Geometry of $SU(3)$ Manifolds

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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2008

Abstract

(Differential Geometry)

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Abstract

I study the differential geometry of 6-manifolds endowed with various $SU(3)$ structures from three perspectives. The first is special Lagrangian geometry; The second is pseudo-Hermitian-Yang-Mills connections or, more generally, ω-anti-self dual instantons; The third is pseudo-holomorphic curves.

For the first perspective, I am interested in the interplay between $SU(3)$-structures and their special Lagrangian submanifolds. More precisely, I study $SU(3)$-structures which locally support as 'nice' special Lagrangian geometry as Calabi-Yau 3-folds do. Roughly speaking, this means that there should be a local special Lagrangian submanifold tangent to any special Lagrangian 3-plane. I call these $SU(3)$-structures *admissible*. By employing Cartan-Kähler machinery, I show that locally such admissible $SU(3)$-structures are abundant and much more general than local Calabi-Yau structures. However, the moduli space of the compact special Lagrangian submanifolds is not so well-behaved in an admissible $SU(3)$-manifold as in the Calabi-Yau case. For this reason, I narrow attention to *nearly Calabi-Yau* manifolds, for which the special Lagrangian moduli space is smooth. I compute the local generality of nearly Calabi-Yau structures and find that they are still much more general than Calabi-Yau structures. I also discuss the relationship between nearly Calabi-Yau and half-flat $SU(3)$-structures. To construct complete or compact admissible examples, I study the twistor spaces of Riemannian 4-manifolds. It turns out that twistor spaces over self-dual Einstein 4-manifolds provide admissible and nearly Calabi-Yau manifolds. I also construct some explicit special Lagrangian examples in nearly Kähler $\mathbf{CP}^3$ and the twistor space of H^4.

For the second perspective, we are mainly interested in pseudo-Hermitian-Yang-

Mills connections on nearly Kähler six manifolds. Pseudo-Hermitian-Yang-Mills connections were introduced by R. Bryant in [4] to generalize Hermitian-Yang-Mills concept in Kähler geometry to almost complex geometry. If the $SU(3)$-structure is nearly Kähler, I show that pseudo-Hermitian-Yang-Mills connections (or, more generally, ω-anti-self-dual instantons) enjoy many nice properties. For example, they satisfy the Yang-Mills equation and thus removable singularity results hold for such connections. Moreover, they are critical points of a Chern-Simons functional. I derive a Weitzenböck formula for the deformation and discuss some of its application. I construct some explicit examples that display interesting singularities.

For the third perspective, I study pseudo-holomorphic curves in nearly Kähler $\mathbf{CP}^3$. I construct a one-to-one correspondence between *null torsion* curves in the nearly Kähler $\mathbf{CP}^3$ and contact curves in the Kähler $\mathbb{CP}^3$ (considered as a complex contact manifold). From this, I derive a Weierstrass formula for all *null torsion* curves by employing a result of R. Bryant in [9]. In this way, I classify all pseudo-holomorphic curves of genus 0.

Contents

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Introduction

0.1 $SU(3)$-structure on vector spaces

Let V be a 6-dimensional vector space. Let $(\Omega, \phi) \in \Lambda^2 V^* \oplus \Lambda^3 V^*$ be a pair of forms with Ω nondegenerate. Clearly the full linear group $GL(V)$ acts on such pairs by pullback. It is interesting to ask what the orbits and stabilizer groups are like. This problem was solved first by Banos in [3]. Later on, Bryant gave a simplified proof in [4].

There are two ways to this problem in the literature. One is to normalize Ω first and then consider the orbits of ϕ under the symplectic group $Sp(\Omega)$. Note that under the action of $Sp(\Omega)$, $\Lambda^3 V^*$ decomposes into two irreducible pieces: $\Omega \wedge V^*$ and $\Lambda^3_0 V^*$ of *effective forms* annihilated by wedge product by Ω. Since $Sp(V)$ acts transitively on $V^* \setminus \{0\}$, $\Omega \wedge V^*$ has only two obvious orbits. The difficult part is the orbits in $\Lambda^3_0 V^*$. This approach was adopted in [3].

The other is just the opposite, namely to classify ϕ orbits under $GL(V)$ and then to normalize Ω under different stabilizer groups of ϕ. The second turns out to be simpler and was adopted in [4]. We cite the result as follows

Proposition 0.1.1 ([3],[4]). *Suppose $(\omega, \Psi) \in \Lambda^2 V^* \oplus \Lambda^3 V^*$. Assume ω is nondegenerate and Ψ is primitive (i.e., $\omega \wedge \Psi = 0$). Then under the action of $GL(V)$, the pair (ω, ϕ) can be normalized to one of the following with the corresponding stabilizer group $G(\Omega, \phi)$:*

Ω	ϕ	$G(\Omega, \phi)$
$e^{12} + e^{25} + e^{36}$	$\mu(e^{123} + e^{456})$	$SL(3, \mathbf{R})$
$e^{14} + e^{25} + e^{36}$	$\mu(e^{123} - e^{156} - e^{246} - e^{345})$	$SU(3)$
$e^{14} + e^{25} + e^{36}$	$\mu(e^{123} - e^{156} + e^{246} + e^{345})$	$SU(1, 2)$
$e^{14} + e^{25} + e^{36}$	$e^{156} + e^{264} + e^{345}$	$\mathbf{R}^5 \times SO(3)$
$e^{14} + e^{25} + e^{36}$	$e^{156} - e^{264} - e^{345}$	$\mathbf{R}^5 \times SO(1, 2)$
$e^{14} + e^{25} + e^{36}$	$e^{123} \pm e^{345}$	
$e^{14} + e^{25} + e^{36}$	e^{123}	
$e^{14} + e^{25} + e^{36}$	0	

where $\mu > 0$ *and we have denoted* $e^{12} = e^1 \wedge e^2$ *and* $e^{123} = e^1 \wedge e^2 \wedge e^3$*, etc.*

In the list of Proposition 0.1.1, the only compact stabilizer group is $SU(3)$, the case we are mainly interested in. We assume $\mu = 1$. By introducing a complex basis

$$\omega^1 = e^1 + ie^4, \quad \omega^2 = e^2 + ie^5, \quad \omega^3 = e^3 + ie^6$$

we rewrite

$$\Omega = \frac{i}{2}(\omega^1 \wedge \overline{\omega^1} + \omega^2 \wedge \overline{\omega^2} + \omega^3 \wedge \overline{\omega^3})$$

and

$$\phi = \mathrm{Re}\Psi$$

with

$$\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3.$$

Thus, by specifying Ω and ϕ (or Ψ), we fix an $SU(3)$ structure on V such that ω^i are special unitary complex basis. It is easy to verify the normalization condition

$$\frac{1}{8}\omega^3 = \frac{i}{6}\Psi \wedge \overline{\Psi}.$$

0.2 $SU(3)$-structures on 6-manifolds

We briefly review G-structures and then focus on $SU(3)$-structures. A good reference for general theory of G-structures is [5].

0.2.1 *G-structure*

Let $V = \mathbf{R}^n$ be the n-dimensional vector space of column vectors. The linear transformation group $GL(V)$ may be considered as invertible $n \times n$ matrices acting by multiplication on vectors. Suppose that M is an n-dimensional smooth manifold. Let $x : \mathcal{F} \to M$ be the total coframe bundle of M. Explicitly,

$$\mathcal{F} = \{(x, u) : u : T_xM \xrightarrow{\simeq} V\}.$$

It is a principal $GL(V)$-bundle over M with the right action

$$(x, u) \circ g = (x, g^{-1}u)$$

for $g \in GL(V)$.

On $\mathcal{F}$, there is a *tautological 1-form* ω defined by

$$\omega_{(x,u)}(V) = u(x_*(V))$$

for any $V \in T_{(x,u)}\mathcal{F}$. Clearly, ω vanishes on all vertical vectors, so the n components of ω form a basis for the semi-basic 1-forms. It also satisfies the $GL(V)$-equivariance property,

$$g^*\omega = g^{-1}\omega$$

for all $g \in GL(V)$. Moreover, it has an interesting *reproducing* property: For any local section s of $\mathcal{F}$, $s^*\omega = s$. This follows from the definition.

Let G be a Lie subgroup of $GL(V)$.

Definition 0.2.1. *A G-structure on M is a reduction of the total coframe bundle $\mathcal{F}$ to a principal G-subbundle* $\mathbf{F}$.

We still denote by ω the tautological 1-form restricted to $\mathbf{F}$. Then ω is G-equivariant. Pick a connection θ on $\mathbf{F}$ (by the general theory of principal bundles, one always exists). From the equivariance of ω, we have *the first structure equation*

$$d\omega = -\theta \wedge \omega + \frac{1}{2}T(\omega \wedge \omega) \tag{1}$$

where $\omega \wedge \omega$ is a $\Lambda^2 V$-valued 2-form and T is $\Lambda^2 V^* \otimes V$-valued. Due to the G-equivariance of both ω and θ, T is also G-equivariant. In other words, T defines a section of the vector bundle $\mathbf{F} \times_G (\Lambda^2 V^* \otimes V)$. We usually call T the *apparent torsion.*

Now any other connection $\hat{\theta}$ on $\mathbf{F}$ differs from θ by a G-equivariant semibasic $\mathfrak{g} = Lie(G)$-valued 1-form, i.e., a section a of $\mathbf{F} \times_G (V^* \times \mathfrak{g})$. The tensor $\hat{T}$ in (1) with θ replaced by $\hat{\theta}$ changes from T to

$$\hat{T}(\omega \wedge \omega) = T(\omega \wedge \omega) + 2a \wedge \omega. \tag{2}$$

The discussion so far is illustrated by the following sequence of G-modules

$$0 \to \mathfrak{g}^{(1)} \to \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^* \to H^{(0,2)}(\mathfrak{g}) \to 0. \tag{3}$$

Here, $\mathfrak{g}$ is regarded as a subspace of $\mathfrak{gl}(V) = V \otimes V^*$. The map δ skew-symmetrizes the two V^* factors in $\mathfrak{g} \otimes V^* \subset V \otimes V^* \otimes V^*$. The kernel $\mathfrak{g}^{(1)}$ of δ is called the *first prolongation* of $\mathfrak{g}$ and the cokernel is denoted by $H^{(0,2)}(\mathfrak{g})$. By associating the various spaces in the sequence with the principal bundle $\mathbf{F}$, we get an exact sequence of vector bundles. We see that a is a section of $\mathbf{F} \times_G (\mathfrak{g} \otimes V^*)$ and T, $\hat{T}$ are sections of $\mathbf{F} \times_G (\Lambda^2 V^* \otimes V)$. The equation (2) says that a different choice of connection only changes T by the δ image of $a \in \mathbf{F} \otimes_G (\mathfrak{g} \otimes V^*)$. Thus, its equivalence class $[T] \in \mathbf{F} \times_G H^{0,2}(\mathfrak{g})$ is independent of the connection chosen. The tensor $[T]$ is called the *intrinsic torsion* of the G-structure $\mathbf{F}$. For this reason, $H^{(0,2)}(\mathfrak{g})$ is sometimes called the *essential torsion space*. The first prolongation $\mathfrak{g}^{(1)}$ measures the freedom we have in choosing the connections θ once we fix a representative T of the torsion $[T]$.

If all finite-dimensional representations of G are completely reducible, we can identify $H^{(0,2)}(\mathfrak{g})$ as a (not necessarily unique) subspace of $V \otimes \Lambda^2 V^*$. Then we can always modify θ so that the apparent torsion T falls in $H^{(0,2)}(\mathfrak{g})$. This is called absorbing non-essential torsion. If, further, $\mathfrak{g}^{(1)} = 0$, e.g., when G is a subgroup of $O(g)$ for some nondegenerate symmetric bi-linear form g, then the connection θ will be unique once we require the torsion is fixed inside $V \otimes \Lambda^2 V^*$. The connection is canonical in the sense that it is preserved by diffeomorphisms.

Example 0.2.2. *Let g be a Riemannian metric on M and let $\mathbf{F}$ be the orthogonal coframe bundle of g. Then $\mathbf{F}$ is an $O(n)$-structure on M. In the sequence (3), when G is replaced by $O(n)$, both $\mathfrak{g}^{(1)}$ and $H^{(0,2)}(\mathfrak{g})$ vanish. Thus the sequence degenerates to $\delta_o : \mathfrak{o}(n) \otimes V^* \xrightarrow{\simeq} V \otimes \Lambda^2 V^*$. It follows that there exists on $\mathbf{F}$ a unique connection θ such that $T = 0$. The connection is usually called the Levi-Civita connection.*

0.2.2 $SU(3)$-structures

From now on, let us assume $n = 6$ and $G = SU(3)$. An $SU(3)$ structure on a 6-manifold M is equivalent to specifying a pair of forms $(\Omega, \phi) \in \Lambda^2 T^* M \oplus \Lambda^3 T^* M$ such that, at each point $x \in M$, there exists a basis of 1-forms $\{e^i\}_{i=1}^6$ so that

$$\Omega_x = e^{14} + e^{25} + e^{36}, \quad \phi_x = e^{123} - e^{156} - e^{246} - e^{345}.$$

To see this, suppose first $\pi : \mathbf{F} \to M$ is an $SU(3)$-structure. For any $(x, u) \in \mathbf{F}$, let

$$\Omega_x = u^*(\Omega_0), \quad \phi_x = u^*(\phi_0)$$

with $\Omega_0 = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2} + dz_3 \wedge d\overline{z_3})$ and $\phi_0 = \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3)$ are forms on $\mathbf{R}^6 \simeq \mathbf{C}^3$. Since $(x, u) \circ g = (x, g^{-1}u)$ for all $g \in SU(3)$, Ω_x and ϕ_x are independent of u. Conversely, suppose such Ω and ϕ exist. Let

$$\mathbf{F} = \{(x, u) \in \mathfrak{F} : u^*\Omega_0 = \Omega_x, u^*\phi_0 = \phi_x\}.$$

By Proposition 0.1.1, $\mathbf{F}$ is an $SU(3)$ principal bundle, i.e., an $SU(3)$-structure on M.

Specially adapted for $SU(3)$ structures, we will use complex tautological 1-forms. Define

$$\omega_i|_{(x,u)} = u^*(dz_i), \quad i = 1, 2, 3$$

where dz_i is the standard i-th complex coordinate on $\mathbf{C}^3$. Then it follows from the above discussion that

$$\pi^*\Omega = \frac{\sqrt{-1}}{2}(\omega_1 \wedge \overline{\omega_1} + \omega_2 \wedge \overline{\omega_2} + \omega_3 \wedge \overline{\omega_3}), \quad \pi^*\phi = \mathrm{Re}(\omega_1 \wedge \omega_2 \wedge \omega_3).$$

Zeroth order invariants: complex volume form, metric, and almost complex structure

The existence of an $SU(3)$-structure on M usually requires topological conditions, e.g., certain characteristic classes vanish. We do not pursue this but are more interested in the geometric consequences of an $SU(3)$-structure. We first discuss invariants of zeroth order, i.e., without differentiating the defining forms Ω and ϕ.

Besides Ω and ϕ themselves, another obvious one is the complex 3-form Ψ which, when pulled back to $\mathbf{F}$ by π, has the form $\omega_1 \wedge \omega_2 \wedge \omega_3$.

Since $SU(3)$ is compact, an $SU(3)$ structure determines a metric g on M, with the property

$$\pi^*g = \omega_1 \circ \overline{\omega_1} + \omega_2 \circ \overline{\omega_2} + \omega_3 \circ \overline{\omega_3}.$$

Since $SU(3)$ is a subgroup of $GL(3, \mathbf{C})$, an $SU(3)$-structure determines an almost complex structure J on M. At each point x, $J = u^{-1} \circ J_0 \circ u : T_xM \to T_xM$ where J_0 is the standard complex structure on $\mathbf{C}^3$. It is clearly independent of the coframe

$(x, u) \in \mathbf{F}_x$ chosen. With respect to J, Ψ is of type $(3, 0)$. We call it the complex volume form.

First order invariants: connection and torsion

Zeroth order invariants do not distinguish $SU(3)$ structures. To go further, we need study the first structure equations. First let us take a closer look at the sequence (3). As mentioned before, $\mathfrak{su}(3)^{(1)} = 0$. Moreover,

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{su}(3) \otimes V^* & \xrightarrow{\delta} & V \otimes \Lambda^2 V^* & \to & H^{(0,2)}(\mathfrak{su}(3)) & \to & 0 \\ & & \| & & \uparrow \delta_o & & \uparrow \bar{\delta}_o & & \\ 0 & \to & \mathfrak{su}(3) \otimes V^* & \to & \mathfrak{so}(6) \otimes V^* & \to & \mathfrak{so}(6)/\mathfrak{su}(3) \otimes V^* & \to & 0 \end{array}$$

is a commutative diagram where δ_o is introduced in Example 0.2.2 and $\bar{\delta}_o$ is induced from it. Since δ_o is an $SU(3)$ isomorphism, so is $\bar{\delta}_o$. The second exact sequence has an obvious splitting by identifying $\mathfrak{so}(6)/\mathfrak{su}(3)$ as the orthogonal complement $\mathfrak{su}(3)^\perp$ in $\mathfrak{so}(6)$ with respect to the Killing metric. In this way, we obtain a splitting of the first exact sequence by identifying $H^{(0,2)}(\mathfrak{su}(3))$ as δ_o-image of $\mathfrak{su}(3)^\perp \otimes V^*$. This splitting uniquely determines an $\mathfrak{su}(3)$ connection θ on $\mathbf{F}$ by requiring its apparent torsion lie entirely in $\delta_o(\mathfrak{su}(3)^\perp \otimes V^*)$.

One way to think about θ is to relate it to the Levi-Civita connection θ_0 on the orthogonal coframe bundle $F \cdot O(n)$. Restricting θ_0 to $\mathbf{F}$,

$$\theta_0 = \theta \oplus (-\tau)$$

according to $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp$. The $\mathfrak{su}(3)^\perp$ component $-\tau$ is clearly semibasic and hence takes values in $\mathfrak{su}(3)^\perp \otimes V^*$. Since the Levi-Civita connection θ_0 is torsion free, the apparent torsion of θ is $\delta_o(\tau)$.

For later use, we need to write the components of torsion more explicitly. First for complexified tautological 1-form ω, the first structure equation reads

$$d \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix} = -\theta_0 \wedge \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix},$$

$$\theta_0 = \begin{pmatrix} \theta + \frac{\sqrt{-1}}{3}\mu & \beta \\ \overline{\beta} & \overline{\theta} - \frac{\sqrt{-1}}{3}\mu \end{pmatrix}$$

where $\overline{\theta} + \theta^t = tr(\theta) = 0$, μ is real and β is complex and skew-symmetric. On $\mathbf{F}$, μ and β are semibasic and hence are linear combinations of ω and $\overline{\omega}$. Based on this, we expand the structure equations out and rearrange terms to get

$$d\omega_i = -\theta_{i\bar{j}} \wedge \omega_j + \frac{1}{2} S_{ij}\overline{\epsilon_{jkl}}\omega_k \wedge \omega_l + \frac{1}{2} N_{i\bar{j}}\epsilon_{jkl}\overline{\omega_k \wedge \omega_l} + \frac{\sqrt{-1}}{3}(\overline{\lambda_{\bar{k}}}\overline{\omega_k} + \lambda_{\bar{k}}\omega_k) \wedge \omega_i. \quad (4)$$

The quantities S, N and λ change tensorially along the fiber and thus are well-defined tensors over M. These are called the torsion of the $SU(3)$-structure $\mathbf{F}$. For example, N is just the famous Nijenhuis tensor and it vanishes if and only if the almost structure J implied by the $SU(3)$-structure is integrable. If all torsion vanishes, the connection θ coincides with the Levi-Civita connection θ_0. Thus the holonomy of the Riemannian metric is contained in $SU(3)$. In other words, the $SU(3)$-structure is Calabi-Yau.

These tensors also show up in the covariant differentiation of the defining forms Ω and ϕ. In fact, one can show that another characterization of Calabi-Yau is $d\Omega = d\Psi = 0$. For our later use, we compute the differential of Ω and $\psi = \mathrm{Im}(\Psi)$:

$$\pi^*(d\Omega) = \frac{\sqrt{-1}}{2}(N_{i\bar{i}}\overline{\Psi} - \overline{N_{i\bar{i}}}\Psi) + \frac{\sqrt{-1}}{4}(S_{il}\overline{\epsilon_{ljk}\omega_i} \wedge \omega_j \wedge \omega_k - \overline{S_{il}}\epsilon_{ljk}\omega_i \wedge \overline{\omega_j \wedge \omega_k}); \quad (5)$$

and

$$\pi^* d\Psi = -\frac{\sqrt{-1}}{8}\overline{\epsilon_{ijk}}\epsilon_{lpq}(N_{i\bar{l}} - \overline{N_{l\bar{i}}})\overline{\omega_p \wedge \omega_q} \wedge \omega_j \wedge \omega_k + \frac{1}{2}(\overline{\lambda_{\bar{l}}}\overline{\omega_l} \wedge \Psi + \lambda_{\bar{l}}\omega_l \wedge \overline{\Psi}). \quad (6)$$

1

Special Lagrangian and $SU(3)$-Structures

1.1 Special Lagrangian geometry and admissible $SU(3)$ structures

A submanifold $L^3 \subset M$ is called *special Lagrangian* if it is $\phi = \mathrm{Re}\Psi$ calibrated, i.e., if $L^*(\phi)$ is the volume form. The form ϕ is a calibration if it is closed. In this case L is minimal. Assume L is orientable. Then L is special Lagrangian with one of its orientations if and only if $\Omega|_L = \psi|_L = 0$ where $\psi = \mathrm{Im}\Psi$. In other words it is an integral manifold of the differential ideal $\mathcal{I}$ generated algebraically by Ω and ψ.

A generic $SU(3)$-structure will not admit any special Lagrangian submanifolds at all, even locally. For example, if

$$d\Omega \equiv a\phi \mod (\Omega, \psi)$$

for some non-vanishing function a, then any special Lagrangian submanifold has to annihilate $d\Omega$ and hence ϕ. Thus no special Lagrangian submanifold exists. We are interested to know which $SU(3)$-structures support as many local special Lagrangian submanifolds as the flat $\mathbf{C}^3$ does. If the $SU(3)$-structure is real analytic, so is the ideal $\mathcal{I}$. Now if $d\Omega \in \mathcal{I}$, then we may invoke the Cartan-Kähler theorem to show that $\mathcal{I}$ is involutive and has the same Cartan characters as the ideal in $\mathbf{C}^3$. Here we need not care about whether or not $d(\mathrm{Im}\Psi)$ is in $\mathcal{I}$ because $d(\mathrm{Im}\Psi)$ is of degree $3+1$ and hence vanishes automatically on any 3-dimensional submanifold.

1.1.1 Admissible $SU(3)$-structures

We make the following definition.

Definition 1.1.1. *(Admissible $SU(3)$-structures) An $SU(3)$-structure (Ω, Ψ) on M is called admissible if there exist a 1-form θ and a real function a such that*

$$d\Omega = \theta \wedge \Omega + a\psi. \tag{1.1}$$

Suppose condition (1.1) is satisfied with

$$\theta = u_{\bar{i}}\omega_i + \overline{u_{\bar{i}}\omega_i}.$$

Then it also holds that

$$d\Omega = \frac{\sqrt{-1}}{4}(u_{\bar{i}}\delta_{\bar{j}k} - u_{\bar{j}}\delta_{\bar{i}k})\omega_i \wedge \omega_j \wedge \overline{\omega_k} - \frac{\sqrt{-1}}{4}(\overline{u_{\bar{i}}}\delta_{\bar{k}j} - \overline{u_{\bar{j}}}\delta_{\bar{k}i})\overline{\omega_i \wedge \omega_j} \wedge \omega_k + a\psi.$$

Comparing with (5), we get

$$\overline{\epsilon_{ljk}}S_{il} = u_{\bar{j}}\delta_{i\bar{k}} - u_{\bar{k}}\delta_{i\bar{j}}$$

and $a = N_{i\bar{i}}$. We summarize the discussion so far as follows:

Lemma 1.1.2. *Suppose (M, Ω, Ψ) is analytic. The ideal $\mathcal{I}$ is involutive and every Ω-Lagrangian analytic 2-submanifold in M^6 can be thickened uniquely to a special Lagrangian submanifold if there exists a (necessarily unique) connection α so that*

$$d\omega_i = -\alpha_{i\bar{j}} \wedge \omega_j + \beta \wedge \omega_i + \frac{1}{2}N_{i\bar{j}}\epsilon_{jkl}\overline{\omega_k \wedge \omega_l} \tag{1.2}$$

where β is a complex 1-form and the trace of the Nijenhuis tensor, $N_{i\bar{i}}$, is real.

Remark 1.1.3. *It is easy to see that the condition (1.2) in Lemma 1.1.2 essentially gives an equivalent definition of admissible $SU(3)$-structures. For this reason, we will also call an $SU(3)$-structure satisfying (1.2)*admissible.

Remark 1.1.4. *The same result for $\mathbf{C}^3$ was shown by Harvey and Lawson in [21]. Lemma 1.1.2 says an admissible $SU(3)$-manifold supports as nice a local special Lagrangian geometry as $\mathbf{C}^3$ does. This is the best situation one can hope.*

Out of these we pick a class of special interest and call it *nearly Calabi-Yau.*

Definition 1.1.5 (Nearly Calabi-Yau)**.** *An $SU(3)$-structure (M^6, Ω, Ψ) is called* n*early Calabi-Yau if for some real function U,*

$$d\Omega = d(e^U \mathrm{Im}\Psi) = 0.$$

From (6), nearly Calabi-Yau condition amounts to $N_{i\bar{j}} - \overline{N_{j\bar{i}}} = 0$ and $\sqrt{-1}\lambda_{\bar{l}} = U_{\bar{l}}$ where we have denoted $\pi^*(dU) = U_{\bar{l}}\omega_l + \overline{U_{\bar{l}}}\overline{\omega_l}$.

In terms of structure equations we have

Proposition 1.1.6. *An $SU(3)$-structure is nearly Calabi-Yau for a real funciton U if and only if there exists a (necessarily unique) connection α so that*

$$d\omega_i = -\alpha_{i\bar{j}} \wedge \omega_j + \frac{1}{2} N_{i\bar{j}} \epsilon_{jkl} \overline{\omega_k \wedge \omega_l} + \frac{1}{3}(U_{\bar{k}}\omega_k - \overline{U_{\bar{k}}}\overline{\omega_k}) \wedge \omega_i, \tag{1.3}$$

where $dU = U_{\bar{k}}\omega_k + \overline{U_{\bar{k}}}\overline{\omega_k}$, $N_{i\bar{j}} - \overline{N_{j\bar{i}}} = 0$ and $tr(N) = N_{i\bar{i}} = 0$.

In other words, the Nijenhuis tensor is Hermitian symmetric and trace free.

Remark 1.1.7. *This definition is motivated by two reasons. First, as we will see shortly, it generalizes the concept of almost Calabi-Yau in that the underlying almost complex structure may not be integrable. Second, the moduli space of special Lagrangian submanifolds of a nearly Calabi-Yau manifold is well-behaved. We will prove this in §1.4.*

Remark 1.1.8 (The case $U = 0$)**.** *When the defining function vanishes identically, a nearly Calabi-Yau structure is also* half-flat*. A half-flat* $SU(3)$*-structure is defined so that* $\Omega \wedge d\Omega = d\text{Im}\Psi = 0$*. This is in general not admissible. However, it has been used by string physicists to study heterotic string compactifications for a long time. In this context, half-flat manifolds are as important as Calabi-Yau structures. Mathematically, it has been studied by S. Chiossi and S. Salamon in its relation with* G_2*-structures. They showed that this structure behaves well under Hitchin's flow equation [22]. Half-flat structures were also studied in [28].*

In spite of much interest, few non-Calabi-Yau examples are known. In the next subsection, we will analyze the local existence of half-flat nearly Calabi-Yau. We will see that local half-flat nearly Calabi-Yau structures are much more general than Calabi-Yau. In §1.2 we will construct a class of complete or even compact half-flat nearly Calabi-Yau but non-Calabi-Yau examples.

1.1.2 First examples

Calabi-Yau

Calabi-Yau is clearly nearly Calabi-Yau. It is the case of most interest so far.

Almost Calabi-Yau

A more general $SU(3)$-structure is called *almost Calabi-Yau* in [23]. It is also considered by R. Bryant [10] and E. Goldstein [17]. An almost Calabi-Yau manifold is a Kähler manifold with a nowhere vanishing holomorphic 3-form $\hat{\Phi}$ specified. Let B be the unique function such that

$$\frac{1}{8}\sqrt{-1}\Psi \wedge \overline{\Psi} = \frac{1}{6}B^2\Omega^3.$$

It is clear that $(\Omega, \frac{\Phi}{B}, B)$ defines a nearly Calabi-Yau structure on M. Thus, an almost Calabi-Yau structures is nearly Calabi-Yau.

