

$$\begin{aligned} S_{\tilde{S}^5} = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \Big[& \theta'^2 + G\cos^2\theta {\varphi'_1}^2 + G\sin^2\theta \dot{\varphi}_2^2 \\ & - 2i\hat{\gamma} G \sin^2\theta \cos^2\theta (\dot{\varphi}_1 \varphi'_2 - \varphi'_1 \dot{\varphi}_2) \Big] \\ = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \Big[& \theta'^2 - G\cos^2\theta {\varphi'_1}^2 + m^2 G \sin^2\theta - 2m\hat{\gamma} G \sin^2\theta \cos^2\theta \varphi'_1 \Big] \end{aligned} \tag{5.0.4}$$

where in the last equality we have performed a Wick rotation $\varphi_1 \to i\varphi_1$ in order to obtain a real configuration, and the G factor is given again by (4.5.10).

First, since φ_1 is cyclic, we can express it in term of his conserved momentum

$$-G\cos^2\theta \varphi'_1 - m\hat{\gamma} G \sin^2\theta \cos^2\theta = \pi_1 \tag{5.0.5}$$

but, for the surface to be closed, we must set $\pi_1 = 0$ and we have

$$\varphi'_1 = -m\hat{\gamma} \sin^2\theta. \tag{5.0.6}$$

Substituting this equation in the $\tilde{S}^5$ part of the Virasoro constraint we get an equation for θ

$$\begin{aligned} \theta'^2 - G\cos^2\theta {\varphi'_1}^2 - m^2 G \sin^2\theta = a^2 \\ \Rightarrow \theta'^2 - m^2 \sin^2\theta = a^2. \end{aligned} \tag{5.0.7}$$

In some cases, for example when we consider the expectation value of a single Wilson loop, the constant is $a^2 = 0$, but in the cases involving the correlator of two loops, $a^2 \neq 0$. For $a^2 > 0$ the angle θ will be a monotonous function of σ, and it is still possible to integrate (5.0.7) in terms of elliptic integrals of the first kind with argument θ and modulus im/a

$$\sigma + \sigma_i = \pm \frac{1}{a} F\left(\theta \Big| i\frac{m}{a}\right). \tag{5.0.8}$$

Let us suppose that the surface starts at $\sigma = 0$ on the first loop, then σ_i is fixed by

$$\sigma_i = \pm \frac{1}{a} F\left(\theta_i \Big| i\frac{m}{a}\right) \tag{5.0.9}$$

and the range of the σ variable is given by

$$\delta\sigma = \frac{1}{a} \left| F\left(\theta_f \Big| i\frac{m}{a}\right) - F\left(\theta_i \Big| i\frac{m}{a}\right) \right|. \tag{5.0.10}$$

We will have to check this with the AdS part of the ansatz.

Substituting (5.0.6) back into (5.0.4) we find for the action

$$\begin{aligned} S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma m^2 \sin^2\theta = \frac{T\sqrt{\lambda}}{2\pi}\int d\theta \frac{m^2\sin^2\theta}{|\theta'|} \\ &= \frac{T\sqrt{\lambda}}{2\pi} a\left[E\left(\theta\Big| i\frac{m}{a}\right) - F\left(\theta\Big| i\frac{m}{a}\right)\right]_{\theta_i}^{\theta_f} \end{aligned} \qquad (5.0.11)$$

For $a^2 < 0$ there will be an extremum for θ at some value θ_m, where $m^2 \sin^2\theta_m = -a^2$. Let us express the above integrals in a more appropriate form[1] for this case

$$\sigma + \sigma_i = \pm\frac{1}{b} F\left(\arccos\frac{\cos\theta}{\cos\theta_m}\Big| i\cot\theta_m\right) \qquad (5.0.12)$$

where we have introduced $b^2 = -a^2$ and $\sin\theta_m = b/m$. Again the value of σ_i is fixed by plugging in the boundary value θ_i.

This solution (5.0.12) has turning points at $\theta = \theta_m$ and $\theta = \pi - \theta_m$, and in order to describe this solution we must take the two branches with the opposite sign.

The full range of σ is now given by

$$\delta\sigma = \frac{1}{b}\left|F\left(\arccos\frac{\cos\theta_f}{\cos\theta_m}\Big| i\cot\theta_m\right) \pm F\left(\arccos\frac{\cos\theta_i}{\cos\theta_m}\Big| i\cot\theta_m\right)\right|. \qquad (5.0.13)$$

When there is a turning point along the worldsheet we have to add the contributes of both branches, so we have to pick the positive sign, while the negative sign is taken when there is no turning point.

Finally, the action is

$$S_{\tilde{S}^5} = \frac{T\sqrt{\lambda}}{2\pi} b\left[E\left(\arccos\frac{\cos\theta_f}{\cos\theta_m}\Big| i\cot\theta_m\right) \pm E\left(\arccos\frac{\cos\theta_i}{\cos\theta_m}\Big| i\cot\theta_m\right)\right] \qquad (5.0.14)$$

where the choice of the sign is the same as in (5.0.13).

Finally, we notice that if we set $\theta_i = \theta_f = \pi/2$, it is possible a solution [2] with constant $\theta = \pi/2$ and the kinetic part of the action is proportional to the range of the worldsheet coordinate σ, which expression is given in terms of complete elliptic integrals.

5.0.4 AdS_5 part

We consider the following anstaz

$$r_0 = r_0(\sigma), \quad r_1 = r_1(\sigma), \quad v = v(\sigma), \quad \phi_1 = k\tau, \quad r_2 = \phi_2 = 0. \qquad (5.0.15)$$

[1] We note that the solutions (5.0.8), (5.0.10) and (5.0.11) are formally also valid for a^2 negative.

[2] We notice that this solution is unstable, but it will play an interesting role when it will be coupled with the AdS_5 part.

