Impact of Prices on Inventory Systems: Theory and Emerging Issues

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Business Administration
in the Graduate School of Duke University
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Abstract

Firms' inventory or production decisions are influenced by a variety of factors, including both the selling price of the end products and the purchasing cost of raw materials. In most cases, there is a strong connection between purchasing costs and selling prices. In my dissertation, I study the impact of prices on a firm's inventory decisions, particularly in systems with delivery lead time and environmental concerns. The findings are reported in three studies. The first study analyzes the joint inventory and pricing problem with lead time, which is known to be difficult to solve due to its computational complexity. We develop a simple heuristic to resolve the computational issue and reveal the impact of lead time on the joint decisions. In the second study, we extend the heuristic approach in the previous study to systems with both positive lead time and fixed ordering costs. The effectiveness of the heuristic in both studies are verified through both theoretical bounds and numerical experiments. In the third study, we examine the effect of the procurement cost and its volatility on a firm's profit. This allows us to study under what conditions a firm can profitably operate an eco-friendly supply chain. Our study also helps the firms to understand what type of products would better absorb the higher costs associated with an eco-friendly production system.

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Symbols

Notations for Chapter 2:

T	Planning horizon.
α	Discount factor.
L	Lead time.
x_t	Net inventory level in period t.
$\mathbf{w_t}$	$(w_{1,t}, \cdots, w_{L-1,t})$.
$w_{l,t}$	Pipeline inventory to be delivered in l periods in period t.
p_t	Selling price in period t.
λ_t, μ_t	Demand parameters in period t.
d_t	Expected demand in period t.
ϵ_t	Demand perturbation in period t.
$F_t(\cdot)$	Distribution function of ϵ_t.
$f_t(\cdot)$	Density function of ϵ_t.
Ω_t	The action space of d_t.
c_t	Unit purchasing cost in period t.
h_t	Unit holding cost in period t.
b_t	Unit backorder cost in period t.
$R_t(d_t)$	Revenue function in period t.

$G(x_t, d_t)$	Inventory cost function in period t.
$V_t(x_t, \mathbf{w_t})$	Maximum expected discounted profit from period t to the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$.
$J_t(x_t, \mathbf{w_t}, q_t, d_t)$	Total expected discounted profit from period t to the end of the planning horizon given state vector $(x_t, \mathbf{w_t})$ and decisions (q_t, d_t).
q_t^*	Optimal replenishment decision in period t.
d_t^*	Optimal expected demand decision in period t.
$d_t^M(x_t)$	Myopic expected demand in period t given x_t.
$p_t^M(x_t)$	Myopic price in period t given x_t.
C_t^*	Asymptotic slope of $d_t^M(x_t)$ under multiplication demand.
$\widetilde{d}_t(x_t)$	Linear approximation of $d_t^M(x_t)$.
δ_t	Slope of $\widetilde{d}_t(x_t)$.
κ_t	Intercept of $\widetilde{d}_t(x_t)$.
$\widetilde{V}_t(x_t, \mathbf{w_t})$	Maximum expected discounted profit from period t to the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$ and demand decisions taken as $\widetilde{d}_i(x_i)$ for all $i = t, \cdots, T$.
$\widetilde{J}_t(x_t, \mathbf{w_t}, q_t)$	Total expected discounted profit from period t to the end of the planning horizon given initial state vector $(x_t, \mathbf{w_t})$, replenishment decision q_t and demand decisions taken as $\widetilde{d}_i(x_i)$ for all $i = t, \cdots, T$.
$\overline{x}_t$	Price-deflated inventory position.
$\epsilon[t, t+L)$	Total weighted errors in periods $t, \cdots, t+L-1$.
$\overline{s}_t$	Base-stock level associated with $\overline{x}_t$.
$W_t(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T) - W_t(x_t, \mathbf{w_t}, q_t, d_t)$	Penalty function at period t with sample path $\{\epsilon\}_t^T$.
$V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$	Maximum expected discounted profit from t to the end of the planning horizon for the upper bound system with sample path $\{\epsilon\}_t^T$.
d_t^D	Optimal demand decision for the upper bound system in period t.

q_t^D	Optimal replenishment decision for the upper bound system in period t.

Notations for Chapter 3:

$I(u)$	Indicator function of $u > 0$.
K	Fixed ordering cost.
$V_t^K(x_t, \mathbf{w_t})$	Maximum expected discounted profit from period t to the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$ and fixed ordering cost K.
$\widetilde{V}_t^K(x_t, \mathbf{w_t})$	Maximum expected discounted profit from period t to the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$, fixed ordering cost K and demand decisions taken as $\widetilde{d}_i(x_i)$ for all $i = t, \cdots, T$.
$\widetilde{J}_t^K(x_t, \mathbf{w_t}, q_t)$	Total expected discounted profit from period t to the end of the planning horizon given initial state vector $(x_t, \mathbf{w_t})$, fixed ordering cost K replenishment decision q_t and demand decisions taken as $\widetilde{d}_i(x_i)$ for all $i = t, \cdots, T$.
$\overline{s}_t$	Reorder point associated with $\overline{x}_t$.
$\overline{S}_t$	Order-up-to level associated with $\overline{x}_t$.

Notations for Chapter 4:

$i = O, E$

T	Planning horizon.
α	Discount factor.
L	Lead time.
C_t^i	Unit procurement cost of input i in period t.
δ^i	Cost trend of input i.
e_t^i	Cost uncertainty of input i in period t.
σ_t^i	Standard deviation of e_t^i.

ρ_t	Proportion of eco-friendly input E in the final product in period t.
$P(t)$	Price in period t.
β	Mark-up level in the mark-up pricing policy.
λ, μ	Demand parameters.
ϵ_t	Demand perturbation in period t.
x_t	Inventory position in period t.
$V_t(x_t, c_t)$	Maximum expected discounted profit from period t to the end of the planning horizon with initial inventory position x_t and unit purchasing cost realized at c_t.

Abbreviations

Abbreviations for Chapter 2:

BLS	Bernstein, Li and Shang.
FH	Federgruen and Heching.
W&W	Wagner and Whitin.

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1

Introduction

Firms' inventory or production decisions are always determined to minimize the mismatch between demand and supply. In practice, there are a variety of factors that influence these decisions and their outcomes. Among these factors, the selling price of the final products and the purchasing cost of raw materials are two non-neglect ones, as the selling price has a strong impact on the demand and the purchasing cost affects the supply quantity. In many cases, these two factors are also strongly connected and influenced by each other. In my dissertation, I study the impact of the selling price and the purchasing cost on a firm's inventory decisions, particularly in systems with delivery lead time and environmental concerns. The findings are reported in three studies.

We first look into the joint inventory and pricing problem for systems with positive lead time. Specifically, we consider a single stage, periodic-review model. At the beginning of each period, the price and the replenishment quantity are simultaneously determined, where demand in each period is stochastic and depends on the price. There is a positive lead time between an order is placed until when it is received. The objective is to maximize the total expected discounted profit over a finite horizon. In practice, companies often integrate inventory and pricing decisions to match demand with supply more efficiently. A company would typically offer a discounted price when there is excess inventory or raise its price when the inventory level is low. The problem of joint control of price and inventory has attracted significant attention in academia. However, almost all papers assume a zero delivery lead time, because the problem becomes intractable when lead time is positive. Given the growing number of companies that source from low-cost countries, which comes at the expense of longer lead times, it is crucial to understand how to coordinate inventory and pricing decisions when lead times are present. Characterizing the optimal joint inventory and pricing policy in systems with positive lead time is extremely difficult as one has to keep track of the price decisions in each of the lead time periods, giving rise to the curse of dimensionality.

Given the difficulty of the exact problem, we develop a simple heuristic that resolves the dimensionality issue. The heuristic provides a practical solution of coordinating inventory and pricing decisions, and reveals insights that can help understand the impact of lead time on managing such a system. In the heuristic, we use a myopic pricing policy as the heuristic pricing policy that generates each period's price as a function of the initial inventory level. The heuristic replenishment policy is a base-stock policy. In each period, the firm monitors its so-called *price-deflated* inventory position and places an order to reach a target base-stock level. The price-deflated inventory position is a weighted sum of the on-hand inventory and all pipeline inventories, which is different from the inventory position in a standard inventory system in that it depreciates the amount of on-hand inventory and pipeline inventories according to a factor that measures the sensitivity of price to the inventory quantity. In summary, the heuristic we propose involves a base-stock policy for inventory replenishment and a myopic pricing policy that determines each period's price according to the initial inventory level.

To verify the effectiveness of our heuristic, we develop a theoretical upper bound to the optimal profit based on the idea of information relaxation, a general framework proposed by Brown et al. (2012). A key feature of the framework is the development of a penalty function for the information relaxation. The upper bound is tight provided that the penalty function is chosen appropriately – and this choice is problem-specific. In our study, we craftily utilizes the proposed heuristic to construct the penalty function, and derive an efficient algorithm to solve the upper bound system. By comparing the heuristic cost to the upper bound profit, our heuristic is proved to be near-optimal. To our knowledge, this is the first theoretical bound developed for the dynamic pricing problem with positive lead time.

We use our heuristic policy to obtain insights regarding the impact of lead time on the pricing policy. We find that, under both demand types, a shorter lead time leads to a more stable pricing policy. Intuitively, when the lead time is shorter, it is easier to predict future demand and therefore control the inventory available at the beginning of each period. This translates into a more stable pricing policy. In particular, our findings suggest that increased responsiveness (e.g., through a reduction in the procurement/production lead time) reduces the need to adjust demand through prices in order to balance supply and demand.

In our second study, we extend the heuristic approach described above to systems with fixed ordering costs. Chen and Simchi-Levi (2004 a,b) study this joint inventory and pricing problem with fixed ordering costs in the context of zero lead time. They show the (s, S, p) policy and the (s, S, A, p) policy are optimal under additive demand and multiplicative demand, respectively, by proving the objective functions follow the structures of $K-$concavity and $K-$ symmetric concavity. However, as a positive lead time presents in the system, it not only rises the computational complexity, but also makes the objective function lose the $K-$concavity structures. To the best of our knowledge, this is the first study that addresses the joint inventory and pricing problem in a setting with both positive lead time and fixed ordering costs. In the heuristic for such model, we use the myopic pricing policy as the heuristic pricing policy, which only depends on the on-hand inventory level. Then, based on the linear approximation of the myopic demand function, we show that the remaining inventory problem is $K-$concave in the *price-deflated* inventory position defined in our first study under both additive and multiplicative demand functions. This implies that the heuristic replenishment policy follows the form of a (s, S) policy. The effectiveness of the heuristic is confirmed by a numerical study with different values of the fixed ordering costs.

In the last study, we examine the profitability of an eco-friendly supply chain. To that end, we consider a manufacturer that decides on the extent of dependence on petroleum-based components used for production. The procurement cost depends on the price of oil, which is itself volatile. This volatility can be alleviated by using eco-friendly components (whose price tend to be more stable) or production/distribution techniques that rely less on oil. However, any of these alternatives may lead to higher procurement costs. We model the price of oil as a time-correlated process, consistent with observed data. The manufacturer uses a mark-up pricing policy to determine the product's selling price, and demand is a function of price. The first goal is to determine the optimal production policy (i.e., mix of standard and eco-friendly components, and their quantities) that maximizes profit in a finite horizon setting. We characterize conditions under which a state-dependent myopic policy is optimal. This allows us to shed light on the effect of procurement cost volatility on the firm's profit. Our results suggest that operating an eco-friendly production system may not undermine the firm's profitability because the benefits of a less volatile procurement cost may outweigh the increased procurement cost. The magnitude of this benefit is determined by the firm's mark-up and the product's price elasticity. In particular, our results provide guidelines to understand what type of products would better absorb the higher costs associated with an eco-friendly production system.

The remainder of this thesis is organized as follows. In Chapter 2, we study the joint inventory and pricing problems for systems with a positive lead time and introduce a heuristic for such problem. The heuristic idea is extended to a system with a fixed ordering cost in Chapter 3. In Chapter 4, we study the profitability issue for an eco-friendly production system by accessing the effects of the procurement cost and its volatility. All the proofs are contained in the Appendix.

2

Joint Inventory and Pricing Problems with Lead Time

2.1 Introduction

This chapter studies a joint inventory and pricing problem for systems with a positive lead time. Specifically, we consider a periodic-review system in which demand in each period is stochastic and depends on the pricing decision. Two price-dependent demand forms are considered: additive and multiplicative. Unfulfilled demand at the end of each period is fully backlogged, and linear holding and backorder costs are charged. The price and the replenishment quantity are simultaneously determined at the beginning of each period. There is a positive delivery lead time. The objective is to maximize the total expected discounted profit over a finite horizon.

In practice, companies often integrate inventory and pricing decisions to match demand with supply more efficiently. For instance, a company may offer a discounted price when there is excess inventory or raise the price when the inventory level is low. Pricing serves as a lever to reduce the incidence of supply and demand mismatches. The problem of joint control of price and inventory has attracted significant

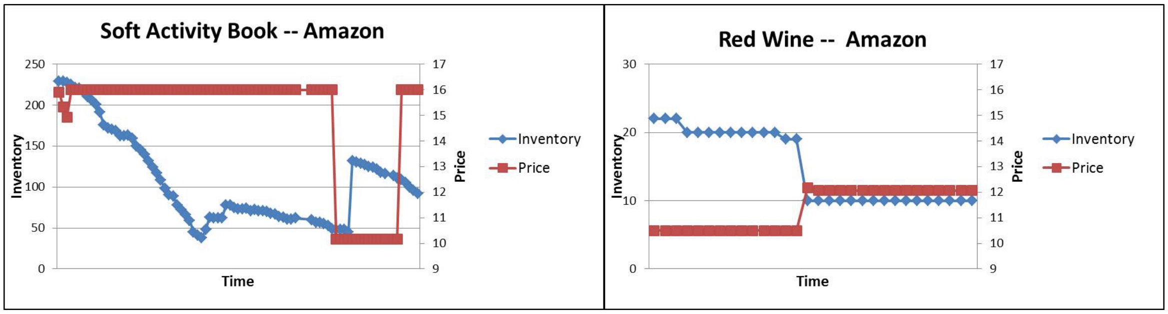


FIGURE 2.1: Tracking inventory and prices – from mid-November 2012 to mid-March 2013

attention in the field – many papers have characterized the optimal joint pricing and replenishment policy in various settings involving both the additive and multiplicative demand forms. However, almost all of those papers assume a zero delivery lead time. While these structural results are of practical use for companies with a negligible lead time (e.g., that source locally), the determination of an efficient policy for firms that experience a longer procurement lead time is of interest. This is particularly the case given the growing number of companies that source from low-cost countries, at the expense of a longer lead time. For example, Murphy (2012) reports on Abercrombie & Fitch's shift from air to ocean delivery to save in shipping costs. The teen-apparel retailer sources products from China and this shift has resulted in a significant increase in the delivery lead time – from days to weeks. As described in the article, the longer lead time increases the chances of supply and demand mismatches, requiring frequent price discounts to liquidate inventory. Figure 1 exhibits prices and inventory levels for two products sold through amazon.com.[1] The activity book[2] is made in China, while the red wine[3] is elaborated in California. The graphs show how prices fluctuate over time and how these price adjustments correlate with the prevailing inventory levels. It is therefore crucial to understand how to coordi-

[1] The authors thank Seyed Emadi for suggestions on how to track inventory levels at amazon.com.

[2] http://www.amazon.com/gp/product/B0001NEAD4

[3] http://www.amazon.com/Red-Verjus-Fusion-750-ml/dp/B0029AFEHI

nate inventory and pricing decisions when a lead time is present. Unfortunately, it is difficult to characterize the optimal joint pricing and replenishment policy in settings with a positive lead time. It is well known that the problem is intractable due to its computational complexity. Indeed, to characterize the optimal policy in systems with a positive lead time, one has to keep track of the pricing decisions in each of the lead time periods, giving rise to the curse of dimensionality.

In this chapter, we propose a heuristic to determine a close-to-optimal joint pricing and inventory replenishment policy for systems with a positive lead time. The idea of the heuristic begins with the observation that the shape of the myopic price (as a function of the initial inventory level) is close to that of the optimal pricing policy. (The myopic price is the price that maximizes the single-period profit based on the initial inventory level in that period, without consideration of future outcomes.) Therefore, as a first step, we propose the myopic price as the heuristic pricing policy. Although the myopic price simplifies the pricing policy, it does not reduce the computational complexity for the inventory policy. To resolves this, we further propose a linear approximation of the myopic expected demand (the expected demand evaluated at the myopic price) as a function of the initial inventory level. This approximation is supported by the observation that the myopic expected demand tends to be linear over a wide range of initial inventory levels. Substituting this linear approximation into the profit function allows us to aggregate all inventory states (i.e., net inventory and pipeline inventory) into a single state variable, which we refer to as *price-deflated inventory position.* The price-deflated inventory position plays the same role as the inventory position (= net inventory + pipeline inventory) in the traditional inventory problem with exogenous prices. However, unlike the inventory position, the price-deflated inventory position is defined as a *weighted* sum of the net inventory and pipeline inventory, where the weights are related to the sensitivity of the myopic optimal price to the initial inventory level. This new state variable assigns more weight to the inventory that is farther away from the system. We then prove that a base-stock policy is optimal in this approximated system.

Based on these results, we propose a heuristic policy that decouples the pricing and inventory decisions. For the pricing policy, we use the myopic price that depends on the initial inventory level. For the inventory policy, a base-stock policy is implemented. That is, at the beginning of each period, the system places an order if the price-deflated inventory position is lower than the target base-stock level (and the order is equal to the difference between the base-stock level and the price-deflated inventory position), and it does not order, otherwise. To test the effectiveness of our heuristic, we develop a theoretical upper bound to the optimal profit in the exact system. The upper bound is based on the information-relaxation approach proposed in Brown et al. (2012). The authors use a duality argument to show that a general class of penalty functions can compensate for the relaxed state space. One key enabler of Brown et al. (2012)'s method is the design of a problem-specific penalty function that effectively compensates for the relaxed state space and that allows for a tractable computation of the upper bound. We construct the penalty function based on our proposed heuristic and show how to efficiently solve the resulting upper bound problem. In addition, this penalty function yields a tight upper bound.

We find that our heuristic policy is near-optimal. We compare the performance of the heuristic to that of the upper bound in an extensive numerical study including settings with lead times ranging from 1 to 5 periods. The average percentage error between the heuristic and the upper bound is 3.5% for the case of additive demand and 3.7% for the case of multiplicative demand. For systems with relatively shorter lead times, i.e., no longer than 2 periods, we find that the performance of our heuristic is very close to that of the optimal solution – an average of 0.55% for additive demand and 0.73% for multiplicative demand – and that the gap between the exact solution and the upper bound contributes to a large portion of the average percentage errors between the heuristic and the upper bound. Moreover, our heuristic consistently outperforms the one proposed in Federgruen and Heching (1999). In settings with zero lead time, Federgruen and Heching (1999) characterize the optimal policy when there are no fixed ordering costs and Chen and Simchi-Levi (2004) characterize the optimal policy for systems with fixed order costs (see Section 2 for further details on these policies). Our heuristic delivers close-to-optimal decisions in systems with zero lead time as well. From a computational perspective, obtaining the exact optimal policy in settings with zero lead time requires solving a dynamic program with two decision variables whereas our heuristic involves the solution to a single-period problem for the price and a single-variable dynamic program for the inventory policy.

We use our heuristic policy to obtain insights regarding the impact of lead time on pricing decisions. We find that, under both demand types, a shorter lead time leads to a more stable pricing policy. Intuitively, when the lead time is shorter, it is easier to predict future demand and therefore control the inventory available at the beginning of each period. This translates into a more stable pricing policy. In particular, our findings suggest that increased responsiveness (e.g., through a reduction in the procurement/production lead time) reduces the need to adjust demand through prices in order to balance supply and demand. We also find that price discounts may be offered in anticipation of the arriving inventory in-transit, even if the net inventory level is relatively low.

2.2 Literature Review

The study of joint inventory and pricing problems can be traced back to Whitin (1955) and Mills (1959), who studied this problem in a single-period model. Pertruzzi and Dada (1999) provide a comprehensive review on this stream of literature. The authors show that the cases of additive and multiplicative demand lead to distinctively different results regarding the optimal solution. In a multi-period setting, Federgruen and Heching (1999) study a single-stage system with zero lead time. The authors prove that a so-called list-price-base-stock policy is optimal for both the nonstationary finite-horizon and stationary infinite-horizon problems. This policy involves two parameters – a list price and a base-stock level. When the inventory level at the beginning of a period is below the base-stock level, an order is placed to reach the target level and the price is set equal to the list-price. Otherwise, no order is placed and price is discounted so that a higher initial inventory level leads to a deeper discount. The authors also suggest a heuristic for systems with positive lead time and additive demand. In their heuristic, the price is fixed within the lead time, which allows them to reduce the state space of the problem. Chen and Iyengar (2012) propose an approach to generate policies for the joint inventory and pricing problem, which involves solving a collection of linear programs. Numerically, the performance of their approach is efficient. However, it only applies to systems with zero lead time.

The literature studying joint dynamic pricing and replenishment decisions with positive lead time is scarce. Pang et al. (2012) study this problem under additive demand. The authors show that the problem is $L^{\natural}$-concave, which guarantees the existence of a state-dependent optimal policy. However, they do not characterize the structure of the optimal policy. From a computational perspective, it remains difficult to obtain the global optimal solution due to the high dimensionality of the state space. Nevertheless, they show that the initial inventory level has a larger impact on the optimal pricing decision than the inventory in the pipeline. This finding supports the use of a myopic pricing policy in our heuristic. Yu (2012) discusses another approach to solve this problem for the case of additive demand, with additional conditions on the ordering behavior.

Another related stream of research considers joint inventory and pricing decisions with uncertain supply. Li and Zheng (2006) study joint inventory and pricing control in a setting with stochastic production yield and demand. Feng (2010) considers a pricing and inventory problem in a model with uncertain capacity supply. In both papers, the demand function is additive and the optimal replenishment policy is of a threshold type, i.e., a positive amount is ordered only when the inventory level is below a critical point. Readers are referred to Yano and Gilbert (2002), Elmaghraby and Keskinocak (2003) and Chan et al. (2004) for comprehensive reviews of the literature on dynamic joint pricing and inventory control.

In the inventory management literature, there is a number of papers developing theoretical bounds to examine the effectiveness of heuristics. Zheng (1992) considers a continuous-review, single-stage system with fixed order costs. The author shows that the economic order quantity obtained from the deterministic counterpart is 72.5% effective. Janakiraman et al. (2008) consider a periodic-review, lost-sales model with positive lead time. The authors show that a dual-balancing policy guarantees a 200% effectiveness. For multi-echelon inventory models, it is fairly common to examine the effectiveness of a heuristic by comparing the heuristic cost with a theoretical bound. For example, Chen and Zheng (1998) compare a heuristic (r, nQ) policy with a lower bound to the optimal cost for serial systems with fixed order costs. Shang and Zhou (2010) employ the same approach to verify the effectiveness of the heuristic (s, T) policy for a serial system. In this chapter, we assess the effectiveness of our proposed heuristic by comparing its performance to that of an upper bound. To our knowledge, this is the first theoretical bound developed for the dynamic joint pricing and inventory problem.

