Single-track Vehicle Dynamics and Stability

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mechanical Engineering and Materials Science
in the Graduate School of Duke University
2014
ABSTRACT

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Abstract

This work is concerned with the dynamics and stability of nonlinear systems that roll in a single track, including holonomic and nonholonomic systems. First the classic case of Euler’s disk is introduced as an example of a nonholonomic system in three dimensions, and the methodology for deriving equations of motion that is used throughout this work is demonstrated, including use of Lagrange’s equations, accommodating constraints with both Lagrange multipliers and with Gauss’s Principle.

Next, a disk in two dimensions with an eccentric center of mass is explored. The disk is assumed to roll on a cubic curve, creating the possibility of well-escape behavior, which is examined analytically and numerically, showing regions of multiperodicity and chaos. This theoretical system is compared to an experiment designed to demonstrate the same behavior.

The remainder of the present document is concerned with the stability of a bicycle, both on flat ground, and on a type of trainer known as “rollers.” The equations of motion are derived using Lagrange’s equations with nonholonomic constraints, then the equations are linearized about a constant forward velocity, and a straight path, yielding a two degree of freedom system for the roll and steer angles. Stability is then determined for a variety of different parameters, exploring the roll of bicycle geometry and rider position, along with the effect of adding a steering torque, taking the form of different control laws.

Finally, the system is adapted to that of a bicycle on rollers, and the related
equations of motion are derived and linearized. Notable differences with the classic bicycle case are detailed, a new eigenvalue behavior is presented, and configurations for optimal drum spacing are recommended.
In sleeping dreams I played with other chaps
But really envied nothing—save perhaps
The miracle of a lemniscate left
Upon wet sand by nonchalantly deft
Bicycle tires.

*Pale Fire, Canto 1*
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Nonlinear dynamical systems are important for describing almost all physical phenomena. Nonlinear systems where components roll have interesting implications in vehicle dynamics and accommodate a variety of analytical techniques. This work looks at four related systems that lend themselves to varied analyses. Euler’s disk, a classic nonholonomic system, is a useful test case for different approaches for determining nonlinear or linearized equations of motion. A disk rolling in two dimensions on a curved surface is a holonomic system that can represent a well-escape problem and accommodates approximate analytical approaches to interpreting nonlinear behavior. In contrast, bicycle stability is a complex nonholonomic problem that is often linearized because full, closed form nonlinear equations have never been offered in print. The ability of this system to steer itself at certain velocities has fascinated many mathematicians and scientists for over a century. Finally, the newly considered system of a bicycle on rollers provides more complex nonholonomic constraints and additional factors that influence stability over the classic bicycle.

The goal of this introduction is to outline the present work and provide relevant background material. Each section introduces a system, presents the most salient
prior work, and highlights new aspects of the research. A more thorough literature review appears in the corresponding chapters.

1.1 Euler’s disk

Chapter 2 is a “tutorial” chapter on a classic problem that relates to the other main systems in the dissertation. This problem has been used to illustrate a non-holonomic system in several advanced dynamics texts [1]. While the solutions in this work are not new, they are independently derived using the approach that will be applied later in the document. This problem sets the stage for the rest of the work by demonstrating two approaches for deriving equations of motion with Lagrange’s equations [2]. Constraint forces are incorporated with both Lagrange multipliers and Gauss’s Principle. The former is the same method used for the bicycle in Chapters 5 and 7.

Analysis includes comparing simulation of the nonlinear equations of motion with the linearized solutions and giving eigenvalues of the linearized system.
1.2 Eccentric disk

Figure 1.2: Line drawing of the rolling disk with an eccentric mass.

A two dimensional disk with an eccentric center of mass is investigated in Chapters 3–4. The former derives equations of motion for a disk with a mass imbalance in two dimensions for a general curve, then specifies the curve to be cubic, and therefore allow potential well escape [3–5]. Equations of motion are nondimensionalized for units of length

\[ \ddot{\xi}(p^2 + q) + \frac{c}{m} \dot{\xi} + \dot{\xi}^2 (p \dot{p} + \frac{1}{2} q \xi) + b^2 R g r \xi = -b \ddot{d}, \]  

(1.1)

where

\[ p(\xi) = 1 + \frac{e}{R} \sqrt{1 + \eta^2 \cos \theta} - \frac{b \eta R \xi}{(1 + \eta^2)^{3/2}} \]  

(1.2)

\[ q(\xi) = \left( \eta - \frac{e}{R} \sqrt{1 + \eta^2 \sin \theta} - \frac{b \eta R \xi \eta}{(1 + \eta^2)^{3/2}} \right)^2 + \frac{I_G}{m R^2} \left( 1 + \eta^2 \right) \]  

(1.3)

\[ r(\xi) = \frac{\eta}{b R} + \frac{e}{R} \cos \theta + \frac{1}{\sqrt{1 + \eta^2}}. \]  

(1.4)

Harmonic balance results are given for a special “bead on a wire” case. Numerical simulations varying frequency and amplitude for different characteristic eccentricities
and basins of attraction for meshes of initial conditions and forcing parameters are plotted.

While disks have been explored in many contexts (notably Euler’s disk), targeting the inherent nonlinearities of a large mass eccentricity is novel, as is the generalization of the curve on which the disk rolls. The equations of motion can also be extended to a curve of any shape, providing many options beyond the well-escape behavior explored here.

The experimental comparisons to this analytical and numerical work are detailed in Chapter 4. This chapter introduces the experimental setup with photographs and describes the image processing techniques used to gather information about the motion of the experimental system. Potential experimental shortcomings are also suggested.

1.3 Classic bicycle

Chapter 5 discusses the derivation of the equations of motion for the classical four frame bicycle by giving kinematic definitions for bicycle parts, and using them to derive system energies. Holonomic and nonholonomic constraints are given, and equations of motion are derived using Lagrange’s equations and generalized constraint forces joined by Lagrange multipliers.

When determining equations of motion starting from the general form of d’Alembert’s principle

\[
\left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} - Q^{NP} \right] \delta q = Q^c \delta q = 0,
\]

where the Lagrangian \( L \equiv T - V \), there are multiple options for techniques that will account for generalized forces of constraint. Holonomic constraints can be directly embedded in the system energies. For holonomic or semi-holonomic (exactly integrable) constraints, the Lagrangian can be augmented with the constraint and La-
grange multiplier because the displaced paths are geometrically possible [6]. However, for general nonholonomic constraints, the constraints can be adjoined to d’Alembert’s principle, once conditions on the virtual displacements are determined.

The linear-acceleration form of the constraints leads to the tangency condition for possible states. Decomposing the constraints into independent and dependent parts and using the variation of dependent displacements provides conditions on the displacements under the general velocity constraints. Adjoining these restrictions to d’Alembert’s principle and requiring that the displacement $\delta q$ be arbitrary gives the equations of state for a system with holonomic and nonholonomic constraints [7],

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j \frac{\partial h_j}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial \dot{q}_i},$$

(1.6)

where $i$, $j$, and $k$ are the indices of the generalized coordinates, holonomic constraints, and nonholonomic constraints, respectively, $\lambda$ is the Lagrange multiplier of each
holonomic term, and $\mu$ is the Lagrange multiplier for each nonholonomic term.

After expanding the Lagrangian to the second order, these equations are used to derive the linearized equations, given in the text with state space coefficients in terms of bicycle parameters. The linearized equations of bicycle motion have been published before elsewhere [8, 9], but this method of derivation using Lagrange’s equations has not been published to the author’s knowledge. Another derivation assumed small angles in the presentation of the position vectors [10].

![Figure 1.4: Eigenvalues and stability of benchmark system, with positive real parts of eigenvalues (red), negative real parts of eigenvalues (blue), and imaginary parts of eigenvalues (green).](image)

To confirm that these equations agree with those derived using Newton’s second law, an eigenvalue plot is presented that agrees with that given in [9], using their benchmark values for bicycle parameters.
The equations take the form
\[
\begin{bmatrix}
M_{\phi\phi} & M_{\phi\beta} \\
M_{\beta\phi} & M_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\beta}
\end{bmatrix}
+ v
\begin{bmatrix}
0 & C_{\phi\beta} \\
C_{\beta\phi} & C_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\beta}
\end{bmatrix}
+ \left( g \begin{bmatrix}
K_{\phi\phi}^1 & K_{\phi\beta}^1 \\
K_{\beta\phi}^1 & K_{\beta\beta}^1
\end{bmatrix} + v^2 \begin{bmatrix}
0 & K_{\phi\beta}^2 \\
0 & K_{\beta\beta}^2
\end{bmatrix}\right)
\begin{bmatrix}
\dot{\phi} \\
\dot{\beta}
\end{bmatrix}
= \begin{bmatrix} 0 \\ T_\beta(\phi, \dot{\phi}) \end{bmatrix},
\] (1.7)
where \( T_\beta \) is the input torque representing the effect of control on the steering assembly.

These linearized equations of motion are then used in Chapter 6 to perturb the location of the rear center of mass to show effect of cyclist position. Steering torque control laws are used to show how handlebar torque location affects stability. An instantaneous control law,
\[
T_\beta = k_1 \phi(t) + k_2 \dot{\phi}(t),
\] (1.8)
is proportional to roll and roll rate. To better approximate a human rider, we also introduce a delayed control law,
\[
T_\beta = k_1 \phi(t - \tau) + k_2 \dot{\phi}(t - \tau),
\] (1.9)
where the presence of a time delay \( \tau \) makes necessitates a more sophisticated numerical approach, so semi-analytical methods are used.

These questions of how bicycle geometry affects stability, independent of aerodynamics are novel in that they compare the geometrical considerations between time trial and road bicycles. Bicycle steering is also a new application of delay differential equations.

1.4 Bicycle on rollers

Chapter 7 provides an analytical explanation for the decreased stability experienced by a cyclist riding on rollers. No attempt has been made in the literature to fully derive the equations of motion for a bicycle on rollers, and so both the derivation
and stability investigations are novel. The priorities of Chapter 7 are to alter the derivation of bicycle EOM to account for tire contact with rollers and give the linearized equations of motion, comparing the coefficients to those of a bicycle on flat surface. The equations take the form

\[
\begin{bmatrix}
M_{\phi\phi} & M_{\phi\beta} & M_{\phi\psi} \\
M_{\beta\phi} & M_{\beta\beta} & M_{\beta\psi} \\
M_{\psi\phi} & M_{\psi\beta} & M_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\beta} \\
\ddot{\psi}
\end{bmatrix}
+ v
\begin{bmatrix}
C_{\phi\phi} & C_{\phi\beta} & C_{\phi\psi} \\
C_{\beta\phi} & C_{\beta\beta} & C_{\beta\psi} \\
C_{\psi\phi} & C_{\psi\beta} & C_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\beta} \\
\dot{\psi}
\end{bmatrix}
+ g
\begin{bmatrix}
K_{\phi\phi} & K_{\phi\beta} & K_{\phi\psi} \\
K_{\beta\phi} & K_{\beta\beta} & K_{\beta\psi} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\beta \\
\psi
\end{bmatrix}
= 0,
\]

where a third degree of freedom \(\psi\) is necessary to describe the stability behavior, and the third equation comes from the acceleration form of one of the nonholonomic constraints on the front wheel.

Once the linearized equations of motion are derived, this chapter investigates the stability of this system by plotting eigenvalues and contours of the largest real eigenvalue, as roller parameters are varied over a range of forward velocities. Rec-
ommendations are made about how to configure rollers for optimal stability, based on cyclist and bicycle parameters.

These investigations will be of both academic interest to the dynamicist and practical interest for a cyclist who wishes to achieve the most stable configuration of his or her rollers.
2 Euler’s disk

2.1 Introduction

This chapter investigates a classic problem with relevance to the other systems studied in this thesis. Euler’s disk, or the “rolling coin” problem, has been used to illustrate a classic nonholonomic system that rolls without slipping [1]. The nonlinear equations of motion have been derived and used in a bifurcation and stability analysis [11–13], and in conjunction with a model for dissipation [14, 15]. Equations of motion have also been found using a Lagrangian approach [2].

The methodology used to rework these previously derived results will illustrate the approaches used later in the document in a simplified form. It is the purpose of this derivation to demonstrate the use of Euler angles in three dimensions to specify position vectors, the derivation of equations of motion by both Gauss’s principle and Lagrange’s equations modified for nonholonomic constraints, and the derivation of linearized equations of motion by Taylor expanding the Lagrangian about a constant forward velocity.
2.2 System description

We consider a thin disk rolling on a plane with contact point \( \mathbf{r}_p = x\hat{I} + y\hat{J} \), and yaw, roll, and pitch \( \psi, \phi, \) and \( \theta \) respectively, as shown in Fig. 2.1. To move between \( (\hat{I}, \hat{J}, \hat{K}) \) coordinates that track with the heading of the disk and fixed coordinates \( (\hat{I}_1, \hat{J}_1, \hat{K}_1) \), the transformation matrix \( T_{\psi} \),

\[
\begin{bmatrix}
\hat{I} \\
\hat{J} \\
\hat{K}
\end{bmatrix} = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{I}' \\
\hat{J}' \\
\hat{K}'
\end{bmatrix}
\]

(2.1)
is used. To move between \( (\hat{i}, \hat{j}, \hat{k}) \) coordinates that roll with the disk and \( (\hat{I}', \hat{J}', \hat{K}') \) coordinates, we have the transformation matrix, \( T_{\phi} \).

\[
\begin{bmatrix}
\hat{I}' \\
\hat{J}' \\
\hat{K}'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

(2.2)
The product $T_\psi T_\phi$ converts between the body-fixed coordinate system and the globally fixed coordinate system.

\[
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} = \begin{bmatrix}
\cos \psi & -\sin \psi \cos \phi & \sin \psi \sin \phi \\
\sin \psi & \cos \psi \cos \phi & -\cos \psi \sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

(2.3)

### 2.3 Kinematics

#### 2.3.1 Position and velocity

A position vector to the center of mass $r_O = r_p + r_{O/P}$ is defined in terms of a vector to contact point $P$ and the relative vector to the disk’s center.

\[
r_p = x\hat{\textbf{i}} + y\hat{\textbf{j}}
\]

(2.4)

\[
r_{O/P} = -R\hat{\textbf{k}}.
\]

(2.5)

where $R$ is the disk radius. In fixed coordinates, this becomes

\[
r_O = x\hat{\textbf{i}} + y\hat{\textbf{j}} + T_\psi T_\phi \begin{bmatrix}
0 \\
0 \\
-R
\end{bmatrix}
\]

\[
= \begin{bmatrix}
x - R\sin \psi \sin \phi \\
y + R\cos \psi \sin \phi \\
-R\cos \phi
\end{bmatrix},
\]

(2.6)

and the corresponding velocity vector is

\[
\dot{r}_O = \begin{bmatrix}
\dot{x} - R\dot{\psi} \cos \psi \sin \phi - R\dot{\phi} \sin \psi \cos \phi \\
\dot{y} - R\dot{\psi} \sin \psi \sin \phi + R\dot{\phi} \cos \psi \cos \phi \\
R\dot{\phi} \sin \phi
\end{bmatrix}.
\]

(2.7)

The angular velocities are

\[
\Omega_\psi = \dot{\psi}\hat{\textbf{k}}
\]

(2.8a)

\[
\Omega_\phi = \dot{\phi}\hat{\textbf{i}}
\]

(2.8b)
and the total angular velocity of the system in body fixed coordinates is

\[ \Omega = T^T_\phi T^T_\psi \Omega_\phi + \Omega_\theta \]

\[ = \left\{ \begin{array}{c}
\dot{\phi} \\
\dot{\psi} \\
\dot{\theta} + \psi \sin \phi \\
\psi \cos \phi
\end{array} \right\}. \]  

(2.9)

2.3.2 System energies

The kinetic and potential energies of the system are

\[ T = \frac{1}{2} m \dot{r}_O \cdot \dot{r}_O + \frac{1}{2} \Omega^T \Omega \]

\[ = \frac{1}{2} m \left[ (\dot{x} - R \dot{\psi} \cos \psi \sin \phi - R \dot{\phi} \sin \psi \cos \phi)^2 \\
+ (\dot{y} - R \dot{\psi} \sin \psi \sin \phi + R \dot{\phi} \cos \psi \cos \phi)^2 + R^2 \dot{\phi}^2 \sin \phi^2 \right] \\
+ \frac{1}{2} \left[ I_{xx} \dot{\phi}^2 + I_{yy} (\dot{\theta} + \psi \sin \phi)^2 + I_{zz} \psi^2 \cos \phi^2 \right] \]  

(2.10)

\[ V = mgR \cos \phi, \]  

(2.11)

where \( m \) is the disk mass and

\[ I = \begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix}, \]  

(2.12)

where \( I_{yy} = 1/2mR^2 \) is measured about the rotating axis, and \( I_{xx} = I_{zz} = 1/4mR^2 \) for a thin disk.

2.3.3 Constraints

To enforce the roll without slip constraint, the net instantaneous velocity of the contact point must be zero. This relates the spin of the disk to the velocity of the
contact point as follows,

\[ g_1 = \dot{x} + R\dot{\theta}\cos\psi = 0 \]  \hspace{1cm} (2.13a)

\[ g_2 = \dot{y} + R\dot{\theta}\sin\psi = 0. \]  \hspace{1cm} (2.13b)

2.4 Equations of motion

2.4.1 Lagrange’s equations

When determining equations of motion starting from the general form of d’Alembert’s principle

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} - Q^{NP} \right] \delta q = Q^C \delta q = 0, \]  \hspace{1cm} (2.14)

where the Lagrangian \( L = T - V \), there are multiple options for techniques that will account for generalized forces of constraint. Holonomic constraints can be directly embedded in the system energies. For holonomic or semiholonomic (exactly integrable) constraints, the Lagrangian can be augmented with the constraint and Lagrange multiplier because the displaced paths are geometrically possible [6]. However, for general nonholonomic constraints, the constraints can be adjoined to d’Alembert’s principle, once conditions on the virtual displacements are determined.

The linear-acceleration form of the constraints leads to the tangency condition for possible states. Decomposing the constraints into independent and dependent parts and using the variation of dependent displacements provides conditions on the displacements under the general velocity constraints. Adjoining these restrictions to d’Alembert’s principle and requiring that the displacement \( \delta q \) be arbitrary gives the equations of state for a system with holonomic and nonholonomic constraints [7],

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_k \mu_k \frac{\partial g_k}{\partial q_i}, \]  \hspace{1cm} (2.15)

where \( i \) and \( k \) are the indices of the generalized coordinates and nonholonomic constraints, respectively, and \( \mu \) is the Lagrange multiplier for each nonholonomic term.
For generalized coordinates $x, y, \psi, \phi, \theta,$ and Lagrange multipliers $\mu_1$ and $\mu_2$, this yields seven unknowns for five Lagrange’s equations and two constraint equations.

