The Hyodo-Kato theorem for rational homotopy types

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1 Introduction

Let A be a complete discrete valuation ring with perfect residue field k of characteristic $p > 0$ and fraction field F. Denote by W the ring of Witt vectors of k and by K the fraction field of W. Endow A with the log structure $A - 0 \hookrightarrow A$ and let X be a proper smooth connected fine saturated log scheme over A with generic fiber X^* and special fiber Y which we assume to be of Cartier type ([5] 2.12). The Hyodo-Kato theorem ([5] Theorem 5.1) says

$$H_{DR}^i(X^*) \simeq H_{cr}^i(Y/W) \otimes_W F$$

where the cohomology groups appearing in the statement are algebraic De Rham cohomology on the left and the crystalline cohomology of Y with respect to W on the right.

We will prove a version of this theorem in the context of the unipotent rational homotopy types defined in [8]. The definitions will be reviewed in the next section, but let us state the main result here. If $A_{cr}(Y)$ denotes the crystalline rational homotopy type of Y and $A_{DR}(X^*)$ the De Rham rational homotopy type of X^*, then

Theorem 1
$$A_{DR}(X^*) \simeq A_{cr}(Y) \otimes F$$

in the homotopy category of commutative differential graded algebras (CDGA's) over F.

In [8], we proved this result essentially when A has the trivial log structure. Let x be a point of X, x^* the correponding point on the generic fiber, and y the reduction of x to the special fiber. It is simple to check that the augmentations induced by x^* and y on the homotopy types are compatible. Hence, we get

Corollary 1
$$\pi_1^{dr}(X^*, x^*) \simeq \pi^{cr}(Y, y) \otimes_K F$$

as pro-unipotent algebraic groups over F.

We also get results of the Artin-Mazur type [1] on higher homotopy groups of simply connected varieties. For this, let X_1^* and X_2^* be proper smooth geometrically-connected varieties over a number field E equipped with normal crossing divisors D_1^* and D_2^*. Denote by $\mathcal{O}_{E,v}$ the localization at a prime v of the ring of integers $\mathcal{O}_E$ of E. Assume that X_i^* extends to a proper flat scheme X_i over $\mathcal{O}_{E,v}$ with special fibers Y_i and that D_i^* extends to a divisor D_i on X_i that is relatively of normal crossing. Further assume that $D_i + Y_i$ is a strict normal crossing divisor. View X_i as a log scheme with the log structures given by $D_i + Y_i$. Y_i also has the induces log structure.

Corollary 2 *Suppose $Y_1 \simeq Y_2$ as log schemes. Then for every embedding $\sigma : E \hookrightarrow \mathbf{C}$ such that the $(X_i^* - D_i^*) \otimes_E \mathbf{C}$ are simply-connected, their higher rational homotopy groups are isomorphic.*

The hypothesis of the corollary is satisfied for example if

$$X_1 \otimes \mathcal{O}_{E,v}/m_v^2 \simeq X_2 \otimes \mathcal{O}_{E,v}/m_v^2$$

by a map that preserves the divisors $D_i + Y_i$.

The proof of the theorem consists of choosing explicit CDGA representatives for the homotopy types and a few other intermediate complexes using embedding systems for crystalline cohomology, constructing explicit multiplicative maps between them and, thereby, 'exorcising the derived category' from the proof of the usual Hyodo-Kato isomorphism.

It is perhaps useful to think about the theorem in the following general context: Let $\mathcal{A}$ be the homotopy category of non-negatively graded CDGA's over some field and consider the forgetful functor

$$R : \mathcal{A} \rightarrow \mathcal{C}$$

to the derived category of complexes. Following the ideas of rational homotopy theory, one should think of objects A in $\mathcal{A}$ as being rationally nilpotent topological spaces and $R(A)$ as being like the homotopy groups of the space. Given two objects A, B and an isomorphism $g : R(A) \simeq R(B)$, we can then ask if there is in fact an isomorphism $f : A \rightarrow B$ such that $g = R(f)$. That is, we wish to prove a Whitehead-type theorem showing that an isomorphism of homotopy groups is induced by an actual isomorphism of spaces. In our situation, the usual Hyodo-Kato isomorphism says that $R(A_{DR}(X^*)) \simeq R(A_{cr}(Y) \otimes_K F)$ and our theorem says that this map can be lifted to a map of 'spaces.'

In the process of making maps explicit, one encounters the problem of assembling a collection of maps in (an increasing sequence of) finite characteristics into a single map of algebras. This is because the Hyodo-Kato isomorphism depends on the fact that repeated Frobenius twists of complexes computing crystalline cohomology w.r.t. two different log structures become increasingly close, i.e., isomorphic modulo increasingly high powers of p. That is, an inverse system of complexes related by the relative Frobenius and reduction mediates the Hyodo-Kato isomorphism. To deal with this difficulty we use a differential graded algebra version of homotopy direct limits, namely, the mapping telescope of an inverse system of maps. In fact, it will be necessary to, employ the notion of a 'twisted' inverse limit introduced by Ogus to define 'twisted' mapping telescopes and the key definition of an 'infinitely twisted' mapping telescope (obtained by taking a direct limit of twisted mapping telescopes). Once one has this machinery in place, the isogenies that occur in the usual proof fall into place nicely as isomorphisms of ∞-twisted mapping telescopes and the rest of the argument becomes quite short.

We remark that the proof of theorem 1 appears to clarify the cohomological Hyodo-Kato isomorphism as well: The rational homotopy types considered as complexes compute the usual cohomology groups (crystalline and De Rham). However, the proof given here, while just a modification of those of Hyodo-Kato and Ogus, possesses an advantage over them in that the maps involved are made completely explicit. Furthermore, the proof makes clear that the essence of the argument is local, although we do not present it in an overtly local fashion.