The AdS_5 part of the action is

$$S_{AdS_5} = \frac{L^2}{4\pi\alpha'} \int d\sigma d\tau \, [-{r'_0}^2 + {r'_1}^2 + r_0^2 v'^2 + r_1^2 k_1^2 + \Lambda(-r_0^2 + r_1^2)] \quad (5.0.16)$$

where v is cyclic, so we can replace it with its conserved momentum

$$v' = \frac{p}{r_0^2}. \quad (5.0.17)$$

The first integral of motion is simple the diagonal AdS contribution to the stress-energy tensor

$$-{r'_0}^2 + {r'_1}^2 + \frac{p^2}{r_0^2} - k^2 r_1^2 + a^2 = 0. \quad (5.0.18)$$

The constant a^2 is the same as on the $\tilde{S}^5$ part of the action, so that together the Virasoro constraint is satisfied.

We use the identity $r_0^2 = 1 + r_1^2$ so (5.0.18) turns in an equation for r_1

$${r'_1}^2 = -a^2 - p^2 + (k^2 - a^2) r_1^2 + k^2 r_1^4 \quad (5.0.19)$$

or, in terms of $z = 1/r_1$

$$z'^2 = k^2 + (k^2 - a^2) z^2 - (a^2 + p^2) z^4. \quad (5.0.20)$$

Here we discuss two cases: the first is for $a^2 + p^2 > 0$, when the equation has a turning point, that is a maximal value for z beyond that the surface goes back to the boundary, while the other one [3] is for $a^2 + p^2 < 0$, when the surface reaches $z = \infty$ and we will have to analytically continue beyond that point to get the second part of the string. In both cases, the surface will reach the boundary twice, so it will correspond to the correlator of two Wilson loops.

Let us start with the case $a^2 + p^2 > 0$: the solution is easily written in terms of elliptic integrals of the first kind

$$\sigma = \frac{z_+}{k} F\left(\arcsin \frac{z}{z_+} \middle| \frac{z_+}{z_-} \right) \quad (5.0.21)$$

where $z_\pm^2$ are the two roots of the polynomial on the right side of (5.0.20)

$$z_\pm^2 = \frac{k^2 - a^2 \pm \sqrt{(a^2 + k^2)^2 + 4k^2 p^2}}{2(a^2 + p^2)}. \quad (5.0.22)$$

[3] We note that a^2 could be negative

The full range of the worldsheet coordinate σ is given in terms of the complete elliptic integrals

$$\delta\sigma = \frac{2z_+}{k} K\left(\frac{z_+}{z_-}\right). \tag{5.0.23}$$

The next step is to integrate v

$$\begin{aligned} v - v_i = p\int d\sigma\, \frac{1}{r_0^2} &= p\int \frac{dz}{z'}\frac{z^2}{1+z^2} \\ &= \frac{pz_+}{k}\left[F\left(\arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right) - \Pi\left(-z_+^2, \arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right)\right] \end{aligned} \tag{5.0.24}$$

where v_i is the initial value at $z = 0$. Since this expression covers only half the worldsheet, in order to obtain the other branch we have to shift this expression by the complete elliptic integrals

$$\begin{aligned} v - v_i = \frac{pz_+}{k}&\left[2K\left(\frac{z_+}{z_-}\right) - 2\Pi\left(-z_+^2\Big|\frac{z_+}{z_-}\right)\right. \\ &\left. -F\left(\arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right) + \Pi\left(-z_+^2, \arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right)\right]. \end{aligned} \tag{5.0.25}$$

On the second branch the surface reaches the final value v_f, so the total range is

$$\delta v = v_f - v_i = \frac{2pz_+}{k}\left[K\left(\frac{z_+}{z_-}\right) - \Pi\left(-z_+^2\Big|\frac{z_+}{z_-}\right)\right]. \tag{5.0.26}$$

The last step it to evaluate the action. To do it rightfully, we must include the two branch and regularize the divergence near the boundary by the cutoff z_0

$$\begin{aligned} S_{AdS_5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\; k^2 r_1^2 = \sqrt{\lambda}k^2\int dz\;\frac{1}{z^2 z'} \\ &= -\sqrt{\lambda}\,2k\left[\frac{1}{z}\sqrt{(1-\frac{z^2}{z_+^2})(1-\frac{z^2}{z_-^2})}\right. \\ &\quad \left. -\frac{1}{z_+}F\left(\arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right) + \frac{1}{z_+}E\left(\arcsin\frac{z}{z_+}\Big|\frac{z_+}{z_-}\right)\right]_{z_0}^{z_+} \\ &= \sqrt{\lambda}\,2k\left[\frac{1}{z_0} - \frac{1}{z_+}E\left(\frac{z_+}{z_-}\right) + \frac{1}{z_+}K\left(\frac{z_+}{z_-}\right)\right]. \end{aligned} \tag{5.0.27}$$

The divergent term, as usual, is canceled by a boundary term.

The $a^2 + p^2 < 0$ case again describes the correlator of two circles, but the orientation of the circles will be the opposite of the previous case. In fact, now the two circles are orientated in the same direction.

Since the roots $z_\pm$ defined in (5.0.22) now are imaginary, it is more convenient to define the positive constant $b^2 = -a^2$ and use

$$\tilde{z}_\pm^2 = \frac{k^2 + b^2 \pm \sqrt{(b^2 - k^2)^2 + 4k^2p^2}}{2(b^2 - p^2)} = -z_\pm^2. \tag{5.0.28}$$

Following the same procedure as above we find the solution in term of elliptic integrals with a different modulus

$$\sigma = \frac{\tilde{z}_+}{k} F\left(\arcsin\frac{z}{\tilde{z}_+}\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right). \tag{5.0.29}$$

We notice that, after we have analytically continued the solution beyond $z = \infty$, the surface will reach the boundary at $z = 0$ again at $\arctan z/z_+ = \pi$. Again the full range of σ is twice the complete elliptic integrals

$$\delta\sigma = \frac{2\tilde{z}_+}{k} K\left(\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right). \tag{5.0.30}$$