2.3 Model and Preliminaries

We consider a single-stage system with a positive delivery lead time of L periods. The net inventory level at the beginning of period t before replenishment is x_t and the pipeline inventory is represented by a vector $\mathbf{w_t} = (w_{1,t}, \cdots, w_{L-1,t})$, where $w_{l,t}$ denotes the replenishment quantity to be delivered in l periods, $l = 1, \cdots, L-1$. At the beginning of each period t, the replenishment quantity q_t and the selling price p_t are determined simultaneously in order to maximize the total expected discounted profit through the end of a finite horizon with T periods. Let α denote the discount factor. The demand in each period is stochastic and depends on the current period's price p_t. Unfulfilled demand at the end of each period is fully backlogged. The backlogging assumption is consistent with the models in other papers studying joint pricing and inventory decisions, e.g., Federgruen and Heching (1999), Chen and Simchi Levi (2004a,b), etc. While a lost sales assumption may be more appropriate in retail settings, the backorder model can efficiently approximate the corresponding lost-sales system when the service level is high. In fact, Huh et al. (2009) show that the base-stock policy is asymptotically close to the lost sales model when the backorder cost increases to infinity. As reported in Nagarajan and Rajagopalan (2009), there are numerous studies (e.g., Anderson Consulting 1996, Gruen et al. 2002, Smith and Agrawal 2000) that suggest that service levels are typically high in retailing – around 95% according to some estimates.[4]

We consider the cases of additive and multiplicative demand functions.

Additive demand: $D_t(p_t, \epsilon_t) = \mathcal{D}_t(p_t) + \epsilon_t = \lambda_t - \mu_t p_t + \epsilon_t,\ \lambda_t > 0,\ \mu_t > 0,$ where ϵ_t is a random variable with $\mathsf{E}[\epsilon_t] = 0$, support on $[-A_t, B_t]$, and $\mathcal{D}_t(p_t) = \lambda_t - \mu_t p_t$.

Multiplicative demand: $D_t(p_t, \epsilon_t) = \mathcal{D}_t(p_t)\,\epsilon_t = \lambda_t p_t^{-\mu_t}\epsilon_t,\ \lambda_t > 0,\ \mu_t > 1$, where ϵ_t is a random variable with $\mathsf{E}[\epsilon_t] = 1$ and $\mathcal{D}_t(p_t) = \lambda_t p_t^{-\mu_t}$.

Different forms of the demand function $D_t(p_t, \epsilon_t)$ are usually associated with different shapes of the mean demand $\mathcal{D}_t(p_t)$. In this study, we assume a linear mean

[4] During the tracking period, no stockouts were observed for the products displayed in 2.1.

demand under additive demand and an iso-elastic mean demand curve under multiplicative demand. Both of these curves are commonly used in the literature. (Nevertheless, as discussed later in the chapter, our results continue to hold under other shapes of the mean demand functions.) Price is allowed to change bi-directionally (decrease or increase) from period to period. We impose constraints on the pricing decisions to avoid negative expected demand:

$$p_t \in \begin{cases} \left[0, \frac{\lambda_t}{\mu_t}\right], & \text{additive demand,} \\ (0, +\infty), & \text{multiplicative demand.} \end{cases}$$

To ensure positive demand for some range of prices under additive demand, we further assume $-A_t > -\lambda_t$. For both types of demand functions, the perturbations ϵ_t are assumed to be independent from period to period. Each random variable ϵ_t has a continuous cdf $F_t(\cdot)$ and pdf $f_t(\cdot)$, which are smooth enough to guarantee the continuity and differentiability of all functions.

For both types of demands, the inverse demand function $\mathcal{D}_t^{-1}(d_t) \triangleq p_t(d_t)$ exists, where d_t represents the expected demand volume:

$$p_t(d_t) = \begin{cases} \frac{\lambda_t - d_t}{\mu_t}, & \text{additive demand,} \\ \left(\frac{\lambda_t}{d_t}\right)^{1/\mu_t}, & \text{multiplicative demand.} \end{cases}$$

The problem can be equivalently formulated as selecting the optimal replenishment quantity q_t and the expected demand d_t in each period t (as opposed to the order quantity and the price p_t).

We define the demand function in terms of the expected demand d_t and the random variable ϵ_t:

$$D_t(d_t, \epsilon_t) = \begin{cases} d_t + \epsilon_t, & \text{additive demand,} \\ d_t \epsilon_t, & \text{multiplicative demand.} \end{cases}$$

The action space is denoted by Ω_t, and from the constraints on the prices, we have that

$$\Omega_t = \begin{cases} [0, \lambda_t], & \text{additive demand,} \\ (0, +\infty), & \text{multiplicative demand.} \end{cases}$$

Let c_t denote the unit purchasing cost, h_t the unit holding cost and b_t the unit backorder cost in period t. The retailer pays a total purchasing cost $c_t q_t$ and incurs inventory-related costs given by

$$G_t(x_t, d_t) \triangleq \mathsf{E}\left[h_t(x_t - D_t(d_t, \epsilon_t))^+ + b_t(D_t(d_t, \epsilon_t) - x_t)^+\right]$$

in each period t. We assume that the revenue collected in each period depends on the demand volume instead of the sales amount, i.e., consumers pay upon arrival when backorders happen. This is a standard assumption in the related literature. Based on this assumption, the expected revenue in period t only depends on the expected demand volume d_t, i.e.,

$$R_t(d_t) \triangleq \mathsf{E}[p_t D_t(p_t, \epsilon_t)] = \mathsf{E}[\mathcal{D}_t^{-1}(d_t) D_t(d_t, \epsilon_t)] = p_t(d_t)\,\mathsf{E}[D_t(p_t, \epsilon_t)] = p_t(d_t)\,d_t.$$

$R_t(d_t)$ is concave in d_t for both types of demand functions.

We index time counting forward, i.e., $t = 1$ represents the beginning of the planning horizon and $t = T$ represents the last period; $t = T + 1$ represents the end of the planning horizon. The sequence of events in each period t is as follows: (1) If $t \geqslant L + 1$, then the replenishment order placed L periods ago, namely q_{t-L}, is delivered; (2) The expected demand volume d_t (or the selling price p_t) and the replenishment quantity q_t are determined simultaneously; (3) Demand occurs; (4) Costs and revenue are calculated at the end of the period.

Let $V_t(x_t, \mathbf{w_t})$ denote the maximum expected discounted profit from period t until the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$. Then, $V_t(x_t, \mathbf{w_t})$,

$t = 1, \cdots, T+1$, satisfy the following recursive equations: Let $V_{T+1}(x_{T+1}, \mathbf{w_{T+1}}) = 0$;

$$\begin{aligned} V_t(x_t, \mathbf{w_t}) &= \max_{q_t \geqslant 0, d_t \in \Omega_t} \left\{ R_t(d_t) - c_t q_t - G_t(x_t, d_t) + \alpha \mathsf{E} V_{t+1}(x_{t+1}, \mathbf{w_{t+1}}) \right\} \quad (2.1) \\ &= \max_{q_t \geqslant 0, d_t \in \Omega_t} \left\{ J_t(x_t, \mathbf{w_t}, d_t, q_t) \right\}. \end{aligned}$$

where the state dynamics are $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t), w_{2,t}, \cdots, w_{L-1,t}, q_t)$.

Proposition 1. *Under both the additive and multiplicative demand forms, $V_t(x_t, \mathbf{w_t})$ is jointly concave in $(x_t, \mathbf{w_t})$ and $J_t(x_t, \mathbf{w_t}, d_t, q_t)$ is jointly concave in $(x_t, \mathbf{w_t}, d_t, q_t)$ for all t*[5].

From Proposition 1, the optimal joint decisions, denoted by (q_t^*, d_t^*), can be obtained from the first order conditions of $J_t(x_t, \mathbf{w_t}, d_t, q_t)$. The optimal decisions in each period depend on the L-dimensional state vector $(x_t, \mathbf{w_t})$, i.e., $(q_t^*, d_t^*) = (q_t^*(x_t, \mathbf{w_t}), d_t^*(x_t, \mathbf{w_t}))$. Computing this state-dependent optimal policy is computationally infeasible due to the curse of dimensionality. This is in contrast to the traditional inventory problem with exogenous demand, in which the inventory states (pipeline inventory and on-hand inventory) can be aggregated into a single inventory variable – the inventory position. Thus, the objective of this chapter is to provide a simple heuristic for the joint pricing and replenishment problem with a positive lead time.

2.4 Heuristic

In this section, we introduce and discuss the heuristic. The development of the heuristic consists of three steps. In Section 2.4.1, we introduce the myopic expected demand policy and characterize its properties. Based on the myopic demand's structural findings, in Section 2.4.2 we construct a linear function to further approximate

[5] Pang et al. (2010) show the concavity of the exact problem for the additive demand case. Here, we show the result for both cases.

the myopic expected demand as a function of the initial inventory level. In Section 2.4.3, we study the remaining inventory problem and show that a base-stock policy is optimal.

2.4.1 Myopic Expected Demand

We introduce the myopic expected demand function. Denote $d_t^M(x_t)$ as the myopic expected demand that solves the following maximization problem:

$$\max_{d_t \in \Omega_t} \{R_t(d_t) - G_t(x_t, d_t)\}.$$

The myopic expected demand policy is a function of the net inventory level x_t only, i.e., it ignores the effect of pipeline inventory. Pang et al. (2010) show that the initial inventory level has the largest impact on the optimal price for the additive demand case. In addition, Propositions 3 and 5 below show properties of the myopic demand policy that suggest that the myopic and exact optimal policies share a similar structure. These results provide support for the use of the myopic policy to approximate the optimal pricing policy.

Additive Demand Case

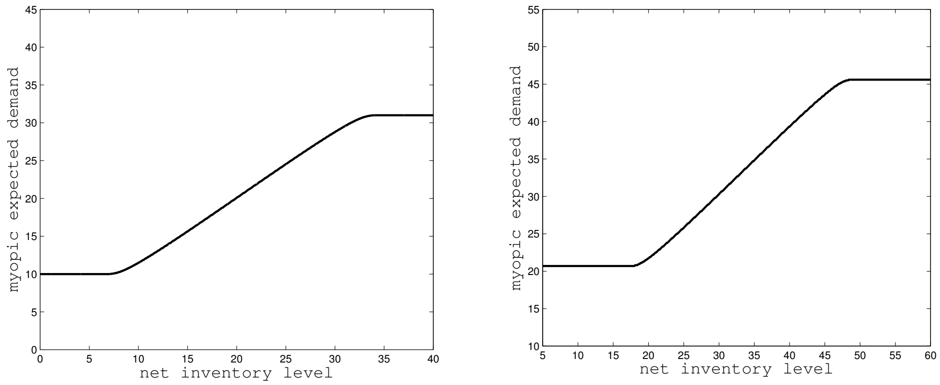
Proposition 2. *Under an additive demand function,*

(i) *The myopic expected demand $d_t^M(x_t)$ is non-decreasing with slope between* 0 *and* 1*;*

(ii) $\lim_{x_t \to +\infty} d_t^M(x_t) = (\lambda_t + \mu_t h_t)/2$*;* $\lim_{x_t \to -\infty} d_t^M(x_t) = (\lambda_t - \mu_t b_t)/2$.

Figure 2.2 presents an example of the shape of the myopic expected demand function in the case of additive demand[6]. The corresponding myopic price function

[6] Although we work with the linear and exponential demand functions introduced in Section 3 (for the cases of additive and multiplicative demand, respectively), our results – and therefore the



(a) $\mathcal{D}(p) = \lambda - \mu p$ (b) $\mathcal{D}(p) = (\lambda - \mu p)^{\theta}$

FIGURE 2.2: Myopic expected demand – additive demand, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 1.5$, $\theta = 1.1$ and $\epsilon \sim \text{Normal}(0, 1)$.

is given by $p_t^M(x) = (\lambda_t - d_t^M(x))/\mu_t$. As shown in the graph, $d^M(x)$ increases in x. More specifically, the myopic demand function is relatively flat when x is either small or large (and asymptotically approaches the bounds) and increases when x is in an intermediate range of initial inventory levels. To explain the flat shape of the expected demand curve when x is large, note that the increase in expected demand to offset an increase in inventory when x is large hurts the profit because the revenue decreases at an increasing marginal rate (due to the concavity of the revenue function) while the marginal inventory cost remains nearly constant at h. A similar logic applies when x takes small values. The structure of $d^M(x)$ also indicates that the myopic demand (pricing) policy is a less effective lever to control demand for large or small inventory states, where the system becomes less profitable from the relatively higher increases in inventory/backorder costs.

Proposition 3. *Consider a setting with additive demand and $L = 1$. Then,*

(i) $d_t^*(x_t) \leqslant d_t^M(x_t)$ *and equality holds if* $c_t = 0$*;*

---

applicability of our heuristic – are valid under different shapes of the mean demand functions $\mathcal{D}(p)$, even when their curvature is more pronounced. Figure 2.2(b) illustrates the shape of the myopic expected demand for another family of mean demand functions in the case of additive demand. Figure 3 below similarly illustrates the shape of the myopic expected demand for the exponential demand function introduced in Section 3 as well as for another functional form.

(*ii*) $d_t'^*(x_t) = d_t'^M(x_t)$;

(*iii*) $\lim_{x_t \to +\infty} d_t^*(x_t) = (\lambda_t + \mu_t h_t - \mu_t c_t)/2$; $\lim_{x_t \to -\infty} d_t^*(x_t) = (\lambda_t - \mu_t b_t - \mu_t c_t)/2$.

Consider now a setting with additive demand, $L > 1$, and $h_t = 0$ for all t. Then,

(*iv*) $d_t^*(x_t, \mathbf{w_t}) \leqslant d_t^M(x_t)$;

(*v*) $\lim_{w_{1,t} \to +\infty} d_t^*(x_t, \mathbf{w_t}) = d_t^M(x_t)$ *for all* t.

Proposition 3 states that the myopic expected demand is an upper bound to the exact expected demand for any x_t and that both functions increase at the same rate when $L = 1$. The difference between these two policies increases in the unit purchasing cost c_t. When c_t increases, the optimal policy involves a lower expected demand and a higher price. The order $d_t^*(x_t, \mathbf{w}_t) \leqslant d_t^M(x_t)$ is not preserved in general for systems with $L > 1$. Nevertheless, when $h_t = 0$, we prove that this relationship holds. Moreover, as shown in part (*v*), the myopic demand is optimal if there is ample pipeline inventory. Numerically, we observe that the myopic expected demand function $d_t^M(x_t)$ has a similar structure as the exact optimal expected demand function for any level of pipeline inventory and any value of h_t.

Multiplicative Demand

Proposition 4. *Under a multiplicative demand function,*

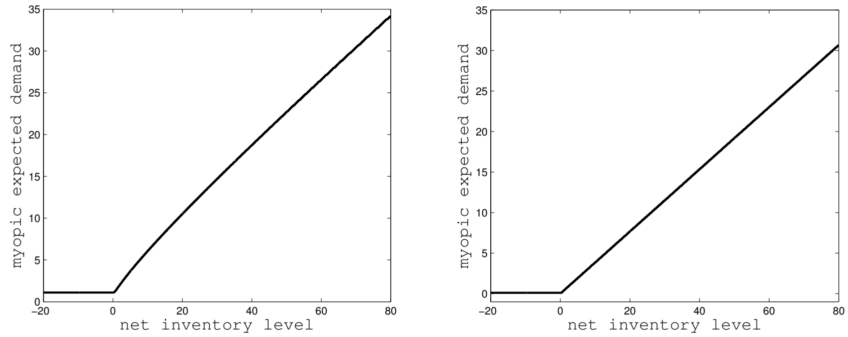
(*i*) *The myopic expected demand $d_t^M(x)$ is non-decreasing in x;*

(*ii*) *When $h_t \neq 0$, $b_t \neq 0$, $d_t^M(x)$ has an asymptotic slope C_t^* that solves*

$$\int_{1/C}^{+\infty} \epsilon dF(\epsilon) = \frac{h_t}{h_t + b_t}.$$

Moreover, $C_t^* \leqslant 1$ when ϵ_t follows a gamma distribution with mean 1.[7]

[7] This also holds with other distributions, e.g., all normal distributions with mean 1.



(a) $\mathcal{D}(p) = \lambda p^{-\mu}$ (b) $\mathcal{D}(p) = (\theta + \lambda p)^{-\mu}$

FIGURE 2.3: Myopic expected demand – multiplicative demand, $h = 1$, $b = 20$, $\lambda = 500$, $\mu = 1.5$, $\theta = 500$ and $\epsilon \sim \text{Gamma}(2, 0.5)$.

(iii) When $x \leqslant 0$, $d_t^M(x)$ is equal to a constant positive value

$$C_{0t} = \lambda_t \left[\frac{(1 - 1/\mu_t)}{b_t} \right]^{\mu_t}.$$

Figure 2.3 provides an example of the shape of the myopic demand function under multiplicative demand. The structure of the myopic demand function follows from the fact that the revenue function is concave increasing in demand. This implies that the myopic expected demand function increases to infinity as the inventory level goes to infinity. As demand gets progressively larger, the rate of increase of the revenue approaches zero. Thus, the asymptotic slope C_t^* is independent of the demand parameters and is only determined by the critical fractile $b_t/(h_t + b_t)$. As shown in Proposition 4(iii), the myopic expected demand equals a positive value C_{0t} when backorders occur. This value C_{0t} balances the revenue and the backorder cost. Moreover, the value of C_{0t} decreases in the backorder cost rate and increases in the demand parameters.

Proposition 5 shows that the myopic expected demand is an upper bound for the exact optimal expected demand under a multiplicative demand form, consistent with the case of additive demand.

Proposition 5. *Consider a setting with multiplicative demand and $L = 1$. Then*

(i) $d_t^*(x_t) \leqslant d_t^M(x_t)$.

Suppose now that $L > 1$, and $h_t = 0$ for all t. Then,

(ii) $d_t^*(x_t, \mathbf{w_t}) \leqslant d_t^M(x_t)$*;*

(iii) $\lim_{w_{1,t} \to +\infty} d_t^*(x_t, \mathbf{w_t}) = d_t^M(x_t)$ *for all t.*

2.4.2 Linear Approximation

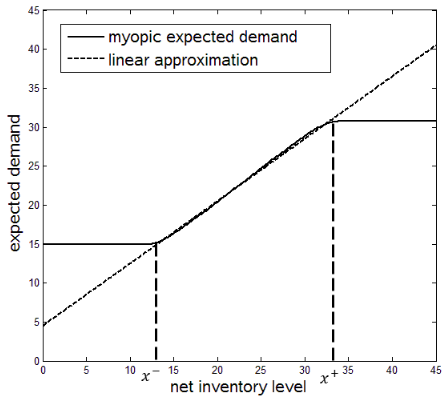
Under the myopic pricing policy, we can transform the original problem into an inventory problem where the future expected demand depends on the beginning inventory level through the myopic price. Despite this simplification, the resulting inventory problem continues to be L-dimensional. To resolve this issue, we propose a linear approximation $\widetilde{d}_t(x_t)$ for the myopic demand $d_t^M(x_t)$, where

$$\tilde{d}_t(x_t) = \delta_t x_t + \kappa_t.$$

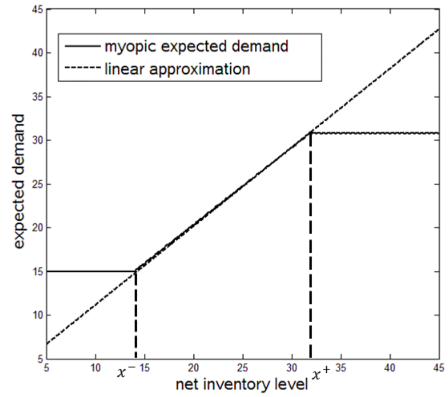
We omit the time index t in the remainder of this section. We next discuss the derivation of the parameters δ and κ for the additive and multiplicative demand cases, respectively.

Additive Demand

Our goal is to find two points, $(x^+, d^M(x^+))$ and $(x^-, d^M(x^-))$, to construct the linear function that approximates the myopic expected demand for the intermediate range of inventory levels. A natural choice for x^+ is the point beyond which the myopic demand curve starts turning flat. This turning point x^+ is determined by one of two values, x^u or x^{ub}. The point x^u is such that $d^M(x^u)$ is sufficiently close to the upper bound, that is, x^u is the solution to $d^M(x) = (\lambda + \mu h)/2 - \zeta$



(a) $\epsilon \sim$ Normal$(0,1)$



(b) $\epsilon \sim$ Uniform$(-1,1)$

FIGURE 2.4: Linear approximation – additive demand case, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 2$.

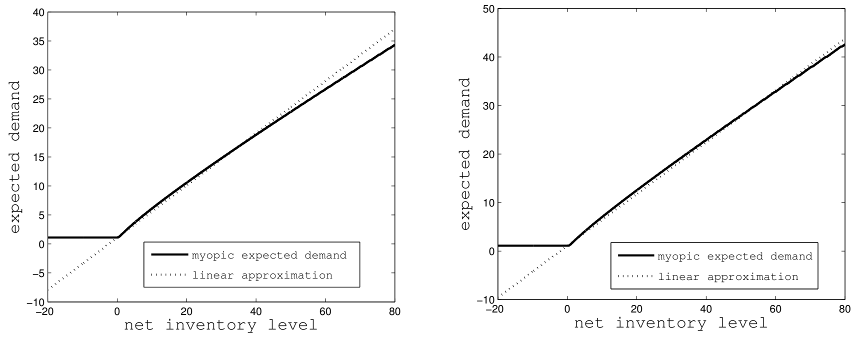
for small ζ. (In the numerical study, we set $\zeta = 0.05$.) Because under additive demand, the expected demand in each period is constrained to the interval $[0, \lambda]$, we define $x^{ub} = \min\left\{x : d^M(x) = \lambda\right\}$ and let $x^+ = \min\{x^u, x^{ub}\}$. We similarly define $x^- = \max\{x^l, x^{lb}\}$, with x^l the solution to $d^M(x) = (\lambda - \mu b)/2 + \zeta$ and $x^{lb} = \max\left\{x : d^M(x) = 0\right\}$.[8]

Define $\hat{x} = (x^+ + x^-)/2$ and the parameters of the linear approximation as $\delta = d'^M(\hat{x})$ and $\kappa = -\delta(\hat{x}) + d^M(\hat{x})$. Figure 2.4 shows an example of the linear approximation under additive demand. It follows from Proposition 2(i) that $0 < \delta < 1$.