### 2.4.2 Gauss’s principle

Alternatively, the unconstrained equations of motion can be adjoined to the acceleration form of the constraint equations using Gauss’s principle of least action. For unconstrained equations of the form

$$M(q, t)\ddot{q} = Q(q, \dot{q}, t), \quad (2.16)$$

where $M$ and $Q$ are matrices and $q$ is the vector of generalized coordinates, the unconstrained acceleration $a = \ddot{q} = M^{-1}Q$. If the system is subject to constraints of the form

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t), \quad (2.17)$$

then the constrained equations of motion are subject to an addition constraint force $Q_c$ and take the form

$$M\ddot{q} = Q(q, \dot{q}, t) + Q_c(q, \dot{q}, t). \quad (2.18)$$

According to Gauss’s principle, the Gaussian

$$\mathcal{G} = [\ddot{q} - a]^T M [\ddot{q} - a] \quad (2.19)$$

is minimized over the $\ddot{q}$ that satisfy the constraints. This gives the generalized force of constraint

$$Q_c = K(b - AM^{-1}Q), \quad (2.20)$$

where $K = A^T(AM^{-1}A^T)^+ = M^{1/2}(AM^{-1/2})^+$, and the superscript $+$ denotes the Moore-Penrose generalized inverse [16].
2.4.3 Linearization

To linearize the equations of motion about the upright position, small angular disturbances and a constant velocity $v$ are assumed. Hence,

\begin{align*}
x & \to vt + \epsilon x \\
y & \to \epsilon y \\
\psi & \to \epsilon \psi \\
\phi & \to \epsilon \phi \\
\theta & \to \epsilon \theta - \frac{vt}{R}
\end{align*}

Expanding the Lagrangian to the second order,

$$\mathcal{L} = \left[ -mgR + \frac{1}{2}mv^2 + \frac{1}{2} \frac{v^2}{R^2} I_{yy} \right] + \epsilon \left[ mv\dot{x} - \frac{v}{R} I_{yy} \dot{\theta} \right]$$

$$+ \epsilon^2 \left[ \frac{1}{2} m\ddot{x}^2 + \frac{1}{2} m\ddot{y}^2 + mR \dddot{\psi} + \frac{1}{2} I_{zz} \dot{\psi}^2 - \left( mRv - \frac{v}{R} I_{yy} \right) \dot{\psi} \dot{\phi} + \frac{1}{2} (mR^2 + I_{xx}) \dddot{\phi}^2 \\
+ mRv \dddot{\psi} + \frac{1}{2} I_{yy} \dddot{\theta}^2 + \frac{1}{2} mgR \dddot{\phi}^2 \right] + \mathcal{O} (\epsilon^3).$$

A similar expansion for the constraints yields,

$$g_1 = \epsilon \left( \dot{x} + R \dot{\theta} \right) + \epsilon^2 \frac{1}{2} v \psi^2 + \mathcal{O} (\epsilon^3)$$

$$g_2 = \epsilon \left( \dot{y} - v \psi \right) + \epsilon^2 R \dot{\theta} \psi + \mathcal{O} (\epsilon^3).$$

Using either of the above methods for finding the constrained equations of motion, we find yaw and roll to a linear approximation,

$$I_{zz} \dddot{\psi} - \frac{v}{R} I_{yy} \dddot{\phi} = 0$$

$$(mR^2 + I_{xx}) \dddot{\phi} + \left( mRv + \frac{v}{R} I_{yy} \right) \dot{\psi} - mgR \dot{\phi} = 0$$
Recalling that $I_{xx} = I_{zz} = \frac{1}{4}mR^2$ and $I_{yy} = \frac{1}{2}mR^2$ for a thin disk, the system can be written,

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\dot{\psi} \\
\dot{\phi} \\
\end{bmatrix}
+ \begin{bmatrix}
0 & -2\alpha \\
\frac{6}{5}\alpha & 0 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{\psi} \\
\ddot{\phi} \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & -\frac{4}{5}\beta \\
\end{bmatrix}
\begin{bmatrix}
\psi \\
\phi \\
\end{bmatrix}
= 0, \quad (2.25)
\]

where $\alpha \equiv \frac{v}{R}$ and $\beta \equiv \frac{g}{R}$.

2.5 Analysis

It is well established that steady motions arise when $\phi = \phi_0$ is constant, and paths are circular (when the angular velocity about $\hat{k}$ is nonzero) [11]. Figures 2.2–2.4 give comparisons of the numerically simulated nonlinear and linearized equations of motion for a small initial disturbance.

![Figure 2.2: Time series plot of Euler's disk for $\psi$ and $\phi$ (rad) vs. $t$ (s), where parameters are $m = 1$ kg, $R = 0.3$ m, $v = 2$ m/s, and $\phi_0 = \pi/20$ rad, solid blue line (nonlinear), dashed red line (linear approximation).](image)

Figure 2.2 shows behavior for $\psi$ and $\phi$ as the disk completes one circular path. The linear approximation is fairly close to the nonlinear simulation, but it slightly
underestimates the frequency of oscillation. For an initial angular displacement of \( \phi = \pi/20 \), the disk oscillates about a roll angle of \( \phi = 0.2 \).

The plot of \( \dot{x} \) vs. \( \dot{y} \) in Fig. 2.3 shows that the velocity remains a constant 2 m/s for the linear case, but increases slightly for the nonlinear case as the disk gains a little kinetic energy as the disk roll angle increases with each oscillation. (Note that the linear velocities were computed using simulated values of \( \psi \), such that \( \dot{x} = v \cos \psi \) and \( \dot{y} = v \sin \psi \).)

Figure 2.4 shows the path of the disk for both linear and simulated nonlinear equations. Because the velocity increases each oscillation for the nonlinear equations, the approximately circular path is slightly longer than in the linear approximation.

The eigenvalues in Fig. 2.5 indicate that for \( R = 0.3 \) m, the linear system is stable for \( v > 1 \) m/s.
Figure 2.4: Plot of $x$ and $y$ (m), with same parameters as Fig. 2.2.

Figure 2.5: Eigenvalues of linearized Euler’s disk equations for $R = 0.3$ m, with real parts (blue circles) and imaginary parts (red dots).
3 Equations of motion for an eccentric disk

3.1 Introduction

While extensions of Euler’s disk have been explored by many authors, the existing literature focuses on a disk that rolls on a horizontal surface. They are concerned with both a rough, roll without slip surface [12, 17], and with cases where slipping occurs [11]. Euler’s disk can also be viewed as a control problem, where feedback is used to stabilize the motion [18, 19]. However, new complexities arise when the surface is allowed to curve and a mass imbalance is introduced.

The present chapter seeks to examine a related, but fundamentally different problem in which the disk motion is constrained in two dimensions, but the surface may be curved, and the center of mass may be eccentric. With the addition of a curved surface, there is a possibility of escape from a potential well, a problem that is characterized by a potential barrier, where escape to a neighboring attractor or to infinity is possible. Such a transition is widely present in natural phenomena and can be induced either deterministically or stochastically [4]. This problem has been investigated for the jump response of a Duffing oscillator, the capsizing of a naval vessel,
snap-through buckling of arches, and bistable regions in atomic force microscopy [5, 20, 21]. Much of this work has been dedicated to predicting when an escape will occur [3, 5]. A cubic potential well, as we choose here, has also been examined, in exploring chaos [22, 23]. Many of these investigations have used Melnikov analysis to predict potential well escape [4, 24]. This paper seeks to examine the effect a mass eccentricity has on the escape from a potential well, and more generally the effect on the dynamic behavior. Rather than attempting to predict escape, qualitative changes in the system dynamics are of interest here. While studies of rolling disks and potential well escape are numerous, the inclusion of a mass imbalance provides an interesting and new feature. The eccentricity is shown to have a significant impact on the dynamical behavior of the system.

The organization of this paper is as follows. Section 3.2 states the specific problem to be explored and gives the system parameters. The mathematical model used to describe the system is detailed in Sec. 3.3. Kinematic relationships are derived and used with Lagrange’s equations to find the equation of motion for the system in Sec. 3.4. A simplified equation of motion for the bead on a wire example is also given. Section 3.5 analyzes the system through approximate analytical and numerical techniques. The bead on a wire case is investigated by generating frequency responses from the harmonic balance method and a numerical frequency sweep. For the full eccentric system, the linearized natural frequencies as a function of eccentricity, initial position basins of attraction, forcing parameter basins of attraction, frequency sweeps, and amplitude sweeps are investigated. Finally, concluding remarks are provided in Sec. 3.6.

3.2 System description

Figure 3.1 shows a surface, described by the function $y(x)$ and subject to a base excitation $d(t)$. The disk rolls through an angle $\theta$, measured from the vertical, and
the center of mass $G$ is a distance $e$ from the geometric center of a disk of radius $R$. Point $A$ describes the center of the body-fixed reference frame attached to the surface, and $C$ is the contact point of the disk.

### 3.3 Kinematics

This section describes the intermediate steps necessary to derive equations of motion for the system, including the constraint equations that reduce the degrees of freedom to one. The position and velocity vectors to the center of mass are determined first, keeping a general expression for the curvature of the surface. Then the expressions for $\theta$ and $\dot{\theta}$ utilizing a roll without slip condition are given, allowing the energy expressions to be written solely in terms of the position $x$. Lagrange’s equations are then used to derive the equation of motion, given in a general form that depends on three complex functions of $x$. 

---

**Figure 3.1**: Line drawing of the rolling disk with an eccentric mass.
3.3.1 Position and velocity

The position vector to the center of mass is given by the sum of three vectors,

\[ \mathbf{r}_G = \mathbf{r}_A + \mathbf{r}_{C/A} + \mathbf{r}_{C/C}. \]  

(3.1)

Using the geometry of this problem, these become

\[ \mathbf{r}_G = \left( d + x + e \sin \theta - \frac{R y_x}{\sqrt{1 + y_x^2}} \right) \hat{i} + \left( y(x) + e \cos \theta + \frac{R}{\sqrt{1 + y_x^2}} \right) \hat{j}, \]  

(3.2)

where \( y_x \) is the partial derivative of \( y \) with respect to \( x \). The velocity of the center of mass is found by taking the time derivative and is given by

\[ \dot{\mathbf{r}}_G = \left( \dot{d} + \dot{x} + e \dot{\theta} \cos \theta - \frac{R \dot{x} y_{xx}}{(1 + y_x^2)^{3/2}} \right) \hat{i} + \left( \dot{y}_x - e \dot{\theta} \sin \theta - \frac{R \dot{x} y_x y_{xx}}{(1 + y_x^2)^{3/2}} \right) \hat{j}, \]  

(3.3)

where an overdot indicates a derivative with respect to time.

3.3.2 System energies

From the position and velocity vectors given in Subsection 3.3.1, the kinetic and potential energies of the system are given by

\[ T = \frac{1}{2} m \left[ \left( \dot{d} + \dot{x} + e \dot{\theta} \cos \theta - \frac{R \dot{x} y_{xx}}{(1 + y_x^2)^{3/2}} \right)^2 + \left( \dot{y}_x - e \dot{\theta} \sin \theta - \frac{R \dot{x} y_x y_{xx}}{(1 + y_x^2)^{3/2}} \right)^2 \right] + \frac{1}{2} I_G \dot{\theta}^2, \]  

(3.4)

\[ V = m g \left( y + e \cos \theta + \frac{R}{\sqrt{1 + y_x^2}} \right). \]  

(3.5)
Knowing $\theta$ can be written as a function of $x$, and using Eqn. (3.12) to eliminate $\dot{\theta}$ from the energies, they become

$$\mathcal{T} = \frac{1}{2}m \left[ (\dot{\theta} + \dot{x} p(x))^2 + \dot{x}^2 q(x) \right]$$  \hspace{1cm} (3.6)$$

$$\mathcal{V} = mg R r(x), \hspace{1cm} (3.7)$$

where

$$p(x) = 1 + \frac{e}{R} \sqrt{1+y_x^2} \cos \theta - \frac{R y_{xx}}{(1+y_x^2)^{3/2}}, \hspace{1cm} (3.8)$$

$$q(x) = \left( y_x - \frac{e}{R} \sqrt{1+y_x^2} \sin \theta - \frac{R y_x y_{xx}}{(1+y_x^2)^{3/2}} \right)^2 + \frac{I_G}{m R^2 (1+y_x^2)}, \hspace{1cm} (3.9)$$

$$r(x) = \frac{y}{R} + \frac{e}{R} \cos \theta + \frac{1}{\sqrt{1+y_x^2}}, \hspace{1cm} (3.10)$$

### 3.3.3 Constraints

The holonomic roll without slip constraint can be found by equating the distance along the path with the corresponding portion of the disk circumference, $\int \sqrt{1+y_x^2} dx = R \theta$. In this 2D problem, only in-plane motion is considered. The $\theta$ coordinate tracks with the $x$ coordinate and provides a holonomic constraint. Solving for $\theta$ and its derivative,

$$\theta = \frac{1}{R} \int \sqrt{1+y_x^2} dx \hspace{1cm} (3.11)$$

$$\dot{\theta} = \frac{\dot{x}}{R} \sqrt{1+y_x^2}. \hspace{1cm} (3.12)$$

The expression for $\theta$ contains an elliptic integral, which will be approximated in the following section, but the expression for $\dot{\theta}$ can be substituted exactly.
3.4 Equations of motion

3.4.1 Lagrange’s equations

Defining the Lagrangian \( \mathcal{L} = T - V \), Lagrange’s equation for a single generalized coordinate \( x \) is

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = Q,
\]

which can now be applied to obtain the equation of motion. In the simplified form, the Lagrangian is,

\[
\mathcal{L} = \frac{1}{2} m \left( \dot{x}^2 + \dot{p}(x)^2 + \dot{q}(x)^2 \right) - m g R r(x).
\]

(3.14)

Letting \( Q = -c \dot{x} \), to approximate the dissipation in the physical system, the equation of motion is

\[
\ddot{x}(p^2 + q) + \frac{c}{m} \dot{x} + \dot{x}^2 (p p_x + \frac{1}{2} q_x) + g R r_x = -\ddot{d},
\]

(3.15)

where \( p_x, q_x, \) and \( r_x \) are partial derivatives of these functions with respect to \( x \).

Eqn. 7.54 is the general equation of motion for any curve \( y(x) \), but in this work, the case where \( y(x) = a x^3 + b x^2 \) was considered. In Section 3.5, a harmonic balance approximate solution is proposed, and numerical frequency and amplitude sweeps are used to explore the behavior of this system.

3.4.2 Nondimensionalization

It is convenient to write the equation of motion using a nondimensional coordinate \( \xi \), which we nondimensionalize by the parameter \( b \), having units of 1/length. Letting \( \xi \equiv bx \) and \( \eta \equiv by \), the cubic curve becomes \( \eta = \alpha \xi^3 + \xi^2 \), where \( \alpha \equiv a/b^2 \).

The parameter \( \alpha \) determines the shape of the cubic function, and for mathematical convenience, we choose \( \alpha = -2/3 \), placing the local maximum of \( \eta \) at \( \xi = 1 \).
The equation of motion in nondimensional form is given by

$$\ddot{\xi}(p^2 + q) + \frac{c}{m} \dot{\xi} + \xi^2 (pp + \frac{1}{2} q) + b^2 R g r_{\xi} = -b \ddot{d},$$  \hspace{1cm} (3.16)$$

where

$$p(\xi) = 1 + \frac{e}{R} \sqrt{1 + \eta^2} \cos \theta - \frac{bR \eta \xi}{(1 + \eta^2)^{3/2}}$$  \hspace{1cm} (3.17)$$

$$q(\xi) = \left( \eta - \frac{e}{R} \sqrt{1 + \eta^2} \sin \theta - \frac{bR \eta \xi \eta}{(1 + \eta^2)^{3/2}} \right)^2 + \frac{I_G}{mR^2} (1 + \eta^2)$$  \hspace{1cm} (3.18)$$

$$r(\xi) = \frac{\eta}{bR} + \frac{e}{R} \cos \theta + \frac{1}{\sqrt{1 + \eta^2}}$$  \hspace{1cm} (3.19)$$

In order to conveniently use Eqn. (3.11) to eliminate $\theta$ and reduce the degrees of freedom to one, the value of the integral must be considered. For the case when $e = 0$, all terms containing $\theta$ disappear, and the energies can be written exactly. However, when $e$ is nonzero, the elliptic integral can be numerically approximated. In nondimensional form, the constraint equation is

$$\theta = \frac{1}{bR} \int \sqrt{1 + \eta^2} \, d\xi.$$  \hspace{1cm} (3.20)$$

Because the section of the curve of physical interest was on the interval $[-1, 2]$, the integrand was expanded about $\xi_0 = 1/2$. Integrating the first order Taylor expansion gives the approximation

$$\theta \approx \frac{\sqrt{5}}{2bR} \xi,$$  \hspace{1cm} (3.21)$$

which closely matches the exact result on the interval of interest, as illustrated in Fig. 3.2.

3.5 Analysis

This section uses numerical and approximate analytical techniques to explore the system behavior. Harmonic balance was used in the point mass system to approximate
the response to a range of frequencies. This was then compared with numerically simulated amplitude and frequency sweeps. Basins of attraction were also plotted for unforced and forced cases.

3.5.1 Reduction to point mass case

The system can be made much simpler by considering the case where $e \to 0$ and $R \to 0$. This represents the case of a bead on a wire, where there is only one degree of freedom $\xi$. The simplified equation of motion for this case is

$$
\ddot{\xi} (1 + \eta \xi^2) + \frac{c}{m} \dot{\xi} + \dot{\xi}^2 \eta \xi + b g \eta \xi = -b \ddot{d},
$$

which is nonlinear for all curved surfaces. Harmonic balance is a convenient approximate analytical technique that can be used to estimate the frequency response of the system. Assuming a solution form $\xi = \tilde{a} \cos \Omega t + \tilde{b} \sin \Omega t$, when $\ddot{d} = \Gamma \sin \Omega t$, and substituting into the equation of motion, the terms multiplying the first harmonics can be collected. Squaring and adding the coefficients of the cosine and sine terms
Figure 3.3: Approximate amplitude response from harmonic balance results for \( \Gamma = 0.29 \, \text{m/s}^2 \), \( c = 0.05 \, \text{N}\cdot\text{s/m} \), and \( m = 0.1 \, \text{kg} \).

