Aspects of the Seiberg-Witten equations on manifolds with cusps

A Dissertation presented

by

Luca Fabrizio Di Cerbo

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2011

Stony Brook University

The Graduate School

Luca Fabrizio Di Cerbo

We, the dissertation committee for the above candidate

for the Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Claude R. LeBrun – Dissertation Advisor

Professor, Mathematics

Michael T. Anderson – Chairperson of Defense

Professor, Mathematics

H. Blaine Lawson Jr.

Professor, Mathematics

Martin Rocek

Professor, Institute for Theoretical Physics

This dissertation is accepted by the Graduate School.

Lawrence Martin

Dean of the Graduate School

Abstract of the Dissertation

Aspects of the Seiberg-Witten equations on manifolds with cusps

by

Luca Fabrizio Di Cerbo

Doctor of Philosophy

in

Mathematics

Stony Brook University

2011

In this work we study several geometrical and analytical aspects arising from the study of the Seiberg-Witten equations on manifolds with cusps.

We study the classification of smooth toroidal compactifications of nonuniform ball quotients in the sense of Kodaira and Enriques. Moreover, several results concerning the Riemannian and complex algebraic geometry of these spaces are given. In particular we show that there are compact complex surfaces which admit Riemannian metrics of nonpositive curvature, but which do not admit Kähler metrics of nonpositive curvature. An infinite class of such examples arise as smooth toroidal compactifications of ball quotients. The proof of these results use a Riemannian cusps closing technique developed by Hummel

and Schroeder.

Using a construction due to Biquard, we derive an obstruction to the existence of cuspidal Einstein metrics on finite-volume complex surfaces. This generalizes a theorem of LeBrun for compact complex surfaces. As in the compact case, such a result relies on a Seiberg-Witten scalar curvature estimate. Then, the obstruction is made explicit on some examples.

Finally, we study the Seiberg-Witten equations on noncompact manifolds which are diffeomorphic to the product of two hyperbolic Riemann surfaces. By extending some constructions of Biquard and Rollin, we show how to construct irreducible solutions of the Seiberg-Witten equations for *any* metric of finite volume which has a "nice" behavior at infinity. We conclude by giving the finite volume generalization of some celebrated results of LeBrun.

Contents

Acknowledgements

I would like to express my gratitude to several people and institutions.

First, I would like to express my gratitude to Professor Claude LeBrun for his constant support during my studies here at Stony Brook University. Moreover, I would like to thank him for suggesting the line of research developed in this thesis. I also would like to thank him for pointing me in the right direction twice during the preparation of this thesis.

Second, I would like to thank Professors Michael Anderson, Blaine Lawson, Claude LeBrun, Lowell Jones, Samuel Grushevsky, Dror Varolin and Aleksey Zinger for all the Mathematics I learned attending their classes.

Third, I would like to thank Professors Michael Anderson, Blaine Lawson and Martin Rocek for being in my thesis committee.

Special thanks go to Meredith Ash, Donna McWilliams, Lucille Meci, Nancy Rohring, Pat Tonra and Barbara Wichard for their constant help during these years.

Finally, I am grateful to the Mathematics Department and the Simons Foundation for their generous support during my years at Stony Brook University.

Chapter 1

Introduction

1.1 Introduction

In this chapter, we collect some of the results needed in the rest of this work. Moreover, we fix the notation.

An outline of the chapter follows. Section 1.2 contains the essential results about the L^2 cohomology of noncompact manifolds. These results are crucially used in Chapter 3 and Chapter 4.

In Section 1.3 we show that, up to a change of gauge, the Seiberg-Witten equations are a nonlinear coupled elliptic system of the first order. This fact is essential for the bootstrap argument contained in Chapter 4.

Finally, in Section 1.4 we give a detailed proof of an unpublished result by J. Milnor regarding the Euler number of a Riemannian 4-manifold whose sectional curvatures along perpendicular plane elements have the same sign. This result plays a role in the proof of Theorem A in Section 2.2.

1.2 L^2 cohomology

Let us start with the definition of the L^2-cohomology and its relation to the space of L^2-harmonic forms. For further details we refer to [2], [44] and the bibliography therein. The results and the notation given in this section will be widely used in this thesis.

Given a orientable noncompact manifold (M, g) we have, when the differential d is restricted to an appropriate dense subset, a Hilbert complex

$$... \longrightarrow L^2\Omega_g^{k-1}(M) \longrightarrow L^2\Omega_g^k(M) \longrightarrow L^2\Omega_g^{k+1}(M) \longrightarrow ...$$

where the inner products on the exterior bundles are induced by g. Define the maximal domain of d, at the k-th level, to be

$$Dom^k(d) = \{\alpha \in L^2\Omega_g^k(M), d\alpha \in L^2\Omega_g^{k+1}(M)\}$$

where $d\alpha \in L^2\Omega_g^{k+1}(M)$ has to be intended in the distributional sense. In other words, $d\alpha \in L^2\Omega_g^{k+1}(M)$ if there exists a positive constant C such that for any $\beta \in C_c^\infty\Omega^{k+1}(M)$ we have

$$|\langle\alpha, d^*\beta\rangle| \leq C\|\beta\|_2.$$

The (reduced) L^2-cohomology groups are then defined to be

$$H_2^k(M) = Z_g^k(M)/\overline{dDom^{k-1}(d)},$$

where

$$Z_2^k(M) = \{\alpha \in L^2\Omega_g^k(M), d\alpha = 0\}.$$

Thus, the groups $Z_2^k(M)$ are simply the spaces of L^2 weakly closed forms.

Let us consider the formally self-adjoint operator

$$D : C_c^\infty(\Omega_g^*(M)) \longrightarrow C_c^\infty(\Omega_g^*(M))$$

where $D = d + d^*$ is the usual Gauss-Bonnet operator. Let us consider the minimal self-adjoint extension

$$D : L_1^2(\Omega_g^*(M)) \longrightarrow L^2(\Omega_g^*(M))$$

where $L_1^2(\Omega_g^*(M))$ is the Sobolev space obtained by completing $C_c^\infty(\Omega^*(M))$ with respect to the inner product $(\cdot,\cdot)_{L^2} + (D\cdot, D\cdot)_{L^2}$. We then obtain that

$$L^2(\Omega_g^*(M)) = \overline{Im(D)} \oplus \overline{Im(D)}^\perp = \overline{Im(D)} \oplus Ker(D).$$

This orthogonal decomposition, considered in each degree k, gives precisely the so-called Hodge-Kodaira decomposition:

$$L^2\Omega_g^k(M) = \mathcal{H}_g^k(M) \oplus \overline{dC_c^\infty\Omega^{k-1}} \oplus \overline{d^*C_c^\infty\Omega^{k+1}},$$

where

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2(\Omega_g^k(M)) |\ d\alpha = d^*\alpha = 0\}.$$

Let (M, g) be a complete Riemannian manifold. It is well-known that in this case the Gauss-Bonnet operator $D = d + d^*$ is essentially self-adjoint, i.e., the minimal and the maximal extensions coincide [44]. Similarly, the maximal and minimal domain of d coincide. We then have the following basic result.

Theorem 1 *Let (M, g) be a complete Riemannian manifold. Then, we have the isomorphism $H_2^k(M) = \mathcal{H}_2^k(M)$.*

It is interesting that, similarly to the compact case, in the complete case harmonic forms can be characterized as forms that are in the kernel of the Hodge laplacian.

Theorem 2 *Let (M, g) be complete, then*

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2(\Omega_g^k(M))|\ (dd^* + d^*d)\alpha = 0\}.$$

See [44].

Finally, we end this section by recalling that the L^2 cohomology groups are quasi-isometric invariants, see [44]. Recall that two metrics g and $g^{'}$ are said to be *quasi-isometric* if there exist positive constants C_1 and C_2 such that

$$C_1 g \leq g^{'} \leq C_2 g.$$

Theorem 3 *Let (M, g) and $(M, g^{'})$ be a quasi-isometric Riemannian manifold and assume g to be complete. Then, we have the isomorphism*

$$\mathcal{H}_g^k(M) \simeq \mathcal{H}_{g^{'}}^k(M)$$

for any k.

1.3 PDE point of view on the Seiberg Witten equations

The main result of this section is to show that, up to a change of gauge, the Seiberg-Witten equations are a nonlinear coupled elliptic system of the first order. Here and in the rest of this thesis we follow the notation of [15]. The SW equations are given by the following system

$$\begin{cases} \mathcal{D}_A \psi = 0 \\ F_A^+ = q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} Id \end{cases}$$

where ψ is a plus $Spin^c$ spinor, and A is a hermitian connection on the determinant line bundle of the $Spin^c$ structure under consideration. The configuration space for these equations is the space of L^2_2 hermitian connection on the determinant line bundle $\mathcal{L}$ and of L^2_2 plus spinors

$$\mathcal{C}(P^c) = \mathcal{A}_{L^2_2}(\mathcal{L}) \times L^2_2(\mathcal{S}^+(P^c)).$$

The group $\mathcal{G} := Map(X, S^1)$ acts on the configuration space

$$f \cdot (A, \psi) = (A - 2\frac{df}{f}, f\psi)$$

preserving the space of solutions. The second of the SW equations is clearly invariant under the gauge transformation described above, for what regard the equation for the spinor one computes

$$\mathcal{D}_{A-2\frac{df}{f}} f\psi = -df \cdot \psi + \mathcal{D}_A f\psi = -df \cdot \psi + df \cdot \psi + f\mathcal{D}_A\psi = 0.$$

To preserve the regularity of the configuration space we then consider the space of L^2_3 changes of gauge denoted by $L^2_3(\mathcal{G})$. Fix a background C^∞ hermitian connection A_0 on $\mathcal{L}$, any other connection $A \in \mathcal{A}_{L^2_2}(\mathcal{L})$ can then be written as $A = A_0 + \alpha$ with $\alpha \in L^2_2(T^*X \otimes i\mathbb{R})$. By the Hodge decomposition we can write

$$\alpha = \alpha_0 + df + d^*\beta$$

with α_0 harmonic, $f \in L^2_3(\Omega^0 \otimes i\mathbb{R})$ and $\beta \in L^2_3(\Omega^3 \otimes i\mathbb{R})$. Consider the change of gauge given by $\sigma = exp(f/2)$, then

$$A - 2\frac{d\sigma}{\sigma} = A_0 + a$$

where $a := \alpha_0 + d^*\beta$ is such that $d^*a = 0$. Thus, given a solution of the SW equations $(A, \phi) \in \mathcal{C}(P^c)$, the equations satisfied by the pair $exp(f/2) \cdot (A, \phi)$, where we define $\psi = exp(f/2)\phi$, can be written as follows

$$\begin{cases} \mathcal{D}_{A_0}\psi = -\frac{1}{2}a \cdot \psi \\ (d^* + d^+)a = q(\psi) - F^+_{A_0} \end{cases} \tag{1.1}$$

This is an elliptic system of the first order with C^∞ coefficients. This remarkable property can be used to show that any solution of the SW equations is gauge equivalent to a C^∞ solution. Recall that $a \in L^2_2$, $\psi \in L^2_2$ and $F^+_A \in L^2_1$. Thus, $a, \psi \in L^p$ for any p and then by the generalized Hölder inequality $a \cdot \psi \in L^p$ for any p. By the elliptic regularity $\psi \in L^p_1$ for any p. Now, by the Sobolev multiplication $L^2_2 \otimes L^4_1 \to L^3_1$ the right hand side of the first equation in 1.1 is in L^3_1, and then by the elliptic regularity $\psi \in L^3_2$. By the Sobolev multiplication $L^2_2 \otimes L^3_2 \to L^2_2$, the same argument shows that $\psi \in L^2_3$. The Sobolev multiplication $L^2_3 \otimes L^2_3 \to L^2_3$ shows that $q(\psi) \in L^2_3$ and therefore by elliptic regularity $a \in L^2_4$. These arguments can now be iterated, by using the Sobolev multiplication $L^2_k \otimes L^2_k \to L^2_k$ $(k \geq 3)$, to show that the pair $(a, \psi) \in L^2_k$ for any k. Now, by the Sobolev embedding theorem $L^2_k \hookrightarrow C^{k-3}$, we conclude that $(A, \psi) \in C^\infty$.

1.4 A computation of Milnor

Here we present a computation by J. Milnor regarding the Euler characteristic of nonpositively curved 4-manifolds. We follow the presentation given in a old paper by S. S. Chern [20].

Recall that

$$\Omega^l_k = \frac{1}{2} R_{ijkl} \theta^i \wedge \theta^j$$

and then

$$\Omega^l_k \wedge \Omega^p_q = \frac{1}{4} R_{ijkl} R_{lmqp} \theta^i \wedge \theta^j \wedge \theta^l \wedge \theta^m.$$

Following [20], for any point on our 4-dimensional manifold we can find an orthonormal frame such that

$$R_{1231} = R_{1241} = R_{1232} = R_{1242} = R_{1332} = R_{1341} = 0.$$

It follows that

- $\Omega^1_2 = R_{1221}\theta^1 \wedge \theta^2 + R_{3421}\theta^3 \wedge \theta^4$;

- $\Omega^1_3 = R_{1331}\theta^1 \wedge \theta^3 + R_{2431}\theta^2 \wedge \theta^4$;

- $\Omega^1_4 = R_{1441}\theta^1 \wedge \theta^4 + R_{2341}\theta^2 \wedge \theta^4$.

As a result

- $\Omega^1_2 \wedge \Omega^3_4 = (R_{1221}R_{3443} + R_{1243}R_{3421})\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$;

- $\Omega^1_3 \wedge \Omega^2_4 = -(R_{1331}R_{2442} + R_{1342}R_{2431})\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$;

- $\Omega^1_4 \wedge \Omega^2_3 = (R_{1441}R_{2332} + R_{1234}R_{4321})\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$.

We then conclude that

$$\begin{aligned} Pf(\Omega) &= \Omega^1_2 \wedge \Omega^3_4 - \Omega^1_3 \wedge \Omega^2_4 + \Omega^1_4 \wedge \Omega^2_3 \\ &= \{R_{1221}R_{3443} + R_{1243}R_{3421} + R_{1331}R_{2442} + R_{1342}R_{2431} \\ &\quad + R_{1441}R_{2332} + R_{1234}R_{4321}\}\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4. \end{aligned}$$

Notice that by the symmetries of the curvature tensor

- $R_{1234}R_{4321} = R^2_{1234}$;

- $R_{1342}R_{2431} = R^2_{1342}$;

- $R_{1243}R_{3421} = R^2_{1243}$.

Theorem 4 (Milnor) *Let M be a closed orientable Riemannian manifold of dimension 4. If its sectional curvatures along perpendicular plane elements always have the same sign, then its Euler characteristic satisfies $\chi(M) \geq 0$. In particular if the sectional curvature is semi-definite then the Euler characteristic is positive semi-definite.*

The Euler integrand is pointwise positive semi-definite.

Chapter 2

On the toroidal compactifications of finite volume complex hyperbolic surfaces

2.1 Introduction

Let $\tilde{M}$ be a symmetric space of noncompact type, and let $\mathrm{Iso}_0(\tilde{M})$ denote the connected component of the isometry group of $\tilde{M}$ containing the identity. Recall that $\mathrm{Iso}_0(\tilde{M})$ is a semi-simple Lie group. A discrete subgroup $\Gamma \subset \mathrm{Iso}_0(\tilde{M})$ is a *lattice* in $\tilde{M}$ if $\tilde{M}/\Gamma$ is of finite volume. When Γ is torsion free, then $\tilde{M}/\Gamma$ is a finite volume manifold or a locally symmetric space. A lattice Γ is *uniform* (*nonuniform*) if $\tilde{M}/\Gamma$ is compact (noncompact).

The theory of compactifications of locally symmetric spaces or varieties has been extensively studied, see for example [5]. In fact, locally symmetric varieties of noncompact type often occur as moduli space in algebraic geometry and number theory, see [1]. For technical reasons this beautiful theory is mainly developed for quotients of symmetric spaces or varieties by *arithmetic* subgroups. For arithmetic subgroups of semi-simple Lie groups a nice reduction theory is available [5]. Among many other things, the aforementioned theory can be used to deduce their finite generation, the existence of finitely many conjugacy classes of maximal parabolic subgroups, and the existence of *neat* subgroups of finite index.

The celebrated work of Margulis [42] implies that lattices in any semi-simple Lie group of real rank greater or equal than two are arithmetic subgroups. This important theorem does not cover many interesting cases such as lattices in the complex hyperbolic space $\mathbb{C}\mathcal{H}^n$, where non-arithmetic lattices are known to exist by the work of Mostow and Mostow-Deligne; see [49] and the bibliography therein.

It is thus desirable to develop a theory of compactifications of locally symmetric varieties modeled on $\mathbb{C}\mathcal{H}^n$ regardless of the arithmeticity of the defining torsion free lattices. A compactification of finite-volume complex-hyperbolic manifolds as a complex spaces with isolated normal singularities was obtained by Siu and Yau in [37]. This compactification may be regarded as a generalization of the Baily-Borel compactification defined for arithmetic lattices in $\mathbb{C}\mathcal{H}^n$. A *toroidal* compactification for finite-volume complex-hyperbolic manifolds was described by Hummel and Schroeder in connection with cusps closing techniques arising from Riemannian geometry [22]; see also the preprint by Mok [45] and the classical reference [1] for what concerns the arithmetic case.

The constructions of both Siu-Yau and Hummel-Schroeder rely on the theory of nonpositively curved Riemannian manifolds. The key point here is that the structure theorems for finite-volume manifolds of negatively pinched curvature, or more generally for *visibility* manifolds [11], can be used as a substitute of the reduction theory for arithmetic subgroups.

In this chapter we study torsion-free nonuniform lattices in the complex hyperbolic plane $\mathbb{C}\mathcal{H}^2$ and their toroidal compactifications. Let Γ be a lattice as above and let $\overline{\mathbb{C}\mathcal{H}^2/\Gamma}$ denote its toroidal compactification. When $\overline{\mathbb{C}\mathcal{H}^2/\Gamma}$ is smooth, it is a compact Kähler surface [22]. It is then of interest to place these smooth Kähler surfaces in the framework of the *Kodaira-Enriques* classification of complex surfaces [18]. The main purpose of this chapter is to prove the following:

Theorem A *Let Γ be a nonuniform torsion-free lattice in $\mathbb{C}\mathcal{H}^2$. There exists a finite subset $\mathcal{F}^{'} \subset \Gamma$ of parabolic isometries for which the following holds: for any normal subgroup $\Gamma^{'} \lhd \Gamma$ with the property that $\mathcal{F}^{'} \cap \Gamma^{'}$ is empty, then $\overline{\mathbb{C}\mathcal{H}^2/\Gamma'}$ is a surface of general type with ample canonical line bundle. Moreover, $\overline{\mathbb{C}\mathcal{H}^2/\Gamma'}$ admits Riemannian metrics of nonpositive sectional curvature but it cannot support Kähler metrics of nonpositive sectional curvature.*

An outline of the chapter follows. Section 2.2 starts with a summary of the results of Hummel and Schroeder [22]. Such results are then combined with the Kodaira-Enriques classification to prove that when the lattice Γ is sufficiently small then $\overline{\mathbb{C}\mathcal{H}^2/\Gamma}$ is a surface of general type with ample canonical bundle.

In Section 2.3 we present some examples of a surfaces of general type which do not admit any nonpositively curved Kähler metric, but whose underlying smooth manifolds admit Riemannian metrics of nonpositive curvature. Finally the proof of Theorem A is given.

In Section 2.4 we show how Theorem A, combined with the theory of semi-stable curves on algebraic surfaces [41], can be used to address the problem of the projective-algebraicity of minimal compactifications (Siu-Yau) of finite-volume complex-hyperbolic surfaces. The results of section 2.4 are then summarized in Theorem B.