Proposition 6. *In the case of additive demand, $d'^M(x)$ is increasing in μ. Furthermore, if $h \leqslant b$ and ϵ has a symmetric and unimodal density function, then δ is increasing in μ.*

Proposition 6 implies that, as μ increases, the corresponding myopic price becomes more sensitive to an increase in the initial inventory level. That is, for a larger

[8] From the discussion before Proposition 3, note that $\frac{\partial R_t}{\partial d}|_{d=d_t^M(x^u)} \approx -h_t P(x^u > d_t^M(x^u) + \epsilon_t)$ and $\frac{\partial R_t}{\partial d}|_{d=d_t^M(x^l)} \approx b_t P(x^l < d_t^M(x^l) + \epsilon_t)$.



(a) $\epsilon \sim \text{Gamma}(2, 0.5)$ (b) $\epsilon \sim \text{Gamma}(4, 0.25)$

FIGURE 2.5: Linear approximation – multiplicative demand case, $h = 1$, $b = 20$, $\lambda = 500$, $\mu = 1.5$.

μ, an increase in x will result in a steeper reduction of the retail price under the linear approximation to the myopic policy.

Multiplicative Demand

For the case of multiplicative demand, we use the results obtained in Proposition 4 to derive the linear approximation. Specifically, we set the intercept to be $\kappa = C_0$. Let $x^M = \min\{x \geqslant 0 : d^M(x) = x\}$ be the smallest point at which the myopic expected demand crosses the 45-degree line. This point is guaranteed to exist because $d^M(0) = C_0 > 0$ and $\lim_{x\to\infty} d'^M(x) = C^* < 1$. On one hand, the inventory-related cost is small when the net inventory level is in the vicinity of x^M. On the other hand, as x increases, the myopic expected demand increases as well to maximize revenue (recall that the revenue function is concave increasing). We therefore define the slope of the linear approximation as an average of the asymptotic slope C^* and $d'^M(x^M)$. That is, $\delta = (C^* + d'^M\left(x^M\right))/2$. Figure 2.5 illustrates the linear approximation under a multiplicative demand function.

The linear approximation has some important properties. First, one can verify that $d'^M(x^M) < 1$, which guarantees that $0 < \delta < 1$ in the case of multiplicative demand. Moreover, we show that δ increases with μ. Again, note that μ is the price sensitivity of the mean demand function and δ measures the price sensitivity to the inventory level.

Proposition 7. *Under multiplicative demand, $d'^M\left(x^M\right)$ is increasing in μ. In turn, this implies that the slope $\delta = (C^* + d'^M\left(x^M\right))/2$ is increasing in μ.*

2.4.3 State Space Reduction and Heuristic Policy

In this section, we describe how to determine the heuristic replenishment and pricing policy. We first substitute each period's expected demand by the linear approximation $\widetilde{d}_t\left(x_t\right) = \delta_t x_t + \kappa_t$ and change the accounting scheme in the remaining inventory problem by calculating the profit L periods forward. Since $\widetilde{d}$ can take negative values, we extend the definition of the revenue function to ensure that the problem is concave. Under additive demand, we let $\widetilde{R}_t\left(d_t\right) = d_t p_t\left(d_t\right),\ d_t \in (-\infty, +\infty)$. In the case of multiplicative demand, we let

$$\widetilde{R}_t\left(d_t\right) = \begin{cases} d_t p_t\left(d_t\right), & \text{if } d_t \geqslant \xi, \\ d_t R_t'\left(\xi\right), & \text{if } d_t < \xi, \end{cases}$$

where ξ is a small positive value.[9] For both types of demand, we have that $\widetilde{R}_t\left(d_t\right)$ is concave in d_t. The optimal inventory policy can now be determined from the following recursion: Let $\widetilde{V}_{T+1} \equiv 0$, and

$$\begin{aligned} \widetilde{V}_t\left(x_t, \mathbf{w}_t\right) &= \max_{q_t \geqslant 0} \mathsf{E}\Big\{\alpha^L \widetilde{R}_{t+L}(\widetilde{d}_{t+L}\left(x_{t+L}\right)) - c_t q_t - \alpha^L G_{t+L}(x_{t+L}, \widetilde{d}_{t+L}\left(x_{t+L}\right)) \\ &\qquad + \alpha \widetilde{V}_{t+1}\left(x_{t+1}, \mathbf{w}_{t+1}\right)\Big\}\Big\}, \qquad (2.2) \\ &\triangleq \max_{q_t \geqslant 0} \widetilde{J}_t\left(x_t, \mathbf{w}_t, q_t\right), \qquad (2.3) \end{aligned}$$

where $(x_{t+1}, \mathbf{w}_{t+1}) = (x_t + w_{1,t} - D_t(\widetilde{d}_t\left(x_t\right), \epsilon_t), w_{2,t}, \cdots, q_t)$, and $x_{t+L} = x_t + \sum_{l=1}^{L-1} w_{l,t} + q_t - \sum_{l=0}^{L-1} D_{t+l}(\widetilde{d}_{t+l}(x_{t+l}), \epsilon_{t+l})$.

[9] Note that $R_t'(0) = +\infty$, so ξ cannot take value at 0.

The functions $\widetilde{V}_t(x_t, \mathbf{w}_t)$ and $\widetilde{J}_t(x_t, \mathbf{w}_t, q_t)$ are concave because of the linearity of $\widetilde{d}_t(x_t)$. Because the single-period's profit function depends only on x_{t+L}, the dimension of the state variable x_{t+L} determines the dimension of the approximate problem. Below we show that x_{t+L} can be expressed as a weighted sum of the variables x_t and $\mathbf{w_t}$. To facilitate our discussion, we define

$$\begin{aligned}\overline{x}_t &= \text{price-deflated inventory position at the beginning of period } t,\\ &= \nu_{0,t}x_t + \sum_{l=1}^{L-1} \nu_{l,t}\left(w_{l,t} - \kappa_{t+l-1}\right) - \kappa_{t+L-1}, \qquad (2.4)\end{aligned}$$

where $\nu_{l,t} = \prod_{k=l}^{L-1}(1-\delta_{t+k}),\ l = 0, 1, ..., L-1$, and

$$\begin{aligned}\epsilon[t, t+L) &= \text{total weighted errors in periods } t,\ t+1, ..., \ t+L-1,\\ &= \begin{cases}\sum_{l=0}^{L-1} \nu_{l+1,t}\ \epsilon_{t+l}, \text{ additive demand,}\\ \sum_{l=0}^{L-1} \nu_{l+1,t}\ \kappa_{t+l}\left(\epsilon_{t+l}-1\right), \text{ multiplicative demand.}\end{cases}\end{aligned}$$

(Note that $E\epsilon[t, t+L) = 0$ for both additive and multiplicative demands.)

We now derive the state variable x_{t+L}. For the case of **additive demand**, we have

$$\begin{aligned}x_{t+L} &= x_t + \sum_{l=1}^{L-1} w_{l,t} + q_t - \sum_{l=0}^{L-1}(\delta_{t+l}x_{t+l} + \kappa_{t+l} + \epsilon_{t+l})\\ &= \prod_{k=0}^{L-1}(1-\delta_t)x_t + \sum_{l=1}^{L-1}\prod_{k=l}^{L-1}(1-\delta_{t+k})(w_{l,t} - \kappa_{t+l-1} - \epsilon_{t+l-1}) + q_t - \kappa_{t+L-1} - \epsilon_{t+L-1}\\ &= \overline{x}_t + q_t - \epsilon[t, t+L).\end{aligned}$$

For the case of **multiplicative demand**,

$$\begin{aligned}x_{t+L} &= x_t + \sum_{l=1}^{L-1} w_{l,t} + q_t - \sum_{l=0}^{L-1}(\delta_{t+l}x_{t+l} + \kappa_{t+l})\epsilon_{t+l}\\ &= \prod_{k=0}^{L-1}(1-\delta_t\epsilon_t)x_t + \sum_{l=1}^{L-1}\prod_{k=l}^{L-1}(1-\delta_{t+k}\epsilon_{t+k})(w_{l,t} - \kappa_{t+l-1}\epsilon_{t+l-1}) + q_t - \kappa_{t+L-1}\epsilon_{t+L-1}\end{aligned}$$

Under the multiplicative demand form, the coefficients of x_t and $\mathbf{w}_t$ depend on the random variables $\{\epsilon_{t+l}\}_{l=1}^{L-1}$. In order to aggregate the terms corresponding to x_t and $\mathbf{w}_t$, we further approximate x_{t+L} by using the mean values of those random coefficients (= 1), obtaining

$$\begin{aligned} x_{t+L} &= = \prod_{k=0}^{L-1}(1-\delta_t)x_t + \sum_{l=1}^{L-1}\prod_{k=l}^{L-1}(1-\delta_{t+k})(w_{l,t} - \kappa_{t+l-1}\epsilon_{t+l-1}) + q_t - \kappa_{t+L-1}\epsilon_{t+L-1} \\ &= \overline{x}_t + q_t - \epsilon[t, t+L). \end{aligned}$$

We can now express the optimality equations in (2.3) using the new state variable $\overline{x}_t$ and the decision variable $\overline{y}_t = \overline{x}_t + q_t$. That is, let $\widetilde{V}_{T+1} \equiv 0$, and

$$\begin{aligned} \widetilde{V}_t(\overline{x}_t) &= c_t\overline{x}_t + \max_{\overline{y}_t \geqslant \overline{x}_t} \mathsf{E}\Big\{\alpha^L \widetilde{R}_{t+L}\left(\widetilde{d}_{t+L}(x_{t+L})\right) - c_t\overline{y}_t - \alpha^L G_{t+L}\left(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L})\right) \\ &\quad + \alpha\widetilde{V}_{t+1}(\overline{x}_{t+1})\Big\}, \end{aligned} \tag{2.5}$$

where $x_{t+L} = \overline{y}_t - \epsilon[t, t+L)$ and $\overline{x}_{t+1} = (1-\delta_{t+L})[\overline{y}_t - \epsilon[t, t+1)] - \kappa_{t+L}$. Define

$$\widetilde{J}_t(\overline{y}_t) = \mathsf{E}\{\alpha^L \widetilde{R}_{t+L}\left(\widetilde{d}_{t+L}(x_{t+L})\right) - c_t\overline{y}_t - \alpha^L G_{t+L}\left(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L})\right) + \alpha\widetilde{V}_{t+1}(\overline{x}_{t+1})\}.$$

Theorem 8. *The value function $\widetilde{V}_t(\overline{x}_t)$ in (2.5) is concave in $\overline{x}_t$ and $\widetilde{J}_t(\overline{y}_t)$ is concave in $\overline{y}_t$ for all t. Let $\overline{s}_t = \arg\max \widetilde{J}_t(\overline{y}_t)$. The optimal inventory policy for the approximated problem in (2.5) is a base-stock policy with parameters $\{\overline{s}_t\}_{t=1}^T$.*

The optimal base-stock policy derived in Theorem 8 serves as the heuristic replenishment policy for the original system. Specifically, our proposed heuristic for the joint inventory and pricing problem is as follows. In each period t, given the state $(x_t, \mathbf{w}_t)$:

(i) Set the price equal to the myopic price, i.e., $p_t^M = p_t\left(d_t^M(x_t)\right)$;

(ii) Calculate $\overline{x}_t$ as in (2.4) and $\overline{s}_t = \arg\max \widetilde{J}_t(\overline{y}_t)$. An order quantity $q_t = \overline{s}_t - \overline{x}_t$ is placed if $\overline{x}_t \leqslant \overline{s}_t$, and no order is placed otherwise.

We now discuss the physical meaning of $\overline{x}_t$. Note that demand in each lead-time period is composed of two parts, controllable and non-controllable. The controllable portion refers to the demand determined by the linear approximation (i.e., by the pricing decisions), while the non-controllable portion of demand refers to the remaining random terms. The value of $\overline{x}_t$, given in (2.4), can be interpreted as the expected inventory level at the beginning of period $t+L$, equal to the sum of the net inventory x_t and all the pipeline inventory terms $w_{l,t}$ minus the total controllable demand during the lead time. It follows from the properties of the slopes δ_t that the weights $\nu_{l,t}$ in $\overline{x}_t$ satisfy $\nu_{0,t} < \nu_{1,t} < \ldots < \nu_{L-1,t}$. This suggests that the price-deflated inventory position assigns a lower weight to the inventory that is closer to the system. To see this, consider the inventory state at the beginning of period t, $(x_t, \mathbf{w_t})$. Since x_t is present in the system before the arrival of the pipeline inventory $\mathbf{w_t}$, the quantity x_t will be used to satisfy the entire controllable demand over the lead time. In contrast, the pipeline inventory will be used to satisfy only a portion of that lead time demand. Thus, the proportion of x_t that is expected to be available at the end of the lead time will be smaller than that of the pipeline inventory. This implies that the weights of the pipeline inventory in $\overline{x}_t$ are progressively higher.

The order quantity q_t that arises from the optimal policy in Theorem 8 equals one period of future controllable demand plus the sum of past errors (uncontrollable portion of demand) during the lead time, i.e., $\epsilon[t, t+L)$. Consider an environment with stationary parameters and $L = 2$. We assume that the system starts with $\overline{s}_0 = \overline{x}_0$ so no order is placed in period 0. At the beginning of period 0, the manager observes x_0 and $w_{1,0}$ and anticipates demand according to the linear approximation $\widetilde{d}_0(x_0)$. Let $\overline{D}_0 = \widetilde{d}_0(x_0)$. Based on x_0, $w_{1,0}$ and $\overline{D}_0$, the manager can further antic-

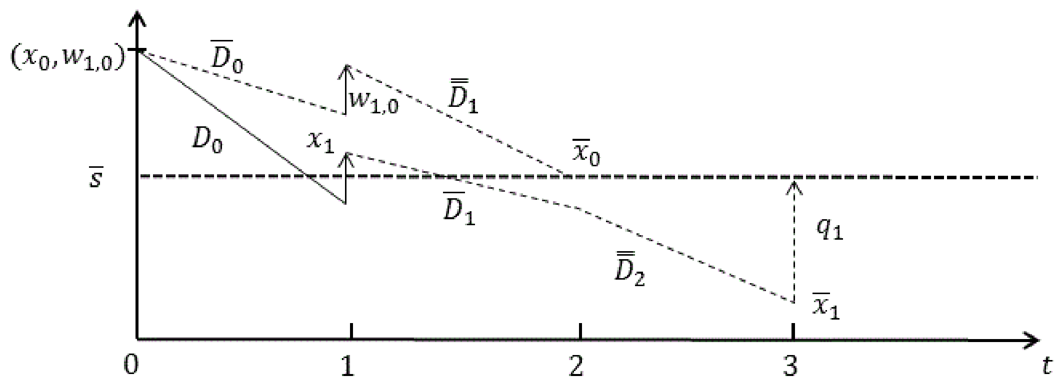


FIGURE 2.6: Heuristic replenishment decision.

ipate the net inventory level at the beginning of period 1, given by $x_0 + w_{1,0} - \overline{D}_0$, and therefore anticipate a demand quantity $\overline{\overline{D}}_1 = \tilde{d}_1\left(x_0 + w_{1,0} - \overline{D}_0\right)$ in period 1. One period later, demand D_0 is realized, so the manager can update $\overline{\overline{D}}_1$ by a new estimate $\overline{D}_1 = \tilde{d}_1\left(x_0 + w_{1,0} - D_0\right)$ that accounts for the actual initial net inventory level in period 1. At that point, the manager also estimates the demand in period 2 based on the linear approximation of demand in that period. It follows that $q_1 = \overline{\overline{D}}_2 + \left(D_0 - \overline{D}_0\right) + \left(\overline{D}_1 - \overline{\overline{D}}_1\right) = \overline{\overline{D}}_2 + \epsilon[0, 2)$. That is, the order quantity in each period equals the estimate of next-period's demand (through the linear approximation of demand in that period) plus a sum of correction terms of past demand estimations given the updated demand realized in the latest period (= the sum of the uncontrollable demand). Figure 2.6 illustrates this ordering behavior. [10].

2.5 Upper Bound

In this section, we derive an upper bound to the expected profit of the exact problem. The idea behind the upper bound is based on the duality approach proposed in Brown et al. (2010). The construction of the upper bound involves relaxing the information

[10] This ordering behavior bears resemblance to the ordering pattern in a system in which demand follows an MMFE model and ordering decisions are based on updates of the demand forecast. See, for example, Chen and Lee (2010).

set and imposing a penalty cost to compensate for the information relaxation. A key feature of Brown et al.s framework is the development of a problem-specific penalty cost that closes the gap created by the relaxed information set and that results in a problem that can be solved efficiently. We next proceed to generate a penalty function for our problem and show how to solve the resulting upper bound.

First, we relax the information set by considering the joint pricing and replenishment problem under deterministic demand. We define the sample-path dependent counterparts of the functions defined in Section 2.3. Given a demand sample path $\{\epsilon\}_1^T$, let $R_t(d_t \mid \epsilon_t) = p_t(d_t)D_t(d_t, \epsilon_t)$ and $G_t(x_t, d_t \mid \epsilon_t) = h_t(x_t - D_t(d_t, \epsilon_t))^+ + b_t(D_t(d_t, \epsilon_t) - x_t)^+$. Define the recursion $V_{T+1} \equiv 0$, and

$$\begin{aligned} V_t(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) &= \max_{q_t \geqslant 0, d_t \in \Omega_t} \Big\{ R_t(d_t \mid \epsilon_t) - c_t q_t - G_t(x_t, d_t \mid \epsilon_t) \\ &\qquad + \alpha V_{t+1}(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \Big\} \\ &= \max_{q_t \geqslant 0, d_t \in \Omega_t} J(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T), \end{aligned}$$

where $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t), w_{2,t}, \cdots, w_{L-1,t}, q_t)$. Then,

$$\mathsf{E} V_t(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) \geqslant V_t(x_t, \mathbf{w_t}) \quad \text{for} \quad t = 1, \cdots, T+1,$$

due to Jensen's inequality and the concavity of the value function, where the expectation is the average value of $V_t(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$ over all sample paths. Intuitively, a higher profit is achieved if demand is revealed before making decisions.

Next, following Proposition 2.2 of Brown et al. (2010), we construct the penalty cost function for each period t and each sample path $\{\epsilon\}_t^T$ as

$$W_t(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T) - W_t(x_t, \mathbf{w_t}, q_t, d_t).$$

Thus, the upper bound is obtained by taking the average over all sample paths of

$$\begin{aligned} V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) &= \max_{q_t \geqslant 0, d_t \in \Omega_t} \Big\{ R_t(d_t \mid \epsilon_t) - c_t q_t - G_t(x_t, d_t \mid \epsilon_t) \\ &\quad - \left[W_t(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T) - W_t(x_t, \mathbf{w_t}, q_t, d_t) \right] \\ &\quad + \alpha V_{t+1}^D (x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \Big\}, \end{aligned} \tag{2.6}$$

where $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t), w_{2,t}, \cdots, w_{L-1,t}, q_t)$. (The superscript D refers to the deterministic value function that involves the penalty cost.)

The penalty function needs to capture as much of the value of demand information as possible. To that end, we use the value function obtained from our heuristic in (2.5), with the appropriate corresponding definitions to denote the dependence on a given sample path. That is, the penalty function is constructed using $\widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right)$ and $\widetilde{J}_t(\overline{y}_t)$, where $\widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right)$ is the sample-path dependent counterpart of $\widetilde{J}_t(\overline{y}_t)$, and is defined as

$$\begin{aligned} \widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right) &= \Big\{ \alpha^L \widetilde{R}_{t+L}\left(\widetilde{d}_{t+L}(x_{t+L}) \mid \epsilon_{t+L}\right) - c_t \overline{y}_t - \alpha^L G_{t+L}\left(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L}) \mid \epsilon_{t+L}\right) \\ &\quad + \alpha \widetilde{V}_{t+1}\left(\overline{x}_{t+1} \mid \{\epsilon\}_{t+1}^T\right) \Big\}. \end{aligned}$$

Because these functions are concave, the expression to maximize in (2.6) is not necessarily concave. We therefore further approximate $\widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right)$ and $\widetilde{J}_t(\overline{y}_t)$ by their first-order Taylor expansions around $\overline{s}_t = \arg\max \widetilde{J}_t(\overline{y}_t)$. More specifically, let

$$\begin{aligned} W_t(\overline{x}_t, q_t \mid \{\epsilon\}_t^T) &= \widetilde{J}_t\left(\overline{s}_t \mid \{\epsilon\}_t^T\right) + \frac{\partial \widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right)}{\partial \overline{y}_t}\Big|_{\overline{y}_t = \overline{s}_t} (\overline{y}_t - \overline{s}_t), \\ W_t(\overline{x}_t, q_t) &= \widetilde{J}_t(\overline{s}_t). \end{aligned} \tag{2.7}$$

[11] Incorporating this penalty function into the recursion defined in (2.6) results in a

[11] Denote $\overline{s}_t(\{\epsilon\}_t^T) = arg\max \widetilde{J}_t(\overline{y}_y | \{\epsilon\}_t^T)$. Then we can alternatively construct the penalty function

dynamic program with value function

$$V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) = \max_{q_t \geqslant 0, d_t \in \Omega_t} \Big\{ R_t(d_t, \epsilon_t) - (c_t + \bar{c}_t) q_t - \bar{c}_t \bar{x}_t - G_t(x_t, d_t, \epsilon_t) + \alpha V_{t+1}^D(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \Big\}, \tag{2.8}$$

where $\bar{c}_t = \frac{\partial \widetilde{J}_t(\bar{y}_t | \{\epsilon\}_t^T)}{\partial \bar{y}_t}\Big|_{\bar{y}_t = \bar{s}_t}$.

Theorem 9. *The average value of* $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_t^T)$ *taken over all sample paths is an upper bound to the optimal profit* $V_1(x_1, \mathbf{w_1})$ *of the joint pricing and replenishment problem given in* (2.1).

Calculating the optimal value $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$ involves solving a multi-state dynamic program. We next show that the problem is, in fact, related to the dynamic lot-sizing problem studied in Wagner and Whitin (1958). The following result sets the stage for this connection.