Many interesting almost Calabi-Yau examples are provided by degree 5 smooth varieties in $\mathbf{C}P^4$. It is well-known that the canonical line bundle of such a variety is trivial. Thus a nowhere vanishing holomorphic 3-form exists. However, the induced Fubini-Study metric is not Calabi-Yau in general. These provide many almost Calabi-Yau but non-Calabi-Yau examples.

Nearly Kähler

The fundamental forms Ω and Ψ satisfy

$$d\Omega = 3\mathrm{Im}\Psi$$

and

$$d\Psi = 2\Omega^2.$$

It is well-known that the underlying metric is Einstein by [16]. It also follows that such structures are real analytic, in, say, coordinates harmonic for the metric. The ideal $\mathcal{I}$ is clearly differentially closed. Hence the almost special Lagrangian geometry of nearly Kähler 6 manifolds is well-behaved locally.

The underlying almost complex structure structure is non-integrable. In fact, the Nijenhuis tensor $N_{\bar{i}\bar{j}}$ is the identity matrix. Nearly Kähler but non-Calabi-Yau Examples include S^6 with the standard metric and almost complex structure, $S^3 \times S^3$, the flag manifold $SU(3)/T^2$ and the projective space $\mathbf{CP}^3$ (with an unusual almost complex structure, however).

Remark 1.1.9. *It should be cautioned that a nearly Calabi-Yau manifold, unless it is Calabi-Yau, is NOT nearly Kähler.*

1.1.3 Generalities

Calabi-Yau and nearly Kähler provide first examples of admissible $SU(3)$-structures. However, we are about to show that they are only a 'closed' subset of the moduli of local admissible $SU(3)$-structures. For this, we need to study the local generality of admissible $SU(3)$-structures.

Let $p : \mathcal{F} \to M^6$ be the total coframe bundle of M^6. Thus $\mathcal{F}$ is a principal $GL(6, \mathbf{R})$-bundle over M^6. The fiber $\mathcal{F}_x$ over x consists of the linear isomorphisms $u : T_xM \to \mathbf{R}^6$. We consider the quotient $\mathcal{F}/SU(3)$ which is $34(= 6 + 6^2 - 8)$ dimensional and projects onto M with fibers diffeomorphic to $GL(6, \mathbf{R})/SU(3)$. An $SU(3)$-structure over M may be regarded as a section of $\overline{p} : \mathcal{F}/SU(3) \to M^6$. In fact, if P is an $SU(3)$-structure, then P is a subbundle of $\mathcal{F}$ and every fiber P_x determines a unique $SU(3)$ orbit of $\mathcal{F}_x$ and thus a section of $\mathcal{F}/SU(3) \to M^6$. Conversely, if $\sigma : M \to \mathcal{F}/SU(3)$ is such a section, let P be the preimage of $\sigma(M)$ under the projection $\mathcal{F} \to \mathcal{F}/SU(3)$. Then P is the needed $SU(3)$-structure.

Remark 1.1.10. *There is a more concrete realization of $\mathcal{F}/SU(3)$. Let $\Lambda^2_+M \oplus \Lambda^3_+M$ be the subbundle of $\Lambda^2 M \oplus \Lambda^3 M$ consiting of the pairs of positive forms (ρ^2, ρ^3) in the sense that there exists a linear isomorphism $u : T_xM \to \mathbf{R}^6$ such that $\rho^2 = u^*\Omega_0$ and $\rho^3 = u^*\mathrm{Re}(\Psi_0)$. We clearly have a projection $\mathcal{F} \to \Lambda^2_+M \oplus \Lambda^3_+M$. Since the isotropy group of $(\Omega_0, \mathrm{Re}\Psi_0)$ is $SU(3) \subset GL(6, \mathbf{R})$, this is indeed a principal $SU(3)$-bundle. For similar discussions concerning G_2-structures and Spin(7)-structures, consult Bryant's work [7]. In fact, the work directly inspired the discussion in this section.*

We will write structures equations for $\mathcal{F}$. For notational conventions on linear algebra see Introduction. Let $(\omega_1, \omega_2, \omega_3, \overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3})$ denote complexified tautological 1-forms. Thus, for example, $\omega_{1u} = u^*(dz_1) \circ p_*$. We fix a $\mathfrak{gl}(6, \mathbf{R})$-valued connection form

$$\begin{pmatrix} \alpha + \frac{\sqrt{-1}}{3}\mu + \kappa & \beta \\ \overline{\beta} & \overline{\alpha} - \frac{\sqrt{-1}}{3}\mu + \overline{\kappa} \end{pmatrix}$$

on $\mathcal{F}$ where α takes value in $\mathfrak{su}(3)$, β is $\mathfrak{gl}(3, \mathbf{C})$-valued, μ is a real form, and κ satisfies

$$\overline{\kappa}^t = \kappa.$$

We have the following structure equations on $\mathcal{F}$:

$$d\begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix} = -\begin{pmatrix} \alpha + \frac{\sqrt{-1}}{3}\mu + \kappa & \beta \\ \overline{\beta} & \overline{\alpha} - \frac{\sqrt{-1}}{3}\mu + \overline{\kappa} \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix}. \tag{1.4}$$

Note that the forms $\Omega = \frac{\sqrt{-1}}{2}(\omega_1 \wedge \overline{\omega_1} + \omega_2 \wedge \overline{\omega_2} + \omega_3 \wedge \overline{\omega_3})$ and $\Psi = \omega \wedge \omega_2 \wedge \omega_3$ are $SU(3)$-invariant on $\mathcal{F}$, so they descend to $\mathcal{F}/SU(3)$. We use the same letters to denote the forms on $\mathcal{F}/SU(3)$ and let $\psi = \mathrm{Im}\Psi$ and $\phi = \mathrm{Re}\Psi$. We will use Cartan-Kähler machinery to study local admissible $SU(3)$-structures and nearly Calabi-Yau structures. For background material on exterior differential systems, see the standard text [12].

Generalities of admissible $SU(3)$-structures

We introduce a new manifold $\mathcal{M} = (\mathcal{F} \times \mathbf{C}^3)/SU(3) \times \mathbf{R}$, where $SU(3)$ acts on $\mathbf{C}^3$ in the obvious way. We use $(u_{\bar{1}}, u_{\bar{2}}, u_{\bar{3}})$ as the coordinate on $\mathbf{C}^3$ and a as the coordinate on $\mathbf{R}$. $\mathcal{M}$ is a vector bundle of rank 7 over $\mathcal{F}/SU(3)$. Let $\theta = u_{\bar{i}}\omega_i + \overline{u_{\bar{i}}\omega_i}$. Then θ is another well-defined differential form on $\mathcal{M}$ besides Ω and Ψ. On $\mathcal{M}$ define a differential ideal

$$\begin{aligned} \mathbf{I} &= \langle \Pi_3 = d\Omega - \theta \wedge \Omega - a\psi \rangle_{diff} \\ &= \langle \Pi_3 = d\Omega - \theta \wedge \Omega - a\psi, \Pi_4 = d\theta \wedge \Omega + (da - a\theta) \wedge \psi + ad\psi \rangle_{alg}. \end{aligned}$$

We are interested in 6-dimensional (local) integral manifolds of this ideal which are also local sections of $\mathcal{M} \to M^6$. Such a section pulls back Ω and Ψ to M

which satisfies the condition (1.1) and thus defines an admissible $SU(3)$-structure. A section satisfies $\sqrt{-1}\Psi\wedge\overline{\Psi}\neq 0$. Conversely a 6-dimensional submanifold of $\mathcal{M}$ on which $\sqrt{-1}\Psi\wedge\overline{\Psi}\neq 0$ is locally a section. Hence we will consider the integral manifolds of $\mathbf{I}$ with the *independence condition* $\sqrt{-1}\Psi\wedge\overline{\Psi}\neq 0$.

Theorem 1.1.11. *The differential system* $(\mathbf{I},\sqrt{-1}\Psi\wedge\overline{\Psi}\neq 0)$ *on the dense open set* $\mathcal{M}\setminus\{a=0\}$ *is involutive with Cartan characters*

$$(s_0,s_1,s_2,s_3,s_4,s_5,s_6)=(0,0,1,3,6,10,15).$$

Proof. Since the system contains no forms of degree 2 or less, we have $c_0=c_1=0$. On the other hand, $c_1=1$ and $c_6=35$. We need to compute the remaining three characters c_3, c_4 and c_5. For this we pass up to $\mathcal{F}\times\mathbf{C}^3\times\mathbf{R}$. For effective computations we set

$$Da=da-a\theta,$$

and

$$Du_{\bar{i}}=du_{\bar{i}}-u_{\bar{j}}\alpha_{\bar{j}\bar{i}}-u_{\bar{j}}\kappa_{\bar{j}\bar{i}}-\frac{\sqrt{-1}}{3}u_{\bar{i}}\mu-\overline{u_{\bar{j}}}\overline{\beta_{ji}}.$$

Relative to the projection $\mathcal{F}\times\mathbf{C}^3\times\mathbf{R}\to\mathcal{M}$, the forms $\omega,\kappa,\beta,\mu,Da,Du$ form a basis for semibasic 1-forms. In terms of these forms we have

$$\begin{aligned}\Pi_3 \;=\;& -\tfrac{\sqrt{-1}}{2}\kappa_{i\bar{j}}\wedge\omega_j\wedge\overline{\omega_i}+\tfrac{\sqrt{-1}}{2}\overline{\kappa_{i\bar{j}}}\wedge\overline{\omega_j}\wedge\omega_i\\ &-\tfrac{\sqrt{-1}}{4}(\beta_{ij}-\beta_{ji})\wedge\overline{\omega_j}\wedge\overline{\omega_i}+\tfrac{\sqrt{-1}}{4}(\overline{\beta_{ij}}-\overline{\beta_{ji}})\wedge\omega_j\wedge\omega_i\\ &-\tfrac{\sqrt{-1}}{2}(u_{\bar{i}}\omega_i+\overline{u_{\bar{i}}\omega_i})\wedge\omega_j\wedge\overline{\omega_j}\\ &-\tfrac{\sqrt{-1}}{2}a(\overline{\omega_1\wedge\omega_2\wedge\omega_3}-\omega_1\wedge\omega_2\wedge\omega_3)\end{aligned}$$

and

$$\begin{aligned}\Pi_4 &= \tfrac{\sqrt{-1}}{2}(Du_{\bar i}\wedge\omega_i+\overline{Du_{\bar i}}\wedge\overline{\omega_i})\wedge\omega_j\wedge\overline{\omega_j}\\ &+\tfrac{\sqrt{-1}}{2}(Da-a\kappa_{i\bar i})\wedge(\overline{\omega_1\wedge\omega_2\wedge\omega_3}-\omega_1\wedge\omega_2\wedge\omega_3)\\ &-\tfrac{a}{2}\mu\wedge(\overline{\omega_1\wedge\omega_2\wedge\omega_3}+\omega_1\wedge\omega_2\wedge\omega_3)\\ &-\tfrac{\sqrt{-1}}{4}a\epsilon_{ijk}\overline{\beta_{il}}\wedge\omega_l\wedge\overline{\omega_j\wedge\omega_k}\\ &+\tfrac{\sqrt{-1}}{4}a\overline{\epsilon_{ijk}}\beta_{il}\wedge\overline{\omega_l}\wedge\omega_j\wedge\omega_k.\end{aligned}$$

A six dimensional subspace E_6 of the tangent plane on which $\Psi\wedge\overline{\Psi}\neq 0$ is defined by the following relations

$$\begin{cases}\kappa_{i\bar j} = A_{i\bar j k}\overline{\omega_k}+\overline{A_{j\bar i k}}\omega_k;\\ \beta_{ij} = B_{ijk}\overline{\omega_k}+C_{ij\bar k}\omega_k,\\ Du_{\bar i} = U_{\overline{ij}}\omega_j+U_{\bar i j}\overline{\omega_j},\\ Da = a_{\bar i}\omega_i+\overline{a_{\bar i}\omega_i},\\ \mu = b_{\bar i}\omega_i+\overline{b_{\bar i}}\overline{\omega_i},\end{cases}\tag{1.5}$$

where $A_{i\bar j k}, B_{ijk}, C_{ij\bar j}, U_{\overline{ij}}, U_{\bar i j}$ and $a_{\bar i}, b_{\bar i}$ are free parameters. In order that E_6 be an integral element of $\mathbf{I}$, it must annihilate Π_3 and Π_4. This amounts to the following equations on the parameters in (1.5),

$$\begin{cases}\epsilon_{ijk}\overline{B_{ijk}} = a,\\ 2(A_{k\bar j i}-A_{i\bar j k})+C_{ik\bar j}-C_{ki\bar j}+\overline{u_{\bar i}}\delta_{\bar j k}-\overline{u_{\bar k}}\delta_{\bar j i} = 0,\\ \sqrt{-1}(U_{\overline{ij}}\epsilon_{jik}+\overline{a_{\bar k}}-C_{ik\bar i})-a(\overline{b_{\bar k}}-\sqrt{-1}A_{i\bar i k}) = 0,\\ -(U_{\bar i j}\delta_{k\bar l}+U_{\bar l k}\delta_{j\bar i}-U_{\bar l j}\delta_{k\bar i}-U_{\bar i k}\delta_{j\bar l})\\ +(\overline{U_{\bar k l}}\delta_{j\bar i}-\overline{U_{\bar j l}}\delta_{k\bar i}-\overline{U_{\bar k i}}\delta_{j\bar l}+\overline{U_{\bar j i}}\delta_{k\bar l})\\ -a\overline{\epsilon_{pil}}B_{pkj}+a\overline{\epsilon_{pil}}B_{pjk}+a\epsilon_{pjk}\overline{B_{pli}}-a\epsilon_{pjk}\overline{B_{pil}} = 0.\end{cases}\tag{1.6}$$

By inspection, we have $35 = (2 + 3 \times 3 \times 2 + 3 \times 2 + 3 \times 3)$ linearly independent affine equations in (1.6) (note that the last equations are real, while the others are complex) . The solution space is smooth, even where $a = 0$. We pick $E_5 = \text{span}\{e_1, e_2, e_3, e_4, e_5\} \subset E_6$ where e_1 is dual to $\text{Re}(\omega_1)$ and e_4 is dual to $\text{Im}(\omega_1)$, etc. First note that

$$c_5 \leq \binom{5}{2} + \binom{5}{3} = 20.$$

We will show that the equality holds, i.e., the polar equations of E_5 has the largest possible rank 20. It then follows that if we pick any flag $E_3 \subset E_4 \subset E_5$ we have

$$c_3 = \binom{3}{2} + \binom{3}{3} = 4$$

and

$$c_4 = \binom{4}{2} + \binom{4}{3} = 10.$$

Since $c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = 1 + 4 + 10 + 20 = 35$ we apply Cartan's Test to finish the proof.

The verification that the polar equations of E_5 have rank 20 is a lengthy linear algebra exercise. First, by translating $\kappa, \beta, Du, Da, \mu$ we may assume these forms vanish on E_6 since Π_3 and Π_4 are affine linear in these forms. Now the rank of polar equations are the number of linearly independent forms in $\{(e_i \wedge e_j) \lrcorner \Pi_3, (e_i \wedge e_j \wedge e_k) \lrcorner \Pi_4\}$. We omit the messy details but only point out the following facts (an unsatisfied reader may consult the computations in the proof of Theorem (1.1.14) and make necessary modifications by himself). The forms $\{(e_i \wedge e_j) \lrcorner \Pi_3\}$ pick out 10 linearly independent forms from linear combinations of $\text{Re}(\kappa_{i\bar{j}})$, $\text{Im}(\kappa_{i\bar{j}})$, $\text{Re}(\beta_{ij} - \beta_{ji})$ and $\text{Im}(\beta_{ij} - \beta_{ji})$. The 6 forms $(e_i \wedge e_{i+3} \wedge e_j) \lrcorner \Pi_4$ pick out real and imaginary parts of $\{Du_{\bar{j}} + \text{linear combinations of } \beta\}_{j=1}^{2} \bigcup \{\text{Re}(Du_{\bar{3}} - a\beta_{21}), \text{Re}(Du_{\bar{3}} + a\beta_{12})\}$. The 4 forms $(e_i \wedge e_j \wedge e_k) \lrcorner \Pi_4$ with $i \in \{1, 4\}$, $j \in \{2, 5\}$ and $k = 3$ give us non-degenerate linear combinations of the forms $Da - a\sum\kappa_{i\bar{i}}$, $a\mu$, $a\mathrm{Re}(\beta_{ii})$ and $a\mathrm{Im}(\beta_{ii})$. These three classes of equations are clearly independent from each other if we assume $a \neq 0$. $\square$

Remark 1.1.12. *There does not exist any regular flag over the locus $\{a = 0\}$ for simple reasons. When $a = 0$, only 7 independent forms $Du_{\bar{i}}$ and Da in Π_4 could contribute to the polar equations. Thus $c_5 \leq 17$.*

Remark 1.1.13. *The last nonzero character is $s_6 = 15$. Modulo diffeomorphisms, which depend on 6 functions of 6 variables, we still have 9 functions of 6 variables of local generality of admissible $SU(3)$-structures. Both local Calabi-Yau and nearly Kähler structures depend on 2 functions of 5 variables. Thus local admissible $SU(3)$-structures are much more general than Calabi-Yau and nearly Kähler.*

Generality of nearly Calabi-Yau

Nearly Calabi-Yau is a subclass of admissible $SU(3)$-structures. One would expect nearly Calabi-Yau to be less general than an admissible $SU(3)$-structure. We will show this is indeed the case. Now the differential system $\mathbb{I}$ is defined on $\mathcal{F}/SU(3)\times\mathbf{R}$ and generated algebraically by the 3-form $d\Omega$ and the 4-form $\Upsilon = d\psi + dU \wedge \psi$ with the independence condition $\sqrt{-1}\Psi \wedge \overline{\Psi}$. This system is better-behaved than $\mathbf{I}$ for admissible $SU(3)$-structures in that it is involutive on the whole $\mathcal{F}/SU(3) \times \mathbf{R}$.

Theorem 1.1.14. *The differential system $\mathbb{I}$ on $\mathcal{F}/SU(3) \times \mathbf{R}$ is involutive with Cartan characters $(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 1, 3, 6, 9, 10)$.*

Proof. Since the system contains no forms of degree 2 or less, we have $c_0 = c_1 = 0$. Moreover, it is easy to see $c_2 = 1$. To use Cartan's Test, we need compute the other 3 characters c_3, c_4 and c_5 and the codimension of the space of 6-dimensional integral elements. For this we pass up to $\mathcal{F}$ where

$$\begin{aligned} d\Omega &= -\tfrac{\sqrt{-1}}{2}\kappa_{i\bar{j}}\omega_j \wedge \overline{\omega_i} + \tfrac{\sqrt{-1}}{2}\overline{\kappa_{i\bar{j}}\omega_j} \wedge \omega_i \\ &\quad -\tfrac{\sqrt{-1}}{2}\beta_{ij} \wedge \overline{\omega_j} \wedge \overline{\omega_i} + \tfrac{\sqrt{-1}}{2}\overline{\beta_{ij}} \wedge \omega_j \wedge \omega_i, \end{aligned}$$

and

$$\begin{aligned}\Upsilon = d\psi + dU \wedge \psi &= -\tfrac{1}{2}[\mu + \sqrt{-1}(\kappa_{i\bar{i}} - dU)] \wedge \overline{\omega_1 \wedge \omega_2 \wedge \omega_3} \\ &\quad -\tfrac{1}{2}[\mu - \sqrt{-1}(\kappa_{i\bar{i}} - dU)] \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \\ &\quad -\tfrac{\sqrt{-1}}{4}\epsilon_{ijk}\overline{\beta_{il}} \wedge \omega_l \wedge \overline{\omega_j} \wedge \overline{\omega_k} + \tfrac{\sqrt{-1}}{4}\overline{\epsilon_{ijk}}\beta_{il} \wedge \overline{\omega_l} \wedge \omega_j \wedge \omega_k.\end{aligned}$$

A 6-dimensional integral element E_6 on which $\sqrt{-1}\Psi \wedge \overline{\Psi} \neq 0$ is parametrized the equations similar to (1.5) for κ, β, μ and dU,

$$\begin{cases} \kappa_{i\bar{j}} = A_{i\bar{j}k}\overline{\omega_k} + \overline{A_{j\bar{i}k}}\omega_k; \\ \beta_{ij} = B_{ijk}\overline{\omega_k} + C_{ij\bar{k}}\omega_k, \\ \mu = b_{\bar{i}}\omega_i + \overline{b_{\bar{i}}\omega_i}, \\ dU = u_{\bar{i}}\omega_i + \overline{u_{\bar{i}}\omega_i} \end{cases} \tag{1.7}$$

but now the quantities A, B, u and b satisfy the following equations

$$\begin{cases} \overline{\epsilon_{ijk}}B_{ijk} = 0 \\ 2A_{k\bar{j}i} - 2A_{i\bar{j}k} + C_{ik\bar{j}} - C_{ki\bar{j}} = 0 \\ C_{ik\bar{i}} - A_{i\bar{i}k} + \overline{u_{\bar{k}}} - \sqrt{-1}\overline{b_{\bar{k}}} = 0 \\ -\overline{\epsilon_{pil}}B_{pkj} + \overline{\epsilon_{pil}}B_{pjk} + \epsilon_{pjk}\overline{B_{pli}} - \epsilon_{pjk}\overline{B_{pil}} = 0. \end{cases} \tag{1.8}$$

The last equations are real while the others are all complex. Moreover, the last equations imply the imaginary part of the first equation. Thus the total rank of these linear equations is $1 + 3 \times 3 \times 2 + 3 \times 2 + 3 \times 3 = 34$. The forms $\mathrm{Re}(\omega_i)$ and $\mathrm{Im}(\omega_i)$ restrict to E_6 to be a dual basis. Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the basis of E_6 for which e_1 is dual to $\mathrm{Re}(\omega_1)$ and e_4 is dual to $\mathrm{Im}(\omega_1)$, etc. Again by translating we may assume $A = B = b = C = u = 0$. Let $E_3 = \mathrm{span}\{e_1, e_2, e_3\}$, $E_4 = \mathrm{span}\{e_1, e_2, e_3, e_4\}$, and $E_5 = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$.

The polar space of E_3 consists of vectors annihilating the following 1-forms

$$\left\{ \begin{array}{rcl} (e_1 \wedge e_2) \lrcorner d\Omega & = & -2\mathrm{Im}(\kappa_{1\bar{2}}) - \mathrm{Im}(\beta_{12} - \beta_{21}) \\ (e_1 \wedge e_3) \lrcorner d\Omega & = & -2\mathrm{Im}(\kappa_{1\bar{3}}) - \mathrm{Im}(\beta_{13} - \beta_{31}) \\ (e_2 \wedge e_3) \lrcorner d\Omega & = & -2\mathrm{Im}(\kappa_{2\bar{3}}) - \mathrm{Im}(\beta_{23} - \beta_{32}) \\ (e_1 \wedge e_2 \wedge e_3) \lrcorner \Upsilon & = & \mu + \mathrm{Im}(\beta_{11} + \beta_{22} + \beta_{33}) \end{array} \right\}. \tag{1.9}$$

Consequently $c_3 = 4$ and $s_3 = 3$.

The polar space of E_4 consists of vectors annihilating the forms in (1.9) as well as the following forms

$$\left\{ \begin{array}{rcl} (e_1 \wedge e_4) \lrcorner d\Omega & = & -2\kappa_{1\bar{1}} \\ (e_2 \wedge e_4) \lrcorner d\Omega & = & -2\mathrm{Re}(\kappa_{1\bar{2}}) - \mathrm{Re}(\beta_{12} - \beta_{21}) \\ (e_3 \wedge e_4) \lrcorner d\Omega & = & -2\mathrm{Re}(\kappa_{1\bar{3}}) - \mathrm{Re}(\beta_{13} - \beta_{31}) \\ (e_1 \wedge e_2 \wedge e_4) \lrcorner \Upsilon & = & -2\mathrm{Re}(\beta_{31}) \\ (e_1 \wedge e_3 \wedge e_4) \lrcorner \Upsilon & = & 2\mathrm{Re}(\beta_{21}) \\ (e_2 \wedge e_3 \wedge e_4) \lrcorner \Upsilon & = & -\sum_i \kappa_{i\bar{i}} + dU + \mathrm{Re}(-\beta_{11} + \beta_{22} + \beta_{33}) \end{array} \right\}. \tag{1.10}$$

These forms are independent among themselves and also independent from forms in (1.9). Thus $s_4 = 6$. The polar space for E_5 consists of vector annihilating forms in (1.9), (1.10) as well as the following forms

$$\left\{ \begin{array}{rcl} (e_1 \wedge e_5) \lrcorner d\Omega & = & -2\mathrm{Re}(\kappa_{1\bar{2}}) + \mathrm{Re}(\beta_{12} - \beta_{21}) \\ (e_2 \wedge e_5) \lrcorner d\Omega & = & -2\kappa_{2\bar{2}} \\ (e_3 \wedge e_5) \lrcorner d\Omega & = & -2\mathrm{Re}(\kappa_{3\bar{2}}) + \mathrm{Re}(\beta_{32} - \beta_{23}) \\ (e_4 \wedge e_5) \lrcorner d\Omega & = & -2\mathrm{Im}(\kappa_{1\bar{2}}) + \mathrm{Im}(\beta_{12} - \beta_{21}) \\ (e_1 \wedge e_2 \wedge e_5) \lrcorner \Upsilon & = & -2\mathrm{Re}(\beta_{32}) \\ (e_1 \wedge e_3 \wedge e_5) \lrcorner \Upsilon & = & -\sum_i \kappa_{i\bar{i}} + dU + \mathrm{Re}(-\beta_{11} + \beta_{22} - \beta_{33}) \\ (e_1 \wedge e_4 \wedge e_5) \lrcorner \Upsilon & = & 2\mathrm{Im}(\beta_{31}) \\ (e_2 \wedge e_3 \wedge e_5) \lrcorner \Upsilon & = & -2\mathrm{Re}(\beta_{12}) \\ (e_2 \wedge e_4 \wedge e_5) \lrcorner \Upsilon & = & 2\mathrm{Im}(\beta_{32}) \\ (e_3 \wedge e_4 \wedge e_5) \lrcorner \Upsilon & = & -\mu + \mathrm{Im}(\beta_{11} + \beta_{22} - \beta_{33}) \end{array} \right\}. \tag{1.11}$$

Note that

$$(e_2 \wedge e_4) \lrcorner d\Omega - (e_1 \wedge e_5) \lrcorner d\Omega - (e_2 \wedge e_3 \wedge e_5) \lrcorner \Upsilon - (e_1 \wedge e_3 \wedge e_4) \lrcorner \Upsilon = 0.$$

No other relations exist among the forms in (1.9), (1.10) and (1.11). Thus $s_5 = 9$ and $s_6 = 10$. Since $6s_0 + 5s_1 + 4s_2 + 3s_3 + 2s_4 + s_5 = 34$, Cartan's test is satisfied and the proof is complete. □

Thus, modulo diffeomorphisms, local nearly Calabi-Yau structures depend on 4 functions of 6 variables.

Remark 1.1.15 (Generality of half-flat nearly Calabi-Yau structures)**.** *The EDS for half-flat nearly Calabi-Yau structures is defined on* $\mathcal{F}/SU(3)$ *and algebraically generated by* $d\Omega$ *and* $d\mathrm{Im}\Psi$*. Similar analysis shows that this system is also involutive with Cartan characters*

$$(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 1, 3, 6, 9, 9).$$

Thus, modulo diffeomorphisms, these structures depend on 3 functions of 6 variables locally. They are much more general than Calabi-Yau structures. It would be interesting to also analyze the half flat system generated by $\Omega \wedge d\Omega$ *and* $d\mathrm{Im}\Psi$*.*

Remark 1.1.16 (Generality of almost Calabi-Yau structues)**.** *The EDS for almost Calabi-Yau structures is defined on* $\mathcal{F}/SU(3) \times \mathbf{R}$ *and generated algebraically by* $d\Omega$ *and* $d\Psi + dU \wedge \Psi$*. We conjecture this system is involutive with the last non-zero Cartan character to be* $s_6 = 7$*. We leave this for the interested reader to investigate.*

1.2 Examples from twistor spaces of Riemannian four-manifolds

There has been an extensive literature on twistor theory. Suppose (M^4, ds^2) is a Riemannian 4-manifold. A twistor at $x \in M$ is an orthogonal complex structure $\mathtt{j} : T_xM \to T_xM$, $\mathtt{j}^2 = -1$ and $\mathtt{j}^*(ds_x^2) = ds_x^2$. The space of twistors at points of M forms a smooth manifold $\mathcal{J}$ called *twistor space* of M. It is well-known that $\mathcal{J}$ has an almost complex structure. It is moreover complex if M has constant sectional curvatures. However, we will not use this usual almost complex structure in this paper. Instead, we will 'reverse' the almost complex structure on the fibers and obtain an $SU(3)$-structure on $\mathcal{J}$. By doing so, we will lose the possible integrability of the almost complex structures in some cases.