Next we integrate the coordinate v

$$\begin{aligned} v - v_i = \frac{p\tilde{z}_+^3}{k(\tilde{z}_+^2 - 1)} \Bigg[& F\left(\arcsin\frac{z}{\tilde{z}_+}\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right) \\ & -\Pi\left(1 - \tilde{z}_+^2, \arcsin\frac{z}{\tilde{z}_+}\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right)\Bigg]. \end{aligned} \tag{5.0.31}$$

This expression is valid only for the first branch, while for the second half of the worldsheet we have

$$\begin{aligned} v - v_i = \frac{p\tilde{z}_+^3}{k(\tilde{z}_+^2 - 1)} \Bigg[& 2K\left(\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right) - 2\Pi\left(1 - \tilde{z}_+^2\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right) \\ & -F\left(\arcsin\frac{z}{\tilde{z}_+}\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right) + \Pi\left(1 - \tilde{z}_+^2, \arcsin\frac{z}{\tilde{z}_+}\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right)\Bigg]. \end{aligned} \tag{5.0.32}$$

The total change is

$$\delta v = v_f - v_i = \frac{2p\tilde{z}_+^3}{k(\tilde{z}_+^2 - 1)} \left[K\left(\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right) - \Pi\left(1 - \tilde{z}_+^2\Big|\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right)\right]. \tag{5.0.33}$$

Our last step is to evaluate the kinetic part of the action

$$S_{AdS_5} = 2S_{AdS_5}^{kinetic} = -\sqrt{\lambda}\,\frac{2k}{\tilde{z}_+} E\left(\sqrt{1 - \frac{\tilde{z}_+^2}{\tilde{z}_-^2}}\right). \tag{5.0.34}$$

The final result for the total action is given by the sum of (5.0.34) and (5.0.14), and it reproduce exactly the value obtained for $\mathcal{N} = 4$ SYM theory.

Chapter 6

Supersymmetric Wilson loops and pure spinors

In the previous chapters we have studied a number of supersymmetric Wilson loop, all of which are captured by two classes: the loops of arbitrary shape on $\mathbb{R}^4$ found by Zarembo [20] and the loops of arbitrary shape on a three sphere $S^3 \subset \mathbb{R}^4$, also known as DGRT loops [21]. It has not been clear whether these two classes contain all possible supersymmetric Wilson loop, until Pestun and Dymarsky [23] gave a systematic answer to this question. Indeed, they find all possible Wilson loop operators W that are invariant at least under one superconformal symmetry Q. Moreover, they classify the interesting subclasses of pairs (Q, W) modulo equivalence under the action of the superconformal group of $\mathcal{N} = 4$ SYM.

They found new supersymmetric Wilson loops which in many cases, involve complex scalar couplings that clearly distinguishes them from the previously studied cases. The crucial ingredient in their construction are the ten-dimensional pure spinors. In fact, the space-time dependent spinor ϵ that parametrizes the superconformal transformations of $\mathcal{N} = 4$ SYM, can be viewed as a reduction of a chiral ten-dimensional spinor.

Our aim is to study the dual gravity solution of this new class of loops in the case in which the ten-dimensional supersymmetry generator spinor $\epsilon(x)$ is not pure. In this case, the tangent vector to the curve and the scalar couplings are completely fixed. The curve we will consider are (p, q) Lissajous figures where $\frac{p}{q} \in \mathbb{Q}$ is rational[1].

[1] In the following, we will indicate such a ratio as q. This will not lead to confusion, since neither p, nor q will appear singularly.

6.1 Lissajous figures

The curve in terms of cartesian coordinates is given by

$$x^1 = \sin\frac{\vartheta}{2}\cos\tau, \quad x^2 = \cos\frac{\vartheta}{2}\sin q\tau, \quad x^3 = \cos\frac{\vartheta}{2}\cos q\tau, \quad x^4 = \sin\frac{\vartheta}{2}\sin\tau. \tag{6.1.1}$$

The normalized scalar coupling is given by

$$\begin{aligned}
\theta^0 &= -\frac{i}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}} \\
\theta^5 &= -\frac{i}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}\sqrt{1-q^2}\cos\frac{\vartheta}{2}\cos q\tau \\
\theta^9 &= \frac{i}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}\sqrt{1-q^2}\cos\frac{\vartheta}{2}\sin q\tau.
\end{aligned} \tag{6.1.2}$$

In order to find the dual string configuration we parametrize as usual

$$\mu_1 = \cos\theta, \quad \mu_2 = \sin\theta\cos\alpha, \quad \mu_3 = \sin\theta\sin\alpha \tag{6.1.3}$$

and we set

$$\begin{aligned}
\theta^7 + i\theta^0 = \mu_1 e^{i\varphi_1} &= \frac{1}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}} \\
\theta^9 + i\theta^5 = \mu_2 e^{\varphi_2} &= \frac{\sqrt{1-q^2}\cos\frac{\vartheta}{2}}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}\Big(\cos q\tau + i\sin q\tau\Big) \\
\theta^6 + i\theta^8 = \mu_3 e^{i\varphi_3} &= 0.
\end{aligned} \tag{6.1.4}$$

The boundary values are $\varphi_2 = q\tau$ and $\alpha = \varphi_1 = \varphi_3 = 0$, while

$$\sin\theta_0 = \frac{\sqrt{1-q^2}\cos\frac{\vartheta}{2}}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}. \tag{6.1.5}$$

The σ model again decouples and we discuss the S^5 and the AdS_5 parts of the action separately. Let us begin with the action of the five-sphere, and afterward, we will see that the AdS part is quite straightforward.

Our S^5 ansatz is

$$\theta = \theta(\sigma), \quad \varphi_2 = q\tau. \tag{6.1.6}$$

The S^5 part of the action is

$$\begin{aligned}
S_{S^5} &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[\theta'^2 + \sin^2\theta\dot{\varphi}_2^2\Big] \\
&= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[\theta'^2 + q^2\sin^2\theta\Big].
\end{aligned} \tag{6.1.7}$$

The diagonal component of the stress-energy tensor gives us an equation for θ

$$\theta'^2 - q^2 \sin^2\theta = 0 \tag{6.1.8}$$

together with the boundary condition (6.1.5).

