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# Discounting the distant future: how much do uncertain rates increase valuations?

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## Abstract

We demonstrate that when the future path of the discount rate is uncertain and highly correlated, the distant future should be discounted at significantly lower rates than suggested by the current rate. We then use two centuries of US interest rate data to quantify this effect. Using both random walk and mean-reverting models, we compute the “certainty-equivalent rate” that summarizes the effect of uncertainty and measures the appropriate forward rate of discount in the future. Under the random walk model we find that the certainty-equivalent rate falls continuously from 4% to 2% after 100 years, 1% after 200 years, and 0.5% after 300 years. At horizons of 400 years, the discounted value increases by a factor of over 40,000 relative to conventional discounting. Applied to climate change mitigation, we find that incorporating discount rate uncertainty almost doubles the expected present value of mitigation benefits.

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## 1. Introduction

Implicit in any long-term cost-benefit analysis is the idea that costs and benefits can be compared across long periods of time using appropriate discount rates. Yet positive discount rates lead us to place very little weight on events in the distant future, such as potential calamities arising from global warming. For example, a dollar invested now yields \$51 after 100 years if the risk-free market return is 4%. Conversely, a promise to pay someone \$1 in 100 years—with complete certainty—is worth only two cents today at a 4% rate of discount. And a promise to pay someone (or rather, his or her descendants) \$1 in 200 years is worth only 4/100 of one cent today.

To bankers, financiers, or economists—people trained in the art of geometric discounting—this is a dull result. From an intuitive perspective, however, it is a bit more surprising. Individuals

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consistently demonstrate the use of a declining discount rate in the future [1]. This effect can be particularly evident when valuations relate to an individual's own lifetime versus future generations [21]. Use of a declining rate of discount is frequently referred to as hyperbolic discounting (in contrast to conventional, geometric discounting). However, the use of a deterministic declining rate, though perhaps consistent with individual preferences, produces time-inconsistent decisions.<sup>1</sup>

An alternative to using continuously declining rates is to simply use a lower rate for long-term projects. Notable economists such as Ramsey [41], for example, have argued that applying a positive rate of pure time preference to discount values across generations is “ethically indefensible.”<sup>2</sup> More recently, Arrow et al. [2] describe normative arguments for lower future discount rates under the rubric of a “prescriptive” approach to discounting, in contrast to a “descriptive” approach that relies fully on historical market rates of return to measure discount rates. In practice, policymakers have, in some cases, begun applying lower discount rates to long-term, intergenerational projects [8]. Unfortunately, this approach comes close to causing the same time-consistency problems as long-term projects in the present become near-term projects in the future.

In contrast, we consider an alternative explanation of declining rates that fits squarely within the standard framework of geometric discounting based on market-revealed rates. The only distinction with most applications of geometric discounting is that we explicitly acknowledge that the discount rate itself is *uncertain*. As a direct consequence of this discount rate uncertainty, there is an increase in the expected net present value of future payoffs.<sup>3</sup> This under appreciated result, is a straightforward implication of Jensen's inequality. Because discounted values are a convex function of the discount rate, the expected discounted value will be greater than the discounted value computed using an average rate. Put differently, the variable over which we should take expectations is not the discount *rate* ( $r$ ), as is typically done, but rather the discount *factor* ( $e^{-rt}$ ), which enters expectations linearly [47].

Suppose, for example, that we are evaluating a project that yields \$1000 in benefits 200 years in the future. The rate is thought to be 4% on average, but is uncertain: it could be 1%

<sup>1</sup> For example, suppose one uses a 5% rate for 100 years and 0% afterward. In the year 2000, a choice that trades a \$1 loss in 2150 for a \$2 gain in 2200 is desirable, because there is no discounting between these periods. After 2050, this choice begins to look worse and worse, as the interval between 2150 and 2200 begins to be discounted. Eventually, an optimizing decision maker will want to reverse the initial decision, instead choosing the \$1 in 2150. Recognizing that this will assuredly happen represents a time inconsistency. That is, the mere passage of time will make the decision maker want to change the initial choice. For further discussion, see Cropper and Laibson [22] and Heal [26].

<sup>2</sup> Such ethical concerns have been raised in the context of climate change (our motivating example) by Batie and Shugart [7]. They more broadly argue against discounted cost-benefit analysis and advocate an approach that ensures the survival of species, habitats, and ecosystems unless the costs of doing so are “unacceptably large.” Despite healthy interest in alternatives, it is hard to argue that a well-executed discounted cost-benefit analysis is not an important component of any decision-making process. On the one hand, it provides a consistent framework for trading off similarly long-lived social efforts, such as curing disease, providing defense, alleviating poverty, and protecting the environment. On the other hand, an alternative like the one proposed by Batie and Shugart requires “a social decision to be made through the political process” in order to define unacceptably large costs—but does not define what information goes into that political process. Presumably, discounted cost-benefit analysis is precisely the kind of information that would be useful. Finally, Cooper [18] reveals that cost-benefit analysis continues to attract the support of most economists. No one should pretend that cost-benefit analysis alone could ever be the sole basis for decision-making—but neither should one ignore its value as a consistent and relatively objective tool.