2.2 Toroidal Compactifications and the Kodaira-Enriques Classification

Let $\mathrm{PU}(1,2)$ denote the connected component of $\mathrm{Iso}(\mathbb{C}\mathcal{H}^2)$ containing the identity. Let Γ be a nonuniform torsion-free lattice of holomorphic isometries of the complex hyperbolic plane $\mathbb{C}\mathcal{H}^2$, i.e., $\Gamma \leq \mathrm{PU}(1,2)$. Recall that the locally symmetric space $\mathbb{C}\mathcal{H}^2/\Gamma$ has finitely many cusp ends $A_1, ..., A_n$ which are in one to one correspondence with conjugacy classes of the maximal parabolic subgroups of Γ [13]. The set of all parabolic elements of Γ can be written as a disjoint union of subsets Γ_x, where Γ_x is the set of all parabolic elements in Γ having x as unique fixed point. Here x is a point in the natural point set compactification of $\mathbb{C}\mathcal{H}^2$ obtained by adjoining points at infinity corresponding to asymptotic geodesic rays. Thus, given a cusp A_i, let us consider the associated maximal parabolic subgroup $\Gamma_{x_i} \leq \Gamma$ and the horoball HB_{x_i} stabilized by Γ_{x_i}. We then have that $\mathrm{HB}_{x_i}/\Gamma_{x_i}$ is naturally identified with A_i.

Recall that after choosing an Iwasawa decomposition [11] for $\mathrm{PU}(1,2)$, we get a identification of ∂HB with the three dimensional Heisenberg Lie group N. Moreover, N comes equipped with a left invariant metric and then we may view Γ_{x_i} as a lattice in Iso(N). The cusps $A_1, ..., A_n$ are then identified with $N/\Gamma_{x_i} \times [0, \infty)$, for $i = 1, ..., n$.

The isometry group of N is isomorphic to the semi-direct product Iso(N) = $N \rtimes U(1)$. We say that a lattice in Iso(N) is rotation free if it is a lattice in N, i.e., if it is a lattice of left translations. A parabolic isometry $\phi \in \Gamma$ is called *unipotent* if it acts as a translation on its invariant horospheres.

We now briefly summarize some of the results of Hummel [22] and Hummel-Schroeder [21].

Theorem 5 (Hummel-Schroeder) *Let Γ be a nonuniform torsion-free lattice in $\mathbb{C}\mathcal{H}^2$. Then, there exists a finite subset $\mathcal{F} \subset \Gamma$ of parabolic isometries such that for any normal subgroup $\Gamma^{'} \lhd \Gamma$ with the property that $\mathcal{F} \cap \Gamma^{'}$ is empty, then $\overline{\mathbb{C}\mathcal{H}^2/\Gamma^{'}}$ is smooth and Kähler.*

Furthermore, using a cusp closing technique arising from Riemannian Geometry they were able to prove:

Theorem 6 (Hummel-Schroeder) *Let Γ be a nonuniform torsion-free lattice in $\mathbb{C}\mathcal{H}^2$. Then, there exists a finite subset $\mathcal{F}^{'} \subset \Gamma$ of parabolic isometries such that $\mathcal{F}^{'} \supseteq \mathcal{F}$ for which the following holds. For any normal subgroup $\Gamma^{'} \lhd \Gamma$ with the property that $\mathcal{F}^{'} \cap \Gamma^{'}$ is empty, then $\overline{\mathbb{C}\mathcal{H}^2/\Gamma^{'}}$ admits a Riemannian metric of nonpositive sectional curvature.*

A few remarks about these results. A nonuniform torsion-free lattice in $\mathbb{C}\mathcal{H}^2$ admits a smooth toroidal compactification if its parabolic isometries are all unipotent. In the arithmetic case this is achieved by choosing a neat subgroup of finite index [1]. It is also interesting to observe that we have plenty of normal subgroups satisfying the requirements of Theorem 5 and 6, in fact $\mathrm{PU}(1,2)$ is linear and then *residually finite* by a fundamental result of Mal'cev [39]. Finally, it is interesting to notice that in general one expects the strict inclusion $\mathcal{F}^{'} \supset \mathcal{F}$ to hold. Explicit examples can be derived from the construction of Hirzebruch [48].

For simplicity, a compactification as in Theorem 6 will be referred as toroidal *Hummel-Schroeder* compactification.

Proposition 2.2.1 *Let M be a finite-volume complex-hyperbolic surface which admits a toroidal Hummel-Schroeder compactification. Then the Euler number of $\overline{M}$ is strictly positive.*

The idea for the proof goes back to an unpublished result of J. Milnor about the Euler number of closed four dimensional Riemannian manifolds having sectional curvatures along perpendicular planes of the same sign; see the paper by S. S. Chern [20]. Let $(\overline{M}, g)$ be the Riemannian manifold obtained by closing the cusps of M under the condition of nonpositive curvature [21]. Let Ω be its curvature matrix. We can always choose [20] a orthonormal frame $\{e_i\}_{i=1}^{4}$ such that:

$$R_{1231} = R_{1241} = R_{1232} = R_{1242} = R_{1332} = R_{1341} = 0.$$

It follows that

$$\begin{aligned} Pf(\Omega) &= \Omega^1_2 \wedge \Omega^3_4 - \Omega^1_3 \wedge \Omega^2_4 + \Omega^1_4 \wedge \Omega^2_3 \\ &= \{R_{1221}R_{3443} + R^2_{1243} + R_{1331}R_{2442} + R^2_{1342} \\ &\quad + R_{1441}R_{2332} + R^2_{1234}\} d\mu_g, \end{aligned}$$

where $Pf(\Omega)$ is the Pfaffian of the skew symmetric matrix Ω. The statement is now a consequence of Chern-Weil theory.

We can now use the Kodaira-Enriques classification of closed smooth surfaces [18] to derive the following theorem. The proof is in the spirit of the theory of nonpositively curved spaces.

Theorem 2.2.2 *Let M be a finite-volume complex-hyperbolic surface which admits a toroidal Hummel-Schroeder compactification. Then $\overline{M}$ is a surface of general type without rational curves.*

Since $\overline{M}$ admits a Riemannian metric of nonpositive sectional curvature, the Cartan-Hadamard theorem [12] implies that the universal cover of $\overline{M}$ is diffeomorphic to the four dimensional euclidean space. Consequently, $\overline{M}$ is aspherical and then it cannot contain rational curves. Moreover, the second Betti number of $\overline{M}$ is even since by construction it admits a Kähler metric. By the Kodaira-Enriques classification [18] we conclude that the Kodaira dimension of $\overline{M}$ cannot be negative.

From Proposition 2.2.1, we know that the Euler number of $\overline{M}$ is strictly positive. The minimal complex surfaces with Kodaira dimension equal to zero and positive Euler number are simply connected or with finite fundamental group. Since $\pi_1(\overline{M})$ is infinite, the Kodaira dimension of $\overline{M}$ is bigger or equal than one.

The fundamental group of an elliptic surface with positive Euler number is completely understood in terms of the *orbifold* fundamental group of the base of the elliptic fibration. More precisely, denoting by $\pi : S \longrightarrow C$ the elliptic fibration, if S has no multiple fibers then π induces an isomorphism $\pi_1(S) \simeq \pi_1(C)$. In the case where we allow multiple fibers we have the isomorphism

$\pi_1(S) \simeq \pi_1^{Orb}(C)$. For these results we refer to [16]. We are now ready to show that $\overline{M}$ cannot be an elliptic surface. When S has multiple fibers, the group $\pi_1(S)$ has always torsion and then it cannot be the fundamental group of a nonpositively curved manifold. If we assume $\pi_1(\overline{M}) \simeq \pi_1(C)$, the fact that $\pi_1(\overline{M})$ grows exponentially [3] forces the genus of the Riemann surface C to be bigger or equal than two. Since all closed geodesics in a manifold of nonpositive curvature are essential in π_1, we have that the fundamental group of the flats introduced in the compactification injects in $\pi_1(\overline{M})$ and then by assumption in $\pi_1(C)$. By elementary hyperbolic geometry this would imply that $\mathbb{Z} \oplus \mathbb{Z}$ acts as a discrete subgroup of $\mathbb{R}$, which is clearly impossible.

Corollary 2.2.3 *A toroidal Hummel-Schroeder compactification has ample canonical line bundle.*

By Theorem 2.2.2 we know that $\overline{M}$ is a minimal surface of general type without rational curves. The corollary follows from Nakai's criterion for ampleness of divisors on surfaces [18].

In the arithmetic case, part of the results contained in Theorem 2.2.2 can be derived from a theorem of Tai, see [1]. Furthermore, similar results for the so-called *Picard* modular surfaces are obtained by Holzapfel in [47].

2.3 Examples

In this section we present examples of surfaces of general type which do not admit nonpositively curved Kähler metrics, but such that their underlying smooth manifolds do admit Riemannian metrics with nonpositive Riemannian curvature. In order to do this one needs to understand the restrictions imposed by the nonpositive curvature assumption on the holomorphic curvature tensor.

Thus, define

$$p = 2Re(\xi), \quad q = 2Re(\eta)$$

where

$$\xi = \xi^\alpha \partial_\alpha, \quad \eta = \eta^\alpha \partial_\alpha.$$

In real coordinates we have

$$R(p,q,q,p) = R_{hijk} p^h q^i q^j p^k$$

while in complex terms

$$R(\xi+\overline{\xi}, \eta+\overline{\eta}, \eta+\overline{\eta}, \xi+\overline{\xi}) = R(\xi,\overline{\eta},\eta,\overline{\xi}) + R(\xi,\overline{\eta},\overline{\eta},\xi)$$
$$+ R(\overline{\xi},\eta,\eta,\overline{\xi}) + R(\overline{\xi},\eta,\overline{\eta},\xi).$$

We then have

$$\begin{aligned}
R_{hijk} p^h q^i q^j p^k &= R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha \eta^{\overline{\beta}} \eta^\gamma \xi^{\overline{\delta}} + R_{\alpha\overline{\beta}\overline{\gamma}\delta} \xi^\alpha \eta^{\overline{\beta}} \eta^{\overline{\gamma}} \xi^\delta + R_{\overline{\alpha}\beta\gamma\overline{\delta}} \xi^{\overline{\alpha}} \eta^\beta \eta^\gamma \xi^{\overline{\delta}} \\
&+ R_{\overline{\alpha}\beta\overline{\gamma}\delta} \xi^{\overline{\alpha}} \eta^\beta \eta^{\overline{\gamma}} \xi^\delta \\
&= R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha \eta^{\overline{\beta}} \eta^\gamma \xi^{\overline{\delta}} - R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha \eta^{\overline{\beta}} \eta^{\overline{\delta}} \xi^\gamma - R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^{\overline{\beta}} \eta^\alpha \eta^\gamma \xi^{\overline{\delta}} \\
&+ R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^{\overline{\beta}} \eta^\alpha \eta^{\overline{\delta}} \xi^\gamma \\
&= R_{\alpha\overline{\beta}\gamma\overline{\delta}} \{ \xi^\alpha \eta^{\overline{\beta}} \eta^\gamma \xi^{\overline{\delta}} - \xi^\alpha \eta^{\overline{\beta}} \eta^{\overline{\delta}} \xi^\gamma - \xi^{\overline{\beta}} \eta^\alpha \eta^\gamma \xi^{\overline{\delta}} + \xi^{\overline{\beta}} \eta^\alpha \eta^{\overline{\delta}} \xi^\gamma \} \\
&= R_{\alpha\overline{\beta}\gamma\overline{\delta}} (\xi^\alpha \eta^{\overline{\beta}} - \eta^\alpha \xi^{\overline{\beta}}) \overline{(\xi^\delta \eta^{\overline{\gamma}} - \eta^\delta \xi^{\overline{\gamma}})}.
\end{aligned}$$

If we assume the Riemannian sectional curvature to be nonpositive we have

$$R_{hijk} p^h q^i q^j p^k = R_{\alpha\overline{\beta}\gamma\overline{\delta}} (\xi^\alpha \eta^{\overline{\beta}} - \eta^\alpha \xi^{\overline{\beta}}) \overline{(\xi^\delta \eta^{\overline{\gamma}} - \eta^\delta \xi^{\overline{\gamma}})} \leq 0.$$

Let us write this equation in detail when the dimension is $n = 2$:

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}}(\xi^\alpha\eta^{\overline{\beta}} - \eta^\alpha\xi^{\overline{\beta}})\overline{(\xi^\delta\eta^{\overline{\gamma}} - \eta^\delta\xi^{\overline{\gamma}})} = R_{1\overline{1}1\overline{1}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})}$$

$$+ R_{1\overline{1}2\overline{1}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})} + R_{1\overline{1}2\overline{2}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}$$

$$+ R_{2\overline{2}1\overline{1}}(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})} + R_{1\overline{2}1\overline{2}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})}$$

$$+ R_{2\overline{1}2\overline{1}}(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})} + R_{1\overline{1}1\overline{2}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})}$$

$$+ R_{2\overline{2}2\overline{1}}(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})} + R_{1\overline{2}2\overline{2}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}$$

$$+ R_{2\overline{1}2\overline{2}}(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})} + R_{2\overline{1}1\overline{1}}(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})}$$

$$+ R_{2\overline{1}1\overline{2}}(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})} + R_{1\overline{2}2\overline{1}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})}$$

$$+ R_{1\overline{2}1\overline{1}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})} + R_{2\overline{2}1\overline{2}}(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})}$$

$$+ R_{2\overline{2}2\overline{2}}(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}.$$

Now

$$R_{2\overline{2}1\overline{1}} = R_{1\overline{1}2\overline{2}}, \quad R_{1\overline{2}2\overline{1}} = R_{1\overline{1}2\overline{2}}, \quad R_{2\overline{1}1\overline{2}} = R_{1\overline{1}2\overline{2}},$$

thus

$$R_{1\overline{1}2\overline{2}}\{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})} + (\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})}$$

$$+ (\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})} + (\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})}\}$$

$$= 2R_{1\overline{1}2\overline{2}}\{Re(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})} + |\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2\}.$$

Now

$$\overline{R_{1\overline{2}1\overline{2}}} = R_{2\overline{1}2\overline{1}}$$

thus

$$R_{1\overline{2}1\overline{2}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})} + R_{2\overline{1}2\overline{1}}(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})\overline{(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})}$$

$$= 2Re\{R_{1\overline{2}1\overline{2}}(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}})\overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})}\}.$$

Now

$$R_{1\bar{1}1\bar{2}} = R_{1\bar{2}1\bar{1}}, \quad R_{1\bar{1}2\bar{1}} = R_{2\bar{1}1\bar{1}}, \quad \overline{R_{1\bar{1}1\bar{2}}} = R_{1\bar{1}2\bar{1}},$$

thus

$$\begin{aligned}
&R_{1\bar{1}1\bar{2}}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})} + R_{1\bar{2}1\bar{1}}(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})\overline{(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})} \\
&\quad + R_{1\bar{1}2\bar{1}}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})} + R_{2\bar{1}1\bar{1}}(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})\overline{(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})} \\
&\quad = 2R_{1\bar{1}1\bar{2}}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})} + 2R_{1\bar{1}2\bar{1}}\overline{(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})}(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}}) \\
&\quad = 4Re\{R_{1\bar{1}1\bar{2}}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\}.
\end{aligned}$$

Now

$$R_{2\bar{2}1\bar{2}} = R_{1\bar{2}2\bar{2}}, \quad R_{2\bar{1}2\bar{2}} = R_{2\bar{2}2\bar{1}}, \quad \overline{R_{2\bar{2}1\bar{2}}} = R_{2\bar{2}2\bar{1}},$$

thus

$$\begin{aligned}
&R_{2\bar{2}1\bar{2}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})} + R_{1\bar{2}2\bar{2}}(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})} \\
&\quad + R_{2\bar{2}2\bar{1}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})} + R_{2\bar{1}2\bar{2}}(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})} \\
&\quad = 2R_{2\bar{2}1\bar{2}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})} + 2R_{2\bar{2}2\bar{1}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})} \\
&\quad = 4Re\{R_{2\bar{2}1\bar{2}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\}.
\end{aligned}$$

In summary we have the equality:

$$\begin{aligned}
&R_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha\eta^{\bar{\beta}} - \eta^\alpha\xi^{\bar{\beta}})\overline{(\xi^\delta\eta^{\bar{\gamma}} - \eta^\delta\xi^{\bar{\gamma}})} \\
&= R_{1\bar{1}1\bar{1}}|\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}}|^2 + 4Re\{R_{1\bar{1}1\bar{2}}(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\} \\
&+ 2R_{1\bar{1}2\bar{2}}\{|\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}}|^2 + Re(\xi^1\eta^{\bar{1}} - \eta^1\xi^{\bar{1}})\overline{(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})}\} \\
&+ 2Re\{R_{1\bar{2}1\bar{2}}(\xi^1\eta^{\bar{2}} - \eta^1\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\} \\
&+ 4Re\{R_{2\bar{2}1\bar{2}}(\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}})\overline{(\xi^2\eta^{\bar{1}} - \eta^2\xi^{\bar{1}})}\} \\
&+ R_{2\bar{2}2\bar{2}}|\xi^2\eta^{\bar{2}} - \eta^2\xi^{\bar{2}}|^2.
\end{aligned}$$

Following Siu-Mostow [36], we choose the ansatz

$$\xi^1 = ia, \quad \xi^2 = -i, \quad \eta^1 = a, \quad \eta^2 = 1$$

where a is a real number. We get the inequality

$$R_{1\bar{1}1\bar{1}}4a^4 - 2R_{1\bar{1}2\bar{2}}4a^2 + R_{2\bar{2}2\bar{2}}4 \leq 0.$$

Since nonpositive Riemannian sectional curvature implies nonpositive holomorphic sectional curvature, we conclude that

$$(R_{1\bar{1}2\bar{2}})^2 \leq R_{1\bar{1}1\bar{1}}R_{2\bar{2}2\bar{2}}. \tag{2.1}$$

Theorem 2.3.1 *A toroidal Hummel-Schroeder compactification does not admit any Kähler metric with nonpositive Riemannian sectional curvature.*

Let us proceed by contradiction. Consider one of the elliptic divisors added in the compactification. By the properties of submanifolds of a Kähler manifold [19], we have that the holomorphic sectional curvature tangent to the elliptic divisor has to be zero. Let us denote such a holomorphic sectional curvature by $R_{1\bar{1}1\bar{1}}$. By the inequality (2.1), we conclude that $R_{1\bar{1}2\bar{2}} = 0$. As a result, the Ricci curvature tangent to the elliptic divisor has to be zero. We conclude that

$$K_{\overline{M}} \cdot \Sigma = \int_\Sigma c_1(K_{\overline{M}}) = 0,$$

which contradicts the ampleness of $K_{\overline{M}}$, see corollary 2.2.3.

Combining Theorems 2.2.2 and 2.3.1 with Corollary 2.2.3, we have thus proved Theorem A.

2.4 Projective-algebraicity of minimal compactifications

Let $\overline{M}$ be a smooth toroidal compactification of a finite-volume complex-hyperbolic surface M and let Σ denote the compactifying divisor. The set Σ is exceptional and it can be blown down. The resulting complex surface, with isolated normal singularities, it is usually referred as the minimal compactification of M [37]. In this section we address the problem of the projective-algebraicity of minimal compactifications of finite-volume complex-hyperbolic surfaces. This is motivated by a beautiful example of Hironaka, see [14] page 417, which shows that by contracting a smooth elliptic divisor on an algebraic surface one can obtain a nonprojective complex space. For the sake of readability, we present this example at the end of the current section. In the arithmetic case, the projective-algebraicity of minimal compactifications of finite-volume complex-hyperbolic surfaces it is known by the work of Baily and Borel, see [5].