One issue we do not deal with in this paper is the dependence of the isomorphism stated in the theorem on the choice of a uniformizer for A. Although it is more or less clear that one gets the same kind of dependence as in the cohomological theorem, we prefer to discuss this in a subsequent paper together with a more detailed study of the monodromy operator and period isomorphisms.

A few words about our convention: For the most part we leave the log structure implicit and do not introduce separate notation to indicate their presence. The important exception of course is in the proof of theorem 1. A *multiplicative quasi-isomorphism* of CDGA's is just a quasi-isomorphism at the level of complexes that respects the multiplicative structure. A *quasi-equivalence*, on the other hand, is an isomorphism in the homotopy category of CDGA's.

2   Brief review

For precise definitions, we refer the reader to [8] sections 3 and 4, and the references therein. In this section, we will just recall at the superficial level some basic notation, the objects that we will need,

and their main properties.

For concreteness, we concentrate on the situation described in the introduction. Therefore, k is endowed with the log structure of the punctured point which is associated to the pre-log structure $\mathbf{N} \to k$ that sends 1 to 0. Y, then, is a smooth proper fine saturated log scheme of Cartier type over k and $y : \mathrm{Spec}(k) \to Y$ is a point of Y. (We remind the reader that this means in particular that it is a map of log schemes.)

The *unipotent crystalline rational homotopy type* $A_{cr}(Y)$ of Y is defined by the following formula:

$$A_{cr}(Y) = TW(W\omega_y) := s_{TW}((\varprojlim \Gamma(G(W\omega_Y))) \otimes_W K)$$

The notation is that given a pro-sheaf L, $G(L)$ is a cosimplicial Godement resolution [3] for the étale topology, $\varprojlim$ goes from the category of inverse systems $(M_n)_{n \in \mathbf{N}}$, where M_n is a (cosimplicial) W_n-algebra, to the category of complete cosimplicial W-algebras, and finally, s_{TW} is Navarro-Aznar's 'simple Thom-Whitney algebra' functor [10]. $W\omega_Y$ is the pro-sheaf of CDGA's consisting of the De Rham-Witt differential forms of Y. It is probably best at this point not to worry about the precise construction and just remember that

$$TW(\cdot) = s_{TW}((\varprojlim \Gamma(G(\cdot)) \otimes)_W K)$$

is a functor from the isogeny category of pro-sheaves of CDGA's over W (that is, systems (L_n) where a given level L_n is a sheaf of CDGA's over W_n) to the homotopy category of CDGA's over K. As motivation for the language of homotopy types, we refer the reader to [9].

A basic proprty of the functor s_{TW} is that for any cosimplicial CDGA C, $s_{TW}(C) \simeq s(C)$ in the derived category of complexes, where $s(\cdot)$ denotes the usual 'associated simple object' functor. Ostensibly, the definition depends on the choice of a surjective system of geometric points on Y. Denote for a moment A_S the rational homotopy type constructed from a specific system S of geometric points. Given two different systems S_1 and S_2, we can find a third system S_3 dominating both

$$
\begin{array}{ccc}
 & S_3 & \\
\swarrow & & \searrow \\
S_1 & & S_2
\end{array}
$$

which therefore gives us multiplicative quasi-isomorphisms

$$
\begin{array}{ccc}
 & A_{S_3} & \\
\nearrow & & \nwarrow \\
A_{S_1} & & A_{S_2}
\end{array}
$$

Hence, the isomorphism class of $A_{cr}(Y)$ in the homotopy category of CDGA's is well-defined. Furthermore, if S_4 were chosen instead of S_3, then we could dominate these two by a third S_5 fitting into a commutative diagram

$$
\begin{array}{ccccc}
 & & S_3 & & \\
 & \swarrow & \uparrow & \searrow & \\
S_1 & \leftarrow & S_5 & \rightarrow & S_2 \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & S_4 & &
\end{array}
$$

leading us to a commutative diagram

$$
\begin{array}{ccccc}
 & & A_{S_3} & & \\
 & \nearrow & \downarrow & \nwarrow & \\
A_{S_1} & \rightarrow & A_{S_5} & \leftarrow & A_{S_2} \\
 & \searrow & \uparrow & \nearrow & \\
 & & A_{S_4} & &
\end{array}
$$

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Therefore, the isomorphism $A_{S_1} \to A_{S_2}$ is canonical in the homotopy category. This justifies omitting the system of points from the notation for the homotopy type. We will omit similar obvious arguments in a few other places of the papers.

The (unipotent) De Rham rational homotopy type of X^* is defined by

$$A_{DR}(X) = TW(\Omega_{X^*/F}) := s_{TW} \Gamma(G(\Omega_{X^*/F}))$$

where the Godement resolution is now taken on X^*. But GAGA implies that

$$A_{DR}(X) \simeq s_{TW}((\varprojlim \Gamma(G(\Omega_{\hat{X}/A}))) \otimes_A F)$$

where the formal completion $\Omega_{\hat{X}/A}$ of the De Rham complex of X/A is regarded as a pro-sheaf on Y. It is the latter object that we will compare to A_{cr}.