<sup>3</sup> If there is an opportunity to postpone a project, a second “option value” effect of interest rate uncertainty arises and creates a value of waiting to see whether interest rates rise or fall [24,28].

or 7% with equal probability. If we simply use the average rate of 4% for discounting, then the present value benefit of this project ( $\$1000e^{-0.04 \times 200}$ ) is 34 cents. On the other hand, if we take expectations properly, we find that the expected value of the project's benefits are now seen to be \$68, which is *200 times higher* than when we improperly used average rates ( $0.5(\$1000e^{-0.01 \times 200}) + 0.5(\$1000e^{-0.07 \times 200}) = \$68$ ).

As shown by Weitzman [47], a striking corollary of this result is that the effective “certainty-equivalent” or forward rate of discount (corresponding to the expected discount factor) will decline over time. And this decline in the effective rate is especially dramatic as  $t$  becomes large. In fact, in the limit as  $t$  approaches infinity, Weitzman finds that the effective rate will decline to the *minimum possible* discount rate when a fixed but uncertain rate persists forever. Intuitively, the only relevant scenario in the limit is the one with the lowest possible interest rate, because all other possible higher interest rates have been rendered insignificant by comparison through the power of compounding over time. The same intuition holds in the intervening years—lower potential rates dominate as one moves further into the future because higher rates receive less and less weight as they are discounted away. As illustrated above, that result has potentially huge implications for the valuation of benefits in the distant future—such as those associated with mitigation of climate change, long-lived infrastructure, reduction of hazardous and radioactive waste, and biodiversity—benefits that are discounted to a pittance when the discount rate is treated as if it is exactly known.<sup>4</sup>

It turns out that a crucial condition underlying the above results is that the discount rate is not only uncertain but also highly persistent. In the simple example above (and in Weitzman [48], described below) the true but unknown rate is assumed to be fixed forever. Uncertainty without persistence will have little consequence for the effective rate if a high rate in one period is likely to be offset by low rates in subsequent periods. For this effect to have real punch, our expectation must be that periods of low rates will tend to be followed by more periods of low rates; the same goes for high rates. This is one of the key questions we will look to historical data to answer.

In order to quantify discount rate uncertainty, Weitzman [48] conducts an email survey of more than 2000 economists, asking them to *state* their “professionally considered gut feeling” about the appropriate real discount rate for valuing environmental projects. Here, uncertainty represents a *current* lack of consensus about the correct discount rate for all *future* time periods.<sup>5</sup> We take a different approach: We assume there is a reasonable consensus about the correct discount rate today based on market rates, but that this rate is likely to change over time. We further assume that historical patterns of interest rate changes *reveal* the likely patterns of changes in the future.<sup>6</sup>

<sup>4</sup>In a concrete application to climate policy, Pizer [39] shows through simulations that uncertainty about future discount rates leads to the use of lower-than-average effective rates.

<sup>5</sup>The difference of opinion expressed by the surveyed economists could be explained by different assumptions of each respondent about the incidence of environmental costs and benefits on consumption versus investment, taxes, and inflation—even though the survey twice specifies that this discount rate will be applied to “expected-consumption-equivalent real dollars.” In an unrelated survey of economists, Ballard and Fullerton [5] find that conventional economic beliefs often supercede the specific features of a question presented for quick response.

<sup>6</sup>The use of historical data to assess the likelihood of future events is a basic tool of scientific inquiry. While we believe there are few practical alternatives to assuming that broad historical patterns will repeat themselves, we would argue that even if one chooses to pursue an alternative, predictions based on historical data provide an important point of comparison.

Two important features therefore motivate this work and distinguish this effort from previous work on discounting. First, future rates decline in our model because of dynamic uncertainty about future events, not static disagreement over the correct rate, nor an underlying belief or preference for deterministic declines in the discount rate. This circumvents concerns about time inconsistency previously associated with the use of declining rates.<sup>7</sup> Second, we use market data to quantify uncertainty about and persistence in future discount rates. This has been an important focus of the literature on the term-structure of interest rates, with recent work corroborating Weitzman's qualitative point that forward rates should eventually decline [13]. However, this literature has not sought to perform valuations over the kinds of long-term time horizons that motivate our work, and has instead focused on forecasts of and arbitrage among various short-term investments.

We find significant empirical evidence that historical rates are indeed uncertain and persistent. For example, there has been a secular decline in interest rates that goes back at least 200—if not more than 1000—years, overlain with persistent deviations of 1–2% lasting 20–40 years. Using this evidence to predict future rate behavior, we find that the certainty-equivalent rate falls gradually from 4% to below 1% at horizons beyond 400 years. However, the particular path of certainty-equivalent rates crucially depends on whether we believe that interest rates are a random walk or are mean reverting—a determination that is ambiguous in our data where the point estimate is 0.98 for the largest autoregressive root.