For completeness, we recall the theory of semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps as developed by Sakai in [41].

Let $\overline{M}$ be a smooth projective surface. Let Σ be a reduced divisor having simple normal crossings on $\overline{M}$.

Definition 2.4.1 *The pair $(\overline{M}, \Sigma)$ is called minimal if $\overline{M}$ does not contain an exceptional curve E of the first kind such that $E \cdot \Sigma \leq 1$.*

We consider the logarithmic canonical line bundle $\mathcal{L} = K_{\overline{M}} + \Sigma$ associated to Σ. Given any integer k, define $\overline{P}_m = \dim H^0(\overline{M}, \mathcal{O}(m\mathcal{L}))$. If $\overline{P}_m > 0$, we define the m-th *logarithmic canonical* map $\Phi_{m\mathcal{L}}$ of the pair $(\overline{M}, \Sigma)$ by

$$\Phi_{m\mathcal{L}}(x) = [s_1(x), ..., s_N(x)],$$

for any $x \in \overline{M}$ and where $s_1, ..., s_N$ is a basis for the vector space $H^0(\overline{M}, \mathcal{O}(m\mathcal{L}))$. At this point one introduces the notion of logarithmic Kodaira dimension exactly as in the closed smooth case. We denote this numerical invariant by $\overline{k}(M)$ where $M = \overline{M} \backslash \Sigma$. We refer to [46] for further details.

Definition 2.4.2 *A curve Σ is semi-stable if has only normal crossings and each smooth rational component of Σ intersects the other components of Σ in more than one point.*

The following proposition gives a numerical criterion for a minimal semi-stable pair $(\overline{M}, \Sigma)$ to be of log-general type. For the proof we refer to [41].

Proposition 2.4.3 *Given a minimal semi-stable pair $(\overline{M}, \Sigma)$ we have that $\overline{k}(M) = 2$ if and only if $\mathcal{L}$ is numerically effective and $\mathcal{L}^2 > 0$.*

We can now state one of the main results contained in [41]. In what follows, we denote by $\mathcal{E}$ the set of irreducible curves E in $\overline{M}$ such that $\mathcal{L} \cdot E = 0$.

Theorem 7 (Sakai) *Let $(\overline{M}, \Sigma)$ be a minimal semi-stable pair of log-general type. The map $\Phi_{m\mathcal{L}}$ is then an embedding modulo $\mathcal{E}$ for any $m \geq 5$.*

It is then necessary to characterize the irreducible divisors in $\mathcal{E}$. In particular, we need the following proposition.

Proposition 2.4.4 *Let $(\overline{M}, \Sigma)$ be a minimal semi-stable pair with $\overline{k}(M) = 2$. Let E be an irreducible curve such that $\mathcal{L} \cdot E = 0$. If E is not contained in Σ then $E \simeq \mathbb{C}P^1$ and $E \cdot E = -2$.*

Under these assumptions we know that $\mathcal{L}^2 > 0$. By the Hodge index theorem

$$\mathcal{L}^2 > 0, \quad \mathcal{L} \cdot E = 0 \quad \Longrightarrow \quad E^2 < 0.$$

But now $\mathcal{L} \cdot E = 0$ which implies

$$K_{\overline{M}} \cdot E = -\Sigma \cdot E \leq 0.$$

We then have $K_{\overline{M}} \cdot E = 0$ if and only if E does not intersect Σ. In this case $p_a(E) = 0$ and then $E \simeq \mathbb{C}P^1$ and $E^2 = -2$. Assume now that $K_{\overline{M}} \cdot E < 0$, then $K_{\overline{M}} \cdot E = E^2 = -1$ and therefore E is an exceptional curve of the first kind such that $E \cdot \Sigma = 1$. This contradicts the minimality of the pair $(\overline{M}, \Sigma)$.

We are now ready to prove the main results of this section. Let $\mathbb{C}\mathcal{H}^2/\Gamma$ be a finite-volume complex-hyperbolic surface that admits a smooth toroidal compactification as in Theorem 2.2.2. We then have that $\overline{\mathbb{C}\mathcal{H}^2/\Gamma}$ is a surface of general type with compactification divisor consisting of smooth disjoint elliptic curves.

Proposition 2.4.5 *Let $\overline{M}$ be a minimal surface of general type. Let Σ be a reduced divisor whose irreducible components consist of disjoint smooth elliptic curves. Then, $(\overline{M}, \Sigma)$ is a minimal semi-stable pair with $\overline{k}(M) = 2$.*

Recall that the canonical divisor of any minimal complex surface of non-negative Kodaira dimension is numerically effective [18]. It follows that the adjoint divisor $\mathcal{L}$ is numerically effective. An elliptic curve on a minimal surface of general type has negative self intersection. Moreover, for a minimal surface of general type it is known that the self-intersection of the canonical divisor is strictly positive [18]. By the adjunction formula

$$\mathcal{L}^2 = K_{\overline{M}}^2 - \Sigma^2 > 0.$$

By Proposition 2.4.3, we conclude that $\overline{k}(M) = 2$.

Let $\mathbb{C}\mathcal{H}^2\backslash\Gamma_1$ be a finite-volume complex-hyperbolic surface which admits a smooth toroidal compactification $\overline{M}_1$. Let $(\overline{M}_1, \Sigma_1)$ be the associated minimal semi-stable pair. By Theorem A, we can find a normal subgroup of finite index $\Gamma_2 \lhd \Gamma_1$ such that the toroidal compactification $\overline{M}_2$ of $\mathbb{C}\mathcal{H}^2/\Gamma_2$ is a minimal surface of general type with compactification divisor Σ_2. Since

$$\pi : \mathbb{C}\mathcal{H}^2/\Gamma_2 \longrightarrow \mathbb{C}\mathcal{H}^2/\Gamma_1$$

is an unramified covering we conclude that $\overline{k}(M_1) = \overline{k}(M_2)$ [46]. But by proposition 2.4.5 we know that $\overline{k}(M_2) = 2$, it follows that $(\overline{M}_1, \Sigma_1)$ is a minimal semi-stable pair of log-general type. Let us summarize this argument into a proposition.

Proposition 2.4.6 *Let $(\overline{M}, \Sigma)$ be a smooth pair arising as the toroidal compactification of a finite-volume complex-hyperbolic surface. The pair $(\overline{M}, \Sigma)$ is minimal and log-general.*

The following theorem is the main result of the present section.

Theorem B *Let $(\overline{M}, \Sigma)$ be a smooth pair arising as the toroidal compactification of a finite-volume complex-hyperbolic surface. Then, the associated minimal compactification is projective algebraic.*

By Proposition 2.4.6, the minimal pair $(\overline{M}, \Sigma)$ is log-general. By Theorem 7 we know that $\Phi_{m\mathcal{L}}$ is an embedding modulo $\mathcal{E}$ for any $m \geq 5$. We clearly have that Σ is contained in $\mathcal{E}$. We claim that there are no other divisors in $\mathcal{E}$. Assume the contrary. By Proposition 2.4.4, any other curve in $\mathcal{E}$ must be a smooth rational divisor E with self-intersection minus two. The adjunction formula gives $K_{\overline{M}} \cdot E = 0$ which implies $\Sigma \cdot E = 0$. This is clearly impossible. By Theorem 7 for $m \geq 5$, the map

$$\Phi_{m\mathcal{L}} : \overline{M} \longrightarrow \mathbb{C}P^{N-1}$$

gives a realization of the minimal compactification as a projective-algebraic variety.

For an approach to the projective-algebraicity problem through L^2-estimates for the $\overline{\partial}$-operator we refer to [45].

As promised, we end this section with the details of the example by Hironaka.

Example 1 (Hironaka) *Let Y_0 be a nonsingular cubic in $\mathbb{C}P^2$. Let p_0 be an inflection point. Thus, there exists an hyperplane H such that $H \cap Y_0 = p_0$ with multiplicities three. Choose p_0 as the origin for the multiplication law of Y_0. In other words we consider $Y_0 \simeq \mathbb{C}\backslash\Gamma$ for a lattice Γ. Choose points $p_1, ..., p_{10}$ on Y_0 such that $\sum_{i=1}^{10} n_1 p_i \notin \Gamma$ for any $n_i \in \mathbb{Z}$. Let Y be the proper transform of Y_0 in $\mathbb{C}P^2 \sharp 10 \overline{\mathbb{C}P^2}$ after we blow up the points $\{p_i\}$. Y is then a smooth elliptic curve with self intersection -1. By the Grauert's contractibility criterion we can then blow down this exceptional curve and obtain a normal complex space. Let X_0 be the complex space so obtained. We know want to show that X_0 is not projective algebraic. Let $p \in X_0$ be the contraction of Y. If we assume X_0 to be projective algebraic we can consider an open affine neighborhood U around p. Let $\tilde{C}_0$ be a curve not containing p, and let C_0 be its closure in X_0. Let C be the inverse image of C_0 in $\mathbb{C}P^2 \sharp 10 \overline{\mathbb{C}P^2}$. Thus, C is a curve not intersecting Y. Let $\tilde{C}$ be the image of C in $\mathbb{C}P^2$. By the Bézout theorem, we can write*

$$\tilde{C} \cdot Y_0 = \sum_{i=1}^{10} n_i p_i$$

with $n_i \geq 0$ *and* $\sum_{i=1}^{10} n_i = 3d$ *where* $d = deg\tilde{C}$. *Since* $\tilde{C} \simeq dH$, *we have that* $H \cdot Y_0 \simeq 3p_0$. *As a result we obtain that* $\sum_{i=1}^{10} n_i p_i = 0 \in \Gamma$ *which is a contradiction.*

Chapter 3

Finite volume complex hyperbolic surfaces without cuspidal Einstein metrics

3.1 Introduction

The aim of modern Riemannian geometry is the study of the interplay between curvature and topology. The study of *Einstein* metrics is currently one of the main themes of research in the field; see [4] and the more recent survey [30]. Recall that a Riemannian manifold (M, g) is said to be Einstein if its Ricci tensor is proportional to the metric:

$$Ric_g = \lambda g.$$

In real dimension 2 and 3 this condition is equivalent to the constancy of the sectional curvature. The two dimensional case is classically understood in terms of the so called "uniformization" theorem for Riemann surfaces, while in dimension three one has to refer to the recent works of Thurston, Hamilton and Perelman [43].

Despite the considerable efforts, this condition remains rather obscure in dimension $n \geq 5$. In fact, no uniqueness or non-existence results are known!

In real dimension 4, the geometric and topological meaning of the Einstein condition is considerably better understood. This is essentially due to the facts that Chern-Weil theory enjoys, in this dimension, several special features and to deep gauge theoretic results genuinely concerning this special dimension. For example, Chern-Weil theory can be easily applied to derive the elegant Hitchin-Thorpe inequality [38], which gives a necessary condition for the existence of an Einstein metric on a closed 4-manifold. This important inequality was later extended in several directions. First, it was refined by Gromov [8] using its *simplicial volume* and then by Sambusetti [33] building up on results of Besson, Courtois and Gallot [7]. Another direction in the study of Einstein metrics on 4-manifolds is through a system of nonlinear elliptic PDEs of the first order: the *Seiberg-Witten* equations. Seiberg-Witten theory can in fact be used to derive non-existence results for Einstein metrics on closed 4-manifolds, when the SW invariant of the underlying smooth manifold is nontrivial [15]. This approach was pioneered and pursued by LeBrun in a long series of papers; see [31] and the bibliography therein.

In this chapter we present an obstruction for *cuspidal* Einstein metrics on blow ups of finite volume complex hyperbolic surfaces; see the next section for the technical definition of cuspidal metric. This result extends to the finite volume setting the obstructions found by LeBrun in [27]. The proof is based on a construction due to Biquard [24] which ensures the existence of irreducible solutions of the Seiberg-Witten equations on certain finite volume 4-manifolds.

3.2 Cuspidal metrics and Biquard's argument

In what follows, let M be a noncompact 4-manifolds with finitely many ends. Assume that each end of the manifold is diffeomorphic to a product $\mathbb{R}^+ \times N$, where N is a compact 3-dimensional manifold. A smooth metric g on M is called *cuspidal* if it has bounded curvature and on each end can be written as $dt^2 + g_t$ with the following additional properties:

- the diameter of N_t goes to zero as t goes to infinity;

- the mean curvature h_t of N_t is bounded below: $h_t \geq h_0 > 0$.

Here the mean curvature is defined as the negative logarithmic derivative of the volume of N_t, thus the second hypothesis in the definition above implies that (M, g) has finite volume. The definition of cuspidal metric clearly abstracts some of the nice properties at "infinity" of complete finite volume metrics with pinched negative sectional curvature, see [11].

We are now ready to briefly review the results of Biquard [24]. The main problem with Seiberg-Witten theory on noncompact manifold is that the analysis becomes much harder. The idea of Biquard to solve the SW system of nonlinear equations is very elegant and geometric in nature. Let us consider a finite volume 4-manifold M which admits a "natural" compactification $\overline{M}$ as smooth manifold or orbifold. Assume moreover that $\overline{M}$ fits in the known classes of 4-manifolds (orbifolds) with nontrivial SW invariant [15]. We can then always solve the SW equations on $\overline{M}$ and try to produce an irreducible solution on M as limit of solutions on $\overline{M}$. From the metric point of view, starting with (M, g) where g is assumed to be of finite volume and with a "nice" behavior at infinity, e.g. cuspidal, one has to construct a sequence $(\overline{M}, g_j)$ of metric compactifications that approximate (M, g) as j goes to infinity.

Let us describe in a more detailed way this approach when M is a finite volume complex hyperbolic surface. It is known [24] that this manifold can always be toroidally compactified as a Kähler orbifold. When the compactification $\overline{M}$ is smooth we have a pair $(\overline{M}, \Sigma)$, where the compactification divisor Σ is composed of smooth disjoint elliptic curves. It can be shown that, by eventually passing to a finite regular cover, the toroidal compactification of a finite volume complex-hyperbolic surface is a smooth minimal surface of general type, see Theorem A. Seiberg-Witten theory on minimal surfaces of general type and their blow-ups is well understood. In fact, the underlying smooth manifold of a surface of general type always admits $Spin^c$ structures with nontrivial SW invariant [15]. Furthermore, we have an explicit classification of all finitely many $Spin^c$ structures with nontrivial invariant, see [17].

We can now state the following existence theorem which can be extracted from the work of Biquard [24].

Theorem 8 (Biquard) *Let M be a finite volume complex hyperbolic surface such that its toroidal compactification $(\overline{M}, \Sigma)$ is a smooth minimal surface of general type. Let $M^{'}$ be obtained from M by blowing up k points. Fix a $Spin^c$ structure on $\overline{M}\sharp k\overline{\mathbb{C}P^2}$ with nontrivial SW invariant and determinant line bundle L. Let g be a smooth cuspidal Einstein metric on $M^{'}$, and let $\{g_j\}$ be the sequence of metrics on $\overline{M}\sharp k\overline{\mathbb{C}P^2}$ that approximate g. Let $\{(A_j, g_j)\}$ be the sequence of solutions of the SW equations with perturbations $\{F^{+}_{B_j}\}$ on*

$\{(\overline{M}\sharp k\overline{\mathbb{C}P^2}, g_j)\}$. Then, up to gauge transformations, the solutions $\{(A_j, \psi_j)\}$ converge, in the C^∞ topology on compact sets, to a solution (A, ψ) of the unperturbed SW equations on $(M^{'}, g)$ such that

- *$A=C+a$ where C is a fixed smooth connection on $L\otimes\mathcal{O}(-\Sigma)$, $d^*a = 0$ and $a \in L^2_1\Omega^1_g(M^{'})$;*

- *$\psi \in L^2_1(M^{'}, g)$ and there exists $K > 0$ such that $sup_{x\in M'}|\psi(x)| \leq K$.*

For the explicit construction of the metrics $\{g_j\}$, the perturbations $\{F^+_{B_j}\}$ and the proof, we refer to the original paper of Biquard [24].

A few final remarks. We decided to state the main analytical theorem of [24] as above to avoid several technical difficulties. In fact, when the toroidal compactification is not smooth, one has to deal with the SW equations on orbifolds where for example one has to use index theory extended to this wider context [24]. Furthermore, even in the case when the compactification is smooth, some extra care is required if we are not working on surfaces of general type. Note that smooth toroidal compactifications of Kodaira dimension zero do exist [48].

Finally, the restriction in Theorem 8 on the toroidal compactification to be minimal of general type is not restrictive. In fact, as mentioned above one can show that, up to a finite regular cover, any complex-hyperbolic surface satisfies this requirement. This geometric fact can then be applied to somewhat simplify the proof of the rigidity result for cuspidal Einstein metrics on finite volume complex-hyperbolic surfaces, proved by Biquard in [24].

3.3 The obstruction

Recall from Chapter 1.1 that given a orientable noncompact manifold (M, g) we have, when the differential d is restricted to an appropriate dense subset, a Hilbert complex

$$... \longrightarrow L^2\Omega_g^{k-1}(M) \longrightarrow L^2\Omega_g^k(M) \longrightarrow L^2\Omega_g^{k+1}(M) \longrightarrow ...$$

where the inner products on the exterior bundles are induced by g. Define the maximal domain of d, at the k-th level, to be

$$Dom^k(d) = \{\alpha \in L^2\Omega_g^k(M), d\alpha \in L^2\Omega_g^{k+1}(M)\}$$

where $d\alpha \in L^2\Omega_g^{k+1}(M)$ has to be intended in the distributional sense. The (reduced) L^2-cohomology groups are then defined to be

$$H_2^k(M) = Z_g^k(M)/\overline{dDom^{k-1}(d)},$$

where

$$Z_2^k(M) = \{\alpha \in L^2\Omega_g^k(M), d\alpha = 0\}.$$

On (M, g) there is a Hodge-Kodaira decomposition

$$L^2\Omega_g^k(M) = \mathcal{H}_g^k(M) \oplus \overline{dC_c^\infty\Omega^{k-1}} \oplus \overline{d^*C_c^\infty\Omega^{k+1}},$$

where

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2\Omega_g^k(M), d\alpha = 0, d^*\alpha = 0\}.$$

Moreover, if we assume (M, g) to be complete the maximal and minimal domain of d coincide. In other words

$$\overline{dDom^{k-1}(d)} = \overline{dC_c^\infty\Omega^{k-1}},$$

which implies

$$H_2^k(M) = \mathcal{H}_g^k(M).$$

Here the completeness assumption is crucially used to show that given $\alpha \in L^2\Omega_g^k(M)$ with $d\alpha \in L^2\Omega_g^{k+1}(M)$, we can always generate a sequence $\{\alpha_n\} \in C_c^\infty\Omega^k(M)$ such that $\|\alpha - \alpha_n\|_{L^2} + \|d\alpha - d\alpha_n\|_{L^2} \to 0$.

Summarizing, if the manifold is complete, the harmonic L^2-forms compute the reduced L^2-cohomology. Moreover, in this case the L^2 harmonic forms can be characterized as follows

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2\Omega_2^k(M), (dd^* + d^*d)\alpha = 0\}.$$

Finally, the orientability of M gives a duality isomorphism via the Hodge $*$ operator

$$\mathcal{H}_g^k(M) \simeq \mathcal{H}_g^{n-k}(M).$$

If the manifold M has dimension $4n$ it then makes sense to talk about L^2 selfdual and anti-selfdual forms on $L^2\Omega_g^{2n}(M)$. If $\mathcal{H}_g^{2n}(M)$ is finite dimensional, the concept of L^2-signature is well defined.