1.2.1 Four dimensional Riemannian geometry

We formulate Riemannian geometry of four-manifolds in moving frames. Let $\pi : \mathcal{F} \to M$ be the oriented orthonormal coframe bundle over M. Thus $\mathcal{F}_x$ consists of orientation preserving isometries $u : T_xM \to \mathbf{R}^4$. Let η be the $\mathbf{R}^4$-valued tautological form on $\mathcal{F}$. By the fundamental theorem of Riemannian geometry, there exists a unique $\mathfrak{so}(4,\mathbf{R})$-valued one-form θ so that

$$d\eta = -\theta \wedge \eta.$$

Denote $\omega_1 = \eta_1 + \sqrt{-1}\eta_3$ and $\omega_2 = \eta_2 + \sqrt{-1}\eta_4$. We write the structrure equation as

$$d\begin{pmatrix} \omega_1 \\ \omega_2 \\ \overline{\omega_1} \\ \overline{\omega_2} \end{pmatrix} = -\begin{pmatrix} \alpha_{i\bar{j}} & \overline{\beta_{\bar{i}\bar{j}}} \\ \beta_{\bar{i}\bar{j}} & \overline{\alpha_{i\bar{j}}} \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \overline{\omega_1} \\ \overline{\omega_2} \end{pmatrix} \tag{1.12}$$

where $\alpha^t + \overline{\alpha} = 0$ and $\beta^t + \beta = 0$. The Riemannian curvature is of course

$$R = d\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} + \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \wedge \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}.$$

This is a $\mathfrak{so}(4,\mathbf{R})$-valued 2-form. Corresponding to the decomposition $\mathfrak{so}(4,\mathbf{R}) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ we decompose $R = R_+ + R_-$, where

$$R_+ = d\begin{pmatrix} \alpha_0 & 0 \\ 0 & \overline{\alpha_0} \end{pmatrix} + \begin{pmatrix} \alpha_0 & 0 \\ 0 & \overline{\alpha_0} \end{pmatrix} \wedge \begin{pmatrix} \alpha_0 & 0 \\ 0 & \overline{\alpha_0} \end{pmatrix}$$

and

$$R_- = d\begin{pmatrix} \frac{1}{2}tr(\alpha)I & \overline{\beta} \\ \beta & \overline{\frac{1}{2}tr(\alpha)I} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}tr(\alpha)I & \overline{\beta} \\ \beta & \overline{\frac{1}{2}tr(\alpha)I} \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2}tr(\alpha)I & \overline{\beta} \\ \beta & \overline{\frac{1}{2}tr(\alpha)I} \end{pmatrix}.$$

for which $\alpha_0 = \alpha - \frac{1}{2}\mathrm{tr}(\alpha)I$ takes value in $\mathfrak{su}(2)$. We are mainly interested in R_- so we examine this part more carefully. Write

$$\beta = \begin{pmatrix} 0 & \omega_3 \\ -\omega_3 & 0 \end{pmatrix},$$

$$(R_-)_1 = \frac{1}{2} dtr(\alpha) + \omega_3 \wedge \overline{\omega_3},$$

and

$$(R_-)_2 = d\omega_3 - tr(\alpha) \wedge \omega_3.$$

The forms

$$\Theta_1 = \omega_1 \wedge \omega_2, \quad \Theta_2 = \overline{\omega_1 \wedge \omega_2}, \quad \Theta_3 = \frac{\sqrt{-1}}{2}(\omega_1 \wedge \overline{\omega_1} + \omega_2 \wedge \overline{\omega_2})$$

give a basis for anti-self dual complex forms at x, while

$$\Sigma_1 = \omega_1 \wedge \overline{\omega_2}, \quad \Sigma_2 = \overline{\omega_1} \wedge \omega_2, \quad \Sigma_3 = \frac{\sqrt{-1}}{2}(\omega_1 \wedge \overline{\omega_1} - \omega_2 \wedge \overline{\omega_2})$$

form a basis for self dual forms at x. Since R_- is semibasic,

$$(R_-)_1 = A\Theta_1 - \overline{A}\Theta_2 + \sqrt{-1}a\Theta_3 + B\Sigma_1 - \overline{B}\Sigma_2 + \sqrt{-1}b\Sigma_3,$$

and

$$(R_-)_2 = C_1\Theta_1 + C_2\Theta_2 + C_3\Theta_3 + D_1\Sigma_1 + D_2\Sigma_2 + D_3\Sigma_3$$

where A, B, C_i, D_i are complex and a, b are real. For our purposes, we view R_- as a $(2,2)$ tensor. Using ds^2, we write R as

$$\begin{aligned} R_- &= 2(R_-)_1 \otimes (E_{\bar{1}} \wedge \overline{E_{\bar{1}}} + E_{\bar{2}} \wedge \overline{E_{\bar{2}}}) + 2(R_-)_2 \otimes \overline{E_{\bar{1}} \wedge E_{\bar{2}}} + 2\overline{(R_-)}_2 \otimes E_{\bar{1}} \wedge E_{\bar{2}} \\ &= 2\sqrt{-1}(R_-)_1 \otimes \Theta_3^* + 2(R_-)_2 \otimes \Theta_2^* + 2\overline{(R_-)}_2 \otimes \Theta_1^* \end{aligned}$$

where, by abuse of notation, we use $E_{\bar{i}}$ to denote the tangent vector dual to ω_i. In this way we may regard R_- as a linear map $R_- : \Lambda^2_- \to \Lambda^2 = \Lambda^2_- \oplus \Lambda^2_+$. Relative to the basis Θ and Σ we write the matrix representation

$$R_-(\Theta_1, \Theta_2, \Theta_3) = 2(\Theta_1, \Theta_2, \Theta_3, \Sigma_1, \Sigma_2, \Sigma_3) \begin{pmatrix} \overline{C_2} & C_1 & \sqrt{-1}A \\ \overline{C_1} & C_2 & -\sqrt{-1}A \\ \overline{C_3} & C_3 & -a \\ \overline{D_2} & D_1 & \sqrt{-1}B \\ \overline{D_1} & D_2 & -\sqrt{-1}B \\ \overline{D_3} & D_3 & -b \end{pmatrix}.$$

It is well-known that R_- decomposes as $Z + W^- + \frac{s}{12}Id$ (see [6], p. 51) where Z is the traceless Ricci curvature, W^- is the anti-self-dual part of the Weyl curvature and s is the scalar curvature. In our notation, Z is represented by the matrix

$$2\begin{pmatrix} \overline{D_2} & D_1 & \sqrt{-1}B \\ \overline{D_1} & D_2 & -\sqrt{-1}B \\ \overline{D_3} & D_3 & -b \end{pmatrix},$$

$$s = 8(C_2 + \overline{C_2} - a),$$

and W^- is represented by

$$2\begin{pmatrix} \overline{C_2} & C_1 & \sqrt{-1}A \\ \overline{C_1} & C_2 & -\sqrt{-1}A \\ \overline{C_3} & C_3 & -a \end{pmatrix} - \frac{2}{3}(C_2 + \overline{C_2} - a)I.$$

The metric with $W^- = 0$ is called *self-dual*. If in addition, ds^2 is Einstein, then s is necessarily constant. In our notations,

Proposition 1.2.1. *The metric ds^2 is self-dual and Einstein if and only if $b = A = B = C_1 = C_3 = D_1 = D_2 = D_3 = 0$ and $C_2 = \overline{C_2} = -a = \frac{s}{24}$. In this case, a part of the structure equation simplifies greatly*

$$d\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\begin{pmatrix} \alpha & 0 \\ 0 & -tr(\alpha) \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \overline{\omega_2 \wedge \omega_3} \\ \overline{\omega_3 \wedge \omega_1} \\ \frac{s}{24}\overline{\omega_1 \wedge \omega_2} \end{pmatrix}. \qquad (1.13)$$

Self-dual Einstein metrics will play important roles in our following constructions. There are not many compact examples with $s \geq 0$ due to the classification by Hitchin (see [6], p 376):

Theorem 1.2.2. *Let M be be self-dual Einstein manifold. Then*

(1) If $s > 0$, M is isometric to S^4 or $\mathbf{C}P^2$ with their canonical metrics.

(2) If $s = 0$, M is either flat or its universal covering is a $K3$ surface with the Calabi-Yau metric.

The proof uses Bochner Technique, which, however does not work well when $s < 0$. No similar results are available for self-dual Einstein metrics with negative scalar curvature.

1.2.2 Twistor spaces of self-dual Einstein 4-manifolds

We fix a complex structure J_0 on $\mathbf{R}^4$ by requiring $dz_1 = dx_1 + \sqrt{-1}dx_3$ and $dz_2 = dx_2 + \sqrt{-1}dx_4$ be complex linear. We define a map $\mathtt{j} : \mathcal{F} \to \mathcal{J}$ as follows

$$\mathtt{j}(u) = u^{-1} \circ J_0 \circ u.$$

Since $SO(4)$ acts transitively on the orthogonal complex structures on $\mathbf{R}^4$ and the isotropy group of J_0 is $U(2)$, $\mathtt{j}$ makes $\mathcal{F}$ a principal $U(2)$-bundle over $\mathcal{J}$. This defines a $U(2)$-structure on $\mathcal{J}$. It in turn determines an $SU(3)$-structure on $\mathcal{J}$ by the standard embedding of $U(2)$ into $SU(3)$. Relative to $\mathtt{j}$, β becomes semi-basic. The almost structure on $\mathcal{J}$ determined by this $SU(3)$-strucure is such that ω_1, ω_2 and ω_3 are complex linear.

Let us now concentrate on self-dual Einstein manifolds. The equations in (1.13) are the first structure equations on $\mathcal{J}$. It clearly satisfies the condition of Lemma 1.1.2.

Lemma 1.2.3. *The twistor space of a self-dual Einstein 4-manifold carries an admissible $SU(3)$-structure.*

$s > 0$

In this case, we scale the metric so that $s = 24$. Now the structure equation (1.13) indicates that the twistor space is actually nearly Kähler. By the aforementioned Hitchin's result, the only two possibilities are $M = S^4$ and $M = \mathbf{CP}^2$. The corresponding twistor spaces are two familiar nearly Kähler examples, $\mathbf{CP}^3$ and the flag manifold $SU(3)/T^2$.

$s = 0$

Again, by Hitchin's result we have two examples: one is the flat case, the other is K_3 surfaces.

$s < 0$

This is the most interesting case in many respects. We scale the metric to make $s = -48$. Now the structure equation (1.13) reads

$$d \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = - \begin{pmatrix} \alpha & 0 \\ 0 & -tr(\alpha) \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \overline{\omega_2 \wedge \omega_3} \\ \overline{\omega_3 \wedge \omega_1} \\ -2\overline{\omega_1 \wedge \omega_2} \end{pmatrix}. \tag{1.14}$$

The only torsion is the Nijenhuis tensor, in local unitary basis,

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Comparing with (1.3) (with $U = 0$ being understood), we have

Theorem 1.2.4. *The twistor space of a self-dual Einstein manifold of negative scalar curvature is half-flat nearly Calabi-Yau but non-Calabi-Yau.*

The simplest example of this category is, of course, the twistor space of the hyperbolic space H^4. Compact examples can be obtained from the quotients of H^4 by certain discrete isometry groups. It remains open to construct complete or compact nearly Calabi-Yau manifolds that are not half-flat.

1.3 Complete special Lagrangian examples

In this section we will construct some complete special Lagrangian submanifolds in the twistor spaces $\mathcal{J}(S^4) = \mathbf{CP}^3$ and $\mathcal{J}(H^4)$ considered in the previous section. Our method is based on the following observation due to Robert Bryant [8]. Suppose on an $SU(3)$ manifold (M, Ω, Ψ), there is a real structure, i.e., an involution c such that

$$c^*\Omega = -\Omega, \quad c^*\Psi = \overline{\Psi},$$

and the set N_c of points fixed under c is a smooth submanifold. Then it is easy to see that N_c, with one of its two possible orientations, is a special Lagrangian submanifold of M . Thus our major task is to construct such involutions for $\mathcal{J}(S^4) = \mathbf{CP}^3$ and $\mathcal{J}(H^4)$.

1.3.1 An example in $\mathcal{J}(S^4) = \mathbf{CP}^3$

We need a more explicit description of the twistor fibration $T : \mathbf{CP}^3 \to S^4$. We follow the discussion in [9]. However, as aforementioned, we will use a different almost complex structure on $\mathbf{CP}^3$.

Let $\mathbf{H}$ denote the real division algebra of quaternions. An element of $\mathbf{H}$ can be written uniquely as $q = z + jw$ where $z, w \in \mathbf{C}$ and $j \in \mathbf{H}$ satisfies

$$j^2 = -1, \quad zj = j\bar{z}$$

for all $z \in \mathbf{C}$. The quaternion multiplication is thus given by

$$(z_1 + jz_2)(z_3 + jz_4) = z_1z_3 - z_2\overline{z_4} + j(z_2z_3 + \overline{z_1z_4}). \tag{1.15}$$

We define an involution $C : \mathbf{H} \to \mathbf{H}$ by $C(z_1 + jz_2) = \overline{z_1} + j\overline{z_2}$. It can be easily checked via the product rule (1.15) that this is in fact an algebra automorphism, i.e., $C(pq) = C(p)C(q)$ for $p, q \in \mathbf{H}$.

We regard $\mathbf{C}$ as subalgebra of $\mathbf{H}$ and give $\mathbf{H}$ the structure of a complex vector space by letting $\mathbf{C}$ act on the right. We let $\mathbf{H}^2$ denote the space of pairs (q_1, q_2) where $q_i \in \mathbf{H}$. We will make $\mathbf{H}^2$ into a quaternion vector space by letting $\mathbf{H}$ act on the right

$$(q_1, q_2)q = (q_1q, q_2q).$$

This automatically makes $\mathbf{H}^2$ into a complex vector space of dimension 4. In fact, regarding $\mathbf{C}^4$ as the space of 4-tuples (z_1, z_2, z_3, z_4), we make the explicit identification

$$(z_1, z_2, z_3, z_4) \sim (z_1 + jz_2, z_3 + jz_4). \tag{1.16}$$

This specific isomorphism is the one we will always mean when we write $\mathbf{C}^4 = \mathbf{H}^2$.

If $v \in \mathbf{H}^2 \backslash (0,0)$ is given, let $v\mathbf{C}$ and $v\mathbf{H}$ denote, respectively, the complex line and the quaternion line spanned by v. As is well-known, $\mathbf{HP}^1$, the space of quaternion lines in $\mathbf{H}^2$, is isometric to S^4. For this reason, we will speak interchangeably of S^4 and $\mathbf{HP}^1$. The assignment $v\mathbf{C} \to v\mathbf{H}$ is exactly the twistor mapping $T : \mathbf{CP}^3 \to \mathbf{HP}^1$. The fibres of T are $\mathbf{CP}^1$'s. Thus, we have a fibration

$$\begin{array}{ccc} \mathbf{CP}^1 & \to & \mathbf{CP}^3 \\ & & \downarrow \\ & & \mathbf{HP}^1 \end{array} \tag{1.17}$$

This is the famous twistor fibration. In order to study its geometry more thoroughly, we will now introduce the structure equations of $\mathbf{H}^2$. First we endow $\mathbf{H}^2$ with a quaternion inner product $\langle , \rangle : \mathbf{H}^2 \times \mathbf{H}^2 \to \mathbf{H}$ defined by

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1 p_1 + \bar{q}_2 p_2.$$

We have identities

$$\langle v, wq \rangle = \langle v, w \rangle q, \quad \overline{\langle v, w \rangle} = \langle w, v \rangle , \quad \langle vq, w \rangle = \bar{q} \langle v, w \rangle .$$

Via the identification (1.16),

$$\begin{aligned} \langle (q_1, q_2), (p_1, p_2) \rangle \; &= \; \overline{z_1} w_1 + \overline{z_2} w_2 + \overline{z_3} w_3 + \overline{z_4} w_4 \\ &\quad + j(z_1 w_2 - z_2 w_1 + z_3 w_4 - z_4 w_3) \end{aligned} \tag{1.18}$$

for $q_1 = z_1 + jz_2, \quad q_2 = z_3 + jz_4, \quad p_1 = w_1 + jw_2, \quad p_2 = w_3 + jw_4$. In other words, $\langle , \rangle$ essentially consists of two parts: one is the standard Hermitian product $dz_1 \circ d\overline{z_1} + \cdots + dz_4 \circ d\overline{z_4}$; the other is the standard complex symplectic form $dz_1 \wedge dz_2 + dz_3 \wedge dz_4$.

Let $\mathfrak{F}$ denote the space of pairs $f = (e_1, e_2)$ with $e_i \in \mathbf{H}^2$ satisfying

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \quad \langle e_1, e_2 \rangle = 0.$$

We regard $e_i(f)$ as functions on $\mathfrak{F}$ with values in $\mathbf{H}^2$. Clearly $e_1(\mathfrak{F}) = S^7 \subset \mathbf{E}^8 = \mathbf{H}^2$. It is well-known that $\mathfrak{F}$ may be canonically identified with $Sp(2)$ up to a left translation in $Sp(2)$. There are unique quaternion-valued 1-forms $\{\phi^a_b\}$ so that

$$de_a = e_b \phi^b_a, \tag{1.19}$$

$$d\phi^a_b + \phi^a_c \wedge \phi^c_b = 0, \tag{1.20}$$

and

$$\phi^a_b + \overline{\phi^b_a} = 0. \tag{1.21}$$

We define a map $\mathfrak{F} \to \mathbf{CP}^3$ by sending (e_1, e_2) to the complex line spanned by e_1. We will denote this map by $\mathtt{j}$ by a slight abuse of notation. The composition $\pi = T \circ \mathtt{j}$ is actually a spin structure on S^4. In fact the oriented coframe bundle $\mathbf{F}$ of S^4 may be identified with $SO(5)$ up to a left translation in $SO(5)$. Thus $\mathfrak{F}$ double covers $\mathbf{F}$ as $Sp(2)$ double covers $SO(5)$.

We now write structure equations for the map $\mathtt{j}$. First we immediately see that $\mathtt{j}$ gives $\mathfrak{F}$ an $S^1 \times S^3$-structure over $\mathbf{CP}^3$ where we have identified S^1 with the unit complex numbers and S^3 with the unit quaternions. The action is given by

$$f(z, q) = (e_1, e_2)(z, q) = (e_1 z, e_2 q)$$

where $z \in S^1$ and $q \in S^3$. If we set

$$\begin{bmatrix} \phi^1_1 & \phi^1_2 \\ \phi^2_1 & \phi^2_2 \end{bmatrix} = \begin{bmatrix} i\rho_1 + j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & i\rho_2 + j\tau \end{bmatrix}$$

where ρ_1 and ρ_2 are real 1-forms while ω_1, ω_2, ω_3 and τ are complex valued, we may rewrite one part of the structure equation (3.3) relative to the $S^1 \times S^3$-structure on $\mathbf{CP}^3$ as

$$d\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = -\begin{pmatrix} i(\rho_2-\rho_1) & -\bar{\tau} & 0\\ \tau & -i(\rho_1+\rho_2) & 0\\ 0 & 0 & 2i\rho_1\end{pmatrix}\wedge\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} + \begin{pmatrix}\overline{\omega_2\wedge\omega_3}\\ \overline{\omega_3\wedge\omega_1}\\ \overline{\omega_1\wedge\omega_2}\end{pmatrix}. \tag{1.22}$$

The nearly Kähler structure on $\mathbf{CP}^3$ is defined by setting ω_1, ω_2 and ω_3 to be complex linear.

Via the algebra automorphism C we define an involution on $\mathbf{H}^2$ by $(p,q) \mapsto (C(p), C(q))$. We denote this map still by C. This map in turn induces an involution on $\mathfrak{F}$, still denoted C, by $C(e_1, e_2) = (C(e_1), C(e_2))$. From (1.18) we see that the defining equations for $\mathfrak{F}$ are preserved and the involution is well-defined. The map C further descends to an involution c on $\mathbf{CP}^3$ as well as an involution $\bar{c}$ on S^4 by $e_1\mathbf{C} \mapsto C(e_1)\mathbf{C}$ and $e_1\mathbf{H} \mapsto C(e_1)\mathbf{H}$ respectively. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{C} & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathbf{CP}^3 & \xrightarrow{c} & \mathbf{CP}^3 \\ \downarrow & & \downarrow \\ \mathbf{HP}^1 & \xrightarrow{\bar{c}} & \mathbf{HP}^1 \end{array}.$$

Apply the automorphism to the structure equations (3.2) and we get

$$dC(e_a) = C(e_b)C(\phi^b_a).$$

Thus in particular we have on $\mathfrak{F}$

$$C^*\omega_i = \overline{\omega_i}$$

for $i = 1, 2, 3$. Consequently

$$C^*\mathtt{j}^*\Omega = -\mathtt{j}^*\Omega, \quad C^*\mathtt{j}^*\Psi = \overline{\mathtt{j}^*\Psi}.$$

Since $\mathtt{j}C = c\mathtt{j}$ and $\mathtt{j}^*$ is injective, we have on $\mathbf{CP}^3$

$$c^*\Omega = -\Omega, \quad c^*\Psi = \overline{\Psi}.$$

Thus by the general principle the fixed set of c is a special Lagrangian submanifold of $\mathbf{CP}^3$. Moreover it is easy to see that this locus is just the usual $\mathbf{RP}^3$.

Theorem 1.3.1. *The real projective space* $\mathbf{RP}^3 = \{[x_1 : x_2 : x_3 : x_4] : x_i \in \mathbf{R}\} \subset \mathbf{CP}^3$ *is a special Lagrangian submanifold of the nearly Kähler* $\mathbf{CP}^3$*.*

The twistor map $T : \mathbf{CP}^3 \to \mathbf{HP}^1$ restricted to the real projective space $\mathbf{RP}^3$ now looks like

$$[x_1 : x_2 : x_3 : x_4] \mapsto [x_1 + jx_2 : x_3 + jx_4].$$

Thus the image is a $\mathbf{CP}^1 \subset \mathbf{HP}^1$ and T is the Hopf fibration

$$\begin{array}{ccc} S^1 & \to & \mathbf{RP}^3 \\ & & \downarrow \\ & & \mathbf{CP}^1 \end{array} .$$

A dual construction for $\mathcal{J}(H^4)$ will follow.

1.3.2 An example in $\mathcal{J}(H^4)$

Let $\mathbf{H}^2$ and the involution C be as before. But now we endow $\mathbf{H}^2$ with a $(1,1)$ quaternion inner product $\langle , \rangle : \mathbf{H}^2 \times \mathbf{H}^2 \to \mathbf{H}$ defined by

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \overline{q_1} p_1 - \overline{q_2} p_2.$$

We still have the identities

$$\langle v, wq \rangle = \langle v, w \rangle q, \quad \overline{\langle v, w \rangle} = \langle w, v \rangle, \quad \langle vq, w \rangle = \bar{q} \langle v, w \rangle.$$

Via the identification (1.16),

$$\begin{aligned} \langle (q_1, q_2), (p_1, p_2) \rangle \;=\;& \overline{z_1} w_1 + \overline{z_2} w_2 - \overline{z_3} w_3 - \overline{z_4} w_4 \\ & + j(z_1 w_2 + z_2 w_1 - z_3 w_4 - z_4 w_3) \end{aligned} \tag{1.23}$$

for $q_1 = z_1 + jz_2, \quad q_2 = z_3 + jz_4, \quad p_1 = w_1 + jw_2, \quad p_2 = w_3 + jw_4$. In other words, $\langle , \rangle$ essentially consists of two parts: one is the $(2,2)$-Hermitian product $dz_1 \circ d\overline{z_1} + dz_2 \circ d\overline{z_2} - dz_3 \circ d\overline{z_3} - dz_4 \circ d\overline{z_4}$; the other is a complex symplectic form $dz_1 \wedge dz_2 - dz_3 \wedge dz_4$. Denote the pseudo-sphere in $\mathbf{H}^2$ by

$$\Psi S^7 = \{(p,q) \in \mathbf{H}^2 : \overline{p}p - \overline{q}q = 1\}.$$

This is a connected non-compact smooth hypersurface in $\mathbf{H}^2$. The group S^3 acts on ΨS^7 by

$$(p,q) \cdot r = (pr, qr)$$

where $r \in S^3$ is a unit quaternion number. This action is clearly free. Thus the quotient space $\Psi\mathbf{HP}^1 = \Psi S^7/S^3$ is smooth. Indeed $\Psi\mathbf{HP}^1 = H^4$. Similarly if we regard S^1 as a subgroup of S^3 consisting of unit complex numbers, the quotient space $\Psi\mathbf{CP}^3 = \Psi S^7/S^1$ is smooth. The clearly well-defined map $T : \Psi\mathbf{CP}^3 \to H^4$ is exactly the twistor fibration of H^4. We have the following commutative diagram of fibrations

$$\begin{array}{ccc} S^1 & \hookrightarrow & \Psi S^7 \\ & & \downarrow \\ \mathbf{CP}^1 & \hookrightarrow & \Psi\mathbf{CP}^3 \\ & & \downarrow \\ & & \Psi\mathbf{HP}^1 \end{array}.$$

Let $\mathfrak{F}$ denote the space of pairs $f = (e_1, e_2)$ with $e_i \in \mathbf{H}^2$ satisfying

$$\langle e_1, e_1 \rangle = 1, \quad \langle e_2, e_2 \rangle = -1, \quad \langle e_1, e_2 \rangle = 0.$$

We regard $e_i(f)$ as functions on $\mathfrak{F}$ with values in $\mathbf{H}^2$. Clearly $e_1(\mathfrak{F}) = \Psi S^7 \subset \mathrm{E}^{(4,4)} = \mathbf{H}^2$. It is well-known that $\mathfrak{F}$ maybe canonically identified with $Sp(1,1)$ up to a left translation in $Sp(1,1)$, where

$$Sp(1,1) = \{A \in \mathfrak{gl}(2, \mathbf{H}) : \overline{A} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .\}$$

There are unique quaternion-valued 1-forms $\{\phi^a_b\}$ so that

$$de_a = e_b \phi^b_a, \tag{1.24}$$

$$d\phi^a_b + \phi^a_c \wedge \phi^c_b = 0, \tag{1.25}$$

and

$$\overline{\phi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi^t = 0. \tag{1.26}$$

We have a canonical map $\mathfrak{F} \to \Psi\mathbf{CP}^3$ by sending (e_1, e_2) to the coset $e_1 \cdot S^1$. We will denote this map by $\mathtt{j}$ by a slight abuse of notation. The composition $\pi = T \circ \mathtt{j}$ is actually a spin structure on H^4. In fact, the oriented coframe bundle $\mathbf{F}$ of H^4 may be identified with $SO^0(4,1)$, the identity component of $SO(4,1)$, up to a left translation in $SO^0(4,1)$. Thus $\mathfrak{F}$ double covers $\mathbf{F}$ as $Sp(1,1)$ double covers $SO^0(4,1)$ (see Harvey [20], p. 272 for the isomorphism $Sp(1,1) \cong Spin^0(4,1)$ where he used the notation $HU(1,1)$ for $Sp(1,1)$).

We now write structure equations for the map $\mathtt{j}$. First we immediately see that $\mathtt{j}$ gives $\mathfrak{F}$ an $S^1 \times S^3$-structure over $\Psi\mathbf{CP}^3$ where we have identified S^1 with the unit complex numbers and S^3 with the unit quaternions. The action is given by

$$f(z,q) = (e_1, e_2) \cdot (z, q) = (e_1 z, e_2 q)$$

where $z \in S^1$ and $q \in S^3$. If we set

$$\begin{bmatrix} \phi^1_1 & \phi^1_2 \\ \phi^2_1 & \phi^2_2 \end{bmatrix} = \begin{bmatrix} i\rho_1 + j\overline{\omega_3} & \overline{\omega_1} - j\omega_2 \\ \omega_1 + j\omega_2 & i\rho_2 + j\tau \end{bmatrix}$$

where ρ_1 and ρ_2 are real 1-forms while ω_1, ω_2, ω_3 and τ are complex valued, we may rewrite one part of the structure equation (3.3) relative to the nearly Calabi-Yau structure on $\Psi\mathbf{CP}^3$ as

$$d\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = -\begin{pmatrix} i(\rho_2-\rho_1) & -\bar{\tau} & 0\\ \tau & -i(\rho_1+\rho_2) & 0\\ 0 & 0 & 2i\rho_1\end{pmatrix}\wedge\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} + \begin{pmatrix}\overline{\omega_2\wedge\omega_3}\\ \overline{\omega_3\wedge\omega_1}\\ -2\overline{\omega_1\wedge\omega_2}\end{pmatrix}. \tag{1.27}$$

The nearly Calabi-Yau structure on $\Psi\mathbf{CP}^3$ is defined by taking ω_1, ω_2 and ω_3 to be complex linear.