Figure 3.4: Forward (green line) and reverse (blue line) numerical frequency sweeps, for same parameters as Fig. 3.3.

\[
\frac{729}{64} \alpha^4 \Omega^4 \bar{r}^{10} + \frac{27}{2} \alpha^2 \Omega^4 \bar{r}^{8} + \left( \frac{27}{4} \alpha^2 \Omega^4 + 4 \Omega^4 - \frac{27}{2} b g \alpha^2 \Omega^2 \right) \bar{r}^{6} + \left( 4 \Omega^4 - 8 b g \Omega^2 \right) \bar{r}^{4} + \left( \Omega^4 + \frac{c^2}{m^2} \Omega^2 - 4 b g \Omega^2 + 4 b^2 g^2 \right) \bar{r}^{2} = 0,
\]

where nondimensional amplitude response \( \bar{r}^2 = \tilde{a}^2 + \tilde{b}^2 \). Numerically solving for the roots of the tenth order polynomial for varying \( \Omega \) yields the frequency response shown in Fig. 3.3 and can be compared to the numerical simulation result in Fig. 3.4.

3.5.2 Eccentric disk case

This section considers the added complexities of a mass imbalance and the rigid body rotation of the disk. Figure 3.5 shows the potential energy curves for different eccentricities. While the case where \( e = 0 \) has just one local minimum, it can be seen that as the eccentricity increases, multiple local minima develop, and the position that corresponds to \( x = \theta = 0 \) loses stability. For the geometry of the experimental system, there are three statically stable equilibria when \( e = 0.5R \). Note that the
Figure 3.5: Potential energy of system when $e = 0$ (blue solid line), $e = 0.1R$ (green dashed line), and $e = 0.5R$ (red dash-dot line) for $m = 0.1$ kg and $R = 9.5$ mm.

radius was chosen so that the unstable equilibrium of the non-eccentric case would stabilize.

The linearized natural frequencies can be found by evaluating the linearized stiffness and inertial term at the stable equilibria. Since the linearized stiffness is given by the second derivative in $x$ of the potential energy $V$, the linearized natural frequencies are given by,

$$
\omega_n = \frac{1}{b} \sqrt{\frac{g R r \xi_e}{p^2 + q}}_{\xi = \xi_e}.
$$

(3.24)

Figure 3.6 shows the values of the linearized natural frequencies as the eccentricity is increased. This figure shows the linear natural frequencies increase, and new stable equilibria form as the eccentricity increases relative to the radius. For $e/R < 0.15$, the natural frequency of a single stable equilibrium decreases to a minimum, then two stable equilibria with very close natural frequencies form. This is an effect of the potential energy well essentially widening, as illustrated by the green dashed line in Fig. 3.5. After $e/R > 0.25$, there are three distinct increasing linearized natural
Figure 3.6: Linearized natural frequency as a function of eccentricity for $m = 0.1$ kg and $R = 9.5$ mm.

frequencies corresponding to the three potential wells for these eccentricity values.

Figure 3.7 shows the basins of attraction for the unforced system. For these system parameters, there are three clear basins of attraction depending on the initial position and velocity of the contact point. Alternatively, for a large enough magnitude of the initial position and velocity, escape to infinity occurs, as indicated by black points on the plot. For certain regions, the end behavior is very sensitive to the initial conditions, resulting in the striations between the two potential wells nearest zero. It is interesting to note that there are initial conditions that result in escape between the basin centered at 1.00 and the other two basins. In other words, at this initial position, there is a range of escape velocities between two basins.

Basins of attraction for the forced system are shown in Fig. 3.8, where the forcing amplitude and frequency are varied. Each simulation is plotted with a colored point, black for escape, blue for period-one behavior, red for period-two behavior, and yellow for other behavior (including quasi-periodic and chaotic behavior) that does not escape. The period-one behavior gives way to chaotic behavior for the smallest
amplitude of forcing around 3.5 Hz, corresponding with the resonant frequency also shown in Fig. 3.16 c). For higher frequencies, around 7.5 Hz, a period doubling bifurcation is indicated in the transition from period-one to period-two behavior as the forcing amplitude is increased beyond about 2 m/s². In most regions, the end behavior of the system is sensitive to slight variations in the forcing parameters.

Figs. 3.9–3.12 show amplitude sweeps where the forcing amplitude was increased gradually to simulate quasi-static behavior. The plots show a definite softening behavior, consistent with the point mass case discussed earlier, where jump bifurcations occur at higher frequencies in the forward sweep than in the reverse. In general, a small eccentricity, as in Fig. 3.10, is a destabilizing effect, as escape occurs at much lower frequencies near resonance, than when there is no mass imbalance. Increasing the eccentricity further, however, as in Fig. 3.12, has a stabilizing effect, allowing the system to oscillate without escaping for much higher frequencies. The zero eccentricity case displays a period doubling bifurcation for an excitation frequency of
Figure 3.8: Basins of attraction for forced system, for $e = 0.5R$, $R = 9.5$ mm, $m = 0.1$ kg, and $c = 0.05$ N·s/m, where behaviors include escape (black), period-one (blue), period-two (red), and other (yellow).

2.5 Hz. The period-two behavior then returns to period-one, as the amplitude is further increased. The $e = 0.1R$ system also displays a period doubling bifurcation, but also has some short periods of quasi-periodicity, which were not present in the $e = 0$ system. Much richer dynamical behavior is present in the $e = 0.5R$ system. At resonant frequencies, shown in Fig. 3.12 b) and c), period-two, period-three, chaotic behavior, and even higher periods are present. Plot d) in the forward sweep even shows that oscillation in the potential well centered at 0.1 m is possible. Additionally, well crossover occurs between the wells closest to zero in b) and c).

Interrogating the model with a frequency sweep clearly demonstrates the softening behavior of the balanced system. Fig. 3.13 shows different peak amplitudes at different frequencies. In the forward frequency sweep, for the greatest forcing amplitude c), the system jumps to a peak response at 1.9 Hz, whereas for the backward frequency sweep, the amplitude reaches a maximum at about 1.3 Hz, with an amplitude of 1, while the forward sweep maximum response amplitude is about half that. The addition of a mass imbalance has interesting implications for the dynamical
Figure 3.9: Amplitude sweeps for $e = 0$ system for forcing frequencies 0.5, 1.1, 2.0, and 2.5 Hz.

Figure 3.10: Amplitude sweeps for $e = 0.1R$ system for forcing frequencies 1.0, 1.5, 2.0, and 2.5 Hz.

behavior of the system.

Figure 3.14 shows that when $e = 0.1R$, more complicated jump phenomena occur, owing to a flattening of the local minimum in the potential energy (Fig. 3.5). In the forward sweep part c), a jump occurs at 1.7 Hz, with a strong response continuing until 1.9 Hz, but for the reverse sweep, a jump occurs at 1.6 Hz. This is in contrast to the clear softening behavior for the case where the center of mass is balanced.
When the eccentricity is increased to $e = 0.2R$, two local minima in the potential energy curve form. Fig. 3.15 shows the response when the disk starts in the deepest potential well. Rich dynamical behavior is present at this eccentricity, including periods of chaos and inter-well solutions not present when $e = 0.1R$, but the shape of the system response remains similar. Period-three behavior occurs in the reverse sweeps, and also briefly in the forward sweep part c).
Increasing the eccentricity further to $e = 0.5R$ has a great impact on the resonant frequencies of the system, shifting them to between 3 and 5 Hz, indicated in both Fig. 3.6 and Fig. 3.16. The forward sweep in part c) shows a jump and brief interwell behavior at 3.4 Hz. Parts b) and c) of the reverse sweep show a jump from the equilibrium at $\xi = 0.262$ to that at $\xi = -0.235$, as well as a smaller secondary resonant response at 1.3 Hz.
3.6 Conclusion

The nonlinear dynamics of a two dimensional disk rolling on a cubic curve were investigated here for a disk with a mass imbalance. The general equation of motion for an eccentric disk rolling on a curved surface were derived. Next, a specific curved surface was examined. Potential energy curves and basins of attraction were plotted for the unforced system, showing three distinct basins for the specific pa-
rameters chosen. Amplitude sweeps were conducted both forward and backward for forcing frequencies near resonance. Forward and backward frequency sweeps were also conducted for several amplitudes that best displayed varying behaviors. These numerical studies were performed for the cases when the mass eccentricity was zero, 0.1R, 0.2R, and 0.5R to better understand the effect of slight and more significant mass imbalances on the system. The system was shown to be highly nonlinear and exhibit softening behavior in the frequency sweeps. Intermittent chaos, period-two, and period-three behavior were also observed, notably when $e = 0.2R$.

The introduction of a mass eccentricity increased the number of fixed points in the system and substantially changed the dynamic behavior, by causing periods of chaos and shifting the resonant frequencies of the system.
Experimental investigations of an eccentric disk

4.1 Introduction

To assess the validity of the proposed analytical model in Chapter 3, an experimental investigation was performed. A disk and guides roll on a cubic track without slipping, and both unforced and sinusoidally forced cases were considered. Image processing software was used to track the location and angle of the disk, and also the relative location of the shaker base and platform.

For the curve defined by $\eta = \alpha \xi^3 + \xi^2$, the equation of motion for the disk is

$$\ddot{\xi}(p^2 + q) + \frac{c}{m} \dot{\xi} + \dot{\xi}^2 (p p_\xi + \frac{1}{2} q_\xi) + b^2 R g r_\xi = -b \ddot{d},$$

(4.1)

where

$$p(\xi) = 1 + \frac{e}{R} \sqrt{1 + \eta^2} \cos \theta - \frac{b R \eta \xi}{(1 + \eta^2)^{3/2}}$$

(4.2)

$$q(\xi) = \left( \eta - \frac{e}{R} \sqrt{1 + \eta^2} \sin \theta - \frac{b R \eta \eta \xi}{(1 + \eta^2)^{3/2}} \right)^2 + \frac{I_G}{m R^2} (1 + \eta^2)^2$$

(4.3)

$$r(\xi) = \frac{\eta}{b R} + \frac{c}{R} \cos \theta + \frac{1}{\sqrt{1 + \eta^2}}$$

(4.4)
Over the range of the curve that is of interest, for the experimental parameters used,

\[ \theta \approx \frac{1}{bR} \xi. \]  

(4.5)

Table 4.1 gives the experimental parameters used in this chapter.

4.2 Experimental design

Figure 4.1 shows the experimental set up from different angles. A center disk provides the bulk of the system mass, while spools connected by a threaded rod contact a cubically curved track. Because the theoretical system only has two dimensions, the track, while in three dimensions, allows only in-plane motion. The roll-without-slip constraint is realized by means of a rubber tape applied to both track and spools. Both the disk and spools were constructed from ABS plastic using a rapid prototyping machine.

The mass imbalance is provided by means of brass weights that can be inserted in one of three circular extruded cuts and secured with a set screw. Figure 4.2 shows the different radial positions available for additional masses. This setup allows eccentricities from $0.1R$ to $0.5R$. 

Figure 4.1: Pictures of the eccentric disk experimental set up.
4.3 Experimental considerations

![Figure 4.3: Picture of the eccentric disk contact surface between track and rolling apparatus](image)

There are a few necessary considerations in adapting the theoretical model to the experiment. Important among these are the theoretical roll-without-slip condition and the two dimensional nature of the model. The interface between the track and
spool is shown in Fig. 4.3. While rubber tape has been adhered to both faces to encourage rolling without slipping, further testing will be required to determine if the static friction is sufficient for the desired forcing. The hyperbolic shape of the spools is designed to restrict the motion to two dimensions, while minimizing the viscous damping. However, the damping in the experimental setup is much higher than in the theoretical analysis of the previous chapter.

4.4 Image tracking

![Experimental setup with the image tracking points.](image)

**FIGURE 4.4:** Experimental setup with the image tracking points.

Data was collected with a Flip video camera with 30 frames per second. Image tracking was performed with a package for MATLAB provided by the Hedrick Lab at UNC [25]. Use of the software was fairly straightforward, since the apparatus approximately moves in only two directions. First the data was corrected for camera movement with the stationary dot on the far right in Fig. 4.4. Next, the movement of the shaker platform was used to isolate the curve in the $\xi$-$\eta$ plane.
The remaining two dots, at the geometric center of the disk and toward the edge were plotted to calculate the position of the contact point and the angle of rotation. To do this, the local minimum of the motion minus a disk radius was used as the center of the coordinate plane, the $\xi$ positions were flipped to reflect the curve from the previous chapter, and the pixel positions were converted to meters, then dimensionless units of distance. The contact point was calculated to be the projection of the center point onto the curve.

4.5 Experimental Results

![Figure 4.5](image)

**Figure 4.5**: Data from image processing of the unforced case with the calculated curve (black), tracked center point (blue), tracked edge point (green), and projected contact point (red), for experimental parameters given in Tab. 4.1.

Figure 4.5 shows a plot of the corrected and calculated points associated with the center, edge, and contact points. It can be seen that for the unforced case, the disk remains in the potential well and comes to rest (blue point) right of center, consistent with the off-center potential minima caused by the eccentric center of mass.

A comparison of the unforced experiment with the simulated model is shown in Fig. 4.6, where the simulation parameters have been chosen to most closely agree
Figure 4.6: Comparison of experiment with theoretical simulation for $\theta_0 = 3/4\pi$, with simulation parameters $c = 2.2$ kg/s, $\xi_0 = -0.1673$, $\dot{\xi}_0 = 0$ s$^{-1}$, and those given in Tab. 4.1.

with the experiment. While the amplitude of oscillations agrees well, the natural frequency of the simulation is slightly greater than the experiment. The experiment also decays faster for later times, even when the damping coefficient of the simulation is chosen to approximate the physical system as closely as possible for the first few oscillations.

Figure 4.7 shows the shifted potential energy curve for $\theta_0 = 3/4\pi$ and experimental parameters. Notably the potential well at $\xi = 0.042$ differs from the theoretical analysis in Chapter 3, since $\theta_0$ depends on the precise initial placement of the disk. For this comparison, $\theta_0 = 3/4\pi$ fit the data best.

Figures 4.8–4.9 illustrate a forced experiment at $f = 0.7$ Hz, near the resonant frequency of the system. The spread of the blue and green experimental paths in Fig. 4.8 show that the roll-without-slip condition was not perfectly met. In a perfect demonstration, the data would form a single curve. There is also probable error from the image processing, since the tracking points blur at the frame rate of the camera used.
The comparison with a simulation in Fig. 4.9 shows slight differences in the amplitude and frequency of the response. The simulation shows more variation in amplitude and a slightly lower frequency.

To improve on this experiment, precise determination of $\theta_0$ would be beneficial. Using a camera with a faster frame rate would help smooth the data points and reduce the blur that required manual image tracking of the points at high velocities. More accurate forcing parameters could also be used in the simulation if information from an accelerometer or the data points from the shaker base were analyzed and applied, to see if the actual forcing parameters agree with the signal input. Further experimental cases including frequency sweeps and multi-frequency excitation would be interesting extensions.
**Figure 4.8:** Data from image processing of the $f = 0.7$ Hz forced case with the calculated curve (black), tracked center point (blue), tracked edge point (green), and projected contact point (red) for experimental parameters given in Tab. 4.1.

**Figure 4.9:** Comparison of experiment with theoretical simulation for $f = 0.7$ Hz and $\theta_0 = 3/4\pi$, with simulation parameters $c = 2.2$ kg/s, $\xi_0 = 0.1438$, $\xi_0 = 0.3$ /s, $\Gamma = 1.0$ m/s$^2$, and those given in Tab. 4.1.
### Appendix: Experimental parameters

Table 4.1: Experimental eccentric disk parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
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<td></td>
</tr>
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</tr>
<tr>
<td>scale</td>
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<td>disk parameters</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$R$</td>
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</tr>
<tr>
<td>eccentricity ratio</td>
<td>$e/R$</td>
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</tr>
<tr>
<td>mass</td>
<td>$m$</td>
<td>0.4427 (\text{kg})</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>$I_G$</td>
<td>6.7287e-04 (\text{kg m}^2)</td>
</tr>
</tbody>
</table>
5

Bicycle equations of motion

5.1 Introduction

While many people are capable of riding a bicycle without difficulty, the reasons for a bicycle’s self-stability are difficult to intuit. For this reason, the issue of bicycle stability has continued to be of interest to scientists [10], cyclists, and motorcycle enthusiasts, ever since the first description of the equations of motion in 1899 independently by F. J. W. Whipple and Emmanuel Carvallo [8]. Since then, new considerations, including human control and tire and frame compliance have been studied [26–29]. Although the nonlinear equations of motion and their linearization about a forward velocity have long been established, the presence of many physical parameters makes stability a complicated issue. These physical parameters have been measured experimentally, and experimental bicycle behavior was compared with the linear theory [30]. In general, a bicycle can be self stable independent of gyroscopic or caster effects, if the front assembly turns in the direction of a fall [31]. Incorporating human control models is useful for determining a more realistic stability model. Bicycle handling has been explained as the need to lean a bike into a turn
by means of either counter-steering or hip thrusts, with the bike initially turning in
the opposite direction [32].

Whipple’s derivation involved resolving forces, then applying D’Alembert’s prin-
ciple and summing moments [8]. Interest in stability continued in the latter half of
the twentieth century [33–35]. Useful kinematical considerations and roll without
slip constraints were elucidated by Kane [36]. Döhring, then Weir also derived lin-
earized equations of motion using Newton’s second law [10]. Hand used a Lagrangian
approach to derive linearized equations of motion, but his derivation differs from the
present one in the linearization process and selection of generalized coordinates [10].
The present approach seeks to show the derivation of the fully nonlinear Lagrangian
expression, using Euler transformation angles to give the complex relationships be-
tween reference frames, before expanding to the second order for linearization pur-
poses. In this way, the present derivation is self-contained and can be used to derive
the full nonlinear Lagrangian expression.