We find that under the random walk assumption, the certainty-equivalent rate falls from 4%, to 2% after 100 years, 1% after 200 years, and 0.5% after 300 years. In contrast, a mean-reverting model indicates certainty-equivalent rates that remain above 3% for the next 200 years, falling to 1% after 400 years. After 400 years, the cumulative effect of these rates is to raise valuations by a factor of more than 40,000 under the random walk model and a factor of 130 under the mean-reverting model. When we apply these rates to a real problem—the potential path of future damages due to current carbon dioxide emissions—we find that uncertainty almost doubles the present value of those damages based on the random walk model, while raising the present value a more modest 14% based on the mean-reverting model. Despite the ambiguity of statistical tests, we argue that the random walk model is more credible because the mean rate over the distant past is less informative than the recent past when we forecast at any horizon in the future. We explore this distinction in detail and discuss ways that both approaches can be combined.

## 2. A model of discounting with uncertain rates

There are two natural approaches to modeling interest rate behavior. The first is a structural approach, which view interest rates as an endogenous outcome of a dynamic general equilibrium model. For example, the interest rate equilibrates demand for new capital and the supply of savings in a Ramsey growth model. Versions of this model with uncertainty, including either single or multiple stochastic factors, have been well studied (e.g., [11,46,19]). In these models, the exogenous, stochastic variables describing technology and growth determine the behavior of

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<sup>7</sup>In an uncertain world there is always the possibility that ex ante good decisions turn out to be regrettable ex post, once nature has revealed herself—much like the purchase of insurance seems wasteful once the risk has uneventfully passed. This stands in contrast to time-inconsistent behavior where we know *with certainty* that our choice now opposes our choice in the future.

multiple interest rates (i.e., the term-structure), as well as other endogenous variables. Parameters are estimated by examining the relationships among the variables.

The second approach—which we adopt—models the time-series behavior of a single interest rate in a reduced-form manner. For example, we could specify that the spot interest rate follows a first-order autoregressive process. This approach does not model the underlying determinants of the interest rate or impose structural restrictions. However, the reduced-form approach provides a transparent connection between historic data and forecast values. More importantly, the apparent strength of a structural model—the representation of underlying economic relations—can turn out to be a weakness because such models are typically based on restrictive, simplifying assumptions. This is particularly true for long-range forecasting, where it is difficult to forecast the explanatory variables over long horizons and the potential for variation in underlying “structural” parameters greatly increases. On the other hand, structural models are well-suited to provide internally consistent estimates and forecasts of inter-related economic variables over shorter horizons.

Empirical work in the term-structure literature highlight problems with the structural approach over longer time horizons. A term-structure model estimated from a single cross-section of returns on assets with varying maturities can be used to predict an appropriate rate for any maturity. In practice, however, Brown and Dybvig [14] find that such an approach overestimates the long-term rate for the full sample period they investigate, and provides mixed results over different historical subperiods—sometimes over-predicting and other times under-predicting actual long-term rates. These authors also find that the interest rate volatility implied by estimating the widely-used Cox et al. [20] term-structure model on cross-sectional data, is at times zero or negative. In addition, estimates of the parameters underlying the term-structure model vary significantly depending on the time period analyzed [12].

While most interest rate research focuses on near-term forecasting, valuation, and arbitrage, and can ignore intermittent regime shifts and prediction errors in long-term rates—we cannot. Our focus is precisely the very-long-term interest rate. To properly capture interest rate uncertainty over such long horizons, we need to incorporate exactly those types of regime shifts that make structural parameters unstable. While we cannot forecast the precise magnitude and timing of individual future events that might influence the interest rate—such as wars<sup>8</sup> or shifts in macroeconomic policy<sup>9</sup>—we can treat these events in a stochastic manner based on a long history of experience.

All of this leads us to a flexible, reduced-form model estimated from a single interest rate series. This allows us to develop a much longer data series and therefore enhances our ability to capture a larger number of historical, structure-changing events in the estimated volatility.

### 2.1. *The model*

We begin by specifying the following stochastic model of interest rate behavior. The interest rate in period  $t$ ,  $r_t$ , is uncertain, and its uncertainty has both a permanent component  $\eta$ , and a

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<sup>8</sup> For example, Barro [6] finds that government purchases associated with wars account for significant fluctuations in interest rates, with the magnitude of their influence varying from conflict to conflict.

<sup>9</sup> For example, evidence suggests that changes in Federal Reserve policy led to a once-and-for-all shift in the behavior short-term interest rates after 1982 [3]. This has led many researchers of term structure to work only with data after 1982 [10].

more transitory component  $\varepsilon_t$ :

$$r_t = \eta + \varepsilon_t. \quad (1)$$

The permanent component  $\eta$  has a normal distribution with mean  $\bar{\eta}$  and variance  $\sigma_\eta^2$ . The other random component,  $\varepsilon_t$ , is an autocorrelated mean-zero shock to the discount rate governed by

$$\varepsilon_t = \rho \cdot \varepsilon_{t-1} + \xi_t, \quad (2)$$

where the correlation parameter  $\rho$  describes the persistence of deviations from the mean rate, and  $\xi_t$  is an independent identically distributed (iid), mean-zero, normally distributed random variable with variance  $\sigma_\xi^2$ . The two uncertain components  $\eta$  and  $\varepsilon_t$  are assumed to be independent. Thus, the mean of  $r_t$  is  $\bar{\eta}$ .<sup>10</sup>

To summarize, the interest rate has some mean value  $\bar{\eta}$ , which is itself uncertain, as well as transitory deviations from this mean rate. A value of  $\rho$  near one means that the interest rate can persistently deviate from the mean rate, staying consistently above or below it for many periods. In this case, the best guess of next year's rate is about equal to last year's rate. A value of  $\rho$  near zero, in contrast, means that a period of abnormally high interest rates may be followed by rates above or below the mean with equal probability, so that the next year's expected rate will be about equal to the mean rate. As we noted earlier, allowing for this persistence is essential. Uncertainty without persistence will have little consequence for discounting the distant future.