Let (M, g) be a complete finite-volume 4-manifold. Let $\mathcal{L}$ be a complex line bundle on M. By extending the Chern-Weil theory for compact manifolds, we can define the L^2-Chern class of $\mathcal{L}$. More precisely, given a connection A on $\mathcal{L}$ such that $F_A \in L^2\Omega_g^2(M)$, we may define

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F_A]_{L^2}$$

where with F_A we indicate the curvature of the given connection. It is an interesting corollary of the L^2 cohomology theory that, on complete manifolds, such an L^2 cohomology element is connection independent as long as we allow connections that differ by a 1-form in the maximal domain of the d operator. More precisely, let $A^{'}$ be a connection on $\mathcal{L}$ such that $A^{'} = A + \alpha$ with $\alpha \in L^2_1\Omega^1_g(M)$. We then have $F_{A'} = F_A + d\alpha$ and therefore by the Hodge-Kodaira decomposition we conclude that $\frac{i}{2\pi}[F_A]_{L^2} = \frac{i}{2\pi}[F_{A'}]_{L^2}$.

The associated L^2 Chern number $c_1^2(\mathcal{L})$ is also well defined. In fact, $\alpha \in Dom^1(d)$ and then we can find a sequence $\{\alpha_n\} \in C_c^\infty\Omega^k(M)$ such that $\|\alpha - \alpha_n\|_{L^2} + \|d\alpha - d\alpha_n\|_{L^2} \to 0$. This implies that

$$\begin{aligned}\int_M F_{A'} \wedge F_{A'} d\mu_g &= \lim_{n\to\infty} \int_M (F_A + d\alpha_n) \wedge (F_A + d\alpha_n) d\mu_g \\ &= \int_M F_A \wedge F_A d\mu_g.\end{aligned}$$

The following lemma is an easy consequence of the Hodge-Kodaira decomposition.

Lemma 3.3.1 *Given $\mathcal{L}$ and A as above, we have*

$$\int_M |F_A^+|^2 d\mu_g \geq 4\pi^2 (c_1^+(\mathcal{L}))^2$$

where $c_1^+(\mathcal{L})$ is the selfdual part of the g-harmonic L^2 representative of $[c_1(\mathcal{L})]$.

We have

$$\begin{aligned}\int_M |F_A^+|^2 d\mu_g &= 2\pi^2 \int_M c_1(\mathcal{L}) \wedge c_1(\mathcal{L}) d\mu_g + \frac{1}{2}\int_M |F_A|^2 d\mu_g \\ &= 2\pi^2 c_1^2(\mathcal{L}) + \frac{1}{2}\int_M |F_A|^2 d\mu_g.\end{aligned}$$

By Hodge-Kodaira decomposition, given any L^2 cohomology class, we have a unique harmonic representative that minimizes the L^2 norm. Thus, given $F_A \in L^2\Omega^2_g(M)$, let us denote by φ its harmonic representative. We then have

$$\frac{1}{2}\int_M |F_A|^2 d\mu_g \geq \frac{1}{2}\int_M |\varphi|^2 d\mu_g$$

which implies

$$\int_M |F_A^+|^2 d\mu_g \geq 2\pi^2 c_1(\mathcal{L})^2 + \frac{1}{2}\int_M |\varphi|^2 d\mu_g$$
$$= \int_M |\varphi^+|^2 d\mu_g = 4\pi^2 (c_1^+(\mathcal{L}))^2.$$

We can now formulate the L^2 analogue of the scalar curvature estimate discovered in [26] for compact manifolds.

Theorem 3.3.2 *Let (M^4, g) be a finite volume cuspidal manifold. Let $(A, \psi) \in L_1^2(M, g)$ be an irreducible solution of the SW equations associated to a Spin^c structure $\mathfrak{c}$ with determinant line bundle $\mathcal{L}$. Then*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2 (c_1^+(\mathcal{L}))^2$$

with equality if and only if g has constant negative scalar curvature, and is Kähler with respect to a complex structure compatible with $\mathfrak{c}$.

By the twisted Lichnerowicz formula [6], we know that

$$0 = \int_M Re(\nabla_A^* \nabla_A \psi, \psi) + (\frac{s}{4}\psi, \psi) + \frac{1}{4}|\psi|^4 d\mu_g. \tag{3.1}$$

Recall that the spinor ψ is in L_1^2. Since (M, g) is complete we can generate a sequence $\psi_n \in C_c^\infty$ such that $\|\psi - \psi_n\|_{L^2} + \|\nabla_A \psi - \nabla_A \psi_n\|_{L^2} \to 0$. As a result we can integrate by parts in (3.1) to obtain

$$0 = \int_M |\nabla_A \psi|^2 + s|\psi|^2 + |\psi|^4 d\mu_g$$

so that $\int_M (-s)|\psi|^2 d\mu_g \geq \int_M |\psi|^4 d\mu_g$, with equality if and only if ψ is parallel. By the Schwartz inequality we have

$$(\int_M s^2 d\mu_g)^{\frac{1}{2}} (\int_M |\psi|^4 d\mu_g)^{\frac{1}{2}} \geq \int_M (-s)|\psi|^2 d\mu_g,$$

and then

$$\int_M s^2 d\mu_g \geq \int_M |\psi|^4 d\mu_g,$$

with equality if and only if ψ is parallel and s is constant. But now $|F_A^+|^2 = \frac{1}{8}|\psi|^4$, which implies

$$\int_M s^2 d\mu_g \geq 8 \int_M |F_A^+|^2 d\mu_g.$$

Therefore we have

$$\int_M s^2 d\mu_g \geq 32\pi^2 (c_1^+(\mathcal{L}))^2,$$

with equality if and only if ψ is parallel and s_g is a negative constant. Under the equality assumption, the non zero constant length self dual 2-form F_A^+ can now be used to produce a g-compatible almost complex structure J. Since $\nabla_A \psi = 0$ we have that $\nabla F_A^+ = 0$. Thus, the almost complex structure J is integrable and g is Kähler. Finally, $\mathfrak{c}$ is the $Spin^c$ determined by J.

The following theorem gives an obstruction to the existence of cuspidal Einstein metrics on blow-ups of finite volume complex hyperbolic surfaces. The strategy of the proof follows [27]. Nevertheless, in the finite volume setting one has to employ the results of [24], see theorem 8.

Theorem C *Let M be a finite volume complex hyperbolic surface such that its toroidal compactification $(\overline{M}, \Sigma)$ is a smooth minimal surface of general type. Let $M^{'} = M \# k\overline{\mathbb{C}P^2}$ be obtained from M by blowing up $k > 0$ points. If*

$$k \geq \frac{2}{3}(c_1^2(\overline{M}) - \Sigma^2),$$

then $M^{'}$ does not admit cuspidal Einstein metrics.

Let $\{E_i\}_{i=1}^k$ denote the exceptional divisors in $M^{'}$. By a result of Morgan and Friedman [17], we know that the manifold $\overline{M}\sharp k\overline{\mathbb{C}P^2}$ admits 2^k different $Spin^c$ structures with determinant line bundles

$$L = K_{\overline{M}}^{-1} \pm E_1 \pm ... \pm E_k,$$

for which the SW equations have irreducible solutions for each metric. Since

$$\begin{aligned}(c_1(L)^+)^2 &= (c_1(\overline{M})^+ \pm E_1^+ \pm ... \pm E_k^+)^2 \\ &= (c_1(\overline{M})^+)^2 + 2\sum_i c_1(\overline{M})^+ \cdot (\pm E_i^+) + (\sum_i \pm E_i^+)^2\end{aligned}$$

we can chose a $Spin^c$ structure whose determinant line bundle satisfies

$$(c_1(L)^+)^2 \geq (c_1^+(\overline{M}))^2 \geq c_1^2(\overline{M}).$$

We can now apply Biquard's construction [24] for any of the above $Spin^c$ structures and with respect to the cuspidal Einstein metric g on $M^{'}$. We then construct 2^k irreducible solutions $(A, \psi) \in L_1^2(M^{'}, g)$, where $A = C + a$ with C a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$ and $a \in L_1^2\Omega_g^1(M^{'})$. By choosing the $Spin^c$ structure as above and using theorem 3.3.2 we compute

$$\begin{aligned}\frac{1}{32\pi^2}\int_{M'} s^2 d\mu_g &\geq (c_1(L \otimes \mathcal{O}(-\Sigma))^+)^2 \\ &\geq c_1^2(\overline{M}) - \Sigma^2\end{aligned}$$

where the last inequality holds by the adjunction formula. We then obtain

$$\begin{aligned}\frac{1}{32\pi^2}\int_{M'} s^2 d\mu_g &\geq c_1^2(\overline{M}) - \Sigma^2 \\ &= 2\chi(\overline{M}) + 3\sigma(\overline{M}) - \Sigma^2.\end{aligned}$$

Following [24], when the toroidal compactification is smooth, we have the finite volume analogue of the 4-dimensional Gauss-Bonnet and Hirzebruch formulas.

More precisely, given $M^{'}$ equipped with a cuspidal metric g one has

$$\chi(M^{'}, g) = \chi(\overline{M}) + k$$
$$\sigma(M^{'}, g) = \sigma(\overline{M}) - \frac{1}{3}\Sigma^2 - k.$$

where $\chi(M^{'}, g)$ and $\sigma(M^{'}, g)$ are defined by the usual curvature integrals. Thus, if we assume g to be Einstein

$$\begin{aligned} c_1^2(\overline{M}) - \Sigma^2 - k &= 2\chi(M^{'}) + 3\sigma(M^{'}) \\ &= \frac{1}{4\pi^2}\int_{M'} 2|W_+|^2 + \frac{s^2}{24}d\mu_g \\ &\geq \frac{1}{96\pi^2}\int_{M'} s^2 d\mu_g \\ &\geq \frac{1}{3}(c_1^2(\overline{M}) - \Sigma^2) \end{aligned}$$

so that

$$\frac{2}{3}(c_1^2(\overline{M}) - \Sigma^2) \geq k.$$

In other words if

$$k > \frac{2}{3}(c_1^2(\overline{M}) - \Sigma^2),$$

we cannot have a cuspidal Einstein metric on $M\sharp k\overline{\mathbb{C}P^2}$. The equality case can also be included and the proof goes as in the compact case. For further details, see [29]. This completes the proof.

It is natural to ask what is the sign of the contribution coming from the self-intersection of the compactification divisor Σ.

Thus, let $\overline{M}$ be a surface of general type as above and let Σ_i be a smooth elliptic divisor contained in Σ. The canonical line bundle $K_{\overline{M}}$ of a minimal surface of general type is numerically effective and has positive self intersection

[10]. By the Hodge index theorem [10], if a divisor E is such that $K_{\overline{M}} \cdot E = 0$ then E is a smooth rational curve with $E \cdot E = -2$. Since $K_{\overline{M}} \cdot \Sigma_i > 0$ and

$$p_a(\Sigma_i) = 1 + \frac{K_{\overline{M}} \cdot \Sigma_i + \Sigma_i \cdot \Sigma_i}{2} = 1$$

we conclude that $\Sigma_i \cdot \Sigma_i < 0$ for any irreducible components of the compactification divisor Σ.

3.4 Final remarks

In [27], LeBrun showed the following theorem:

Theorem 9 (LeBrun) *Let X be a minimal surface of general type, and let $M = X \# k\overline{\mathbb{C}P^2}$ be its blow-up at $k > 0$ points. If $k \geq \frac{2}{3}c_1^2(X)$, then M does not admit Einstein metrics.*

In light of this result and several subsequent developments, it is conjectured [29] that non-minimal surfaces of general type do not admit Einstein metrics. Unfortunately, all of the obstructions known at present, depend on the Chern number c_1^2 of the associated minimal model. Therefore, these results are not strong enough to provide an obstruction that is *uniform* in the number of blow-ups. In fact, one can simply consider a sequence of Fermat hypersurfaces V_n in $\mathbb{C}P^3$. Recall that such hypersurfaces are defined as the zero locus of the homogeneous polynomials $z_0^n + z_1^n + z_2^n + z_3^n$ in $\mathbb{C}P^3$. Moreover, for $n \geq 5$ they are of general type since they have ample canonical line bundle. For these surfaces one has $c_1^2(V_n) = n(4-n)^2$, which diverges as n goes to infinity.

In the noncompact case it is somewhat more complicated to construct explicit examples. Nevertheless, one can use the constructions of Holzapfel [47] and Hirzebruch [48]. For example, Hirzebruch [48] constructed a sequence of ball quotients X_n $(n \geq 2)$ which have smooth toroidal compactifications $(\overline{X}_n, \Sigma_n)$ of general type, where Σ_n is the disjoint union of $4n^4$ smooth elliptic divisors with self intersection $-n$, and where the Chern numbers of $\overline{X}_n$ are given by

$$c_2(\overline{X}_n) = n^7,$$
$$c_1^2(\overline{X}_n) = 3n^7 - 4n^5.$$

By Theorem C, it follows that $X_n \sharp k\overline{\mathbb{C}P^2}$ does not admit cuspidal Einstein metrics if $k \geq 2n^7$. Similar examples can also be constructed from the ball-quotients considered by Holzapfel [47]. Details are left to the interested reader.

In conclusion, as in the compact case, Theorem C is far from providing an obstruction that is uniform in the number of blow-ups.

Chapter 4

Seiberg-Witen Equations on finite volume manifolds with cusps

4.1 Introduction

In this chapter we study the Seiberg-Witten equations on product manifolds $M = \Sigma \times \Sigma_g$, where Σ is a finite volume hyperbolic Riemann surface and Σ_g a compact Riemann surface of genus g.

The main problem with Seiberg-Witten theory on noncompact manifold is the lack of a satisfactory existence theory. Following Biquard [24], we solve the SW equations on M by working on the compactification $\overline{M}$. Here the compactification $\overline{M}$ is the obvious one coming from the compactification of Σ. More precisely, we produce an irreducible solution of the unperturbed SW equations on M as limit of solutions of the perturbed SW equations on $\overline{M}$. From the metric point of view, starting with (M, g) where g is assumed to be of finite volume and with a "nice" behavior at infinity, one has to construct a sequence $(\overline{M}, g_j)$ of metric compactifications that approximate (M, g) as j goes to infinity. The irreducible solution of the SW equations on (M, g) is then constructed by a bootstrap argument with the solutions of the SW equations on $(\overline{M}, g_j)$ with suitably constructed perturbations.

When $M = \Sigma \times \mathbb{C}P^1$ this construction was carried out by Yann Rollin in [40].

An outline of the chapter follows. Section 4.2 describes explicitly the metric compactifications $(\overline{M}, g_j)$. These metrics are completely analogous to the one used by Rollin and Biquard in [40] and [24]. Furthermore, few results concerning the scalar curvatures and volumes of the spaces $(\overline{M}, g_j)$ are given.

In Section 4.3 we compute the L^2 cohomology of $(\Sigma \times \Sigma_g, g)$ when g is a metric C^0 asymptotic to a product metric $g_{-1} + g_2$, where g_{-1} is a hyperbolic metric on Σ and g_2 any metric on Σ_g.

Sections 4.4 and 4.5 contain the proofs of the uniform Poincaré inequalities on functions and 1-forms needed for the bootstrap argument. Moreover the convergence, as j goes to infinity, of the harmonic forms on $(\overline{M}, g_j)$ is studied in detail. These results are then combined with the results given in Section 4.3 to provide an explicit isomorphism between the DeRham cohomology of $\overline{M}$ and the L^2 cohomology of (M, g).

In Section 4.6 the bootstrap argument is carefully worked out. The existence result so obtained is summarized in Theorem D.

In Section 4.7, Theorem D is applied to derive several geometrical consequences. First, we give the *sharp* minimization of the Riemannian functional $\int s_g^2 d\mu_g$ on M. Second, an obstruction to the existence of Einstein metrics on blow-ups of M is given. These results are summarized in Theorem E and Theorem F. These theorems are the finite volume generalization of some celebrated and well-known results of Claude LeBrun for closed four manifolds, see for example [29] and the bibliography therein.

4.2 The Metric Compactifications

Let Σ be a finite volume hyperbolic Riemann surface and denote with Σ_g a compact Riemann surface of genus g. In this chapter, we study the Seiberg-Witten equations on manifolds that topologically are products of the form $\Sigma \times \Sigma_g$. Recall that Σ is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$ with a finite number of points removed, say $\{p_1, ..., p_l\}$, satisfying the condition that $2g(\overline{\Sigma}) - 2 + l > 0$. Conversely, given a compact Riemann surface $\overline{\Sigma}$ and points $\{p_1, ..., p_l\}$ such that $2g(\overline{\Sigma}) - 2 + l > 0$, the open Riemann surface $\Sigma = \overline{\Sigma} \backslash \{p_1, ..., p_l\}$ admits a finite volume real hyperbolic metric. In summary, a finite volume hyperbolic Riemann surface (Σ, g_{-1}) is a manifolds with finitely many cusps corresponding to the marked points of the associated compactification $\overline{\Sigma}$. Our hyperbolic cusps are modeled on $\mathbb{R}^+ \times S^1$ with the metric $g_{-1} = dt^2 + e^{-2t} d\theta^2$. We can now fix a metric g_2 on the compact Riemann surface Σ_g and consider the Riemannian product $(\Sigma \times \Sigma_g, g_{-1} + g_2)$. For simplicity we define $M = \Sigma \times \Sigma_g$. It is then clear that M is a complete finite volume manifold with cusp ends modeled on $\mathbb{R}^+ \times S^1 \times \Sigma_g$ with the metric $g_t = dt^2 + e^{-2t} d\theta^2 + g_2$.

Definition 4.2.1 *A metric $\tilde{g}$ on M of the form $g_{-1} + g_2$ will be called a* standard model.

We now want to study the natural compactification of M. It is clear that each of the cusp end of M can be closed topologically as a manifold by adding a compact genus g Riemann surface. Let us denote by N the disjoint union of these embedded curves. Denoted with $\overline{M}$ the compactification of M, we then have $\overline{M}\backslash N \simeq M$. If we know consider Σ and Σ_g as complex manifolds, it is clear that M can be compactified as a complex manifold by adding a finite number of genus g divisors with trivial self intersection. According to the genus of $\overline{\Sigma}$, we have the following classification for $\overline{M}$ as a complex manifold:

- if $g(\overline{\Sigma}) = 0 \Longrightarrow \overline{M}$ is rational ruled with $c_1^2(\overline{M}) = 4\chi(\Sigma_g)$;

- if $g(\overline{\Sigma}) = 1 \Longrightarrow \overline{M}$ is elliptic with $c_1^2(\overline{M}) = 0$;

- if $g(\overline{\Sigma}) \geq 2 \Longrightarrow \overline{M}$ is of general type with $c_1^2(\overline{M}) = 2\chi(\overline{\Sigma}) \cdot \chi(\Sigma_g)$.