It is the goal of this chapter to use the bicycle system to provide an illustrative ex-
ample relevant to the teaching of physics of how to implement Lagrange’s equations,
modified for the inclusion of holonomic and nonholonomic constraints. It is worth
noting that simpler, more fundamental examples of nonholonomic systems exist, such
as Euler’s disk [2] or the Chaplygin sleigh [37]. This problem additionally provides
an interesting example of representing the kinematics in terms of Euler angle trans-
formation matrices, greatly streamlining the mathematics. Taylor expanding the
Lagrangian allows the linearized equations of state to be derived, which can then be
reduced to the two degrees of freedom that determine a bicycle’s self-stability, the roll
and steer angles. Finally, the simplified equations are presented in state-space form,
with mass, damping, and stiffness matrices consisting of complex combinations of the
physical parameters, and a sample eigenvalue plot and time series representations of
behavior are presented.
5.2 System description

This section serves to orient the reader to the angular displacements and relevant points on the illustration. Figure 5.1 shows an illustration of a bicycle with four frame components. The model assumes the two wheels make knife-edge roll-without-slip contact with a flat plane. The position vector \( \mathbf{r}_A = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} \) tracks the contact point of the rear wheel, \( A \), in the globally fixed coordinates \( \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}} \). The authors found it convenient to consider the rear wheel in body-fixed coordinates \( \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \), that yaw and pitch with the rear wheel, as shown in Fig. 5.2. The rear frame and wheel yaw with angle \( \psi \), a rotation about \( \hat{\mathbf{K}} \), and roll with angle \( \phi \), a rotation about \( \hat{\mathbf{i}} \), which points in the direction headed by the rear wheel. As shown in Fig. 5.1, the rear frame pitches with angle \( \gamma \), and the rear wheel pitches (spins) with angle \( \theta_O \), both about \( \hat{\mathbf{j}} \). The coordinates \( \mathbf{i'}, \mathbf{j'}, \mathbf{k'} \) pitch with the rear frame. The front assembly is described in a different set of body-fixed coordinates shown in Fig. 5.3, \( \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}} \), defined with
\( \hat{w} \) as the axis of rotation for the steering angle \( \beta \), and \((\hat{u}', \hat{v}', \hat{w}')\), shifted by the caster angle \( \alpha \), so that \( \hat{u}' \) points in the direction of the front wheel precession. The front wheel spins with angular displacement \( \theta_P \), about \( \hat{v} \).

### 5.3 Kinematics

This section introduces a series of transformation matrices before deriving the position and velocity vectors. We find it convenient to give position vectors in the globally fixed coordinates and angular velocities in the plane of motion. The following Euler angle transformation matrices are useful in converting between coordinate systems. The transformation matrices, \( T_\psi \), \( T_\phi \), and \( T_\gamma \), for the rear assembly are given in the following expressions,

\[
T_\psi = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad T_\phi = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix},
\]

\[
T_\gamma = \begin{bmatrix}
\cos \gamma & 0 & \sin \gamma \\
0 & 1 & 0 \\
-\sin \gamma & 0 & \cos \gamma
\end{bmatrix}.
\]

\[ (5.1) \]
These transformations are used to move between the rear frame coordinate systems
(shown in Fig. 5.2) as follows,

$$\begin{align*}
\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} &= T_\psi T_\phi \begin{bmatrix} i \\ j \\ k \end{bmatrix}, \\
\begin{bmatrix} i \\ j \\ k \end{bmatrix} &= T_\gamma \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix}.
\end{align*}$$

(5.2)

The transformation matrices, $T_\alpha$ and $T_\beta$, for the front frame coordinates are,

$$T_\alpha = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}, \quad T_\beta = \begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.

(5.3)

These are used to rotate into the front frame coordinate system (shown in Fig. 5.3),
and rotated by $\alpha$ again to align $\hat{u}'$ with the direction the front wheel is heading and $\hat{w}'$ in the direction of the ground contact point,

$$\begin{align*}
\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} &= T_\alpha T_\beta \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix}, \\
\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix} &= T_\alpha^T \begin{bmatrix} \hat{u}' \\ \hat{v}' \\ \hat{w}' \end{bmatrix}.
\end{align*}$$

(5.4)
5.3.1 Position and velocity

![Bicycle Diagram]

**Figure 5.4:** Illustration of bicycle with dimensions and masses indicated.

Noting the dimensions in Fig. 5.4 for a bicycle in the reference position where all angular rotations are zero, the position vectors in global coordinates to points on the rear assembly are given as follows,

\[
\mathbf{r}_A = \begin{cases} x \\ y \\ 0 \end{cases} \tag{5.5}
\]

\[
\mathbf{r}_O = \mathbf{r}_A + \mathbf{r}_{O/A} = \mathbf{r}_A + \mathbf{T}_\psi \mathbf{T}_\phi \begin{cases} 0 \\ 0 \\ -R_O \end{cases} \tag{5.6}
\]

\[
\mathbf{r}_R = \mathbf{r}_O + \mathbf{r}_{R/O} = \mathbf{r}_O + \mathbf{T}_\psi \mathbf{T}_\phi \mathbf{T}_\gamma \begin{cases} l_R \\ 0 \\ -(h_R - R_O) \end{cases} \tag{5.7}
\]

\[
\mathbf{r}_B = \mathbf{r}_O + \mathbf{r}_{B/O} = \mathbf{r}_O + \mathbf{T}_\psi \mathbf{T}_\phi \mathbf{T}_\gamma \begin{cases} w + c \\ 0 \\ R_O \end{cases}, \tag{5.8}
\]

where \( \mathbf{r}_O \) and \( \mathbf{r}_R \) give the center of mass locations of the rear wheel and rear frame,
respectively. The vector $\mathbf{r}_B$ indicates where the steering axis intersects the ground plane in the reference position, when all angular displacements are zero, as shown in Fig. 5.4. This point remains the same in rear reference coordinates even as the steering angle changes, making it a useful point from which to determine points on the front assembly,

$$\mathbf{r}_P = \mathbf{r}_B + \mathbf{r}_{P/B} = \mathbf{r}_B + T_\psi T_\phi T_\gamma T_\alpha T_\beta T^T \begin{bmatrix} -c \\ 0 \\ -R_P \end{bmatrix}$$

$$\begin{equation} (5.9) \end{equation}$$

$$\mathbf{r}_F = \mathbf{r}_B + \mathbf{r}_{F/B} = \mathbf{r}_B + T_\psi T_\phi T_\gamma T_\alpha T_\beta T^T \begin{bmatrix} -(w + c) + l_F \\ 0 \\ -h_F \end{bmatrix}$$

$$\begin{equation} (5.10) \end{equation}$$

$$\mathbf{r}_C = \mathbf{r}_B + \mathbf{r}_{C/B} = \mathbf{r}_B + T_\psi T_\phi T_\gamma T_\alpha T_\beta T^T \begin{bmatrix} -c \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{equation} (5.11) \end{equation}$$

where $\mathbf{r}_P$ and $\mathbf{r}_F$ give the center of mass locations for the front wheel and the front frame, and $c$ is the trail. Point $C$ is the location where the front tire makes contact with the ground. Now, each position vector can be differentiated with respect to time to find the velocity vectors in the global coordinate system.

To allow compatibility with body-fixed moment of inertia tensors, the angular velocity vectors are found in body-fixed coordinates by using the transpose of each transformation matrix. (For transformation matrices the transpose is equivalent to the inverse because the cofactor matrix is identical to the original matrix, and the
determinant is one.) The angular velocities are given as follows,
\[ \Omega_O = \begin{pmatrix} 0 \\ \dot{\theta}_O \end{pmatrix} + T_{\phi}^T \left( \begin{pmatrix} \dot{\phi} \\ 0 \end{pmatrix} + T_{\psi}^T \begin{pmatrix} 0 \\ \dot{\psi} \end{pmatrix} \right) \] (5.12)
\[ \Omega_R = \begin{pmatrix} 0 \\ \dot{\gamma} \end{pmatrix} + T_{\gamma}^T \left( T_{\phi}^T \left( \begin{pmatrix} \dot{\phi} \\ 0 \end{pmatrix} + T_{\psi}^T \begin{pmatrix} 0 \\ \dot{\psi} \end{pmatrix} \right) \right) \] (5.13)
\[ \Omega_P = \begin{pmatrix} 0 \\ \dot{\theta}_P \end{pmatrix} + T_{\alpha} \left( \begin{pmatrix} 0 \\ \dot{\beta} \end{pmatrix} + T_{\beta}^T T_{T}^T \Omega_R \right) \] (5.14)
\[ \Omega_F = T_{\alpha} \left( \begin{pmatrix} 0 \\ \dot{\beta} \end{pmatrix} + T_{\beta}^T T_{T}^T \Omega_R \right). \] (5.15)

5.3.2 System energies

Summing energy expressions over all four frame parts, the kinetic and potential energies of the system are
\[ T = \frac{1}{2} m_O \ddot{r}_O \cdot \dot{r}_O + \frac{1}{2} m_R \ddot{r}_R \cdot \dot{r}_R + \frac{1}{2} m_P \ddot{r}_P \cdot \dot{r}_P + \frac{1}{2} m_F \ddot{r}_F \cdot \dot{r}_F \]
\[ + \frac{1}{2} \Omega_O^T I_O \Omega_O + \frac{1}{2} \Omega_R^T I_R \Omega_R + \frac{1}{2} \Omega_P^T I_P \Omega_P + \frac{1}{2} \Omega_F^T I_F \Omega_F \] (5.16)
\[ V = -m_O g r_O \cdot \ddot{K} - m_R g r_R \cdot \ddot{K} - m_P g r_P \cdot \ddot{K} - m_F g r_F \cdot \ddot{K}, \] (5.17)

where the mass moment of inertia tensors are given by
\[ I_O = \begin{bmatrix} I_{xxO} & 0 & 0 \\ 0 & I_{yyO} & 0 \\ 0 & 0 & I_{zzO} \end{bmatrix}, \quad I_R = \begin{bmatrix} I_{xxR} & 0 & 0 \\ 0 & I_{yyR} & 0 \\ I_{zzR} & 0 & I_{zzR} \end{bmatrix}, \] (5.18)
\[ I_P = \begin{bmatrix} I_{xxP} & 0 & 0 \\ 0 & I_{yyP} & 0 \\ 0 & 0 & I_{zzP} \end{bmatrix}, \quad I_F = \begin{bmatrix} I_{xxF} & 0 & 0 \\ 0 & I_{yyF} & 0 \\ I_{zzF} & 0 & I_{zzF} \end{bmatrix}, \] (5.19)

and the inertia tensors have already been simplified by the products of inertia becoming zero due to planes of symmetry in the object based on the location of a given reference frame.

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5.3.3 Constraints

One of the most interesting aspects of the bicycle problem is the need to treat both holonomic (geometric) and nonholonomic (nonintegrable) constraints. Requiring that the front wheel touches the ground provides a holonomic constraint, since the requirement is geometric and independent of the bicycle’s path. Conversely, assuming that the wheels roll without slipping provides two nonholonomic vector constraint equations, indicating that the ground velocity corresponds to the angular velocity of each wheel. Because these roll-without-slip constraints depend on the velocities and cannot be exactly integrated or integrated by means of an integrating factor, they are not semiholonomic [6]. The holonomic constraint requires

\[ h_1 \equiv r_C \cdot \dot{K} = 0, \]  \hspace{1cm} (5.20)

wherein the vertical component of \( r_C \) is required to be zero, ensuring that the front contact point moves on the contact plane.

The nonholonomic constraints give the two vector equations (or four scalar equations, \( g_1, g_2, g_3, \) and \( g_4 \), since only the first two components of each equation are of interest),

\[
\begin{align*}
\dot{r}_A + \omega_O \times r_{A/O} &= \dot{r}_A + T_\psi T_\phi \left\{ \begin{array}{c} R_O \dot{\theta}_O \\ 0 \\ 0 \end{array} \right\} = 0 \hspace{1cm} (5.21) \\
\dot{r}_C + \omega_P \times r_{C/P} &= \dot{r}_C + T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \left\{ \begin{array}{c} R_P \dot{\theta}_P \\ 0 \\ 0 \end{array} \right\} = 0. \hspace{1cm} (5.22)
\end{align*}
\]

The constraints in Eqns. 5.21–5.22 enforce that the velocity of the ground contact point correspond to the angular velocity of the wheel such that the wheel rolls without slipping. This is often stated as “the velocity at the contact point must be zero,” but it is clarifying to consider that the sum of the velocity due to the moving contact
point and the velocity at that point due to the rotation of the wheel must be zero because the point of contact and surface do not move relative to one another, as they would if “slipping” were to occur.

5.4 Equations of motion

5.4.1 Lagrange’s equations

When determining equations of motion starting from the general form of d’Alembert’s principle

$$\left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} - Q^{NP} \right] \delta q = Q^c \delta q = 0, \quad (5.23)$$

where the Lagrangian $L \equiv T - V$, there are multiple options for techniques that will account for generalized forces of constraint. Holonomic constraints can be directly embedded in the system energies. For holonomic or semiholonomic (exactly integrable) constraints, the Lagrangian can be augmented with the constraint and Lagrange multiplier because the displaced paths are geometrically possible [6]. However, for general nonholonomic constraints, the constraints can be adjoined to d’Alembert’s principle, once conditions on the virtual displacements are determined.

The linear-acceleration form of the constraints leads to the tangency condition for possible states. Decomposing the constraints into independent and dependent parts and using the variation of dependent displacements provides conditions on the displacements under the general velocity constraints. Adjoining these restrictions to d’Alembert’s principle and requiring that the displacement $\delta q$ be arbitrary gives the equations of state for a system with holonomic and nonholonomic constraints [7],

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) - \frac{\partial L}{\partial q_i} \delta q_i = \sum_j \lambda_j \frac{\partial h_j}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial q_i}, \quad (5.24)$$

where $i, j, k$ are the indices of the generalized coordinates, holonomic constraints, and nonholonomic constraints, respectively, $\lambda$ is the Lagrange multiplier of each
holonomic term, and \( \mu \) is the Lagrange multiplier for each nonholonomic term. In this system, using the generalized coordinates, \( x, y, \psi, \phi, \gamma, \beta, \theta_O, \) and \( \theta_P \), there are eight Lagrange’s equations, five constraint equations, and five unknown Lagrange multipliers, for a total of thirteen equations and thirteen unknowns.

5.4.2 Linearization

While the fully nonlinear equations could be found from Eqns. 5.16–5.17 and 5.20–5.24, as an approximation to bicycle motion, we choose to find the linearized equations of motion for a bicycle with forward velocity \( v \) and small disturbances about its displacements.

\[
x \rightarrow vt + \epsilon x \\
y \rightarrow \epsilon y \\
\psi \rightarrow \epsilon \psi \\
\phi \rightarrow \epsilon \phi \\
\gamma \rightarrow \epsilon \gamma \\
\beta \rightarrow \epsilon \beta \\
\theta_O \rightarrow \epsilon \theta_O - \frac{vt}{R_O} \\
\theta_P \rightarrow \epsilon \theta_P - \frac{vt}{R_P}
\]

Each constraint equation and the Lagrangian is now expanded in its Taylor series, to \( \mathcal{O}(\epsilon^2) \). Lagrange’s equation is applied, yielding an equation in each generalized coordinate. To a linear approximation, the holonomic constraint \( h_1 \) requires,

\[-w\gamma = 0, \quad (5.33)\]
so the rear frame pitch and its derivatives can be set equal to zero. The nonholonomic constraints result in the following linearized equations,

\[ \dot{x} + R_O \dot{\theta}_O = 0 \]  
\[ \dot{y} - v \psi = 0 \]  
\[ \dot{x} + R_P \dot{\theta}_P = 0 \]  
\[ w \dot{\psi} - c \cos \alpha \dot{\beta} - v \cos \alpha \beta = 0, \]  

where \( \dot{\gamma} \) has been set equal to zero and \( g_2 \) has been used to condense the \( \dot{y} \) and \( \psi \) terms out of \( g_4 \). Together with equations \( g_1 \) and \( g_3 \), the Lagrange equations with respect to \( x, \theta_O, \) and \( \theta_P \), called \( L_x, L_{\theta_O}, \) and \( L_{\theta_P} \), require \( \ddot{x} = 0, \ddot{\theta}_O = 0, \ddot{\theta}_P = 0, \) and \( \mu_1 = \mu_3 = 0 \). The Lagrange equation, \( L_{\gamma} \), can be solved for \( \lambda_1 \), giving,

\[ \lambda_1 = -\frac{g}{w} (l_R m_R + l_F m_F + w m_P). \]  

Constraints \( g_2 \) and \( g_4 \) can be solved for \( \dot{y} \) and \( \dot{\psi} \). Inserting these, their derivatives, and \( \lambda_1 \) into \( L_y, L_\psi, L_\phi, \) and \( L_\beta \) gives four equations with unknowns \( \phi, \beta, \mu_2, \) and \( \mu_4 \). \( L_\psi \) can be solved for \( \mu_4 \), then \( L_y \) for \( \mu_2 \). Now \( L_\phi \) and \( L_\beta \) completely describe the linearized behavior of a bicycle in a two degree of freedom, second order, linear ordinary differential equation. The equations describing the roll (lean) and steer of a bicycle can be written,

\[ \begin{bmatrix} M_{\phi\phi} & M_{\phi\beta} \\ M_{\beta\phi} & M_{\beta\beta} \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} 0 & C_{\phi\beta} \\ C_{\beta\phi} & C_{\beta\beta} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} K_{\phi\phi}^1 & K_{\phi\beta}^1 \\ K_{\beta\phi}^1 & K_{\beta\beta}^1 \end{bmatrix} + v^2 \begin{bmatrix} 0 & K_{\phi\beta}^2 \\ 0 & K_{\beta\beta}^2 \end{bmatrix} \begin{bmatrix} \phi \\ \beta \end{bmatrix} = 0, \]  

where \( \phi \) and \( \beta \) are the roll and steer angles respectively, \( v \) is the forward velocity, and the coefficients are given by different functions of the bicycle parameters given by Eqns. 5.41–5.43. This result agrees with previous derivations [9], although it was derived using different methods. In the nomenclature of this work, the benchmark values used are given in Table 5.1 [9].
5.5 Stability and behavior

In this section, the behavior of the system is examined by plotting the eigenvalues for the linearized system and explaining the relevant bicycle modes.

For the solution vector $\mathbf{y}$ consisting of $\phi$, $\beta$, exponential form $\mathbf{y} = \mathbf{c} e^{\lambda t}$ can be assumed. Inserting this solution into Eqn. 5.39 gives the characteristic equation

$$\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K} = 0,$$

with four corresponding eigenvalues, which depend on the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$.