A_{cr} can be computed using the 'crystalline complex' associated to an embedding system ([5] 2.18): Let $(Y_., Z_.)$ be a pair of simplicial schemes that fit into a diagram

$$\begin{array}{ccc} Y_. & \hookrightarrow & Z_. \\ \downarrow & & \\ Y & & \end{array}$$

where $Y_.$ is a simplicial hypercovering (that satisfies cohomological descent for the étale topology) which we equip with the log structure pulled back from that of Y, $Y_. \hookrightarrow Z_.$ is a closed embedding of formal log schemes, and $Z_.$ is smooth formal log scheme over W. Then the associated crystalline complex is by definition

$$C(Y_., Z_.) := \Omega_{Z_./W} \otimes_{\mathcal{O}_{Z_.}} D_{Y_.}(Z_.)$$

Here, $D_{Y_.}(Z_.)$ is the divided power envelop of $Y_.$ in $Z_.$. Regard $C(Y_., Z_.)$ as a simplicial pro-CDGA on $Y_.$. Then we also defined $TW(C(Y_., Z_.))$ in this setting and we have a quasi-equivalence (i.e., an isomorphism in the homotopy category):

$$TW(C(Y_., Z_.)) \simeq TW(W\omega_Y)$$

The functor TW (Thom-Whitney) can be defined in a more general setting, as we have already done in the definition of A_{DR} for example, but also for crystalline complexes over more general bases. So if Y/S is a fine saturated smooth log scheme over an affine base $S = \mathrm{Spec}(R)$ and $S \hookrightarrow T = \mathrm{Spec}(B)$ is an exact immersion of formal log schemes where B is of characteristic zero, then to any embedding system $(Y_., Z_.)$ as above with $Z_.$ smooth over B, we can associate the crystalline complex $C(Y_., Z_.)$ and the Thom-Whitney algebra $TW(C(Y_., Z_.))$ which ends up as a CDGA over $B \otimes \mathbf{Q}$. This algebra is in fact canonically independent of the embedding system: Any two embedding systems can be dominated by a third giving rise to a quasi-equivalence which, in turn, is independent in the homotopy category of the dominating system.

3   Mapping telescopes

Now we will go on to define the *(twisted) mapping telescopes* of inverse systems of complete W-algebras.

First, if $f : A \to B$ is a map of (complete) CDGA's over W, we define the *mapping cylinder* that fits into a diagram

$$\begin{array}{ccc} & & Cyl(f) \\ & \nearrow & \downarrow \\ A & \to & B \end{array}$$

as follows: Denote by I the formal divided power De Rham complex of $W << x >>$, which therefore is concentrated in degrees 0 and 1. Consider the two augmentation maps $e_0, e_p : I \to \mathbf{Z}$ that evaluate functions in I^0 at 0 and p, respectively, and sends I^1 to zero. Hence $B \otimes I$ is likewise equipped with

two maps to B which we also denote by e_0 and e_p. Here and henceforward, all tensor products are topological. The mapping cylinder $Cyl(f)$ of f is defined to be the subalgebra of $A \oplus B \otimes I$ consisting of elements (a, b) such that $f(a) = e_0(b)$. Notice that the map $Cyl(f) \rightarrow B$ induced by $e_p : B \otimes I \rightarrow B$ becomes surjective after tensoring with $\mathbf{Q}$: for any $b \in B$, $(0, xb/p)$ is in $Cyl(f) \otimes \mathbf{Q}$ and maps to b. On the other hand, integrally, we can only say that if $b \in B$, then pb is in the image of e_p. The map $A \rightarrow Cyl(f)$ given by $a \mapsto (a, f(a))$ induces a q.i. In fact, the projection $Cyl(f) \rightarrow A$ to the first component is a chain homotopy inverse. Now, given an inverse system of maps

$$A_{\cdot} : \cdots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1$$

indexed by the positive integers, it is clear how to construct the mapping telescope $Tel(A_{\cdot})$. It is the inverse limit of the inverse system defined inductively by putting $Tel_1 = A_1$ and

$$Tel_{i+1} := Cyl(A_{i+1} \rightarrow Tel_i)$$

where the map $A_{i+1} \rightarrow Tel_i$ is the composite $A_{i+1} \rightarrow A_i \rightarrow Tel_i$. The construction of Tel is clearly functorial for inverse systems of CDGA's. In particular, one can apply the construction to inverse systems of cosimplicial CGDA's or bi-cosimplicial CDGA's. Also, there is a map of inverse systems $A_{\cdot} \rightarrow Tel_{\cdot}$ giving rise to a functorial map $\varprojlim A_{\cdot} \rightarrow Tel$.

We will also need the 'twisted' inverse limit construction of Ogus ([11] p.203): Given an inverse system $L_{\cdot}$ of W-modules indexed by $\mathbf{N}$ and an integer m, let $\varprojlim^m L_{\cdot}$ be the collection of elements (a_i), $a_i \in L_i$, such that $a_i \mapsto p^m a_{i-1}$. If $n \geq m$, then we have natural maps $\varprojlim^m L_{\cdot} \rightarrow \varprojlim^n L_{\cdot}$ given by $(a_i) \mapsto (p^{(n-m)i} a_i)$. That is, the twisted inverse limits form a directed system.

Using these, we can also construct the $m-$twisted telescope

$$Tel^m(A_{\cdot}) = \varprojlim^m Tel_i(A_{\cdot})$$

as well as the infinitely twisted telescope

$$Tel^{\infty}(A_{\cdot}) = \varinjlim_m Tel^m(A_{\cdot})$$

Clearly, we have functorial maps $\varprojlim^m A_{\cdot} \rightarrow Tel^m(A_{\cdot})$ and

$$\varinjlim_m \varprojlim^m A_{\cdot} \rightarrow Tel^{\infty}(A_{\cdot})$$

Lemma 1 *Let $L_{\cdot} \rightarrow K_{\cdot}$ be a map of inverse systems of ((bi-)cosimplicial) CDGA's which is a quasi-isomorphism at each level. That is, each $L_n \rightarrow K_n$ is a quasi-isomorphism. Suppose, furthermore, that the transition maps for both systems (denoted π) have the following property: there exists some m such that for each element x of level i, there exists an element y of level $i + 1$ such that $\pi(y) = p^m x$. Then*

$$\varinjlim_n \varprojlim^n L_{\cdot} \rightarrow \varinjlim_n \varprojlim^n K_{\cdot}$$

is a quasi-isomorphim.