Discounting future consequences in period  $t$  back to the present is typically computed as the discount factor  $P_t$ , where

$$P_t = \exp\left(-\sum_{s=1}^t r_s\right).$$

Because  $r$  is stochastic, the *expected* discounted value of a dollar delivered after  $t$  years (i.e., the certainty-equivalent discount factor) will be<sup>11</sup>

$$E[P_t] = E\left[\exp\left(-\sum_{s=1}^t r_s\right)\right]. \quad (3)$$

<sup>10</sup> Many interest rate models (e.g., [20]) ignore uncertainty about the permanent component ( $\eta$ ) and specify that the volatility of the transitory component ( $\varepsilon$ ) is time-varying and proportional to the square root of the interest rate,  $\sigma_{\xi,t} \propto \sqrt{r_t}$ . This latter assumption captures the notion that rates near zero are likely to fluctuate less, and is similar to our own adjustment in the empirical section.

<sup>11</sup> This expected discount factor ignores the potential for risk aversion. That is, Eq. (3) assumes that an investor (borrower) is indifferent between a certain debit (credit) now and an uncertain credit (debit) in the future with the same expected value. Household needs such as housing, college and retirement (and similar business needs) suggest that this assumption is not exactly correct. The general idea that preferences for assets and obligations with specific maturities can affect asset prices vis-à-vis Eq. (3) is referred to as the "preferred habitat theory" [34]. A similar result arises in structural interest rate models driven by stochastic factors (e.g., [20,46]) where a no-arbitrage condition leads to a "market price of risk" or "liquidity premium" associated with longer-term issues (if wealth is negatively correlated with

Following Weitzman (1998), we define the corresponding certainty-equivalent forward rate for discounting between adjacent periods at time  $t$  as equal to the rate of change of the expected discount factor:

$$\tilde{r}_t = -\frac{dE[P_t]/dt}{E[P_t]}. \quad (4)$$

Note that  $\tilde{r}_t$  is the instantaneous period-to-period rate at time  $t$  in the future—not an average rate for discounting between period  $t$  and the present.

Evaluating Eq. (3) using Eq. (1), one can show that the expected discount factor for period  $t$  is (see [36] for derivation):

$$E[P_t] = \exp\left(-\bar{\eta}t + \frac{t^2\sigma_\eta^2}{2}\right) \exp\left(\frac{\sigma_\xi^2}{2(1-\rho)^2} \left(t - \frac{2(\rho - \rho^{t+1})}{1-\rho} + \frac{\rho^2 - \rho^{2t+2}}{1-\rho^2}\right)\right), \quad (5)$$

assuming  $|\rho| < 1$ . The corresponding certainty-equivalent discount rate at time  $t$  is given by the rate of change of  $E[P_t]$ , as shown in Eq. (4); it is equal to

$$\tilde{r}_t = \bar{\eta} - t\sigma_\eta^2 - \sigma_\xi^2\Omega(\rho, t), \quad (6)$$

where  $\Omega(\rho, t)$  is the effect of autocorrelation in the interest rate shocks and is given by

$$\Omega(\rho, t) = \frac{1 - \rho^2 + 2 \log(\rho)\rho^{t+1}(1 + \rho - \rho^{t+1})}{2(1 - \rho)^3(1 + \rho)},$$

assuming  $|\rho| < 1$ <sup>12</sup> with  $\lim_{t \rightarrow \infty} \Omega(\rho, t) = 1/2(1 - \rho)^2$ .

## 2.2. Implications of the model

Eq. (6) gives the basic result of the model of discount rate uncertainty. The certainty-equivalent rate declines from the mean rate with increases in the forecast period ( $t$ ), uncertainty in the mean rate ( $\sigma_\eta^2$ ), uncertainty in deviations from the mean rate ( $\sigma_\xi^2$ ), and the degree of persistence in those deviations ( $\rho$ ). The second term of Eq. (6) captures the influence of uncertainty in the mean interest rate, and the third term captures the effect of persistent deviations away from the mean. Positive correlation reduces the certainty-equivalent discount rate because  $\Omega(\rho, t)$  will always be positive; this effect is increasing in  $\rho$  and  $t$ .

Thus, the certainty-equivalent discount rate will become lower the further one forecasts into the future, the greater the uncertainty in the mean interest rate, and the greater the variance and persistence (correlation) of shocks in the interest rate. As mentioned earlier, the degree of

(footnote continued)

the spot interest rate). In this paper, we set aside the issue of market risk and preferred habitats because there are neither markets nor market arbitrage opportunities to establish an appropriate premium over the horizons we consider. If anything, we believe the appropriate thought experiment for our motivating example—where the government compares climate mitigation efforts to compensation of future climate change losses through market lending—would depress longer-term rates, as the government might find itself demanding a fixed return on its lending over hundreds of years with few if any willing borrowers. Under these assumptions, our estimated certainty equivalent rates are too high and our valuations of future consequences are too low.