![Figure 5.5: Eigenvalues and stability of benchmark system, with positive real parts of eigenvalues (red), negative real parts of eigenvalues (blue), and imaginary parts of eigenvalues (green).](image)

Figure 5.5 shows the eigenvalues of the unforced benchmark system, and they agree with those in the benchmark study [9]. The imaginary parts of the eigenvalues are shown in blue and represent the frequency response of the system; the real parts...
Figure 5.6: Weave mode illustrated by $\phi$ (solid blue line) and $\beta$ (dashed green line) when $v = 4$ m/s: time series a) and phase portrait b) for an initial displacement of 2 deg in $\phi$ and time series c) and phase portrait d) for an initial displacement of 2 deg in $\beta$.

are shown in red (positive) and green (negative). Since the system is stable when all of the real parts are negative, the stable velocity region has been highlighted from 4.3–6.0 m/s. For this hands-free system, eigenvalues associated with the phenomena of weave, capsize, and castering are shown.

Figures 5.6–5.7 are presented to demonstrate by means of time series and phase portraits the characteristic behavior of each mode and provide physical intuition. Weave refers to the mode of slow oscillations illustrated in Fig. 5.6. As shown in the plot, regardless of whether the initial perturbation is in the roll or steer angle, oscillations grow, with the steer angle lagging slightly behind the roll angle. Capsize refers to a nonoscillatory slow toppling that is easily controlled by a rider, shown in Fig. 5.7. Although the steering angle veers slightly in the direction of the fall, the effect is not enough to right the vehicle. Castering refers to the effect of the front
Figure 5.7: Capsize mode illustrated by $\phi$ (solid blue line) and $\beta$ (dashed green line) when $v = 7$ m/s: time series a) and phase portrait b) and an initial displacement of 5 deg in $\phi$ and time series c) and phase portrait d) for an initial displacement of 5 deg in $\beta$.

wheel aligning with the direction of travel [38]. The capsize phenomenon becomes slightly unstable at 6 m/s, but can be easily stabilized by a human rider. The weave phenomenon stabilizes once the bicycle is going sufficiently fast, and the castering effect is a stabilizing effect for all velocities shown.

5.6 Conclusion

Using a Lagrangian framework for the bicycle derivation has several benefits. The Lagrangian formulation is not tied to a coordinate system, of particular importance given the number of natural coordinate systems for each frame part, but rather uses convenient generalized coordinates. Euler angle transformation matrices are particularly useful in describing this system with rigid bodies hinging on one another. This problem provides a thorough example of employing this technique.
To treat the constraints of the system, there is an elegant method for using variational principles to include the action of constraints in holonomic systems, where the constraints are geometrically possible [6]. However, this nonholonomic system with nonlinear constraints requires a different approach valid for general velocity constraints [7]. The bicycle problem encourages understanding how to treat both types of constraints, an interesting nuance of Lagrangian mechanics, which is important to many physical systems. While the present work derives the linearized bicycle equations of motion, the presentation of all of the position vectors and system energies from a clearly-defined starting point, in a self-contained form will facilitate the derivation of the nonlinear Lagrangian.

Finally, in plotting the eigenvalues of the system, a connection can be made between the stability of different bicycle modes and physical parameters. Furthermore, the time series and phase portrait plots of weave and capsize behavior demonstrate the dominant phenomena at different forward velocities, thus connecting the angular motions to the behavior found in the stability analysis. This system provides a thorough and systematic understanding of deriving equations of state for complex three-dimensional rigid bodies and their associated constraints in a convenient Lagrangian framework.
Appendix: Definition of terms in equations of motion

\[ M_{\phi \phi} = I_{xxR} + I_{xxF} + I_{xxO} + I_{xxP} + m_R h_R^2 + m_F h_F^2 + m_O R_O^2 + m_P R_P^2 \]

\[ M_{\phi \beta} = \left[ I_{zzF} + m_F (l_F - c - w) h_F \right] \cos \alpha \]

\[ + \left[ I_{zzF} + I_{xxF} + m_F h_F^2 + m_P R_P^2 \right] \sin \alpha \]

\[ M_{\beta \phi} = M_{\phi \beta} \]

\[ M_{\beta \beta} = \frac{1}{2} \left( \frac{c}{w} \right)^2 \left( I_{xxO} + I_{xxP} + I_{zzR} + I_{zzF} + m_R l_R^2 + m_F l_F^2 \right) \]

\[ + \frac{c}{w} \left( I_{xxP} + I_{zzF} + m_F l_F (l_F - c) \right) + I_{xxP} + m_F (cw - 2cl_F - wl_F) \]

\[ + \frac{1}{2} \left( I_{xxF} + I_{zzF} + m_F (l_F^2 + h_F^2 + c^2 + w^2) + m_P R_P^2 \right) \]

\[ + \left[ \frac{1}{2} \left( \frac{c}{w} \right)^2 \left( I_{xxO} + I_{xxP} + I_{zzR} + I_{zzF} + m_F (l_F^2 + w^2 - 2wl_F) \right) \right. \]

\[ + m_R l_R^2 \left. + \frac{c}{w} \left( I_{xxP} + I_{zzF} + m_F (l_F - w)^2 \right) \right] \] \cos 2\alpha

\[ + \frac{c + w}{w} \left[ I_{zzF} + m_F (l_F - w) h_F \right] \sin 2\alpha \] \hspace{1cm} (5.41)
\[ C_{\phi\beta} = \left[ \frac{I_{yyP}}{R_P} + \frac{I_{zzF} + I_{zzR}}{w} + m_P R_P + \frac{m_R l_R h_R + m_F l_F h_F}{w} \right] \cos \alpha \]
\[ + \frac{c}{w} \left( \frac{I_{yyO}}{R_O} + \frac{I_{yyP}}{R_P} + m_R h_R + m_F h_F + m_O R_O + m_P R_P \right) \cos \alpha \]
\[ C_{\beta\phi} = -\left[ \frac{I_{yyP}}{R_P} + \frac{c}{w} \left( \frac{I_{yyO}}{R_O} + \frac{I_{yyP}}{R_P} \right) \right] \cos \alpha \]
\[ C_{\beta\beta} = -\frac{1}{2} m_F (c + l_F) + \frac{1}{2} \frac{c}{w^2} (I_{xxO} + I_{xxP} + I_{zzR} + I_{zzF}) \]
\[ + m_R l_R^2 + m_F l_F^2 \right) + \frac{1}{2} \left( \frac{c}{w} \right)^2 (m_R l_R + m_F l_F) \]
\[ + \frac{1}{2} \frac{c}{w} (I_{xxP} + I_{zzF} + m_F (l_F^2 - c^2)) \]
\[ + \left[ -\frac{1}{2} m_F (l_F + c) + \frac{1}{2} \frac{c}{w^2} (I_{xxO} + I_{xxP} + I_{zzR} + I_{zzF}) \]
\[ + m_R l_R^2 + m_F l_F^2 \right) + \frac{1}{2} \left( \frac{c}{w} \right)^2 (m_R l_R + m_F l_F) \]
\[ + \frac{1}{2} \frac{c}{w} (I_{xxP} + I_{zzF} + m_F (l_F^2 - c^2)) \right] \cos 2\alpha \]
\[ + \left[ \frac{1}{2} m_P R_P + \frac{1}{2} \frac{1}{w} (I_{xxF} + m_F (l_F + c) h_F + m_P c R_P) \right] \sin 2\alpha \]
(5.42)
\[ K_{\phi\phi}^1 = -(m_R h_R + m_F h_F + m_O R_O + m_P R_P) \]

\[ K_{\phi\beta}^1 = \left[ m_F (c + w - l_F) - \frac{c}{w} (m_R l_R + m_F l_F) \right] \cos \alpha \]

\[ - \left[ m_F h_F + m_P R_P \right] \sin \alpha \]

\[ K_{\phi\beta}^1 = K_{\phi\phi}^1 \]

\[ K_{\beta\phi}^1 = -\frac{1}{2} \left( m_F h_F + m_P R_P \right) + \frac{1}{2} \left[ m_F h_F + m_P R_P \right] \cos 2\alpha \]

\[ + \frac{1}{2} \left[ m_F (c + w - l_F) - \frac{c}{w} (m_R l_R + m_F l_F) \right] \sin \alpha \]

\[ K_{\phi\phi}^2 = \frac{1}{w} \left[ \frac{I_{\gamma\gamma} O}{R_O} + \frac{I_{\gamma\gamma} P}{R_P} + m_R h_R + m_F h_F + m_O R_O + m_P R_P \right] \cos \alpha \]

\[ K_{\beta\beta}^2 = \frac{1}{2} \left( -m_F + \frac{m_F l_F}{w} + \frac{c}{w} \left( \frac{m_F l_F}{w} + \frac{m_R l_R}{w} - m_F \right) \right) \]

\[ + \frac{1}{2} \left[ -m_F + \frac{m_F l_F}{w} + \frac{c}{w} \left( \frac{m_R l_R + m_F l_F}{w} - m_F \right) \right] \cos 2\alpha \]

\[ + \frac{1}{2} \frac{1}{w} \left[ \frac{I_{\gamma\gamma} P}{R_P} + m_F h_F + m_P R_P \right] \sin 2\alpha \] (5.43)
### Appendix: Bicycle parameters

**Table 5.1: Benchmark bicycle parameters.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>wheel base</td>
<td>( w )</td>
<td>1.02 m</td>
</tr>
<tr>
<td>trail</td>
<td>( c )</td>
<td>0.08 m</td>
</tr>
<tr>
<td>caster angle</td>
<td>( \alpha )</td>
<td>( \pi/10 ) (18°)</td>
</tr>
<tr>
<td>rear wheel radius</td>
<td>( R_O )</td>
<td>0.3 m</td>
</tr>
<tr>
<td>mass</td>
<td>( m_O )</td>
<td>2 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_{zzO} )</td>
<td>0.0603 kg m²</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_{yyO} )</td>
<td>0.12 kg m²</td>
</tr>
<tr>
<td>rear frame horizontal distance</td>
<td>( l_R )</td>
<td>0.3 m</td>
</tr>
<tr>
<td>from ( A )</td>
<td>( h_R )</td>
<td>0.9 m</td>
</tr>
<tr>
<td>mass</td>
<td>( m_R )</td>
<td>85 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_R )</td>
<td>\begin{bmatrix} 9.2 &amp; 0 &amp; 2.4 \ 0 &amp; 11 &amp; 0 \ 2.4 &amp; 0 &amp; 2.8 \end{bmatrix} \text{ kg m}^2</td>
</tr>
<tr>
<td>front wheel radius</td>
<td>( R_P )</td>
<td>0.35 m</td>
</tr>
<tr>
<td>mass</td>
<td>( m_P )</td>
<td>3 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_{zzP} )</td>
<td>0.1405 kg m²</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_{yyP} )</td>
<td>0.28 kg m²</td>
</tr>
<tr>
<td>front frame horizontal distance</td>
<td>( l_F )</td>
<td>0.06976 m</td>
</tr>
<tr>
<td>from ( B )</td>
<td>( h_F )</td>
<td>0.1302 m</td>
</tr>
<tr>
<td>mass</td>
<td>( m_F )</td>
<td>4 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>( I_F )</td>
<td>\begin{bmatrix} 0.05892 &amp; 0 &amp; -0.00756 \ 0 &amp; 0.06 &amp; 0 \ -0.00756 &amp; 0 &amp; 0.00708 \end{bmatrix} \text{ kg m}^2</td>
</tr>
</tbody>
</table>
6

Bicycle geometry and stability

6.1 Introduction

Bicycle stability in the presence of human control is crucial to road races, where either road bicycles or time trial/triathlon (TT) bicycles are commonly used. Road bicycles are often favored for their superior handling; however, aerodynamic considerations are also vitally important [39], and time trial bicycles are often preferred for this advantage. It is known empirically that time trial bicycles are more difficult to steer and experience longer transients when the path is perturbed. Nonetheless, their altered geometry is sometimes desirable for the aerodynamic benefits. In addition to a steeper seat tube angle, aero bars on the handlebar assembly provide an option to steer with the forearms in an aggressive position, so that the cyclist’s torso forms a smaller cross sectional area. Typical aero bar pad placement is 8–16 cm, or 16–20 cm for the “British position.” The closer together the pads are, the smaller the chest cavity created, which results in a more aerodynamic cross section [40].

This chapter considers the geometrical differences between both riding styles, notably changes in the cyclist’s center of mass and the length of fulcrum from the
steering axis. A linearized model with human control is used to identify the stability
differences between road bicycles and time trial bicycles. The linearized equations
of motion and associated eigenvalue problem are given in Sec. 6.2, while Sec. 6.3
demonstrates the effect of varying rider center of mass on the uncontrolled system.
The controlled system, with both instantaneous and delayed feedback is presented
in Sec. 6.4. Stability charts indicate the changing stability boundaries for different
gains, bicycle parameters, and reaction times.

6.2 Linearized equations of motion

Recall from Chapter 5 the linearized equations of motion,

\[
\begin{bmatrix}
M_{\phi\phi} & M_{\phi\beta} \\
M_{\beta\phi} & M_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\beta}
\end{bmatrix}
+ v
\begin{bmatrix}
0 & C_{\phi\beta} \\
C_{\beta\phi} & C_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
\phi \\
\beta
\end{bmatrix}
+ \left(g
\begin{bmatrix}
K_{\phi\phi}^1 & K_{\phi\beta}^1 \\
K_{\beta\phi}^1 & K_{\beta\beta}^1
\end{bmatrix}
+ v^2
\begin{bmatrix}
0 & K_{\phi\beta}^2 \\
0 & K_{\beta\beta}^2
\end{bmatrix}
\right)
\begin{bmatrix}
\phi \\
\beta
\end{bmatrix}
= \begin{bmatrix}
0 \\
T_\beta(\phi, \dot{\phi})
\end{bmatrix},
\]  

(6.1)

where \(\phi\) and \(\beta\) are the roll and steer angles respectively, \(v\) is the forward velocity,
\(T_\beta\) is the steering torque, a function of roll and roll rate, and the coefficients are
given by different functions of the bicycle parameters given by Eqns. 5.41–5.43 in
the previous chapter. Parameter values and definitions are given in Tab. 6.1. Some
of the geometric parameters are shown in Fig. 6.1. The only external torque we
consider is that about the steering axis. For the solution vector \(y\) consisting of \(\phi, \beta\),
exponential form \(y = ce^{\lambda t}\) can be assumed. Inserting this solution into Eqn. 6.1
gives the characteristic equation

\[
\lambda^2 M + \lambda C + K = 0,
\]  

(6.2)

with four corresponding eigenvalues, which depend on the mass matrix \(M\), damping
matrix \(C\), and stiffness matrix \(K\).
Figure 6.1: Line drawing of the bicycle configuration and mass and dimension parameters.

Figure 6.2 shows the eigenvalues of the unforced system for the parameters used in this chapter. (Values are the same as in the benchmark study [9], except that front wheel parameters have been changed to match those of the rear wheel.) The imaginary parts of the eigenvalues are shown in blue and represent the frequency response of the system; the real parts are shown in red (positive) and green (negative). Since the system is stable when all of the real parts are negative, the stable velocity region has been highlighted between about 5.2 and 8.5 m/s. For this hands-free system, eigenvalues associated with the phenomena of weave, capsize, and castering are shown. See Chapter 5 for an explanation of these modes.

6.3 Uncontrolled system

One of the most noticeable changes when a cyclist goes down on the aero bars is the shift in the cyclist’s center of mass. Generally, it moves forward and lower. To explore this effect, the eigenvalues of the unforced system were calculated for different
positions of the cyclist center of mass. For each calculation, benchmark parameter values were used, except that the front wheel parameters were changed to match those of the back wheel.

The result of shifting the cyclist’s center of mass by 10 cm forward or backward is shown in Fig. 6.3. Increasing the longitudinal distance to the rear center of mass $l_R$ has a stabilizing effect, shown where the real parts of the eigenvalues describing both weave and capsize become more negative for the higher velocities, increasing the range of stable velocities. Decreasing $l_R$ destabilizes the weave effect for higher velocities, but stabilizes it slightly for lower velocities. Ultimately moving the center of mass forward results in a wider range of stable velocities. However, despite the larger range of stable velocities when $l_R$ is moved forward, the weave eigenvalue in the unstable range increases, indicating that when control is considered, it may be more difficult to stabilize the bicycle at lower velocities.
Figure 6.3: Real parts of eigenvalues vs. forward velocity for 10 cm change in $l_R$, where arrows indicate eigenvalue trends for increasing $l_R$.

Figure 6.4 shows the same analysis for moving the center of mass 20 cm higher or lower. Increasing the vertical location of the cyclist’s center of mass $h_R$ has a very slight destabilizing effect, as shown in Fig. 6.4. Decreasing $h_R$ 20 cm increases the stable range of velocities, but the effect is less pronounced than when varying $l_R$. However, there is also a change in the imaginary parts of the eigenvalues.

In general, lowering and moving forward the center of mass results in a small increase in stability, although the displacement analyzed is exaggerated compared to a cyclist. This consideration alone does not explain the decreased stability experienced on a time trial bike. It is implied, however, by the change in imaginary weave eigenvalues that a control law may be useful in showing the stability changes that result from moving the cyclist’s COM.
Humans balance by several sensory systems, visual, vestibular ("gyro"), and somatosensory (position and orientation). It is believed that the central nervous system is capable of using information from the somatosensory system to extrapolate the future position of a body’s center of mass from its acceleration [41]. Goodworth and Peterka found that in response to a surface tilt, a human’s pelvis tilted in the same direction so the upper body would generate a torque proportional to the angular displacement and velocity with no time delay before the lower body responds with delayed control mechanisms [42]. Many simple models for human balance have been found to agree with various experiments. Examples include a linear model with position and velocity intrinsic and delayed feedback pathways [43], and a neural controller with input torque proportional to the angular position, velocity, and time integral of angular position (although this model only accounts for movement in the
sagittal plane, or anterior-posterior direction) [44].

In response to these findings, this section explores the effect of a control torque proportional to the roll and roll rate, using both instantaneous and delayed feedback. While delayed feedback represents the actual human reaction, the effect of immediate response is first shown to separately demonstrate the effect of body position changes only for the perfect reaction.

6.4.1 Instantaneous feedback

Control law Based on the indicated physiology, under the assumption that human perception to lean on a bicycle is similar to that of lean while standing or walking, a simple way to model human control of a bicycle would be to add a steering torque proportional to the roll angle and roll rate of the bicycle (and cyclist),

\[ T_\beta = k_1 \phi + k_2 \dot{\phi}, \]  

(6.3)

where \( k_1 \) is the gain of the roll term in N·m/rad, \( k_2 \) is the gain of the roll rate term in N·m·s/rad, and the cyclist is assumed to steer the front assembly in the direction of roll.