Lemma 10. $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T) = \Gamma(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T) + \widehat{V}_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$, *where* $\Gamma(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$ *is independent of the decision variables* $(q_t, d_t)_{t=1,\cdots,T}$ *and the second term is defined recursively as* $\widehat{V}_{T+1}^D \equiv 0$, *and*

$$\widehat{V}_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) = \max_{q_t \geqslant 0, d_t \in \Omega_t} \Big\{ R_t(d_t, \epsilon_t) + \widehat{c}_t^d d_t - \widehat{c}_t q_t - G_t(x_t, d_t, \epsilon_t) + \alpha \widehat{V}_{t+1}^D(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \Big\}, \quad \textit{with} \tag{2.9}$$

$$\widehat{c}_t = c_t + \bar{c}_t + \sum_{k=t+1}^{T-L} \alpha^{k-1} \bar{c}_k \Pi_{j=0}^{L-1}(1 - \delta_{k+j}),$$

$$\widehat{c}_t^d = \begin{cases} \sum_{k=t+1}^{T-L} \alpha^{k-1} \bar{c}_k \Pi_{j=0}^{L-1}(1 - \delta_{k+j}), & \textit{additive demand} \\ \sum_{k=t+1}^{T-L} \alpha^{k-1} \bar{c}_k \Pi_{j=0}^{L-1}(1 - \delta_{k+j}) \epsilon_k, & \textit{multiplicative demand,} \end{cases}$$

---

by taking the first order Taylor expansion around $\bar{s}_t(\{\epsilon\}_t^T)$, i.e., $W_t(\bar{x}_t, q_t | \{\epsilon\}_t^T) = \widetilde{J}_t(\bar{s}_t(\{\epsilon\}_t^T) | \{\epsilon\}_t^T)$, $W_t(\bar{x}_t, q_t) = \mathsf{E}\Big[\widetilde{J}_t(\bar{s}_t(\{\epsilon\}_t^T)) + \frac{\partial \widetilde{J}_t(\bar{y}_t)}{\partial \bar{y}_t}\big|_{\bar{y}_t = \bar{s}_t(\{\epsilon\}_t^T)}\Big]$. Numerically, the performance of the resulting upper bound is similar to the one with the penalty function 2.7.

for $t \leqslant T-L-1$*, and* $\widehat{c}_t = c_t + \overline{c}_t$*,* $\widehat{c}_t^d = 0$ *for* $T-L \leqslant t \leqslant T$*.*

Lemma 10 states that for a given deterministic path $\{\epsilon\}_1^T$, we can recursively write the dynamic program in (2.8) as the sum of two terms. Because the first term $\Gamma(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$ is independent of the decision variables, the optimal solution for $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_t^T)$ is the same as that for the dynamic program given by $\widehat{V}_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$. For a given sample path $\{\epsilon\}_1^T$, the problem in (2.9) is equivalent to the dynamic lot-sizing problem studied in Wagner and Whitin (1958). The authors provide an algorithm to solve the dynamic lot-sizing problem. The output of this algorithm consists of the optimal ordering time for a unit of demand in each period t, together with the corresponding inventory holding and backorder costs associated with that ordering time. We propose a similar algorithm to compute the optimal ordering times, and we then use these ordering times and resulting holding and backorder costs as inputs to the demand optimization problem. This leads to the optimal value $\widehat{V}_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$. Note that we need to solve the first L periods in the initialization stage before running the algorithm. We provide the details of the optimization algorithm below.

Algorithm for $\widehat{V}_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$

(Initialization) Compute the optimal demands (equivalently, prices) for the first L periods: $d_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T), \cdots, d_L^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$.

1. (Wagner and Whitin) For a unit of demand in a given period $t > L$, determine the optimal ordering period $\tau^*(t)$ by comparing the costs obtained from all possible order periods τ. This can be determined by comparing:

   (*i*) If $\tau = t - L$, no inventory cost is incurred. The resulting (purchase) cost incurred in period t is $\widehat{c}_{t-L}/\alpha^L$;

(*ii*) If $\tau = 1, 2, \cdots, t-L-1$, the order is delivered before demand occurs. Therefore, a holding cost is incurred in periods $\tau + L, \cdots, t-1$. The resulting cost incurred in period t is $\widehat{c}_\tau/\alpha^{t-\tau} + \sum_{k=\tau+L}^{T-1} h_k/\alpha^{t-k}$;

(*iii*) If $\tau = t-L+1, \cdots, T-L$, the order is delivered after demand occurs. Therefore, a backorder cost is incurred in periods $t, \cdots, \tau + L - 1$. The resulting cost incurred in period t is $\widehat{c}_\tau/\alpha^{t-\tau} + \sum_{k=t}^{\tau+L-1} \alpha^{k-t} b_k$.

The optimal ordering time $\tau^*(t)$ is the period τ with the smallest cost.[12]

2. Denote by $\widehat{c}_t^*$ the cost associated with the optimal ordering time $\tau^*(t)$ determined in Step 1. The optimal expected demand in each period t is then computed as follows:

$$d_t^D(\{\epsilon\}_{t=1}^T) = \arg\max_{d_t \in \Omega_t} \left[R_t(d_t \mid \epsilon_t) + \widehat{c}_t^d d_t - \widehat{c}_t^* d_t \right].$$

The corresponding order quantity for period t is given by

$$q_t^D(\{\epsilon\}_{t=1}^T) = \sum_{j=1}^T d_j^D(\{\epsilon\}_{t=1}^T) \ \mathbf{1}_{\{\tau^*(j)=t\}},$$

where $\mathbf{1}_{\{\tau^*(j)=t\}}$ is the indicator function that equals 1 if $\tau^*(j) = t$ and equals 0, otherwise.

After determining the optimal solution $(q_t^*, d_t^*)_{t=1,\cdots,T}$ for each demand sample path following the algorithm described above, we compute the upper bound $\mathsf{E}[V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)]$ by averaging $V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$ over all sample paths.

2.6 Numerical Study

In this section, we present the results of our numerical study. We first report the performance of our heuristic and then discuss insights related to the impact of lead

[12] Note that $\{\widehat{c}_t\}$ depends on the sample path, so these parameters may not be stationary even if the original problem has stationary parameters.

time on the joint pricing and inventory decisions.

2.6.1 Performance of the Heuristic

We examine the effectiveness of our heuristic, which we denote BLS, by comparing the resulting profit to that of the exact optimal solution (when lead time is short) and to the profit derived from the upper bound. We also compare our heuristic with the heuristic proposed by Federgruen and Heching (1999), which we denote FH. The FH heuristic determines a price in each period and assumes that this price will be maintained over the next lead-time periods. Based on this assumption, the inventory states can be aggregated into a single variable and the heuristic policy takes the form of a list-price base-stock policy. In their paper, this heuristic is examined in settings with additive demand, so we extend it to the multiplicative demand case in our numerical study.

We consider a total of 3,360 problem instances, including a set of 3,240 instances with stationary parameters (648 instances with each $L \in \{1, 2, 3, 4, 5\}$) and a set of 120 problem instances with non-stationary parameters. The planning horizon is set at $T = 20$ and $\alpha = 0.95$. For the instances with stationary parameters, we consider scenarios with $c \in \{1.5, 2, 2.5\}$, $h \in \{0.4, 1, 4\}$, and $b \in \{10, 20, 50, 90\}$. For additive demand, we set $\lambda \in \{60, 90, 120\}$, $\mu \in \{0.5, 1, 1.5\}$, and $\epsilon \sim$ Normal$(0, 1)$; for multiplicative demand, we set $\lambda \in \{500, 700, 900\}$, $\mu \in \{1.1, 1.25, 1.5\}$, and $\epsilon \sim$ Gamma$(2, 0.5)$. These system parameters cover a wide range of scenarios, including scenarios with service levels ($= b/(b + h)$) ranging from 71% to 99%.

For instances with $L \leqslant 2$, we compute the exact optimal solution and use simulation to generate the expected optimal profit. For all instances, we compute the expected profit generated by the upper bound. We compute the BLS and FH heuristic policies for all instances and use simulation to obtain the corresponding expected profits. The simulation is conducted by generating $10,000$ randomly generated sample paths for each problem instance. For each sample path, we first calculate the average value of inventory states over all periods and then re-compute the profit associated with that sample path by taking the initial on-hand and pipeline inventory states equal to those average values.[13] We calculate the following percentage ratios to evaluate the performance of the BLS heuristic:

$$\frac{\{\text{upper bound, optimal}\}\text{ expected profit - BLS heuristic expected profit}}{\{\text{upper bound, optimal}\}\text{ expected profit}} \times 100\%.$$

This ratio represents the percentage error with respect to the optimal (upper bound) profit. Similarly, we examine the performance of the FH heuristic. Table 2.1 provides a summary of the results.

The BLS heuristic is near-optimal when $L \leqslant 2$. The average percentage error between the heuristic profit and the optimal profit is 0.61% among all instances with $L \leqslant 2$, with a maximum gap of 1.70%. The BLS heuristic significantly outperforms the FH heuristic. In general, our heuristic performs slightly better for the additive demand case. The performance of the BLS heuristic tends to deteriorate when the lead time becomes longer. For settings with $L \geqslant 3$, we compare the BLS heuristic with the upper bound. While the percentage gap increases as L becomes larger, this increase occurs at a decreasing rate. More specifically, the change in average percentage gap systematically decreases from 1.7 (= 2.19%/1.31%) when L goes from 1 to 2, to 1.3 (= 6.17%/4.83%) when L goes from 4 to 5. This observation suggests that the quality of the BLS heuristic will not deteriorate significantly for long lead times. For $L = 5$, the average percentage gap remains satisfactory at 6.17%. The gap between the optimal profit and the BLS heuristic profit is about one third of that between the upper bound and the heuristic when $L = 1$ and $L = 2$. (This proportion remains similar for a subset of experiments with $L = 3$ under which we

[13] With stationary parameters, the inventory states tend to be stationary after the initial warm-up periods.

Table 2.1: Average percentage errors – stationary cases

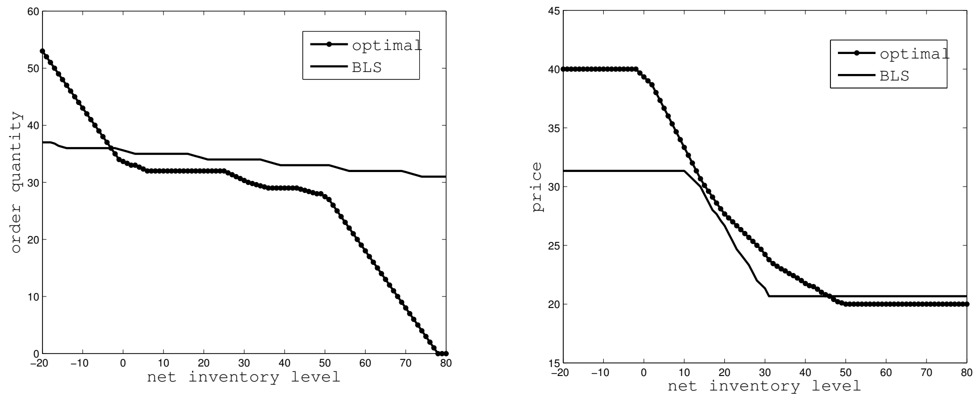
	Comparison to exact system				Comparison to upper bound		
	Additive		Multiplicative		Additive	Multiplicative	Average
	BLS	FH	BLS	FH	BLS	BLS	BLS
$L = 1$	0.32%	4.93%	0.52%	7.86%	1.34%	1.27%	1.31%
$L = 2$	0.61%	10.38%	0.71%	11.85%	2.08%	2.30%	2.19%
$L = 3$	–	–	–	–	3.49%	3.47%	3.48%
$L = 4$	–	–	–	–	4.91%	4.74%	4.83%
$L = 5$	–	–	–	–	6.36%	5.98%	6.17%

computed the profit of the exact system as well.) This suggests that the gap between our heuristic and the upper bound significantly overestimates the gap between the BLS heuristic and the exact system.

The following observations also arise from the numerical study. (1) The heuristic is more effective when the purchase cost c is relatively small because the myopic price is independent of c. Thus, when c is small, the gap between the myopic price and the optimal price is small as well. (2) The heuristic is generally more effective for large values of λ and small values of μ, as both result in larger revenues. (3) The performance of the heuristic is, in general, not sensitive to h and b.

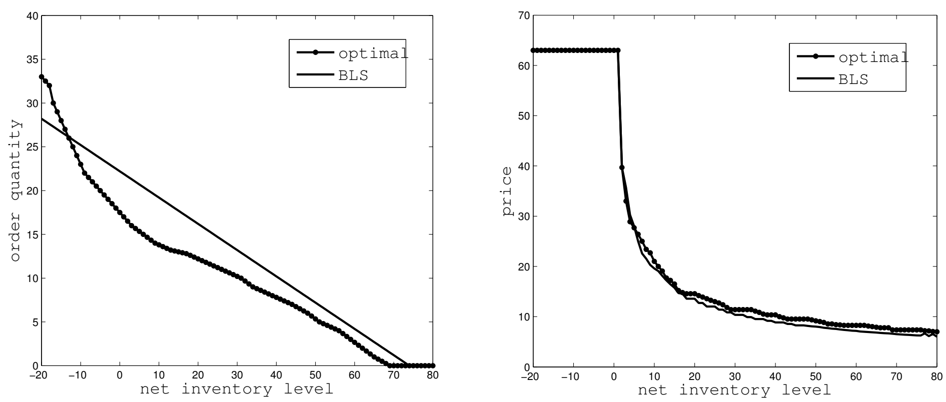
To further illustrate the performance of our heuristic, we compare the prices and order quantities in the BLS heuristic and the optimal policy. See Figures 2.7 and 2.8. The decisions under our heuristic are very close to the optimal decisions. The larger gaps occur for very low (negative) and very high net inventory levels under additive demand. These gaps are mainly the result of the linear approximation used to reduce the state space. Because these larger gaps occur for inventory levels that lead to relatively high backorder or inventory costs, they are less likely to be observed and therefore do not greatly affect the performance of our heuristic.

Finally, we report the results of the numerical study for settings with non-stationary parameters. We consider base cases with $c = 2$, $h = 1$, $b = 50$, and $\lambda = 60$, $\mu = 0.5, 1$ for additive demand, and $\lambda = 500$, $\mu = 1.25, 1.5$ for multiplica-



(a) order quantity (b) price

FIGURE 2.7: Policy comparison – additive demand, $L = 2$, $w_{1,0} = 10$, $T = 20$, $c = 2$, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 1.5$.



(a) order quantity (b) price

FIGURE 2.8: Policy comparison – multiplicative demand, $L = 2$, $w_{1,0} = 10$, $T = 20$, $c = 2$, $h = 1$, $b = 20$, $\lambda = 500$, $\mu = 1.5$.

tive demand. The cost and demand parameters are varied one at a time, following one of six patterns: increasing; decreasing; jump up / jump down (the cost takes a constant value for the first half of the planning horizon, then increases [decreases] to a higher [lower] level and remains at that level); and seasonal up / seasonal down (the cost increases [decreases] gradually in the first half of the planning horizon and then decreases [increases] gradually in the second half). For each instance with non-stationary parameters, we set the initial states to equal those that are used for

Table 2.2: Average percentage error - non-stationary cases

	Comparison to exact system		Comparison to upper bound	
	Additive	Multiplicative	Additive	Multiplicative
$L = 1$	0.36%	0.66%	1.14%	1.74%
$L = 2$	0.89%	1.03%	2.29%	2.52%
$L = 3$	–	–	3.41%	3.68%
$L = 4$	–	–	4.67%	4.91%
$L = 5$	–	–	5.39%	6.03%

the (stationary) base case. Table 2.2 reports the profit difference between the BLS heuristic and the exact system for $L \in \{1, 2\}$, and the profit gap with the upper bound for all values of L. The numerical results suggest that the our heuristic is also very effective for non-stationary systems.

2.6.2 Pricing and Responsiveness

The order quantity and price functions in Figures 2.7 and 2.8 suggest that price tends to be more sensitive to changes in the net inventory level than the order quantity. In this section, we examine the value of responsiveness (e.g., through the implementation of a quick response strategy) vis-à-vis the use of price controls to balance supply and demand. We base our findings on the numerical study with non-stationary parameters. In particular, we consider settings in which the unit purchasing cost c_t follows one of the following patterns: increasing, decreasing, jump up, or jump down. To avoid the impact of discounting, we set $\alpha = 1$. From this study, we conclude that price reacts more to the change in cost as lead time increases. More precisely, the range of prices charged over the planning horizon increases as lead time increases. This suggests that, as lead time becomes longer, pricing becomes a more useful lever to balance supply and demand due to the delay in receiving shipments. We also examine the effect of price sensitivity μ (of the mean demand functions) on the range of prices charged over the planning horizon. As the price sensitivity increases, the length of the price-range decreases. A large value of μ means that

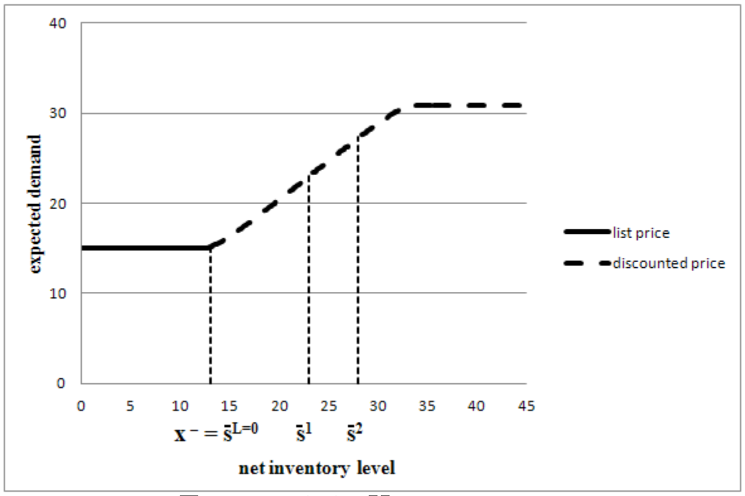


FIGURE 2.9: Heuristic price

demand is more sensitive to price changes. Therefore, a relatively smaller adjustment in price is sufficient to control demand as the cost (and therefore the order quantity and inventory levels) changes.

We next study how a change in the lead time impacts the replenishment policy. The next result shows that the base-stock level increases with the lead time.

Theorem 11. *Consider two systems with different lead times L_1 and L_2, identical stationary parameters, and $\alpha = 1$. Let the terminal condition for system L_i be defined as $\widetilde{V}_{T+1} = c\,\overline{x}_{T-L_i+1}$ and let $\overline{s}^i$ denote the corresponding base-stock level derived from Theorem 8. Then, $L_1 \leqslant L_2$ implies that $\overline{s}^1 \leqslant \overline{s}^2$.*

When $L = 0$ and demand is additive, $\overline{s}^{L=0}$ coincides with x^- (defined in Section 4.2) and the myopic price is a good approximation of the list-price defined in Federgruen and Heching (1999). Under the base-stock list-price policy, the price is discounted when the inventory level is higher than the base-stock level $\overline{s}^{L=0}$. As the lead time increases, Theorem 11 shows that the price is discounted even if the inventory level is below the base-stock level, in contrast to a setting with zero lead time. This is illustrated in Figure 2.9. In the case of positive lead time, it may be necessary to discount the price even if the on-hand inventory is below the base-stock level in anticipation of the inventory in-transit that will arrive in subsequent periods.

We finally use the numerical study to explore the impact of demand uncertainty on prices. The additive and multiplicative demand forms exhibit a different dependence on the random terms. Specifically, the demand variance is independent of price under additive demand, but it is decreasing in price under multiplicative demand. This results in different findings regarding the change of prices driven by changes in demand uncertainty under the different demand forms. Under additive demand, we find that the range of prices charged over the planning horizon increases in demand variance, i.e., when demand is more variable, prices fluctuate more in order to balance supply and demand. Under multiplicative demand, the range of prices shifts upwards as demand variability increases, i.e., a relatively higher price is charged in a more variable system. In this case, an increase in price leads to a reduction in mean demand, counteracting the increased variance of the random term.

2.7 Conclusion

In this chapter, we study the joint inventory and pricing control problem for a single-stage system with positive lead time. Demand is price sensitive and is either of the additive or of the multiplicative form. A replenishment decision and a selling price are determined simultaneously in each period. The computation of the optimal policy in this system is computationally intractable. We provide a simple and effective heuristic for this problem in which the decisions about the pricing and the replenishment strategies are separated. First, the price (or, equivalently, the expected demand) decision is determined by solving a series of single-period problems, leading to a myopic pricing policy. We then propose a linear approximation to the myopic policy as a function of the inventory level. This approximation allows us to reduce the dimension of the state space. The single state variable is the so-called price-deflated inventory position, which is a weighted sum of inventories in the system.

The weights are determined by the slopes of the linear approximation, which measure the sensitivity of price to the inventory level. We show that a base-stock policy is optimal in the approximated system. We also derive an upper bound to the exact system and show how to efficiently compute the resulting expected profit.

We evaluate the performance of our heuristic by comparing it to the exact system (for $L \leqslant 2$) and to the upper bound. Under both types of demand functions, our heuristic is nearly optimal when compared to the profit of the exact system. The relative gaps with the upper bound suggest that the heuristic performs very well for longer lead times. Finally, we discuss the impact of lead time on the pricing and inventory decisions. We find that, under both types of demand functions, price becomes an efficient lever to balance supply and demand as lead time increases. We also find that price discounts may be offered in anticipation of the upcoming inventory in-transit.

3

Joint Inventory and Pricing Problems with Lead Time and Fixed Ordering Costs

3.1 Introduction

In this chapter, we still focus on the integration of inventory control and pricing decisions. Different from the previous chapter, we consider systems where a fixed ordering cost occurs whenever an order is placed. In standard inventory problems (with price fixed), the studies with and without fixed ordering costs are parallel, as these two phenomena co-exist in practice. However, when it comes to joint inventory and pricing problems, especially with positive lead time, there is little study for the settings with fixed ordering costs. This is because, as both the fixed ordering cost and the positive lead time present in the system, the problem is computationally intractable, and the structure of objective function becomes unclear.

We show in this chapter that the same myopic pricing policy, linear approximation and resulting state reduction derived in Chapter 2 can be applied to construct the heuristic pricing and replenishment policies in a setting with fixed ordering costs. Specifically, we still use the myopic pricing policy, which depends only on the on-hand inventory level, as the heuristic pricing policy. Using the linear approximation of the myopic demand function, we show that an (s, S) policy is optimal based on the *price-deflated* inventory position defined in Chapter 2. That is, if the *price-deflated* inventory position is below s, then an order is placed to bring it up to S; otherwise, no order is placed. To our knowledge, this is the first heuristic for joint inventory and pricing optimization in a setting with fixed ordering costs.

3.2 Literature Review

In a periodic-review system with fixed ordering costs and fixed price (standard inventory system), Scarf (1960) shows that the (s, S) policy is optimal for general lead time, i.e, if the inventory position is below s, an order is placed to reach S; otherwise, no order should be placed. In this paper, he also first introduces the concept of $K-$concavity. In a continuous-review setting, the (r, Q) policy is proved to be optimal by Galliher et al. (1959), where r is the reorder point and Q is the lot size.

In the literature of joint inventory and pricing decisions, Chen and Simchi-Levi (2004 a,b) study a single-stage, periodic-review, multi-period planning model with fixed ordering costs. They assume the delivery lead time is zero and prove that an (s, S, p) policy and an (s, S, A, p) policy are optimal under additive demand and a general form of demand (involving both a multiplicative and an additive demand term), respectively. For the infinite-horizon model, they prove that the (s, S, p) policy is optimal for general demand functions. Extensions and refinements of this problem include Chen et al. (2006), Huh and Janakiraman (2008), and Song et al. (2009). Chen et al. (2006) and Song et al. (2009) extend the optimality of the (s, S, p) and (s, S, A, p) policies to lost-sale models with additive and multiplicative demands, respectively, under mild assumptions on the demand function and randomness. Huh and Janakiraman (2008) consider a multi-dimensional demand control approach in both backorder and lost-sales models. The authors show the optimality of a so called $(s,S)-$type policy with stationary parameters. The $(s,S)-$type policy is a generalization of the (s,S,p) policy.