Let us now consider a standard model $\tilde{g}$ on M. We want to construct a sequence of metrics $\{\tilde{g}_j\}$ on $\overline{M}$ that approximate $(M, \tilde{g})$. More precisely, choose coordinates on the cusp ends of M such that the metric $\tilde{g}$ is given by $g_t = dt^2 + e^{-2t}d\theta^2 + g_2$ for $t > 0$. Define then

$$\tilde{g}_j = dt^2 + \varphi_j^2(t)d\theta^2 + g_2$$

where $\varphi_j(t)$ is a smooth warping function such that:

1. $\varphi_j(t) = e^{-t}$ for $t \in [0, j+1]$;

2. $\varphi_j(t) = T_j - t$ for $t \in [j+1+\epsilon, T_j]$;

where ϵ is a fixed number that can be chosen to be small, and T_j is an appropriate number bigger than $j + 1 + \epsilon$. Because of the second condition above, $\tilde{g}_j$ is a smooth metric on $\overline{M}$ for any j. Moreover the metrics $\{\tilde{g}_j\}$ are by construction isometric to $\tilde{g}$ on bigger and bigger compact sets of M. For later convenience we want to prescribe in more details the behavior of $\varphi_j(t)$ in the interval $t \in [j+1, j+1+\epsilon]$. We require that $\partial_t^2\varphi_j(t)$ decreases from $e^{-(j+1)}$ to 0 in the interval $[j+1, j+1+\delta_j]$ where δ_j is a positive number less than ϵ. Then for $t \in [j+1+\delta_j, \epsilon]$, we make $\partial_t^2\varphi_j$ very negative in order to decrease $\partial_t\varphi_j$ to -1 and smoothly glue $\varphi_j(t)$ to the function $T_j - t$. Moreover, by eventually letting the parameters δ_j go to zero as j goes to infinity, we require $\frac{|\partial_t\varphi_j|}{\varphi_j}$ to be increasing in the interval $[j+1, j+1+\delta_j]$. Finally, we require $\frac{|\partial_t\varphi_j|}{\varphi_j}$ to be bounded from above uniformly in j.

In summary, given a standard model $\tilde{g}$ for M we can always generate a sequence of metrics $\{\tilde{g}_j\}$ on $\overline{M}$ approximating $(M, \tilde{g})$. A similar argument shows that this is indeed the case for any metric g on M, that is asymptotic to a standard model. For later convenience, we restrict ourself to metrics that are asymptotic to a standard model at least in the C^2 topology. More precisely, if g is such a metric we set

$$g_j = (1-\chi_j)g + \chi_j\tilde{g}_j$$

where $\chi_j(t)$ is a sequence of smooth increasing functions defined on the cusps of M such that $\chi_j(t) = 0$ if $t \leq j$ and $\chi_j(t) = 1$ if $t \geq j+1$.

Proposition 4.2.2 *The scalar curvature of the metrics $\{g_j\}$ can be expressed as*

$$s_{g_j} = s^b_{g_j} - 2\chi_j\frac{\partial_t^2\varphi_j}{\varphi_j}$$

where $s^b_{g_j}$ is a smooth function on $\overline{M}$ that can be bounded uniformly in j.

For $t \leq j$, the metrics g_j and g are isometric and therefore $s_{g_j} = s_g$. If $t \in [j, j+1]$, the metric g_j is close in the C^2 topology to g and then $s_{g_j} \approx s_g$. Finally if $t \geq j+1$, the scalar curvature function is explicitly given by

$$s_{g_j} = s_{\tilde{g}_j} = s_{g_2} - 2\frac{\partial_t^2 \varphi_j}{\varphi_j}.$$

We conclude this section with a proposition regarding the volumes of the Riemannian manifolds $(\overline{M}, g_j)$.

Proposition 4.2.3 *There exists a constant $K > 0$ such that*

$$Vol_{g_j}(\overline{M}) \leq K$$

for any j.

4.3 L^2 cohomology of products

Let (Σ, g_{-1}) be a finite volume hyperbolic Riemann surface. Furthermore, let (Σ_g, g_2) be a genus g compact Riemann surface equipped with a fixed metric. Let us consider $(\Sigma \times \Sigma_g, g_{-1} + g_2)$, where by $g_{-1} + g_2$ we denote the product metric. We then want compute the L^2 cohomology of $(\Sigma \times \Sigma_g, g)$ when g is a metric "asymptotic" to the product metric $g_{-1} + g_2$. Following the definition of section 4.2 a metric of the from $g_{-1} + g_2$ is referred as standard metric or model. For simplicity let us define $M := \Sigma \times \Sigma_g$. Let us start by computing the L^2 cohomology of M when equipped with a standard metric.

Regarding the L^2 cohomology of $(M, g_{-1} + g_2)$, an L^2-Künneth formula argument [50] reduces the problem to the computation of the L^2 cohomology of a hyperbolic Riemann surface of finite topological type. This computation can be achieved by using the following classical theorem.

Theorem 10 (Huber) *Let (Σ, g) be a complete finite volume Riemann surface with bounded curvature. Then Σ is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$ with a finite number of points removed.*

See [35].

Corollary 4.3.1 *Let (Σ, g_{-1}) be a complete finite volume hyperbolic Riemann surface. Then we have the isomorphism*

$$H_2^*(\Sigma, g_{-1}) \simeq H^*(\overline{\Sigma}).$$

We clearly just have to prove that $H_2^1(\Sigma) \simeq H^1(\overline{\Sigma})$. Since Σ is complete, the space of L^2 harmonic forms computes the L^2 cohomology. Let $(\overline{\Sigma}\backslash\{p_1, ..., p_l\}, \overline{g})$ as in theorem 10, where $\overline{g} = e^{2u}g$. Since $\overline{g}$ and g_{-1} define the same global L^2 inner product on 1-forms and $d^{*_{\overline{g}}} = e^{-2u}d^{*_{g_{-1}}}$, we have that $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, ..., p_l\}) \simeq \mathcal{H}^1_{g_{-1}}(\Sigma)$. But now one can show that any harmonic field in $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, ..., p_l\})$ can be smoothly extended across the cusps points. For the proof of this simple analytical fact see [44]. We therefore have $\mathcal{H}^1_{\overline{g}}(\overline{\Sigma}\backslash\{p_1, ..., p_l\}) \simeq \mathcal{H}^1_{\overline{g}}(\overline{\Sigma})$. The corollary is now a consequence of the classical Hodge theorem for closed manifolds.

We can now formulate the main result of this section.

Proposition 4.3.2 *In the notation above, consider (M, g) where g is a Riemannian metric C^0 asymptotic a standard model. Then we have the isomorphism*

$$H_2^*(M) \simeq H^*(\overline{M}; \mathbb{R}).$$

The L^2 cohomology is a quasi-isometric invariant.

4.4 Poincaré inequalities and convergence of 1-forms

We need to show that, given the sequence of metrics $\{g_j\}$, we can find a uniform Poincaré inequality on functions. We have the following lemma.

Lemma 4.4.1 *Consider the metric $g = dt^2 + g_t$ on the product $[0, \infty) \times N$, such that the mean curvature of the cross section N is uniformly bounded from below by a positive constant h_0. Then, for any function f we have*

$$\int |\partial_t f|^2 d\mu_g \geq h_0^2 \int |f|^2 d\mu_g + h_0 \int_{t=T} |f|^2 d\mu_{g_t} - h_0 \int_{t=0} |f|^2 d\mu_{g_t}.$$

See Lemma 4.1 in [24].

Using this lemma, we can now derive the desired uniform Poincaré inequality.

Proposition 4.4.2 *There exists a positive constant c, independent of j, such that*

$$\int_{\overline{M}} |df|^2 d\mu_{g_j} \geq c \int_{\overline{M}} |f|^2 d\mu_{g_j}$$

for any function f on $\overline{M}$ such that $\int_{\overline{M}} f d\mu_{g_j} = 0$.

Assume the existence of a sequence of functions $\{f_j\}$ such that $\int_{\overline{M}} f_j d\mu_{g_j} = 0$, $\|f_j\|_{L^2(g_j)} = 1$, and for which $\|df_j\|_{L^2(g_j)} \to 0$. Fixed a compact set K, the sequence $\{f_j\}$ is bounded in $L^2_1(K, g_j)$. We can then extract a weak limit $f_j \rightharpoonup f$ where $f \in L^2_1(K, g)$. By the compactness of the embedding $L^2_1(K, g) \hookrightarrow L^2(K, g)$, we have $f_j \to f$ in $L^2(K, g)$. Since $\|df\|_{L^2} = 0$, f is constant. Given $\epsilon > 0$, we can always chose K big enough such that $Vol_{g_j}(\overline{M} \backslash K) \leq \epsilon$. Since

$$\int_K f d\mu_g = \lim_{j \to \infty} \int_K f_j d\mu_{g_j}$$

and $\int_K f_j d\mu_{g_j} = -\int_{\overline{M}\backslash K} f_j d\mu_{g_j}$, we conclude that

$$\Big| \int_{\overline{M}\backslash K} f_j d\mu_{g_j} \Big| \leq Vol_{g_j}(\overline{M}\backslash K)^{\frac{1}{2}} \|f_j\|_{L^2_{g_j}} = Vol_{g_j}(\overline{M}\backslash K)^{\frac{1}{2}}.$$

In other words $\int_K f d\mu_g$ can be made arbitrarily small and then f has to be equal to zero. Fix a big compact set K and a smooth cut off function χ such that $\chi = 1$ on K and equal to zero outside an open set containing K. We then compute

$$\frac{1}{2}\int_{\overline{M}} |\chi f_j + (1-\chi) f_j|^2 d\mu_{g_j} \leq \int_{\overline{M}} |\chi f_j|^2 d\mu_{g_j} + \int_{\overline{M}} |(1-\chi) f_j|^2 d\mu_{g_j}$$

but by lemma 4.4.1 we know that

$$\int_{\overline{M}} |(1-\chi) f_j|^2 d\mu_{g_j} \leq h_0^{-2} \int_{\overline{M}} |\partial_t (1-\chi) f_j|^2 d\mu_{g_j} \leq h_0^{-2} \int_{\overline{M}} |d(1-\chi) f_j|^2 d\mu_{g_j}$$

and therefore

$$\begin{aligned} \frac{1}{2}\int_{\overline{M}} |\chi f_j + (1-\chi) f_j|^2 d\mu_{g_j} \leq \int_{\overline{M}} |\chi f_j|^2 d\mu_{g_j} + 2h_0^{-2} \int_{\overline{M}} |d\chi f_j|^2 d\mu_{g_j} \\ + 2h_0^{-2} \int_{\overline{M}} |df_j|^2 d\mu_{g_j}. \end{aligned}$$

Since the elements in the right hand side of the inequality above can be made arbitrarily small by taking K and j big enough, we conclude that $\|f_j\|_{L^2(g_j)} \to 0$ which is a contradiction.

The uniform Poincaré inequality proved in proposition 4.4.2, can now be used to derive a Poincaré inequality for the limiting open manifold (M, g). This result will not be used anywhere in this thesis, we thus omit the proof.

Corollary 4.4.3 *There exists a positive constant c such that*

$$\int_M |df|^2 d\mu_g \geq c \int_M |f|^2 d\mu_g$$

for any function f such that $\int_M f d\mu_g = 0$.

We now have to derive a uniform Poincaré inequality on 1-forms for the sequence of metrics $\{g_j\}$. On the products $[0,T_j)\times S^1\times\Sigma_g$ consider the metrics

$$g_j = dt^2 + \varphi_j(t)^2 d\theta^2 + g_2,$$

where g_2 is a fixed metric on Σ_g and $\varphi_j(t)$ is a smooth warping function as in section 4.2 . For simplicity we denote by N the product $S^1\times\Sigma_g$. Thus, given a 1-form α we want to compute $\nabla\alpha$. First, let us write

$$\nabla\alpha = dt\otimes\nabla_{\partial_t}\alpha + \nabla_{|_N}\alpha$$

where $\nabla_{|_N}\alpha$ is an element in $T^*N\otimes\Omega^1([0,T)\times N)$. Write $\alpha = f dt + \alpha_1$ where α_1 is such that $i_{\partial_t}\alpha_1 = 0$. Thus

$$\nabla_{|_N}\alpha = d_N f\otimes dt + f\nabla_{|_N}dt + \nabla_{|_N}\alpha_1.$$

Given a metric of the form dt^2+g_t and denoted by II its second fundamental form, we have $\mathrm{II} = -\frac{1}{2}\partial_t g$. As a result

$$\mathrm{II}_{g_j} = -\frac{\partial_t\varphi_j}{\varphi_j}(\varphi_j d\theta)^2.$$

Next we compute that

$$\nabla_X\partial_t = \frac{\partial_t\varphi_j}{\varphi_j}X, \quad \nabla_Y\partial_t = 0,$$

if X and Y are respectively tangent to S^1 and Σ_g. In other words $\nabla_{|_N}dt$ has components in the direction of the S^1 factor only. Thus, given X_1 and X_2 tangent to S^1, we compute

$$\begin{aligned}(\nabla_{X_1}dt)X_2 &= X_1 dt(X_2) - dt(\nabla_{X_1}X_2) = -dt(\nabla_{X_1}X_2)\\ &= -g_j(\nabla_{X_1}X_2,\partial_t) = g_j(X_2,\nabla_{X_1}\partial_t)\\ &= \frac{\partial_t\varphi_j}{\varphi_j}(\varphi_j d\theta)^2(X_1,X_2) = -\mathrm{II}_{g_j}\,(X_1,X_2),\end{aligned}$$

and then

$$f\nabla_{|_N} dt = -f\mathrm{II}(\cdot,\cdot).$$

It remains to study the term $\nabla_{|_N}\alpha_1$. The component in the direction of N is clearly $\nabla^N\alpha_1$, where with ∇^N we indicate the Levi-Civita connection induced by g_j on N. Let us compute the component of $\nabla_{|_N}\alpha_1$ in the direction of dt. Let X be tangent to N, since $g_j(\nabla_X\alpha_1, dt) = -g_j(\alpha_1, \nabla_X dt)$, we can assume X to be tangent to the S^1 factor. We then have

$$g_j(\nabla_X\alpha_1, dt) = -\frac{\partial_t\varphi_j}{\varphi_j} g(\alpha_1, X) = \mathrm{II}(\alpha_1, X)$$

which implies

$$\nabla_{|_N}\alpha_1 = \nabla^N\alpha_1 + \mathrm{II}(\alpha_1,\cdot)\otimes dt$$

where here α_1 has to be considered as a vector. In summary

$$\nabla\alpha = dt\otimes\nabla_{\partial_t}\alpha + d^N f\otimes dt - f\mathrm{II}(\cdot,\cdot) + \mathrm{II}(\alpha_1,\cdot)\otimes dt + \nabla^N\alpha_1.$$

Next we can decompose α_1 as follows

$$\alpha_1 = f_1\varphi_j d\theta + \alpha_2$$

where $i_{v_\theta}\alpha_2 = 0$. Since N is a product manifold we compute

$$\begin{aligned}\nabla^N\alpha_1 &= d^N f_1\otimes\varphi_j d\theta + f_1\nabla^N(\varphi_j d\theta) + \nabla^N\alpha_2\\ &= d^{\Sigma_g} f_1\otimes\varphi_j d\theta + \varphi_j^{-1}\partial_\theta f_1(\varphi_j d\theta)^2 + \nabla^{S^1}\alpha_2 + \nabla^{\Sigma_g}\alpha_2.\end{aligned}$$

Recall that given a 1-form α, we have the identity

$$(L_X\alpha)(Y) = X\alpha(Y) - \alpha[X,Y] = (\nabla_X\alpha)Y + \alpha(\nabla_X Y)$$

thus

$$\nabla^{S^1}\alpha_2 = \varphi_j d\theta \otimes L_{\varphi_j^{-1}\partial_\theta}\alpha_2$$

which implies the identity

$$\nabla^N \alpha_2 = d^{\Sigma_g} f_1 \otimes \varphi_j d\theta + \varphi_j^{-1}\partial_\theta f_1 \varphi_j d\theta \otimes \varphi_j d\theta + \nabla^{\Sigma_g}\alpha_2 + \varphi_j d\theta \otimes L_{\varphi_j^{-1}\partial_\theta}\alpha_2.$$

In summary we obtain

$$\begin{aligned}\nabla\alpha = {} & dt \otimes \nabla_{\partial_t}\alpha + \{\varphi_j^{-1}\partial_\theta f - \frac{\partial_t \varphi_j}{\varphi_j} f_1\}\varphi_j d\theta \otimes dt + \{\varphi_j^{-1}\partial_\theta f_1 + \frac{\partial_t\varphi_j}{\varphi_j} f\}\varphi_j d\theta \otimes \varphi_j d\theta \\ & + d^{\Sigma_g} f \otimes dt + d^{\Sigma_g} f_1 \otimes \varphi_j d\theta + \varphi_j d\theta \otimes L_{\varphi_j^{-1}\partial_\theta}\alpha_2 + \nabla^{\Sigma_g}\alpha_2.\end{aligned}$$

The Ricci curvature of a doubly warped product can be computed explicitly [12], in our case we obtain

$$\mathrm{Ric}^{g_j}(\alpha,\alpha) = \mathrm{Ric}^{\Sigma_g}(\alpha_2,\alpha_2) - \frac{\partial_t^2\varphi_j}{\varphi_j}\{|f|^2 + |f_1|^2\}$$

which implies

$$\begin{aligned}\int_N |\nabla\alpha|^2 + \mathrm{Ric}^{g_j}(\alpha,\alpha) d\mu_{g_t} \geq {} & \int_N |\nabla_{\partial_t}\alpha|^2 + \Big\{\frac{\partial_t \varphi_j^2}{\varphi_j^2} - \frac{\partial_t^2\varphi_j}{\varphi_j}\Big\}\{|f|^2 + |f_1|^2\} d\mu_{g_t} \\ & + \int_N |\varphi_j^{-1}\partial_\theta f|^2 + |\varphi_j^{-1}\partial_\theta f_1|^2 d\mu_{g_t} \\ & - 2\int_N (\varphi_j^{-1}\partial_\theta f, \frac{\partial_t\varphi_j}{\varphi_j} f_1) d\mu_{g_t} \\ & + 2\int_N (\varphi_j^{-1}\partial_\theta f_1, \frac{\partial_t\varphi_j}{\varphi_j} f) d\mu_{g_t}.\end{aligned}$$

If now $t \leq j+1$, by construction we know that $\varphi_j(t) = e^{-t}$. Under this assumption want to show that the mixed terms $(\varphi_j^{-1}\partial_\theta f, f_1)_{L^2(N_t)}$ and $(\varphi_j^{-1}\partial_\theta f_1, f)_{L^2(N_t)}$ are controlled, for big t big enough, by the terms $\|\varphi_j^{-1}\partial_\theta f\|^2_{L^2(N_t)}$ and $\|\varphi_j^{-1}\partial_\theta f_1\|^2_{L^2(N_t)}$. More precisely, integrating by parts with respect to θ, we can always assume that f and f_1 are such that

$$\int_{S^1} f d\theta = \int_{S^1} f_1 d\theta = 0.$$

The Wirtinger (Poincaré) inequality can be used to obtain

$$\int_N |\partial_\theta f|^2 d\mu_{g_t} \geq \int_N |f|^2 d\mu_{g_t}, \quad \int_N |\partial_\theta f_1|^2 d\mu_{g_t} \geq \int_N |f_1|^2 d\mu_{g_t}.$$

By the Cauchy inequality

$$\int_N |(\varphi_j^{-1}\partial_\theta f, f_1)| d\mu_{g_t} \leq \frac{1}{\varphi_j}\|f_1\|_{L^2}\|\partial_\theta f\|_{L^2}, \quad \int_N |(\varphi_j^{-1}\partial_\theta f_1, f)| d\mu_{g_t} \leq \frac{1}{\varphi_j}\|f\|_{L^2}\|\partial_\theta f_1\|_{L^2}.$$

and then

$$\frac{1}{\varphi_j}\{\|f_1\|_{L^2}\|\partial_\theta f\|_{L^2} + \|f\|_{L^2}\|\partial_\theta f_1\|\} \leq \frac{1}{\varphi_j}\{\|\partial_\theta f\|^2_{L^2} + \|\partial_\theta f_1\|^2_{L^2}\}.$$

Since $\varphi_j(t) \to 0$ we have that for $t \to \infty$

$$\int_N |\varphi_j^{-1}\partial_\theta f|^2 d\mu_{g_t} + \int_N |\varphi_j^{-1}\partial_\theta f_1|^2 d\mu_{g_t} \geq 2\int_N (\varphi_j^{-1}\partial_\theta f, f_1) - (\varphi_j^{-1}\partial_\theta f_1, f) d\mu_{g_t}.$$

In conclusion there exists $T > 0$ such that

$$\int_N |\nabla\alpha|^2 + \mathrm{Ric}^{g_j}(\alpha,\alpha) d\mu_{g_t} \geq \int_N |\nabla_{\partial_t}\alpha|^2 d\mu_{g_t}. \tag{4.1}$$

for any $t \in [T, j+1]$.