This involution C on $\mathbf{H}^2$ induces an involution on $\mathfrak{F}$, still denoted by C, by $C(e_1, e_2) = (C(e_1), C(e_2))$. From (1.23), we see that the defining equations for $\mathfrak{F}$ are preserved and the involution is well-defined. The map C further descends to an involution c on $\Psi\mathbf{CP}^3$ as well as an involution $\bar{c}$ on H^4 by $e_1 \cdot S^1 \mapsto C(e_1)\cdot S^1$ and $e_1 \cdot S^3 \mapsto C(e_1)\cdot S^3$ repectively. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{C} & \mathfrak{F} \\ \downarrow & & \downarrow \\ \Psi\mathbf{CP}^3 & \xrightarrow{c} & \Psi\mathbf{CP}^3 \\ \downarrow & & \downarrow \\ H^4 & \xrightarrow{\bar{c}} & H^4 \end{array}.$$

Apply the automorphism to the structure equations (1.24) and we get

$$dC(e_a) = C(e_b)C(\phi_a^b).$$

Thus in particular we have on $\mathfrak{F}$

$$C^*\omega_i = \overline{\omega_i}$$

for $i = 1, 2, 3$. Consequently

$$C^*\mathtt{j}^*\Omega = -\mathtt{j}^*\Omega, \quad C^*\mathtt{j}^*\Psi = \overline{\mathtt{j}^*\Psi}.$$

Since $\mathtt{j}C = c\mathtt{j}$ and $\mathtt{j}^*$ is injective, we have on $\Psi\mathbf{CP}^3$

$$c^*\Omega = -\Omega, \quad c^*\Psi = \overline{\Psi}.$$

Thus by the general principle the fixed set of c is a special Lagrangian submanifold of $\Psi\mathbf{CP}^3$. It is easy to see that this manifold is the pseudo-projective 3-space $\Psi\mathbf{RP}^3$, defined as the quotient of the pseudo 3-sphere $\Psi S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$ by $\mathbb{Z}_2$.

Theorem 1.3.2. *The real pseudo-projective space $\Psi\mathbf{RP}^3 \subset \Psi\mathbf{CP}^3$ is a special Lagrangian submanifold of the nearly Calabi-Yau $\Psi\mathbf{CP}^3$.*

The image under T of this pseudo-sphere is easily seen to be the hyperbolic 2-space $H^2 \subset H^4$. Thus we have the following fibration

$$\begin{array}{ccc} S^1 & \rightarrow & \Psi\mathbf{RP}^3 \\ & & \downarrow \\ & & H^2 \end{array} .$$

1.4 Compact special Lagrangian submanifolds in nearly Calabi-Yau manifolds

We discuss compact special Lagrangian submanifolds in nearly Calabi-Yau manifolds. We answer two questions:

1. Let N be a compact special Lagrangian 3-fold in a fixed nearly Calabi-Yau manifold (M, Ω, Ψ, U). Let $\mathcal{M}_N$ be the moduli space of special Lagrangian deformations of N, that is, the connected component of the set of special Lagrangian 3-folds containing N. What can we say about $\mathcal{M}_N$? Is it a smooth manifold? What is its dimension?

2. Let $\{(M, \Omega_t, \Psi_t, U_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of nearly Calabi-Yau manifolds. Suppose N_0 is an SL-3$-$fold. Under what conditions can we extend N_0 to a smooth family of special Lagrangian 3-folds N_t in $(M, \Omega_t, \Psi_t, U_t)$?

These questions concern the deformations of special Lagrangian 3-folds and obstructions to their existence respectively. In the Calabi-Yau case, the first question is answered by R. McLean in [24], and the second is answered by D. Joyce in [23]. Moreover, [23] also answers these questions for almost Calabi-Yau manifolds. We show that their proofs generalize to nearly Calabi-Yau manifolds. The argument uses a rescaling trick, communicated to me by R. Bryant. I would like to thank him for this.

1.4.1 Deformations of compact special Lagrangian 3-folds

We have the following result similar to one in [24]

Theorem 1.4.1. *Let (M, Ω, Ψ, U) be a nearly Calabi-Yau 3-fold, and N a compact special Lagrangian 3-fold in M. Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

Proof. We emphasize the difference from the arguments in [24]. Let ν_N be the normal bundle of N. Nearby submanifolds of M can be viewed as small sections of ν_N. Thus we consider the following map between Banach spaces

$$F : C^{1,\alpha}(\nu_N) \to dC^{1,\alpha}(\Lambda^1(N)) \times dC^{1,\alpha}(\Lambda^2(N))$$

defined by

$$V \mapsto (\mathtt{exp}_V^*(\Omega), \mathtt{exp}_V^*(e^U \mathrm{Im}\Psi)).$$

As in [24], it is easy to see from standard Hodge theory and elliptic regularity that this map is well-defined. The kernel of F consists of sections V on whose $\mathtt{exp}$-image Ω and $e^U\mathrm{Im}\Psi$ vanishes. Thus $\mathtt{exp}(V)$ is a special Lagrangian submanifold.

Now the differential F_* of F at 0 may computed in a way similar to [24],

$$\begin{aligned} F_*(V) &= (\mathcal{L}_V\Omega|_N, \mathcal{L}_V(e^U \mathrm{Im}\Psi)|_N) \\ &= (d(V \lrcorner \Omega), d(e^U V \lrcorner \mathrm{Im}\Psi)). \end{aligned}$$

Note that the map $V \mapsto v = V \lrcorner \Omega$ gives a bundle isomorphism between ν_N and T^*N. Via this correspondence $V \lrcorner \mathrm{Im}\Psi = - * v$, where it should be cautioned that $*$ is defined by the induced metric on N. Thus F_* may be equivalently viewed as a map

$$C^{1,\alpha}(\Lambda^1(N) \to dC^{1,\alpha}(\Lambda^1(N)) \times dC^{1,\alpha}(\Lambda^2(N))$$

by

$$v \mapsto (dv, -d * (e^U v)).$$

Here we see the major difference. The second term is no longer $-d * v$ as in the Calabi-Yau situation. To overcome this difficulty, we rescale the induced metric by a proper factor (in terms of U) so that the new Hodge star operator $\hat{*} = e^U *$. This is clearly possible. Then the map is

$$v \mapsto (dv, -d\hat{*}v).$$

By Hodge Theory, the map F_* is surjective and the kernel consists of harmonic 1-forms (with the rescaled metric being understood). The proof is finished as in [24] by employing the Implicit Function Theorem for smooth maps between Banach spaces . □

Remark 1.4.2 (S. Salur's result). *A similar deformation theorem was proved in [25] for slightly differently defined special Lagrangian submanifolds in a more general class of sympletic manifolds.*

1.4.2 Obstructions to the existence of compact SL 3-folds

We address Question 2 above. Let $\{(M, \Omega_t, \Psi_t, U_t)\}$ be a smooth 1-parameter family of nearly Calabi-Yau manifolds. Suppose N_0 is a special Lagrangian 3-fold of $(M, \Omega_0, \Psi_0, U_0)$ and N_t is an extension. Then we can view N_t as a family of embeddings of $\mathbf{i}_t : N_0 \to M$ such that $\mathbf{i}_t^*(\Omega_t) = \mathbf{i}_t^*(e^{U_t}\mathrm{Im}\Psi_t) = 0$. Since the cohomology classes $[\mathbf{i}_s^*(\Omega_t)]$ and $[\mathbf{i}_s^*(e^{U_t}\mathrm{Im}\Psi_t)]$ do not vary with s, we have $[\mathbf{i}_0^*(\Omega_t)] = [\mathbf{i}_0^*(e^{U_t}\mathrm{Im}\Psi_t)] = 0$. Thus, a necessary condition for such an extension of N_0 to exist is

$$[\Omega_t|_{N_0}] = [e^{U_t}\mathrm{Im}\Psi_t|_{N_0}] = 0.$$

Actually this is also sufficient:

Theorem 1.4.3. *Let $\{(M, \Omega_t, \Psi_t, U_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of nearly Calabi-Yau 3-folds. Let N_0 be a compact SL 3-fold in (M, Ω_0, Ψ_0), and suppose $[\Omega_t|_{N_0}] = 0$ in $H^2(N_0, \mathbf{R})$ and $[e^{U_t}\mathrm{Im}\Psi_t|_{N_0}] = 0$ in $H^3(N_0, \mathbf{R})$ for all $t \in (-\epsilon, \epsilon)$. Then N_0 extends to a smooth 1-parameter family $\{N_t : t \in (-\delta, \delta)\}$ for some $0 < \delta \leq \epsilon$ and N_t is a compact SL 3-fold in (M, Ω_t, Ψ_t).*

Again, the proof combines the rescaling trick and the argument for the Calabi-Yau case as in [23]. However, since the details are not readily available, we write them down.

Proof. Let ν_{N_0} be the normal bundle of N_0 in (M, Ω_0, Ψ_0). Denote by $\mathtt{exp}$ the exponential map of (M, Ω_0, Ψ_0). For a vector bundle E over N_0, we use $C^{1,\alpha}(E)$ and $C^{0,\alpha}(E)$ to denote the sections of E of class $C^{1,\alpha}$ and $C^{0,\alpha}$ respectively. We define a map

$$F : C^{1,\alpha}(\nu_{N_0}) \times (-\epsilon, \epsilon) \to dC^{1,\alpha}(\Lambda^1(N_0)) \times dC^{1,\alpha}(\Lambda^2(N_0))$$

by

$$F(V, t) = (\mathtt{exp}_V^*(\Omega_t), \mathtt{exp}_V^*(e^{U_t}\mathrm{Im}\Psi_t)).$$

We need show this is well-defined. The maps $F(sV,t)(0 \le s \le 1)$ provide a homotopy between $F(0,t) = (\Omega_t|_{N_0}, e^{U_t}\mathrm{Im}\Psi|_{N_0})$ and $F(V,t)$. Since $[\Omega_t|_{N_0}] = 0$, $[\mathtt{exp}_V^*(\Omega_t)] = 0$. Thus, $\mathtt{exp}_V^*(\Omega_t) = d\tau$ for some τ. Moreover, by the standard Hodge theory the form τ can be chosen to be in $C^{2,\alpha}$ because V is $C^{1,\alpha}$ and so is $\mathtt{exp}_V^*(\Omega_t)$. A similar argument shows that $\mathtt{exp}_V^*(e^{U_t}\mathrm{Im}\Psi_t)$ lies in $dC^{1,\alpha}(\Lambda^2(N_0))$.

Now we compute the tangent map of F at the point $(0,0)$,

$$F_* : \mathbf{R} \times C^{1,\alpha}(\nu_{N_0}) \to dC^{1,\alpha}(\Lambda^1(N_0)) \times dC^{1,\alpha}(\Lambda^2(N_0)).$$

First

$$\begin{aligned} F_*(\tfrac{\partial}{\partial t}, 0) &= \tfrac{d}{dt}|_{t=0,V=0}(\mathtt{exp}_V^*\Omega_t, \mathtt{exp}_V^*\mathrm{Im}\Psi_t) \\ &= (\Omega'|_{N_0}, \mathrm{Im}(e^U\Psi)'|_{N_0}). \end{aligned}$$

where

$$\Omega' = \frac{d}{dt}|_{t=0}\Omega_t, \quad (e^U\Psi)' = \frac{d}{dt}|_{t=0}(e^{U_t}\Psi_t).$$

Second

$$\begin{aligned} F_*(0,V) &= \tfrac{d}{ds}|_{s=0}(\mathtt{exp}_{sV}^*\Omega_0, \mathtt{exp}_{sV}^*(e^{U_0}\mathrm{Im}\Psi)) \\ &= (\mathcal{L}_V\Omega_0|_{N_0}, \mathcal{L}_V(e^{U_0}\mathrm{Im}\Psi_0|_{N_0})) \\ &= ((V\lrcorner d\Omega_0 + d(V\lrcorner\Omega_0))|_{N_0}, (V\lrcorner d(e^{U_0}\mathrm{Im}\Psi_0) + d(e^{U_0}V\lrcorner\mathrm{Im}\Psi_0))|_{N_0}) \\ &= (d(V\lrcorner\Omega_0)|_{N_0}, d(e^{U_0}V\lrcorner\mathrm{Im}\Psi_0)|_{N_0}) \end{aligned}$$

where $\mathcal{L}_V$ is the Lie derivative in the V direction and the Cartan formula is used. Actually, in order to take the Lie derivative, one must extend the normal vector field V to an open neighborhood first. It is easy to see the result is independent of this extension.

Note that the mapping $V \mapsto v = V\lrcorner\Omega_0$ gives a bundle isomorphism between T^*N_0 and ν_{N_0}. Translated via this correspondence $V\lrcorner\mathrm{Im}\Psi_0 = - * v$ as is shown in [24], where the Hodge star $*$ is defined by the induced metric. As before, we rescale the metric by a conformal factor so that $\hat{*} = e^{U_0}*$. By Hodge theory, $F_*(0, V)$ runs over every element in $dC^{1,\alpha}(\Lambda^1(N_0)) \times dC^{1,\alpha}(\Lambda^2)(N_0)$. Thus F_* is surjective. We can also compute the kernel

$$F_*^{-1}(0,0) = \{(r\frac{\partial}{\partial t}, V) : r\dot{\Omega}|_{N_0} = -dv, r\mathrm{Im}(\dot{\Psi})|_{N_0} = d\hat{*}v, r \in \mathbf{R}\},$$

where v relates to V as above. Since $[\Omega_t|_{N_0}] = 0$ and $[e^{U_t}\mathrm{Im}(\Psi_t)|_{N_0}] = 0$ we have $[\dot{\Omega}|_{N_0}] = 0$ and $[\mathrm{Im}(\dot{\Psi})|_{N_0}] = 0$. Thus, $\dot{\Omega}|_{N_0}$ and $\mathrm{Im}(\dot{\Psi})|_{N_0}$ are exact. Again, by Hodge theory, $F_*^{-1}(0,0)$ is nonempty and finite-dimensional, with dimension $b^1(N_0)+1$. By the Implicit Function Theorem for smooth maps between Banach spaces, $F^{-1}(0,0)$ is a smooth manifold with its tangent space at $(0,0)$ equal to $F_*^{-1}(0,0)$. The $C^{1,\alpha}(\nu_{N_0})$ components of elements of $F^{-1}(0,0)$ are in fact smooth sections by the elliptic regularity theorem. Note that the projection map t restricted to $F^{-1}(0,0)$ is nondegenerate at $(0,0)$. Thus the manifold $F^{-1}(0,0)$ is a local smooth fibration over $(-\epsilon, \epsilon)$. Pick a local section (t, V_t) of such a local fibration where $-\delta \leq t \leq \delta$ for some $0 < \delta \leq \epsilon$. Then $N_t = \mathtt{exp}_{V_t}(N_0)$ are the desired 1-parameter family of smooth special Lagrangian manifolds. □

Remark 1.4.4 (on the proof)**.** *Strictly speaking, the domain of F is not a Banach space because of the $(-\epsilon, \epsilon)$ part. This minor difficulty can be overcome by either using a cut-off function of t or reparametrizing t by a diffeomorphism between $(-\epsilon, \epsilon)$ and* $\mathbf{R}$ *preserving* 0.

2

Instantons on Nearly Kähler 6-Manifolds

The notion of anti-self-dual instantons plays an important role in Donaldson's theory of 4-manifolds ([14]). This concept has been generalized to higher dimensions (e.g., [15] and [27]). To motivate the generalization, we first recall the 4-dimensional theory.

Suppose M is an oriented 4-dimensional Riemannian 4-manifold. It is well known that the space of 2-forms splits into self-dual and anti-self-dual parts, corresponding respectively to ± 1-eigenspaces of Hodge $*$ operator. A connection A on a certain principal bundle over M is said to be an *anti-self-dual* instanton if its curvature F, when viewed as a vector-bundle valued two-form, satisfies $*F = -F$. Of course, this definition does not generalize directly to higher dimensions. If, moreover, M is almost Hermitian, i.e., endowed with an almost complex structure compatible with the Riemannian structure, we can formulate the notion in another way. This is based on the observation that anti-self-dual 2-forms are exactly ω-trace free $(1,1)$-forms. Thus, in the almost Hermitian case, we can equally define anti-self-dual instantons to be those connections A satisfying

$$F^{2,0} = \mathrm{tr}_\omega F = 0. \tag{2.1}$$

The latter description obviously allows generalizations to higher dimensional almost Hermitian manifolds. We will also call connections satisfying (2.1) *pseudo-Hermitian-Yang-Mills* by slight abuse of terminology (compare [4], for example).

When the dimension is 6, we can formulate (2.1) in yet another way. Notice that the operator $*(\omega \wedge \cdot)$ maps the space of two forms into itself. It can also be shown that the space of ω-trace free $(1,1)$-forms is exactly the -1 eigenspace of $*(\omega \wedge \cdot)$. Thus, we can rewrite the equation (2.1) as

$$\omega \wedge F = - * F. \tag{2.2}$$

For this reason, we also call pseudo-Hermitian-Yang-Mills connections ω-anti-self-dual instantons.

Now, (2.2) makes sense in even more general contexts. Suppose that M is endowed with an $n-4$ form Ω. Then the operator $*(\Omega \wedge \cdot)$ maps 2-forms into 2-forms. We can define Ω-*anti-self-dual instantons* to be those connections A whose curvatures F satisfies

$$\Omega \wedge F = - * F. \tag{2.3}$$

This definition behaves the best when M has a special structure such as $SU(3)$, G_2 or $Spin(7)$. In this situation, Ω is naturally defined, i.e., Ω is the Kähler form for an $SU(3)$-structure, the defining 3-form for a G_2-structure, or the defining 4-form for a $Spin(7)$-structure.

However, even when Ω is parallel, (2.3) is in general overdetermined. It is natural to ask when (2.3) has solutions, even locally, and how general they are. In dimension 6, R. Bryant showed in [4] that there is a large class of almost Hermitian structures, called *quasi-integrable*, for which the differential system for pseudo-Hermitian-Yang-Mills $SU(n)$-connections is involutive. Thus the theory behaves well in quasi-integrable case. It is interesting to ask under what conditions other instanton differential systems will be involutive.

In this chapter, we are mainly interested in ω-anti-self-dual instantons on a nearly Kähler 6-manifold and Ω-anti-self-dual instantons on its G_2-cone. We first show that ω-anti-self-dual instantons are automatically Yang-Mills, i.e., are critical points of the Yang-Mills functional. We prove the involutivity of the ω-anti-self-dual instanton system. We construct a Chern-Simons type functional on nearly Kähler 6-manifold. This is an $\mathbf{R}$-valued functional, rather than $\mathbf{R}/\mathbf{Z}$-valued as in 3-manifold case. We show that its critical connections are exactly the ω-anti-self-dual instantons. We compute its gradient flow and discuss its relation with Ω-instantons on the G_2-cone. Second, we derive a Weitzenböck formula for an elliptic operator on nearly Kähler manifolds and apply it to study deformations of ω-anti-self-dual instantons. Finally, we construct a class of instantons on S^6 and $\mathbf{R}^7$ that display interesting singularities.

2.1 Some linear algebra in 6 and 7 dimensions

In this section, we clarify notational convention of inner product spaces in 6 and 7 dimensions with emphasis on representation theory of $SU(3)$ and G_2. The interplay between Hodge star operations will be important in later discussions.

Suppose V is an n-dimensional oriented inner product space and let $\{e_i\}_{i=1}^n$ be a oriented orthonormal basis. The inner product on V induces an inner product $\langle,\rangle$ on its dual V^* with the dual basis denoted by $\{dx_i\}$. By taking the convention that $\{dx_{i_1}\wedge\cdots\wedge dx_{i_k}\}$ be orthonormal, we make Λ^*V^* an inner product space. We define *Hodge star* $*$ on Λ^*V^* by the following rule. Let $\phi\in\Lambda^*V^*$ and its Hodge star $*\phi$ is determined by

$$*\,\phi\wedge\psi=\langle\phi,\psi\rangle\mathrm{vol}_V. \tag{2.4}$$

for any $\psi\in\Lambda^*V^*$ where $\mathrm{vol}_V=dx_1\wedge\cdots\wedge dx_n$ is the volume form on V.

Remark 2.1.1. *Through the inner product, we identify vectors and 1-forms. We will not distinguish between them. Thus for example, an linear operator defined on vectors may be thought of as an operator on 1-forms. No confusion should be caused.*

2.1.1 Dimension 6

In dimension 6, we suppose further that V is endowed with a complex structure and a complex volume form Ψ. The complex structure coupled with the inner product determines a symplectic form ω on V. We normalize these quantities so that $\frac{1}{6}\omega^3 = \frac{i}{8}\Psi \wedge \overline{\Psi} = \mathrm{vol}_V$. It is now natural to complexify V^* and its various exterior powers. Denote $V^*_{\mathbf{C}}$ the space of complex linear forms on V. Then $V^* \otimes \mathbf{C} = V^*_{\mathbf{C}} \oplus \overline{V^*_{\mathbf{C}}}$. We extend the inner product and Hodge star operation complex linearly to $V \otimes \mathbf{C}$.

We pick an orthonormal basis $\{dx_i, dy_i\}_{i=1}^3$ for V^* such that $dz_i = dx_i + \sqrt{-1}dy_i$ is complex linear and that

$$\omega = \frac{\sqrt{-1}}{2}(dz_1 \wedge \overline{dz_1} + dz_2 \wedge \overline{dz_2} + dz_3 \wedge \overline{dz_3}), \quad \Psi = dz_1 \wedge dz_2 \wedge dz_3.$$

$SU(3)$-representations

The subgroup of $SO(6)$ preserving both ω and ψ is the special unitary group $SU(3)$. Under the action of $SU(3)$, $\Lambda^* V^* \otimes \mathbf{C}$ may be decomposed into irreducible pieces

$$V^* \otimes \mathbf{C} = V^*_{\mathbf{C}} \oplus \overline{V^*_{\mathbf{C}}}$$

$$\Lambda^2 V^* \otimes \mathbf{C} = \wedge^2 V^*_{\mathbf{C}} \oplus \wedge^2 \overline{V^*_{\mathbf{C}}} \oplus \mathbf{C} \cdot \omega \oplus V^{(1,1)}$$

$$\Lambda^3 V^* \otimes \mathbf{C} = \mathbf{C} \cdot \Psi \oplus \mathbf{C} \cdot \overline{\Psi} \oplus V^{(2,0)} \oplus V^{(0,2)} \oplus V^*_{\mathbf{C}} \wedge \omega \oplus \overline{V^*_{\mathbf{C}}} \wedge \omega$$

$$\Lambda^4 V^* \otimes \mathbf{C} = \overline{V^*_{\mathbf{C}}} \wedge \Psi \oplus V^*_{\mathbf{C}} \wedge \overline{\Psi} \oplus \mathbf{C}\omega^2 \oplus V^{(1,1)}_{\mathbf{C}} \wedge \omega$$

$$\Lambda^5 V \otimes \mathbf{C} = V^*_{\mathbf{C}} \wedge \omega^2 \oplus \overline{V^*_{\mathbf{C}}} \wedge \omega^2,$$

where $V^{(1,1)}$ denotes the representation of the highest weight $(1,1)$, which consists of $(1,1)$-forms whose inner product with ω is zero, $V^{(0,2)} \simeq \mathrm{sym}^2 V^*_{\mathbf{C}}$ is the representation of the highest weight $(0,2)$ and $V^{(0,2)} \simeq \overline{V^{(2,0)}}$. The decomposition of 2-forms and 4-forms will be the most important for us. Note that the wedge product with ω gives an isomorphism between the irreducible pieces in Λ^2 and Λ^4 as outlined above. Another isomorphism is given by Hodge star. These two isomorphisms will be fundamental in the definition of anti-self-dual instantons later, so we examine their relation carefully below.

Hodge star

It is easy to compute that

$$*(dz_1 \wedge dz_2) = \frac{\sqrt{-1}}{2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_3}$$

$$*(dz_2 \wedge dz_3) = \frac{\sqrt{-1}}{2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1}$$

and

$$*(dz_3 \wedge dz_1) = \frac{\sqrt{-1}}{2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_2}.$$

Also

$$\omega \wedge dz_1 \wedge dz_2 = \frac{i}{2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_3}$$

and similarly for $dz_2 \wedge dz_3$, $dz_3 \wedge dz_1$. Thus we have

$$* \, \alpha = \omega \wedge \alpha \tag{2.5}$$

for any $\alpha \in \wedge^2 V_{\mathbf{C}}^* \oplus \wedge^2 \overline{V_{\mathbf{C}}^*}$.

Moreover,

$$* \, \omega = \frac{1}{2}\omega^2. \tag{2.6}$$

On the other hand,

$$*(dz_1 \wedge d\overline{z_2}) = -\frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3} = -\omega \wedge (dz_1 \wedge d\overline{z_2}).$$

More generally, we have

$$* \alpha = -\omega \wedge \alpha \tag{2.7}$$

for any $\alpha \in V^{(1,1)}$.

To conclude, the irreducible (real) $SU(3)$-modules in $\Lambda^2 V^*$ are indexed by the eigenvalues of the operator $*(\omega\wedge)$ (note $*^2 = 1$ on 2-forms).

The other chain of isomorphic $SU(3)$-representations consists of $V^*_{\mathbf{C}}$, $\wedge^2 \overline{V^*_{\mathbf{C}}}$ and various Hodge star images. Again, there are many isomorphisms among these spaces given by compositions of Hodge star, wedge product with the Ψ and with ω. We exploit some of them.

First, we compute that

$$*(dz_3) = \frac{\sqrt{-1}}{4} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2},$$

and thus,

$$\frac{\sqrt{-1}}{4} * (dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2}) = -dz_3.$$

On the other hand

$$\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2} = \frac{\sqrt{-1}}{2}(dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2}).$$

Thus we have

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2}) = -\sqrt{-1} d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge dz_3.$$

It is easy to see

$$\omega \wedge d\overline{z_1} \wedge d\overline{z_2} = -\frac{\sqrt{-1}}{2} \overline{dz_1 \wedge dz_2 \wedge dz_3} \wedge dz_3$$

and thus

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2}) = 2\omega \wedge d\overline{z_1} \wedge d\overline{z_2}.$$

Because $\wedge^2\overline{V^*_{\mathbf{C}}}$ is an irreducible $SU(3)$-representation, it must hold that

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge \alpha) = 2\omega \wedge \alpha. \tag{2.8}$$

Some linear operators

We use the $SU(3)$-representation theory to describe several useful linear operators. Some of them are standard, but we hope to fix notation.

First, we describe $\lrcorner$. For any 1-form $v \in V^*$, $v\lrcorner : \Lambda^k V^* \to \Lambda^{k-1}V^*$ is defined as

$$v\lrcorner(\alpha_1 \wedge \cdots \alpha_k) = \sum_i (-1)^{i-1}\langle v, \alpha_i\rangle \alpha_1 \wedge \cdots \hat{\alpha}_i \wedge \cdots \wedge \alpha_k$$

where $\langle\rangle$ is the inner product. Note that this is adjoint to the wedge product in the sense that

$$\langle v\lrcorner\alpha, \beta\rangle = \langle \alpha, v \wedge \beta\rangle.$$

We extend $\lrcorner$ complex linearly to $V^* \otimes \mathbf{C}$ and $\Lambda^*_{\mathbf{C}}$. Something must be cautioned. For instance

$$dz_1 \lrcorner dz_1 = 0.$$

Next, we use $\lrcorner$ to identify $\Lambda^* V^*$ inside $\mathfrak{so}(V^*)$ by

$$\beta : \alpha \mapsto \alpha\lrcorner\beta.$$

The inverse map is given by for any $A \in \mathfrak{so}(V^*)$

$$A :\mapsto \frac{1}{2}\sum \omega_i \wedge A(\omega_i)$$

where ω_i is an orthonormal basis.