3.3 Model

The basic model and notation is the same as those in Chapter 2. We define the function $I(u)$ as $I(u)=1$, if $u>0$, and $I(u)=0$ if $u\leqslant 0$. Assume that a fixed ordering cost K is incurred whenever an order is placed. That is, the total purchasing cost in period t is given by $KI(q_t)+c_tq_t$, with q_t the order quantity in that period. Let $V_t^K(x_t,\mathbf{w_t})$ denote the maximum expected discounted profit from period t to the end of the planning horizon with initial state vector $(x_t,\mathbf{w_t})$. Then, $V_t^K(x_t,\mathbf{w_t})$, $t=1,\cdots,T+1$, satisfy the following value-function recursion:

$$
\begin{aligned}
V_{T+1}^K &\equiv 0 \\
V_t^K(x_t,\mathbf{w_t}) &= \max_{q_t\geqslant 0, d_t\in\Omega_t}\Big\{R_t(d_t)-KI(q_t)-c_tq_t-G_t(x_t,d_t) \\
&\qquad + \alpha \mathsf{E} V_{t+1}^K(x_{t+1},\mathbf{w_{t+1}})\Big\}, \qquad (3.1)
\end{aligned}
$$

where the state dynamics are $(x_{t+1},\mathbf{w_{t+1}})=(x_t+w_{1,t}-D_t(d_t,\epsilon_t),w_{2,t},\cdots,w_{L-1,t},q_t)$.

3.4 Heuristic

We define the myopic demand $d_t^M(x_t)$ function as the solution to

$$\max_{d_t\in\Omega_t}\{R_t(d_t)-G_t(x_t,d_t)\}.$$

Follow the same procedure in Section 2.4.2, we construct the linear approximation $\widetilde{d}_t(x_t)=\delta_tx_t+\kappa_t$, both in the cases of additive and multiplicative demands as follows:

Additive demand: Define

$$\delta_t = d_t'^M(\hat{x}_t), \quad \kappa_t = -\delta_t(\hat{x}_t)+d_t^M(\hat{x}_t),$$

$\hat{x}_t$ is the average of two points x_t^+ and x_t^-, where $x_t^+ = \min\{x_t^u, x_t^{ub}\}$, $x_t^- = \min\{x_t^l, x_t^{lb}\}$. x_t^u is the solution to $d_t^M(x_t) = (\lambda_t + \mu_t h_t)/2 - \zeta$ and $x_t^{ub} = \min\left\{x_t : \; d_t^M(x_t) = \lambda_t\right\}$. x_t^l is the solution to $d_t^M(x_t) = (\lambda_t - \mu_t b_t)/2 + \zeta$ and $x_t^{lb} = \max\left\{x_t : \; d_t^M(x_t) = 0\right\}$.

Multiplicative demand: Define

$$\delta_t = (C_t^* + d_t'^M\left(x_t^M\right))/2, \quad \kappa_t = C_{0t},$$

where $x_t^M = \min\{x_t \geqslant 0 : d_t^M(x_t) = x_t\}$, C_t^* that solves $\int_{1/C}^{+\infty} \epsilon dF(\epsilon) = \frac{h_t}{h_t+b_t}$ and $C_{0t} = \lambda_t \left[\frac{(1-1/\mu_t)}{b_t}\right]^{\mu_t}$.

By substituting the expected demand decision d_t in 3.1 with $\widetilde{d}_t(x_t) = \delta_t x_t + \kappa_t$, we get an approximate inventory problem with fixed ordering costs. Since the pricing (demand) decisions have been fixed as the linear approximation, we can further change the accounting scheme in this inventory problem by calculating the profit L period forward and obtain the following value functions:

$$\begin{aligned}
\widetilde{V}_{T+1}^K &\equiv 0 \\
\widetilde{V}_t^K(\overline{x}_t) &= c_t\overline{x}_t + \max_{\overline{y}_t \geqslant \overline{x}_t} \mathsf{E}\{\alpha^L \widetilde{R}_{t+L}(\widetilde{d}_{t+L}(x_{t+L})) - KI(\overline{y}_t - \overline{x}_t) - c_t\overline{y}_t \\
&\quad -\alpha^L G_{t+L}(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L})) + \alpha \widetilde{V}_{t+1}^K(\overline{x}_{t+1})\}, \qquad (3.2)
\end{aligned}$$

The dimension of the state space in 3.2 depends on the dimension of x_{t+L} since the single-period's profit depends on x_{t+L} only. Follow a similar analysis in Section 2.4.3, we can express x_{t+L} in terms of the price-deflated inventory position $\overline{x}_t$, i.e.,

$$x_{t+L} = \overline{y}_t - \epsilon[t, t+L) = \overline{x}_t + q_t,$$

and $\overline{x}_{t+1} = (1 - \delta_{t+L})[\overline{y}_t - \epsilon[t, t+1)] - \kappa_{t+L}$, $\epsilon[t, t+l)$ is defined as in Section 2.4.3. Therefore, the state space of the approximate inventory problem is reduced to one dimension.

Table 3.1: Average percentage error – Fixed ordering cost

	Additive demand		Multiplicative demand	
	$K = 50$	$K = 100$	$K = 50$	$K = 100$
$L = 1$	0.35%	0.39%	0.85%	0.93%
$L = 2$	0.59%	0.68%	1.25%	1.29%
$L = 3$	0.90%	0.97%	1.61%	1.70%

Lemma 12. *(Scarf 1960) Let $K \geqslant 0$. We say that $H(x)$ is $K-$convex if*

$$K + H(\beta + x) - H(x) - \beta \frac{H(x) - H(x - \gamma)}{\gamma} \geqslant 0$$

for all positive β and γ and all x.

Note that if $H(x)$ is $K-$convex, then $-H(x)$ is $K-$concave.

Define

$$\widetilde{J}_t^K(\overline{y}_t) = \mathsf{E}\{\alpha^L \widetilde{R}_{t+L}(\widetilde{d}_{t+L}(x_{t+L})) - c_t \overline{y}_t - \alpha^L G_{t+L}(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L})) + \alpha \widetilde{V}_{t+1}^K(\overline{x}_{t+1})\}.$$

Theorem 13. *The function $\widetilde{J}_t^K(\overline{y}_t)$ and the value function $\widetilde{V}_t^K(\overline{x}_t)$ are K-concave in $\overline{y}_t$ for all t. Let $\overline{S}_t$ be the smallest value of $\overline{y}_t$ that maximizes $\widetilde{J}_t^K(\overline{y}_t)$. Let $\overline{s}_t$ be the largest value of $x \leqslant \overline{S}_t$ satisfying $\widetilde{J}_t^K(x) = K + \widetilde{J}_t^K(\overline{S}_t)$. The optimal replenishment policy in each period t takes the form of an $(\overline{s}_t, \overline{S}_t)$ policy, i.e., if $\overline{x}_t \leqslant \overline{s}_t$, then order $\overline{S}_t - \overline{x}_t$; otherwise, do not order.*

The optimal replenishment policy is combined with the myopic price and it is operationalized as described in Theorem 13. In the numerical section we report the performance of this heuristic.

3.5 Numerical Performance

We examine the performance of the heuristic in settings with a fixed ordering cost by considering 192 instances (96 each for additive and multiplicative demand) with stationary parameters. The parameters under consideration are: $c \in \{1.5, 2\}$, $h = 1$,

$b \in \{20, 50\}$, and $\lambda \in \{60, 70\}$, $\mu \in \{1, 2\}$ for additive demand, and $\lambda \in \{300, 500\}$, $\mu \in \{1.25, 1.5\}$ for multiplicative demand. Table 3.1 provides a summary of the results. The heuristic is also effective for systems with fixed ordering costs. The average percentage gap between our heuristic and the optimal policy is below 1% under additive demand and below 2% under multiplicative demand. The maximum [minimum] gap is 1.21% [0.11%] under additive demand and 2.10% [0.34%] under multiplicative demand.

4

On the Profitability of an Eco-Friendly Production System

In this chapter, we seek to examine the effect of the procurement cost and its volatility on a firm's profit. This allows us to study under what conditions a firm can be profitably operate an eco-friendly supply chain. To this end, we consider a manufacturer who can decide the extent of dependence on petroleum-based components for production. The procurement cost depends on the price of oil, which is itself quite volatile. This volatility can be alleviated by using eco-friendly components or production/distribution techniques that rely less on oil. Any of these alternatives may lead to a higher procurement cost. We model the price of oil as a time-correlated process, consistent with observed data. The manufacturer uses a mark-up pricing policy to determine the product's selling price, and demand is a function of price. The first goal is to determine the optimal production policy (i.e., mix and quantity) that maximizes profit in a finite horizon setting. We characterize conditions under which a state-dependent myopic policy is optimal. This allows us to shed light on how the effect of the procurement cost on the firm's profit. Our results suggest that operating an eco-friendly production system may not undermine the firm's profitability because the benefits of a less volatile procurement cost may outweigh the increased procurement cost. The magnitude of this benefit is determined by the firm's mark-up pricing policy and the customer's price elasticity. In particular, our results provide guidelines to understand what type of products would better absorb the higher costs associated with an eco-friendly production system.

4.1 Introduction

Society's growing concern for the environment has led many firms to integrating sustainable practices into their business processes. Eco-friendly practices and efforts have been made to various stages of a supply chain, including the procurement process, the distribution process and the consumption stage. For example, some firms utilize reused or renewable materials in their production processes. Hybrid fleet vehicles are used in the distribution system to save petroleum consumption. Wal-Mart encourages its suppliers to reduce the weight of packages, to lower oil consumption in the transportation process. Electric cars are promoted to reduce petroleum consumption of consumers. From the environmental point of view, these strategies lead to systems less dependent on oil, therefore reducing their environmental footprint. In practice, the implementation of these eco-friendly strategies is limited by the cost associated with their implementation and by the potential response from the market.

Our study focuses on the procurement stage of a production and distribution system. In particular, we evaluate the profitability of implementing a sustainable procurement strategy. This procurement strategy involves the substitution of petroleum-based inputs with eco-friendly, sustainable components. This practice is increasingly prevalent in several industries. From 2010, the Coca-Cola company started to replace the traditional plastic bottles, which are entirely made with components based on petroleum and other fossil fuels, with a so-called 'Plant Bottle'. This 'Plant Bottle' uses up to 30% of plant-based materials, which are mainly produced from sugar cane. In contrast to crude oil, sugar cane is easy to plant and fast growing, therefore making the system more sustainable. This kind procurement strategy is also common in the automobile industry. More than 30 parts of Mercedes-Benz's plastic components have been replaced by bio-plastic components made from natural fibers. In both examples, the look and the functionality of the products remain the same, but the use of eco-friendly components reduces the carbon footprint of the companies' production systems. From the cost perspective (see Figure 4.1), the price of standard PET plastic is highly correlated with the price of crude oil, which has an increasing trend and exhibits high volatility. As a result, products based on standard plastic exhibit similar cost patterns. On the other hand, bio-plastic is less dependent on oil, therefore generally facing a more stable cost pattern. Due to the limitation imposed by the necessary technology and the size of the market that consumes these components, the price of bio-plastic tends to be, as of now, around $2-4$ times higher than that of standard plastic. As the technology evolves and the market for these components expands, it is expected that the price of bio-plastic will decrease over time. Furthermore, the decrease in production costs will be transferred to consumers in the form of lower selling price for the products involved. According to a survey by USDA (US Department of Agriculture) 2012, the price of soft drinks has increased by 2% to 3% due to the increasing cost of the plastic package.

In view of the trade-off between higher procurement costs and lower volatility, we study the optimal procurement policy in an environment with price-dependent consumer demand. In particular, we consider a manufacturer who has access to two substitutable sources of components. One input is highly dependent on oil (for example, traditional plastic); the other input is not (for example, bio-plastic). The manufacturer determines the order quantities for each kind of input in order to maximize the total profit over a finite horizon. The cost of the oil-based component

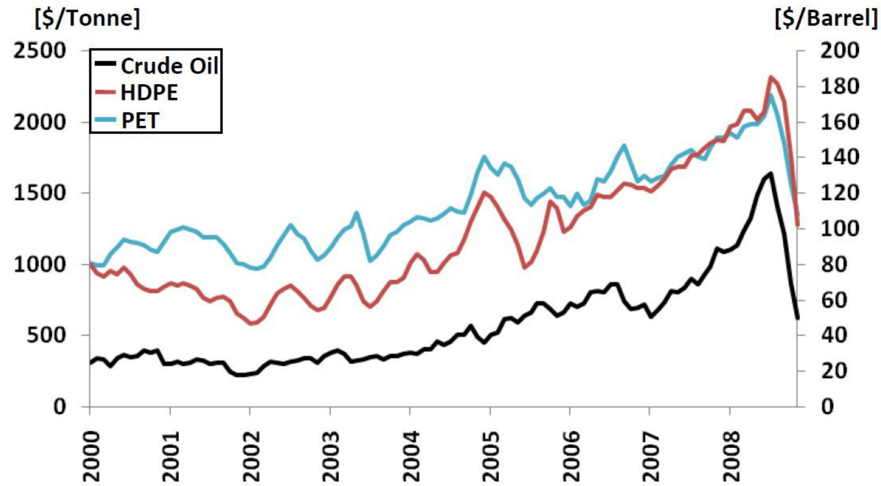


FIGURE 4.1: Price correlation between PET plastic and crude oil

is lower but more volatile fashion than that of the eco-friendly alternative. The firm uses a mark-up pricing policy, so variations in cost are partially transferred over to consumers. We first fix the proportion of traditional versus eco-friendly inputs used in production and determine the optimal associated procurement policy. The optimal procurement policy takes the form of a state-dependent-base-stock policy. To further understand the impact of the cost level and its volatility on the firm's procurement strategy, we focus on a myopic policy. We show that there exists an optimal proportion of traditional versus eco-friendly components that maximizes profit. The proportion of eco-friendly component used in production decreases with its cost and with the product's price-sensitivity. Given the volatility associated with the price of oil, a sustainable strategy can serve as a tool for operational hedging.

4.2 Literature Review

In the operations management field, there is a increasing number of papers addressing different issues related to the sustainability of supply chain operations. Plambeck and Taylor (2012) study how the feedstock intensity will affect the clean tech manufactures' profit when both the feedstock and output prices are uncertain. Benjaafar et al. (2013) study an inventory management problem with concerns of carbon emission. The paper also analyzes the effect of different emission regulations on supply chain collaboration. Agrawal et al. (2012) investigate the the environmental impact of leasing comparing to selling. Swamidas and Newell (1987) consider the influence of the manufacture strategy using an empirical approach. The authors show that the environmental uncertainty influences manufacturing strategy variables such as manufacturing flexibility. Goyal and Netessine (2011) use a theoretical framework to study the value of volume flexibility under an uncertain demand environment. Our study contributes to the stream of literature on sustainable operations by focusing on the impact of a procurement strategy that substitutes oil-based inputs with eco-friendly counterparts. Avci et al (2013) study the consumer adoption problem of electric vehicles and access the impact of a switching-station-based solution for the battery range limitation.

From the modeling perspective, our paper is also related to the work involving stochastic cost and demand processes. Fabian et al. (1959) present a solution to the problem of determining inventory decisions when the purchasing price of the raw material fluctuates from period to period. Sethi and Cheng (1997) and Chen and Song (2001) study an inventory problem under a Markov-Modulated demand process. Aviv (2003) and Chen and Lee (2009) incorporate more general time-series models to describe the demand process. In our paper, both the purchasing cost of the raw material and the demand process are stochastic and they are correlated. Moreover, the cost of the oil-based component is related to the oil price. DeMiguel et al. (2003) provide a framework to forecast the price of oil using a time-series models.

4.3 Model

We consider a system with a single manufacturer who faces a multi-period production planning problem with planning horizon T. The production lead time is denoted as L. The discount factor is α.

Procurement Process:

The manufacturer has access to two types of components: the oil-based input, which we denote with a subscript O, and the eco-friendly input, which we denote with a subscript E. The cost of each input varies stochastically over time. Specifically, we model the cost processes as AR(1) processes with trend. Let $C^i(t)$ denote the unit cost of input i at time t, $i = O, E$. Then, $C^O(t)$ and $C^E(t)$ are expressed as:

$$C^O_{t+1} = aC^O_t + \theta^O + \delta^O t + e^O_t$$
$$C^E_{t+1} = aC^E_t + \theta^E - \delta^E t + e^E_t$$

where $\delta^i \geqslant 0$ and e^i_t is a Normal random variable with mean 0 and standard deviation σ^i, $i = O, E$.

The cost of the oil-based input is positively correlated with the cost of the oil price, which generally exhibits an increasing trend (so $\delta^O \geqslant 0$). On the other hand, the eco-friendly input follows a decreasing trend due to the advances in technology and the rate of adoption in the market (so $\delta^E \geqslant 0$). The cost of the oil-based input is generally lower than that of the eco-friendly component. However, the price of oil is more volatile. We model this by assuming that $\mathsf{E}C^O(t) \leqslant \mathsf{E}C^E(t)$, for all t, and $\sigma^O \geqslant \sigma^E$. In this paper, we focus on the trade-off between a more volatile oil-based input and a more costly eco-friendly component.

The manufacturer chooses the proportion of eco-friendly components used in production, which we denote as ρ_t. Let $C(t)$ be the total unit procurement cost at time t. That is,

$$C(t) = \rho_t C^E(t) + (1 - \rho_t) C^O(t).$$

The cost $C(t)$ also follows an AR(1) process with trend, i.e.,

$$C(t+1) = aC(t) + \theta_t - \delta_t t + e_t,$$

where $\delta_t = \rho_t \delta^E - (1 - \rho_t)\delta^O$, $\theta_t = \rho_t \theta^E + (1 - \rho_t)\theta^O$ and $e_t \sim \text{Normal}(0, \sigma_t)$ with $\sigma_t = \sqrt{(\rho_t \sigma^E)^2 + ((1 - \rho_t)\sigma_O)^2}$. Note that θ_t measures the cost level of a product, while σ_t measures its volatility. As ρ_t increases with $\rho_t \leqslant \sigma^E/(\sigma^O + \sigma^E)$, we have that θ_t increases and σ_t decreases. That is, an increased proportion of the eco-friendly component leads to a more costly, but also more stable, cost process.

Demand Process:

In practice, manufacturers frequently adjust the selling price of the final product according to the change in their production costs. To capture the relationship between the production cost and the selling price, we assume that the manufacturer follows a mark-up pricing policy. Let $P(t)$ be the selling price at time t. Then,

$$P(t) = (1 + \beta)C(t),$$

where β is the mark-up level. Demand occurring in period t, denoted as $D(t)$, is stochastic and depends on price according to the following linear and additive relationship:

$$D(t) = \lambda - \mu P(t) + \epsilon_t,$$

where $\{\epsilon_t\}$ are i.i.d. Normal random variables with zero mean and standard deviation ξ.

From the relationship between $P(t)$ and $C(t)$, we can write the demand process as a function of the procurement cost, as follows:

$$D(t) = \lambda - \mu(1 + \beta)C(t) + \epsilon_t.$$

Then, $D(t)$ incorporates both the intrinsic uncertainty associated with demand and the volatility associated with the cost of components. The total production cost paid in each period is linear in the production quantity. A holding cost h and a backorder cost b are paid for each unit of leftover inventory and backorder in each period. We assume that these cost rates are constant over time.

In what follows, we first fix the proportions $(\rho_1, \rho_2, \cdots, \rho_T)$ and determine the optimal procurement policy in each period that maximizes the total expected profit through period T. Based on these results, we next examine the impact of incorporating eco-friendly components on the profitability of the firm. Note that in our model, the proportion of eco-friendly components used in production can be adjusted in every period. However, in practice, a firm may not have the flexibility to alter this proportion in every period; instead, it is usually fixed over time, i.e., there is a single decision about the proportion of eco-friendly components used in production at the beginning of the planning horizon which does not change over time. This scenario is a special case of our model. All the results in Section 4 carry through if we assume that ρ is constant over time.

4.4 Main Results

4.4.1 Optimality

Given $(\rho_1, \rho_2, \cdots, \rho_T)$, the optimal procurement decision can be solved from the following dynamic program. Let $V_t(x_t, c_t)$ denote the maximum expected profit from period t and through the end of the planning horizon, given that the cost at time t is c_t and that the initial inventory position is x_t. Then, $V_t(x_t, c_t)$ satisfies the following

Bellman equations:

$$\begin{aligned} V_{T-L} &= 0 \\ V_t(x_t, c_t) &= \max_{y_t \geqslant x_t} \mathsf{E}\{R(D(t)) - C(t)(y_t - x_t) - \alpha^L G(y_t - D^L(t)) \\ &\quad + \alpha \mathsf{E} V_{t+1}(y_t - D(t), C(t+1)) | C(t) = c_t\} \end{aligned}$$

where $R(D(t)) = \mathsf{E}D(t)P(t)$ and $G(\cdot) = \mathsf{E}\left[h(\cdot)^+ + b(\cdot)^-\right]$.

The next result characterizes the optimal replenishment policy.

Theorem 14. *Given an exogenous set of values $(\rho_1, \rho_2, \cdots, \rho_T)$, the optimal replenishment policy in each period is a state-dependent base-stock policy. Denote $s_t(c_t)$ as the base-stock level in period t given that the unit procurement cost is $C(t) = c_t$. If $x_t \leqslant s_t(c_t)$, then order $s_t(c_t) - x_t$; otherwise, do not order. Furthermore, we have that $s_t(c_t)$ is decreasing in c_t.*

Theorem 14 shows that the optimal ordering decision depends on the realization of the current period procurement cost. As the realized unit procurement cost increases, it becomes more costly to order. Then the optimal order-up-to level decreases, i.e., the chance of replenishment or the order quantity gets smaller.

4.4.2 Myopic Problem

To study the impact of incorporating eco-friendly components on the production system, we now study the trade-off between utilizing the eco-friendly and the oil-based components. These components differ in terms of their cost and volatility. To this end, we focus on the following myopic problem:

$$\max_{y_t \geqslant x_t} \mathsf{E}\{R(D(t)) - C(t)(y_t - x_t) - \alpha^L G(y - D^L(t)) + \alpha C(t+1)(y_t - D(t)) | C(t) = c_t\}$$

Below we examine in what settings the myopic policy is optimal.