Let us study the range $t \in [j+1, j+1+\delta_j]$. By construction the term $\frac{\partial_t \varphi_j^2}{\varphi_j^2} - \frac{\partial_t^2 \varphi_j}{\varphi_j}$ is nonnegative. We then have to study the mixed terms

$$(\varphi_j^{-1}\partial_\theta f, \frac{\partial_t \varphi_j}{\varphi_j} f_1)_{L^2(N_t)}, \quad (\varphi_j^{-1}\partial_\theta f_1, \frac{\partial_t \varphi_j}{\varphi_j} f)_{L^2(N_t)}.$$

We have the estimates

$$\int_N |(\varphi_j^{-1}\partial_\theta f, \frac{\partial_t\varphi_j}{\varphi_j} f_1)| d\mu_{g_t} = \frac{|\partial_t \varphi_j|}{\varphi_j^2}\int_N |(\partial_\theta f, f_1)| d\mu_{g_t} \leq \frac{|\partial_t\varphi_j|}{\varphi_j^2}\|\partial_\theta f\|_{L^2}\|f_1\|_{L^2}$$
$$\int_N |(\varphi_j^{-1}\partial_\theta f_1, \frac{\partial_t\varphi_j}{\varphi_j} f)| d\mu_{g_t} \leq \frac{|\partial_t\varphi_j|}{\varphi_j^2}\|\partial_\theta f_1\|_{L^2}\|f\|_{L^2}.$$

Moreover

$$\frac{|\partial_t\varphi_j|}{\varphi_j^2}\|\partial_\theta f\|_{L^2}\|f_1\|_{L^2} \leq \frac{|\partial_t\varphi_j|}{2\varphi_j^2}\{\|f_1\|^2_{L^2} + \|\partial_\theta f\|^2_{L^2}\} \leq \frac{|\partial_t\varphi_j|}{2\varphi_j^2}\{\|\partial_\theta f_1\|^2_{L^2} + \|\partial_\theta f\|^2_{L^2}\}$$

and similarly

$$\frac{|\partial_t \varphi_j|}{\varphi_j^2} \|\partial_\theta f_1\|_{L^2} \|f\|_{L^2} \leq \frac{|\partial_t \varphi_j|}{2\varphi_j^2} \{ \|\partial_\theta f_1\|_{L^2}^2 + \|\partial_\theta f\|_{L^2}^2 \}.$$

In conclusion

$$\begin{aligned}
&\int_N |\varphi_j^{-1}\partial_\theta f|^2 + |\varphi_j^{-1}\partial_\theta f_1|^2 d\mu_{g_t} - 2\int_N (\varphi_j^{-1}\partial_\theta f, \frac{\partial_t \varphi_j}{\varphi_j} f_1) - (\varphi_j^{-1}\partial_\theta f_1, \frac{\partial_t \varphi_j}{\varphi_j} f) d\mu_{g_t} \geq \\
&\quad \frac{1}{\varphi_j^2}\{ \|\partial_\theta f\|_{L^2}^2 + \|\partial_\theta f_1\|_{L^2}^2 \} - 2\int_N |(\varphi_j^{-1}\partial_\theta f, \frac{\partial_t \varphi_j}{\varphi_j} f_1)| + |(\varphi_j^{-1}\partial_\theta f_1, \frac{\partial_t \varphi_j}{\varphi_j} f)| d\mu_{g_t} \geq \\
&\quad \frac{1}{\varphi_j^2}\{ \|\partial_\theta f\|_{L^2}^2 + \|\partial_\theta f_1\|_{L^2}^2 \} - 2\frac{|\partial_t \varphi_j|}{\varphi_j^2}\{ \|\partial_\theta f\|_{L^2}^2 + \|\partial_\theta f_1\|_{L^2}^2 \}.
\end{aligned}$$

Since $\varphi_j(t) \to 0$ and $\frac{|\partial_t \varphi_j|}{\varphi_j}$ is bounded between two fixed positive real numbers for any j, there exists a a number $T > 0$ such that

$$\int_N |\nabla \alpha|^2 + \mathrm{Ric}^g(\alpha, \alpha) d\mu_{g_t} \geq \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t} \tag{4.2}$$

for any $t \in [j+1, j+1+\delta_j] \cap [T, \infty)$.

By construction, in the interval $t \in [j+1+\delta_j, j+1+\epsilon]$ we have $-\partial_t^2 \varphi_j \backslash \varphi_j \geq 0$, since

$$\int_N |\nabla \alpha|^2 + \mathrm{Ric}^{g_j}(\alpha, \alpha) d\mu_{g_t} \geq -\frac{\partial_t^2 \varphi_j}{\varphi_j}\{ \|f\|_{L^2}^2 + \|f_1\|_{L^2}^2 \} + \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t}$$

we conclude that

$$\int_N |\nabla \alpha|^2 + \mathrm{Ric}^{g_j}(\alpha, \alpha) d\mu_{g_t} \geq \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t}. \tag{4.3}$$

It remains to study the interval $t \in [j+1+\epsilon, T_j]$. In this range $-\partial_t^2 \varphi_j = 0$, and then by the same argument above we conclude that

$$\int_N |\nabla \alpha|^2 + \mathrm{Ric}^{g_j}(\alpha, \alpha) d\mu_{g_t} \geq \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t}. \tag{4.4}$$

In summary combining 4.1 up to 4.4 we have the following lemma.

Lemma 4.4.4 *There exists $T > 0$ such that*

$$\int_N |\nabla\alpha|^2 + Ric^{g_j}(\alpha,\alpha)d\mu_{g_t} \geq \int_N |\nabla_{\partial_t}\alpha|^2 d\mu_{g_t}$$

for any $t \in [T, T_j)$.

We can now prove the desired uniform Poincaré inequality on 1-forms for the metrics $\{g_j\}$. First, observe that for $[t_1, t_2] \subset [T, T_j]$

$$\begin{aligned}\int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} &= \int_{[t_1,t_2]\times N} \partial_t(|\alpha|^2 d\mu_{g_t})dt = \int_{[t_1,t_2]\times N} \partial_t|\alpha|^2 d\mu_{g_t} dt \\ &+ \int_{[t_1,t_2]\times N} |\alpha|^2 \partial_t d\mu_{g_t} dt \\ &= \int_{[t_1,t_2]\times N} \partial_t|\alpha|^2 d\mu_{g_j} - 2\int_{[t_1,t_2]\times N} h|\alpha|^2 d\mu_{g_j}.\end{aligned}$$

We then obtain

$$\int_{[t_1,t_2]\times N} \partial_t|\alpha|^2 d\mu_{g_j} \geq \int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} + 2h_0 \int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}.$$

where h_0 is a uniform lower bound for the mean curvature. But now

$$\partial_t|\alpha|^2 = 2(\alpha, \nabla_{\partial_t}\alpha) \leq 2|\alpha||\nabla_{\partial_t}\alpha| \leq h_0|\alpha|^2 + \frac{1}{h_0}|\nabla_{\partial_t}\alpha|^2$$

which then implies

$$\int_{[t_1,t_2]\times N} |\nabla_{\partial_t}\alpha|^2 d\mu_{g_j} \geq h_0 \int_{\partial\{[t_1,t_2]\times N\}} |\alpha|^2 d\mu_{g_j} + h_0^2 \int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}. \quad (4.5)$$

We summarize the discussion above into the following lemma.

Lemma 4.4.5 *There exist positive numbers $c > 0$, $T > 0$ such that*

$$\int_{[t_1,t_2]\times N} |d\alpha|^2 + |d^{*g_j}\alpha|^2 d\mu_{g_j} \geq c\int_{[t_1,t_2]\times N} |\alpha|^2 d\mu_{g_j}$$

for any $[t_1, t_2] \subset [T, T_j)$ and α with support contained in $[t_1, t_2] \times N$.

Combining lemma 4.4.4 and 4.5, the lemma follows from the well know Bochner formula for 1-forms.

Let us study the convergence of harmonic 1-forms.

Proposition 4.4.6 *Let $[a] \in H^1_{dR}(\overline{M})$ and $\{\alpha_j\}$ be the sequence of harmonic representatives with respect the metrics $\{g_j\}$. Then $\{\alpha_j\}$ converges, with respect to the C^∞ topology on compact sets, to a harmonic 1-form $\alpha \in L^2\Omega^1_g(M)$.*

Let β be a closed smooth representative for $[a] \in H^1_{dR}(\overline{M})$. Given g_j, by the Hodge decomposition theorem, we can write $\alpha_j = \beta + df_j$ with α_j harmonic and f_j a C^∞ function. Without loss of generality we can assume that $\int_{\overline{M}} f_j d\mu_j = 0$. Furthermore, we have

$$0 = d^*\alpha_j = d^*\beta + d^*df_j \Longrightarrow \Delta^{g_j}_H f_j = -d^*\beta.$$

By construction of the metrics $\{g_j\}$, it is clear that $\|\beta\|_{L^2(\overline{M},g_j)}$ is bounded independently of j. By Proposition 4.4.2 we have

$$\begin{aligned} c\int_{\overline{M}} |f_j|^2 d\mu_{g_j} \leq \int_{\overline{M}} |df_j|^2 d\mu_{g_j} &= (df_j, df_j)_{L^2(g_j)} \\ &= (f_j, d^*df_j) = -(d^*\beta, f_j) \\ &= -(\beta, df_j) \leq \|df_j\|_{L^2(g_j)} \|\beta\|_{L^2(g_j)} \end{aligned}$$

and then

$$c\int_{\overline{M}} |f_j|^2 d\mu_{g_j} \leq \|df_j\|^2_{L^2(g_j)} \leq \|df_j\|_{L^2(g_j)} \|\beta\|_{L^2(g_j)} \leq \|\beta\|^2_{L^2(g_j)}. \qquad (4.6)$$

Since $\|\beta\|_{L^2(\overline{M},g_j)}$ is uniformly bounded in j, the inequality 4.6 allows us to conclude that the sequence of functions $\{f_j\}$ is uniformly bounded in $L^2_1(\overline{M}, g_j)$. By using a standard diagonal argument we can extract a subsequence, that for simplicity we still denote with $\{f_j\}$, converging on compact sets of M to a weak limit $f \in L^2_1(M, g)$. To show that $f \in L^2_1(M, g)$ one uses the lower semicontinuity of the Hilbert space norm with respect to the weak convergence. Indeed using 4.6 one obtain that

$$c\|f\|_{L^2} \leq \|\beta\|_{L^2} \quad \|df\|_{L^2} \leq \|\beta\|_{L^2}.$$

Finally, we want to show

$$\int_M f d\mu_g = 0, \quad \Delta^g_H f = -d^*\beta.$$

By the Cauchy inequality, given a compact set K, we have

$$0 \leq \int_{M\setminus K} |f| d\mu_g \leq Vol_g(M\setminus K)^{\frac{1}{2}} \{ \int_{M\setminus K} |f|^2 d\mu_g \}^{\frac{1}{2}} \leq Vol_g(M\setminus K)^{\frac{1}{2}} \|f\|_{L^2}$$

therefore given $\epsilon > 0$ we can always find a compact set K_ϵ such that

$$\int_{M\setminus K_\epsilon} |f| d\mu_g \leq \epsilon. \tag{4.7}$$

At the same time we can arrange K_ϵ to satisfy, for j big enough, the following sequence of inequalities

$$\int_{M\setminus K_\epsilon} |f_j| d\mu_{g_j} \leq \epsilon. \tag{4.8}$$

By the compactness of the embedding $L^2_1(K,g) \hookrightarrow L^2(K,g)$, we have that $f_j \to f$ strongly in $L^2(K,g)$. We therefore obtain that

$$\int_K f d\mu_g = \lim_{j\to\infty} \int_K f_j d\mu_j = -\lim_{j\to\infty} \int_{\overline{M}\setminus K} f_j d\mu_{g_j}$$

and

$$|\int_M f d\mu_g| \leq |\int_K f d\mu_g| + |\int_{M\setminus K} f d\mu_g| \leq |\lim_{j\to\infty} \int_{\overline{M}\setminus K} f_j d\mu_{g_j}| + |\int_{M\setminus K} f d\mu_g|$$

which combined with the estimates in 4.7 and 4.8 allows us to conclude that $\int_M f d\mu_g = 0$. A standard bootstrapping argument now shows that $f_j \to f$ in the C^∞ topology on compact sets and therefore $\Delta^g_H f = -d^*\beta$. In conclusion $\{\alpha_j\}$ converges to a L^2 harmonic 1-form for (M,g).

It is now possible to refine Proposition 4.4.6 and analyze the convergence in more details. Notice that β can be chosen as follows

$$\beta = \beta_c + \gamma$$

where β_c is a smooth closed 1-form with support not intersecting the cusp points $\{p_1, ..., p_l\}$ and $\gamma \in H^1(\Sigma_g; \mathbb{R})$. The metric g is C^2 asymptotic to a standard model, as a result

$$\lim_{t\to\infty} d^{*_g}\gamma = 0$$

since γ can be chosen to be harmonic with respect to the metric g_2. Furthermore, given $\epsilon > 0$ we can find T big enough such that $\lim_{j\to\infty} \|d^{*_j}\gamma\|_{L^2_{g_j}(t\geq T)} \leq \epsilon$. In other words we proved

Lemma 4.4.7 *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\geq T} |d^*\beta|^2 d\mu_g \leq \epsilon, \quad \int_{t\geq T} |d^{*_j}\beta|^2 d\mu_{g_j} \leq \epsilon.$$

We can now prove

Lemma 4.4.8 *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t\geq T} |\alpha|^2 d\mu_g \leq \epsilon, \quad \int_{t\geq T} |\alpha_j|^2 d\mu_{g_j} \leq \epsilon.$$

Recall that by construction $\alpha_j = \beta + df_j$, thus

$$\begin{aligned}
\int_{t\geq T} |df_j|^2 d\mu_{g_j} &= \int_{t\geq T} df_j \wedge *df_j = \int_{t\geq T} d(f_j \wedge *df_j) - \int_{t\geq T} f_j \wedge d * df_j \\
&= \int_{t=T} f_j \wedge *df_j - \int_{t\geq T} (*d * df_j, f_j) d\mu_{g_j} \\
&= \int_{t=T} f_j \wedge *df_j - \int_{t\geq T} (d^* df_j, f_j) d\mu_{g_j}.
\end{aligned}$$

But now

$$d^{*_j}\alpha_j = d^{*_j}\beta + d^{*_j} df_j = 0 \Longrightarrow d^{*_j} df_j = -d^{*_j}\beta,$$

thus

$$\int_{t\geq T} |df_j|^2 d\mu_{g_j} = \int_{t=T} f_j \wedge *df_j + \int_{t\geq T} (d^*\beta, f_j) d\mu_{g_j}.$$

By the Cauchy inequality

$$\int_{t\geq T}(d^*\beta, f_j)d\mu_{g_j} \leq \|f_j\|_{L^2_{g_j}}\|d^{*_j}\beta\|_{L^2_{g_j}(t\geq T)} \tag{4.9}$$

and then this term can be made arbitrarily small. It remains to study the term $\int_{t=T} f_j \wedge *df_j$. Recall that $f_j \to f$ in the C^∞ topology on compact sets. Thus, for a fixed T

$$\int_{t=T} f_j \wedge *df_j \to \int_{t=T} f \wedge *df.$$

It remains to show that $\int_{t=T} f \wedge *df$ can be made arbitrarily small by taking T big enough. Define the function $F(s) = \int_{t=s} f * df$, since $f \in L^2_1$ we have $F(s) \in L^1(\mathbb{R}^+)$ and then we can find a sequence $\{s_k\} \to \infty$ such that $F(s_k) \to 0$.

Proposition 4.4.9 *There exists $c > 0$ independent of j such that*

$$\int_{\overline{M}} |d\alpha|^2 + |d^{*g_j}\alpha|^2 d\mu_{g_j} \geq c\int_{\overline{M}} |\alpha|^2 d\mu_{g_j}$$

for any $\alpha \perp \mathcal{H}^1_{g_j}$.

Let us proceed by contradiction. Assume the existence of a sequence $\{\alpha_j\} \in (\mathcal{H}^1_{g_j})^\perp$ such that $\|\alpha_j\|_{L^2(g_j)} = 1$ and for which

$$\int_{\overline{M}} |d\alpha_j|^2 + |d^{*g_j}\alpha_j|^2 d\mu_{g_j} \longrightarrow 0$$

as $j \to \infty$. By eventually passing to a subsequence, a diagonal argument shows that $\{\alpha_j\}$ converges, with respect to the C^∞ topology on compact sets, to a 1-form $\alpha \in L^2\Omega^1_g(M)$. By construction $\alpha \in \mathcal{H}^1_g(M)$. On the other hand, Lemma 4.4.8 combined with the isomorphism $H^1_2(M) \simeq H^1(\overline{M})$ gives that $\alpha \in (\mathcal{H}^1_g)^\perp$. We conclude that $\alpha = 0$. Lemma 4.4.5 can now be easily applied to derive a contradiction.

4.5 Convergence of two-forms

In this section we have to study the convergence of 2-forms. The first result is completely analogous to the case of 1-forms.

Proposition 4.5.1 *Let* $[a] \in H^2_{dR}(\overline{M})$ *and* $\{\alpha_j\}$ *be the sequence of harmonic representatives with respect the sequence of metrics* $\{g_j\}$*. Then* $\{\alpha_j\}$ *converges, with respect to the* C^∞ *topology on compact sets, to a harmonic 2-forms* $\alpha \in L^2\Omega^2_g(M)$*.*

Given an element $a \in H^2_{dR}(\overline{M})$, take a smooth representative of the form $\beta = \beta_c + \gamma$ where β_c is a closed 2-form with support not intersecting the cusp points and $\gamma \in H^2(\Sigma_g; \mathbb{R})$. Given g_j, let α_j be the harmonic representative of the cohomology class determined by a. By the Hodge decomposition theorem we can write $\alpha_j = \beta + d\sigma_j$ with $\sigma_j \in (\mathcal{H}^1_{g_j})^\perp$ such that $d^{*_j}\sigma_j = 0$. Thus

$$0 = d^{*_j}\beta + d^{*_j}d\sigma_j \Longrightarrow d^*d\sigma_j = -d^{*_j}\beta.$$

Taking the global L^2 inner product of $d^*d\sigma_j$ with σ_j we obtain the estimate

$$\begin{aligned}(d^*d\sigma_j, \sigma_j)_{L^2(g_j)} = \|d\sigma_j\|^2_{L^2} &= -\int_{\overline{M}} (\sigma_j, d^*\beta) d\mu_{g_j} \qquad (4.10)\\ &\leq \|\sigma_j\|_{L^2(g_j)} \|d^*\beta\|_{L^2(g_j)}.\end{aligned}$$

By Proposition 4.4.9, we conclude that

$$\|\sigma_j\|^2_{L^2(g_j)} \leq c\|d\sigma_j\|^2_{L^2(g_j)}. \qquad (4.11)$$

Combining 4.10 and 4.11 we then obtain

$$\|\sigma_j\|^2_{L^2(g_j)} \leq c\|d\sigma_j\|^2_{L^2(g_j)} \leq c\|\sigma_j\|_{L^2(g_j)} \|d^{*_j}\beta\|_{L^2(g_j)}.$$

Since $\|d^{*_j}\beta\|_{L^2(g_j)}$ is bounded independently of j, we conclude that the same is true for $\|\sigma_j\|_{L^2(g_j)}$ and $\|d\sigma_j\|_{L^2(g_j)}$. By the elliptic regularity, we conclude that

$\|\sigma_j\|_{L^2_1(g_j)}$ is uniformly bounded. Now a standard diagonal argument allows us to conclude that, up to a subsequence, $\{\sigma_j\}$ weakly converges to an element $\sigma \in L^2_1$. Using the elliptic equation

$$\Delta_H^{g_j} \sigma_j = -d^{*_j} \beta$$

and a bootstrapping argument it is possible to show that $\sigma_j \to \sigma$ in the C^∞ topology on compact sets. This proves the proposition.