The solution is given by

$$\sin\theta = \frac{1}{\cosh[q(\sigma_0 \pm \sigma)]}. \tag{6.1.9}$$

The sign choice corresponds to a surface wrapping the northern or the southern hemisphere and the integration constant σ_0 is chosen so that at $\sigma = 0$ it reaches the boundary value

$$\sin\theta_0 = \frac{1}{\cosh(q\sigma_0)} = \frac{\sqrt{1-q^2}\cos\frac{\vartheta}{2}}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}. \tag{6.1.10}$$

The last step is to integrate the action, where for the curve to be closed, the period over τ is $2\pi/q$. Hence

$$\begin{aligned} S_{S^5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau \Big(q^2\sin^2\theta\Big) \\ &= \sqrt{\lambda}\, q\int_0^\infty d\sigma\, \frac{1}{\cosh^2[q(\sigma_0\pm\sigma)]} \\ &= \sqrt{\lambda}\,\Big[\pm\tanh[q(\sigma_0\pm\sigma)]\Big]_0^\infty \\ &= \sqrt{\lambda}\,\Big(1\pm\tanh[q\sigma_0]\Big) \\ &= \sqrt{\lambda}\,\left(1\pm\frac{1}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}\right). \end{aligned} \tag{6.1.11}$$

In order to find out the AdS_5 contribute for the action, we note that the loop described in (6.1.1) is a special case of toroidal loops that we have already discussed in section 4.5. Indeed, we recover our present loop by substituting $k_1 = 1$ and $k_2 = q$ in (4.5.36). As a consequence we follow the same procedure and we obtain the AdS_5 kinetic part of the action

$$S_{AdS_5} \simeq -\sqrt{\lambda}\left(\frac{1}{q} + 1 - \frac{1}{\sqrt{\sin^2\frac{\vartheta}{2} + q^2\cos^2\frac{\vartheta}{2}}}\right) \tag{6.1.12}$$

where we have already removed the usual linear divergence.

Putting together (6.1.12) and (6.1.11) we obtain the total action

$$\begin{aligned} S &= \sqrt{\lambda}\left(-\frac{1}{q}+\frac{1}{\sqrt{\sin^2\frac{\vartheta}{2}+q^2\cos^2\frac{\vartheta}{2}}}\pm\frac{1}{\sqrt{1-(1-q^2)\cos^2\frac{\vartheta}{2}}}\right) \\ &= \sqrt{\lambda}\left(-\frac{1}{q}+(1\pm 1)\frac{1}{\sqrt{1-(1-q^2)\cos^2\frac{\vartheta}{2}}}\right). \end{aligned} \tag{6.1.13}$$

We notice that an exact cancellation between the S^5 part of the action and a piece of the AdS_5 one take places.

6.1.1 $q = 0$ case

In this case the curve takes the form of a latitude

$$x^1 = \sin\frac{\vartheta}{2}\cos\tau, \quad x^2 = 0, \quad x^3 = \cos\frac{\vartheta}{2}, \quad x^4 = \sin\frac{\vartheta}{2}\sin\tau \tag{6.1.14}$$

and the scalar couplings simplify to

$$\begin{aligned} \theta^0 &= -\frac{i}{\sqrt{1+\cos^2\frac{\vartheta}{2}}} \\ \theta^5 &= -\frac{i}{\sqrt{1+\cos^2\frac{\vartheta}{2}}}\cos\frac{\vartheta}{2} \\ \theta^9 &= 0. \end{aligned} \tag{6.1.15}$$

We note that the time dependence disappears here, therefore the scalar couplings are constants.

To find the dual string configuration, we parametrize the μ_i coordinates as in (6.1.3) and we note that

$$\begin{aligned} \theta^7 + i\theta^0 &= \mu_1 e^{i\varphi_1} = \frac{1}{\sqrt{1+\cos^2\frac{\vartheta}{2}}} \\ \theta^9 + i\theta^5 &= \mu_2 e^{\varphi_2} = \frac{\cos\frac{\vartheta}{2}}{\sqrt{1+\cos^2\frac{\vartheta}{2}}} \\ \theta^6 + i\theta^8 &= \mu_3 e^{i\varphi_3} = 0. \end{aligned} \tag{6.1.16}$$

At the boundary, the dual string configuration must satisfy $\alpha = \varphi_1 = \varphi_2 = \varphi_3 = 0$, while

$$\sin\theta_0 = \frac{\cos\frac{\vartheta}{2}}{\sqrt{1+\cos^2\frac{\vartheta}{2}}} \tag{6.1.17}$$

For the AdS_5 part of he σ model we use the target space metric

$$ds^2 = L^2 \left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho(d\chi^2 + \cos^2\chi\, d\psi^2 + \sin^2\chi\, d\phi^2\right).$$

Since the circle will follow the coordinate ψ on a latitude of the S^3 on the boundary of AdS_5, we use the ansatz

$$\rho = \rho(\sigma)\,, \qquad \psi(\tau) = \tau\,, \qquad t = \chi = \phi = 0 \tag{6.1.18}$$

that yields to the kinetic AdS action in conformal gauge

$$S_{AdS_5} = \frac{L^2}{4\pi\alpha'}\int d\sigma\, d\tau\left[\rho'^2 + \sinh^2\rho\right]. \tag{6.1.19}$$

The equation of motion for ρ is

$$\rho'' = \sinh\rho\cosh\rho \tag{6.1.20}$$

from which we obtain a first integral

$$\rho'^2 - \sinh^2\rho = c. \tag{6.1.21}$$

In order to get a surface that corresponds to a single circle and not the correlator or two one has to set $c = 0$, so the solution is

$$\sinh\rho(\sigma) = \frac{1}{\sinh\sigma} \tag{6.1.22}$$

where an integration constant in this equation that shifts σ was set to zero so the boundary of the world-sheet at $\sigma = 0$ is at the boundary of AdS_5.