Proof. It is straightforward to check that the twisted inverse limit commutes with the cone construction, so that, for each n, we have an exact triangle

$$0 \rightarrow \varprojlim^n L_{\cdot} \rightarrow \varprojlim^n K_{\cdot} \rightarrow \varprojlim^n C_{\cdot}$$

where $C_{\cdot}$ is the cone of $L_{\cdot} \rightarrow K_{\cdot}$. Since direct limits commute with cohomology, we need only show that

$$\varinjlim H(\varprojlim^n C_{\cdot}) = 0$$

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In fact, the transition map

$$H(\varprojlim^{n} C_{\cdot}) \to H(\varprojlim^{n+m} C_{\cdot})$$

is zero. To see this, let the cocycle (c_i) represent an element of $H(\varprojlim^{n} C_{\cdot})$. We will show that $(p^{mi} c_i) \in \varprojlim^{n+m} C_{\cdot}$ is a coboundary. Construct an element $(b_i) \in \varprojlim^{n+m} C_{\cdot}$ such that $d(b_i) = (p^{mi} c_i)$ inductively as follows: assume we have constructed up to b_j. That is, for $i \leq j$, $db_i = p^{mi} c_i$ and $\pi(b_j) = p^{n+m} b_{j-1}$. Since each C_i is acyclic, there exists an $x_{j+1} \in C_{j+1}$ such that $dx_{j+1} = c_{j+1}$. Hence, $d\pi(p^{mj} x_{j+1}) = p^{mj} \pi(c_{j+1}) = p^{mj+n} c_j = p^n db_j$ and $d(\pi(p^{mj} x_{j+1}) - p^n b_j) = 0$. Again by acyclicity in level j, we can then find an a_j such that $da_j = \pi(p^{mj} x_{j+1}) - p^n b_j$. Multiplying by p^m, we get

$$\pi(p^{(j+1)m} x_{j+1}) - p^{n+m} b_j = d(p^m a_j) = d\pi(a_{j+1})$$

for some a_{j+1}. Now put

$$b_{j+1} := p^{(j+1)m} x_{j+1} - da_{j+1}$$

Then $\pi(b_{j+1}) = p^{n+m} b_j$ and $db_{j+1} = p^{(j+1)m} c_{j+1}$, so we are done.

We emphasize that Tel^{∞} is a functor from inverse systems of ((bi-)cosimplicial) CDGA's to ((bi-)cosimplicial) CDGA's and that the maps

$$\varprojlim A_{\cdot} \to \varinjlim_{m} \varprojlim^{m} A_{\cdot} \to Tel^{\infty}(A_{\cdot})$$

are multiplicative.

4 Proof of theorem

In this section, it will be useful to employ the following notation: Given a p-adic formal scheme S and an integer m, S/m denotes $S \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^{v_p(m)}$, where v_p is the p-adic valuation for which $v_p(p) = 1$.

We denote by S_1 and S_2 the scheme $\mathrm{Spec} W[t]$ equipped with the log structures associated to the pre-log structures $\mathbf{N} \to W[t]$ that send 1 to 0 and t, respectively. So both restrict to the log structure of the punctured point on $\mathrm{Spec}(k)$. $W[t]$ carries the Frobenius σ given by the usual Frobenius on W and sending t to t^p. Denote by $W << t >>$, the p-adic completion of the divided power polynomial algebra $W < t >$ over W which therefore also carries log structures induced by those of S_1 and S_2. Denote these formal log schemes by $\hat{S}_1$ and $\hat{S}_2$. The Frobenius σ of $W[t]$ naturally extends to $W << t >>$. We will denote by $S_i^{(n)}$ the scheme $\mathrm{Spec}(W[t])$ equipped with the log structure of S_i pulled back by the n-th iterate of the Frobenius. Hence, we see that $S_1^{(n)} \simeq S_1$ while the log structure on $S_2^{(n)}$ is associated to the pre-log structure $\mathbf{N} \to W[t]$ that sends 1 to t^{p^n}. We will also use the notation $\hat{S}_i^{(n)}$ in the obvious manner.

The difficult part of the Hyodo-Kato isomorphism says that if Y is a smooth fine proper log scheme of Cartier type, then

$$H^i(Y/S_1) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq H^i(Y/S_2) \otimes_{\mathbf{Z}} \mathbf{Q}$$

that is, the crystalline cohomology is the same for the two log structures, up to isogeny.

We recall the proof, taking care to make some maps more explicit.