We know want to obtain a refinement of Proposition 4.5.1. We begin with the following simple lemma.

Lemma 4.5.2 *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |d^{*_g} \beta|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |d^{*_j} \beta|^2 d\mu_{g_j} \leq \epsilon.$$

Since $\beta = \beta_c + \gamma$ with γ a fixed element in $H^2(\Sigma_g; \mathbb{R})$, the lemma follows from the definition of the metrics $\{g_j\}$.

An analogous result holds for the 2-forms $\{d\sigma_j\}$.

Lemma 4.5.3 *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |d\sigma|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |d\sigma_j|^2 d\mu_{g_j} \leq \epsilon.$$

The first inequality follows easily from the fact that $\alpha \in L^2\Omega^2_g(M)$. By Lemma 4.5.2, given $\epsilon > 0$ we can find T such that

$$\|\sigma_j\|_{L^2(g_j)} \Big\{ \int_{t \geq T} |d^* \beta|^2 d\mu_{g_j} \Big\}^{\frac{1}{2}} \leq \frac{\epsilon}{2} \qquad (4.12)$$

independently of the index j. Now

$$
\begin{aligned}
\int_{t\geq T} |d\sigma_j|^2 d\mu_{g_j} &= \int_{t\geq T} d\sigma_j \wedge *_j d\sigma_j = \int_{t\geq T} d(\sigma_j \wedge *_j d\sigma_j) + \int_{t\geq T} \sigma_j \wedge d *_j d\sigma_j \\
&= \int_{t=T} \sigma_j \wedge *_j d\sigma_j - \int_{t\geq T} d *_j d\sigma_j \wedge \sigma_j \\
&= \int_{t=T} \sigma_j \wedge *_j d\sigma_j - \int_{t\geq T} (*_j d *_j d\sigma_j, \sigma_j) d\mu_j \\
&= \int_{t=T} \sigma_j \wedge *_j d\sigma_j - \int_{t\geq T} (d^{*_j} d\sigma_j, \sigma_j) d\mu_{g_j}
\end{aligned}
$$

but $d^{*_j} d\sigma_j = -d^{*_j}\beta$, thus

$$
\begin{aligned}
\int_{t\geq T} |d\sigma_j|^2 d\mu_{g_j} &\leq \Big| \int_{t=T} \sigma_j \wedge *_j d\sigma_j \Big| + \Big| \int_{t\geq T} (d^{*_j}\beta, \sigma_j) d\mu_{g_j} \Big| \\
&\leq \Big| \int_{t=T} \sigma_j \wedge *_j d\sigma_j \Big| + \|\sigma_j\|_{L^2(g_j)} \Big\{ \int_{t\geq T} |d^{*_j}\beta|^2 d\mu_{g_j} \Big\}^{\frac{1}{2}} \\
&\leq \frac{\epsilon}{2} + \Big| \int_{t=T} \sigma_j \wedge *_j d\sigma_j \Big|.
\end{aligned}
$$

Since $\sigma_j \to \sigma$ in the C^∞ topology on compact sets, we have that $\int_{t=T} \sigma_j \wedge *_j d\sigma_j \to \int_{t=T} \sigma \wedge *_g d\sigma$. But now $\sigma \in L^2_1(g)$ and therefore we can conclude the proof of the proposition.

Lemma 4.5.4 *σ is orthogonal to the harmonic 1-forms on (M, g).*

By construction we have $\sigma_j \in (\mathcal{H}^1_{g_j})^\perp$. Recall that fixed a cohomology element $[a] \in H^1_{dR}(\overline{M})$, denoted by $\{\gamma_j\}$ the sequence of the harmonic representatives with respect to the $\{g_j\}$, given $\epsilon > 0$ we can chose T such that $\int_{t\geq T} |\gamma_j|^2 d\mu_{g_j} \leq \epsilon$. Now, given $\gamma \in \mathcal{H}^1_g$ we want to show that $(\sigma, \gamma)_{L^2(g)} = 0$. Since $H^1_{dR}(\overline{M}) = \mathcal{H}^1_g(M)$, we can find a sequence of harmonic 1-forms $\{\gamma_j\}$ such that $\gamma_j \to \gamma$ in the C^∞ topology on compact sets. Let K be a compact set in M, then

$$
\Big| \int_{\overline{M}\setminus K} (\sigma_j, \gamma_j) d\mu_j \Big| \leq \|\sigma_j\|_{L^2_{g_j}} \|\gamma_j\|_{L^2_{g_j}(\overline{M}\setminus K)} \tag{4.13}
$$

can be made arbitrarily small by choosing the compact K big enough. Since $(\sigma_j, \gamma_j)_{L^2(\overline{M}, g_j)} = 0$, we have

$$\int_K (\sigma_j, \gamma_j) d\mu_{g_j} = -\int_{\overline{M}\setminus K} (\sigma_j, \gamma_j) d\mu_{g_j} \tag{4.14}$$

and then the integral $\int_K (\sigma_j, \gamma_j) d\mu_{g_j}$ can be made arbitrarily small. On the other hand

$$\begin{aligned} \Big| \int_M (\sigma, \gamma) d\mu_g \Big| &\leq \Big| \int_K (\sigma, \gamma) d\mu_g \Big| + \Big| \int_{M\setminus K} (\sigma, \gamma) d\mu_g \Big| \\ &\leq \Big| \int_K (\sigma, \gamma) d\mu_g \Big| + \|\sigma\|_{L^2(M,g)} \|\gamma\|_{L^2_g(M\setminus K)}. \end{aligned}$$

Since $\gamma \in L^2\Omega^1_g(M)$ we conclude that $\sigma \in (\mathcal{H}^1_g)^\perp$.

We now want to study the intersection form of $(\overline{M}, g_j)$ and eventually show the convergence to the L^2 intersection form of (M, g). Recall the isomorphism $H^2_{dR}(\overline{M}) \simeq \mathcal{H}^2(M)$, moreover given $[a] \in H^2_{dR}(\overline{M})$ we can generate $\{\alpha_j\} \in \mathcal{H}^2_{g_j}(\overline{M})$ that converges in the C^∞ topology on compact sets to a $\alpha \in \mathcal{H}^2_g(M)$. We also have that, fixed a compact set K, then $*_j = *_g$ for j big enough. As a result

$$\mathcal{H}^{+_{g_j}} \oplus \mathcal{H}^{-_{g_j}} \to \mathcal{H}^{+_g} \oplus \mathcal{H}^{-_g}. \tag{4.15}$$

Indeed

$$\alpha_j = \alpha_j^{+_j} + \alpha_j^{-_j} = \frac{\alpha_j + *_j \alpha_j}{2} + \frac{\alpha_j - *_j \alpha_j}{2} \to \alpha^{+_g} + \alpha^{-_g} = \alpha_g. \tag{4.16}$$

4.6 Biquard's construction

In this section we show how to construct an irreducible solution of the Seiberg-Witten equations on (M, g), *for any* metric g asymptotic to a standard model $\tilde{g}$.

Fix a $Spin^c$ structure on $\overline{M}$, with determinant line bundle L, and let g be a cuspidal metric on $\overline{M}\backslash\Sigma$ that is assumed to be C^2 asymptotic to a standard model. Let $\{g_j\}$ be the sequence of metrics on $\overline{M}$ approximating (M, g) constructed in Section 4.2. Let (A_j, ψ_j) be a solution of the perturbed Seiberg-Witten equations on $(\overline{M}, g_j)$

$$\begin{cases} \mathcal{D}_{A_j}\psi_j = 0 \\ F^+_{A_j} + i2\pi\omega^+_j = q(\psi_j) \end{cases}$$

where $\omega_j = \frac{i}{2\pi}F_{B_j}$ and B_j is the connection 1-form on the line bundle $\mathcal{O}_{\overline{M}}(\Sigma)$ given by

$$B_j = d - i\chi_j(\partial_t\varphi_j)d\theta.$$

The idea is to show that, up to gauge transformations, the (A_j, ψ_j) converge in the C^∞ topology on compact sets to a solution of the unperturbed Seiberg-Witten equations

$$\begin{cases} \mathcal{D}_A\psi = 0 \\ F^+_A = q(\psi) \end{cases}$$

on (M, g), where $A = C + a$ with C is a fixed smooth connection on $L\otimes\mathcal{O}(-\Sigma)$, and $a \in L^2_1(\Omega^1_g(M))$ with $d^*a = 0$.

Lemma 4.6.1 *We have the decomposition*

$$s_{g_j} = s^b_{g_j} - 2\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}$$
$$F_{B_j} = -i\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}dt\wedge\varphi_j d\theta + F^b_j$$

with $s^b_{g_j}$ and F^b_j bounded independently of j

See Proposition 4.2.2.

Since $i2\pi\omega_j = -F_{B_j}$, we can rewrite the perturbed Seiberg-Witten equations as follows

$$\begin{cases} \mathcal{D}_{A_j}\psi_j = 0 \\ \\ F^+_{A_j} - F^+_{B_j} = q(\psi_j). \end{cases}$$

Recall that in the case under consideration, the twisted Licherowicz formula [6] reads as follows

$$\mathcal{D}^2_{A_j}\psi_j = \nabla^*_{A_j}\nabla_{A_j}\psi_j + \frac{s_{g_j}}{4}\psi_j + \frac{1}{2}F^+_{A_j}\cdot\psi_j.$$

By using the SW equations we have

$$0 = \nabla^*_{A_j}\nabla_{A_j}\psi_j + \frac{s_{g_j}}{4}\psi_j + \frac{|\psi_j|^2}{4}\psi_j + \frac{1}{2}F^+_{B_j}\cdot\psi_j.$$

Keeping into account the decomposition given in Lemma 4.6.1 we obtain

$$0 = \nabla^*_{A_j}\nabla_{A_j}\psi_j + P_j\psi_j + P^b_j\psi_j + \frac{|\psi_j|^2}{4}\psi_j$$

where

$$P_j\psi_j = -\frac{1}{2}\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}\psi_j - \frac{i}{2}\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}(dt\wedge\varphi_j d\theta)^+\cdot\psi_j$$

with P^b_j uniformly bounded in j. Now, it can be explicitly checked that for a metric of the form $dt^2 + \varphi_j^2 d\theta^2 + g_2$ the self-dual form $(dt\wedge\varphi_j d\theta)^+$ acts by Clifford multiplication with eigenvalues $\pm i$. The eigenvalues of the operator P_j are then given by 0 and $-\chi_j\frac{\partial^2_t\varphi_j}{\varphi_j}$.

Lemma 4.6.2 *There exists a constant $K > 0$ such that*

$$|\psi_j(x)|^2 \leq K$$

for every j and $x \in \overline{M}$.

Given a point $x \in \overline{M}$ choose an orthonormal frame $\{e_i\}$ centered at x such that $\nabla_{e_j} e_{i_{|_x}} = 0$. We then compute

$$-\sum_i e_i(e_i\langle\psi_j,\psi_j\rangle)_x = -\sum_i e_i\{\langle\nabla_{e_i}\psi_j,\psi_j\rangle + \langle\psi_j,\nabla_{e_i}\psi_j\rangle\}$$
$$= -\sum_i\{\langle\nabla_{e_i}\nabla_{e_i}\psi_j,\psi_j\rangle + 2\langle\nabla_{e_i}\psi_j,\nabla_{e_i}\psi_j\rangle + \langle\psi_j,\nabla_{e_i}\nabla_{e_i}\psi_j\rangle\}.$$

Since $\nabla^2_{e_i,e_i}\psi_j = \nabla_{e_i}\nabla_{e_i}\psi_j$ and $\nabla^*_{A_j}\nabla_{A_j} = -\sum_i \nabla^2_{e_i,e_i}$ we have

$$\Delta|\psi_j|^2 + 2|\nabla_{A_j}\psi_j|^2 = \langle\nabla^*_{A_j}\nabla_{A_j}\psi_j,\psi_j\rangle + \langle\psi_j,\nabla^*_{A_j}\nabla_{A_j}\psi_j\rangle$$
$$= 2Re\langle\nabla^*_{A_j}\nabla_{A_j}\psi_j,\psi_j\rangle.$$

Thus, if x_j is a maximum point for $|\psi_j|^2$ we have $\Delta|\psi_j|^2_{x_j} \geq 0$ and therefore $Re\langle\nabla^*_{A_j}\nabla_{A_j}\psi_j,\psi_j\rangle \geq 0$. In conclusion

$$0 = Re\langle\nabla^*_{A_j}\nabla_{A_j}\psi_j,\psi_j\rangle_{x_j} + Re\langle\{P_j + P^b_j\}\psi_j,\psi_j\rangle_{x_j} + \frac{|\psi_j|^4_{x_j}}{4}$$
$$\geq Re\langle\{P_j + P^b_j\}\psi_j,\psi_j\rangle_{x_j} + \frac{|\psi_j|^4_{x_j}}{4}.$$

By construction the operator $P_j + P^b_j$ is uniformly bounded from below, the proof is then complete.

Since $F^+_{A_j} - F^+_{B_j} = q(\psi_j)$ and by Lemma 4.6.2 the norms of the ψ_j are uniformly bounded, a similar estimate holds for $F^+_{A_j} - F^+_{B_j}$.

Lemma 4.6.3 *There exists a constant $K > 0$ such that*

$$\|\nabla_{A_j}\psi_j\|_{L^2(\overline{M},g_j)} \leq K$$

for any j.

We have

$$0 = \int_{\overline{M}} Re\langle \nabla^*_{A_j}\nabla_{A_j}\psi_j, \psi_j\rangle d\mu_{g_j} + \int_{\overline{M}} Re\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j}$$
$$+ \frac{1}{2}\int_{\overline{M}} Re\langle q(\psi_j)\psi_j, \psi_j\rangle d\mu_{g_j}$$
$$= \|\nabla_{A_j}\psi_j\|^2_{L^2(\overline{M},g_j)} + \int_{\overline{M}} Re\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j} + \frac{1}{4}\int_{\overline{M}} |\psi_j|^4 d\mu_{g_j}$$

but now

$$\int_{\overline{M}} Re\langle \{P^b_j + P_j\}\psi_j, \psi_j\rangle d\mu_{g_j} \geq -k\|\psi_j\|^2_{L^2(\overline{M},g_j)}$$

which then implies

$$\|\nabla_{A_j}\psi_j\|^2_{L^2(\overline{M},g_j)} \leq k\|\psi_j\|^2_{L^2(\overline{M},g_j)} - \frac{1}{4}\|\psi_j\|^4_{L^2(\overline{M},g_j)}$$
$$\leq k\|\psi_j\|^2_{L^2(\overline{M},g_j)}.$$

Since by Proposition 4.2.3 the volumes of the Riemannian manifolds $(\overline{M}, g_j)$ are uniformly bounded, the lemma follows from Lemma 4.6.2.

Define $C_j = A_j - B_j$ and let C be a fixed smooth connection on the line bundle $L \otimes \mathcal{O}(-\Sigma)$. By the Hodge decomposition theorem we can write

$$C_j = C + \eta_j + \beta_j$$

where η_j is g_j-harmonic and $\beta_j \in (\mathcal{H}^1_{g_j})^\perp$. Thus

$$F^+_{C_j} = q(\psi_j) = F^+_C + d^+\beta_j.$$

Since C is a fixed connection 1-form, $\|F_C\|_{L^2(\overline{M},g_j)}$ is uniformly bounded in the index j. As a result, there exists $K > 0$ such that

$$\|d^+\beta_j\|_{L^2(\overline{M},g_j)} \leq K$$

for any j. By the Stokes' theorem

$$\begin{aligned}\|d^+\beta_j\|^2_{L^2(\overline{M},g_j)} &= \int_{\overline{M}} d\beta_j \wedge d\beta_j d\mu_{g_j} + \|d^-\beta_j\|^2_{L^2(\overline{M},g_j)} \\ &= \int_{\overline{M}} d(\beta_j \wedge d\beta_j) d\mu_{g_j} + \|d^-\beta_j\|^2_{L^2(\overline{M},g_j)} \\ &= \|d^-\beta_j\|^2_{L^2(\overline{M},g_j)}\end{aligned}$$

and we then obtain an uniform upper bound on $\|d\beta_j\|_{L^2(\overline{M},g_j)}$. By Gauge fixing we can always assume $d^*\beta_j = 0$. The Poincaré inequality given in Proposition 4.4.9 can then be used to conclude that

$$c\|\beta_j\|^2_{L^2(\overline{M},g_j)} \leq \|d\beta_j\|^2_{L^2(\overline{M},g_j)} \leq 2K.$$

By a diagonal argument we can now extract a weak limit

$$\beta_j \rightharpoonup \beta$$

with $\beta \in L^2_1(M,g)$. Similarly we extract a weak limit

$$\eta_j \rightharpoonup \eta$$

with $\eta \in L^2(M,g)$ and harmonic with respect to g, see Proposition 4.4.6.

Define $a_j = \eta_j + \beta_j$ that by construction satisfies $d^*a_j = 0$. If we fix a compact set $K \subset M$, there exists j_0 such that for any $j \geq j_0$ the connection 1-form B_j restricted to K is zero. Thus, for any $j \geq j_0$ we have $A_j = C_j$ and then $C = A_j - a_j$. We know that a_j is uniformly bounded in $L^2(\overline{M}, g_j)$, by using Lemma 4.6.3 we conclude that $\|\nabla_C\psi_j\|^2_{L^2(K,g_j)}$ is bounded independently of j. On this compact set K we can therefore extract a weak limit of the sequence $\{\psi_j\} \rightharpoonup \psi$. By using a diagonal argument and recalling that in a Hilbert space the norm is lower semicontinuous with respect the weak convergence, we obtain a weak limit $\psi \in L^2_1(M,g)$.