Now, if a linear map commutes with the complex structure on V^*, i.e., maps $V^*_{\mathbf{C}}$ to itself, then it is easy to see that viewed as a 2-form, A lies in the space $\Lambda^{1,1}$. In fact, the corresponding 2-form is given by

$$\frac{1}{2}d\overline{z_i} \wedge A(dz_i).$$

Since Ψ is $SU(3)$-invariant, any linear combination r of maps $v \mapsto v \lrcorner \mathrm{Re}(\Psi)$ and $v \mapsto v \lrcorner \mathrm{Im}\Psi$ gives an $SU(3)$-equivariant map $V^* \to \wedge^2 V^*$. The image $r(v)$ may be viewed as a map $V^* \to V^*$. Skewsymmetrizing $r(v)$ gives a map $\wedge^2 V^* \to \wedge^2 V^*$

$$r(v) : \alpha \wedge \beta \mapsto r(v)(\alpha) \wedge \beta + \alpha \wedge r(v)(\beta).$$

We still denote the map by $r(v)$. From $SU(3)$-equivariance of r, we see that

$$r(g(v))(\alpha) = g(r(v)(g^{-1}\alpha)), \tag{2.9}$$

for any $g \in SU(3)$, $v \in V$ and $\alpha \in \wedge^2 V^*$. We define

$$\begin{aligned} \pi_{(2,0)}(\beta) &= \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_2}\rangle dz_1 \wedge dz_2 + \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_2\rangle \overline{dz_1 \wedge dz_2} \\ &\quad + \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_3}\rangle dz_1 \wedge dz_3 + \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_3\rangle \overline{dz_1 \wedge dz_3} \\ &\quad + \tfrac{1}{4}\langle \beta, \overline{dz_2 \wedge dz_3}\rangle dz_2 \wedge dz_3 + \tfrac{1}{4}\langle \beta, dz_2 \wedge dz_3\rangle \overline{dz_2 \wedge dz_3}, \end{aligned} \tag{2.10}$$

dually

$$\begin{aligned} -\sqrt{-1}\pi_{(0,2)}(\beta) &= \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_2}\rangle dz_1 \wedge dz_2 - \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_2\rangle \overline{dz_1 \wedge dz_2} \\ &\quad + \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_3}\rangle dz_1 \wedge dz_3 - \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_3\rangle \overline{dz_1 \wedge dz_3} \\ &\quad + \tfrac{1}{4}\langle \beta, \overline{dz_2 \wedge dz_3}\rangle dz_2 \wedge dz_3 - \tfrac{1}{4}\langle \beta, dz_2 \wedge dz_3\rangle \overline{dz_2 \wedge dz_3}. \end{aligned} \tag{2.11}$$

and

$$\pi_\omega(\beta) = \frac{1}{3}\langle \beta, \omega\rangle \omega \tag{2.12}$$

where the bracket is the complex extension of the inner product. Note that both $\pi_{(2,0)}$ and $\pi_{(0,2)}$ are real operators. Also define the projection onto ω-trace free 2-forms

$$\pi_0^{1,1} = I - \pi_{(2,0)} - \pi_\omega. \tag{2.13}$$

Note that $\pi_{(2,0)}$ are identity on forms of type $(2,0)$ and type $(0,2)$. While $\pi_{(0,2)}$ is multiplication by $\sqrt{-1}$ on $(2,0)$ forms and $-\sqrt{-1}$ on $(0,2)$ forms. Both of them are clearly $SU(3)$ equivariant. In fact, if we think of the diagonal elements in $\Lambda^2V^*\oplus\Lambda^2\overline{V^*}$ as a real representation of $SU(3)$, the space of $SU(3)$ equivariant homomorphisms is real 2-dimensional, spanned by $\pi_{(2,0)}$ and $\pi_{(0,2)}$. They satisfy the relation

$$\pi_{(2,0)}^2 = \pi_{(2,0)}, \quad \pi_{(0,2)}^2 = -\pi_{(2,0)}. \tag{2.14}$$

In particular, $\pi_{(2,0)}$ is a projection but $\pi_{(0,2)}$ is not.

Denote

$$P = \lambda\pi_{(2,0)} + \mu\pi_\omega \tag{2.15}$$

where λ and μ are real constants. Clearly, P is a real operator and commutes with the action of $SU(3)$. Moreover, $P^2 = \lambda^2\pi_{(2,0)} + \mu^2\pi_\omega$.

Let $\{v_i\}_{i=1}^6$ be a orthonormal basis of V and ω_i dual basis. We define a map by

$$B(\alpha) = \sum_i \omega_i \lrcorner [r(v_i), P^2](\alpha) \tag{2.16}$$

where $\alpha \in \wedge^2V^*$. Note that the definition of B does not depend on the choice of the orthonormal basis. We have the following result concerning the B.

Proposition 2.1.2. *The operator B factors through a (possibly complex) linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$.*

Proof. For any $\alpha \in \wedge^2V^*$ and $g \in SU(3)$, we have

$$\begin{aligned} B(\alpha) &= \textstyle\sum_i \omega_i \lrcorner [r(v_i), P^2](g\alpha) \\ &= \textstyle\sum_i \omega_i \lrcorner g([r(g^{-1}(v_i)), P^2](\alpha)) \\ &= g(\textstyle\sum_i g^{-1}(\omega_i) \lrcorner ([r(g^{-1}(v_i)), P^2](\alpha)) \\ &= g(B(\alpha)). \end{aligned}$$

where the second equality is due to (2.9) as well as the commutativity of P and $SU(3)$, the third is because $g(v\lrcorner\alpha) = g(v)\lrcorner g(\alpha)$ and the last is because of the independence of orthonormal coframes in the definition of B. So B gives a $SU(3)$-equivariant map from $\wedge^2 V^* \to V^*$. Since as a $SU(3)$-space, $\wedge^2 V^*$ contains only a copy of the irreducible representation isomorphic to V^*, namely, $(\wedge^2 V^*_{\mathbf{C}} \oplus \overline{\wedge^2 V^*_{\mathbf{C}}})_{\mathbf{R}}$, we know from Schur's Lemma, B must factor through a linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$. $\square$

Next, for each i, j, we consider the operator on V^*,

$$L(\omega_i, \omega_j)(\alpha) = \omega_i \wedge (\omega_j \lrcorner \alpha) + \omega_i \lrcorner P^2(\omega_j \wedge \alpha). \tag{2.17}$$

We have the following result concerning L.

Proposition 2.1.3. *Let $\lambda = \sqrt{2}$ and $\mu = \sqrt{3}$ and thus*

$$P = \sqrt{2}\pi_{(2,0)} + \sqrt{3}\pi_\omega. \tag{2.18}$$

Then the operator L satisfies the Clifford relations, i.e.,

$$L(\omega_i, \omega_j) + L(\omega_j, \omega_i) = 2\delta_{ij}.$$

Moreover, we define an operator $M : \wedge^2 \mathbf{R}^6 \to \mathrm{End}(\mathbf{R}^6)$ by linearly extending $L(\omega_i, \omega_j)$ for $i \neq j$. Then M is an $SU(3)$ equivariant map from $\Lambda^2 V^*$ to $V \otimes V^*$. In fact, we have

$$M(\beta)(v) = v\lrcorner(-2\pi_0^{1,1}\beta + \pi_\omega \beta).$$

Proof. Since L is real, it suffices to prove the proposition for $(1,0)$ forms. Without loss of generality, we check for $L(dx_1, dx_1)$, $L(dx_1, dy_1)$ and $L(dx_1, dx_2)$. Let $\alpha_i = dx_i + \sqrt{-1}dy_i$. These form a basis for $V^*_{\mathbf{C}}$. Then

$$\begin{aligned} L(dx_1, dx_1)(\alpha_1) &= dx_1 \wedge (dx_1 \lrcorner \alpha_1) + dx_1 \lrcorner P^2(dx_1 \wedge \alpha_1) \\ &= dx_1 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2}\overline{dz_1} \wedge dz_1) \\ &= dx_1 + dx_1 \lrcorner \tfrac{\sqrt{-1}}{3}\mu^2 \omega \\ &= dx_1 + \sqrt{-1}dy_1 \end{aligned}$$

because $\mu^2 = 3$.

For $i = 2, 3$

$$\begin{aligned}
L(dx_1, dx_1)(\alpha_i) &= dx_1 \wedge (dx_1 \lrcorner \alpha_i) + dx_1 \lrcorner P^2(dx_1 \wedge \alpha_i) \\
&= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2}(dz_1 + d\overline{z_1}) \wedge dz_i) \\
&= dx_1 \lrcorner \lambda^2 \tfrac{1}{2} dz_1 \wedge dz_i \\
&= \tfrac{\lambda^2}{2} dz_i \\
&= \alpha_i
\end{aligned}$$

because $\lambda^2 = 2$. This proves the first equality.

Now consider $L(dx_1, dy_1)$. We compute

$$\begin{aligned}
L(dx_1, dy_1)(\alpha_1) &= dx_1 \wedge (dy_1 \lrcorner \alpha_1) + dx_1 \lrcorner P^2(dy_1 \wedge \alpha_1) \\
&= \sqrt{-1} dx_1 + dy_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{-1}{2\sqrt{-1}} d\overline{z_1} \wedge dz_1) \\
&= \sqrt{-1} dx_1 + dx_1 \lrcorner \tfrac{\mu^2}{3} \sqrt{-1} \omega \\
&= \sqrt{-1} dx_1 - \tfrac{1}{3} \mu^2 dx_1 \lrcorner \omega \\
&= \sqrt{-1} dx_1 - dy_1 \\
&= \sqrt{-1}(dz_1)
\end{aligned}$$

and

$$\begin{aligned}
L(dy_1, dx_1)(\alpha_1) &= dy_1 \wedge (dx_1 \lrcorner dz_1) + dy_1 \lrcorner P^2(dx_1 \wedge dz_1) \\
&= dy_1 + dy_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dy_1 + dy_1 \lrcorner \mu^2 \pi_\omega (\tfrac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dy_1 + dy_1 \lrcorner \mu^2 \tfrac{\sqrt{-1}}{3} \omega \\
&= dy_1 - \sqrt{-1} dx_1 \\
&= -\sqrt{-1}(dx_1 + \sqrt{-1} dy_1).
\end{aligned}$$

Thus

$$L(dx_1, dy_1)(\alpha_1) + L(dy_1, dx_1)(\alpha_1) = 0.$$

Also,

$$\begin{aligned} L(dx_1, dy_1)(\alpha_2) &= dx_1 \wedge (dy_1 \lrcorner dz_2) + dx_1 \lrcorner P^2(dy_1 \wedge \alpha_2) \\ &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2\sqrt{-1}}(dz_1 \wedge dz_2 - d\overline{z_1} \wedge dz_2)) \\ &= -\sqrt{-1} dx_1 \lrcorner (dz_1 \wedge dz_2) \\ &= -\sqrt{-1} dz_2 \end{aligned}$$

and

$$\begin{aligned} L(dy_1, dx_1)(\alpha_2) &= dy_1 \wedge (dx_1 \lrcorner dz_2) + dy_1 \lrcorner P^2(dx_1 \wedge dz_2) \\ &= dy_1 \lrcorner (\lambda^2 \pi_{2,0})(\tfrac{1}{2} dz_1 \wedge dz_2) \\ &= \sqrt{-1} dz_2. \end{aligned}$$

Thus

$$L(dx_1, dy_1)(\alpha_2) + L(dy_1, dx_1)(\alpha_2) = 0.$$

Similarly

$$L(dx_1, dy_1)(\alpha_3) + L(dy_1, dx_1)(\alpha_3) = 0.$$

Next we consider $L(dx_1, dx_2)$.

$$\begin{aligned} L(dx_1, dx_2)(dz_1) &= dx_1 \wedge (dx_2 \lrcorner dz_1) + dx_1 \lrcorner P^2(dx_2 \wedge dz_1) \\ &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2} dz_2 \wedge dz_1 + \tfrac{1}{2} d\overline{z_2} \wedge dz_1) \\ &= dx_1 \lrcorner \lambda^2 \pi_{(2,0)}(\tfrac{1}{2} dz_2 \wedge dz_1) \\ &= dx_1 \lrcorner (dz_2 \wedge dz_1) \\ &= -dz_2 \end{aligned}$$

and

$$\begin{aligned} L(dx_2, dx_1)(dz_1) &= dx_2 \wedge (dx_1 \lrcorner dz_1) + dx_2 \lrcorner P^2(dx_1 \wedge dz_1) \\ &= dx_2 + dx_2 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2} d\overline{z_1} \wedge dz_1) \\ &= dx_2 + \mu^2 \pi_\omega (\tfrac{1}{2} d\overline{z_1} \wedge dz_1) \\ &= dx_2 + \sqrt{-1} dx_2 \lrcorner \omega \\ &= dx_2 + \sqrt{-1} dy_2 = dz_2 \end{aligned}$$

Thus

$$L(dx_1, dx_2)(dz_1) + L(dx_2, dx_1)(dz_1) = 0.$$

Similarly

$$L(dx_1, dx_2)(dz_2) + L(dx_2, dx_1)(dz_2) = 0.$$

Moreover,

$$\begin{aligned} L(dx_1, dx_2)(dz_3) &= dx_1 \wedge (dx_2 \lrcorner dz_3) + dx_1 \lrcorner P^2(dx_2 \wedge dz_3) \\ &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\tfrac{1}{2} dz_2 \wedge dz_3 + \tfrac{1}{2} d\overline{z_2} \wedge dz_3) \\ &= dx_1 \lrcorner \lambda^2 \pi_{(2,0)} \tfrac{1}{2} dz_2 \wedge dz_3 \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} L(dx_2, dx_1)(dz_3) &= dx_2 \wedge (dx_1 \lrcorner dz_3) + dx_2 \lrcorner P^2(dx_1 \lrcorner dz_3) \\ &= 0 + dx_2 \lrcorner \lambda^2 \tfrac{1}{2} dz_1 \wedge dz_3 \\ &= 0. \end{aligned}$$

Thus

$$L(dx_1, dx_2)(dz_3) + L(dx_2, dx_1)(dz_3) = 0.$$

So far we have proved that

$$L(dx_1, dx_1) = 1$$

and

$$L(dx_1, dy_1) + L(dy_1, dx_1) = L(dx_1, dx_2) + L(dx_2, dx_1) = 0$$

. By symmetry and the linearity of L we see that

$$L(dx_i, dx_i) = L(dy_i, dy_i) = 1$$

,

$$L(dx_i, dx_j) + L(dx_j, dx_i) = 0, i \neq j$$

and

$$L(dx_i, dy_j) + L(dy_j, dx_i) = 0$$

. For instance, in order to show

$$L(dx_1, dy_2) + L(dy_2, dx_1) = 0$$

we replace dx_2 by dy_2 and dy_2 by $-dx_2$. Then it follows from the calculation on $L(dx_1, dx_2)$.

Now for arbitray orthonormal basis ω_i, the Clifford relations follow from the fact that the orthogonal transformations act transitively on coframes. If they hold for a particular coframe, they hold for all.

Suppose $g \in SU(3)$. We have for any 1-form α,

$$\begin{aligned} L(\omega_i, \omega_j)(g\alpha) &= \omega_i \wedge \omega_j \lrcorner g(\alpha) + \omega_i \lrcorner P^2(\omega_j \wedge g(\alpha)) \\ &= g[(g^{-1}\omega_i) \wedge (g^{-1}\omega_j) \lrcorner \alpha + (g^{-1}\omega_i) \lrcorner P^2(g^{-1}(\omega_j) \wedge \alpha)] \\ &= gL(g^{-1}\omega_i, g^{-1}\omega_j)(\alpha). \end{aligned}$$

Now by the definition of M we have for any $\alpha = a_i\omega_i$ and $\beta = b_j\omega_j$,

$$M(\alpha, \beta) = \frac{1}{2}(a_i b_j - a_j b_i)L(\omega_i, \omega_j).$$

Thus,

$$\begin{aligned} M(g\alpha, g\beta)(v) &= \tfrac{1}{2}(a_ib_j - a_jb_i)L(g\omega_i, g\omega_j)(v) \\ &= \tfrac{1}{2}(a_ib_j - a_jb_i)gL(\omega_i, \omega_j)g^{-1}(v) \\ &= gM(\alpha, \beta)g^{-1}(v) \end{aligned}$$

i.e., M is $SU(3)$ equivariant.

Note from the above computations, $M(\beta)$ maps $(1,0)$ forms to $(1,0)$ forms for any two-form β. Moreover, since P^2 is self-adjoint, $M(\beta)$ also preserves the inner product. Thus $M(\beta)$, when identified as a two-form, takes value in $\Lambda^{1,1}$. Combined with the $SU(3)$-equivariance, M gives a $SU(3)$-equivariant map from Λ^2V^* to $\Lambda^{1,1}$. Since both of the two irreducible components $\Lambda_0^{1,1}$ and $\mathbf{R}\omega$ are real, $Hom_{SU(3)}(\Lambda^2, \Lambda^{1,1})$ is real 2-dimensional. In other words, there exist two constants a, b so that

$$M(\beta) = a\pi_0^{1,1}(\beta) + b\pi_\omega(\beta).$$

It is a matter of computing examples to determine the constants.

If we take $\beta = dx_1 \wedge dy_1$, then by the convention described above,

$$\begin{aligned} M(\beta) = \frac{1}{2}(d\overline{z_1} \wedge M(\beta)(dz_1) + d\overline{z_2} \wedge M(\beta)(dz_2) + d\overline{z_3} \wedge M(\beta)(dz_3)) \\ = \frac{\sqrt{-1}}{2}(d\overline{z_1} \wedge dz_1 - d\overline{z_2} \wedge dz_2 - d\overline{z_3} \wedge dz_3) \\ = -(dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3). \end{aligned}$$

On the other hand

$$\pi_\omega(dx_1 \wedge dy_1) = \frac{1}{3}\omega.$$

and

$$\pi_0^{1,1}(dx_1 \wedge dy_1) = \frac{1}{3}(2dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3).$$

Consequently

$$a = -2, \quad b = 1.$$

Thus M is of the desired form. $\square$

2.1.2 Dimension 7

Now we assume that $\dim V = 7$ and we pick an oriented orthonormal basis for V^* denoted by $\{dx_1, dy_1, dx_2, dy_2, dx_3, dy_3, du\}$. For later use, let

$$dz_i = dx_i + \sqrt{-1} dy_i$$

and define ω and Ψ as in §2.1. We introduce a special three form

$$\begin{aligned} \Omega &= du \wedge \omega + Im\Psi \\ &= du \wedge (dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3) \\ &\quad + dx_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dy_3 \\ &\quad + dy_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dy_2 \wedge dx_3. \end{aligned} \tag{2.19}$$

Due to [7], it is now well-known that the exceptional Lie group G_2 may be defined as the stabilizers of Ω. For this reason, we call Ω the *fundamental 3-form.* We embed $\mathbf{R}^6$ considered in the last section into V to be the hyperplane $du = 0$. We also let $SU(3)$ act on V by identity on the line $dx_i = dy_i = 0$ and the standard action on $du = 0$. Clearly, $SU(3)$ preserves Ω, so it embeds into G_2 as a Lie subgroup.

G_2-representations

A good resource on this part is [11]. We recall some basic facts. The standard V^* is irreducible with the highest weight $(1,0)$. The most important part for us is $\Lambda^2 V^*$. It decomposes as the sum of two irreducible pieces $V^{(1,0)} \oplus V^{(1,1)}$ where $V^{(a,b)}$ is the irreducible representation of G_2 with the highest weight (a,b). The subspace $V^{(1,0)}$ is 7-dimensional, consisting of 2-forms $v \lrcorner \Omega$ for any $v \in V^*$. The other one $V^{(1,1)}$ is isomorphic to the Lie algebra $\mathfrak{g}_2$.

The space $\Lambda^5 V^*$ is isomorphic to Λ^2 as G_2-modules either by wedge product with Ω or by the Hodge star operation. Again the interplay between these two isomorphisms will be important in defining anti-self-dual instantons in dimesion 7.

Hodge star

We only consider the Hodge star on Λ^2. Now we may compute that

$$*(\alpha) = \frac{1}{2}\Omega \wedge (\alpha) \tag{2.20}$$

for all $\alpha \in V^{(1,0)} \subset \Lambda^2$ and that

$$*\alpha = -\Omega \wedge \alpha \tag{2.21}$$

for $\alpha \in \mathfrak{g}_2$. These may be checked for special forms (e. g., $\alpha = du \lrcorner \Omega \in V^{(1,0)}$ and $\alpha = dz_1 \wedge d\overline{z_2} \in \mathfrak{su}(3) \subset \mathfrak{g}_2$). Then, since these spaces are irreducible and both * and $\Omega\wedge$ commutes with G_2 action, we know these relations must be true for the whole spaces. Thus, these irreducible subspaces are indexed by the eigenvalues of $*(\Omega\wedge)$.

2.2 Anti-self-dual instantons on nearly Kähler 6-manifolds and G_2-cones

Let G be a compact Lie group. Suppose X^n is a smooth manifold endowed with an (n-4)-form Υ (for our purposes, $X = M$ is nearly Kähler and Υ is the $(1,1)$-form ω, or $X = N$ has G_2 holonomy and Υ is the fundamental 3-form Ω). Suppose also Υ is a (n-4)-form on M and P is a principal G-bundle over X. A connection A on P is called Υ-*instanton* if its curvature F_A satisfies

$$\Upsilon \wedge F_A = - *_X F_A. \tag{2.22}$$

Remark 2.2.1. *When G is a unitary group, our definition is different from the one used in [27] (see Remark 1 in its §1.2, however). When G is a special unitary group, these two definitions coincide. This is the group we will use mostly.*

Note that if Υ is closed, an Υ-instanton A is Yang-Mills, i.e., it satisfies the Euler-Lagrange equation of the Yang-Mills functional, since

$$d_A *_X F = -d\Upsilon \wedge F \pm \Upsilon \wedge d_A F = 0$$

because of the Bianchi identity. Thus, an Ω-instanton on a manifold with holonomy in G_2 is Yang-Mills since Ω is closed. Remarkably, as we will show later, when $X = M$ is nearly Kähler, although ω is not closed, an ω-anti-self-dual instanton is still Yang-Mills.

2.2.1 Nearly Kähler 6-manifolds

In this subsection, we collect basic facts about nearly Kähler 6-manifolds. The concept was first introduced and studied by A. Gray in [18]. Later on, N. Hitchin [22] found that it is a critical point of a diffeomorphism invariant functional and thus put it in a more natural context.

An $SU(3)$ structure on a 6-manifold M is a reduction of the total coframe bundle to an $SU(3)$ subbundle. It may be specified by a real two-form ω of type $(1,1)$ and a $(3,0)$-form Ψ normalized so that $\frac{1}{6}\omega^3 = \frac{i}{8}\Psi \wedge \overline{\Psi}$. A nearly Kähler structure is an $SU(3)$-structure for which

$$d\omega = 3c\mathrm{Im}\Psi, \quad d\Psi = 2c\omega^2. \tag{2.23}$$

for some real constant c.

When $c = 0$, the underlying almost complex structure is integrable. In fact, M is Calabi-Yau. When $c \neq 0$, by scaling the metric, we can always assume $c = 1$. In this situation, M is usually called *strictly nearly Kähler*. In this chapter, we assume from now on that $c = 1$ and we speak of this as *nearly Kähler* without danger of confusion.

Structure equations

Let $\alpha_i, i = 1, \cdots, 3$ be a local special unitary coframe, i.e., α_i is complex linear and

$$\omega = \frac{\sqrt{-1}}{2}(\alpha_1 \wedge \overline{\alpha_1} + \alpha_2 \wedge \overline{\alpha_2} + \alpha_3 \wedge \overline{\alpha_3}), \quad \Psi = \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

There exists a unique $\mathfrak{su}(3)$-valued 1-form $(\kappa_{i\bar{j}})$ so that

$$d\alpha_i = -\kappa_{i\bar{j}} \wedge \alpha_j + \epsilon_{ijk}\overline{\alpha_j \wedge \alpha_k} \tag{2.24}$$

where summation is understood when repeated barred and unbarred indices appear. Differentiate this and we get the curvature of κ:

$$d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}}\alpha_l \wedge \overline{\alpha_l}) + K_{i\bar{j}p\bar{q}}\alpha_q \wedge \overline{\alpha_p}, \tag{2.25}$$

where $K_{i\bar{j}p\bar{q}} = K_{p\bar{j}i\bar{q}} = K_{i\bar{q}p\bar{j}} = \overline{K_{j\bar{i}q\bar{p}}}$ and $K_{i\bar{i}p\bar{q}} = 0$.

It follows from the structure equations that κ is a pseudo-Hermitian-Yang-Mills connection on the complex tangent bundle of M.

Compact nearly Kähler examples include the standard S^6, the flag manifold $SU(3)/T^2$, $S^3 \times S^3$, and $\mathbf{CP}^3$ (with an unusual almost complex structure). All these examples are homogeneous. On the other hand, it remains open to find non-homogeneous compact examples.

Example 2.2.2 (G_2-invariant S^6)**.** *The standard G_2 invariant almost complex structure on S^6 is perhaps the best known non-integrable almost complex structure. As a subgroup of $SO(7)$, G_2 acts transitively on S^6 and the stabilizer of any point is isomorphic to $SU(3) \subset G_2$. Thus, G_2 preserves an $SU(3)$-structure on S^6. This $SU(3)$ structure is in fact nearly Kähler. Using Maurer-Cartan forms on G_2, we write the nearly Kähler structure equations*

$$d\alpha_i = -\kappa_{i\bar{j}} \wedge \alpha_j + \epsilon_{ijk}\overline{\alpha_j \wedge \alpha_k} \tag{2.26}$$

$$d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}}\alpha_l \wedge \overline{\alpha_l}). \tag{2.27}$$

2.2.2 G_2-cones over a nearly Kähler 6-manifold

A G_2-structure on a 7-manifold N^7 is a reduction of the total coframe bundle to a G_2-subbundle. Suppose N^7 has such a G_2 structure. Then, on N, there exists a fundamental 3-form Ω characterized by the property that at each point x, there exists a linear isomorphism $u : T_xN \to \mathbf{R}^7$ so that $\Omega_x = u^*(\Omega_0)$. Conversely, given such a fundamental 3-form Ω on N, the set of such linear isomorphisms forms a G_2 subbundle of the total coframe bundle and thus defines a G_2-structure on N.

Associated with any G_2-structure, N has a metric g. The Levi-Civita connection of g has its holonomy group contained in G_2 if and only if $d\Omega = d(*\Omega) = 0$. R. Bryant constructed the first metric with holonomy G_2 [7]. It was the cone metric over $\mathbf{R}_+ \times SU(3)/T^2$. It is now well-known that if M^5 is nearly Kähler with the metric g_M, the cone metric on $N = \mathbf{R}_+ \times M^6$ defined as

$$g_N = dt^2 + t^2 g_M$$

has holonomy in G_2. The fundamental 3-form is

$$\Omega = t^2 dt \wedge \omega + t^3 \mathrm{Im}\Psi.$$

Such conical G_2-singularities were used by string physicists recently to construct string models with chiral matter fields (see [2], [1]). For us, the case $M = S^6$ is especially important. Then the cone has a removable singularity and in fact $N = \mathbf{R}^7$. When studying anti-self-dual instantons on manifolds with G_2 holonomy, $\mathbf{R}^7$ plays the natural role of an infinitesimal model.

Hodge star on 2-forms

Suppose $\omega_i (i = 1, \cdots, 6)$ is an oriented local orthonormal coframe for M. Then, $dt, t\omega_i (i = 1, \cdots, 6)$ form an oriented local orthonormal coframe for the cone N.

Denote $*_M$ and $*_N$ Hodge star operations on M and N respectively. It is easy to show that

$$t^2 dt \wedge *_M(\omega_i \wedge \omega_j) = *_N(\omega_i \wedge \omega_j)$$

and

$$*_N(dt \wedge \omega_i) = t^4 *_M (\omega_i).$$

Consequently, if a 2-form α on N satisfies $\frac{\partial}{\partial t} \lrcorner \alpha = 0$, its Hodge star may be computed by

$$*_N\, \alpha = t^2 dt \wedge *_M(\alpha), \tag{2.28}$$

and if $\alpha = dt \wedge \beta$ with $\frac{\partial}{\partial t} \lrcorner \beta = 0$,

$$*_N\, (dt \wedge \beta) = t^4 *_M (\beta), \tag{2.29}$$

where we extend $*_M$ linearly across functions on N. The formula (2.28), (2.29) will be important below.

2.2.3 ω anti-self-dual instantons

If the underlying manifold is almost Hermitian with the Kähler form ω, we may decompose the curvature as

$$F = F^{2,0} + \overline{F^{2,0}} + (F^{\circ})^{1,1} + H\omega,$$

where $F^{2,0}$ is of type $(2,0)$ and $(F^{\circ})^{1,1}$ is of type $(1,1)$ but with zero ω-trace. Now the ω-anti-self-dual instanton condition (2.22) implies $F^{2,0} = H = 0$.

Remark 2.2.3. *In the case G is a special unitary group, the above argument implies that an ω-instanton is the same as a pseudo-Hermitian-Yang-Mills connection on the canonically associated complex vector bundle.*

If, moreover, we are working on a nearly Kähler manifold, this condition may be simplified.

Lemma 2.2.4. *Suppose A is a connection on nearly Kähler M^6 and F is its curvature. The following are equivalent:*

a. $F \wedge \mathrm{Im}\Psi = 0$.

b. $F \wedge \Psi = 0$.

c. $F \wedge \mathrm{Re}\Psi = 0$.

d. A is an ω-anti-self-dual instanton.

Consequently, if F is of type $(1,1)$, A is an ω-anti-self-dual instanton.

Proof. 1. a$\Longrightarrow$b. We write $F = F^{2,0} + \overline{F^{2,0}} + (F^\circ)^{1,1} + H\omega$. Then $F \wedge \mathrm{Im}\Psi = 0$ gives

$$F \wedge (\Psi - \overline{\Psi}) = 0,$$

i.e.,

$$F^{2,0} \wedge \overline{\Psi} = 0.$$

It follows then that

$$F \wedge \overline{\Psi} = 0$$

and hence $F \wedge \Psi = 0$.

2. b$\Longrightarrow$c is obvious.

3. c$\Longrightarrow$d. As mentioned before, A is ω-anti-self-dual if and only if $F^{2,0} = 0$ and $H = 0$ in the above decomposition. Now

$$F \wedge \mathrm{Re}\Psi = \frac{1}{2} F \wedge (\Psi + \overline{\Psi}) = \frac{1}{2} F^{2,0} \wedge \overline{\Psi} + \overline{F^{2,0}} \wedge \Psi.$$

Thus c gives $F^{2,0} = 0$. Differentiating c gives

$$\begin{aligned} 0 &= d_A(F \wedge \mathrm{Re}\Psi) \\ &= d_A F \wedge \mathrm{Re}\Psi + F \wedge d\mathrm{Re}\Psi \\ &= 0 + 2F \wedge \omega^2 \\ &= 2H\omega^3 \end{aligned}$$

where the last equality uses (2.23) and the Bianchi identity. Hence

$$H = 0.$$

4. d$\Longrightarrow$a is obvious.