We start out by choosing embedding systems $Y_{\cdot} \hookrightarrow Z_{\cdot}^1$ and $Y_{\cdot} \hookrightarrow Z_{\cdot}^2$ for Y w.r.t. the two log schemes S_1 and S_2 that admit Frobenius lifts. This notion requires a brief explanation: Denote by $S_a^{(n)}$ the Frobenius twisted log schemes introduced above. There are then maps $S_a \to S_a^{(n)}$ induced from the monoid map $\mathbf{N} \to \mathbf{N}$, $1 \mapsto p^n$. Now, $(Z^a)^{(n)}$ is a smooth simplicial log scheme over $S_a^{(n)}$, and a Frobenius lift refers to a map $F: Z_{\cdot}^a \to (Z_{\cdot}^a)^{(1)}$ that fits into the commutative diagram

$$
\begin{array}{ccc}
Z_{\cdot}^a & \xrightarrow{F} & (Z_{\cdot}^a)^{(1)} \\
\downarrow & & \downarrow \\
S_a & \rightarrow & S_a^{(1)}
\end{array}
$$

Such embedding systems can be constructed, for example, as follows: Y_0 in both cases is just the disjoint union of the elements of an affine open covering of Y and Z_0^1 and Z_0^2 are smooth liftings of Y_0 to S_1 and S_2, respectively. Then the $Y_i = Y_1 \times_Y Y_1 \times \cdots \times Y_1$ ($i+1$-times) embed diagonally into

$$Z_i^a = Z_0^a \times_{S_a} Z_0^a \times_{S_a} \cdots \times_{S_a} Z_0^a$$

($i+1$-times) for $a = 1, 2$ and come together to form a simplicial hypercovering. By the affine smoothness, we easily get the Frobenius lifts in cosimplicial level 0 and then the other levels by taking products.

Let C_1 and C_2 be crystalline complexes for the two different embedding systems, and form the bi-cosimplicial algebras over $W << t >>$

$$B_a := \varprojlim \Gamma(G(C_a))$$

for $a = 1, 2$. Denote by $B_a^{(n)}$ the pull-back of B_a via σ^n, which therefore arise from crystalline complexes for $(Z^a_{\cdot})^{(n)}$, the n-th Frobenius pull-back of $Z^a_{\cdot}$, w.r.t. $S_a^{(n)}$. In particular, when we adjoin divided powers and reduce mod $p^n!$, the two log structures are isomorphic, so we get isomorphisms in the derived category of bi-cosimplicial complexes

$$B_1^{(n)}/(p^n)! \simeq B_2^{(n)}/(p^n)!$$

We can get these maps to fit into commutative diagrams of bi-cosimplicial CDGA's

$$
\begin{array}{ccccc}
B_1^{(n+1)}/(p^n)! & \to & L_{n+1} & \leftarrow & B_2^{(n+1)}/(p^n)! \\
\downarrow \phi & & \downarrow & & \downarrow \phi \\
B_1^{(n)}/(p^n)! & \to & L_n & \leftarrow & B_2^{(n)}/(p^n)!
\end{array}
$$

where the horizontal arrows are multiplicative quasi-isomorphisms of CDGA's (as opposed to maps in the derived category of complexes) constructed as follows. The vertical arrows on either end are induced by the Frobenius lifts. L_n is the crystalline complex for $Y^{(n)}_{\cdot}$ relative to $\hat{S}_1^{(n)}/p^n! = \hat{S}_2^{(n)}/p^n!$ computed using the 'diagonal' embedding system

$$Y_{\cdot}^{(n)} \hookrightarrow (Z^1_{\cdot})^{(n)} \times_{\hat{S}_1^{(n)}/p^n!} (Z^2_{\cdot})^{(n)}$$

The Frobenius lifts for $Z^1_{\cdot}$ and $Z^2_{\cdot}$ determine one for each product giving the maps $L_{n+1} \to L_n$. The horizonal maps are then induced by the projections. We stress that all the maps are given by pull-backs of differential forms, and hence, are multiplicative.

On the other hand, the iterates of the Frobenius induce multiplicative maps of inverse systems

$$B_a^{(\cdot)}/p^\cdot! \to B_a/p^\cdot!$$

where the target is just the inverse system given by the reductions of B_a with the natural projections connecting them.

Lemma 1 from the previous section admits the following corollaries:

Corollary 3 *The following arrows are quasi-isomorphisms:*

$$Tel^\infty(B_1^{(\cdot)}) \to Tel^\infty(L_{\cdot}) \leftarrow Tel^\infty(B_2^{(\cdot)})$$

Corollary 4

$$\varinjlim_m \varprojlim^m (B_a/p^\cdot!) \to Tel^\infty(B_a/p^\cdot!)$$

is a quasi-isomorphism.

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On the other hand, we have the

Proposition 1 *The map*

$$B_a^{(\cdot)}/p^{\cdot}! \to B_a/p^{\cdot}!$$

induces a quasi-isomorphism

$$Tel^\infty(B_a^{(\cdot)}/p^{\cdot}!) \simeq Tel^\infty(B_a/p^{\cdot}!)$$

for $a = 1, 2$.

Proof. We will omit the subscript a from the notation. Hence, S also refers to either S_1 or S_2.

If C_n is the cone of $B^{(n)}/p^n! \to B/p^n!$ then $Tel_n(C_\cdot)$ is the cone of $Tel_n(B^{(\cdot)}/p^{\cdot}!) \to Tel_n(B/p^{\cdot}!)$ That is, we have an exact triangle:

$$0 \to Tel_n(B^{(\cdot)}/p^{\cdot}) \to Tel_n(B/p^{\cdot}!) \to Tel_n(C_\cdot)$$

The key point is the following

Lemma 2 $H^i(C_n) = H^i(Tel_n(C_\cdot))$ *is killed by* p^{2in} *for* $i > 0$ *and* p^n *for* $i = 0$.

Proof of lemma. Since the algebras are constructed out of the stalks of Godements resolutions, the statement is local on Y. So we may assume that the embedding system is just a smooth S lift X and that we also have a Frobenius lift f. Thus, f induces pull-back maps

$$\phi^n : \Omega_{X^{(n)}/S^{(n)}} \to \Omega_{X/S}$$

It suffices to show that the cone C_n of this map mod $p^n!$ has i-th cohomology killed by p^{2in} for $i > 0$ and p^n for $i = 0$. We just give the argument for $i > 0$ since the $i = 0$ case is an obvious modification.