Before solving the myopic problem, we first express the lead time demand $D^L(t)$ as $D^L(t) = \mathsf{E}D^L(t) + Ran(D^L(t))$, where $\mathsf{E}D^L(t)$ is deterministic and all random terms are included in $Ran(D^L(t))$. Define

$$
\begin{aligned}
K(a,t,L) &= \sum_{j=t}^{t+L-1} (j - (a + \cdots + a^j))/(1-a), \\
e(a,t) &= a^{t-1}e_1 + a^{t-2}e_2 + \cdots + e_t.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathsf{E}D^L(t) &= \lambda L - \mu(1+\beta)[LC(0) - K(a,t,L)\delta] \\
Ran(D^L(t)) &= \sum_{j=t}^{t+L-1} [\epsilon_j - \mu(1+\beta)e(a,j)]
\end{aligned}
$$

Proposition 15. *Given $(\rho_1, \rho_2, \cdots, \rho_T)$, the optimal myopic procurement policy in each period is a state-dependent base-stock policy. Denote $s_t^*(c_t)$ as the myopic base-stock level in period t. Then,*

$$
s_t^*(c_t) = \mathsf{E}[D^L(t)|C(t) = c_t] + stdev(Ran(D^L(t))|C(t) = c_t)z^*(c_t),
$$

where $z_t^* = \Phi^0(h + c_t - \alpha\mathsf{E}[C(t+1)|C(t) = c_t]/(h+b))$ and Φ^0 is the ccdf of standard normal distribution.

Proposition 15 shows that the myopic base-stock level can be expressed as the mean demand during the lead time plus a safety stock. The latter depends on both the intrinsic demand uncertainty and the volatility associated with the cost of components.

Note that in the traditional inventory problem, the optimality of the myopic base-stock policy is achieved when the myopic base-stock levels increase over time. Similarly, we require $s_1^*(C(1)) \leqslant s_2^*(C(2)) \leqslant \cdots \leqslant s_T^*(C(T))$ to guarantee that the myopic solution is optimal for the original problem. Note that the cost $C(t)$

in each period t is stochastic. Therefore, whether or not the above condition is satisfied depends on the specific realization of the cost process. Proposition **??** below characterizes a setting where the base-stock levels are increasing in time. In that setting, the myopic policy is optimal.

Proposition 16. *Consider the terminal condition $V_{T-L} = c_{T-L}x_{T-L}$. If the realization of the costs $C(t) = c_t$ and $C(t+1) = c_{t+1}$ satisfy*

$$c_{t+1} - c_t \leqslant \min\left\{\frac{\delta_t}{1+a}, \alpha\frac{\delta_{t+1}(t+1) - \delta_t t}{1-\alpha a}\right\}$$

for all t, then we have $s_1^*(c_1) \leqslant s_2^*(c_2) \leqslant \cdots \leqslant s_T^*(c_T)$. This implies that the myopic procurement policy is optimal.

As stated in Section 4.3, the cost process of the final product has a decreasing trend. However, the realization of the cost may either go up or down from period to period. An increasing cost implies an increasing selling price or a decreasing demand volume. In the traditional inventory problem, the myopic policy is not necessarily optimal if demand is stochastically decreasing over time. The latter can occur in our model if there is a jump in the procurement cost. However, Proposition 16 shows that, as long as the magnitude of the cost increase is constrained in a certain range, the myopic policy can still be optimal. Denote by $U_t = \min\left\{\frac{\delta_t}{1+a}, \alpha\frac{\delta_{t+1}(t+1)-\delta_t t}{1-\alpha a}\right\}$. Note that $C(t+1) - C(t) = a^t e_t - \frac{1-a^t}{1-a}\delta_t$. The probability that the myopic solution is optimal is given by

$$\Pi_{t=1}^{T} Pr(C(t+1) - C(t) \leqslant U_t) = \Pi_{t=1}^{T} Pr\left(e_t \leqslant \frac{U_t + (1-a^t)\delta_t/(1-a)}{a^t}\right).$$

Figure 4.3 provides an example of the discrepancy of the optimal policy and the myopic policy. The cost realization of this example is presented in Figure 4.2

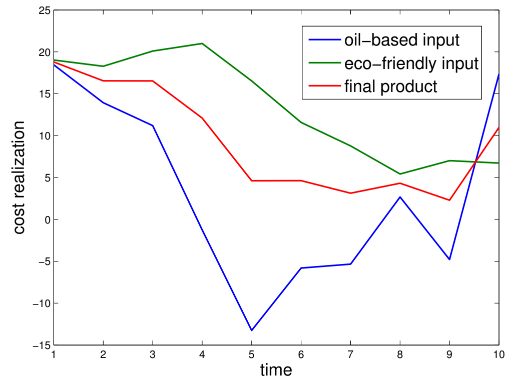
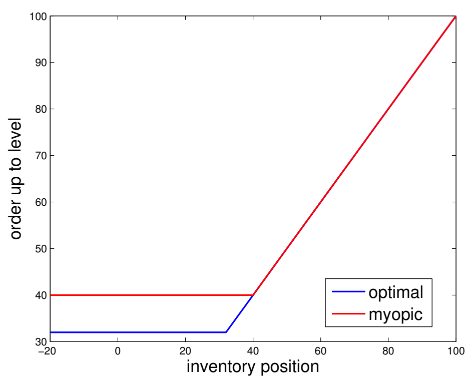
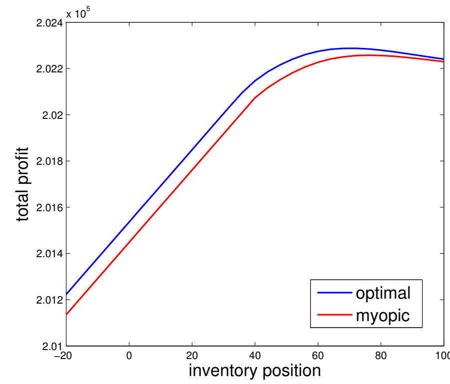


FIGURE 4.2: Cost realization



(a) Compare policies



(b) Compare profits

FIGURE 4.3: Policy and profit comparisons, $L = 2$, $T = 10$, $\rho = 0.6$, $a = 0.8$, $\sigma^O = 10$, $\sigma^E = 3$, $\theta^O = 1$, $\theta^E = 4$, $\delta^O = 0.1$, $\delta^E = 0.15$, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 1.5$.

We now explore the impact of employing an eco-friendly input in the production process. Viewing the problem from the beginning of the planning horizon, we study how a change in the proportion of the eco-friendly component used for production affects the whole production process in terms of the optimal replenishment decisions.

Proposition 17. *For every t, $s_t^*(C(0))$ decreases in ρ_t.*

By increasing the proportion of eco-friendly components used in production, the average unit production cost increases but also the cost process becomes more stable. Therefore, both the mean demand volume and the demand variance decrease. As shown in Proposition 15, the critical fractile for the myopic base-stock level $s_t^*(C(0))$ is defined as $h + \mathsf{E}[C(t) - \alpha C(t+1)|C(0)]/(h+b)$. With a higher cost, the critical fractile gets larger. Then $z_t^* = \Phi^0(h + \mathsf{E}[C_t - \alpha C(t+1)|C(0)]/(h+b))$ also decreases in the procurement cost. Taking these effects into account, the order-up-to level decreases with the proportion of eco-friendly components used in production.

Denote the expected optimal myopic profit accrued in period t viewed from the beginning of the planning horizon as $V_t^M(C(0))$. Then,

$$\begin{aligned} V_t^M(C(0)) \quad = \quad & [\mathsf{E}R(D(t)) - (C(t) - \alpha \mathsf{E}C(t+1))s^*(C(t)) - \alpha \mathsf{E}C(t+1)D(t) \\ & -\alpha^L G(y - D^L(t))|C(0)]. \end{aligned}$$

This function satisfies the properties described in the next result.

Proposition 18. *(i)* $V_t^M(C(0))$ *is concave in* ρ_t *if* $b - [C(t) - \alpha \mathsf{E}C(t+1)] \geqslant \Phi(1)(b+h)$ *(or* $\leqslant \Phi(-1)(b+h)$*). In that case, there exists* ρ_t^* *that maximizes* $V_t^M(C(0))$. *(ii)* ρ_t^* *increases in* t.

When ρ increases, the revenue may either increase or decrease because a higher production cost results in a higher price, but a lower market demand. The total procurement cost may also increase or decrease: while the unit production cost increases, the production quantity decreases. The inventory cost, however, will always decrease as ρ increases. Because an increased dependence on eco-friendly component leads to a more stable system, leftover inventory and backorders are less likely to happen.

The concavity of the myopic profit depends on a set of parameter conditions. Particularly, the condition $b - [C(t) - \alpha \mathsf{E}C(t+1)] \geqslant \Phi(1)(b+h)$ is satisfied when the service level is relatively high ($\frac{b}{b+h} \geqslant \Phi(1) + \frac{[C(t) - \alpha \mathsf{E}C(t+1)]}{h+b}$). In that case, the manufacturer can improve its profit by increasing the proportion of eco-friendly inputs used in production, up to a level ρ_t^*. In view of the expectations regarding a decreasing trend in the cost of eco-friendly components, the optimal proportion increases over time.

Next, we study how the product's price sensitivity affects the firm's procurement strategy.

Proposition 19. *The optimal proportion of eco-friendly components ρ_t^* determined in Proposition 18 decreases in μ.*

When product demand is more sensitive to the selling price, cost has a stronger impact on demand. Proposition 19 shows that with a higher price sensitivity, a lower proportion of eco-friendly components is needed to maintain profitability. This is because a higher price sensitivity increases the effect of the cost process on the demand process. When the cost process becomes more stable by incorporating the eco-friendly component, the demand process is also more stable and this effect is stronger with a higher price sensitivity. On the other hand, the revenue drops more dramatically with the price change when the price sensitivity is high. Therefore, a relatively lower proportion of eco-friendly components is preferred when demand is highly sensitive to price changes.

4.4.3 Numerical Study

In this section, we use several numerical examples to illustrate the results derived in Section 4. Figure 4.4 illustrates the result of Proposition 18. As can be seen in the graph, the myopic profit is concave in the proportion of the eco-friendly component. In later time periods, the optimal proportion of the eco-friendly component increases due to its decreasing cost trend. Moreover, the profit of the manufacturer is gradually increasing over time.

Figure 4.5 illustrates the result of Proposition 19, i.e., the dependence of the optimal proportion of eco-friendly component on the price sensitivity. As the price

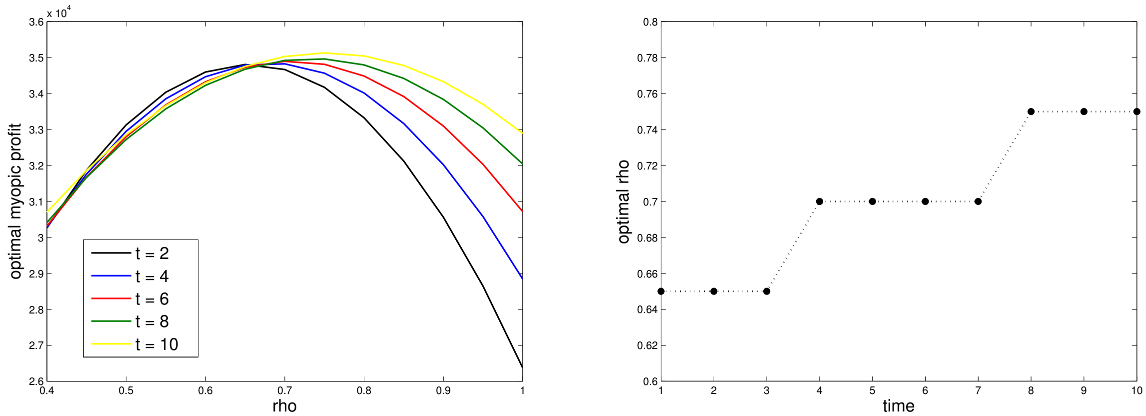


FIGURE 4.4: Impact of ρ on profit, $L = 2$, $T = 10$, $a = 0.8$, $\sigma^O = 10$, $\sigma^E = 3$, $\theta^O = 1$, $\theta^E = 4$, $\delta^O = 0.1$, $\delta^E = 0.15$, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 1.5$.

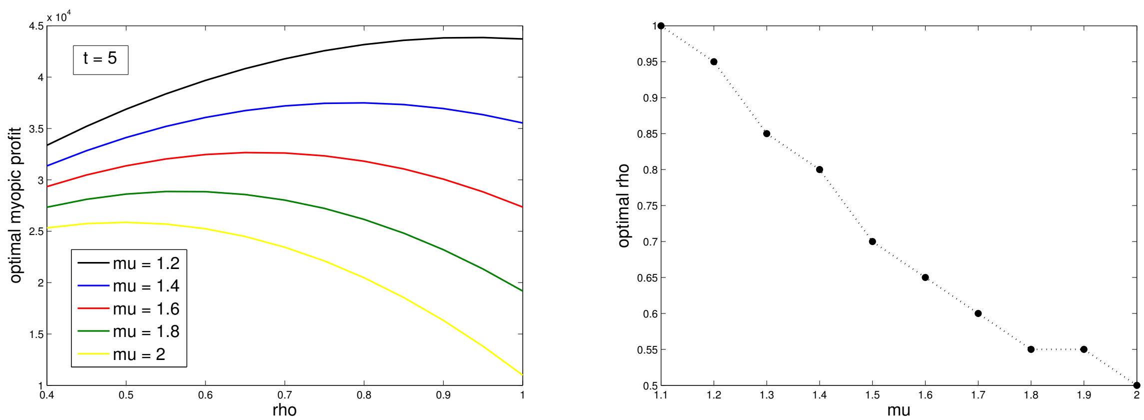


FIGURE 4.5: Impact of ρ on μ, $L = 2$, $T = 10$, $a = 0.8$, $\sigma^O = 10$, $\sigma^E = 3$, $\theta^O = 1$, $\theta^E = 4$, $\delta^O = 0.1$, $\delta^E = 0.15$, $h = 1$, $b = 20$, $\lambda = 60$, $\mu = 1.5$.

sensitivity μ increases, the optimal proportion ρ^* decreases. Moreover, the optimal profit also decreases.

4.4.4 Conclusion

In this chapter, we study the profitability of a sustainable procurement strategy that relies on component materials with less fossil-fuel content. This strategy allows the manufacturer to limit the exposure to procurement cost volatility, but may lead to higher costs. We evaluate the impact of features of the cost process on the manufacturer's procurement decision and overall profit. We find that, in certain settings, the benefit of risk reduction associated with the procurement of eco-friendly components outweighs the cost increase, therefore improving the manufacturer's profitability. Given the current volatility of oil prices, a sustainable procurement strategy can be regarded as a form operational hedging.

Appendix A

Appendix

Proof of Proposition 1. We prove this result by induction. The result holds for $T+1$. Suppose that at $t+1$, $V_{t+1}\left(x_{t+1}, \mathbf{w_{t+1}}\right)$ is jointly concave in $\left(x_{t+1}, \mathbf{w_{t+1}}\right)$ and $J_{t+1}\left(x_{t+1}, \mathbf{w_{t+1}}, d_{t+1}, q_{t+1}\right)$ is jointly concave in $\left(x_{t+1}, \mathbf{w_{t+1}}, d_{t+1}, q_{t+1}\right)$.

The concavity of $V_{t+1}\left(x_{t+1}, \mathbf{w_{t+1}}\right)$ implies that

$$V_{t+1}\left(x_t + w_{1,t} - D_t\left(d_t, \epsilon_t\right), w_{2,t}, \cdots, w_{L-1,t}, q_t\right)$$

is concave in $\left(x_t, \mathbf{w_t}, d_t, q_t\right)$ for any realization of ϵ_t, since

$$\left(x_t + w_{1,t} - D_t\left(d_t, \epsilon_t\right), w_{2,t}, \cdots, w_{L-1,t}, q_t\right)$$

is an affine transformation of $\left(x_t, \mathbf{w_t}, d_t, q_t\right)$ under both additive and multiplicative demand functions. Therefore, $\mathsf{E}V_{t+1}\left(x_t + w_{1,t} - D_t\left(d_t, \epsilon_t\right), w_{2,t}, \cdots, w_{L-1,t}, q_t\right)$ is concave in $\left(x_t, \mathbf{w_t}, d_t, q_t\right)$. Furthermore, the concavity of the revenue function and the convexity of the inventory cost function imply that $J_t\left(x_t, \mathbf{w_t}, d_t, q_t\right)$ is jointly concave in $\left(x_t, \mathbf{w_t}, d_t, q_t\right)$. Concavity is preserved under maximization. Hence, $V_t\left(x_t, \mathbf{w_t}\right)$ is jointly concave in $\left(x_t, \mathbf{w_t}\right)$. $\square$

Proof of Proposition 2. To simplify notation, the subscript t is omitted in this proof.

(i) We need to show that $0 \leqslant \frac{dd^M(x)}{dx} \leqslant 1$ for all x. If $d^M(x)$ lies on the boundary of Ω_t, then $\frac{dd^M(x)}{dx} = 0$. Otherwise, $d^M(x)$ satisfies the first-order condition of $R(d) - G(x,d)$ with respect to d, i.e.,

$$R'(d) - \frac{\partial G(x,d)}{\partial d} |_{d=d^M(x)} = 0.$$

Thus,

$$\frac{dd^M(x)}{dx} = -\frac{-\frac{\partial^2 G(x,d)}{\partial x \partial d}}{R''(d) - \frac{\partial^2 G(x,d)}{\partial^2 d}} = \frac{(h+b) f(x-d)}{\frac{2}{\mu} + (h+b) f(x-d)} \in (0,1).$$

(ii) Using simple algebra, we can write the first order condition of $d^M(x)$ as

$$\frac{\lambda - 2d^M(x)}{\mu} = b - (h+b) F\left(x - d^M(x)\right) \tag{A.1}$$

Since $d^M(x)$ increases in x with rate smaller than 1, $x - d^M(x)$ is increasing in x. The monotonicity of $x - d^M(x)$ implies that $\lim_{x \to +\infty} F\left(x - d^M(x)\right) = 1$, and $\lim_{x \to -\infty} F\left(x - d^M(x)\right) = 0$.

Since equation (A.1) also holds as $x \to \pm\infty$, we have

$$\lim_{x \to +\infty} \frac{\lambda - 2d^M(x)}{\mu} = \frac{\lambda - 2\lim_{x \to +\infty} d^M(x)}{\mu} = b - (h+b) \lim_{x \to +\infty} F\left(x - d^M(x)\right) = -h,$$

$$\lim_{x \to -\infty} \frac{\lambda - 2d^M(x)}{\mu} = \frac{\lambda - 2\lim_{x \to -\infty} d^M(x)}{\mu} = b - (h+b) \lim_{x \to -\infty} F\left(x - d^M(x)\right) = b.$$

Thus, $\lim_{x \to +\infty} d^M(x) = \frac{\lambda + \mu h}{2}$ and $\lim_{x \to +\infty} d^M(x) = \frac{\lambda - \mu b}{2}$. □

Proof of Proposition 3. (i) Under additive demand, the first order conditions of the optimal joint decisions $(d_t^*(x_t), q_t^*(x_t))$ are as follows:

$$\begin{cases} R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d) + \frac{\partial}{\partial d} \mathsf{E} V_{t+1}(x_t + q - d - \epsilon_t) |_{\left(d_t^*, q_t^*\right)} & = 0 \\ -c_t + \frac{\partial}{\partial q} E V_{t+1}(x_t + q - d - \epsilon_t)) |_{\left(d_t^*, q_t^*\right)} & = 0. \end{cases}$$

The second equation of the first-order conditions implies that

$$\frac{\partial}{\partial q}\mathsf{E}V_{t+1}\left(x_t+q-d-\epsilon_t\right)\mid_{\left(d_t^*,q_t^*\right)}=\mathsf{E}V_{t+1}'\left(x_t+q-d-\epsilon_t\right)\mid_{\left(d_t^*,q_t^*\right)}=c_t,$$

and the first equation of the first-order conditions implies that

$$R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)\mid_{\left(d_t^*,q_t^*\right)}=\mathsf{E}V_{t+1}'\left(x_t+q-d-\epsilon_t\right)\mid_{\left(d_t^*,q_t^*\right)}.$$

Thus, the optimal demand policy $d_t^*\left(x_t\right)$ satisfies

$$R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)\mid_{d_t^*}=c_t.$$

The myopic demand policy $d_t^M\left(x_t\right)$ satisfies the first-order condition

$$R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)\mid_{d_t^M}=0.$$

When $c_t>0$,

$$R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)\mid_{\left(d_t^*,q_t^*\right)}=c_t>0=R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)\mid_{d_t^M}.$$

The concavity of the function $R_t\left(d\right)-G_t\left(x_t,d\right)$ implies that $R_t'\left(d\right)-\frac{\partial}{\partial d}G_t\left(x_t,d\right)$ is decreasing in d for each value of x_t. Hence $d_t^M\left(x_t\right)\geqslant d_t^*\left(x_t\right)$.

When $c_t=0$, the first-order conditions of $d_t^M\left(x_t\right)$ and $d_t^*\left(x_t\right)$ are identical, implying that $d_t^M\left(x_t\right)=d_t^*\left(x_t\right)$.

(ii) Applying the implicit function theorem on the first order conditions of $d_t^M\left(x_t\right)$ and $d_t^*\left(x_t\right)$, we obtain the derivative of $d_t^M\left(x_t\right)$ and $d_t^*\left(x_t\right)$ with respect to x_t, as follows:

$$\frac{dd_t^*\left(x_t\right)}{dx_t}=\frac{\left(h_t+b_t\right)f_t\left(x_t-d_t^*\left(x_t\right)\right)}{\frac{2}{\mu_t}+\left(h_t+b_t\right)f_t\left(x_t-d_t^*\left(x_t\right)\right)}=\frac{dd_t^M\left(x_t\right)}{dx_t}\in\left(0,1\right).$$

Thus, the optimal demand policy has the same slope as the myopic demand policy.

(*iii*) From (*i*), $d_t^*(x_t)$ satisfies the first-order condition

$$R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d) \mid_{(d_t^*, q_t^*)} = c_t.$$

Using simple algebra, we can rewrite this equation as

$$\frac{\lambda_t - 2d_t^*(x_t)}{\mu_t} = b_t - (h_t + b_t) F_t(x_t - d_t^*(x_t)) + c_t \tag{A.2}$$

The result in (*ii*) implies that $x_t - d_t^*(x_t)$ increases in x_t. Thus, we have

$$\lim_{x_t \to +\infty} F_t(x_t - d_t^*(x_t)) = 1$$

and

$$\lim_{x_t \to -\infty} F_t(x_t - d_t^*(x_t)) = 0.$$

Equation (A.2) holds as $x_t \to \pm\infty$. Therefore,

$$\begin{aligned}
\lim_{x_t \to +\infty} \frac{\lambda_t - 2d_t^*(x_t)}{\mu_t} &= b_t - (h_t + b_t) \lim_{x_t \to +\infty} F_t(x_t - d_t^*(x_t)) + c_t = -h_t + c_t, \\
\lim_{x_t \to -\infty} \frac{\lambda_t - 2d_t^*(x_t)}{\mu_t} &= b_t - (h_t + b_t) \lim_{x_t \to -\infty} F_t(x_t - d_t^*(x_t)) + c_t = b_t + c_t,
\end{aligned}$$

leading to

$$\lim_{x_t \to +\infty} d_t^*(x_t) = \frac{\lambda_t + \mu_t h_t - \mu_t c_t}{2}$$

and

$$\lim_{x_t \to -\infty} d_t^*(x_t) = \frac{\lambda_t - \mu_t b_t - \mu_t c_t}{2}.$$

(*iv*) and (*v*) We first show by induction that for any t,

$$d_t^M(x_t) = \lim_{w_{1,t} \to +\infty} d_t^*(x_t, w_{1,t}, \cdots, w_{L-1,t})$$

and $\lim_{x_t \to +\infty} V_t'(x_t, w_{1,t}, \cdots, w_{L-1,t}) = 0$ when $h_t = 0$.