□

This lemma says that we could have defined an ω-anti-self-dual as $F^{2,0} = 0$. This reduces the indeterminacy and will be useful later when we construct concrete examples.

Remark 2.2.5. *The same result holds for a more general class of almost complex manifolds, called strictly quasi-integrable in [4]. We leave it for the reader to carry out the details. In fact, this has already been observed in [4] for unitary instantons.*

Generality

We now address the problem of the involutivity problem of the instanton equations. First, the instanton equations may be rephrased as

$$F \wedge \Psi = 0 \tag{2.30}$$

and

$$F \wedge \omega^2 = 0. \tag{2.31}$$

Since the problem is local, we assume that the bundle is trivial, and the connection is simply a $\mathfrak{g}$-valued 1-form A. The differential system we need to analyze is

$$\mathbf{I} = \langle F \wedge \Psi, F \wedge \omega^2 \rangle,$$

defined on $M \times \mathfrak{g}$ where $F = dA + \frac{1}{2}[A, A]$. We have

Lemma 2.2.6. *The system* $\mathbf{I}$ *is involutive with Cartan characters*

$$(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 0, 0, 2d, 3d, d).$$

where $d = \dim G$*.*

Proof. Note that

$$d(F \wedge \Psi) = [A, F] \wedge \Psi + F \wedge d\Psi \equiv 0, \quad \text{mod } \mathbf{I}.$$

because of Bianchi identity and the nearly Kähler condition $d\Psi = 2\omega^2$. It is now routine to check the system is involutive with displayed characters. □

Some remarks are in order.

Remark 2.2.7. *More generally, Lemma 2.2.6 holds for similarly defined* ω*-instantons on a quasi-integrable* $U(3)$*-structure (see [4] for the definition). We leave the details for the interested readers. When* $G = U(r)$ *is a unitary group, it is treated in [4] .*

Remark 2.2.8. *For nearly Kähler* M*, we could have used the differential system* $\langle F \wedge \mathrm{Im}\Psi \rangle$ *by Lemma 2.2.4. The reader can check that this is involutive with the last Cartan character also equal to* d*. The advantage of the original system is that it applies to more general almost complex manifolds.*

Remark 2.2.9. *The last character is* d*, due to the fact that gauge transformations depend on* d *functions of* 6 *variables and that instanton equations are gauge-invariant. We leave for the interested reader to impose a symmetry breaking condition.*

Instantons are Yang-Mills

Now we compute

$$d_A *_M F = -d\omega \wedge F - \omega \wedge d_A F = -3\mathrm{Im}\Psi \wedge F = 0$$

because F is of type $(1,1)$.

Proposition 2.2.10. *An ω-instanton on a nearly Kähler 6-manifold is Yang-Mills.*

A consequence is some removable singularity results for instantons on nearly Kähler 6-manifolds.

Corollary 2.2.11. *Suppose that all representations of $\pi_1(M) \to G$ are trivial and that E is a trivial smooth bundle over M. Assume that A is a ω-instanton on E with a closed singular set S whose $n-4$ Hausdorff measure is locally finite. Then there exists $\epsilon = \epsilon(G, M)$ such that if*

$$\| F_A \|_\infty \leq \epsilon,$$

then the singularity of A is removable.

Corollary 2.2.12. *Suppose that all representations of $\pi_1(M) \to G$ are trivial and that E is a trivial smooth G bundle over M. Assume that A is a ω-instanton on E whose singular set is a closed smooth submanifold of codimension at least 4. Then there exists $\epsilon = \epsilon(G, M)$ such that if*

$$\| F_A \|_{L^{\frac{6}{2}}(M)} \leq \epsilon,$$

then the singularity of A is removable.

Both are proved by employing the results in [29].

Instantons as critical points of a Chern-Simons functional

Consider the functional

$$CS(A) = \int_M \mathrm{tr}(F_A^2) \wedge \omega. \tag{2.32}$$

On a Kähler manifold, since ω is closed, CS is a topological constant. However, on a nearly Kähler manifold, this gives more interesting information.

It is easy to compute that the first variation of CS is

$$\delta CS = 2 \int_M \mathrm{tr}(F_A \wedge d_A \delta A) \wedge \omega.$$

Integration by parts gives

$$\delta CS = 2 \int_M \mathrm{tr}[d_A(F_A \wedge \omega) \wedge \delta A].$$

Thus the Euler-Lagrange equation for CS is

$$d_A(F_A \wedge \omega) = 0.$$

Using Bianchi Identity, we see this is equivalent to

$$F \wedge \mathrm{Im}\Psi = 0. \tag{2.33}$$

It follows from Lemma 2.2.4 that

Proposition 2.2.13. *An ω-anti-self-dual instanton is equivalent to a critical connection of the Chern-Simons functional CS.*

This makes it possible to use variational methods to study ω-anti-self-dual instantons on nearly Kähler 6-manifolds.

It also follows that the gradient flow of CS takes the form

$$\frac{d}{dt} A = *_M (F \wedge \mathrm{Im}\Psi). \tag{2.34}$$

Remark 2.2.14. *To illustrate, we assume that the principal bundle under consideration is topologically trivial. Using* $d\omega = 3\mathrm{Im}\Psi$ *and transgression formula, it can be shown that up to a constant*

$$CS(A) = \int_M \mathrm{tr}(F \wedge A - \frac{1}{3}A \wedge A \wedge A) \wedge \mathrm{Im}\Psi. \tag{2.35}$$

Here we regard G *as a matrix Lie group. This formulation is more similar to the Chern-Simons functional on 3-manifolds.*

Next we compute the second variation Q of CS. Suppose that $A(s,t)$ (for small s,t) are a two parameter family of connections such that $A = A(0,0)$ is an instanton. Let $a = \frac{\partial A}{\partial s}|_{s=0,t=0}, \quad b = \frac{\partial A}{\partial t}|_{s=0,t=0}$.Then by definition

$$Q(a,b) = \frac{\partial^2}{\partial s \partial t}|_{s=0,t=0} CS(A)$$

We have essentially computed that

$$\frac{\partial}{\partial t} CS(A) = -6 \int_M \mathrm{tr}(F_A \wedge \mathrm{Im}\Psi \wedge \frac{\partial}{\partial t} A(s,t)).$$

Thus the second derivative is (remember $F_A \wedge \mathrm{Im}\Psi = 0$)

$$\begin{aligned} Q(a,b) &= -6 \int_M \mathrm{tr}(d_A(\frac{\partial A}{\partial s}|_{s=0,t=0}) \wedge \mathrm{Im}\Psi \wedge \frac{\partial A}{\partial t}|_{s=0,t=0}) && (2.36) \\ &= -6 \int_M \mathrm{tr}(d_A a \wedge \mathrm{Im}\Psi \wedge b) && (2.37) \end{aligned}$$

Clearly, this is a symmetric bilinear form.

The null space of Q consists of a so that

$$d_A a \wedge \mathrm{Im}\Psi = 0.$$

This implies $d_A a$ is of type $(1,1)$ and hence

$$d_A a \wedge \Psi = 0.$$

Differentiating once and using Bianchi Identity gives one more equation

$$d_A a \wedge \omega^2 = 0.$$

This is exactly the infinitesimal deformation of the instanton equation.

2.2.4 Ω-anti-self-dual instantons on the G_2-cone

We investigate the relation between ω-anti-self-dual instantons on M and Ω-instantons on N.

First, note that any principal G-bundle over N is isomorphic to a bundle $P \times \mathbf{R}^+ \to M \times \mathbf{R}^+$ for a G-bundle P over M. Thus, without loss of generality, we assume that the G-bundle we are working on is a pull-back from M and we use the same letter P to denote these two bundles.

Suppose A is an Ω-instanton. A priori, A involves a dt-term $a \cdot dt$. However, we may perform a gauge transformation $A \mapsto g^{-1}Ag + g^{-1}dg$ to eliminate the dt-term. It is easy to see that we can simply take g as a solution to the differential equation

$$g^{-1}agdt + g^{-1}dg = 0.$$

Thus, we assume that A has no dt-term. We regard A as a family of connections on P parametrized by t and denote $\dot{A} = \frac{d}{dt}A$. Now the curvature may be computed

$$F^N = dA + \frac{1}{2}[A, A] = dt \wedge \dot{A} + F^M,$$

where $F^M = d_M A + \frac{1}{2}[A, A]$. The Ω-instanton condition with the formulae (2.28) and (2.29) gives

$$t *_M \alpha = -\mathrm{Im}\Psi \wedge F^M$$

and

$$\omega \wedge F^M - t\mathrm{Im}\Psi \wedge \alpha = - *_M F^M.$$

We denote the $(1,1)$-part (with coefficients depending on t) of F^M by F_0^M and $F_1^M = F^M - F_0^M$. By type decomposition in the above two equations we have

$$t *_M \dot{A} = -\mathrm{Im}\Psi \wedge F_1^M,$$

$$\omega \wedge F_0^M = - *_M F_0^M, \tag{2.38}$$

and

$$\omega \wedge F_1^M - t\mathrm{Im}\Psi \wedge \dot{A} = - *_M F_1^M.$$

By taking Hodge star of both sides, we see that the first equation is equivalent to

$$t\dot{A} = *(\mathrm{Im}\Psi \wedge F_1^M). \tag{2.39}$$

Combining (2.8) and (2.5) we see that the last equation is implied by (2.39).

The equation (2.38) looks very much like the ω-anti-self-dual instanton equation on M. The only problem is that F_0^M is not necessarily the curvature of a well-defined connection.

The equation (2.39) is exactly the gradient flow of the Chern-Simons functional CS. It would be interesting to analyze this equation coupled with (2.38). The first natural question is whether we could evolve through (2.39) in the class of ω-anti-self-dual instantons on M to get a Ω-anti-self-dual instanton on N. Unfortunately, this is impossible. An ω-instanton has its curvature of type $(1,1)$. If $A(t)$ stays ω-anti-self-dual for all t, the evolution equation (2.39) will imply that $\frac{d}{dt}A = 0$, i.e., A is constant in t. On the other hand, if A is constant in t and ω-anti-self-dual, it is Ω-anti-self-dual when pulled back to the cone N. These give a class of special solutions.

Lemma 2.2.15. *Suppose A is an ω-anti-self-dual connection on the nearly Kähler 6-manifold M and extend it to the G_2-cone N by constant in t. Then A is a Ω-anti-self-dual connection on N.*

Remark 2.2.16. *When $M = S^6$, in order that the principal bundle extend through the origin in $\mathbf{R}^7$, P has to be trivial over M. Even when this is true, the extended Ω-anti-self-dual connection on $\mathbf{R}^7 \setminus \{0\}$ described in the above Lemma does not necessarily extend through origin. It is interesting to ask under what condition this singularity is removable after a gauge transformation.*

2.3 A Weitzenböck formula

In this section, we derive a Weitzenböck formula for nearly Kähler 6-manifolds and describe its application to the deformation of ω-anti-self-dual instantons.

2.3.1 The general formula

Let E be a vector bundle over M. Suppose E is equipped with a metric and a metric-compatible connection A. Suppose also that the curvature F_A is an ω-instanton. Consider the following complex

$$0 \to \Gamma(E) \xrightarrow{d_A} \Gamma(E \otimes T^*M) \xrightarrow{Pd_A} \Gamma(E \otimes (\Lambda^{(2,0)}T^*M)_{\mathbf{R}} \oplus \mathbf{R}\omega),$$

where the operator d_A is induced from d and the connection A and P is defined in §2.1.1 the projection onto the orthogonal complement of ω-trace free $(1,1)$-forms. This complex is elliptic at the middle term. It could be extended to an elliptic complex, but we will not need the full sequence.

The 0th cohomology group consists of parallel sections of E. We are mainly interested in the 1st cohomology group. A well-known result in Hodge Theory states that this group can be represented by harmonic sections, i.e., the kernel of the elliptic operator

$$\Delta_A = (d_A^* + Pd_A)^*(d_A^* + Pd_A) = d_A d_A^* + d_A^* P^2 d_A$$

As usual, we will compare Δ_A with a certain rough Laplacian of a connection. Note that, on $E\otimes T^*M$, there are several connections, e.g., A, coupled with the $\mathfrak{su}(3)$-connection on T^*M, denoted by $\hat{D}$ as well as A with the Levi-Civita connection, denoted by D. After many trials, we choose D. However, $\hat{D}$ will be useful.

Suppose $x \in M$ is a fixed point. Let $\{e_i\}_{i=1}^6$ be a local orthonormal frames centered at x whose covariant derivatives with respect to the Levi-Civita connection vanish at x. Let $\{\omega_i\}$ be the coframe. The Hodge Laplacian may be computed

$$\begin{aligned}
\Delta_A &= d_A d_A^* + d_A^* P^2 d_A \\
&= (\textstyle\sum_{i=1}^6 \omega_i \wedge D_{e_i}) \circ (-\sum_{j=1}^6 \omega_j \lrcorner D_{e_j}) \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner D_{e_i}) \circ P^2 \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&= (\textstyle\sum_{i=1}^6 \omega_i \wedge D_{e_i}) \circ (-\sum_{j=1}^6 \omega_j \lrcorner D_{e_j}) \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner \circ P^2 \circ D_{e_i}) \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&= -\textstyle\sum_{i,j=1}^6 (\omega_i \wedge \circ \omega_j \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_j \wedge) D_{e_i} D_{e_j} \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A \\
&= -\textstyle\sum_{i=1}^6 (\omega_i \wedge \circ \omega_i \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_i \wedge) D_{e_i} D_{e_i} \\
&\quad -\textstyle\sum_{i\neq j} (\omega_i \wedge \circ \omega_j \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_j \wedge) D_{e_i} D_{e_j} \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A \\
&= -\textstyle\sum_{i=1}^6 D_{e_i} D_{e_i} \\
&\quad -\textstyle\sum_{i<j} M(\omega_i \wedge \omega_j) \circ (D_{e_i} D_{e_j} - D_{e_j} D_{e_i}) \\
&\quad -(\textstyle\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A,
\end{aligned}$$

where the operator M is defined in §2.1.1.

Recall $De_i|_x = 0$. Thus at x,

$$-\sum_{i=1}^{6} D_{e_i} D_{e_i} = D^* D$$

the rough Laplacian. For the same reason, $D_{e_i}D_{e_j} - D_{e_j}D_{e_i}$ is the curvature on $T^*M \otimes E$. The curvature has two parts $R \otimes Id_E + Id_{T^*M} \otimes F^E$ where R is the Riemannian curvature of M. We write $R = \frac{1}{4} R_{klij} \omega_l \wedge \omega_k \otimes \omega_i \wedge \omega_j$. Given a 1-form α and two vectors X and Y

$$D_X D_Y \alpha - D_Y D_X \alpha - D_{[X,Y]}\alpha = \frac{1}{4} R_{klij} \omega_i \wedge \omega_j (X, Y) \alpha \lrcorner (\omega_l \wedge \omega_k).$$

Now consider the term in the formula $\sum_{i=1}^{6} \omega_i \lrcorner \circ [D_{e_i}, P^2]$. Note that the $\mathfrak{su}(3)$-connection $\hat{D}$ commutes with P^2, i.e., $[\hat{D}_{e_i}, P^2] = 0$ for any e_i. Moreover, the difference $r(e_i) = D_{e_i} - \hat{D}_{e_i}$ is exactly the $\mathfrak{su}(3)$-torsion up to a constant. Here, the nearly Kähler structure plays the central role. By definition, this torsion r is covariantly constant with respect to the $\mathfrak{su}(3)$-connection. Hence, r satisfies (2.9) and the operator

$$\sum_{i=1}^{6} \omega_i \lrcorner \circ [D_{e_i}, P^2] = \sum_{i=1}^{6} \omega_i \lrcorner \circ [r(e_i), P^2] = B$$

factors through a linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$ according to (2.1.2).

In summary, we have the following Weitzenböck formula.

$$\Delta_A = \nabla_A^* \nabla_A \tag{2.40}$$

$$-B \circ d_A \tag{2.41}$$

$$-\frac{1}{2} \sum_{i,j} M(\omega_i \wedge \omega_j) \circ R_{ij} \otimes Id_E - \frac{1}{2} \sum_{ij} M(\omega_i \wedge \omega_j) \otimes F_{ij}. \tag{2.42}$$

A routine consequence of this formula is the following:

Lemma 2.3.1. *Suppose M is a compact nearly Kähler 6-manifold. Suppose the curvature (2.42) is non-negative as an operator on $T^*M \otimes E$. Then the first cohomology group is at most $6 \cdot \mathrm{rank} E$-dimensional. If, moreover, the curvature is positive somewhere, the first cohomology group vanishes.*

Proof. The key observation is that for any harmonic section s of the elliptic sequence representing an element in the first cohomology group, we have

$$Bd_A(s) = 0.$$

The rest of the proof parallels the argument in usual Bochner Technique. □

Remark 2.3.2. *It is not difficult to work out the explicit formula for the curvature term (2.42) using $SU(3)$-representation theory. We will discuss this for $M = S^6$ and leave the general case as an exercise for the interested reader.*

2.3.2 Deformation of ω-anti-self-dual instantons

Suppose $\mathbf{P}$ is a principal G-bundle with G a compact Lie group. As said before, a connection A on P is an ω-anti-self-dual instanton if and only if its curvature F satisfies

$$P(F) = 0. \tag{2.43}$$

Unless G is Abelian, this equation is nonlinear in A. Moreover, it is invariant under the action of the gauge transformations of $\mathbf{P}$.

The linearization of (2.43) at an ω-anti-self-dual instanton A is given by

$$Pd_A\alpha = 0$$

for $\alpha \in T^*M \otimes \mathbf{P} \times_G \mathfrak{g}$. Of course, one would like to divide by the infinitesimal gauge transformation since (2.43) is gauge-invariant. These infinitesimal gauge transformations are given by the image of $d_A : \mathbf{P} \times_G \mathfrak{g} \to T^*M \otimes \mathbf{P} \times_G \mathfrak{g}$. Thus, in fact, the essential infinitesimal deformations of the ω-anti-self-dual instanton A correspond to the elements of the first cohomology group of the following sequence

$$0 \to \Gamma(E) \xrightarrow{d_A} \Gamma(E \otimes T^*M) \xrightarrow{Pd_A} \Gamma(E \otimes (\Lambda^{(2,0)}T^*M)_{\mathbf{R}} \oplus \mathbf{R}\omega) \to 0,$$

where $E = \mathbf{P} \times_G \mathfrak{g}$.

It follows that, all discussion in the previous section applies to instanton deformations. In particular, if the curvature of A is small enough (with a bound depending only on the base manifold M), then A is rigid, i.e., allows no deformation. We will illustrate this by analyzing S^6.

Applications to S^6

For S^6, the Riemannian curvature simplifies greatly

$$R = \frac{1}{2}\omega_j \wedge \omega_i \otimes \omega_i \wedge \omega_j.$$

It may be computed that

$$\begin{aligned} \tfrac{1}{4}\sum M(\omega_i \wedge \omega_j)(\omega_1 \lrcorner(\omega_j \wedge \omega_i)) &= \tfrac{1}{4}\sum M(\omega_i \wedge \omega_j)(\delta_{1j}\omega_i - \delta_{1i}\omega_j) \\ &= \tfrac{1}{4}(\textstyle\sum_{i=1}^{6} M(\omega_i \wedge \omega_1)(\omega_i) \\ &- \textstyle\sum_{j=1}^{6} M(\omega_1 \wedge \omega_j)(\omega_j) \\ &= -\tfrac{5}{2}\omega_1. \end{aligned}$$

By symmetry, it holds that

$$\frac{1}{4}\sum M(\omega_i \wedge \omega_j)(\alpha \lrcorner(\omega_j \wedge \omega_i)) = -\frac{5}{2}\alpha$$

for any one-form α. Thus the first curvature term in the Weitzenbock formula

$$-\frac{1}{2}\sum_{i,j} M(\omega_i \wedge \omega_j) \circ R_{ij} \otimes Id_E = \frac{5}{2}.$$

An easy consequence is

Theorem 2.3.3. *A flat ω-instanton on S^6 is rigid.*

As another application, we consider the $\mathfrak{su}(3)$-connection on the standard structure bundle $G_2 \to S^6$. We need to describe the $\mathfrak{su}(3)$ connection a bit. Recall the connection 1-form $\kappa_{i\bar{j}}$ in (2.26). Through this, the connection on $(1,0)$-forms is

$$D_X \alpha_i = -\alpha_j \kappa_{i\bar{j}}(X)$$

for any vector field X. Correspondingly the curvature is give by

$$D_X D_Y - D_Y D_X - D_{[X,Y]}(\alpha_i) = -\alpha_j (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}})(X, Y).$$

Thus the action of F on $(1,0)$ forms is given by

$$\alpha_i \mapsto -\alpha_j \otimes (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}}).$$

More generally,

$$F : \alpha \mapsto \alpha \lrcorner - \frac{1}{2}(\overline{\alpha_i} \wedge \alpha_j) \otimes (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}}).$$

Denote $\Omega_{i\bar{j}} = d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}} \alpha_l \wedge \overline{\alpha_l})$. For each k, l, F_{kl} in the curvature term is

$$F_{kl} : \alpha \mapsto \alpha \lrcorner - \frac{1}{2}(\overline{\alpha_i} \wedge \alpha_j)\Omega_{i\bar{j}}(e_k, e_l).$$

Then the second curvature term in this context when $E = T^{1,0}$ is

$$\begin{aligned}
&-\frac{1}{2}M(\omega_k \wedge \omega_l)(v) \otimes F_{kl}(\alpha) \\
= -\frac{1}{2}M(\omega_k \wedge \omega_l)(v) \otimes \alpha \lrcorner (-\frac{1}{2})&(\overline{\alpha_i} \wedge \alpha_j)\Omega_{i\bar{j}}(e_k, e_l) \\
&= \frac{1}{4}M(\Omega_{i\bar{j}})(v) \otimes \alpha \lrcorner (\overline{\alpha_i} \wedge \alpha_j) \\
&= -\frac{1}{2}v \lrcorner \Omega_{i\bar{j}} \otimes \alpha \lrcorner (\overline{\alpha_i} \wedge \alpha_j)
\end{aligned}$$

because $\Omega_{i\bar{j}}$ is in $\Lambda_0^{1,1}$. This is not exactly what we want when we study the deformation of $\mathfrak{su}(3)$ connection on G_2. However, all we need is to replace α by a section of $ad_{G_2} \simeq \Lambda_0^{1,1}$ where the identification has been defined before. The action of $\overline{\alpha_i} \wedge \alpha_j$ will be the Lie bracket whose meaning should be clear via the aforementioned identification.

Denote

$$B(s,t) = \langle -\frac{1}{2} M(\omega_i, \omega_j) \otimes F_{ij}(s), t \rangle$$

where s, t are sections of $T^* \otimes ad_{G_2}$. Note that if $s = \phi \otimes X$ and $t = \psi \otimes Y$ we have

$$\begin{aligned} B(s,t) &= -\frac{1}{2} \langle M(\omega_i \wedge \omega_j)(\phi) \otimes [F_{ij}, X], \psi \otimes Y \rangle \\ &= -\frac{1}{2} \langle M(\omega_i \wedge \omega_j)(\phi), \psi \rangle \langle [F_{ij}, X], Y \rangle \\ &= -\frac{1}{2} \langle M(\omega_i \wedge \omega_j), \phi \wedge \psi \rangle \langle F_{ij}, [X, Y]] \rangle \\ &= \langle -\frac{1}{2} M(\omega_i \wedge \omega_j) \otimes F_{ij}, [s,t] \rangle. \end{aligned}$$

where we view $M(\omega_i \wedge \omega_j)$ as a 2-form.

Then since F is $SU(3)$-invariant, B is a $SU(3)$-invariant symmetric bilinear form on $\mathbf{R}^6 \otimes \mathfrak{su}(3)$. We study the space of $SU(3)$ invariant symmetric bilinear forms on $\mathbf{R}^6 \otimes \mathfrak{u}(3)$. One candidate is obvious, the $SU(3)$ invariant inner product, denoted by B_0. For others we apply representation theory of $SU(3)$. Then complexified representation $(\mathbf{R}^6 \otimes \mathfrak{su}(3)) \otimes \mathbf{C} \cong (\mathbf{C}^3 \oplus \overline{\mathbf{C}^3}) \otimes_{\mathbf{C}} sl_3(\mathbf{C})$ decomposes as

$$(V^{(1,0)} \oplus \overline{V^{(1,0)}}) \oplus V^{(2,0)} \oplus \overline{V^{(2,0)}} \oplus V^{(2,1)} \oplus \overline{V^{(2,1)}},$$

where $V^{(a,b)}$ denotes the irreducible complex representation of $SU(3)$ with the highest weight (a,b). The representation $V^{(a,b)}$ is real (i.e., $V^{(a,b)} \cong \overline{V^{(a,b)}}$) if and only if $a = b$. Thus the original $\mathbf{R}^6 \otimes \mathfrak{u}(3)$ decomposes as

$$(V^{(1,0)})_{\mathbf{R}} \oplus (V^{(2,0)})_{\mathbf{R}} \oplus (V^{(2,1)})_{\mathbf{R}},$$

where $V_{\mathbf{R}}$ means the real representation by forgetting the complex structure of V. The irreducible pieces are $6, 12, 30$ dimensional respectively. One of them is known as $V^{(1,0)} = \mathbf{C}^3$. Coupled with the standard inner product, every $SU(3)$-invariant bilinear form on $\mathbf{R}^6 \otimes \mathfrak{u}(3)$ will give rise to a $SU(3)$-invariant endomorphism. The space of such endomorphisms is 6 dimensional, 2 for each irreducible component. Of the two independent bilinear forms on every irreducible component, one can be taken as the $SU(3)$-invariant inner product and the other is symplectic. Thus the space of $SU(3)$-invariant symmetric bilinear forms is 3-dimensional, represented by linear combinations of inner products of various components.

We will construct a basis for this 3-dimensional space. We already have one– the inner product of the whole space B_0. To construct two more, we need more information about the irreducible components.

First consider the map

$$T_1 : \mathbf{R}^6 \otimes \mathfrak{su}(3) \to \mathbf{R}^6$$

defined by $v \otimes \alpha \mapsto v \lrcorner \alpha$. This map is clearly $SU(3)$-equivariant so $B_1(u, v) = \langle T_1 u, T_1 v \rangle$ is clearly $SU(3)$-invariant. Moreover, since $\mathbf{R}^6 \otimes \mathfrak{su}(3)$ contains only one copy of $\mathbf{R}^6$ and T_1 is nonzero, by Schur's Lemma, T_1 and thus B_1 is zero on $(V^{(2,0)})_{\mathbf{R}} \oplus (V^{(2,1)})_{\mathbf{R}}$.

For later estimate, we need the right inverse of T_1. Define the operatore

$$S_1 : \mathbf{R}^6 \to \mathbf{R}^6 \otimes \mathfrak{su}(3)$$

by

$$v \to \frac{3}{16} \sum_i \alpha_i \otimes \pi_0^{1,1}(\overline{\alpha_i} \wedge v) + \frac{3}{16} \sum_i \overline{\alpha_i} \otimes \pi_0^{1,1}(\alpha_i \wedge v).$$

It is clearly $SU(3)$ equivariant. Since $\mathbf{R}^6$ is irreducible, S_1 maps onto the irreducible components $V_{\mathbf{R}}^{(1,0)} \in \mathbf{R}^6 \otimes \mathfrak{su}(3)$.

Then the composition $T_1 \circ S_1$ must be a linear combination of Id and J(the almost complex structure). However, it may be computed that

$$S_1(\alpha_1) = \frac{3}{16}(\alpha_1 \otimes \pi_0^{1,1}(\overline{\alpha_1} \wedge \alpha_1) + \alpha_2 \otimes \pi_0^{1,1}(\overline{\alpha_2} \wedge \alpha_1) + \alpha_3 \otimes \pi_0^{1,1}(\overline{\alpha_3} \wedge \alpha_1)$$

$$= \frac{3}{16}(\alpha_1 \otimes \frac{1}{3}(2\overline{\alpha_1} \wedge \alpha_1 + \alpha_2 \wedge \overline{\alpha_2} + \alpha_3 \wedge \overline{\alpha_3}) + \alpha_2 \otimes \overline{\alpha_2} \wedge \alpha_1 + \alpha_3 \otimes \overline{\alpha_3} \wedge \alpha_1)$$

Thus $T_1 S_1(\alpha_1) = \alpha_1$ and hence $T_1 S_1 = Id$.