Recall that on any compact set K, for j big enough we have $F^+_{A_j} = q(\psi_j)$. Since

$$\nabla F^+_{A_j} = \nabla_{A_j}\psi_j \otimes \psi_j^* + \psi_j \otimes \nabla_{A_j}\psi_j^* - Re\langle \nabla_{A_j}\psi_j, \psi_j\rangle Id$$

we conclude that $\|\nabla F^+_{A_j}\|_{L^2(K,g_j)}$ is uniformly bounded. In summary we have an L^2_1 bound on $F^+_{A_j}$. Consider now the first order elliptic operator $d^+ \oplus d^*$. By the Garding's inequality we obtain

$$\begin{aligned} c\|a_j\|_{L^2_2(K,g_j)} &\leq \|a_j\|_{L^2(K,g_j)} + \|(d^+ \oplus d^*)a_j\|_{L^2_1(K,g_j)} \\ &\leq \|a_j\|_{L^2(K,g_j)} + \|d^+\beta_j\|_{L^2_1(K,g_j)} \end{aligned}$$

which gives us an uniform $L^2_2(K,g_j)$ bound on a_j. Since $C = A_j - a_j$ on K, we can write

$$0 = \mathcal{D}_{A_j}\psi_j = \mathcal{D}_{C+a_j}\psi_j = \mathcal{D}_C\psi_j + \frac{1}{2}a_j \cdot \psi_j,$$

in other words

$$\mathcal{D}_C\psi_j = -\frac{1}{2}a_j \cdot \psi_j. \tag{4.17}$$

Combining the L^2_2 bound on a_j and the L^∞ bound on ψ_j with the Sobolev multiplication $L^2_2(K,g_j) \otimes L^p(K,g_j) \to L^4(K,g_j)$, for p big enough, we obtain a $L^4(K,g_j)$ bound on $-\frac{1}{2}a_j \cdot \psi_j$, that is exactly the forcing term in the first order elliptic equation given in 4.17. By the elliptic L^p estimates we then obtain

$$c\|\psi_j\|_{L^4_1} \leq \|\psi_j\|_{L^4} + \|f\|_4$$

where we define $f = -\frac{1}{2}a_j \cdot \psi_j$. This shows $\psi_j \in L^4_1$ that combined with the Sobolev multiplication $L^2_2 \otimes L^4_1 \to L^3_1$ can be used to obtain a L^3_1 estimate on f. By applying again the elliptic L^p estimate we obtain

$$c\|\psi_j\|_{L^3_2} \leq \|\psi_j\|_{L^2} + \|f\|_{L^3_1}.$$

Now the Sobolev multiplication $L^2_2 \otimes L^3_2 \to L^2_2$ combined with the fact that $\psi_j \in L^3_2$, we obtain a L^2_2 bound on f. Once more the L^p elliptic estimates gives us

$$c\|\psi_j\|_{L^2_3} \leq \|\psi_j\|_{L^2} + \|f\|_{L^2_2}.$$

By using the Sobolev multiplication $L^2_3 \otimes L^2_3 \to L^2_3$ we then obtain a L^2_3 bound on $q(\psi_j)$ and therefore by the Seiberg-Witten equations on $F^+_{A_j}$. But now a L^2_3 estimate on $F^+_{A_j}$ gives us a analogous estimate on d^+a_j. Consider the first order elliptic operator $d^+ \oplus d^*$, remembering that $d^*a_j = 0$ and applying the L^p elliptic estimate we obtain

$$c\|a_j\|_{L^2_4} \leq \|a_j\|_{L^2} + \|d^+a_j\|_{L^2_3}.$$

The uniform L^2_3 bound on d^+a_j gives us a L^2_4 bound. By using the Sobolev multiplication $L^2_3 \otimes L^2_3 \to L^2_3$ we obtain a L^2_3 bound on f. By applying the elliptic L^p estimates to the equation 4.17 we obtain

$$c\|\psi_j\|_{L^2_4} \leq \|\psi_j\|_{L^2} + \|f\|_{L^2_3}.$$

By induction we can assume to have a L^2_k bound on ψ_j and a_j, then combining the Sobolev multiplication $L^2_k \otimes L^2_k \to L^2_k$ and the first order elliptic equation given in 4.17 we obtain

$$c\|\psi_j\|_{L^2_{k+1}} \leq \|\psi_j\|_{L^2} + \|f\|_{L^2_k}$$

and we therefore obtain a uniform estimate on $\|\psi_j\|_{L^2_{k+1}}$. Now by the Sobolev embedding $L^2_k \hookrightarrow C^{k-3}$ we can then conclude that the ψ_j are indeed smooth. A completely analogous argument can now be used to show the C^∞ on compact sets of the $\{\psi_j\}$.

Let us summarize the discussion above into a theorem.

Theorem D *Fix a $Spin^c$ structure on $\overline{M}$ with determinant line bundle L. Let g be a metric on M asymptotic to a standard model in the C^2 topology, and let $\{g_j\}$ the sequence of metrics on $\overline{M}$ that approximate g. Let $\{(A_j, g_j)\}$ be the sequence of solutions of the SW equations with perturbations $\{F^+_{B_j}\}$ on $\{(\overline{M}, g_j)\}$. Then, up to gauge transformations, the solutions $\{(A_j, \psi_j)\}$ converge, in the C^∞ topology on compact sets, to a solution (A, ψ) of the unperturbed SW equations on (M, g) such that*

- *A=C+a where C is a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$, $d^*a = 0$ and $a \in L^2_1(\Omega^1_g(M))$;*
- *$\psi \in L^2_1(M, g)$ and there exists $K > 0$ such that $sup_{x \in M} |\psi(x)| \leq K$.*

4.7 Geometric applications

For a compact oriented 4-manifold N, the Gauss-Bonnet and Hirzebruch theorems state that

$$\chi(N) = \int_N E(g) d\mu_g, \quad \sigma(N) = \int_N L(g) d\mu_g$$

where $E(g)$ and $L(g)$ are respectively the Euler and signature forms associated to the metric g.

For noncompact manifolds the above curvature integrals might be not defined or dependent on the choice of the metric. Nevertheless, if the manifold has finite volume and bounded curvature these curvature integrals are defined. In this case it remains to study their metric dependence. Here, we want to compute

$$\chi(M, g) = \int_M E(g) d\mu_g, \quad \sigma(M, g) = \int_M L(g) d\mu_g$$

when g is a metric C^2 asymptotic to a standard model for M. The idea is to approximate the metric g with the sequence of metrics $\{g_j\}$ on $\overline{M}$. We then have

$$\chi(M,g) = \lim_{j\to\infty} \int_{t\leq j+1} E(g_j) d\mu_{g_j}, \quad \sigma(M,g) = \int_{t\leq j+1} L(g_j) d\mu_{g_j}.$$

Thus

$$\chi(M,g) = \chi(\overline{M}) - \lim_{j\to\infty} \int_{t\geq j+1} E(\tilde{g}_j) d\mu_{\tilde{g}_j}$$

and

$$\sigma(M,g) = \sigma(\overline{M}) - \lim_{j\to\infty} \int_{t\geq j+1} L(\tilde{g}_j) d\mu_{\tilde{g}_j}.$$

In other words, the characteristic numbers of (M,g) are computed in terms of $\chi(\overline{M})$ and $\sigma(\overline{M})$ plus a contribution coming from the cusps. More precisely we have the following proposition.

Proposition 4.7.1 *Let M be equipped with a metric g asymptotic in the C^2 topology to a standard model. Then, we have the equalities*

$$\chi(M,g) = \chi(\overline{M}) - l\chi(\Sigma_g), \quad \sigma(M,g) = \sigma(\overline{M}) = 0,$$

where l is the number of cusp ends of M.

See Proposition 3.4. in [24].

A simple Mayer-Vietoris argument can now be used to show that $\chi(M) = \chi(\overline{M}) - l\chi(\Sigma_g)$. We then conclude that $\chi(M,g) = \chi(M)$. This discussion can then be summarized into the following proposition.

Proposition 4.7.2 *The Gauss-Bonnet theorem is valid on (M,g) for any metric g asymptotic in the C^2 topology to a standard model.*

We can now study the Riemannian functional $\int_M s_g^2 d\mu_g$ restricted to the space of metrics asymptotic to a standard model.

Theorem E *Let M be equipped with a metric g asymptotic to a standard model in the C^2 topology. Then*

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq 2\chi(\Sigma)\cdot\chi(\Sigma_g)$$

with equality if and only if g is, up to scaling, the product of two -1 hyperbolic metrics on Σ and Σ_g.

Let us consider the standard $Spin^c$ structure associated to the complex structure of $\overline{M}$. Theorem D can be used to construct an irreducible solution of the SW equations on (M, g). Furthermore, by applying Theorem 3.3.2 we conclude that

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq (c_1(K_{\overline{M}}^{-1} - \Sigma)^+)^2.$$

By the adjunction formula we have

$$(c_1(K_{\overline{M}}^{-1} - \Sigma)^+)^2 \geq (c_1(K_{\overline{M}}^{-1} - \Sigma))^2 = 2(\chi(\overline{M}) + 2(g-1)l)$$

where l is the number of cusp ends. By Propositions 4.7.1 and 4.7.2

$$\chi(M) = \chi(\overline{M}) + 2l(g-1),$$

and we conclude that

$$\frac{1}{32\pi^2}\int_M s_g^2 d\mu_g \geq 2\chi(\Sigma)\cdot\chi(\Sigma_g)$$

with equality if and only if g is Kähler with constant negative scalar curvature and the harmonic representative of $c_1(\mathcal{L})$ is self dual. The latter condition implies that g is Kähler Einstein. We can now apply Theorem D for a $Spin^c$

structure of complex type compatible the reversed oriented $\overline{M}$. This implies that g must be Kähler Einstein with respect to the commuting complex structures J and $\overline{J}$ on M. This implies that g is, up to scaling, the product of two hyperbolic -1 metrics on Σ and Σ_g.

Finally, we present an obstruction for Einstein metrics on blow-ups.

Theorem F *Let (M, g) as above. Let $M^{'}$ be obtained from M by blowing up k points. If $k \geq \frac{4}{3}\chi(\Sigma)\chi(\Sigma_g)$, then $M^{'}$ does not admit a cuspidal Einstein metric.*

By a result of Morgan-Friedman [17], we know that the manifold $\overline{M}\sharp k\overline{\mathbb{C}P^2}$ admits at least 2^k different $Spin^c$ structures with determinant line bundles

$$L = K_{\overline{M}}^{-1} \pm E_1 \pm ... \pm E_k$$

for which the SW equations have irreducible solutions for each metric. Since

$$\begin{aligned}(c_1(L)^+)^2 &= (c_1(\overline{M})^+ \pm E_1^+ \pm ... \pm E_k^+)^2 \\ &= (c_1(\overline{M})^+)^2 + 2\sum_i c_i(\overline{M})^+ \cdot \pm E_i^+ + (\sum_i \pm E_i^+)^2\end{aligned}$$

we can chose a $Spin^c$ structure whose determinant line bundle satisfies

$$(c_1(L)^+)^2 \geq (c_1(\overline{M})^+)^2 \geq c_1(\overline{M})^2 = c_1^2(\overline{M}).$$

We can now apply Theorem D for any of the $Spin^c$ structure above and with respect to the metric g on $M^{'}$. We then construct 2^k irreducible solutions $(A, \psi) \in L_1^2(M^{'}, g)$, where $A = C + a$ with C a fixed smooth connection on $L\otimes\mathcal{O}(-\Sigma)$ and $a \in L_1^2(\Omega_g^1(M^{'}))$. By appropriately choosing the $Spin^c$ structure and using Theorem 3.3.2 we compute

$$\frac{1}{32\pi^2}\int_{M'} s^2 d\mu_g \geq (c_1(L\otimes\mathcal{O}(-\Sigma))^+)^2$$

$$\geq (c_1(L)^+)^2 + \Sigma^2 + 2K_{\overline{M}}\cdot\Sigma$$

$$\geq c_1^2(\overline{M}) + 2K_{\overline{M}}\cdot\Sigma$$

where in the last inequality we used the fact that Σ has trivial self intersection. By the adjunction formula we have

$$\int_{M'} s^2 d\mu_g \geq c_1^2(\overline{M}) + 4l(g-1)$$

$$= 2\chi(\overline{M}) + 4l(g-1),$$

where k is the number of distinct components of the divisor Σ. By an obvious modification of Proposition 4.7.1 one has

$$\chi(M^{'}, g) = \chi(\overline{M}) + k + 2l(g-1)$$

$$\sigma(M^{'}, g) = -k$$

Thus, if we assume g to be Einstein

$$c_1^2(\overline{M}) + 4l(g-1) - k = 2\chi(M^{'}) + 3\sigma(M^{'})$$

$$= \frac{1}{4\pi^2}\int_{M'} 2|W_+|^2 + \frac{s^2}{24}d\mu_g$$

$$\geq \frac{1}{96\pi^2}\int_{M'} s^2 d\mu_g$$

$$\geq \frac{1}{3}(c_1^2(\overline{M}) + 4l(g-1))$$

so that

$$\frac{2}{3}(c_1^2(\overline{M}) + 4l(g-1)) \geq k.$$

In other words if

$$k > \frac{4}{3}(\chi(\Sigma)\cdot\chi(\Sigma_g))$$

we cannot have a cuspidal Einstein metric on $M\sharp k\overline{\mathbb{C}P^2}$. The equality case can also be included and the proof goes as in the compact case. For more details, see [29]. The proof is then complete.

Bibliography

[1] A. Ash, D. Mumford, M. Rapoport, Y-S. Tai, **Smooth compactifications of locally symmetric varieties**. Cambridge Mathematical Library, Cambridge University press, (2010).

[2] M. Anderson, L^2-harmonic forms on complete Riemannian manifolds. Geometry and Analysis on manifolds, **Lectures Notes in Math. 1339**, Springer-Verlag, (1988).

[3] A. Avez, *Variétés riemanniennes sans point focaux.* **C. R. Acad. Sci. Paris Sér. A 270** 188-191, (1970).

[4] A. Besse, **Einstein Manifolds**. Springer-Verlag, (1987).

[5] A. Borel, L. Ji, **Compactifications of symmetric and locally symmetric spaces**. Mathematics: Theory and Applications, Birkhäuser Boston, (2006).

[6] B. Lawson, M. L. Michelson, **Spin Geometry**. Princeton University Press, (1989).

[7] G. Besson, G. Courtois, S. Gallot *Entropies et rigidités des espaces localment symétriques de curbure strictment négative.* **Geom. Func. Anal. 5**, 731-799, (1995).

[8] M. Gromov, *Volume and bounded cohomology.* **Publ. Math. IHES, 56**, 5-99, (1982).

[9] A. Sambusetti, *An obstruction to the existence of Einstein metrics on 4-manifolds.* **Math. Ann. 311**, 533-547, (1998).

[10] P. Griffiths and J. Harris, **Principles of Algebraic Geometry**. Wiley, New York, (1978).

[11] P. B. Eberlein, **Geometry of Nonpositively Curved Manifolds**. Chicago Lectures in Mathematics, University of Chicago Press, (1996).

[12] P. Petersen, **Riemannian Geometry**. Springer-Verlag, (2006).

[13] P. B. Eberlein, *Lattices in spaces of nonpositive curvature.* **Ann. Math. 111**, 435-476, (1980).

[14] R. Hartshorne, **Algebraic Geometry**. Springer-Verlag, (1977).

[15] J. Morgan, **The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds**. Vol. 44 of Mathematical Notes, Princeton University Press, (1996).

[16] R. Friedman and J. Morgan, **Smooth 4-Manifolds and Complex Surfaces**. Springer-Verlag, (1994).

[17] R. Friedman and J. Morgan, *Algebraic Surfaces and Seiberg-Witten invariants.* **J. Alg. Geom. 6**, 445-479, (1997).

[18] W. Barth, C. Peters and A. Van de Ven, **Compact complex surfaces**. Springer-Verlag, (2003).

[19] S. Kobayashi, K. Nomizu, **Foundations of Differential Geometry**. Volume II, Wiley, New York (1996).

[20] S. S. Chern, *On Curvature and Characteristic Classes of a Riemannian Manifold.* **Abh. Math. Semin. Univ. Hamb. 20**, 117-126, (1955).

[21] C. Hummel, V. Schroeder, *Cusp closing in rank one symmetric spaces.* **Invent. Math. 123**, 283-307 (1996).

[22] C. Hummel, *Rank One Lattices whose Parabolic Isometries have no rotation part.* **Proc. Am. Math. Soc. 126**, 2453-2458 (1998).

[23] A. Kaplan, *Riemannian Nilmanifolds attached to Clifford Modules.* **Geom. Ded. 11**, 127-136, (1981).

[24] O. Biquard, *Metriques d'Einstein a cusps et equations de Seiberg-Witten.* **J. Reine Angew. Math. 490**, 129-154 (1997).

[25] C. LeBrun, *Einstein metrics and Mostow rigidity.* **Math. Res. Lett. 2**, 1-8, (1995).

[26] C. LeBrun, *Polarized 4-Manifolds, Extremal Kahler Metrics, and Seiberg-Witten Theory.* **Math. Res. Lett. 2**, 653-662, (1995).

[27] C. LeBrun, *Four-manifolds without Einstein metrics.* **Math. Res. Lett. 3**, 133-147, (1996).

[28] C. LeBrun, *Kodaira dimension and the Yamabe problem.* **Comm. Anal. Geom. 7**, 133-156, (1999).

[29] C. LeBrun, *Four-Dimensional Einstein Manifolds and Beyond.* **Surveys in Differential Geometry, vol VI: Essays on Einstein Manifolds**, LeBrun and Wang, editors, International Press of Boston, 247–285, (1999).

[30] **Surveys in Differential Geometry, vol VI: Essays on Einstein Manifolds**, LeBrun and Wang editors, International Press of Boston, 247–285, (1999).

[31] C. LeBrun, *Ricci Curvature, Minimal Volumes, and Seiberg-Witten Theory.* **Invent. math. 145**, 279-316, (2001).

[32] J. Milnor, *A note on curvature and the fundamental group.* **J. Diff. Geom. 2**, 1-7, (1968).

[33] A. Sambusetti, *An obstruction to the existence of Einstein metrics on 4-manifolds.* **Math. Ann. 311**, 533-547, (1998).

[34] E. Witten, *Monopoles and four-manifolds.* **Math. Res. Lett. 1**, 769-796, (1994).

[35] A. Huber, *On subharmonic functions and differential geometry in the large.* **Comment. Math. Helv. 32**, 12-72, (1957).

[36] D. Mostow, Y. T. Siu, *A compact Kähler Surface of Negative Curvature not covered by the Ball.* **Ann. of Math. 2**, Vol. 112, 321-360, (1980).

[37] Y. T. Siu, S. T. Yau, Compactification of negatively curved complete Kähler manifolds of finite volume, **Seminars in Differential Geometry**, Annals of Mathematics Studies, Vol. 102, 363-380, (1982).

[38] N. Hitchin, *On compact four-dimensional Einstein manifolds.* **J. Diff. Geom. 9**, 435-442, (1974).

[39] A. I. Mal'cev, *On the faithful representation of infinite groups by matrices.* **Amer. Math. Soc. Transl. (2) 45**, 1-18, (1965).

[40] Y. Rollin, *Surfaces Kählériennes de volume fini et équationes de Seiberg-Witten.* **Bull. Soc. Math. France (3) 130**, 409-456, (2002).

[41] F. Sakai, *Semi-Stable Curves on Algebraic Surfaces and Logarithmic Pluricanonical Maps.* **Math. Ann. 254**, 89-120, (1980).

[42] G. Margulis, *Arithmeticity of the irreducible lattices in semisimple groups of rank gretaer than 1.* **Invent. Math. (1) 76**, 93-120, (1984).

[43] J. Morgan, G. Tian, **Ricci flow and the Poincaré conjecture**. American Mathematical Society, Providence, RI, (2006).

[44] G. Carron, *L^2 harmonic forms on noncompact manifolds.* Lecture notes for the CIMPA summer school "Recent Topics in Geometric Analysis".

[45] N. Mok, *Projective-algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume.* Preprint.

[46] S. Iitaka, **Algebraic Geometry**. Springer-Verlag, (1981).

[47] R. P. Holzapfel, *A class of minimal surfaces in the unknown region of surface geography.* **Math. Nachr. 98**, 211-232 (1980).

[48] F. Hirzebruch, *Chern numbers of algebraic surfaces: an example.* **Math. Ann 266**, 351-356 (1984).

[49] P. Deligne, G. Mostow, **Commensurabilities among lattices in PU(1,n)**. Annals of Mathematics Studies 132, Princeton University press, (1993).

[50] S. Zucker, *L_2 Cohomology of Warped Products and Arithmetic Groups.* **Invent. Math. 70**, 169-218, (1982).