The stabilizing effect of this control law is shown in Fig. 6.5, where the controlled system can be made stable at arbitrarily high velocities by increasing the roll gain (shown here as 0.2 N·m/rad to stabilize a range of reasonable race velocities), but stability is created at the low velocities by increasing the roll rate gain (shown here as 20 N·m·s/rad).

Figure 6.6 shows the effect of roll gain \( k_1 \) alone in blue (when \( k_2 = 0 \)) as well as when \( k_2 = 20 \) N·m·s/rad. (Note that shaded regions are stable, and unshaded regions are unstable for all stability charts in this thesis.) Positive gain indicates the cyclist is steering in the direction of a fall, a stabilizing maneuver. From the figure, the range of stable velocities increases as the gain is increased. A negative
Figure 6.5: Eigenvalues of uncontrolled (circles) vs. instantaneously controlled system (solid line), where $k_1 = 0.2 \text{ N}\cdot\text{m}/\text{rad}$ and $k_2 = 20 \text{ N}\cdot\text{m}\cdot\text{s}/\text{rad}$.

gain indicates the cyclist is steering away from a fall, decreasing the range of stable forward velocities. While the unstable region for higher velocities remains unchanged between the plots, the stability boundary shifts leftward for higher $k_2$, indicating $k_2$ as the dominant means of stabilizing the bicycle at low velocities.

The effect of roll rate gain $k_2$ alone can be seen in Fig. 6.7 (when $k_1 = 0$), along with the effect of $k_2$ at a constant $k_1 = 0.8 \text{ N}\cdot\text{m}/\text{rad}$. The unstable region retains the same shape at lower velocities, but the stability boundary changes drastically with the addition of a roll gain for higher velocities. Figs. 6.5–6.7 combine to show that stability at lower velocities is dependent on $k_2$, while stability at higher velocities is dependent on $k_1$.

Perturbations of $l_R$ To see the effect of control on different bicycle geometries, it is illustrative to perturb the rear center of mass location parameters. From the
stability charts in Figs. 6.8–6.9, it is clear that increasing \( l_R \) results in a larger stable region. For a range of roll gain \( k_1 \), the stability boundary at the upper end of the stable velocity range shifts rightward, drastically so for positive gains, while the left stability boundary remains almost unchanged. For a range of roll rate gain \( k_2 \), the right stability boundary shifts far rightward when the rear center of mass is shifted forward, but there is little change to the lower stable velocity.

**Perturbations of \( h_R \)** Changes to the height of the rear center of mass \( h_R \) produce more subtle changes in the stability charts. Shifting \( h_R \) down 0.2 m creates a slightly larger stable region in Fig. 6.10. In Fig. 6.11, it can be seen that the stability boundary shifts rightward for a lower center of mass. Both of the changes to the rear center of mass that result in greater stability, e.g., forward and downward, are associated with a cyclist positioning himself on aero bars, the opposite of what one might expect,
since many report experiencing decreased stability on a time trial bicycle. It is then likely that the popular use of “stability” is being conflated with the idea of handling.

6.4.2 Delayed feedback

Introducing a reaction time delay into the human feedback creates an interesting and more realistic stability problem. Time delays have been used in modeling balancing a stick on the fingertip [45–47], but less has been done with total body balance. The control torque subject to a reaction time delay is,

\[ T_\beta(t) = k_1 \phi(t - \tau) + k_2 \dot{\phi}(t - \tau), \]

(6.4)

where \( \tau \) is the time delay in seconds.

For a delay differential equation, the stability procedure is more complex. Because the state-space is infinite-dimensional, approximate techniques are often nec-
Figure 6.8: Stability chart for the instantaneously controlled bike and \( l_R \) perturbations in the \( k_1 \) vs. \( v \) plane for \( k_2 = 20 \text{ N} \cdot \text{m} \cdot \text{s} / \text{rad} \) and \( l_R = 0.2, 0.4 \text{ m} \).

Figure 6.9: Stability chart for the instantaneously controlled bike and \( l_R \) perturbations in the \( k_2 \) vs. \( v \) plane for \( k_1 = 0.8 \text{ N} \cdot \text{m} \cdot \text{rad} \) and \( l_R = 0.2, 0.4 \text{ m} \).

A spectral element method as described in reference [49] was used to determine stability in the present work. Interpolation nodes are chosen to be the asymptotically arcsine-distributed Legendre-Gauss-Lobatto (LGL) points, and the trial functions and their derivatives are Lagrange polynomials as calculated using the barycentric formula. Finally, an LGL quadrature is used to evaluate the weighted residual integrals.

The addition of a time delay alters the stability boundaries of roll gain vs. velocity; Fig. 6.12 shows the effect of a 1 s delay when \( k_2 = 0 \), and Fig. 6.13 shows the effect of a 0.027 s delay when \( k_2 = 20 \text{ N} \cdot \text{m} \cdot \text{s} / \text{rad} \). Note than reaction times were chosen to illustrate trends in stability boundary behavior. The highest stable velocity in particular decreases for both cases, indicating that any lag in the human control inputs jeopardizes the stability of the system at high velocities, independent of the control gains.

Similarly for the stability charts of roll rate gain vs. velocity in Figs. 6.14–6.15, a time delay of 0.04 s is enough shrink the stable region, independent of the control gains.
trol gains. Since increasing $k_2$ is the primary means of stabilizing low velocities, these plots show that a delayed reaction can make stabilizing these low velocities impossible, regardless of roll rate gain.

**Perturbations of $l_R$** It is illuminating to combine the effects of a reaction time delay and perturbations of the rear center of mass. Figures 6.16-6.17 show the effects of perturbing the longitudinal location of the rear center of mass. For this delayed case, there is distinct contrast with the instantaneous control case from Figs. 6.8-6.9. The stable region shrinks appreciably more between $l_R = 0.2$ and 0.4 m, indicating that the system is much more sensitive to parameter changes when delayed control is involved.

**Perturbations of $h_R$** Rather than generally smaller stable regions created by decreasing $l_R$, Figs. 6.18-6.19 show a more nuanced effect of perturbing the height of the rear center of mass $h_R$. For instantaneous control, increasing $h_R$ had a destabilizing effect, but for delayed control, the effect on the stable region depends on the range
of gains and velocities of interest. Figure 6.18 shows that the upper velocity stability boundary actually grows as $h_R$ increases, where for instantaneous control, the lowest stabilizing roll gain $k_1$ increased, making the stable region slightly smaller. In contrast, Fig. 6.19 shows the stable region become narrower, but also taller as $h_R$ is increased, giving a narrower range of stable velocities, but with a wider stable range of $k_2$.

6.4.3 Steering axis radius

Having demonstrated that differences in rear center of mass position alone do not account for experienced instability on the time trial bicycle, other geometrical differences must be considered. Steering on a time trial bicycle is achieved with the forearms placed a distance of 16–32 cm apart, while typical road bike handlebars have a breadth of 40 cm or more and are steered from either the drops or the tops of the brake hoods. The torque a rider applies is $T_\beta = rF$, where $r$ is the radial distance from the steering axis to the point of application of force $F$. Assuming individual riders are capable of controlling the magnitude of this force according to
their experience and skill level, we propose the following relationship for handlebar torque,

$$T_\beta = r k_i \phi,$$  \hspace{1cm} (6.5)

where the product of the radius and the individual gain $k_i$ represents the effective gain for a given configuration in N, and $k_e = r k_i$. Assuming $k_i$ represents an individual cyclist’s capability of maneuvering for stability, the smaller $r$ of a time trial bike will always make it more difficult to steer because the effective gain will be smaller.

The difference in stability is illustrated by Fig. 6.20. For the same $k_i$, the stability charts vary slightly for time trial bicycles ($r = 12 \text{ cm}$) and road bicycles ($r = 20 \text{ cm}$). At racing velocities, the road bicycle requires a lower gain to be stable, implying a rider need not be as skilled to maneuver successfully. Although the stable regions mostly overlap, the experience and learning necessary to increase $k_i$ sufficiently may be quite large. This plot can also be thought of as a stretching of the $k_2 = 0$ region of Fig. 6.6, and an argument is made for the physical significance of introducing the stretching factor $r$. This gain is currently a theoretical tool, and experimental
investigations would be required to put its magnitude in a physical context.

6.5 Conclusion

Due to two simple physical parameter differences between road and time trial bicycles, rider COM and steering radius, corresponding differences in stability have been demonstrated. Using a set of linearized equations of motion for roll and steer angles, the cyclist’s center of mass was shifted to reflect the change of position on aero bars. While this had a small effect on the stable range of forward velocities, the effect was stabilizing and did not account for the decreased stability of a time trial bicycle. As another possible explanation, a human control law was introduced.

Assuming a steering torque proportional to the roll angle and roll rate allowed the model to be stabilized over the desired range of velocities. The roll angle gain $k_1$ had the biggest impact stabilizing the bicycle at higher velocities, while the roll rate gain $k_2$ stabilized the bicycle at lower velocities. It was then demonstrated that perturbing the rear center of mass in directions consistent with the aero position
(forward and lower) had a stabilizing effect for both parameters, as shown in stability charts varying both $k_1$ and $k_2$ over a range of velocities.

Since the reaction time of the human rider may be important to the stability of different rider positions, time delays were introduced into the control equation. It was shown that adding a reaction time delay shrinks the stable area, and causes changes in rear COM parameters to have a particularly noticeable response. With a time delay, decreases in $l_R$ caused the stable region to shrink appreciably, while an increase in $h_R$ caused stability boundaries to shift in more subtle ways.

With inconclusive stability responses for parameter changes reflecting time trial rider position, we proposed a modification to the control law to account for the geometry of the steering assembly. The effective gain in this controller was equal to the radial displacement of the control force and an individual gain, indicative of a cyclist’s ability to stabilize a bicycle through steering torque. This model was able to account for decreased stability in time trial bicycles, independent of the cyclist’s skill level. The stability charts indicate that time trial bicycles can also be stabilized
for common racing velocities but require a more skillful rider.

Further proposed additions to the control model include exploring the effect of human learning, to determine how the gain might be enhanced with experience. Since sources disagree about the importance of passive and active control in human balance while standing or walking, there is much room for discussion regarding human control of a bicycle. While some maneuvering by a cyclist seems automatic, there may still be a small delay in the human reaction time that would affect stability. Yet another important physical reality neglected by the current linear model is out of plane motion of the rider center of mass. Empirically, cyclists often steer by shifting their body weight, and this adds another dimension to the stability discussion.
## Appendix: Bicycle parameters

Table 6.1: Bicycle parameters used in this work.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>wheel base</td>
<td>$w$</td>
<td>1.02 m</td>
</tr>
<tr>
<td>trail</td>
<td>$c$</td>
<td>0.08 m</td>
</tr>
<tr>
<td>caster angle</td>
<td>$\alpha$</td>
<td>$\pi/10$ (18°)</td>
</tr>
<tr>
<td>wheels</td>
<td>$R_Q, R_P$</td>
<td>0.3 m</td>
</tr>
<tr>
<td></td>
<td>$m_Q, m_P$</td>
<td>2 kg</td>
</tr>
<tr>
<td></td>
<td>$I_{xxQ}, I_{xxP}$</td>
<td>0.0603 kg m²</td>
</tr>
<tr>
<td></td>
<td>$I_{yyQ}, I_{yyP}$</td>
<td>0.12 kg m²</td>
</tr>
<tr>
<td>rear frame</td>
<td>$l_R$</td>
<td>0.3 m</td>
</tr>
<tr>
<td></td>
<td>$h_R$</td>
<td>0.9 m</td>
</tr>
<tr>
<td></td>
<td>$m_R$</td>
<td>85 kg</td>
</tr>
<tr>
<td></td>
<td>$I_R$</td>
<td>$\begin{bmatrix} 9.2 &amp; 0 &amp; 2.4 \ 0 &amp; 11 &amp; 0 \ 2.4 &amp; 0 &amp; 2.8 \end{bmatrix}$ kg m²</td>
</tr>
<tr>
<td>front frame</td>
<td>$l_F$</td>
<td>0.06976 m</td>
</tr>
<tr>
<td></td>
<td>$h_F$</td>
<td>0.1302 m</td>
</tr>
<tr>
<td></td>
<td>$m_F$</td>
<td>4 kg</td>
</tr>
<tr>
<td></td>
<td>$I_F$</td>
<td>$\begin{bmatrix} 0.05892 &amp; 0 &amp; -0.00756 \ 0 &amp; 0.06 &amp; 0 \ -0.00756 &amp; 0 &amp; 0.00708 \end{bmatrix}$ kg m²</td>
</tr>
</tbody>
</table>
Equations of motion for a bicycle on rollers

7.1 Introduction

Since the first description of the bicycle equations of motion in 1899 independently by F. J. W. Whipple and Emmanuel Carvallo [8], bicycle stability has continued to be of interest [33–35]. Useful kinematical considerations and roll without slip constraints were elucidated by Kane [36], and Döhring, then Weir also derived linearized equations of motion using Newton’s second law [10]. Since then, new considerations, including human control and tire and frame compliance have been studied [26–29]. Although the nonlinear equations of motion and their linearization about a forward velocity have long been established, the presence of many physical parameters makes stability a complicated issue, so a review of bicycle literature and benchmark eigenvalues have been published [9]. Physical parameters have been measured experimentally, and experimental bicycle behavior was compared with the linear theory [30]. In general, a bicycle can be self stable independent of gyroscopic or caster effects, if the front assembly turns in the direction of a fall [31]. Incorporating human control models is useful for determining a more realistic stability model. Bicycle
handling has been explained as the need to lean a bike into a turn by means of either counter-steering or hip thrusts, with the bike initially turning in the opposite direction [32].

There have been many designs in the last decades for an apparatus that prevents longitudinal movement of a bicycle, while allowing the cyclist to train. While bicycle trainers fixing the rear and front wheels are common, some designs seek to provide an experience more akin to riding outdoors. One early patent describes a roller affixed to the rear frame to limit rearward rotation and friction when performing a wheelie [50]. Another such US patent features two rollers, a fan facing the bicycle, and a link connected to seat stays that can move laterally [51]. While rollers have been used in bicycle experiments [52–54], very little analysis has been done to compare the dynamics of a bicycle on rollers with a bicycle on flat ground.

This chapter seeks to provide an analytical explanation for the decreased stability on rollers that cyclists experience by comparing the linearized equations of motion for each system. Section 7.2 describes the configuration of a bicycle on rollers and orients the reader to the typical arrangement of bicycle rollers. A kinematic description is given in Sec. 7.3 with position vectors, angular velocities, system energies, and both holonomic and nonholonomic constraints enforcing roller contact and roll-without-slip. Section 7.4 gives a formulation of Lagrange’s equations and discusses the process of linearization and substitution that reduces the number of degrees of freedom influencing stability to three. A stability analysis of the equations is offered in Sec. 7.5, where a new phenomenon specific to rollers is illustrated, and recommendations for the most stable roller configuration are made.

7.2 System description

This derivation considers a bicycle with four frame parts (two wheels, a rear frame, and a front frame) with no tire or frame compliance and requires that the knife-edge
wheels roll without slipping on a set of three cylindrical rollers. Fig. 7.1 shows such a bicycle on a typical system of rollers. Two of the rollers $R_1$ and $R_2$ support the rear wheel, one of which is belted to a third roller $R_3$ which supports the front wheel. The bicycle contacts the rollers at points $A$, $B$, and $C$, and point $D$ indicates where the steering axis would intersect the ground if the bicycle were on a flat plane. For this work, each roller is assumed to have identical properties to the first.

The rear frame and wheel yaw with angle $\psi$ and roll with angle $\phi$. The rear frame pitches with angle $\gamma$. The front assembly is described in a different set of body-fixed coordinates which rotate about the steering axis by angle $\beta$, inclined by the constant caster angle $\alpha$. 

**Figure 7.1:** Line drawing of bicycle on rollers.
7.3 Kinematics

7.3.1 Position and velocity

Considering the origin of the global coordinate system to be the center of roller $R1$ and point $A$ to be the contact point with the rear wheel, as illustrated in Fig. 7.2, position vectors can be written to the center of mass (COM) of the rear wheel $O$, contact points $A$ and $B$, the COM of the rear frame $R$, and point $D$ on the steering
axis,

\[
r_A = \begin{bmatrix} R_1 \sin \theta_1 \\ y \\ -R_1 \cos \theta_1 \end{bmatrix} \quad \text{(7.1)}
\]

\[
r_{O/A} = \mathbf{T}_\psi \mathbf{T}_\phi \begin{bmatrix} R_O \sin \theta_O \\ 0 \\ -R_O \cos \theta_O \end{bmatrix} \quad \text{(7.2)}
\]

\[
r_{B/O} = \mathbf{T}_\psi \mathbf{T}_\phi \begin{bmatrix} R_O \sin \theta_O \\ 0 \\ R_O \cos \theta_O \end{bmatrix} \quad \text{(7.3)}
\]

\[
r_{R/O} = \mathbf{T}_\psi \mathbf{T}_\phi \mathbf{T}_\gamma \begin{bmatrix} l_R \\ 0 \\ R_O - h_R \end{bmatrix} \quad \text{(7.4)}
\]

\[
r_{D/O} = \mathbf{T}_\psi \mathbf{T}_\phi \mathbf{T}_\gamma \begin{bmatrix} w + c \\ 0 \\ R_O \end{bmatrix}, \quad \text{(7.5)}
\]

where \( R_1 \) is the radius of all rollers, \( \theta_1 \) is the angle contact point \( A \) makes with \( R1 \), \( y \) is the lateral position of \( A \), \( R_O \) is the radius of the rear and front wheels, \( \theta_O \) is the angle \( A \) makes with \( O \), \( l_R \) and \( h_R \) are the longitudinal and vertical distance to the rear frame COM, \( w \) is the wheelbase, and \( c \) is the trail, given in Tab. 7.1. The angles \( \theta_1 \) and \( \theta_O \) must be determined by geometric considerations and are given in Subsection 7.3.3. The transformation matrices are

\[
\mathbf{T}_\psi = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix},
\]

\[
\mathbf{T}_\gamma = \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ -\sin \gamma & 0 & \cos \gamma \end{bmatrix}.
\quad \text{(7.6)}
\]

For the front frame coordinates, the position vectors to front frame \( F \), front wheel
Figure 7.3: Line drawing of front wheel on rollers.