Meanwhile, it is easy to compute that

$$B_0(S_1(\alpha), S_1(\overline{\alpha})) = \frac{3}{4} B_1(S_1(\alpha), S_1(\overline{\alpha})). \tag{2.44}$$

Second consider the map

$$T_2 : \mathbf{R}^6 \otimes \mathfrak{su}(3) \to \wedge^3 \mathbf{R}^6 \to (\mathbf{R}^6 \wedge \omega)^{\perp}.$$

defined by $v \otimes \alpha \mapsto v \wedge \alpha$ followed by the projection onto the orghogonal complement of $\mathbf{R}^6 \wedge \omega$. Define $B_2(u, v) = \langle T_2 u, T_2 v \rangle$. Then T_2 is $SU(3)$ equivariant and B_2 is $SU(3)$ invariant. The image of T_2 lies in the space of type $(2,1) + (1,2)$ forms.

We also need the partial inverse of T_2. Define

$$S_2 : \psi \mapsto \frac{1}{4}(\alpha_i \otimes \pi_0^{1,1}(\overline{\alpha_i} \lrcorner \psi) + \overline{\alpha_i} \otimes \pi_0^{1,1}(\alpha_i \lrcorner \psi)).$$

It is clearly $SU(3)$ equivariant. The image under S_2 of $(2,1)+(1,2)$ forms orthogonal to $\mathbf{R}^6 \wedge \omega$ is $V^{(2,0)}$. It is easy to compute

$$S_2(\alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}) = \frac{1}{4}(2\alpha_1 \otimes \alpha_2 \wedge \overline{\alpha_3} - 2\alpha_2 \otimes \alpha_1 \wedge \overline{\alpha_3}).$$

Consequently, $T_2 S_2(\alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}) = \alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}$. Thus $T_2 S_2 = 1$.

It is also easy to verify that

$$B_0(S_2(\psi), S_2(\overline{\psi})) = \frac{1}{2} B_2(S_2(\psi), S_2(\overline{\psi})). \tag{2.45}$$

On the other hand, it may be computed that

$$T_2 S_1 = 0, \quad T_1 S_2 = 0. \tag{2.46}$$

The 3 symmetric bilinear forms B_0, B_1, B_2 are clearly linearly independent. Thus there exist constants λ_i such that $B = \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2$. We will compute examples to determine these constants.

Set

$$u_1 = \alpha_1 \otimes \sqrt{-1}(2\alpha_1 \wedge \overline{\alpha_1} - \alpha_2 \wedge \overline{\alpha_2} - \alpha_3 \wedge \overline{\alpha_3}),$$

$$u_2 = \alpha_1 \otimes (\alpha_2 \wedge \overline{\alpha_3} + \overline{\alpha_2} \wedge \alpha_3).$$

and

$$u_3 = \alpha_1 \otimes \alpha_1 \wedge \overline{\alpha_2}.$$

It is easy to see that $[u_1, \overline{u_1}] = [u_2, \overline{u_2}] = 0$. Thus

$$0 = B(u_1, \overline{u_1}) = \lambda_0 B_0(u_1, \overline{u_1}) + \lambda_1 B_1(u_1, \overline{u_1}) = (\lambda_0 + \frac{4}{3}\lambda_1) B_0(u_1, \overline{u_1}),$$

$$0 = B(u_2, \overline{u_2}) = \lambda_0 B_0(u_2, \overline{u_2}) + \lambda_2 B_2(u_2, \overline{u_2}) = (\lambda_0 + 2\lambda_2) B_0(u_2, \overline{u_2}),$$

and

$$B(u_3, \overline{u_3}) = \lambda_0 B_0(u_3, \overline{u_3}).$$

Hence

$$\lambda_1 = -\frac{3}{4}\lambda_0, \lambda_2 = -\frac{1}{2}\lambda_0$$

and

$$\lambda_0 = \frac{B(u_3, \overline{u_3})}{B_0(u_3, \overline{u_3})}.$$

The curvature $F = -\frac{1}{8}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{ij}\alpha_l \wedge \overline{\alpha_l}) \otimes_{\mathbf{C}} \overline{\alpha_i} \wedge \alpha_j$. Thus

$$\begin{aligned} B(u_3, \overline{u_3}) &= \langle F, [u_3, \overline{u_3}] \rangle \\ &= \langle F, \alpha_1 \wedge \overline{\alpha_1} \otimes (-2\overline{\alpha_1} \wedge \alpha_1 + 2\overline{\alpha_2} \wedge \alpha_2) \rangle \\ &= 12. \end{aligned}$$

Thus $\lambda_0 = \frac{3}{2}$. Consequently,

$$B = \frac{3}{2}(B_0 - \frac{3}{4}B_1 - \frac{1}{2}B_2).$$

Lemma 2.3.4.

$$B \geq 0.$$

Proof. Let $\psi \in \mathbf{R}^6 \otimes \mathfrak{su}(3)$ be real. Write $\psi = S_1T_1(\psi) + S_2T_2(\psi) + \hat{\psi}$. Note that $\hat{\psi} \in \ker T_1 \cap \ker T_2$. Thus, in fact $\psi \in V_{\mathbf{R}}^{(2,1)}$. These three different components are thus pairwise perpendicular, since they lie in different irreducible pieces. It follows that

$$\frac{2}{3}B(\psi, \psi) = B_0(\hat{\psi}, \hat{\psi}).$$

□

The contribution from the second curvature term is nonnegative. All together the curvature part is strictly positive.

To summarize, we have the following result.

Theorem 2.3.5. *The $\mathfrak{su}(3)$-connection on $G_2 \to S^6$ in (2.26) is a rigid $SU(3)$ instanton.*

2.4 $SO(4)$-invariant examples

We construct cohomogeneity one $SU(2)(S^3)$ anti-self-dual instantons (equivalent to pseudo-Hermitian-Yang-Mills here) on S^6. The idea is to impose symmetries to reduce the instanton equations to ODEs. We regard $SU(2) = S^3$ as the set of unit quaternions whose Lie algebra is the tangent space at 1 consisting of imaginary quaternions for which we use I, J, K to denote the standard basis for imaginary quaternions. A remark on the notation is necessary. Throughout this section, we use $\sqrt{-1}$ to represent complex numbers to avoid confusion with quaternions. It should be cautioned that when complex numbers are regarded as coefficients in the complexified Lie algebra, they commute with I, J, K rather than following the usual rule of multiplication with quaternions. Hopefully, this will be clear from context.

2.4.1 A dense open subset U of S^6

More precisely, $U = S^6 \setminus (S^2 \cup S^3)$ is parametrized by $S^2 \times S^3 \times (0, \frac{\pi}{2})$ as

$$(x, y, t) \mapsto v = (x \cos t, y \sin t))$$

where we think of $x \in S^3 \subset \mathbf{R}^4$ as a unit 4-vector and $y \in S^2 \subset \mathbf{R}^3$ as a unit 3-vector. Actually, if we extend the map to the closed interval $[0, \frac{\pi}{2}]$, we cover the whole S^6. Reverse the picture and we get a map $t : S^6 \to [0, \frac{\pi}{2}]$ which is roughly the distance function from the totally geodesic pseudo-holomorphic $S^2 = \{t = 0\}$. A generic level set is a scaled $S^2 \times S^3$ and $\{t = \frac{\pi}{2}\}$ is a totally geodesic, special Lagrangian S^3.

For later use,

$$S^3 \times S^2 = S^3 \times S^3/S^1$$

as a homogeneous space via $(p, q) \sim (pz, qz)$ for $(p, q) \in S^3 \times S^3$ and $z \in S^1$. Composing this quotient with the map $(x, y, t) \mapsto v$, we have a map $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to U \subset S^6$ by

$$(p, q) \mapsto (pI\overline{p} \cos t, qp^{-1} \sin t).$$

Denote by $\omega = \omega_1 I + \omega_2 J + \omega_3 K$ and $\psi = \psi_1 I + \psi_2 J + \psi_3 K$ the left-invariant Maurer-Cartan forms on the two copies of S^3, respectively. Then, $dt, \omega_2, \omega_3, \psi_2, \psi_3$ and $\tau = \omega_1 - \psi_1$ form a basis of semibasic 1-forms for the projection $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to U$. We use this to describe the nearly Kähler structures on U induced from S^6.

Recall that the G_2-invariant almost complex structure J on $T_v S^6$ is given by the left Cayley multiplication by v when we we regard both v and tangent vectors as Cayley numbers in $\mathbf{R}^8 = \mathbb{O}$. In other words,

$$J : dv \mapsto v \cdot dv. \tag{2.47}$$

The standard metric and J determines the Kähler 2-form $\omega = \langle Jdv, dv \rangle$.

Using (2.47) and Cayley-Dickson rule of Cayley multiplication, we can establish the following

$$\begin{aligned} J(dt) &= \sin t\tau, \\ J(2\cos t\omega_3) &= (2\cos^2 t - \sin^2 t)\omega_2 + \sin^2 t\psi_2, \\ J(-2\cos t\omega_2) &= (2\cos^2 t - \sin^2 t)\omega_3 + \sin^2 t\psi_3. \end{aligned}$$

The Kähler form ω is determined by

$$\begin{aligned} -\omega = \langle v, Jv \rangle &= 2\sin t\psi_1 \wedge dt - 2\sin t\omega_1 \wedge dt \\ &\quad + 2\cos t(9\cos^2 t - 5)\omega_3 \wedge \omega_2 + 6\sin^2 t\cos t\omega_3 \wedge \psi_2 \\ &\quad - 6\cos t\sin^2 t\omega_2 \wedge \psi_3 + 2\sin^2 t\cos t\psi_2 \wedge \psi_3. \end{aligned}$$

2.4.2 Bundle constructions and $SO(4)$-invariant connections

S^3-bundles

We now describe the principal S^3-bundles on which to construct instantons. First, note that $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to S^3 \times S^2 \times (0, \frac{\pi}{2})$ in §2.4.1 is a principal S^1-bundle. The principal S^3-bundles are obtained by extending the structure group through the group homomorphisms

$$z \mapsto z^l$$

for $z \in S^1$. More explicitly, denote

$$B_l = S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) / \sim$$

where

$$(p, q, r, t) \sim (pz, qz, rz^{-l}, t)$$

for any $(p, q, r, t) \in S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$, $t \in (0, \frac{\pi}{2})$ and $z \in S^1$. The structure group S^3 acts on B_l by

$$[p, q, r, t] \mapsto [p, q, rg, t]$$

for any $g \in S^3$. Clearly this is well-defined. Then the projection

$$[p, q, r, t] \mapsto (pIp^{-1} \cos t, qp^{-1} \sin t)$$

makes B_l a principal S^3-bundle over U.

Remark 2.4.1 (on the symmetry of B_l)**.** *Note that if we let* $[g_1, g_2] \in SO(4) = S^3 \times S^3/\mathbb{Z}_2$ *act on* B_l *by*

$$[p, q, r, t] \to [g_1 p, g_2 q, r, t]$$

and on S^6 *by*

$$(x, y) \mapsto (pap^{-1}, qbp^{-1}),$$

this action commutes with the bundle projection. In other words, the principal bundle B_l *over* U *has an* $SO(4)$*-symmetry. It is well-known that the action on* S^6 *is induced from the embedding of* $SO(4)$ *into* G_2 *and has cohomogeneity 1. We will construct* $SO(4)$*-invariant instantons, i.e., instantons of cohomogeneity one.*

Remark 2.4.2 (on the topology of B_l)**.** *A priori,* B_l *is only defined on* U*. However, note that* B_l *is actually the pullback of a* S^3*-bundle from* S^2 *obtained by extending the structure group of a Hopf circle bundle. Since* $\pi_1(S^3)$ *is trivial, every* S^3*-bundle over* S^2 *must be trivial. As a consequence,* B_l *is also trivial. In other words, it is possible to make gauge transformations so that* $B_l \sim U \times S^3$*. Thus this bundle has natural extension to the whole* S^6*, and, for later use, to the whole* $\mathbf{R}^7$*. The former description has the advantage that it makes the* $SO(4)$*-symmetry clear.*

Remark 2.4.3 (on the numbers l)**.** *A priori, this construction only makes sense for integer* l*. However, we will see that it is more interesting if we think of* l *as real valued.*

We will carry out computations on $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$. We will continue with the notation in §2.4.1 on the left-invariant forms on the first two copies of S^3. However, for the last S^3, we need use the *right*- invariant Maurer-Cartan form $drr^{-1} = \beta = \beta_1 I + \beta_2 J + \beta_3 K$. The left invariant Maurer-Cartan form is $r^{-1}dr = r^{-1}\beta r$. Of course, the following Maurer-Cartan equations hold

$$d\omega = -\omega \wedge \omega,$$

$$d\psi = -\psi \wedge \psi,$$

and

$$d\beta = \beta \wedge \beta.$$

More explicitly

$$d\omega_1 = -2\omega_2 \wedge \omega_3, \quad d\omega_2 = -2\omega_3 \wedge \omega_1, \quad d\omega_3 = -2\omega_1 \wedge \omega_2,$$

similarly for ψ_i and

$$d\beta_1 = 2\beta_2 \wedge \beta_3, \quad d\beta_2 = 2\beta_3 \wedge \beta_1, \quad d\beta_3 = 2\beta_1 \wedge \beta_2.$$

The space of semibasic 1-forms for the projection $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ is spanned by $dt, \omega_2, \omega_3, \psi_2, \psi_3, \beta_2, \beta_3, \omega_1 - \psi_1$ and $l\psi_1 + \beta_1$.

Invariant connections

Now suppose A is an $SO(4)$-invariant connection on B_l. We pull back A to $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ and denote it by the same letter. Then, since A is semibasic with respect to the projection $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \to B_l$, we can write

$$A = A_0\tau + A_1(l\psi_1 + \beta_1) + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + C_2\beta_2 + C_3\beta_3 + B_0 dt$$

with A_i, B_i, C_i valued in $Lie(S^3)$. Since A is $SO(4)$-invariant and the 1-forms listed are also $SO(4)$-invariant, the coefficients do not depend on (p, q), i.e., they are functions only in t and r. Moreover, A has to satisfy the following properties:

1. A must be right S^3-equivariant where we let S^3 act on $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ and B_l by right multiplication on the last S^3 factor.

2. A restricts to the last S^3 factor to be the Maurer-Cartan left invariant form $r^{-1}\beta r$.

3. The differential dA must be semibasic.

We investigate the consequences of these conditions.

1. Since all the forms listed in A are S^3 right-invariant, this condition is equivalent to

$$A_i(t, r) = r^{-1}A(t, 1)r, B_i(t, r) = r^{-1}B_i(t, 1)r, C_i = r^{-1}C_i(t, 1)r.$$

   To save notation, we will, from now on, write

$$A = r^{-1}(A_0\tau + A_1(l\psi_1 + \beta_1) + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + C_2\beta_2 + C_3\beta_3 + B_0 dt)r$$

   where A_i, B_i, C_i are functions of t.

2. This condition says that

$$A_1 = I, \quad C_2 = J, \quad C_3 = K.$$

   Thus we may further reduce A to

$$A = r^{-1}(A_0\tau + Il\psi_1 + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + B_0 dt)r + r^{-1}\beta r$$

3. It can be computed from Maurer-Cartan equations that

$$\begin{aligned} rdAr^{-1} \equiv\ & -l[B_0, I]\psi_1 \wedge dt - l[A_0, I]\psi_1 \wedge \tau \\ & -(l[A_2, I] + 2A_3)\psi_1 \wedge \omega_2 - (l[A_3, I] + 2A_2)\psi \wedge \omega_3 \\ & -(l[B_2, I] + 2B_3)\psi_1 \wedge \psi_2 - (l[B_3, I] + 2B_2)\psi_1 \wedge \psi_3 \end{aligned}$$

mod semibasic 2-forms. Thus this condition is equivalent to the following algebraic equations

$$
\begin{aligned}
&l[A_0, I] = 0, && l[B_0, I] = 0, \\
&l[B_2, I] + 2B_3 = 0, && l[B_3, I] + 2B_2 = 0, \\
&l[A_2, I] + 2A_3 = 0, && l[A_3, I] + 2A_2 = 0.
\end{aligned}
\tag{2.48}
$$

Hence, we solve the algebraic equations (2.48). We divide the solutions into several cases according to different values of l.

1. Case $l = 0$. We have $B_2 = B_3 = A_2 = A_3 = 0$ but (2.48) puts no restrictions on A_0 and B_0. Therefore A is reduced to

$$A = r^{-1}(A_0\tau + B_0 dt)r + r^{-1}\beta r.$$

2. Case $l = 1$. We have

$$
\begin{aligned}
A_0 &= a_0 I, \quad B_0 = b_0 I, \\
A_2 &= u_1 J + u_2 K, \quad A_3 = -u_2 J + u_1 K, \\
B_2 &= v_1 J + v_2 K, \quad B_3 = -v_2 J + v_1 K,
\end{aligned}
$$

for a_0, b_0, u_i, v_i functions of t.

3. Case $l = -1$. We have

$$
\begin{aligned}
A_0 &= a_0 I, \quad B_0 = b_0 I, \\
A_2 &= u_1 J + u_2 K, \quad A_3 = u_2 J - u_1 K, \\
B_2 &= v_1 J + v_2 K, \quad B_3 = v_2 J - v_1 K,
\end{aligned}
$$

for a_0, b_0, u_i, v_i functions of t.

4. Case $l \neq 0, \pm 1$. We have

$$A_0 = a_0 I, \quad B_0 = b_0 I,$$

$$A_2 = A_3 = B_2 = B_3 = 0.$$

2.4.3 SO(4)-invariant instantons

Now we take instanton conditions into consideration. As mentioned before, A is an ω-anti-self-dual instanton if and only if its curvature F satisfies

$$F^{2,0} = \mathrm{tr}_\omega F = 0.$$

It is easily seen that, restricted to U, this equivalent to

$$F \wedge \sigma_0 \wedge \sigma_1 \wedge \sigma_2 = 0, \tag{2.49}$$

and

$$F \wedge \omega^2 = 0. \tag{2.50}$$

According to Lemma 2.2.4, (2.50) is implied by (2.49), so we only care about (2.49). This simplifies the problem greatly. We consider four different cases according to the four different values of l in the last section.

$l = 0$

It may be computed that

$$F = r^{-1}\{(\dot{A}_0 + [B_0, A_0])dt \wedge \tau - 2A_0(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3)\}r,$$

where $\dot{A}_0 = \frac{d}{dt}A_0$. The equation (2.49) gives

$$32\sqrt{2}A\cos^2 t \sin t\omega_3 \wedge \omega_2 \wedge \psi_2 \wedge \psi_3 \wedge (dt - \sqrt{-1}\sin t\tau) = 0.$$

The only solution is $A_0 = 0$, which is the trivial connection

$$A = r^{-1}dr.$$

This is a case of little interest.

$l \neq 0, \pm 1$

It may be computed that

$$F \;=\; \{\dot{a_0} dt \wedge \tau - 2a_0(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3) - 2l\psi_2 \wedge \psi_3\} r^{-1} Ir.$$

The equation (2.49) gives

$$8a_0 \cos^2 t + l - 9l \cos^2 t = 0.$$

It is solved by

$$a_0 = \frac{l}{8} \frac{9\cos^2 t - 1}{\cos^2 t}.$$

For safety, one can check that, in fact, a_0 also satisfies the equation (2.50) which, in this case, is

$$-4a_0 + 5l + 8\cos^2 t a_0 - 9\cos^2 t l + 2 \sin t \cos t \dot{a_0}.$$

We arrive at the corresponding instanton, pulled back to $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$,

$$A = r^{-1} l I r \left(\frac{1}{8} \frac{9\cos^2 t - 1}{\cos^2 t} \tau + \psi_1 + b dt \right) + r^{-1} dr. \tag{2.51}$$

Theorem 2.4.4. *(2.51) defines for each $l \in \mathbf{Z}$ a singular Hermitian-Yang-Mill connection on S^6.*

Remark 2.4.5 (on singularity)**.** *The coordinate system is not extendable through the submanifolds $S^2 = \{t = 0\}$ and $S^3 = \{t = \frac{\pi}{2}\}$. However, the connection A has different behavior when t approaches 0 and $\frac{\pi}{2}$. When $t \to \frac{\pi}{2}$, the curvature F blows up. However, for $t = 0$, the connection is bounded. It might be possible to remove the singularity by (2.2.11), we can extend the connection to the locus $t = 0$. In other words, this might be a singularity due to unwise choice of coordinates, rather than a singularity of the instanton A itself.*

Remark 2.4.6 (on reducibility)**.** *A cautious reader may have noticed that, A has its holonomy in* S^1*, so it is reducible. If we restrict the connection to the generic level sets of* t*, we obtain the standard Hopf connection up to a constant.*

Remark 2.4.7 (on b)**.** *Note that* b *is not essential. We could have applied a gauge transformation in the* t *direction to* A *at the beginning to remove the* dt *component. The same remark applies to the next subsection.*

$l = \pm 1$

We only deal with the case $l = 1$. The other case is similar.

According to Case 2 in §5.2.2, the curvature is computed to be

$$\begin{aligned} rFr^{-1} \;=\; & \dot{a_0} I dt \wedge \tau - 2I\psi_2 \wedge \psi_3 \\ & +(\dot{u_1} J + \dot{u_2} K)dt \wedge \omega_2 + (-\dot{u_2} J + \dot{u_1} K)dt \wedge \omega_3 \\ & +(\dot{v_1} J + \dot{v_2} K)dt \wedge \psi_2 + (-\dot{v_2} J + \dot{v_1} K)dt \wedge \psi_3 \\ & -2a_0 I(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3) \\ & -2(u_1 J + u_2 K)\omega_3 \wedge \tau - 2(-u_2 J + u_1 K)\tau \wedge \omega_2 \\ & +2a_0(u_1 K - u_2 J)\tau \wedge \omega_2 + 2a_0(-u_2 K - u_1 J)\tau \wedge \omega_3 \\ & +2a_0(v_1 K - v_2 J)\tau \wedge \psi_2 + 2a_0(-v_2 K - v_1 J)\tau \wedge \psi_3 \\ & +2(u_1^2 + u_2^2) I\omega_2 \wedge \omega_3 + 2(v_1^2 + v_2^2) I\psi_2 \wedge \psi_3 \\ & +2(u_1 v_2 - u_2 v_1) I(\omega_2 \wedge \psi_2 + \omega_3 \wedge \psi_3) \\ & +2(u_1 v_1 + u_2 v_2) I(\omega_2 \wedge \psi_3 - \omega_3 \wedge \psi_2) \end{aligned}$$

where, again, $\dot{}$ means $\frac{d}{dt}$.

A tedious computation shows that the equation (2.49) amounts to the following

$$\sin t(1 - 3\cos^2 t)\dot{v_1} - \sin^3 t\dot{u_1} + 4a\cos t v_1 = 0,$$

$$\sin t(1 - 3\cos^2 t)\dot{v_2} - \sin^3 t\dot{u_2} + 4a\cos t v_2 = 0,$$

$$\sin t\cos t\dot{v_1} + u_1(1 - a)\sin^2 t + a(3\cos^2 t - 1)v_1 = 0,$$

$$\sin t\cos t\dot{v_2} + u_2(1 - a)\sin^2 t + a(3\cos^2 t - 1)v_2 = 0,$$

$$u_1 v_2 = u_2 v_1$$

$$-9\cos^2 t + 1 + 8a\cos^2 t - \sin^2 t(u_1^2 + u_2^2)$$

$$+(9\cos^2 t - 1)(v_1^2 + v_2^2) + (6\cos^2 t - 2)(u_1 v_1 + u_2 v_2) = 0.$$

We may assume that

$$u_2 = \lambda u_1, \quad v_2 = \lambda v_1$$

with λ necessarily constant. It can be shown that by a substitution like $(u_1, v_1) \mapsto \sqrt{1+\lambda^2}(u_1, v_1)$, we may simply assume that $v_2 = u_2 = 0$.

The system reduces to

$$\sin t(1 - 3\cos^2 t)\dot{v}_1 - \sin^3 t\dot{u}_1 + 4a\cos t v_1 = 0,$$

$$\sin t\cos t\dot{v}_1 + u_1(1-a)\sin^2 t + a(3\cos^2 t - 1)v_1 = 0,$$

$$-9\cos^2 t + 1 + 8a\cos^2 t - \sin^2 t u_1^2 + (9\cos^2 t - 1)v_1^2 + (6\cos^2 t - 2)u_1 v_1 = 0$$

which is now determined and thus solvable.

It is easy to see that any solution must be of the form

$$u_1 = U(\sin t), v_1 = V(\sin t), a = W(\sin t),$$

where the functions $U(x), V(x)$ and $W(x)$ defined on $[0,1]$ satisfy

$$x(-2 + 3x^2)\frac{d}{dx}V - x^3\frac{d}{dx}U + 4WV = 0$$

$$x(1 - x^2)\frac{d}{dx}V + x^2 U(1 - W) + (2 - 3x^2)WV = 0$$

$$-8 + 9x^2 + 8(1 - x^2)W - x^2U^2 + (8 - 9x^2)V^2 + (4 - 6x^2)UV = 0.$$

We rewrite the ODEs as

$$x(1 - x^2)\frac{d}{dx}V = -x^2U(1 - W) - (2 - 3x^2)WV \tag{2.52}$$

$$x^3(1 - x^2)\frac{d}{dx}U = x^2(2 - 3x^2)U(1 - W) + (8 - 16x^2 + 9x^4) \tag{2.53}$$

$$-8 + 9x^2 + 8(1 - x^2)W - x^2U^2 + (8 - 9x^2)V^2 + (4 - 6x^2)UV = 0. \tag{2.54}$$

It is clear that the system (2.52), (2.53), (2.54) has many solutions which have possible singularities along $x = 0$ and $x = 1$.