<sup>12</sup>For the case of  $\rho = 1$ ,  $\Omega(\rho, t) = \frac{1}{12}(1 + 6t + 6t^2)$ , while for  $\rho = 0$ ,  $\Omega(\rho, t) = \frac{1}{2}$ .

persistence in discount rate fluctuations turns out to be a critical component of what drives the certainty-equivalent rate down over time and there are dramatic consequences for values of  $\rho$  near one. This can be seen by choosing reasonable values for the mean rate and each of the parameters in Eq. (6), and then varying  $\rho$  to see its effect. For example, letting  $\bar{\eta} = 4\%$ ,  $\sigma_{\xi} = 0.23\%$ ,  $\sigma_{\eta} = 0.52\%$ , and  $\rho = 0.96$ —parameter estimates presented in the next section—we find leads to very little change in the certainty-equivalent rate over 200 years.<sup>13</sup> In contrast, a value of  $\rho = 1$  leads to the certainty-equivalent rate declining to only 1% after just 100 years. In practice, the estimated mean and standard error of  $\rho$  do not rule out the possibility of  $\rho$  equal to or very close to one, as we discuss further below.

While illustrative, this simple model unfortunately implies that the certainty-equivalent rate eventually becomes negative. This is a straightforward feature of our model that fails to rule out negative rates based on the specification given in Eq. (1) with normally distributed errors (which we assume for analytic tractability). Although one could imagine circumstances supporting negative expected rates, none are observed in the millennia of data covered by our primary source on interest rates, Homer and Sylla [27]. Further, common sense about the pure rate of time preference suggests that persistently negative rates are unlikely (see Chapter 7 in [32]). This is a second issue we need to address if we want to model interest rate behavior realistically.

### 3. Estimation of interest rate behavior

Our simple analytical model of interest rate behavior provides a useful guide to the various parameter combinations that make discounting with uncertain rates substantially different from discounting with certain rates. The model parameters are easily estimated from an autoregression of appropriate historical interest rate data. As we previously noted, however, there are several problems with the simple model as a realistic model of historical interest rate behavior.

First, the analytical model does not rule out the possibility of persistently negative discount rates even though rates below 1% have rarely been observed. Second, our analytical results demonstrate that small differences in the estimate of the autoregressive parameter  $\rho$  near one have significant consequences for the certainty-equivalent discount rate. At the same time, a large literature in time-series econometrics—and more recently the term structure of interest rates—emphasizes that ordinary inference is inappropriate and  $\rho$  is biased downward when the true value of  $\rho$  is near or equal to one ([4,23,38,44], Chapter 17 of [25]).<sup>14</sup> Given the statistical error in our estimate of  $\rho$ , which does not rule out  $\rho = 1$ , we need to acknowledge uncertainty in  $\rho$  as well as  $\eta$  and  $\varepsilon$ .

These problems can be overcome by modifying our earlier model. First, we transform the model in a way that prevents negative rates. In particular, we assume

$$r_t = \eta \exp(\varepsilon_t), \quad (7)$$

<sup>13</sup>Plugging these parameters into Eq. (6) yields  $\tilde{r}_t = 3.6\%$  after 100 years and  $\tilde{r}_t = 3.3\%$  after 200 years.

<sup>14</sup>Specifically, the “unit root” literature shows that estimates of  $\rho$  are asymptotically biased downward when  $\rho = 1$ , with a standard 5% *t*-test of the  $\rho = 1$  null hypothesis falsely rejecting 65% of the time [35]. When  $\rho$  is near but not equal to one, there will also be a finite-sample bias. Ball and Torous [4] show that this problem arises in estimates of interest rate dynamics and can substantially bias estimates of long-term yields.

or after taking logs,

$$\ln r_t = \ln \eta + \varepsilon_t, \quad (8)$$

with  $\eta$  now modeled as a log-normally distributed random variable<sup>15</sup> with mean  $\bar{\eta}$  and variance  $\sigma_\eta^2$ , and where we generalize the autoregressive form for  $\varepsilon_t$  given by Eq. (2) to allow for  $\varepsilon_t$  to depend on more than one past value:

$$\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_L \varepsilon_{t-L} + \xi_t, \quad (9)$$

where  $L$  is the number of lagged values included in the model and the  $\rho$ s are autoregressive coefficients.<sup>16</sup> This is called the “mean-reverting” model because, with  $\sum_s \rho_s < 1$ , the series would eventually tend toward its long-run mean.

Second, we test the null hypothesis that historical interest rates are random walks (with  $\sum_s \rho_s = 1$ ) and, if the test fails to reject, we estimate the random walk version of Eq. (8).<sup>17</sup> This is given by

$$\ln r_t = \ln r_0 + \varepsilon_t, \quad (10)$$

where now we impose the constraint that  $\sum \rho_s = 1$ . Note that  $r_0$  in the random walk model replaces  $\eta$  in the mean-reverting model because interest rates are now modeled as an accumulation of permanent innovations from an initial rate ( $r_0$ ), rather than deviations from a long-run mean ( $\eta$ ).