$P$, and front wheel contact point $C$ are

$$
\mathbf{r}_{F/D} = T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix}
l_F - w - c \\
0 \\
-h_F
\end{bmatrix}
$$

(7.7)

$$
\mathbf{r}_{P/D} = T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix}
-c \\
0 \\
-R_O
\end{bmatrix}
$$

(7.8)

$$
\mathbf{r}_{C/P} = T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix}
R_O \sin \theta_P \\
0 \\
R_O \cos \theta_P
\end{bmatrix},
$$

(7.9)

where $l_F$ and $h_F$ are the longitudinal and vertical distances to the front frame $F$, given in the appendix, and $\theta_P$ is the angle contact point $C$ makes with the center of
front wheel $P$, derived in Subsection 7.3.3. The transformation matrices are

$$
\mathbf{T}_\alpha = \begin{bmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha \\
\end{bmatrix}, \quad \mathbf{T}_\beta = \begin{bmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

(7.10)

The angular velocities of the frames, wheels, and rollers are

$$
\Omega_O = \left\{ \begin{array}{c} 0 \\
\dot{\delta}_O \\
0 \\
\end{array} \right\} + \mathbf{T}_\phi^T \left( \left\{ \begin{array}{c} \dot{\phi} \\
0 \\
0 \\
\end{array} \right\} + \mathbf{T}_\psi^T \left\{ \begin{array}{c} 0 \\
0 \\
\dot{\psi} \\
\end{array} \right\} \right) 
$$

(7.11)

$$
\Omega_R = \left\{ \begin{array}{c} 0 \\
\dot{\gamma} \\
0 \\
\end{array} \right\} + \mathbf{T}_\gamma^T \left( \mathbf{T}_\phi^T \left( \left\{ \begin{array}{c} \dot{\phi} \\
0 \\
0 \\
\end{array} \right\} + \mathbf{T}_\psi^T \left\{ \begin{array}{c} 0 \\
0 \\
\dot{\psi} \\
\end{array} \right\} \right) \right) 
$$

(7.12)

$$
\Omega_P = \left\{ \begin{array}{c} 0 \\
\dot{\delta}_P \\
0 \\
\end{array} \right\} + \mathbf{T}_\alpha \left( \left\{ \begin{array}{c} 0 \\
0 \\
\dot{\beta} \\
\end{array} \right\} + \mathbf{T}_\beta^T \mathbf{T}_\alpha^T \Omega_R \right) 
$$

(7.13)

$$
\Omega_F = \mathbf{T}_\alpha \left( \left\{ \begin{array}{c} 0 \\
0 \\
\dot{\beta} \\
\end{array} \right\} + \mathbf{T}_\beta^T \mathbf{T}_\alpha^T \Omega_R \right) 
$$

(7.14)

$$
\Omega_1 = \left\{ \begin{array}{c} 0 \\
\dot{\delta}_1 \\
0 \\
\end{array} \right\} 
$$

(7.15)

$$
\Omega_2 = \left\{ \begin{array}{c} 0 \\
\dot{\delta}_2 \\
0 \\
\end{array} \right\} 
$$

(7.16)

$$
\Omega_3 = \left\{ \begin{array}{c} 0 \\
\dot{\delta}_3 \\
0 \\
\end{array} \right\}, 
$$

(7.17)

where \(\dot{\delta}_O\) and \(\dot{\delta}_P\) are the angular velocities of the rear and front wheels, and \(\dot{\delta}_1, \dot{\delta}_2,\) and \(\dot{\delta}_3\) are the angular velocities of the rollers.
7.3.2 System energies

Summing energy expressions over all four frame parts and the three roller components, the kinetic and potential energies of the system are

\[
T = \frac{1}{2} m_O \dot{\mathbf{r}}_O \cdot \mathbf{\dot{r}}_O + \frac{1}{2} m_R \dot{\mathbf{r}}_R \cdot \mathbf{\dot{r}}_R + \frac{1}{2} m_P \dot{\mathbf{r}}_P \cdot \mathbf{\dot{r}}_P + \frac{1}{2} m_F \dot{\mathbf{r}}_F \cdot \mathbf{\dot{r}}_F
\]

\[
+ \frac{1}{2} \Omega^T \mathbf{I}_O \Omega_O + \frac{1}{2} \Omega^T \mathbf{I}_R \Omega_R + \frac{1}{2} \Omega^T \mathbf{I}_P \Omega_P + \frac{1}{2} \Omega^T \mathbf{I}_F \Omega_F
\]

\[
+ \frac{1}{2} \Omega^T \mathbf{I}_1 \Omega_1 + \frac{1}{2} \Omega^T \mathbf{I}_2 \Omega_2 + \frac{1}{2} \Omega^T \mathbf{I}_3 \Omega_3
\]

(7.18)

\[
V = -m_O g \dot{\mathbf{r}}_O \cdot \mathbf{\dot{K}} - m_R g \dot{\mathbf{r}}_R \cdot \mathbf{\dot{K}} - m_P g \dot{\mathbf{r}}_P \cdot \mathbf{\dot{K}} - m_F g \dot{\mathbf{r}}_F \cdot \mathbf{\dot{K}},
\]

(7.19)

where the mass moment of inertia tensors for the bicycle are those given by

\[
\mathbf{I}_O = \begin{bmatrix}
I_{xxO} & 0 & 0 \\
0 & I_{yyO} & 0 \\
0 & 0 & I_{zzO}
\end{bmatrix}, \quad \mathbf{I}_R = \begin{bmatrix}
I_{xxR} & 0 & I_{xxR} \\
0 & I_{yyR} & 0 \\
I_{xzR} & I_{yyR} & I_{zzR}
\end{bmatrix}, \quad (7.20)
\]

\[
\mathbf{I}_P = \begin{bmatrix}
I_{xxP} & 0 & 0 \\
0 & I_{yyP} & 0 \\
0 & 0 & I_{zzP}
\end{bmatrix}, \quad \mathbf{I}_F = \begin{bmatrix}
I_{xxF} & 0 & I_{xxF} \\
0 & I_{yyF} & 0 \\
I_{xzF} & I_{yyF} & I_{zzF}
\end{bmatrix}, \quad (7.21)
\]

and those for the rollers are given by

\[
\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}_3 = \begin{bmatrix}
I_{xx1} & 0 & 0 \\
0 & I_{yy1} & 0 \\
0 & 0 & I_{zz1}
\end{bmatrix}.
\]

(7.22)

7.3.3 Holonomic constraints

For the bicycle on flat ground, the holonomic constraints included an implicit requirement that the rear wheel touch the ground and one further requirement that the front wheel touch the ground, thereby defining the dependent coordinate \(\gamma\). On rollers, angles of contact between the roller and wheel have been defined to yield dependent coordinates \(\theta_1 = \theta_2, \theta_3, \theta_O, \theta_P, \text{ and } \gamma\) that are defined by the system parameters and independent coordinates. To constrain these angles geometrically, intersection and tangency conditions provide the five holonomic constraints.
Rear wheel

Figure 7.4 shows a diagram of contact point $B$ between the roller and the rear wheel. The position vector to this point can be written through point $O$ or $R_2$. From Eqns. 7.1-7.3,

$$\mathbf{r}_B = \begin{cases} R_1 \sin \theta_1 \\ y \\ -R_1 \cos \theta_1 \end{cases} + \mathbf{T}_\psi \mathbf{T}_\phi \begin{cases} 2R_O \sin \theta_O \\ 0 \\ 0 \end{cases}, \tag{7.23}$$

where $\theta_1$ is the angle from the roller axis to the contact point $B$, and $\theta_O$ is the angle between the contact point and the center of the rear wheel $O$. The vector to the same point through the second roller is

$$\mathbf{r}_B = \begin{cases} d_1 - R_1 \sin \theta_1 \\ Y_1 \\ -R_1 \cos \theta_1 \end{cases}, \tag{7.24}$$

where $Y_1$ is the lateral displacement at point $B$, and $d_1$ is the distance between $R1$ and $R2$. The wheel has a tangent unit vector,

$$\mathbf{t}_O = \mathbf{T}_\psi \mathbf{T}_\phi \begin{cases} -\cos \theta_O \\ 0 \\ \sin \theta_O \end{cases}, \tag{7.25}$$
and the vector normal to the roller is
\[ \mathbf{n}_1 = \begin{pmatrix} -\sin \theta_1 \\ 0 \\ -\cos \theta_1 \end{pmatrix}. \tag{7.26} \]

The constraint equation for the reference position ($\psi = \phi = 0$) will be computed first. For this case, the expression for the rear wheel becomes,
\[ \mathbf{r}_B = \begin{pmatrix} R_1 \sin \theta_1 + 2R_O \sin \theta_O \\ y \\ -R_1 \cos \theta_1 \end{pmatrix} = \begin{pmatrix} d_1 - R_1 \sin \theta_1 \\ Y_1 \\ -R_1 \cos \theta_1 \end{pmatrix}. \tag{7.27} \]

Equating with Eqn. 7.24 gives the intersection condition (the wheel and roller must touch),
\[ 2(R_O \sin \theta_O + R_1 \sin \theta_1) - d_1 = 0, \tag{7.28} \]
and that $Y_1 = y$. Furthermore, the tangent vector to the rear wheel must be orthogonal to the normal vector of the roller (this ensures that the curves only intersect at one point),
\[ \begin{pmatrix} -\cos \theta_O \\ 0 \\ \sin \theta_O \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta_1 \\ 0 \\ -\cos \theta_1 \end{pmatrix} = 0 \tag{7.29} \]
\[ \cos \theta_O \sin \theta_1 - \sin \theta_O \cos \theta_1 = \sin(\theta_1 - \theta_O) = 0, \tag{7.30} \]
indicating either that $\theta_1 - \theta_O = 0$ or $\pi$. For the relevant physical geometry, we choose the first solution, $\theta_1 = \theta_O$. For the reference position,
\[ \sin \theta_1 = \frac{d_1}{2(R_1 + R_O)}. \tag{7.31} \]

For the case where $\psi \neq 0$ and $\phi \neq 0$, the intersection and orthogonality equations are,
\[ 2(R_1 \sin \theta_1 + R_O \sin \theta_O \cos \psi) - d_1 = 0 \tag{7.32} \]
\[ \cos \theta_O \sin \theta_1 \cos \psi - \sin \theta_O \sin \theta_1 \sin \psi \sin \phi - \sin \theta_O \cos \theta_1 \cos \phi = 0. \tag{7.33} \]
It will be useful to find the Taylor expansion for small $y$, $\psi$, and $\phi$ and $\theta_1$ and $\theta_0$ about their reference position values ($\theta_1 \rightarrow \theta_{1c} + \epsilon \theta_1$, $\theta_0 \rightarrow \theta_{1c} + \epsilon \theta_0$), where the result from the reference position can be used to find the constant $\theta_{1c}$, leaving

$$ h_1 = 2\epsilon(R_1 \theta_1 + R_0 \theta_O) \cos \theta_{1c} + \epsilon^2 (-R_1 \theta_1^2 - R_0(\theta_0^2 + \psi^2)) \sin \theta_{1c} = 0 \quad (7.34) $$

$$ h_2 = \epsilon(\theta_1 - \theta_O) + \frac{1}{4} \epsilon^2 (2(-1 + \cos(2\theta_{1c})) \psi \phi + \sin(2\theta_{1c})(\phi^2 - \psi^2)) = 0. \quad (7.35) $$

**Front wheel** Following a similar procedure for the front wheel gives the remaining three equations for $\theta_3$, $\theta_P$, and $\gamma$. The position vector to point $C$ can be written two ways and equated,

$$ \mathbf{r}_C = \begin{bmatrix} R_1 \sin \theta_1 \\
-y \\
-R_1 \cos \theta_1 \end{bmatrix} + T_\psi T_\phi \begin{bmatrix} R_O \sin \theta_O \\
0 \\
-R_O \cos \theta_O \end{bmatrix} + T_\psi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix} w + c \\
0 \\
R_O \end{bmatrix} $$

$$ + T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix} -c + R_P \sin \theta_P \\
0 \\
R_P(-1 + \cos \theta_P) \end{bmatrix} $$

$$ = \begin{bmatrix} d_1 + d_2 - R_1 \sin \theta_3 \\
Y_3 \\
-R_1 \cos \theta_3 \end{bmatrix}, \quad (7.36) $$

where $d_2$ is the distance between $R2$ and $R3$, $\theta_3$ is the angle of the contact point $C$ with $R3$, and $Y_3$ is the lateral location of $C$ from $R3$. The tangent vector to the front wheel is given by,

$$ \mathbf{\hat{t}}_P = T_\psi T_\phi T_\gamma T_\alpha T_\beta T_\alpha^T \begin{bmatrix} -\cos \theta_P \\
0 \\
\sin \theta_P \end{bmatrix} \quad (7.37) $$

and the normal vector to the front roller is,

$$ \mathbf{\hat{n}}_3 = \begin{bmatrix} -\sin \theta_3 \\
0 \\
-\cos \theta_3 \end{bmatrix}. \quad (7.38) $$
In the reference position, the intersection condition in Eqn. 7.36 and tangency condition given by requiring the dot product of Eqn. 7.37 and Eqn. 7.38 be zero give,

\[ R_1 \sin \theta_1 + R_O \sin \theta_O + R_1 \sin \theta_3 + R_O \sin(\gamma + \theta_P) + w \cos \gamma - d_1 - d_2 = 0 \]  
\[ -R_1 \cos \theta_1 + R_O \cos \theta_O + R_1 \cos \theta_3 + R_O \cos(\gamma + \theta_P) - w \sin \gamma = 0 \]  
\[ \sin(\gamma - \theta_3 + \theta_P) = 0. \]  
(7.39)  
(7.40)  
(7.41)

Noting from Eqn. 7.41 that \( \theta_P = \theta_3 - \gamma \) in the reference position, the intersection equations can be written,

\[ (R_1 + R_O)(\sin \theta_1 + \sin \theta_3) + w \cos \gamma - d_1 - d_2 = 0 \]  
\[ (R_1 + R_O)(- \cos \theta_1 + \cos \theta_3) - w \sin \gamma = 0. \]  
(7.42)  
(7.43)

For the case where yaw, roll, and steering angles are nonzero, the equations are much more complex. A second order Taylor expansion of Eqn. 7.36 was used in this work to provide the three holonomic constraints associated with contact point \( C \), given in Eqns. 7.44–7.46. The expansion assumes small \( y, \psi, \phi, \) and \( \beta \) and \( \theta_1, \theta_3, \theta_O, \theta_P, \) and \( \gamma \) about their reference position values \( (\theta_1 \rightarrow \theta_1c + \epsilon \theta_1, \theta_3 \rightarrow \theta_3c + \epsilon \theta_3, \theta_O \rightarrow \theta_1c + \epsilon \theta_O, \theta_P \rightarrow \theta_3c - \gamma_0 + \epsilon \theta_P, \) and \( \gamma \rightarrow \gamma_0 + \epsilon \gamma) \).

\[ h_3 = \epsilon((R_O \cos \theta_3c - w \sin \gamma_0) \gamma + R_1 \cos \theta_1c \theta_1 + R_1 \cos \theta_3c \theta_3 + R_O \cos \theta_1c \theta_O + R_O \cos \theta_3c \theta_P) + \frac{1}{2} \epsilon^2 (\cos(\alpha + \gamma_0)(c \cos \alpha + R_O(- \sin \alpha + \sin(\alpha + \gamma_0 - \theta_3c)))) \beta^2 \]
\[ - (w \cos \gamma_0 + R_O \sin \theta_3c) \gamma^2 - R_1 \sin \theta_1c \theta_1^2 - R_1 \sin \theta_3c \theta_3^2 - R_O \sin \theta_1c \theta_O^2 \]
\[ - 2R_O \sin \theta_3c \gamma \theta_P - R_O \sin \theta_3c \theta_P^2 + 2(c \cos \alpha + R_O(- \sin \alpha + \sin(\alpha + \gamma_0 - \theta_3c))) \beta \psi \]
\[ - 2(R_O \cos \theta_1c - R_O \cos \theta_3c + w \sin \gamma_0) \phi \psi - (w \cos \gamma_0 + R_O \sin \theta_1c + R_O \sin \theta_3c) \psi^2 \]  
\[ = 0 \]  
(7.44)
\[ h_4 = -\epsilon (w \cos \gamma_0 + R_O \sin \theta_{3c}) \gamma - R_1 \sin \theta_{1c} \theta_1 - R_O \sin \theta_{1c} \theta_O + R_O \sin \theta_{3c} \theta_P \]
\[ - \frac{1}{2} \epsilon^2 (\sin(\alpha + \gamma_0)(c \cos \alpha + R_O(- \sin \alpha + \sin(\alpha + \gamma_0 - \theta_{3c}))) \beta^2 \]
\[ + (R_O \cos \theta_{3c} - w \sin \gamma_0) \gamma^2 - R_1 \cos \theta_{1c} \theta_{1c}^2 - R_O \cos \theta_{1c} \theta_O^2 + 2R_O \cos \theta_{3c} \gamma_0 \theta_P \]
\[ + R_O \cos \theta_{3c} \theta_P^2 + 2(c \cos \alpha + R_O(- \sin \alpha + \sin(\alpha + \gamma_0 - \theta_{3c}))) \beta \phi \]
\[ - (R_O \cos \theta_{1c} - R_O \cos \theta_{3c} + w \sin \gamma_0) \phi^2 \]  
\[ = 0 \quad (7.45) \]

\[ h_5 = -\epsilon (\gamma - \theta_3 + \theta_P) + \frac{1}{4} \epsilon^2 (\sin(2(\alpha + \gamma_0 - \theta_{3c})) \beta^2 + \sin(2\theta_{3c}) \phi^2 - 4 \sin^2 \theta_{3c} \phi \psi \]
\[ - \sin(2\theta_{3c}) \psi^2 + 4 \cos(\alpha + \gamma_0 - \theta_{3c}) \beta(\cos \theta_{3c} \phi - \sin \theta_{3c} \psi)) = 0 \quad (7.46) \]

In the reference position, roller parameters \( R_1, d_1, \) and \( d_2 \) determine \( \theta_{1c}, \theta_{3c}, \) and \( \gamma_0. \)

For the special case \( d_2 = w, \theta_{1c} = \theta_{3c}, \) and \( \gamma_0 = 0. \)

### 7.3.4 Nonholonomic constraints

![Figure 7.5: Rear wheel nonholonomic constraints.](image)

Ensuring that the wheels roll without slipping on the rollers yields four nonholonomic constraints, two in the direction tangent to the velocity vectors and two
orthogonal to the velocity and the surface normal. As shown in Fig. 7.5, the constraints require that the velocity of the contact point, and the instantaneous velocity due to the rotation of the roller and the wheel are zero,

\[ \mathbf{v}_A + (\omega_O \times \mathbf{r}_{A/O}) - (\omega_1 \times \mathbf{r}_A) = 0, \quad (7.47) \]

where the position vectors are given in Subsection 7.3.1, and the angular velocities are transformed into global coordinates. Linearizing about \( \theta_1 = \theta_O = \theta_{1c} \) gives,

\[ g_1 = \epsilon(R_1(\dot{\delta}_1 + \dot{\theta}_1) + R_O \dot{\delta}_O) = 0 \quad (7.48) \]
\[ g_2 = \epsilon(\dot{y} - v \cos \theta_{1c} \psi + v \sin \theta_{1c} \phi) = 0. \quad (7.49) \]

Similarly, for the front wheel,

\[ g_3 = \epsilon(R_1(\dot{\delta}_3 - \dot{\theta}_3) + R_O \dot{\delta}_P) = 0 \quad (7.50) \]
\[ g_4 = \epsilon(v(- \cos \theta_{1c} + \cos \theta_{3c}) \psi + v(\sin \theta_{1c} + \sin \theta_{3c}) \phi + R_O(- \cos \theta_{1c} + \cos \theta_{3c}) \dot{\phi} - w \sin \gamma_0 \dot{\phi} - R_O(\sin \theta_{1c} + \sin \theta_{3c}) \dot{\psi} - w \cos \gamma_0 \dot{\psi} + (c \cos \alpha - R_O \sin \alpha) \dot{\beta} + v \cos(\alpha + \gamma_0 - \theta_{3c}) \beta + R_O \sin(\alpha + \gamma_0 - \theta_{3c}) \dot{\beta}) = 0, \quad (7.51) \]

where \( g_2 \) has been used to replace the \( \dot{y} \) term in \( g_4 \).