The bulk part of the classical action is proportional to the area

$$\begin{aligned} S_{AdS_5} &= \sqrt{\lambda}\int d\sigma\ \sinh^2\rho = \int_{\sigma_{\min}}^{\infty} d\sigma\,\frac{1}{\sinh^2\sigma} = \sqrt{\lambda}\,(\coth\sigma_{\min} - 1) \\ &= \sqrt{\lambda}\,(\cosh\rho_{\max} - 1). \end{aligned} \tag{6.1.23}$$

Here $\sigma_{\min}$ is a cutoff on σ and $\rho_{\max}$ the corresponding cutoff on ρ. The first term is exactly the divergent part which is canceled by an extra boundary term in the action.

From the boundary condition (6.1.17) our ansatz involves only the angle $\theta = \theta(\sigma)$, therefore the S^5 part of the action reduces simply to

$$S_{S^5} = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\ \theta'^2. \tag{6.1.24}$$

Since the variable θ is cyclic, we could express its derivative in terms of the conserved momenta, but the S^5 part of the diagonal stress-energy tensor forces it to vanish. Therefore θ results to be constant and its value is given by the boundary condition (6.1.17). As a consequence, the S^5 part of the action vanishes and it does not contribute to the vev of the Wilson loop operator.

From this one finds that the expectation value of the Wilson loop at strong coupling is given by

$$\langle W \rangle = e^{-\sqrt{\lambda}} . \tag{6.1.25}$$

This is the same result we find in section 4.2.

Hence, the expectation value for a generic latitude coupled to scalars in the way proposed by Dymarsky and Pestun is equal to the value for a maximal latitude with the standard scalar couplings.

6.2 β-deformed Lissajous figures

Let us calculate the same Wilson loop considered in section 6.1 in $\mathcal{N} = 1$ β-deformed SYM. The loop and the scalar coupling are defined in the same way.

Our $\tilde{S}^5$ ansatz is

$$\theta = \theta(\sigma), \quad \varphi_2 = q\tau, \quad \varphi_1 = \varphi_1(\sigma). \tag{6.2.1}$$

As usual, due to the presence of the B-field, we must consider an ansatz involving an additional angle φ_1.

The $\tilde{S}^5$ part of the action is

$$\begin{aligned} S_{\tilde{S}^5} &= \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \Big[\theta'^2 + G\cos^2\theta {\varphi_1'}^2 + G\sin^2\theta \dot{\varphi}_2^2 \\ &\qquad\qquad - 2i\hat{\gamma} G \sin^2\theta\cos^2\theta (\dot{\varphi}_1\varphi_2' - \varphi_1'\dot{\varphi}_2) \Big] \\ &= \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \Big[\theta'^2 - G\cos^2\theta {\varphi_1'}^2 + Gq^2 \sin^2\theta - 2q^2\hat{\gamma} G \sin^2\theta\cos^2\theta \varphi_1' \Big] \end{aligned} \tag{6.2.2}$$

where in the last equality we have performed a Wick rotation $\varphi_1 \to i\varphi_1$ in order to obtain a real configuration, and the G factor is given by

$$G^{-1} = 1 + \hat{\gamma}^2 \sin^2\theta \cos^2\theta. \tag{6.2.3}$$

Since φ_1 is cyclic, we introduce his conserved momentum

$$-G\cos^2\theta\varphi_1' - (k_2 - k_1)\hat{\gamma} G \sin^2\theta\cos^2\theta = \pi \tag{6.2.4}$$

but, for the surface to be closed, we must set $\pi = 0$ and we have

$$\varphi_1' = (k_1 - k_2)\hat{\gamma}\sin^2\theta. \tag{6.2.5}$$

Substituting this equation in the $\tilde{S}^5$ part of the Virasoro constraint we get an equation for θ

$$\theta'^2 - G\cos^2\theta\varphi_1'^2 - q^2 G\sin^2\theta = 0$$
$$\Rightarrow \theta'^2 = q^2\sin^2\theta \qquad (6.2.6)$$

together with the boundary condition (6.1.5), that forces the integration constant σ_0 to be non-zero. In fact, the solution is given by

$$\sin\theta = \frac{1}{\cosh[q(\sigma_0 \pm \sigma)]} \qquad (6.2.7)$$

$$\varphi_1 = \hat{\gamma}\big(\tanh(\sigma \pm \sigma_0) \mp \tanh(\sigma_0)\big). \qquad (6.2.8)$$

The sign choice corresponds to a surface wrapping the northern or the southern hemisphere and the integration constant σ_0 is chosen so that at $\sigma = 0$ it reaches the boundary value

$$\frac{1}{\cosh[q\sigma_0]} = \frac{\sqrt{1-q^2}\cos\frac{\vartheta}{2}}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}. \qquad (6.2.9)$$

The last step is to integrate the action

$$\begin{aligned} S_{S^5} &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau\Big(q^2\sin^2\theta\Big) \\ &= \sqrt{\lambda}\, q\int_0^\infty d\sigma\,\frac{1}{\cosh^2[q(\sigma_0 \pm \sigma)]} \\ &= \sqrt{\lambda}\,\Big[\pm\tanh[q(\sigma_0 \pm \sigma)]\Big]_0^\infty \\ &= \sqrt{\lambda}\,\Big(1 \pm \tanh[q\sigma_0]\Big) \\ &= \sqrt{\lambda}\,\left(1 \pm \frac{1}{\sqrt{1+(1-q^2)\cos^2\frac{\vartheta}{2}}}\right). \end{aligned} \qquad (6.2.10)$$

This result matches exactly with the undeformed one.