Finally, we use the estimated covariance matrix of the model parameters, including those describing the degree of autocorrelation, to randomly draw combinations of parameters as well as stochastic shocks  $\xi_t$ . We use these draws to simulate one hundred thousand alternative future paths for the discount rate. We then use these simulations to compute the certainty-equivalent rate numerically, rather than analytically.

<sup>15</sup>A normality test on the data rejects a normal distribution for the data in levels, but does not reject normality once the data have been logged. A logarithmic transformation also has the “empirically relevant” properties described by Cox et al. [20]: (1) negative rates are precluded, (2) if the rate gets close to zero it can subsequently increase, (3) the absolute variance of the interest rate increases with the interest rate itself, and (4) there is a steady state distribution of the interest rate. Relative to the commonly used models of Vasicek [46] and Cox, Ingersoll, and Ross [20], a logarithmic transformation makes interest rate volatility more sensitive to the level of the rate. However, even higher sensitivity has been recommended in an empirical study by Chan et al. [15], who find that such models do a better job predicting future rates—precisely our goal. This particular approach, with a logarithmic transformation, is a straightforward way to insure our discrete-time simulations remain positive and is equivalent to the geometric Brownian motion model used by Black and Scholes [9] and Marsh and Rosenfeld [33]. As suggested by Lubrano [31], there continues to be disagreement over the best empirical model of interest rate behavior.

<sup>16</sup>Note that the typical approach in the finance literature is to include multiple error terms in Eq. (8), each representing different AR(1) “factors” that differentially affect yields with different maturities [30]. Because we estimate a linear model of a single rate, however, it is observationally equivalent to instead use a single error term with a more general ARMA( $p, q$ ) representation, based on the Wold representation theorem.

<sup>17</sup>The standard approach to univariate modeling of time series is to specifically test the random walk hypothesis using one of several approaches. If the model rejects, the series is presumed to be stationary and standard methods can be applied directly to the data. If the model fails to reject, there is an unfortunate ambiguity because these tests have notoriously low power—that is, they are unable to distinguish among true values of  $\rho$  near unity. Thus, either of two approaches may be reasonable. One can impose a unit root and estimate the remaining parameters using standard methods. Or one can continue to estimate the unconstrained model but recognize the likely bias. We do both.

We estimate the model parameters conditional on the initial observations, dropping those for which lagged values are not directly observed. With a single lagged value in the autoregression, this is equivalent to the Cochrane-Orcutt [17] method; with more lagged values, this approach is referred to as conditional maximum likelihood [25]. We pick the number of lagged values in the autoregression based on the Schwarz–Bayes information criterion [42].<sup>18</sup> For comparison, we also estimate the “simple autoregressive model” given by Eqs. (1) and (2), which includes only a first-order autoregressive lag and is estimated on unlogged data.

### 3.1. Data

To estimate the model of interest rate behavior, we compiled a series of market interest rates over the two-century period 1798 through 1999. The question of which rate to use is naturally contentious and has been discussed at length by numerous authors (see e.g., [2,29,40]). Plausible candidates include rates of return from bonds and other debt instruments, equities, or direct investment. Even within these broad categories there are a variety of possibilities with various risk levels, time horizons, and tax characteristics. For the present analysis, we have focused on US market interest rates for long-term, high-quality, government bonds (primarily US Treasury bonds).<sup>19</sup> This decision was based mainly on our desire to construct a very long time series of relatively low-risk rates of return and is described more thoroughly in [36].

We compiled a series of bond yields based on Homer and Sylla’s [27] monumental *History of Interest Rates* and used their assessments to determine the best instrument among high-quality, long-term government bonds available each year. Based on these nominal rates, we create a series of real interest rates by subtracting, after 1950, a measure of expected inflation using the Livingston Survey of professional economists [45], also described in [36]. We then convert these rates to their continuously compounded equivalents. The final data set has 202 observations. We estimate the models using a 3-year moving average of the real interest rate series to smooth very short-term fluctuations, which we are not interested in modeling here and which would otherwise mask the longer-term behavior of the data in which we are interested. The resultant interest rate series are shown in Fig. 1. We also note that our basic results are robust to the market interest rate used, the approach selected for inflation adjustment, and whether the data are smoothed before estimation; see [36] for further detail.

Although a full discussion of the causes of various trends and patterns of these rates is beyond the scope of this paper, a few points are worth noting. There appears to be a fairly steady downward trend from 1800 through at least 1940. Viewed on a grander historical scale, this general trend has been apparent for at least the last millennium.<sup>20</sup> Deviations from any mean or trend are persistent: for example, interest rates over 1860–1870 and 1910–1920 are noticeably

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<sup>18</sup>The Schwarz-Bayes information criterion equals the log-likelihood minus a penalty,  $k/2 \ln(n)$ , where  $k$  is the number of parameters and  $n$  is the number of observations. This criterion reveals the (asymptotic) logged odds ratios for any collection of models, regardless of dimension.

<sup>19</sup>Long-term bond yields in other industrialized countries have historically followed similar trends and general fluctuations, although the magnitude of fluctuations has at times differed substantially [27].