7.4 Equations of motion

7.4.1 Lagrange’s equations

When determining equations of motion starting from the general form of d’Alembert’s principle

\[ \left[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} - Q^{NP} \right] \delta q = Q^C \delta q = 0, \quad (7.52) \]

where the Lagrangian \( \mathcal{L} \equiv \mathcal{T} - V \), there are multiple options for techniques that will account for generalized forces of constraint. Holonomic constraints can be directly embedded in the system energies. For holonomic or semiholonomic (exactly
integrable) constraints, the Lagrangian can be augmented with the constraint and Lagrange multiplier because the displaced paths are geometrically possible [6]. However, for general nonholonomic constraints, the constraints can be adjoined to d’Alembert’s principle, once conditions on the virtual displacements are determined.

The linear-acceleration form of the constraints leads to the tangency condition for possible states. Decomposing the constraints into independent and dependent parts and using the variation of dependent displacements provides conditions on the displacements under the general velocity constraints. Adjoining these restrictions to d’Alembert’s principle and requiring that the displacement δq be arbitrary gives the equations of state for a system with holonomic and nonholonomic constraints [7],

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j \frac{\partial h_j}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial \dot{q}_i},
\]

(7.53)

where \(i, j, \) and \(k\) are the indices of the generalized coordinates, holonomic constraints, and nonholonomic constraints, respectively, \(\lambda\) is the Lagrange multiplier of each holonomic term, and \(\mu\) is the Lagrange multiplier for each nonholonomic term. In this system, using the generalized coordinates, \(y, \psi, \phi, \beta, \delta_O, \delta_1, \delta_2, \delta_3, \delta_P, \theta_1, \theta_O, \theta_3, \theta_P, \) and \(\gamma\), there are thirteen Lagrange’s equations, nine constraint equations, and nine unknown Lagrange multipliers, for a total of twenty-two equations and unknowns.

7.4.2 Linearization

To find the equations of motion linearized about a constant forward velocity \(v\) (the velocity the bicycle would have on flat ground), second order Taylor expansions of the Lagrangian and constraints were used, where \(y, \psi, \phi, \) and \(\beta\) were assumed to be small, \(\delta_O \) and \(\delta_P \) were perturbed about \(-vt/R_O, \delta_1, \delta_2, \) and \(\delta_3\) about \(vt/R_1, \theta_1 \) and \(\theta_O \) about \(\theta_{1c}, \theta_3 \) about \(\theta_{3c}, \gamma \) about \(\gamma_0, \) and \(\theta_P \) about \(\theta_{3c} - \gamma_0.\)

The linearized Lagrange equations for the dependent variables \(L_{\theta_1}, L_{\theta_O}, L_{\theta_3}, L_{\theta_P}, L_{\gamma}, \) can be solved for the holonomic Lagrange multipliers, \(g_2 \) can be solved for \(\dot{y} \) and
differentiated to find $\ddot{y}$. However, solving $g_4$ for $\dot{\psi}$ is not as straightforward as in the classic bicycle case, since $\dot{\psi} = \dot{\psi}(\psi, \phi, \phi, \beta, \beta)$ instead of $\dot{\psi} = \dot{\psi}(\beta, \beta)$, as the constraint yields on a flat surface. For the system on rollers, the yaw of the bicycle has an effect on stability that is not present on a flat surface. For this reason, a third equation must be included. Since, the Lagrange multiplier $\mu_4$ is determined from $L_\psi$, the remaining equation comes from $g_4$. It is convenient to consider the constraint in its acceleration form and include it with $L_\phi$ and $L_\beta$ in the linearized equations of motion,

$$
\begin{align*}
\begin{bmatrix}
M_{\phi\phi} & M_{\phi\beta} & M_{\phi\psi} \\
M_{\beta\phi} & M_{\beta\beta} & M_{\beta\psi} \\
M_{\psi\phi} & M_{\psi\beta} & M_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\beta} \\
\ddot{\psi}
\end{bmatrix}
+ v
\begin{bmatrix}
C_{\phi\phi} & C_{\phi\beta} & C_{\phi\psi} \\
C_{\beta\phi} & C_{\beta\beta} & C_{\beta\psi} \\
C_{\psi\phi} & C_{\psi\beta} & C_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\beta} \\
\dot{\psi}
\end{bmatrix}
&+ g
\begin{bmatrix}
K_{\phi\phi} & K_{\phi\beta} & K_{\phi\psi} \\
K_{\beta\phi} & K_{\beta\beta} & K_{\beta\psi} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\beta \\
\psi
\end{bmatrix}
= 0,
\end{align*}
$$

(7.54)

where the eigenvalues of this system can be used to determine stability.

### 7.5 Stability analysis

As a result of the additional equation of motion, the roller system has two more eigenvalues than the bicycle. The effect of yaw on the rear wheel, or a perturbation in $\psi$, creates a new behavior this work will refer to as “dip.” The parameter $d_O = d_1/2$ will be useful for indicating the distance the middle roller is forward of the rear wheel hub, and $d_P = d_2 + d_O - w$ indicates the distance the front roller is forward of the front wheel hub. Figure 7.6 shows the eigenvalues of the roller system when $d_O = 0$ (dip eigenvalues are zero), and $d_O = 0.1$ m, where the dip phenomenon is illustrated by the eigenvalues that are slightly positive over the whole velocity range, and the weave eigenvalues remain positive, unlike the classic bicycle case or the single rear roller case where $d_O = 0$. In contrast to the weave, capsize, and castering effects of the standard bicycle, dip is always slightly unstable.
Figure 7.6: Eigenvalues of a bicycle on rollers for $R_1 = d_P = 0$ and $d_O = 0$ (circles) and $d_O = 0.1$ m (solid lines).

Figure 7.7 isolates the effect of moving the front contact point forward of the wheel hub by enforcing the single rear roller case (no dip phenomenon). The weave eigenvalues become less stable, while capsize and castering eigenvalues become more stable. These cases demonstrate that there are individual destabilizing effects for perturbing either rear or front contact points.

Since all physically relevant configurations of rollers are unstable, stability charts would be relatively uninteresting. Instead, Figs. 7.8–7.10 give contour plots of the largest real eigenvalue, illustrating stability gradients for different parameter configurations. Figure 7.8 shows by the nearly vertical contour lines that there is little effect of roller radius on stability when the contact points are fixed directly under
Figure 7.7: Eigenvalues of a bicycle on rollers for $R_1 = d_O = 0$ and $d_P = 0$, (circles) and $d_P = 0.05$ m (solid lines).

Figure 7.9 shows a contour plot indicating the most stable values of $d_O$. The lightest spike near zero indicates that the spacing between the rear rollers should ideally be as small as possible for the most stable position. However, the instability for the lowest forward velocities decreases with increasing $d_O$, so there could be a low speed stability benefit to spacing the rollers at $d_O/R_O = 0.1$.

Perhaps most interesting to the cyclist configuring his or her rollers are the results in Fig. 7.10, which indicate the most stable position for the front roller. Manuals and online articles make recommendations about the position of the front drum relative to the front wheel hub; for example, “The axle of your front wheel should be directly
over, or up to 0.5 inches [0.0127 m] behind, the axle of the front roller. Stability will be compromised if the wheel is in front of the roller” [55], or “Check to see that the front drum is exactly underneath your front wheel’s axle or up to 1.5 inches [0.0381 m] ahead of it. Take note: Never ride rollers with your bike’s front axle ahead of the front roller’s axle, or more than 2 inches [0.0508 m] behind” [56]. The plot corroborates some of this advice. The most stable region occurs for values of $d_P/R_O$ between 0 and 0.05 in this model. This puts the front roller up to 0.015 m (0.6 inches) ahead of the front wheel hub, which agrees with the most common advice.

7.6 Conclusion

The derivation of equations of motion for a bicycle on rollers closely resembles that of the classic bicycle, except for the nature of the constraints. In addition to more complex nonholonomic roll-without-slip constraints, the holonomic constraints governing $\theta_1, \theta_O, \theta_3, \theta_P$, and $\gamma$ must be taken into account. The key difference in the
linearized equations of motion is that the yaw $\psi$ has an effect on the stability of the system, while it does not for a bicycle on a flat surface.

These additional eigenvalues, representing the phenomenon referred to as dip, cause the system to have a slight instability over all velocities. Riders have experienced this decreased stability on rollers and have observed that it requires extra diligence in steering to overcome. Finally, recommendations for the most stable roller parameters were made. Some are relevant to manufacturing concerns, and other can be implemented in the assembly of the roller set.

Figure 7.9: Contour plot of largest real eigenvalue for $d_O/R_O$ vs. $v$ with $R_1 = d_P = 0$. 
Figure 7.10: Contour plot of largest real eigenvalue for $d_P/R_O$ vs. $v$ with $R_1 = d_O = 0$. 
## Appendix: Bicycle parameters

Table 7.1: Bicycle parameters used in this work.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>wheel base</td>
<td>$w$</td>
<td>1.02 m</td>
</tr>
<tr>
<td>trail</td>
<td>$c$</td>
<td>0.08 m</td>
</tr>
<tr>
<td>caster angle</td>
<td>$\alpha$</td>
<td>$\pi / 10 \ (18^\circ)$</td>
</tr>
<tr>
<td>wheels</td>
<td>$R_O, R_P$</td>
<td>0.3 m</td>
</tr>
<tr>
<td>mass</td>
<td>$m_O, m_P$</td>
<td>2 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>$I_{xxO}, I_{xxP}$</td>
<td>0.0603 kg m$^2$</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>$I_{yyO}, I_{yyP}$</td>
<td>0.12 kg m$^2$</td>
</tr>
<tr>
<td>rear frame</td>
<td>$l_R$</td>
<td>0.3 m</td>
</tr>
<tr>
<td>horizontal distance from $A$</td>
<td>$h_R$</td>
<td>0.9 m</td>
</tr>
<tr>
<td>vertical distance from $A$</td>
<td>$m_R$</td>
<td>85 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>$I_R$</td>
<td>$\begin{bmatrix} 9.2 &amp; 0 &amp; 2.4 \ 0 &amp; 11 &amp; 0 \ 2.4 &amp; 0 &amp; 2.8 \end{bmatrix}$ kg m$^2$</td>
</tr>
<tr>
<td>front frame</td>
<td>$l_F$</td>
<td>0.06976 m</td>
</tr>
<tr>
<td>horizontal distance from $B$</td>
<td>$h_F$</td>
<td>0.1302 m</td>
</tr>
<tr>
<td>vertical distance from $B$</td>
<td>$m_F$</td>
<td>4 kg</td>
</tr>
<tr>
<td>mass moment of inertia</td>
<td>$I_F$</td>
<td>$\begin{bmatrix} 0.05892 &amp; 0 &amp; -0.00756 \ 0 &amp; 0.06 &amp; 0 \ -0.00756 &amp; 0 &amp; 0.00708 \end{bmatrix}$ kg m$^2$</td>
</tr>
</tbody>
</table>
This research intends to provide insight into several nonlinear systems that roll without slipping. They can either be holonomic systems in the case of the eccentric disk in two dimensions, or nonholonomic as in the case of Euler disk and the bicycle in three dimensions. For these different problems, a variety of analytical techniques are employed, including linearization and approximate analytical techniques. Of particular importance is the stability of these systems in different configurations.

Chapter 1 oriented the reader to the most important prior work on each topic. It served to clarify the goals of this thesis and describe the context of the work. Motivation for portraying the different systems was also described.

A classic nonholonomic system in Chapter 2 provided insight into the more complex problems that follow by deriving equations of motion for Euler’s disk or the “rolling coin” problem. This system was used to illustrate the Euler angle transformation matrices that make establishing kinematics more convenient. The nonlinear and linearized equations of motion were then derived using a Lagrangian approach and two methods for incorporating the constraints. First, a generalized constraint force was included directly in Lagrange’s equations with a Lagrange multiplier, then
an alternative method was offered, whereby Gauss’s principle is used to join the non-holonomic constraint equations in acceleration form. Finally, the linearized equations are compared to a simulation of the nonlinear equations.

8.1 Eccentric disk

Chapter 3 explored the special case of a disk with a mass imbalance, which exhibits very different behavior from Euler’s disk. General equations of motion were derived first, followed by the specification that the curve is cubic, creating a potential well. In addition to escape, interesting nonlinear behavior was exhibited by the system as determined by a variety of approximate analytical and numerical techniques. Basins of attraction plots provided additional insight into the behavior of the system when different parameters are varied.

The numerical results of Chapter 3 were compared to a corresponding experiment in Chapter 4. This chapter explained the construction of the experimental set up and discussed its limitations. Image tracking was used to provide information about the disk’s motion. These points were transformed into the $\xi-\eta$ plane to compare the experimental disk’s motion with the corresponding numerical simulation, achieving reasonable agreement.

8.2 Bicycle

Chapter 5 began the material on bicycle stability by deriving the linearized equations of motion. The kinematics were thoroughly explained with Euler angle transformation matrices and all position vectors given. The system energies were used to find the Lagrangian, expanded to its second order Taylor series. One holonomic and four nonholonomic constraints were then incorporated into Lagrange’s equations with Lagrange multipliers. A discussion followed of the steps taken to reduce the system to roll and steer degrees of freedom. Finally, a plot of the eigenvalues confirmed the
results with existing literature. Full expressions for the coefficients of the equations of motion are given.

These equations of motion were used in Chapter 6 to examine the stability of time trial and road bicycles. The cyclist’s center of mass parameters were perturbed with three different steering conditions: uncontrolled, instantaneous control, and delayed control. The uncontrolled case looked at the effect of COM coordinates on bicycle self stability. A linear control law proportional to roll and roll rate was proposed, and stability chart were shown that demonstrate the effect human control and rear COM perturbation have on stability. Next, to better represent the physical reality of human reaction time, a time delayed control law was incorporated. A stability chart was also “stretched” to represent the effect of a cyclists arms being closer to the steering axis on a time trial bicycle.

8.3 Bicycle on rollers

Chapter 7 extended the classic bicycle problem to that of a bicycle on the type of trainer known as rollers, requiring more dependent coordinates and more holonomic constraints to describe the positions of the contact points. The kinematics were described similarly to Chapter 5. Then the holonomic constraints were determined by requiring intersection and orthogonality of the parametric expressions for the rollers and wheels. The nonholonomic roll-without-slip constraints were also derived, requiring the sum of the contact point velocity and angular velocities of the wheel come to zero at that point. Using these constraints, the linearized equations of motion were found to differ from the bicycle equations because the yaw angle $\psi$ was required to determine stability. Lastly, a brief stability analysis provided a mathematical explanation for the instability empirically experienced by cyclists on rollers, along with recommendations for the most stable configuration, depending on bicycle parameters.
8.4 Future work

Further research into these systems would include more approximate analytical work for the eccentric disk, in addition to the harmonic balance technique applied to the point mass case. Theoretical work on predicting the chaotic transitions would be useful, as well as including multi-frequency, colored noise, and stochastic excitation. Melnikov analysis with these types of forcing would be useful, and the corresponding experiment would be a natural extension of the one in this work.

For the classic bicycle, much work has been done, but extending the delay control laws would be worthwhile, especially gathering experimental evidence relating to how humans steer. It may be that the lateral COM motion is more important to stabilization than handlebar steering, as was investigated in this thesis. Further exploring the method of human balance would be useful, particularly investigating the extent to which our control relies on roll, roll rate, or roll acceleration input. Finding a self-contained expression for the full nonlinear equations of motion would, of course, be the holy grail of bicycle dynamics.

Since the equations for a bicycle on rollers are new, there are many directions the analysis could take, but one of the most practical interest would be using a confirmed control law to see how the system behaves with a rider. The work done in Chapter 7 dealt with self-stability, and incorporating human control would be very telling for interpreting the instability of a rider on rollers.
Bibliography


Genevieve Marie Lipp was born in Orange, California on September 1, 1987. She attended middle and upper school at St. John’s School in Houston, Texas, and it was there that she developed the interest in math and physics that would cause her to matriculate in mechanical engineering at Duke University. She graduated with a B.A. in German and a B.S.E. in mechanical engineering in 2010. Under the supervision of Dr. Brian Mann, she continued in a graduate program and received a M.S. in mechanical engineering in 2013. This work marks the completion of her Ph.D. in mechanical engineering and third, possibly final, degree from Duke University.

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When she isn’t deriving, Genevieve can be found making ice cream with local produce, rock climbing, and singing in the soprano section of the North Carolina Master Chorale.