Recall the map F given by ϕ/p^i in degree i. The definition works just as in the usual case ([6] 0.2.3.3) using the W-flatness of the sheaf of differentials. F induces an injection ([7] III.1.5.4):

$$F : \Omega_{X^{(1)}/S^{(1)}}^i / [p^n(\Omega_{X^{(1)}/S^{(1)}}^i) + pd(\Omega_{X^{(1)}/S^{(1)}}^{i-1})] \hookrightarrow \Omega_{X/S}^i / [p^n(\Omega_{X/S}^i) + d(\Omega_{X/S}^{i-1})]$$

This is an immediate consequence of the Cartier isomorphism, again as in the classical case.

Note the following corollary: given $x \in \Omega_{X^{(1)}/S^{(1)}}^i$, if $F(x) = p^m y$ for some y, then $x = p^m z$ for some z. This is obvious for $m = 1$ from the above injection. Assume it for $m - 1$. $F(x) = p^m y$ in any case implies $x = p^m w + pdu$. But then $F(pdu) = F(x) - F(p^m w) = p^m(y - F(w))$ so $F(du)$ is divisible by p^{m-1}. Therefore, $du = p^{m-1}v$ and x is divisible by p^m.

By iterating the argument, we also see that if $x \in \Omega_{X^{(n)}/S^{(n)}}^i$ and $F^n(x)$ is divisible by p^m, then so is x.

Now, let

$$(a, b) \in \Omega_{X/S}^i \oplus \Omega_{X^{(n)}/S^{(n)}}^{i+1}$$

represent a cocycle in

$$C^i = (\Omega_{X/S}^i \oplus \Omega_{X^{(n)}/S^{(n)}}^{i+1}) \otimes \mathbf{Z}/p^n!$$

Then $\phi^n(b) - da = p^n! x$ and $db = p^n! y$. Write the first equality as

$$F^n(p^{in} b) = p^n! x + da$$

and apply the injection above to get

$$p^{in} b = p^n! w + pds = p^n! w + dc$$

for some w and c. We also get

$$d(\phi^n(c) - p^{in} a) = p^{in} \phi^n(b) - p^n! p^{in} F^n(w) - p^{in} da = p^n! p^{in}(x - F^n(w))$$

That is,
$$\phi^n(c) - p^{in}a \in d^{-1}(p^n! p^{in} \Omega^{i+1}_{X/S})$$

Recall the formula

$$d^{-1}(p^n \Omega^{i+1}_{X/S}) = \Sigma_{0 \le k \le n} p^k F^{n-k}(\Omega^i_{X^{(n-k)}/S^{(n-k)}}) + \sum_{0 \le k \le n-1} F^k(d\Omega^{i-1}_{X^{(k)}/S^{(k)}})$$

whose proof also follows [6] 0.2.3.13 verbatim, as pointed out by Jannsen in the appendix of [4]. Thus, we have
$$\phi^n(c) - p^{in}a = p^n! p^{in} z_0 + p^n! p^{-1} p^{in} F(z_1) + \cdots + p^n! p^{-n} p^{in} F^n(z_n) + \cdots$$
$$F^N(z_N) + du_0 + F(du_1) + \cdots + F^{N-1}(du_{N-1})$$

where $N = v_p(p^n!) + in$. We examine the terms in this equality after multiplying by another p^{in}. We get that $\phi^n(p^{in}c) - p^{2in}a$ is of the form $p^n! p^{in} z + \phi^n(l) + du$. If we apply the differential d, we get that $d\phi^n(l) = \phi^n(dl)$ is divisible by $p^{in} p^n!$. Thus, $F^n(dl)$ is divisible by $p^n!$, and hence, so is dl. So we get $d(p^{in}c - l) = p^{2in}b \pmod{p^n!}$ and $\phi^n(p^{in}c - l) - du = p^{2in}a \pmod{p^n!}$. Hence, the class of $(p^{2in}a, p^{2in}b) \pmod{p^n!}$ is a coboundary. This proves the lemma.

For all our considerations, we may assume that the dimension d of Y is positive. Therefore, we see that all the cohomology of C_n is killed by p^{2dn}. The remainder of the proof of the proposition is as in the proof of Lemma 1 : Let (c_n) be a cocycle in $Tel^m(C.)$. Then $(p^{(2d+1)n}c_n) \in Tel^{(m+2d+1)}(C.)$ is a coboundary.

Remark: The fact that the relative Frobenius is an isogeny has been the source of many important theorems on crystalline cohomology starting with the theorem of Berthelot and Ogus. Here, it is rather mysteriously manifested in the isomorphism of infinitely twisted mapping telescopes.

Thus we have arrived at an explicit isomorphism in the derived category from

$$\varinjlim^m \varprojlim^m B_1/p^{\cdot}!$$

to

$$\varinjlim^m \varprojlim^m B_2/p^{\cdot}!$$

mediated by the following system of arrows all of which are quasi-isomorphisms:

$$Tel^\infty(L.)$$
$$\nearrow \qquad \nwarrow$$
$$Tel^\infty(B_1^{(.)}/p^{\cdot}!) \qquad\qquad\qquad Tel^\infty(B_2^{(.)}/p^{\cdot}!)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$
$$\varinjlim^m \varprojlim^m B_1/p^{\cdot}! \quad \to \quad Tel^\infty(B_1/p^{\cdot}!) \qquad\qquad Tel^\infty(B_2/p^{\cdot}!) \quad \leftarrow \quad \varinjlim^m \varprojlim^m B_2/p^{\cdot}!$$

On the other hand, as has already been pointed out, it is trivial to check that if $L.$ is an inverse system of W-algebras, then $\varinjlim^m \varprojlim^m L.$ is a W algebra, even though each $\varprojlim^m L.$ separately is not. Also, maps in the category of inverse systems of algebras induce algebra homomorphisms for this double limit. Therefore, we see that the isomorphism constructed above is actually an isomorphism in the homotopy category of algebras.