6.3 Circles correlator in the $q = 0$ case

In this section we want to evaluate the correlator of two concentric circles with the scalar coupling given by (6.1.15). Let us write explicitly the two loops γ_1 and γ_2 in cartesian coordinates

$$\begin{aligned} \gamma_1 &: x^1 = \sin\frac{\vartheta_1}{2}\cos\tau, \quad x^2 = 0, \quad x^3 = \cos\frac{\vartheta_1}{2}, \quad x^4 = \sin\frac{\vartheta_1}{2}\sin\tau \\ \gamma_2 &: x^1 = \sin\frac{\vartheta_2}{2}\cos\tau, \quad x^2 = 0, \quad x^3 = \cos\frac{\vartheta_2}{2}, \quad x^4 = \sin\frac{\vartheta_2}{2}\sin\tau \end{aligned} \qquad (6.3.1)$$

and the corresponding non-vanishing scalar coupling are simply obtained by replacing ϑ with ϑ_1, ϑ_2 in (6.1.15)

$$\begin{aligned} \gamma_1 \ &: \ \theta^0 = -\frac{i}{\sqrt{1+\cos^2\frac{\vartheta_1}{2}}} \, , \qquad \theta^5 = -\frac{i\cos\frac{\vartheta_1}{2}}{\sqrt{1+\cos^2\frac{\vartheta_1}{2}}} \\ \gamma_2 \ &: \ \theta^0 = -\frac{i}{\sqrt{1+\cos^2\frac{\vartheta_2}{2}}} \, , \qquad \theta^5 = -\frac{i\cos\frac{\vartheta_2}{2}}{\sqrt{1+\cos^2\frac{\vartheta_2}{2}}} . \end{aligned} \tag{6.3.2}$$

Let us start with the S^5 part of the σ model. It is simple to convince ourselves that the correct ansatz should involve only one angle. The boundary conditions for both circles are simply obtained from (6.1.16) and read $\alpha = \varphi_1 = \varphi_2 = \varphi_3 = 0$ for both, while for the θ coordinate they reach different values on the boundary

$$\begin{aligned} \gamma_1 \ &: \ \sin\theta_0^i = \frac{\cos\frac{\vartheta_1}{2}}{\sqrt{1+\cos^2\frac{\vartheta_1}{2}}} \\ \gamma_2 \ &: \ \sin\theta_0^f = \frac{\cos\frac{\vartheta_2}{2}}{\sqrt{1+\cos^2\frac{\vartheta_2}{2}}} . \end{aligned} \tag{6.3.3}$$

This means that both circles couple to fixed point on S^5, but these points are situated along the same longitude. Therefore, our ansatz involve only one angle $\theta = \theta(\sigma)$ and the S^5 part of the action is simply

$$S_{S^5} \ = \ \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \ \theta'^2 . \tag{6.3.4}$$

Unlike (6.1.24), in this case the action will not vanish, due to different boundary conditions. Again, θ is cyclic, then we can express its derivative in terms of the conserved momenta Π, that is $\theta' = \Pi$. In this case we will keep $\Pi \neq 0$, since we are dealing with correlators, and not expectation values, of Wilson loops. It is simple now to integrate the equation of motion and obtain

$$\theta = \Pi\sigma + \theta_0^i \tag{6.3.5}$$

since $\theta(\sigma = 0) = \theta^i$. Then

$$S_{S^5} \ = \ \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \ \Pi^2 = \frac{\sqrt{\lambda}}{2} \delta\sigma \Pi^2 . \tag{6.3.6}$$

We have already solved the AdS part of the σ model for the correlator of two generic concentric circles in section (5.0.4). First, we notice that the coordinates (5.0.15) of the AdS_5 metric we chose are related to our parameters by

$$\sin\frac{\vartheta_1}{2} = e^{v_i}, \quad \sin\frac{\vartheta_2}{2} = e^{v_f}, \quad a = \Pi, \quad k = 1. \tag{6.3.7}$$

In this picture the angles ϑ_1 and ϑ_2 are the *radii* of the circles, and the third condition is necessary to ensure that the total Virasoro constraint is satisfied.

Using (6.3.7), the two roots of the polynomial on the right side of (5.0.20) are given by

$$\hat{z}_\pm^2 = \frac{1 - a^2 \pm \sqrt{(a^2+1)^2 + 4p^2}}{2(a^2+p^2)}. \tag{6.3.8}$$

Then, the solution for the worldsheet coordinate σ is again given in terms of elliptic integrals of the first kind

$$\sigma = \hat{z}_+ F\left(\arcsin\frac{z}{\hat{z}_+}\middle|\frac{\hat{z}_+}{\hat{z}_-}\right) \tag{6.3.9}$$

and, similarly, the full range

$$\delta\sigma = 2\hat{z}_+ K\left(\frac{\hat{z}_+}{\hat{z}_-}\right). \tag{6.3.10}$$

Now we can find the value of the conserved momenta Π for this configuration

$$\theta(\delta\sigma) = \theta_0^f \quad \Longrightarrow \quad \Pi = \frac{\theta_0^f - \theta_0^i}{\delta\sigma}. \tag{6.3.11}$$

The results for v in both branches are

$$\begin{aligned} v - v_i &= p\hat{z}_+\left[F\left(\arcsin\frac{z}{\hat{z}_+}\middle|\frac{\hat{z}_+}{\hat{z}_-}\right) - \Pi\left(-\hat{z}_+^2, \arcsin\frac{z}{\hat{z}_+}\middle|\frac{\hat{z}_+}{\hat{z}_-}\right)\right] \\ v - v_i &= p\hat{z}_+\left[2K\left(\frac{\hat{z}_+}{\hat{z}_-}\right) - 2\Pi\left(-\hat{z}_+^2\middle|\frac{\hat{z}_+}{\hat{z}_-}\right)\right. \\ &\quad \left. -F\left(\arcsin\frac{z}{\hat{z}_+}\middle|\frac{\hat{z}_+}{\hat{z}_-}\right) + \Pi\left(-\hat{z}_+^2, \arcsin\frac{z}{\hat{z}_+}\middle|\frac{\hat{z}_+}{\hat{z}_-}\right)\right] \end{aligned} \tag{6.3.12}$$

and the total range is

$$\delta v = v_f - v_i = 2p\hat{z}_+\left[K\left(\frac{\hat{z}_+}{\hat{z}_-}\right) - \Pi\left(-\hat{z}_+^2\middle|\frac{\hat{z}_+}{\hat{z}_-}\right)\right] \tag{6.3.13}$$

which is related to the ratio of the radii of the loops by

$$\frac{\sin\frac{\vartheta_1}{2}}{\sin\frac{\vartheta_2}{2}} = e^{\delta v}. \tag{6.3.14}$$