<sup>20</sup>Over the last 1000 years, the range of yields on long-term debt for the United States, England, and several European countries shows highly persistent changes, from rates that averaged near 10% around AD 1000 to rates currently averaging less than 5%. See [36].

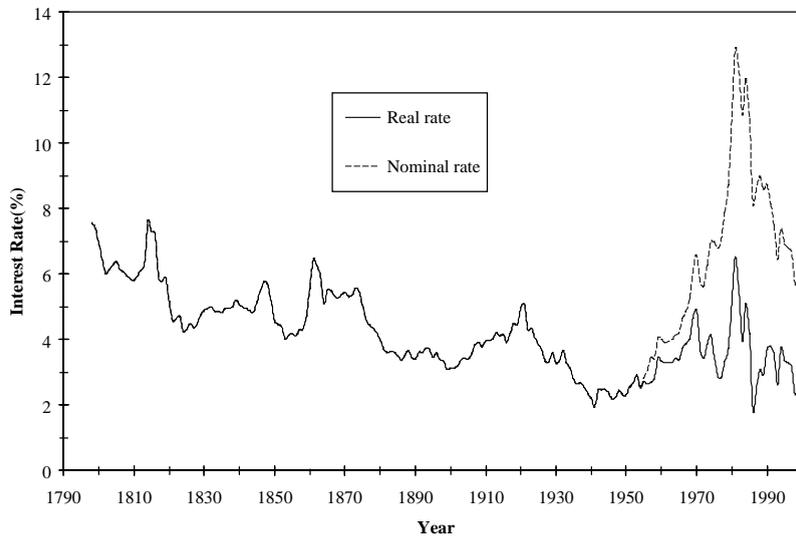


Fig. 1. Market interest rate on US long-term government bonds (1798–1999).

higher than in adjacent periods. Finally, the rates observed during the early 1980s—even adjusted for inflation—are the highest since the early 1800s.

### 3.2. Estimation results

We first test the random walk hypothesis for the historical interest rate data in both levels and logs using the augmented Dickey–Fuller test with appropriate critical values [23].<sup>21</sup> With a point estimate of  $\rho = 0.976$  and standard error of 0.011, we fail to reject the hypothesis of a random walk in the model with logged interest rates and no trend, as well as in other models based on unlogged interest rates and including trends.<sup>22</sup> Because our results are therefore likely to be extremely sensitive to either a downward bias in  $\rho$  (if we estimate without imposing a unit root) or a specific assumption that  $\rho = 1$ , we estimate and present results for both the random walk and the mean-reverting models (Eqs. (10) and (8), respectively). We discuss the implications of these models below.<sup>23</sup> In the end, we believe the only convincing way to decide between random walk and mean-reverting models is to ask whether—having observed unusually low rates for an extended period (say 30–40 years)—one anticipates a return to the longer-run average or a continuation of low rates. We find the latter perspective more compelling.

<sup>21</sup> See [36] for detailed results on the unit root tests.

<sup>22</sup> Unless otherwise noted, we use  $\rho$  as shorthand to refer to the single autoregressive parameter in a simple AR(1) process or the largest root in a more general AR( $L$ ) process.

<sup>23</sup> It is possible using a Bayesian approach to compute the posterior probabilities for the mean-reverting and random walk models [43]. Placing equal prior weight on each model (and a flat prior over the mean-reverting interval  $\rho \in (0.5, 1.0)$ ) yields a posterior likelihood of 59% for the random walk; using Sims' preferred prior of a 20% weight on the random walk model yields a posterior likelihood of 26% for the random walk model. We discuss how one might combine the results of the two models using such probabilities in the next section.

Table 1

Estimation results ( $\ln r_t = \ln \eta + \varepsilon_t$  and  $\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \dots + \rho_L \varepsilon_{t-L} + \zeta_t$ )

	Random walk model ( $\sum \rho_s = 1$ )		Mean-reverting model		Simple unlogged model <sup>a</sup>	
Mean rate ( $\bar{\eta}$ )			3.69 <sup>*,b</sup>	3.95 <sup>*,b</sup>	3.52*	3.92*
Std error ( $\sigma_\eta$ )			0.45	0.23	0.52	0.31
Autoregressive coefficients <sup>c</sup>						
$\rho_1$	1.92* (0.06)	1.92* (0.06)	1.88* (0.07)	1.85* (0.07)	0.96* (0.01)	0.94* (0.02)
$\rho_2$	-1.34* (0.12)	-1.34* (0.12)	-1.31* (0.12)	-1.26* (0.12)		
$\rho_3$	0.43* (0.07)	0.43* (0.07)	0.40* (0.07)	0.36* (0.07)		
Trend		-0.0037 <sup>d</sup> (0.0055)		-0.0033 <sup>*,d</sup> (0.0010)		-0.010 <sup>d</sup> (0.005)
$\sigma_\zeta^2$	0.0015* (0.0002)	0.0015* (0.0002)	0.0015* (0.0002)	0.0015* (0.0002)	0.0522* (0.0052)	0.0517* (0.0052)

\*Significant at the 5% level.