Now consider the natural map

$$\varprojlim B_a/p^{\cdot}! \to \varinjlim^m \varprojlim^m B_a/p^{\cdot}!$$

By Ogus ([11] Lemma 18), we have an isomorphism

$$(\varprojlim B_a/p^{\cdot}!) \otimes \mathbf{Q} \simeq (\varinjlim^m \varprojlim^m B_a/p^{\cdot}!) \otimes \mathbf{Q}$$

Thus, we have constructed a quasi-equivalence from

$$(\varprojlim B_1/p^i!) \otimes \mathbf{Q}$$

to

$$(\varprojlim B_2/p^i!) \otimes \mathbf{Q}$$

Applying the Thom-Whitney simple object functor twice, we get a quasi-equivalence

$$TW(C(Y_., Z_.^1) \simeq TW(C(Y_., Z_.^2))$$

It is straightforward to check that this map is independent (in the homotopy category) of the various choices made.

The rest of the argument is as in [8], section 7, and amounts to a 'Berthelot-Ogus' type argument [2]. Choose a uniformizer π of A which therefore determines a presentation $A \simeq W[t]/(f(t))$, where $f(t)$ is an Eisentein polynomial of degree $e = [F : K]$. Let R be the p-adic completion of the divided power envelop of $(f(t), p)$ inside $W[t]$. Thus R is also the completed DP envelop of the ideal (t^e). We have a natural map $g : R \to W << t >>$. On the other hand, if r is such that $p^r \geq e$, then the map $\sigma^r : W << t >> \to W << t >>$ factors through $W << t >> \to R \to W << t >>$. Composing in the other direction $R \to W << t >> \to R$ is, by definition, the Frobenius map of R. Let $Y' = X \otimes A/p$ with the induced log structure. Also, give $\mathrm{Spec}(R)$ the log structure induces from S_2. We have the commutative diagram

$$
\begin{array}{ccccc}
Y' & \to & Y & \hookrightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(A/p) & \to & \mathrm{Spec}(k) & \hookrightarrow & \mathrm{Spec}(A/p) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(R) & \to & \hat{S}_2 & \to & \mathrm{Spec}(R)
\end{array}
$$

where the composite of the horizontal arrows are all the r-th iterate of the Frobenius. We choose crystalline complexes for Y and Y' as follows. Construct first embedding systems for Y and Y' w.r.t. S_2 that fit into a diagram

$$
\begin{array}{ccccc}
Y_. & \hookrightarrow & Y'_. & \hookrightarrow & Z_. \\
\downarrow & & \downarrow & & \\
Y & \hookrightarrow & Y' & &
\end{array}
$$

where $Z_.$ is smooth over S_2 and the left hand square is cartesian. We can also arrange for $Z_.$ to admit a Frobenius lift compatible with the Frobenius of $W[t]$. Then $C = \Omega_{Z_./W[t]} \otimes W << t >>$ and $C' = \Omega_{Z_./W[t]} \otimes R$ are crystalline complexes for Y and Y' and we can regard both as simplicial sheaves of CDGA's on $Y_.$.

Denote by $C^{(r)}$ (resp. $(C')^{(r)}$) the pull-back of C (resp. C') by the r-th power of the Frobenius map of $W << t >>$ (resp. R). Then the big diagram above implies that

$$(C')^{(r)} \simeq C^{(r)} \otimes_{W<<t>>} R$$

in the homotopy category of sheaves of CDGA's on $Y_.$. On the other hand, the Frobenius lifts induce maps $C^{(r)} \to C$ and $(C')^{(r)} \to C'$. So we have maps of sheaves of CDGA's

$$C \otimes R \leftarrow C^{(r)} \otimes R \simeq (C')^{(r)} \to C'$$

and taking Thom-Whitney algebras, we have multiplicative maps

$$TW(C) \otimes (R \otimes \mathbf{Q}) \leftarrow TW(C^{(r)}) \otimes (R \otimes \mathbf{Q}) \simeq TW((C')^{(r)}) \to TW(C')$$

of CDGA's over $R \otimes \mathbf{Q}$. From the fact that the relative Frobenius is an isogeny ([5] Proposition 2.24), we know that all these maps are quasi-equivalences. Now we tensor with the quotient map $R \otimes \mathbf{Q} \to A \otimes \mathbf{Q} = F$ to get

$$TW(C) \otimes_{W<<t>>\otimes\mathbf{Q}} F \simeq TW(C') \otimes_{R\otimes Q} F$$

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By using the fact that $C' \otimes A$ is the crystalline complex associated to an embedding system for Y' w.r.t. A which is also true of $\Omega_{\hat{X}/A}$, we get

$$TW(C') \otimes F \simeq TW(\Omega_{\hat{X}/A}) \simeq TW(\Omega_{X^*/F})$$

On the other hand, C is quasi-equivalent to $C(Y_., Z_.^2)$ from the previous section, so that $TW(C) \simeq TW(C(Y_., Z_.^2)) \simeq TW(C(Y_., Z_.^1))$ and $C(Y_., Z_.^1)$ is quasi-equivalent to the base change to $W << t >>$ of a crystalline complex for Y w.r.t. W so we get

$$TW(C) \otimes F \simeq TW(W\omega_Y) \otimes_K F$$

giving us the desired quasi-equivalence

$$TW(W\omega_Y) \otimes_K F \simeq TW(\Omega_{X*/F})$$

This is the isomorphism of homotopy types stated in the theorem.