At time T, $q_T^* = 0$,

$$V_T(x_T, w_{1,T}, \cdots, w_{L-1,T}) = V_T(x_T) = \max_{d_T \in \Omega_T} \{R_T(d_T) - G_T(x_T, d_T)\}.$$

The problem that determines d_T^* is the same as the one that determines d_T^M. Therefore,

$$d_T^*(x_T, w_{1,T}, \cdots, w_{L-1,T}) = d_T^*(x_T) = d_T^M(x_T).$$

Furthermore,

$$V_T(x_T) = R_T(d_T^*(x_T)) - G_T(x_T, d_T^*(x_T)),$$

$$V_T'(x_T) = b_T - (h_T + b_T) F(x_T - d_T^*(x_T)).$$

Thus, $\lim_{x_T \to +\infty} V_T'(x_T) = -h_T = 0$, and the results hold at time T.

Suppose that at time $t+1$, $d_{t+1}^M(x_{t+1}) \geqslant d_{t+1}^*(x_t, +\infty, w_{2,t} \cdots, w_{L-1,t})$ and

$$\lim_{x_{t+1} \to +\infty} \frac{\partial V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})}{\partial x_{t+1}} = 0.$$

Consider the problem at time t:

$$\begin{aligned} V_t(x_t, w_{1,t}, \cdots, w_{L-1,t}) \quad = \quad & \max_{q_t \geqslant 0, d_t \in \Omega_t} \{R_t(d_t) - c_t q_t - G_t(x_t, d_t) \\ & + \alpha \mathsf{E} V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})\}, \end{aligned}$$

where $x_{t+1} = x_t + w_{1t} - d_t - \epsilon_t$. The first order condition for $d_t^*(x_t, w_{1,t}, \cdots, w_{L-1,t})$ is

$$R_t'(d_t) - \frac{\partial G_t(x_t, d_t)}{\partial d_t} - \alpha \mathsf{E} \frac{\partial V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})}{\partial x_{t+1}} = 0.$$

Since $x_{t+1} = x_t + w_{1t} - d_t - \epsilon_t$, we have by induction that

$$\begin{aligned} & \lim_{w_{1,t} \to +\infty} \mathsf{E} \frac{\partial V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})}{\partial x_{t+1}} \\ = & \lim_{x_{t+1} \to +\infty} \mathsf{E} \frac{\partial V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})}{\partial x_{t+1}} \\ = & 0. \end{aligned}$$

Hence, the first-order condition for $d_t^*(x_t, +\infty, w_{2,t}, \cdots, w_{L-1,t})$ becomes the same as the first-order condition for $d_t^M(x_t)$, which implies that

$$\lim_{w_{1,t}\to+\infty} d_t^*(x_t, w_{1,t}, \cdots, w_{L-1,t}) = d_t^M(x_t).$$

Furthermore,

$$\begin{aligned}
&\lim_{x_t\to+\infty} \frac{\partial V_t(x_t, w_{1,t}, \cdots, w_{L-1,t})}{\partial x_t} \\
=&\lim_{x_t\to+\infty}\left[-\frac{\partial G_t(x_t, d_t)}{\partial d_t} + \alpha \mathsf{E}\frac{\partial V_{t+1}(x_{t+1}, w_{1,t+1}, \cdots, w_{L-1,t+1})}{\partial x_{t+1}}\right]|_{d_t=d_t^*} \\
=&-h_t + 0 = 0.
\end{aligned}$$

By Pang et al. (2012), $d_t^*(x_t.w_{1,t}, \cdots, w_{L-1,t})$ is non-decreasing in $w_{l,t}$, $l = 1, \cdots, L$. Thus, $\lim_{w_{1,t}\to+\infty} d_t^*(x_t, w_{1,t}, \cdots, w_{L-1,t}) = d_t^M(x_t)$ implies that

$$d_t^*(x_t, w_{1,t}, \cdots, w_{L-1,t}) \leqslant d_t^M(x_t)$$

for any value of $w_{1,t}$. □

Proof of Proposition 4. To simplify notation, subscript t is omitted in this proof.

(i) The myopic demand $d^M(x)$ satisfies the first order condition

$$R'(d) - \frac{\partial G(x, d)}{\partial d} = 0.$$

Thus,

$$\frac{dd^M(x)}{dx} = -\frac{-\frac{\partial^2 G(x,d)}{\partial x \partial d}}{R''(d) - \frac{\partial^2 G(x,d)}{\partial^2 d}} = \frac{(h+b)\frac{x}{d^2}f\left(\frac{x}{d}\right)}{-R''(d) + (h+b)\frac{x^2}{d^3}f\left(\frac{x}{d}\right)}\begin{cases} > 0, & \text{if } x > 0, \\ = 0, & \text{if } x \leqslant 0, \end{cases}$$

i.e., $d^M(x)$ is non-decreasing in x.

(ii) Using simple algebra, the first-order condition of $d^M(x)$ can be written as

$$\lambda^{1/\mu}(1 - 1/\mu)\, d^{-1/\mu}\,|_{d=d^M(x)} + h = (h+b)\int_{x/d}^{+\infty} \epsilon dF(\epsilon)\,|_{d=d^M(x)}. \tag{A.3}$$

The monotonicity of $d^M(x)$ implies that

$$\lim_{x\to+\infty}\left[\lambda^{1/\mu}(1-1/\mu)\,d^{-1/\mu}\mid_{d=d^M(x)}+h\right]=\lambda^{1/\mu}(1-1/\mu)\lim_{x\to+\infty}d^M(x)^{-1/\mu}+h=h,$$

i.e., the left hand side of (A.3) exists, which implies that the limit of the right hand side of equation (A.3) also exists. Since on the right hand side of (A.3) the variable x only appears in the lower limit of the integral in the function $\frac{x}{d^M(x)}$, we have that $\lim_{x\to+\infty}\frac{x}{d^M(x)}$ exists. The value of $\lim_{x\to+\infty}\frac{x}{d^M(x)}$ could be 0, $+\infty$ or some positive constant. We now examine these three possibilities:

(1) If $x/d^M(x)\to 0$ when $x\to+\infty$, then

$$\int_{x/d^M(x)}^{+\infty}\epsilon dF(\epsilon)\to\int_0^{+\infty}\epsilon dF(\epsilon)=\mathsf{E}\epsilon=1.$$

The right hand side of (A.3) converges to $(h+b)$, which does not equal the limit of the left hand side of (A.3), unless $b=0$.

(2) If $x/d^M(x)\to+\infty$ when $x\to+\infty$, then

$$\int_{x/d^M(x)}^{+\infty}\epsilon dF(\epsilon)\to\int_{+\infty}^{+\infty}\epsilon dF(\epsilon)=0.$$

The right hand side of (A.3) converges to 0, which does not equal the limit of the left hand side of (A.3) unless $h=0$.

(3) When $h\neq 0$ and $b\neq 0$, $\lim_{x\to+\infty}\frac{x}{d^M(x)}=$ constant, which implies that $d^M(x)$ has an asymptotic slope. Let C^* be the solution to

$$\int_{1/C}^{+\infty}\epsilon dF(\epsilon)=\frac{h}{h+b}.$$

Thus, C^* is the asymptotic slope of $d^M(x)$.

(*iii*) Since equation (A.3) holds for any value of $x\leqslant 0$, we have

$$\lambda^{1/\mu}(1-1/\mu)\,d^M(x)^{-1/\mu}+h=(h+b)\int_{x/d^M(x)}^{+\infty}\epsilon dF(\epsilon).$$

Since ϵ is nonnegative, we have that $\frac{x}{d^M(x)} \leqslant 0$ for $x \leqslant 0$ implies that

$$\int_{x/d^M(x)}^{+\infty} \epsilon dF(\epsilon) = \int_0^{+\infty} \epsilon dF(\epsilon) = \mathsf{E}\epsilon = 1.$$

For $x \leqslant 0$, we have

$$\lambda^{1/\mu}(1-1/\mu)\, d^M(x)^{-1/\mu} + h = h + b, \quad d^M(x) = \lambda\left[\frac{1-1/\mu}{b}\right]^\mu \triangleq C_0. \quad \square$$

Proof of Proposition 5. (i) Under a multiplicative demand function, the first-order conditions for the optimal joint decisions $(d_t^*(x_t), q_t^*(x_t))$ are:

$$\begin{cases} R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d) + \frac{\partial}{\partial d}\mathsf{E}V_{t+1}(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} & = 0 \\ -c_t + \frac{\partial}{\partial q} EV_{t+1}(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} & = 0. \end{cases}$$

The second equation implies that

$$\frac{\partial}{\partial q}\mathsf{E}V_{t+1}(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} = \mathsf{E}V_{t+1}'(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} = c_t,$$

and the first equation implies that

$$R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d)\,|_{(d_t^*, q_t^*)} = \mathsf{E}V_{t+1}'(x_t + q - D_t(d, \epsilon_t))\,\epsilon_t\,|_{(d_t^*, q_t^*)}.$$

Note that $V_{t+1}'(x_{t+1})$ is a non-increasing function due to the concavity of $V_{t+1}(\cdot)$, so $V_{t+1}'(x_t + q - d\epsilon_t)$ is increasing in ϵ_t, i.e., the random variables $V_{t+1}'(x_t + q - d\epsilon_t)$ and ϵ_t are positively correlated. Hence,

$$\mathsf{E}V_{t+1}'(x_t + q - d\epsilon_t)\,\epsilon_t\,|_{(d_t^*, q_t^*)} \geqslant \mathsf{E}V_{t+1}'(x_t + q - d\epsilon_t)\,\mathsf{E}\epsilon_t\,|_{(d_t^*, q_t^*)} = \mathsf{E}V_{t+1}'(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} = c_t.$$

The myopic demand policy $d_t^M(x_t)$ satisfies the first-order condition $R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d)\,|_{d_t^M} = 0$. Thus, $R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d)\,|_{(d_t^*, q_t^*)} \geqslant c_t \geqslant 0 = R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d)\,|_{d_t^M}$, which implies that $d_t^M(x_t) \geqslant d_t^*(x_t)$.

(ii), (iii) First, we show that $d_t^*(x_t, \mathbf{w_t})$ in nondecreasing in x_t and $w_{1,t}$. Because $d_t^*(x_t, \mathbf{w_t})$ satisfies the first order condition

$$\frac{\partial J_t(x_t, \mathbf{w_t}, q_t, d_t)}{\partial d_t} = R_t'(d) - \frac{\partial}{\partial d} G_t(x_t, d) + \frac{\partial}{\partial d} \mathsf{E} V_{t+1}(x_t + q - d\epsilon_t)\,|_{(d_t^*, q_t^*)} = 0,$$

we have that

$$\frac{\partial d_t^*(x_t, \mathbf{w_t})}{\partial x_t} = -\frac{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)/\partial d_t \partial x_t}{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)/\partial d_t^2}, \quad \frac{\partial d_t^*(x_t, \mathbf{w_t})}{\partial w_{1,t}} = -\frac{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)/\partial d_t \partial w_{1,t}}{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)/\partial d_t^2}.$$

In addition,

$$\begin{aligned}
\frac{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)}{\partial d_t \partial x_t} &= -\frac{\partial^2 G_t(x_t, d_t)}{\partial d_t \partial x_t} - \alpha \mathsf{E} \frac{\partial^2 V_{t+1}(x_t + w_{1,t} - d_t\epsilon_t, \mathbf{w_{t+1}})}{\partial (x_t + w_{1,t} - d_t\epsilon_t)^2}\epsilon_t, \\
\frac{\partial^2 J_t(x_t, \mathbf{w_t}, q_t, d_t)}{\partial d_t \partial w_{1,t}} &= -\alpha \mathsf{E} \frac{\partial^2 V_{t+1}(x_t + w_{1,t} - d_t\epsilon_t, \mathbf{w_{t+1}})}{\partial (x_t + w_{1,t} - d_t\epsilon_t)^2}\epsilon_t.
\end{aligned}$$

Because $\frac{\partial^2 G_t(x_t, d_t)}{\partial d_t \partial x_t} = -(h_t + b_t)\frac{x_t}{d_t^2} f(\frac{x_t}{d_t}) \leqslant 0$, $V_t(x_t, \mathbf{w_t})$ and $J_t(x_t, \mathbf{w_t}, q_t, d_t)$ are concave functions, and ϵ_t is a random variable with positive support, it follows that $\frac{\partial d_t^*(x_t, \mathbf{w_t})}{\partial x_t} \geqslant 0$ and $\frac{\partial d_t^*(x_t, \mathbf{w_t})}{\partial w_{1,t}} \geqslant 0$, i.e., $d_t^*(x_t, \mathbf{w_t})$ is non-decreasing in x_t and $w_{1,t}$.

Next, we show by induction that for any t, $d_t^M(x_t) = \lim_{w_{1,t} \to +\infty} d_t^*(x_t, \mathbf{w_t})$ and $\lim_{x_t \to +\infty} V_t'(x_t, \mathbf{w_t}) = 0$ when $h_t = 0$. In period T, $q_T^* = 0$, $V_T(x_T, \mathbf{w_T}) = V_T(x_T) = \max_{d_T \in \Omega_T} \{R_T(d_T) - G_T(x_T, d_T)\}$. The problem that determines d_T^* is the same as the one that determines d_T^M. Therefore, $d_T^*(x_T, \mathbf{w_T}) = d_T^*(x_T) = d_T^M(x_T)$. Furthermore, we have $V_T(x_T) = R_T(d_T^*(x_T)) - G_T(x_T, d_T^*(x_T))$ and $V_T'(x_T) = b_T - (h_T + b_T) F\left(\frac{x_T}{d_T^*(x_T)}\right)$. As shown in Proposition 4, $d_T^*(x_T)$ increases with asymptotic slope C_t^* as x_T goes to infinity. As $C_t^* < 1$, $\lim_{x_T \to +\infty} x_T/d_T^*(x_T) = +\infty$. Thus, $\lim_{x_T \to +\infty} V_T'(x_T) = -h_T = 0$. Therefore, the results hold in the last period.

Suppose that in period $t+1$, $d_{t+1}^M(x_{t+1}) = d_{t+1}^*(x_t, +\infty, w_{2,t} \cdots, w_{L-1,t})$ and $\lim_{x_{t+1} \to +\infty} \frac{\partial V_{t+1}(x_{t+1}, \mathbf{w_{t+1}})}{\partial x_{t+1}} = 0$. Consider the problem at time t:

$$V_t(x_t, \mathbf{w_t}) = \max_{q_t \geqslant 0, d_t \in \Omega_t} \{R_t(d_t) - c_t q_t - G_t(x_t, d_t) + \alpha \mathsf{E} V_{t+1}(x_{t+1}, \mathbf{w_{t+1}})\}$$

where $x_{t+1} = x_t + w_{1t} - d_t\epsilon_t$. The first order condition for $d_t^*(x_t, \mathbf{w_t})$ is

$$R_t'(d_t) - \frac{\partial G_t(x_t, d_t)}{\partial d_t} - \alpha \mathsf{E}\frac{\partial V_{t+1}(x_{t+1}, \mathbf{w_t})}{\partial x_{t+1}}\epsilon_t = 0.$$

Since $x_{t+1} = x_t + w_{1t} - d_t\epsilon_t$, we have by induction that

$$\lim_{w_{1,t}\to+\infty} \mathsf{E}\frac{\partial V_{t+1}(x_{t+1}, \mathbf{w_{t+1}})}{\partial x_{t+1}} = \lim_{x_{t+1}\to+\infty} \mathsf{E}\frac{\partial V_{t+1}(x_{t+1}, \mathbf{w_{t+1}})}{\partial x_{t+1}} = 0.$$

Then, $\lim_{x_{t+1}\to+\infty} \mathsf{E}\frac{\partial V_{t+1}(x_{t+1},\mathbf{w_{t+1}})}{\partial x_{t+1}}\epsilon_t = \mathsf{E}\lim_{x_{t+1}\to+\infty} \frac{\partial V_{t+1}(x_{t+1},\mathbf{w_{t+1}})}{\partial x_{t+1}}\epsilon_t = 0$. Hence, the first-order condition of $d_t^*(x_t, +\infty, w_{2,t}, \cdots, w_{L-1,t})$ becomes the same as the first-order condition of $d_t^M(x_t)$, which implies that $\lim_{w_{1,t}\to+\infty} d_t^*(x_t, \mathbf{w_t}) = d_t^M(x_t)$. The monotonicity of $d_t^*(x_t, \mathbf{w_t})$ in $w_{1,t}$ implies that $d_t^M(x_t) \geqslant d_t^*(x_t, \mathbf{w_t})$. Furthermore,

$$\lim_{x_t\to+\infty} \frac{\partial V_t(x_t, \mathbf{w_t})}{\partial x_t} = \lim_{x_t\to+\infty} \left[-\frac{\partial G_t(x_t, d_t)}{\partial d_t} + \alpha\mathsf{E}\frac{\partial V_{t+1}(x_{t+1}, \mathbf{w_{t+1}})}{\partial x_{t+1}}\right]|_{d_t=d_t^*} = -h_t + 0 = 0. \square$$

Proof of Proposition 6. In this proof, we explicitly denote the myopic expected demand as a function of both x and μ to study its dependence on μ. To show that δ increases in μ, we need to show that

$$d'^M\left(\frac{x^+(\mu_1) + x^-(\mu_1)}{2}, \mu_1\right) \leqslant d'^M\left(\frac{x^+(\mu_2) + x^-(\mu_2)}{2}, \mu_2\right)$$

for any $\mu_1 < \mu_2$. Define

$$x^{mid}(\mu) = \{x : d^M(x^{mid}(\mu), \mu) = \frac{d^M(x^+(\mu), \mu) + d^M(x^-(\mu), \mu)}{2}\}.$$

Then, one can show that under a symmetric density function of ϵ,

$$x^{mid}(\mu) = \frac{x^+(\mu) + x^-(\mu)}{2}$$

, i.e., we can equivalently define δ as $\delta = d^M(x^{mid}(\mu), \mu)$. Furthermore, we can show the following results (a detailed analysis is available from the authors):

(1) $\frac{\partial d^M(x,\mu)}{\partial x}$ is increasing in μ, which implies that

$$d'^M\left(x^{mid}(\mu_1),\mu_1\right) < d'^M\left(x^{mid}(\mu_1),\mu_2\right),$$

(2) If $h \leqslant b$, then $x^{mid}(\mu)$ decreases in μ, i.e., $x^{mid}(\mu_2) < x^{mid}(\mu_1)$,

(3) Under a symmetric and unimodal density function of ϵ, $d'^M(x,\mu)$ decreases in x when $x \geqslant x^{mid}(\mu)$.

These items imply that $d'^M\left(x^{mid}(\mu_1),\mu_1\right) < d'^M\left(x^{mid}(\mu_2),\mu_2\right)$. □

Proof of Proposition 7. From the proof of Proposition 3, we have that $d^M(x)$ satisfies the first-order condition

$$\lambda^{1/\mu}(1-1/\mu)\,d^{-1/\mu}\,|_{d=d^M(x)} + h = (h+b)\int_{x/d}^{+\infty}\epsilon dF(\epsilon)\,|_{d=d^M(x)},$$

and

$$\frac{dd^M(x)}{dx} = -\frac{-\frac{\partial^2 G(x,d)}{\partial x\partial d}}{R''(d)-\frac{\partial^2 G(x,d)}{\partial^2 d}} = \frac{(h+b)\frac{x}{d^2}f\left(\frac{x}{d}\right)}{-R''(d)+(h+b)\frac{x^2}{d^3}f\left(\frac{x}{d}\right)}$$

for any x, where

$$-R''(d) = \lambda^{1/\mu}(1-\frac{1}{\mu})\frac{1}{\mu}d^{-1/\mu-1}.$$

Then, at $x = x^M$, $d^M(x^M) = x^M$, and the first order condition becomes

$$\lambda^{1/\mu}(1-1/\mu)\,x^{M^{-1/\mu}} + h = (h+b)\int_1^{+\infty}\epsilon dF(\epsilon).$$

Furthermore,

$$\begin{aligned}-R''\left(d^M(x^M)\right) &= \lambda^{1/\mu}(1-\frac{1}{\mu})\frac{1}{\mu}x^{M^{-1/\mu-1}}\\ &= [(h+b)\int_1^{+\infty}\epsilon dF(\epsilon)-h]x^{M^{-1}}\frac{1}{\mu},\end{aligned}$$

which implies that

$$\frac{dd^M(x)}{dx}\Big|_{x=x^M} = \frac{(h+b)\frac{1}{x^M}f(1)}{\left[(h+b)\int_1^{+\infty}\epsilon dF(\epsilon)-h\right]\frac{1}{x^M}\frac{1}{\mu}+(h+b)\frac{1}{x^M}f(1)}$$

is increasing in μ. □

Proof of Theorem 8. We prove this result by induction. The result holds in period $T+1$. Suppose that in period $t+1$, $\widetilde{V}_{t+1}(\overline{x}_{t+1})$ is concave in $\overline{x}_{t+1}$ and $\widetilde{J}_{t+1}(\overline{y}_{t+1})$ is concave in $\overline{y}_{t+1}$. Consider now the problem in period t, where

$$\widetilde{J}_t(\overline{y}_t) = \mathsf{E}\{\alpha^L\widetilde{R}_{t+L}\left(\widetilde{d}_{t+L}(x_{t+L})\right) - c_t\overline{y}_t - \alpha^L G_{t+L}\left(x_{t+L}, \widetilde{d}_{t+L}(x_{t+L})\right) + \alpha\widetilde{V}_{t+1}(\overline{x}_{t+1})\}.$$

Since x_{t+L} is an affine transformation of $\overline{y}_t$, the concavity of $\widetilde{V}_{t+1}(\overline{x}_{t+1})$ implies the concavity of $\mathsf{E}\widetilde{V}_{t+1}(\overline{x}_{t+1})$ in $\overline{y}_t$. Furthermore, x_{t+L} and $\widetilde{d}_{t+L}(x_{t+L})$ are linear functions of $\overline{y}_t$. Then, the concavity of the revenue function and the convexity of the inventory cost function are preserved with respect to $\overline{y}_t$. Therefore, $\widetilde{J}_t(\overline{y}_t)$ is concave in $\overline{y}_t$ and $\widetilde{V}_t(\overline{x}_t) = c_t\overline{x}_t + \widetilde{J}_t(\max\{\overline{s}_t, \overline{x}_t\})$ is concave in $\overline{x}_t$. □

Proof of Theorem 9. The condition to ensure $\mathsf{E}V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$ is an upper bound to the exact system, according to Proposition 2.2 in Brown et al. (2010), is that the functions $W_t(\overline{x}_t, q_t \mid \{\epsilon\}_t^T)$ and $W_t(\overline{x}_t, q_t)$ at each period t depends only on decisions up to time t, which implies that the penalty function $W_t(\overline{x}_t, q_t \mid \{\epsilon\}_t^T) - W_t(\overline{x}_t, q_t)$ is dual feasible. In our case, these two functions only depend on decisions at time t. Thus, $\mathsf{E}V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$ provides an upper bound to the optimal value function. [1]

Denote $\overline{c}_t = \frac{\partial \widetilde{J}_t\left(\overline{y}_t \mid \{\epsilon\}_t^T\right)}{\partial \overline{y}_t}\Big|_{\overline{y}_t=\overline{s}_t}$ and $\gamma_t = \widetilde{J}_t\left(\overline{s}_t \mid \{\epsilon\}_t^T\right) - \widetilde{J}_t(\overline{s}_t) - \overline{c}_t\overline{s}_t$. The penalty function can be written as a linear function of q_t and $\overline{x}_t$:

$$W_t(\overline{x}_t, q_t \mid \{\epsilon\}_t^T) - W_t(\overline{x}_t, q_t) = \gamma_t - \overline{c}_t\overline{x}_t - \overline{c}_t q_t.$$

[1] Note that Brown and Smith (2011) also uses the first order Taylor expansion of an approximate value function to generate the penalty function.