Finally, the action

$$S_{AdS_5} = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\, r_1^2 - \frac{\sqrt{\lambda}}{2}\delta\sigma\Pi^2$$
$$\simeq -\frac{2\sqrt{\lambda}}{\hat{z}_+}\left[E\left(\frac{\hat{z}_+}{\hat{z}_-}\right) - K\left(\frac{\hat{z}_+}{\hat{z}_-}\right)\right] - \frac{\sqrt{\lambda}}{2}\delta\sigma\Pi^2 \qquad (6.3.15)$$

where we have already removed the usual linear divergence. We notice that the second term cancels exactly the full S^5 action, so that the full regularized action is given in terms of complete integrals

$$S = -\frac{2\sqrt{\lambda}}{\hat{z}_+}\left[E\left(\frac{\hat{z}_+}{\hat{z}_-}\right) - K\left(\frac{\hat{z}_+}{\hat{z}_-}\right)\right]. \qquad (6.3.16)$$

Here we have focus on the case for the circles to have opposite orientation, but the procedure is exactly the same if they have the same orientation.

In principle, we could find the explicit dependence of the string action on the radii parameter ϑ_1 and ϑ_2. In fact, we should invert (6.3.13) and, using (6.3.14) we would find the value of the conserved momenta p in terms of the parameters of the loops. Then we should substitute this value into (6.3.8) and evaluate the elliptic integrals in (6.3.15). Although an analytic approach seems quite impossible, it could be done numerically. The most interesting result would be to verify if the Gross-Ooguri phase transition takes place.

6.3.1 Coincident circles

Let us now consider the case of coincident two circles on the boundary, that corresponds to $\vartheta_1 = \vartheta_2$. In this case v does not vary, and we can set it to zero. Therefore, in this case the ansatz involves only the coordinates r_0, r_1 and ϕ_1, which parametrize an AdS_2 subspace of AdS_5.

We will consider the case $a > 0$, corresponding to the correlator of two coincident circles of opposite orientation. Hence the solution is the same as the more general case described in the previous section, providing that now (6.3.8) becomes

$$\hat{z}_+ = \frac{1}{a}, \qquad \hat{z}_- = i. \qquad (6.3.17)$$

Hence

$$\sigma = \frac{1}{a}F\left(\arcsin az \Big| \frac{i}{a}\right). \qquad (6.3.18)$$

We notice that the coordinate z takes values between the boundary of AdS_5, when it is zero, and its maximum $\hat{z}_+$, and then comes down to the boundary of AdS_5 again. Therefore the full range of σ is given by

$$\delta\sigma = \frac{2}{a}K\left(\frac{i}{a}\right) \qquad (6.3.19)$$

It is easy to show that the regularized AdS_5 action (6.3.16) turns into

$$S_{AdS_5} = -2a\sqrt{\lambda}\left[E\left(\frac{i}{a}\right) - K\left(\frac{i}{a}\right)\right] - \frac{\sqrt{\lambda}}{2}\delta\sigma a^2 \tag{6.3.20}$$

where we recall from (6.3.7) that the Virasoro constraint forces $\Pi = a$. However, since the circles are coincident they couple to the same constant point in S^5, since from $\vartheta_1 = \vartheta_2$ follows that $\theta_0^f = \theta_0^i$, and (6.3.11) leads to $\Pi = 0$. The S^5 solution is then described by an angle $\theta = \theta_0^i$ constant, where its value is fixed by the radius of the circles by (6.3.3). As a consequence, the S^5 part of the action vanishes and it does not contribute to the correlator.

Then, we have to find the solution for the AdS_5 part of the model for $a = 0$, which it is very simple as we will see. The equation of motion (5.0.20) for $z = 1/r_1$ in this case reads

$$z'^2 = 1 + z^2 \tag{6.3.21}$$

which it is solved by

$$z = \sinh\sigma. \tag{6.3.22}$$

The action in this case is, after subtracting the divergence

$$S = S_{AdS_5} = -2\sqrt{\lambda}. \tag{6.3.23}$$

This is exactly the value known for the expectation value of two circle Wilson loops in the disconnected phase.

6.4 Phase transition analysis

The result found in (6.3.23) for the two coincident circles suggests us that the connected solution might not exist. This because the value of the string action is equal to the value of the disconnected solution, that is the values of the circles separately. It is worth to ask if the connected solution effectively exists, namely if the Gross-Ooguri phase transition take places. In order to answer such a question, we fix one of the two circles at the equator of S^2, and we keep the radius of the other circles as our free parameter. Due to the new scalar couplings, all the other quantities are fixed by the geometry, once we set the distance between the circles.

As we have already announced, we put one circle at the equator of S^2, so we set $\vartheta_1 = \pi$ and $\vartheta_2 = \vartheta$, while the other circles is defined by the equation

$$\sin\theta_0^f = \frac{\cos\frac{\vartheta}{2}}{\sqrt{1+\cos^2\frac{\vartheta}{2}}}. \tag{6.4.1}$$

We introduce the relevant parameter ξ, which is the ratio of the *radii* of the circles

$$\frac{\sin\frac{\vartheta_1}{2}}{\sin\frac{\vartheta_2}{2}} = \frac{1}{\sin\frac{\vartheta}{2}} = \xi^{-1} = e^{\delta v}. \tag{6.4.2}$$

We can rewrite (6.4.1) in a more suitable form

$$\theta_0^f = \arctan\cos\frac{\vartheta}{2} = \arctan\sqrt{1-\xi^2}. \tag{6.4.3}$$