Proof of corollary 1

We need to discuss basepoints. We start with a careful discussion of the basepoint for $W\omega_Y$. If $y : \mathrm{Spec}(k) \to Y$ is a point of Y, then there is a map $W\omega_Y \to y_*(W)$. It is given by 0 in positive degrees and the canonical map $e_y : W\emptyset_Y \to W(k)$ induced by $\emptyset_Y \to k$. This map induces $TW(W\omega_Y) \to TW(y_*(W))$. However, a simple examination of the definition yields the following description of the degree zero term $TW^0(y_*(W))$: It consists of collections (f_n) where f_n is a function on $\mathbf{A}_K^n$ with the property that $\partial_i(f_{n+1}) = f_n$ for any i. The map $(f_n) \to f_0$ yields a quasi-isomorphism $TW(y_*(W)) \simeq K$. We obtain thereby the augmentation map $TW(W\omega_Y) \to K$. Let's describe this map explicitly. The degree zero term of $TW(W\omega_Y)$ consists of compatible sequences ([8] section 3) (a_n), $a_n \in \prod_{\bar{p}}(W\emptyset_Y)_{\bar{p}} \otimes \emptyset_{\mathbf{A}_K^n}$ where $\bar{p}$ is an $n+1$-tuple of points in Y. Thus, a_0 is just an element of $\prod_p(W\emptyset_Y)_p$, and the map $TW(W\omega_Y) \to K$ is zero in positive degrees while in degree zero it sends (a_n) to $e_y(a_0)$. Now, let $(Y_., Z_.)$ be an embedding system for which Y_0 is the disjoint union of an affine open cover of Y and Z_0 is a smooth lifting of Y_0 which admits a Frobenius lift F. By the construction at the beginning of the section, for example, such embedding systems exist. The base point $y \in Y$ lifts to Y_0 and a W point z of $Z = Z_0$. Locally, we can express $W_n\emptyset_Y$ as the cohomology $\underline{H}^0(\Omega_{Z_n/W_n})$ and then we get the map $\emptyset_Z \to W\emptyset_Y$ that sends a to the sequence $(F^n(a)(\mathrm{mod} p^n))_n$ which is the degree zero component of the quasi-isomorphism $\Omega_{Z/W} \to W\omega_Y$. Let $\Omega_{Z/W} \otimes D_y(Z)$ be the divided power envelop of y in Z. Then we have $(\underline{H}^0(\Omega_{Z_n/W_n} \otimes D_y(Z_n)))_n \simeq W(k)$. The augmentation $e_z : \emptyset_Z \to W$ given by evaluation at z then fits into a commutative diagram

$$
\begin{array}{ccccc}
\emptyset_Z & \to & (\underline{H}^0(\Omega_{Z_n/W_n}))_n & \simeq & W\emptyset_Y \\
\downarrow & & \downarrow & & \downarrow \\
W & \to & (\underline{H}^0(\Omega_{Z_n/W_n} \otimes D_y(Z_n)))_n & \simeq & W
\end{array}
$$

where the first arrow in the bottom row sends $a \in W$ to the inverse system $(\sigma^n(a)(\mathrm{mod} p^n))_n$ This implies that

$$(TW(C(Y_., Z_.)), e_z) \simeq (TW(W\omega_Y), e_y)$$

as augmented algebras.

Now, assume we are in the situation at the end of section 2 where $V \to \mathrm{Spec}(R)$ is a smooth proper connected fine saturated log scheme over an affine base and $\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(B)$ is an exact topologically nilpotent immersion where B is of characteristic zero. Assume also that we are given a point $v : \mathrm{Spec}(R) \to V$. Then we can always find an embedding system $(V_., Z_.)$ with the property that v lifts to V_0 and to a point $z : B \to Z_0$. This point allows us to put an augmentation $TW(C(V, Z_.)) \to B \otimes \mathbf{Q}$ on the Thom-Whitney algebra and a product construction shows that for any two choices of embedding systems, there is a homotopy equivalence between the Thom-Whitney algebras which is compatible with the augmentation. Similarly for different choices of liftings of the point y. From this, applied to

the various embedding systems that occur in the proof of the theorem, one easily deduces that the homotopy equivalence

$$TW(C(Y_\cdot, Z_\cdot)) \otimes F \simeq TW(\Omega_{X^*})$$

takes the augmentation induced by e_z to that induced by evaluation at x^*.

Proof of Corollary 2.

The isomorphism classes of the higher rational homotopy groups are determined by their dimension, and this dimension can be computed in any complex embedding of E or after base change to the completion E_v of E w.r.t. v. The assumptions imply that the special fibers Y_1 and Y_2 are isomorphic smooth log schemes. Thus, $TW(\omega_{Y_2}) \simeq TW(\omega_{Y_2})$, which implies the quasi-equivalence of $TW(\Omega_{X_1^*}(\log D_1^*)) \otimes E_v$ and $TW(\Omega_{X_2^*}(\log D_2^*)) \otimes E_v$. Thus, their bar complexes are quasi-equivalent, giving isomorphisms of their cohomology groups, i.e., the higher De Rham homotopy groups of $X_1^* - D_1^*$ and $X_2^* - D_2^*$ [12].

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