Note that the penalty function is linear in all the state variables and decision variables. Substituting the above penalty function into (2.6), the upper bound to the optimal value function is:

$$\mathsf{E}V_t^D(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) = \gamma_t + \mathsf{E}\Big\{ \max_{q_t \geqslant 0, d_t \in \Omega_t} \Big\{ R_t(d_t, \epsilon_t) - (c_t + \overline{c}_t) q_t - \overline{c}_t \overline{x}_t - G_t(x_t, d_t, \epsilon_t) + \alpha V_{t+1}^D(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \Big\} \Big\}. \tag{A.4}$$

□

Proof of Lemma 10.

$$\begin{aligned} V_1^D(x_1, \mathbf{w}_1 | \{\epsilon\}_1^T) &= \sum_{t=1}^{T} \Big[R_t(d_t, \epsilon_t) - (c_t + \bar{c}_t) q_t - \bar{c}_t x_t - G_t(x_t, d_t, \epsilon_t) \Big] \\ &= \Gamma(x_1, \mathbf{w_1}, \{\epsilon\}_1^T) + \sum_{t=1}^{T} \Big[R_t(d_t, \epsilon_t) - \widehat{c}_t q_t - \widehat{c}_t^d d_t - G_t(x_t, d_t, \epsilon_t) \Big] \\ &= \Gamma(x_1, \mathbf{w_1}, \{\epsilon\}_1^T) + \widehat{V}_1^D(x_1, \mathbf{w}_1 | \{\epsilon\}_1^T) \end{aligned}$$

where

$$\begin{aligned} \Gamma(x_1, \mathbf{w_1}, \{\epsilon\}_1^T) &= \gamma_1 + \sum_{k=1}^{T-L} \overline{c}_k \nu_{0,k} x_1 + \sum_{m=2}^{L} \sum_{k=m}^{T-L} \alpha^{k-1} \overline{c}_k \nu_{0,k} w_{m-1,1} \\ &\quad + \sum_{k=1}^{L-1} \alpha^{k-1} \overline{c}_k \Big[\sum_{l=k}^{L-1} w_{l,1} \Pi_{j=l+1}^{k+L-1} (1 - \delta_j) \Big] - \overline{\kappa}_i - \overline{\epsilon}_i \end{aligned}$$

$$\overline{\kappa}_i = \begin{cases} \sum_{k=1}^{T-L} \alpha^{k-1} \Big(\sum_{l=1}^{L-1} \nu_{l,k} \kappa_{k+l-1} \Big), & i = \text{additive demand} \\ \sum_{k=1}^{T-L} \alpha^{k-1} \Big(\sum_{l=1}^{L-1} \nu_{l,k} (\kappa_{k+l-1} - 1) \Big), & i = \text{multiplicative demand} \end{cases}$$

$$\overline{\epsilon}_i = \begin{cases} \sum_{k=1}^{T-L-1} \sum_{j=k+1}^{T-L-1} \alpha^{j-1} \nu_{0,j} \overline{c}_j \epsilon_k, & i = \text{additive demand} \\ 0, & i = \text{multiplicative demand} \end{cases}$$

□

Proof of Theorem 11. We first prove that for both additive and multiplicative demands, under the terminal condition $\widetilde{V}_{T+1} = \alpha^{T-L_i} c\overline{x}_{T-L_i+1}$, the dynamic problem is identical to the following myopic problem

$$\max_y \mathsf{E}[\alpha^L g(\overline{y} - \epsilon(L)) - c\overline{y} + \alpha c(1-\delta)(\overline{y} - \epsilon(1))],$$

in the sense that the optimal order-up-to level obtained in the myopic problem is the same as the base-stock level of the dynamic problem. Here,

$$g(y) = \widetilde{R}(\widetilde{d}(y)) - G(y, \widetilde{d}(y))$$

and

$$\epsilon(L) = \begin{cases} \sum_{l=1}^{L} (1-\delta)^{L-l}\epsilon, \text{ additive}, \\ \sum_{l=1}^{L} (1-\delta)^{L-l}\kappa(\epsilon - 1), \text{ multiplicative}. \end{cases}$$

To guarantee the optimality of the myopic replenishment policy, two conditions need to be examined: 1) the total profit over the finite horizon needs to be expressed as the summation of a series of identical single-period problems (myopic problem); 2) the optimizer of the myopic problem can be achieved in each period, i.e., the inventory state is regenerated.

We examine the first condition. Denote B as the total profit over the finite horizon T, that is,

$$B = \sum_{t=1}^{T} \alpha^{t-1}[g(x_t) - cq_t] - \alpha^{T-L} c\overline{x}_{T-L+1}.$$

Note that $x_{t+L} = \overline{x}_t + q_t - \epsilon(L) = \overline{y}_t - \epsilon(L)$, and $\overline{x}_t = (1-\delta)(\overline{y}_{t-1} - \epsilon(1)) - \kappa$. Plugging $\overline{x}_t = (1-\delta)(\overline{y}_{t-1} - \epsilon(1)) - \kappa$ into the above expression of B, we obtain

$$\begin{aligned} B &= g(x_1) + \alpha g(x_2) + \cdots + \alpha^{L-1} g(x_L) + c\overline{x}_1 - \textstyle\sum_{t=1}^{T-L} \alpha^L c\kappa \\ &+ \textstyle\sum_{t=1}^{T-L} [\alpha^L g(\overline{y}_t - \epsilon(L)) - c\overline{y}_t + \alpha c(1-\delta)(\overline{y}_t - \epsilon(1))]. \end{aligned}$$

That is, B is expressed as the summation of a term that is independent of any decision variable and $T-L$ identical single-period profit functions. Therefore, the first condition above is satisfied.

Next we verify that the optimizer of the myopic problem can be achieved in each period. Note that the random variables $\epsilon(L)$ and $\epsilon(1)$ can take both positive and negative values. However, their realizations have a finite lower bound. Let $-\tau^L$ be the lowest possible realization of $\epsilon(L)$ and $\epsilon(1)$. We shift the state variable, the decision variable and the random variables by a positive volume τ^L, i.e., define $\widehat{x}_t = \overline{x}_t + \tau^L$, $\widehat{y}_t = \overline{y}_t + \tau^L$, $\widehat{\epsilon}(L) = \epsilon(L) + \tau^L$ and $\widehat{\epsilon}(1) = \epsilon(1) + \tau^L$. Thus, we can equivalently express

$$
\begin{aligned}
B \quad &= g(x_1) + \alpha g(x_2) + \cdots + \alpha^{L-1} g(x_L) + c\overline{x}_1 - \textstyle\sum_{t=1}^{T-L} \alpha^L c\kappa \\
&\quad + \textstyle\sum_{t=1}^{T-L} [\alpha^L g(\widehat{y}_t - \widehat{\epsilon}(L)) - c\widehat{y}_t + \alpha c(1-\delta)(\widehat{y}_t - \widehat{\epsilon}(1))].
\end{aligned}
$$

where $\widehat{\epsilon}(L)$ and $\widehat{\epsilon}(1)$ are all non-negative random variables.

Let $\widehat{s}^L$ be the maximizer of the single-period problem

$$
\max_{\widehat{y}} \mathsf{E}[\alpha^L g(\widehat{y} - \widehat{\epsilon}(L)) - c\widehat{y} + \alpha c(1-\delta)(\widehat{y} - \widehat{\epsilon}(1))].
$$

As long as $\widehat{s}^L$ can be reached at the beginning of the planning horizon, the positivity of the random variables $\widehat{\epsilon}(L)$ and $\widehat{\epsilon}(1)$ will guarantee that $\widehat{s}^L$ is feasible in the remaining periods. The myopic replenishment policy, expressed as a function of $\widehat{s}^L$ is as follows: if $\widehat{x} < \widehat{s}^L$, then order $\widehat{s}^L - \widehat{x}$; otherwise, do not order. Equivalently, this policy can be stated as: if $x < \overline{s}^L$, the order $\overline{s}^L - x$; otherwise, do not order. Here, $\overline{s}^L$ solves

$$
\max_{y} \mathsf{E}[\alpha^L g(\overline{y} - \epsilon(L)) - c\overline{y} + \alpha c(1-\delta)(\overline{y} - \epsilon(1))].
$$

We have proved that the dynamic problem is equivalent to the myopic problem.

Next, to compare the base-stock levels in systems with different lead times, we can focus on the myopic problem:

$$\max_{y} \mathsf{E}[\alpha^L g(\overline{y} - \epsilon(L)) - c\overline{y} + \alpha c(1-\delta)(\overline{y} - \epsilon(1))].$$

We show the result for the case of multiplicative demand and the analysis for the case of additive demand follows similarly.

First, we shift all variables to guarantee the positivity of the random variables. Let $-\overline{\tau}$ be the lowest possible realization of $\epsilon(L_i)$, $\epsilon(1_i)$ and $(1-\delta)\epsilon(L_i) + \kappa(\epsilon - 1)$, $i = 1, 2$ and define $\widehat{\epsilon}(L_i) = \epsilon(L_i) + \overline{\tau}$, $\widehat{\epsilon}(1_i) = \epsilon(1_i) + \overline{\tau}$, $\widehat{x} = \overline{x} + \overline{\tau}$, $\widehat{y} = \overline{y} + \overline{\tau}$. Since we shift the two systems by the same magnitude, this shifting will not affect the order of the solutions to the two systems, i.e., the order of $\overline{s}^{L_1}$ and $\overline{s}^{L_2}$ is the same as the order of $\widehat{s}^{L_1}$ and $\widehat{s}^{L_2}$. The latter are the solutions of

$$\max_{\widehat{y}} \mathsf{E}[\alpha^L g(\widehat{y} - \widehat{\epsilon}(L_i)) - c\widehat{y} + \alpha c(1-\delta)(\widehat{y} - \widehat{\epsilon}(1_i))], i = 1, 2.$$

The comparison of the base-stock levels $\widehat{s}^{L_i}$, $i = 1, 2$, is now similar as in standard base-stock models – refer to Song (1994). (The details of the analysis are available from the authors.) $\square$

Proof of Theorem 13. We prove this result by induction. In period $T+1$, $\widetilde{V}^K_{T+1} \equiv 0$ is K-concave.

Suppose that in period $t+1$, $\widetilde{V}^K_{t+1}$ is K-concave. Then, in period t, from the proof of Theorem 8 and the properties of K-concave functions, we have that $\widetilde{J}^K_t(\overline{y}_t)$ is K-concave in $\overline{y}_t$.

Following the definition of $\overline{S}_t$, it is optimal not to order when $\overline{x}_t \geqslant \overline{S}_t$. Likewise, for $\overline{s}_t \leqslant \overline{x}_t \leqslant \overline{S}_t$, it is optimal not to order since $\widetilde{J}^K_t(\overline{x}_t) \geqslant \widetilde{J}^K_t(\overline{S}_t) + K$.

We now show that for all $\overline{x}_t < \overline{s}_t$, it it optimal to order up to $\overline{S}_t$, i.e., $\widetilde{J}^K_t(\overline{x}_t) \leqslant \widetilde{J}^K_t(\overline{S}_t) + K$. Suppose that at the point $\overline{s}_t - \gamma$, with $\gamma > 0$, we had $\widetilde{J}^K_t(\overline{s}_t - \gamma) >$

$\widetilde{J}_t^K(\overline{S}_t) + K$. Then, we would have $\widetilde{J}_t^K(\overline{S} - \gamma) > \widetilde{J}_t^K(\overline{s}_t)$. Defining $\beta = \overline{S}_t - \overline{s}_t$, we have

$$\widetilde{J}_t^K(\overline{s}_t) + \frac{\beta[\widetilde{J}_t^K(\overline{s}_t) - \widetilde{J}_t^K(\overline{s}_t - \gamma)]}{\gamma} < \widetilde{J}_t^K(\overline{s}_t) = \widetilde{J}_t^K(\overline{S}_t + \beta) + K,$$

which violates the K-concavity of $\widetilde{J}_t^K$.

Next, we prove that $\widetilde{V}_t^K(\overline{x}_t)$ is K-concave. Define $\widetilde{V}_t^{K+} = \widetilde{V}_t^K(\overline{x}_t) - c_t\overline{x}_t$. It is suffices to show that $\widetilde{V}_t^{K+}(\overline{x}_t)$ is K-concave. For any positive numbers γ and β, we have:

Case 1. If $\overline{x}_t + \beta \leqslant \overline{s}_t$ for all $z \leqslant \overline{x}_t + \beta$, then $\widetilde{V}_t^{K+}(z) = \widetilde{J}_t^K(\overline{S}_t) + K$, which is K-concave.

Case 2. If $\overline{x}_t - \gamma \geqslant \overline{s}_t$ for all $z \geqslant \overline{x}_t - \gamma$, then $\widetilde{V}_t^{K+}(z) = \widetilde{J}_t^K(z)$, which is also K-concave.

Case 3. If $\overline{x}_t - \gamma \leqslant \overline{s}_t < \overline{x}_t + \beta$, then $\widetilde{V}_t^{K+}(\overline{x}_t - \gamma) = \widetilde{J}_t^K(\overline{S}_t) + K$ and $\widetilde{V}_t^{K+}(\overline{x}_t + \beta) = \widetilde{J}_t^K(\overline{x}_t + \beta) \leqslant \widetilde{J}_t^K(\overline{S}_t)$. Moreover, if $\widetilde{V}_t^{K+}(\overline{x}_t) \geqslant \widetilde{J}_t^K(\overline{S}) + K$, then

$$\widetilde{V}_t^{K+}(\overline{x}_t) + \frac{\beta[\widetilde{V}_t^{K+}(\overline{x}_t) - \widetilde{V}_t^{K+}(\overline{x}_t - \gamma)]}{\gamma} \geqslant \widetilde{V}_t^{K+}(\overline{x}_t) \geqslant \widetilde{J}_t^K(\overline{S}_t) + K \geqslant \widetilde{V}_t^{K+}(\overline{x}_t + \beta) + K.$$

If $\widetilde{V}_t^{K+}(\overline{x}_t) < \widetilde{J}_t^K(\overline{S}_t) + K$, then $\widetilde{V}_t^{K+}(\overline{x}_t) = \widetilde{J}_t^K(\overline{x})$. Therefore, $\widetilde{V}_t^{K+}(\overline{x}_t)$ is K-concave in $\overline{x}_t$. $\square$

Proof of Theorem 14. Prove by induction. $V_{T-L} = 0$. Theorem holds at time $T - L$. At the beginning of time $T - L - 1$, the current period procurement cost $C(T - L - 1) = c_{T-L-1}$ is first realized. Then the optimal procurement decision is solved from

$$\max_{y_{T-L-1} \geqslant x_{T-L-1}} \mathsf{E}\{R(D(T - L - 1)) - c_{T-L-1}(y_{T-L-1} - x_{T-L-1}) \\ -\alpha^L G(y_{t-L-1} - D^L(t - L - 1))\}$$

which is concave in y_{T-L-1}. Then there exists a base-stock level s_{T-L-1} that maximizes the above objective function. The first order condition of y_{T-L-1} can be expressed as follows:

$$-c_{T-L-1} = \alpha^L \frac{\partial G(y_{t-L-1} - D^L(t-L-1))}{\partial y_{t-L-1}}$$

Then from the convexity of the inventory cost function we can have that s_{T-L-1} is non-increasing in c_t. T Therefore, the results also hold at time $t-L-1$. □

Proof of Proposition 15. The myopic procurement policy in each period t is solved from the following single period maximization problem, which is concave in y_t.

$$\max_{y_t \geqslant x_t} \mathsf{E}\{R(D(t)) - C(t)(y_t - x_t) - \alpha^L G(y - D^L(t)) + \alpha C(t+1)(y_t - D(t)) | C(t) = c_t\}$$

Then optimal order-up-to level is determined from the first order condition:

$$-c_t - \alpha \mathsf{E}[C(t+1)|C(t) = c_t] = \alpha^L \frac{\partial G(y_t - D^L(t))}{\partial y_t}$$

which is similar to the first order condition in the newsvendor model. Furthermore, since we express the lead time demand in the way that separates the mean demand value and the demand randomness, then we can write the myopic base-stock level as the summation of the mean demand during lead time and the safety stock which only depends on the randomness of the demand function. □

Proof of Proposition 16. Since $s_t^*(c_t) = \mathsf{E}[D^L(t)|C(t) = c_t] + std(Ran(D^L(t))|C(t) = c_t)z^*(c_t)$, then to show $s_t^*(c_t)$ increases in t, we analyze each part of $s_t^*(c_t)$ separately. The standard deviation $std(Ran(D^L(t))|C(t) = c_t)$ is independent of c_t, then we only need to take a look at $\mathsf{E}[D^L(t)|C(t) = c_t]$ and $z^*(c_t)$.

The inequality $\mathsf{E}[D^L(t+1)|C(t+1) = c_{t+1}] \geqslant \mathsf{E}[D^L(t)|C(t) = c_t]$ can be expressed as

$$
\begin{aligned}
& 2\lambda - (1+a)\mu(1+\beta)c_{t+1} - \mu(1+\beta)\theta + \mu(1+\beta)\delta_{t+1} \\
\geqslant \quad & 2\lambda - (1+a)\mu(1+\beta)c_t - \mu(1+\beta)\theta + \mu(1+\beta)\delta_t
\end{aligned}
$$

which is equivalent to $(1+a)\mu(1+\beta)(c_{t+1}-c_t) \leqslant \mu(1+\beta)\delta_t$, i.e, $(c_{t+1}-c_t) \leqslant \delta_t/(1+a)$.

To have $z^*(c_t+1) \leqslant z^*(c_t)$, it is equivalen to have

$$
\frac{h + c_t - \alpha \mathsf{E}[C(t+1)|c_t]}{h+b} \leqslant \frac{h + c_{t+1} - \alpha \mathsf{E}[C(t+2)|c_{t+1}]}{h+b}.
$$

Or equivalently,

$$
c_t - \alpha \mathsf{E}[C(t+1)|c_t] \leqslant c_{t+1} - \alpha \mathsf{E}[C(t+2)|c_{t+1}].
$$

$c_t - \alpha \mathsf{E}[C(t+1)|c_t] = (1-\alpha a)c_t - \alpha\theta + \alpha\delta_t t$, then the above inequality is equivalent to

$$
c_{t+1} - c_t \leqslant \alpha \frac{\delta_{t+1}(t+1) - \delta_t t}{1 - \alpha a}
$$

□

Proof of Proposition 17. From the fact that $\mathsf{E}C(t)$ is linearly increasing in ρ_t, it is not hard to show that $\mathsf{E}D^L(t)$ is linearly decreasing in ρ_t, i.e., as more eco-friendly inputs are incorporated into the production process, the production cost tends to be higher and the demand during lead time tends to be dampened since the higher cost is partially transferred to the selling price. On the other hand, with a higher proportion of the eco-friendly input, the cost variance $e(t)$ in each period t is reduced. Then the variance of lead time demand is correspondingly reduced, i.e., $Std(Ran(D^L(t)))$ decreases in ρ_t. □

Proof of Proposition 18.

$$\begin{aligned} V_t^M(C(0)) &= [\mathsf{E}R(D(t)) - (C(t) - \alpha\mathsf{E}C(t+1))s^*(C(t)) \\ &\quad -\alpha\mathsf{E}C(t+1)D(t) - \alpha^L G(y - D^L(t))|C(0)] \\ &= [\mathsf{E}R(D(t)) - (C(t) - \alpha\mathsf{E}C(t+1))\mathsf{E}D^L(t) - \alpha\mathsf{E}C(t+1)D(t) \\ &\quad -\alpha^L(h+b)std(Ran(D^L(t)))\phi(z^*(C(t)))|C(0)] \end{aligned}$$

where

$$\begin{aligned} \mathsf{E}R(D(t)) &= \lambda(1+\beta)\mathsf{E}C(t) - \mu(1+\beta)^2\mathsf{E}C(t)^2 \\ &= \lambda(1+\beta)[C(0) - \delta_t\tfrac{t-(1-a^t)/(1-a)}{1-a}] \\ &\quad -\mu(1+\beta)^2[C(0) - \delta_t\tfrac{t-(1-a^t)/(1-a)}{1-a} + \mathsf{E}e(a,t)^2]^2 \end{aligned}$$

Since both δ_t and $\mathsf{E}e(a,t)^2$ decrease in ρ_t, then $\mathsf{E}R(D(t))$ is increasing, concave in ρ_t.

From the expression of $-(C(t) - \alpha\mathsf{E}C(t+1))\mathsf{E}D^L(t)$ and $-\alpha\mathsf{E}C(t+1)D(t)$, it is easy to tell that they are concave in ρ_t. Next we check the concavity of $-\phi(z^*(c_t))$. Since the density function of the standard normal distribution $\phi(z)$ is convex when $z \leqslant -1$ or $z \geqslant 1$, then we need $z^*(c_t) \leqslant -1$ or $z^*(c_t) \geqslant 1$ to guarantee the concavity of the myopic profit function, which is equivalent to $b-[C(t)-\mathsf{E}C(t+1)] \leqslant \Phi(-1)(b+h)$ and $b - [C(t) - \mathsf{E}C(t+1)] \geqslant \Phi(1)(b+h)$. □

Proof of Proposition 19. To prove ρ^* is decreasing in μ, we only need to show the myopic optimal profit $V_t^M(C(0))$ is sub-modular in (ρ, μ). Since $\mathsf{E}R(D(t))$ is the only term contains both μ and ρ and $\mathsf{E}R(D(t))$ is sub-modular in (ρ, μ). Then ρ^* is decreasing in μ. □

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