Theorem 2.4.8. *Each solution of the ode system (2.52), (2.53), (2.54) and a real number λ determine a unique Hermitian-Yang-Mills connection on the trivial $SU(2)$ bundle over S^6, with possible singularities along submanifolds S^2 and S^3.*

Remark 2.4.9. *It is interesting to ask whether we could apply Corollary (2.2.11) or (2.2.12) to remove the possible singularities along S^2. This is doable by analyzing the singular behavior of the above ODE system along $x = 0$.*

3

Pseudo-Holomorphic Curves in Nearly Kähler $\mathbf{CP}^3$

3.1 Structure equations, projective spaces, and the flag manifold

In this section we collect some facts needed in the next section and formulate them in terms of the moving frame. Let $\mathbf{H}$ denote the real division algebra of quaternions. An element of $\mathbf{H}$ can be written uniquely as $q = z + jw$ where $z, w \in \mathbf{C}$ and $j \in \mathbf{H}$ satisfies

$$j^2 = -1, zj = j\bar{z}$$

for all $z \in \mathbf{C}$. In this way we regard $\mathbf{C}$ as subalgebra of $\mathbf{H}$ and give $\mathbf{H}$ the structure of a complex vector space by letting $\mathbf{C}$ act on the right. We let $\mathbf{H}^2$ denote the space of pairs (q_1, q_2) where $q_i \in \mathbf{H}$. We will make $\mathbf{H}^2$ into a quarternion vector space by letting $\mathbf{H}$ act on the right

$$(q_1, q_2)q = (q_1 q, q_2 q).$$

This automatically makes $\mathbf{H}^2$ into a complex vector space of dimension 4. In fact, regarding $\mathbf{C}^4$ as the space of 4-tuples (z_1, z_2, z_3, z_4) we make the explicit identification

$$(z_1, z_2, z_3, z_4) \sim (z_1 + jz_2, z_3 + jz_4).$$

This specific isomorphism is the one we will always mean when we write $\mathbf{C}^4 = \mathbf{H}^2$. If $v \in \mathbf{H}^2 \setminus (0,0)$ is given, let $v\mathbf{C}$ and $v\mathbf{H}$ denote respectively the complex line and the quaternion line spanned by v. The assignment $v\mathbf{C} \to v\mathbf{H}$ is a well-defined mapping $T : \mathbf{CP}^3 \to \mathbf{HP}^1$. The fibers of T are $\mathbf{CP}^1$'s. So we have a fibration

$$\begin{array}{ccc} \mathbf{CP}^1 & \rightarrow & \mathbf{CP}^3 \\ & & \downarrow \\ & & \mathbf{HP}^1 \end{array} \tag{3.1}$$

This is the famous twistor fibration. In order to study its geometry more thoroughly, we will now introduce the structure equations of $\mathbf{H}^2$. First, we endow $\mathbf{H}^2$ with a quaternion inner product $\langle , \rangle : \mathbf{H}^2 \times \mathbf{H}^2 \to \mathbf{H}$ defined by

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1 p_1 + \bar{q}_2 p_2.$$

We have identities

$$\langle v, wq \rangle = \langle v, w \rangle q, \overline{\langle v, w \rangle} = \langle w, v \rangle , \langle vq, w \rangle = \bar{q} \langle v, w \rangle .$$

Moreover, $Re \langle , \rangle$ is a positive definite inner product that gives $\mathbf{H}^2$ the structure of a Euclidean space $\mathbf{E}^8$. Let $\mathfrak{F}$ denote the space of pairs $f = (e_1, e_2)$ with $e_i \in \mathbf{H}^2$ satisfying

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \langle e_1, e_2 \rangle = 0.$$

We regard $e_i(f)$ as functions on $\mathfrak{F}$ with values in $\mathbf{H}^2$. Clearly, $e_1(\mathfrak{F}) = S^7 \subset \mathbf{E}^8 = \mathbf{H}^2$. It is well known that $\mathfrak{F}$ maybe canonically identified with $Sp(2)$ up to a left translation in $Sp(2)$. There are unique quaternion-valued 1-forms $\{\phi^a_b\}$ so that

$$de_a = e_b \phi^b_a, \tag{3.2}$$

$$d\phi^a_b + \phi^a_c \wedge \phi^c_b = 0, \tag{3.3}$$

and

$$\phi^a_b + \overline{\phi^b_a} = 0. \tag{3.4}$$

We define two canonical maps $C_1 : \mathfrak{F} \to \mathbf{CP}^3$ and $C_2 : \mathfrak{F} \to \mathbb{CP}^3$ by sending $f \in \mathfrak{F}$ to the complex lines spanned by $e_1(f)$ and $e_2(f)$ respectively. Recall that we have denoted the Kähler projective space by $\mathbb{CP}^3$ and the nearly Kähler one by $\mathbf{CP}^3$ whose structure will be explicitly described below. We are mainly interested in $\mathbf{CP}^3$. However, $\mathbb{CP}^3$ will play an important role. We now write structure equations for C_1 and C_2. First, we immediately see that C_1 gives $\mathfrak{F}$ the structure of an $S^1 \times S^3$ bundle over $\mathbf{CP}^3$, where we have identified S^1 with the unit complex numbers and S^3 with the unit quaternions. The action is given by

$$f(z, q) = (e_1, e_2)(z, q) = (e_1 z, e_2 q),$$

where $z \in S^1$ and $q \in S^3$. If we set

$$\begin{bmatrix} \phi^1_1 & \phi^1_2 \\ \phi^2_1 & \phi^2_2 \end{bmatrix} = \begin{bmatrix} i\rho_1 + j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & i\rho_2 + j\tau \end{bmatrix}$$

where ρ_1 and ρ_2 are real 1-forms while ω_1, ω_2, ω_3 and τ are complex-valued, we may rewrite one part of the structure equation (3.3) relative to the $S^1 \times S^3$ structure on $\mathbf{CP}^3$ as

$$d\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\begin{pmatrix} i(\rho_2 - \rho_1) & -\bar{\tau} & 0 \\ \tau & -i(\rho_1 + \rho_2) & 0 \\ 0 & 0 & 2i\rho_1 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \overline{\omega_2 \wedge \omega_3} \\ \overline{\omega_3 \wedge \omega_1} \\ \overline{\omega_1 \wedge \omega_2} \end{pmatrix}. \tag{3.5}$$

This in particular defines a nearly Kähler structure on $\mathbf{CP}^3$ by requiring ω_1, ω_2 and ω_3 to be of type $(1, 0)$ (note that this almost complex structure is nonintegrable, thus different from the usual integrable one). We denote

$$\begin{pmatrix} \kappa_{1\bar{1}} & \kappa_{1\bar{2}} \\ \kappa_{2\bar{1}} & \kappa_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} i(\rho_2 - \rho_1) & -\bar{\tau} \\ \tau & -i(\rho_1 + \rho_2) \end{pmatrix}$$

and $\kappa_{3\bar{3}} = 2i\rho_1$ in the usual notation of a connection. Then the other part of the structure equation (3.3) may be written as the curvature of this nearly Kähler structure

$$d\begin{pmatrix} \kappa_{1\bar{1}} & \kappa_{1\bar{2}} \\ \kappa_{2\bar{1}} & \kappa_{2\bar{2}} \end{pmatrix} + \begin{pmatrix} \kappa_{1\bar{1}} & \kappa_{1\bar{2}} \\ \kappa_{2\bar{1}} & \kappa_{2\bar{2}} \end{pmatrix} \wedge \begin{pmatrix} \kappa_{1\bar{1}} & \kappa_{1\bar{2}} \\ \kappa_{2\bar{1}} & \kappa_{2\bar{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \omega_1 \wedge \overline{\omega_1} - \omega_3 \wedge \overline{\omega_3} & \omega_1 \wedge \overline{\omega_2} \\ \omega_2 \wedge \overline{\omega_1} & \omega_2 \wedge \overline{\omega_2} - \omega_3 \wedge \overline{\omega_3} \end{pmatrix},$$

as well as

$$d\kappa_{3\bar{3}} = -(\omega_1 \wedge \overline{\omega_1} + \omega_2 \wedge \overline{\omega_2} - 2\omega_3 \wedge \overline{\omega_3}). \tag{3.6}$$

In an exactly analogous fashion, C_2 gives $\mathfrak{F}$ a structure of an $S^1 \times S^3$ bundle over $\mathbb{CP}^3$ with the action now given by

$$(e_1, e_2)(q, z) = (e_1 q, e_2 z),$$

where $z \in S^1$ and $q \in S^3$. However, ω_1, ω_2, $\kappa_{1\bar{2}}$ and their complex conjugates become semibasic and ω_3 is not. The usual Kähler structure on $\mathbb{CP}^3$ is defined by requiring $\frac{\overline{\omega_1}}{\sqrt{2}}$, $\frac{\omega_2}{\sqrt{2}}$ and $\kappa_{2\bar{1}}$ to be of type $(1,0)$ and unitary. Relative to this Kähler structure, we may rewrite part of the structure equations as

$$d\begin{pmatrix} \frac{\overline{\omega_1}}{\sqrt{2}} \\ \frac{\omega_2}{\sqrt{2}} \\ \kappa_{2\bar{1}} \end{pmatrix} = -\begin{pmatrix} -\kappa_{1\bar{1}} & \omega_3 & -\frac{\overline{\omega_2}}{\sqrt{2}} \\ -\overline{\omega_3} & \kappa_{2\bar{2}} & \frac{\omega_1}{\sqrt{2}} \\ \frac{\omega_2}{\sqrt{2}} & -\frac{\overline{\omega_1}}{\sqrt{2}} & \kappa_{2\bar{2}} - \kappa_{1\bar{1}} \end{pmatrix} \begin{pmatrix} \frac{\overline{\omega_1}}{\sqrt{2}} \\ \frac{\omega_2}{\sqrt{2}} \\ \kappa_{2\bar{1}} \end{pmatrix}. \tag{3.7}$$

We will also need some properties of the flag manifold $\mathbf{Fl} = \mathfrak{F}/(U(1) \times U(1))$. Equivalently, $\mathbf{Fl}$ consists of pairs of complex lines $([e_1], [e_2])$ with $\langle e_1, e_2 \rangle = 0$. Of course $\mathfrak{F}$ defines a natural $S^1 \times S^1$ structure on $\mathbf{Fl}$ for which the forms $\omega_1, \omega_2, \omega_3, \kappa_{2\bar{1}}$ and their complex conjugates are semibasic. Moreover, we have a double fibration of $\mathbf{Fl}$

over the two projective spaces:

$$\begin{array}{ccc} & \mathfrak{F} & \\ & \downarrow & \\ & \mathbf{Fl} & \\ \swarrow & & \searrow \\ \mathbf{CP}^3 & & \mathbb{CP}^3 \end{array} \tag{3.8}$$

We denote the first fibration by Π_1 and the second fibration by Π_2. Explicitly Π_a $(a = 1, 2)$ sends $([e_1], [e_2]) \in \mathbf{Fl}$ to the complex line $[e_a]$. By requiring $\overline{\omega_1}, \omega_2, \omega_3, \kappa_{2\bar{1}}$ to be complex linear we define an almost complex structure on $\mathbf{Fl}$. It is easy to check from the structure equations that this almost complex structure is integrable and Π_2 is thus a holomorphic projection.

Finally, there are various complex vector bundles associated with $\mathfrak{F}$ that will be important. First, on $\mathbf{CP}^3$, there are two obvious complex bundles, the tautological bundle ϵ and the trivial rank 4 bundle $\mathbf{C}^4$. We view ϵ as the subbundle of $\mathbb{C}^4$ spanned by e_1 and denote the quotient bundle by Q. Using the obvious Hermitian product, we identify Q as a subbundle of $\mathbf{C}^4$ locally spanned by $e_1 j$, e_2 and $e_2 j$. Note that $e_1 j$ itself spans a well-defined line bundle, which is isomorphic to ϵ^*. Denote the quotient Q/ϵ^* by $\tilde{Q}$ which, again, may be regarded as a subbunle of $\mathbf{C}^4$ locally spanned by $e_2, e_2 j$. We write $T\mathbf{CP}^3$ the complex tangent bundle of $\mathbf{CP}^3$. The $S^1 \times S^3$ determines a splitting

$$T\mathbf{CP}^3 = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{H}$ has rank 2 and $\mathcal{V}$ has rank 1. We call $\mathcal{H}$ the horizontal part and $\mathcal{V}$ the vertical part relative to the fibration 3.1. One can show $\mathcal{V}$ is isomorphic to ϵ^2 as a Hermitian line bundle by locally identifying $\frac{1}{\sqrt{2}} e_1 \otimes e_1$ with the complex tangent vector dual to the $(1, 0)$ form ω_1, denoted by $f_{\bar{1}}$. Similarly $\mathcal{H}$ is isomorphic to $\epsilon^* \otimes \tilde{Q}$ with $\frac{1}{\sqrt{2}} e_1^* \otimes e_2$ identified with the tangent vector $f_{\bar{2}}$ dual to ω_2 and $\frac{1}{\sqrt{2}} e_1^* \otimes e_2 j$ identified with $f_{\bar{3}}$ dual to ω_3. Pulled back to **Fl** by Π_1, the bundles $\tilde{Q}$ splits as

$$\Pi_1^*\tilde{Q} = \tilde{\epsilon} \oplus \tilde{\epsilon}^*,$$

where $\tilde{\epsilon}$ is locally spanned by e_2 and $\tilde{\epsilon}^*$ denotes its dual, locally spanned by $e_2 j$. Of course $\Pi_1^*\mathcal{H}$ splits correspondingly as

$$\Pi_1^*\mathcal{H} = \epsilon^* \otimes \tilde{\epsilon} \oplus \epsilon^* \otimes \tilde{\epsilon}^*.$$

Similar constructions apply to $\mathbb{CP}^3$. We will only point out differences and some relations. The tautological bundle on $\mathbb{CP}^3$ becomes $\tilde{\epsilon}$ when pulled back to **Fl**. The complex tangent bundle also splits as a sum of a vertical part $\mathbb{V}$ and a horizontal part $\mathbb{H}$. The vertical part is isomorphic to $(\tilde{\epsilon}^*)^2$ compared with the $\mathbf{CP}^3$ case because of the reversed almost complex structure. The horizontal part, when pulled back by Π_2 to **Fl**, is isomorphic to $\tilde{\epsilon}^* \otimes \epsilon^* \oplus \tilde{\epsilon}^* \otimes \epsilon$. Note this splitting shares a common factor with $\Pi_1^*\mathcal{H}$, which will become important later. In the various isomorphisms, we no longer need the $\frac{1}{\sqrt{2}}$ to make them Hermitian. Moreover, since the almost complex structure on $\mathbb{CP}^3$ is integrable, many of these bundles have holomorphic structures. Among them, the dual of the vertical tangent bundle of $\mathbb{CP}^3$, which we denote by $\mathbb{V}^*$, is particularly important. Locally, $\mathbb{V}^*$ is spanned by $\kappa_{2\bar{1}}$ as a subbundle of the complex cotangent bundle of $\mathbb{CP}^3$. We have the following result due to Bryant [9]

Lemma 3.1.1. *The bundle $\mathbb{V}^*$ is isomorphic to $\tilde{\epsilon}^2$ as a Hermitian holomorphic line bundle. Moreover, it induces a holomorphic contact structure on $\mathbb{CP}^3$.*

The integrals of this holomorphic contact system were thoroughly investigated in [9] (see Section 3).

3.2 Pseudo-holomorphic curves in $\mathbf{CP}^3$

Let M^2 be a connected Riemann surface. A map $X : M^2 \to \mathbf{CP}^3$ is called a pseudo-holomorphic curve if X is nonconstant and the differential of X commutes with the almost complex structures. We let $x : \mathfrak{F}_X \to M^2$, $\mathcal{V}_X \to M^2$ and $\mathcal{H}_X \to M^2$ be the pullback bundles of $\mathfrak{F} \to \mathbf{CP}^3$, $\mathcal{V} \to \mathbf{CP}^3$, and $\mathcal{H} \to \mathbf{CP}^3$ respectively. Thus, for instance, we have

$$\mathfrak{F}_X = \{(x, f) \in M^2 \times \mathfrak{F} | X(x) = C_1(f)\}.$$

Of course, $\mathfrak{F}_X$ is an $S^1 \times S^3$ bundle over M^2 and $\mathcal{V}_X$ and $\mathcal{H}_X$ are Hermitian complex bundles of rank 1 and 2 respectively. Moreover, the natural map $\mathfrak{F}_X \to \mathfrak{F}$ pulls back various quantities on $\mathfrak{F}$, which we still denote by the same letters. For example, $f_{\bar{1}}$, $f_{\bar{2}}$ now denote functions on $\mathfrak{F}_X$ valued in $\mathcal{H}_X$. The structure equations (3.5), (3.6) and (3.6) still hold, on $\mathfrak{F}_X$ now. Also for functions and sections with domains in M^2, we will pull these back up via x^* to $\mathfrak{F}_X$. For example, any section $s : M^2 \to \mathcal{H}_X$ can be written in the form $s = f_{\bar{1}} s_1 + f_{\bar{2}} s_2$ where s_i are complex functions on $\mathfrak{F}_X$. Using this convention, the pullback of κ induces connections on $\mathcal{H}_X$ and $\mathcal{V}_X$ compatible with the Hermitian structures. Namely $\nabla : \Gamma(\mathcal{H}_X) \to \Gamma(\mathcal{H}_X \otimes T^*M^2)$ is given by

$$\nabla(f_{\bar{i}} s_i) = f_{\bar{i}} \otimes (ds_i + \kappa_{i\bar{j}} s_j).$$

Since we are working over a Riemann surface, it is well-known that there are unique holomorphic structures on $\mathcal{H}_X$ and $\mathcal{V}_X$ compatible with these connections. From now on we will regard these two bundles as holomorphic Hermitian vector bundles over M^2.

Another thing to notice is that $\{\omega_i\}$ are semi-basic with respect to $x : \mathfrak{F}_X \to M^2$. Moreover, they are of type $(1, 0)$ since dX is complex linear. Set

$$\mathbf{I}_1 = f_{\bar{1}} \otimes \omega_1 + f_{\bar{2}} \otimes \omega_2, \mathbf{I}_2 = f_{\bar{3}} \otimes \omega_3.$$

It is clear that $\mathbf{I}_1$ and $\mathbf{I}_2$ are well defined sections of $\mathcal{H}_X \otimes T^*M^2$ and $\mathcal{V} \otimes T^*M^2$ respectively where T^*M^2 is the holomorphic line bundle of $(1, 0)$ forms on M^2.

Lemma 3.2.1. *The sections $\mathbf{I}_1$ and $\mathbf{I}_2$ are holomorphic. Moreover, $\mathbf{I}_1$ and $\mathbf{I}_2$ only vanish at isolated points unless $X(M^2)$ is horizontal (when $\mathbf{I}_2$ vanishes identically)* *or vertical (when $\mathbf{I}_1$ vanishes identically and thus $X(M^2)$ is an open set of a fiber $\mathbf{CP}^1$ in (3.1)).*

Proof. We only show $\mathbf{I}_1$ is holomorphic and leave $\mathbf{I}_2$ for the reader. Choose a uniformizing parameter z on a neighborhood of $x_0 \in M$. In a neighborhood of $x^{-1}(x_0)$, there exist functions a_i so that $\omega_i = a_i dz$. It follows that $\omega_i \wedge \omega_j = 0$, so we have $d\omega_i = -\kappa_{i\bar{j}} \wedge \omega_j$. This translates to $(da_i + \kappa_{i\bar{j}} a_j) \wedge dz = 0$ so there exist b_i so that

$$da_i + \kappa_{i\bar{j}} a_j = b_i dz.$$

Thus, when we compute $\overline{\partial}\mathbf{I}_1$ we have

$$\begin{aligned} \bar{\partial}\mathbf{I}_1 &= (\nabla(f_{\bar{i}} a_i) \otimes dz)^{0,1} \\ &= f_{\bar{i}} \otimes dz \otimes (da_i + \kappa_{i\bar{j}} a_j)^{0,1} \\ &= f_{\bar{i}} \otimes dz \otimes (b_i dz)^{0,1} \\ &= 0, \end{aligned}$$

so $\mathbf{I}_1$ is holomorphic. Moreover, by complex analysis, if $\mathbf{I}_1$ or $\mathbf{I}_2$ vanishes at a sequence of points with an accumulation, the section has to be identically 0 since M^2 is connected.

□

Remark 3.2.2. *It is clear that $\mathbf{I}_1$ and $\mathbf{I}_2$ are just horizontal and vertical parts of the evaluation map $X_*(TM) \to T_X$.*

We will call a curve with $\mathbf{I}_1 = 0$ ($\mathbf{I}_2 = 0$) vertical (horizontal). Of course vertical curves are just the fibers $\mathbf{CP}^1$ of T. To study horizontal curves it does no harm to reverse the almost complex structure on the fiber of T. This new complex structure is integrable and actually equivalent to the usual complex structure on the 3 projective space. The horizontal bundle $\mathcal{H}$ turns out to be a holomorphic contact structure under the usual complex structure. The integral curves of this contact system are thoroughly described in [9]. We therefore have a good understanding of horizontal pseudo-holomorphic curves in $\mathbf{CP}^3$.

We now assume $\mathbf{I}_1$ is not identically 0. There exists a holomorphic line bundle $L \subset \mathcal{H}_X$ so that $\mathbf{I}_1$ is a nonzero section of $L \otimes T^*M$. We let R_1 be the ramification divisor of $\mathbf{I}_1$. That is,

$$R_1 = \sum_{p:\mathbf{I}_1(p)=0} ord_p(\mathbf{I}_1)p.$$

R_1 is obviously effective, and we have

$$L = TM \otimes [R_1].$$

Similarly if $\mathbf{I}_2$ does not vanish identically let R_2 be the ramification divisor of $\mathbf{I}_2$. Then R_2 is effective and

$$\mathcal{V}_X = TM \otimes [R_2].$$

Now we adapt frames in accordance with the general theory. We let $\mathfrak{F}_X^{(1)}$ be the subbundle of pairs (x, f) with $f_{\bar{2}} \in L_x$. Then $\mathfrak{F}_X^{(1)}$ is a $U(1) \times U(1)$ bundle over M. The canonical connection on L is described as follows: If $s : M \to L$ is a section, then $s = f_{\bar{2}} s_2$ for some function s_2 on $\mathfrak{F}_X^{(1)}$. Then

$$\nabla s = f_{\bar{2}} \otimes (ds_2 + \kappa_{2\bar{2}} s_2).$$

Similarly the quotient bundle $N_X = \mathcal{H}_X / L$ has a natural holomorphic Hermitian structure. Let $(f_{\bar{1}}) : \mathfrak{F}_X^{(1)} \to N_X$ be the function $f_{\bar{1}}$ followed by the projection $\mathcal{H}_X \to N_X$. If $s : M \to N_X$ is any section, then $s = (f_{\bar{1}}) s_1$ for s_1 on $\mathfrak{F}_X^{(1)}$ and we have

$$\nabla s = (f_{\bar{1}}) \otimes (ds_1 + \kappa_{1\bar{1}} s_1).$$

Note since $\mathbf{I}_1$ has values in $L \otimes T^*M$, we must have $\omega_1 = 0$ on $\mathfrak{F}_X^{(1)}$. If we differentiate this using structure equations (3.5) we have

$$d\omega_1 = -\kappa_{1\bar{2}} \wedge \omega_2 = 0.$$

It follows that $\kappa_{1\bar{2}}$ is of type $(1,0)$.

Lemma 3.2.3. *Let* $\mathbf{II} = (f_{\bar{1}}) \otimes f_2 \otimes \kappa_{1\bar{2}}$ *where* f_2 *is the dual of* $f_{\bar{2}}$*. Then* $\mathbf{II}$ *is a holomorphic section of* $N_X \otimes L^* \otimes T^*M$*.*

Proof. Since $\kappa_{1\bar{2}}$ is of type $(1,0)$, there exists b locally such that $\kappa_{1\bar{2}} = bdz$. The structure equations (3.6) pulled back to $\mathfrak{F}^{(1)}$ gives $d\kappa_{1\bar{2}} = -\kappa_{1\bar{1}} \wedge \kappa_{1\bar{2}} - \kappa_{1\bar{2}} \wedge \kappa_{2\bar{2}} + \omega_1 \wedge \overline{\omega_2} = -(\kappa_{1\bar{1}} - \kappa_{2\bar{2}}) \wedge \kappa_{1\bar{2}}$. This translates into

$$(db + (\kappa_{1\bar{1}} - \kappa_{2\bar{2}})b) \wedge dz = 0.$$

The rest follows exactly as in Lemma 3.2.1. □

We say a curve has *null-torsion* if $\mathbf{II} = 0$. Since $\wedge^2\mathcal{H} \otimes \mathcal{V} \cong \mathbf{C}$ we have

$$N_X \otimes L \otimes \mathcal{V} \cong \mathbf{C}.$$

If $\mathbf{II}$ is not identically 0, we define the *planar divisor* by

$$P = \sum_{p:\mathbf{II}(p)=0} ord_p(\mathbf{II})p.$$

In this case, we have

$$N_X = [P] \otimes L \otimes TM.$$

Theorem 3.2.4. *Let* $M = \mathbf{CP}^1$*. Then any complex curve* $X : M \to \mathbf{CP}^3$ *either is one of the vertical fibers or horizontal or has null-torsion.*

Proof. Assume both $\mathbf{I}_1$ and $\mathbf{I}_2$ are not identically 0. We must show that $\mathbf{II}$ vanishes identically. If not, we have, for $R_1, R_2, P \geq 0$,

$$\mathcal{V}_X = [R_2] \otimes TM, L = [R_1] \otimes TM, N_X = [P] \otimes L \otimes TM,$$

which implies, since $N_X \otimes Ł \otimes \mathcal{V} \cong \mathbf{C}$, that

$$(TM)^3 \otimes [2R_1 + P + R_2] \cong \mathbf{C},$$

thus d$egTM \leq 0$, but d$egTM = 2$ when $M = \mathbf{CP}^1$. □

Remark 3.2.5. *The computation in this theorem actually shows that if M^2 has genus g, then any pseudo-holomorphic curve $X : M \to \mathbf{CP}^3$ with none of $\mathbf{I}_1$, $\mathbf{I}_2$ and* **II** *vanishing identically must satisfy*

$$6(g-1) = 2deg(R_1) + deg(R_2) + deg(P).$$

This puts severe restrictions on the bundles L, $\mathcal{V}_X$ and N_X. For example, if $g = 1$, so that M is elliptic, then a pseudo-holomorphic curve $X : M \to \mathbf{CP}^3$ must satisfy $R_1 = R_2 = P = 0$, so that $\mathcal{V}_X = TM$, $L = TM$ and $N_X = (TM)^2$.

If the pseudo-holomorphic curve $X : M^2 \to \mathbf{CP}^3$ has $\mathbf{I}_1 \neq 0$, we have a lift of X to a map $\hat{X} : M^2 \to \mathbf{Fl}$ defined by $x \mapsto (X(x), N_X(x) \otimes X(x))$. Some clarification may be necessary. The bundle N_X can be viewed canonically as a subbundle of $X^*(\tilde{Q} \otimes \epsilon^*) \subset \mathbf{C}^4 \otimes X^*(\epsilon^*)$. By tensoring with $X^*\epsilon$ and canonically identifying $\epsilon \otimes \epsilon^* = \mathbf{C}$ we see that $N_X(x) \otimes X^*(\epsilon)(x) = N_X(x) \otimes X(x)$ is a complex line in $\mathbf{C}^4$. It is easy to see that this line is Hermitian orthogonal to $X(x) \subset \epsilon$ and $X(x)j \subset \epsilon^*$ and thus $\hat{X}$ is well-defined. Moreover, X has null torsion iff $\hat{X}^*(\kappa_{2\bar{1}}) = 0$. Composed with $\Pi_2 : \mathbf{Fl} \to \mathbb{CP}^3$, $\hat{X}$ induces a map $Y = \Pi_2 \circ \hat{X} : M^2 \to \mathbb{CP}^3$.

Theorem 3.2.6. *The assignment $X \mapsto Y$ establishes a $1-1$ correspondence between null-torsion pseudo-holomorphic curves in $\mathbf{CP}^3$ and nonconstant holomorphic integrals of the holomorphic contact system $\mathbb{V}^*$ on $\mathbb{CP}^3$.*

Proof. It is clear from the structure equations (3.7) that Y is an integral of $\mathbb{V}^*$ if X has null torsion. Conversely, if $Y : M^2 \to \mathbb{CP}^3$ is a nonconstant holomorphic integral of $\mathbb{V}^*$, there exists a unique line bundle $\mathcal{L} \subset \mathbb{H} \subset T\mathbb{CP}^3$ which contains Y_*TM. We lift Y to a map $\hat{Y} : M^2 \to \mathbf{Fl}$ by $x \mapsto ((\mathbb{H}/\mathcal{L})(x) \otimes Y(x), Y(x))$. We define the corresponding map $X = \Pi_1 \circ \hat{Y} : M \to \mathbf{CP}^3$. It is clear from the structure equations(3.5) that such an X has null-torsion. We next show that if we start with null-torsion curve $X : M^2 \to \mathbf{CP}^3$ and run through the procedure

$X \to Y \to X$ of the above constructions, we arrive at the original curve. In fact the frame adaptations we made before show that we can arrange $\{e_a\}$ so that $\Pi_1^* L_X(x)$ is spanned by $\frac{1}{\sqrt{2}} e_2 j \otimes e_1^*$ and $\Pi_1^* N_X(x)$ is spanned by $\frac{1}{\sqrt{2}} e_2 \otimes e_1^*$. Thus by definition $Y(x) = [e_2]$. Since $\overline{\omega_1} = 0$, $\Pi_2^* \mathcal{L}$ is spanned by $e_1 j$ by the structure equations. Therefore $\Pi_2^*(\mathbb{H}/\mathcal{L}) = [e_1]$ from which we see $\Pi_1(\hat{Y}(x)) = X(x)$. We omit the proof that if we start with Y and run the procedure of constructions $Y \to X \to Y$ we get Y back. $\square$

As mentioned before, a powerful construction the integrals of the holomorphic contact system $\mathbb{V}^*$ was provided in [9] (see Section 3). Of course, there are corresponding results about null-torsion pseudo-holomorphic curves in $\mathbf{CP}^3$. We omit the details for most of translation work and only mention some consequences.

Theorem 3.2.7. *Let M be a compact Riemann surface. There always exists a pseudo-holomorphic embedding $M \to \mathbf{CP}^3$ with null torsion.*

This is the translation of Theorem G in [9].

A horizontal pseudo-holomorphic curve $X : M^2 \to \mathbf{CP}^3$ with null torsion corresponds to $Y : M^2 \to \mathbb{CP}^3$ which is simply a projective line $\mathbb{CP}^1$ (see the remark in [9] following Theorem F). In particular, if $M \neq S^2$ and let $X : M \to \mathbf{CP}^3$ be null-torsion, X is neither vertical or horizontal. On the other hand, it is easy to construct a horizontal rational curve $Y : S^2 \to \mathbb{CP}^3$ which is not a complex line using the "Weierstrass formula" in Theorem F in [9]. The corresponding X will be neither vertical nor horizontal. We state these as the following

Corollary 3.2.8. *For any Riemann surface M, there exists a pseudo-holomorphic curve $X : M \to \mathbf{CP}^3$ which is neither vertical nor horizontal.*

A rational pseudo-holomorphic curve is either vertical or horizontal or has null torsion. Both horizontal and null-torsion curves are reduced to integrals of the holomorphic contact system $\mathbb{V}^*$ by Theorem 2.2. By the result in [9], Section 2, such an integral represents a lift of a minimal 2-sphere in S^4. Thus the space of nonvertical rational curves in $\mathbb{CP}^3$ can be regarded as the union of 2 copies of the space of minimal 2-spheres in S^4. These two copies have a nonempty intersection, corresponding to geodesic 2-spheres.

4

Summary

In this summary, we discuss further research directions.

It is an open problem to construct nonhomogeneous nearly Kähler 6-manifolds. One way might be to consider the submanifolds in a G_2 manifold. Since Joyce has constructed many compact manifolds with G_2 holonomy, it is natural to ask what kind of $SU(3)$ structures their submanifolds could have. Note that the normal bundle essentially gives the almost complex structure. So conditions on Nijenhaus tensor translates into conditions on normal bundle. Thus the question is largely on submanifolds with special normal bundle, a subject studied extensively from integrable system point of view.

Anti-self-dual connections on nearly Kähler manifolds have much left to explore. It is natural to ask whether the Chern-Simons functional is Morse-Smale. If this is true, it is natural to ask what kind of invariants can be built out of the study of Morse theory on the moduli. The discussion in this thesis may provide some hint.

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Biography

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