Then, the equations (6.3.8)-(6.3.13) become

$$-\ln\xi = 2p\hat{z}_+\left[K\left(\frac{\hat{z}_+}{\hat{z}_-}\right) - \Pi\left(-\hat{z}_+^2\Big|\frac{\hat{z}_+}{\hat{z}_-}\right)\right] \tag{6.4.4}$$

$$\hat{z}_\pm^2 = \frac{1-a^2\pm\sqrt{(a^2+1)^2+4p^2}}{2(a^2+p^2)} \tag{6.4.5}$$

$$\theta_0^f = 2a\hat{z}_+K\left(\frac{\hat{z}_+}{\hat{z}_-}\right). \tag{6.4.6}$$

We define $t = \frac{\hat{z}_+}{\hat{z}_-}$ and we choose t and $\hat{z}_+$ as our variables. Explicitly

$$t^2 = \frac{1-a^2+\sqrt{(a^2+1)^2+4p^2}}{1-a^2-\sqrt{(a^2+1)^2+4p^2}} \tag{6.4.7}$$

from which we get an expression for p^2

$$p^2 = -\frac{(1+a^2t^2)(a^2+t^2)}{(1+t^2)^2}. \tag{6.4.8}$$

Substituting this back into (6.4.5) we obtain two solution for $\hat{z}_+$

$$\hat{z}_+^2 = -\frac{1+t^2}{1-a^2}, \qquad \hat{z}_+^2 = -\frac{1+t^{-2}}{1-a^2}. \tag{6.4.9}$$

We note that the two solution are related by the transformation $t \to 1/t$. Therefore, we have two possible values for a^2

$$a^2 = \frac{1+t^2+\hat{z}_+^2}{\hat{z}_+^2}, \qquad a^2 = \frac{1+t^{-2}+\hat{z}_+^2}{\hat{z}_+^2}. \tag{6.4.10}$$

Now, from (6.4.6) it is simple to obtain an expression for ξ

$$\xi^2 = 1-\tan^2\left(2\sqrt{1+t^2+\hat{z}_+^2}\;K(t)\right) \tag{6.4.11}$$

where we have chosen the first solutions in (6.4.9) and (6.4.10). Plugging the first expression for a^2 into (6.4.8), the equation (6.4.4) gives the final relation between t and $\hat{z}_+$

$$\ln\left[1-\tan^2\left(2\sqrt{1+t^2+\hat{z}_+^2}\;K(t)\right)\right] = \pm\frac{4}{\hat{z}_+}\sqrt{(1+\hat{z}_+^2)(z_+^2+t^2)}\left[K\left(\frac{\hat{z}_+}{\hat{z}_-}\right)-\Pi\left(-\hat{z}_+^2\middle|\frac{\hat{z}_+}{\hat{z}_-}\right)\right]. \tag{6.4.12}$$

This equation, as the radius of the "free" circle varies, describes a curve in the Gross-Ooguri phase diagram. Our next step will be to determine if this curve intersects the phase transition line or if it lies in only one single region of the diagram, that is no phase transition takes place.

Conclusions and outlooks

In this work, we faced the study of the Wilson loop operator in two supersymmetric gauge theories, namely $\mathcal{N} = 4$ super Yang Mills and a exact marginal deformation of it, known as $\mathcal{N} = 1$ β-deformed super Yang Mills. Within the framework of the AdS/CFT correspondence, the vacuum expectation value of the Wilson loop can be evaluated in string theory in the limit of large gauge group parameter N and large 't Hooft constant λ. More in detail, Wilson loops are evaluated in these limits by finding the classical solution of the string equation of motion in the product space $AdS_5 \times S^5$, and analyzing the corresponding value of the string action.

Within this framework, we checked that the vacuum expectation value for some Wilson loops in the two theories are equal. In particular, we showed that this result does not hold for 1/2 or 1/4 BPS Wilson loop only, the only cases treated in literature so far, but also for operators that preserve a less amount of supersymmetry (for example, the toroidal loop which are 1/8 BPS operators), or even non-BPS operators. This last case corresponds to Wilson loops correlator. In this contest, the same Gross-Ooguri phase transition, namely the transition through the solution described by a connected minimal surface to a disconnected one, is obtained in the two theories. This result reinforces the conjecture that the expectation value of the Wilson loop in $\mathcal{N} = 1$ β-deformed SYM at strong coupling is described by the same matrix model as in the non-deformed theory. It would be worth to investigate if there exist other loops, corresponding to relevant operators in the field theories, that do not share the same expectation value. Another possibility would be to check if the subleading corrections at order $1/N$ differ between the two theories, and if it possible to calculate them at strong coupling with the D-branes techniques we used in this work.

We have then turned to consider toroidal loops with a different scalar coupling on S^5 (Lissajous figures), proposed by Dymarsky and Pestun recently: they proved that the two classes of coupling contain all possible supersymmetric Wilson loops with this type of geometry. In $\mathcal{N} = 4$ SYM theory, the result obtained at strong coupling matches with the computation at weak coupling performed using perturbation theory. The common feature of these solutions is an exact cancelation between the S^5 part of the action and a piece of the AdS_5 part of the action. As a consequence, the expectation value of such operators posses a trivial dependence on the parameters that define the loops. We have shown that even for this new type of Wilson loop the same results are obtained in the $\mathcal{N} = 1$ β-deformed SYM.

Furthermore, the correlator of the circles of generic radii has been evaluated. The result found for the action when the two circles are coincident suggests us that the connected solution might not exist. This because the value of the string action is equal to the value of the disconnected solution, that is the sum of the values of the circles separately. It is worth to ask if the connected solution effectively exists, namely if the Gross-Ooguri phase transition take places. We derived an equation that describes a curve in the Gross-Ooguri phase diagram. Our next step will be to determine if this curve intersects the phase transition line or if it lies in only one single region of the diagram, that is no phase transition takes place. If that would be the case, the solution would be described by the same matrix model of a single circular Wilson loop. Then, it would be interesting to try to evaluate the expectation value at strong coupling of the correlator of two generic Lissajous loops, and see if a matrix model description is possible.

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