<sup>a</sup> Simple unlogged model is  $r_t = \eta + \varepsilon_t$ , where  $\varepsilon_t = \rho \varepsilon_{t-1} + \zeta_t$ .

<sup>b</sup> The mean rates in this table were constructed, for simulation purposes, to reflect a continuously compounded rate. To convert to simple annual rates, simply compute  $100 \times (\exp(\eta/100) - 1)$ , e.g.,  $100 \times (\exp(3.69/100) - 1) = 3.76$ .

<sup>c</sup> Number of autoregressive terms chosen using the Schwarz–Bayes information criterion.

<sup>d</sup> Indicates a linear time trend estimated alongside  $\varepsilon_t$  in each model.

Table 1 presents estimates of the three models. We estimate each model with and without a time trend. The time trend is significant only in the mean-reverting model. However, this apparent significance is suspect because the coefficient will have a nonstandard distribution if logged interest rates follow a random walk, which we cannot reject.<sup>24</sup> Further, it is unclear whether and how to extrapolate an observed historical trend into the future. Any extrapolation will be extremely sensitive to functional form; for example, a trend in the log of interest rates has very different implications than a trend in the level of interest rates. For these reasons, the remainder of our discussion focuses on the models without time trends.

All three models yield remarkably consistent estimates. The mean-reverting and simple unlogged models provide similar estimates of both the mean interest rate and its associated standard error. The random walk and mean-reverting models provide similar estimates of the autocorrelation parameters—not surprising, since we are unable to resolve whether a unit root exists (i.e., whether  $\rho = 1$ ). One can further show that, after appropriate transformation, the estimates of  $\sigma_\zeta^2$  are also consistent across these models (see [36]).

Despite similar parameter estimates, the mean-reverting and random walk models paint starkly different pictures of the future. The mean-reverting forecast has a prediction interval that remains

<sup>24</sup> Nelson and Plosser [35] show that a standard 5% *t*-test will falsely reject the null hypothesis of no trend 30% of the time even when the underlying series is truly a random walk without a trend.

virtually unchanged after 30 or 40 years, centered on the long-run mean to which it regresses. In contrast, the random walk forecast is centered on the last observation and has a prediction interval that continues widening. Comparing prediction intervals to realized rates based on a sub-sample of our data, we find some impetus to favor the random walk model in our application. A split-sample forecast in 1899, for example, yields a 95% prediction interval for the mean-reverting model that lies above the actual realized rate beyond the late 1930s [36]. Similar patterns appear in virtually any out-of-sample forecast. This tendency of the mean-reverting forecasts to over-predict future rates based on a 95% prediction interval is troubling because we know that the lower range of possible interest rates ultimately determines the future certainty-equivalent rate. Because the random walk model has better prediction interval coverage of realized rates in most split samples, we find it more compelling for our application, even though evidence based on standard statistical tests is ambiguous.

#### 4. Forecasts and application of certainty-equivalent discount rates

We now construct a numerical approximation to the certainty-equivalent discount rate based on simulations using the estimated discount rate uncertainty, including uncertainty about the degree of autocorrelation. We simulate 100,000 possible future discount rate paths for each model starting in 2000 and extending 400 years into the future. For each model estimated in Table 1, the simulations begin with point estimates of the parameters and their joint covariance matrix. Then, for each simulated discount rate path, we randomly draw parameter values assuming joint normality.<sup>25</sup> Next, we draw values for the stochastic shocks  $\xi$  and create the  $\varepsilon$  shocks by recursively defining  $\varepsilon$  based on Eq. (9). Finally, we use the simulated values of  $\varepsilon$  to construct a path of simulated discount rates based on Eqs. (1), (8), or (10), depending on the model.<sup>26</sup>

After repeating the process of picking parameters and shocks and computing discount rate paths 100,000 times for each model, we compute the expected discount factor  $E[P_t]$  based on Eq. (3). The certainty-equivalent discount rate is computed as the discrete approximation to Eq. (4), given by  $\tilde{r}_t = E[P_t]/E[P_{t+1}] - 1$ . Table 2 presents the resulting estimates of discount factors associated with the logged models (8) and (10) over the next 400 years based on a 4% rate of return in 2000, along with the constant rate reference case.<sup>27</sup> Fig. 2 shows the certainty-equivalent rates corresponding to the discount factors given in Table 2 as well as rates derived from the simple, unlogged, autoregressive model estimated in Table 1.

<sup>25</sup> In simulations of the mean-reverting and simple unlogged models, we also replace any random draws that result in explosive autoregressive models; i.e., when  $\Sigma\rho_i > 1$ , which would unrealistically imply that current innovations  $\xi$  have an increasing rather than diminishing effect on future discount rates.

<sup>26</sup> We also impose in the models a very small deterministic trend to offset the positive trend that would otherwise exist in the actual (unlogged) rate due to an increasing variance in the logged rate over time. This correction does not affect the parameter estimates. See [36].

<sup>27</sup> We begin with an initial rate of 4%, rather than the actual bond rate of about 3% in 2000, because short-term forecasts of the interest rate suggest it is likely to rise over the next few years—making a fair comparison with a constant interest rate impossible. A rate of 4% reflects the approximate 200-year average in our data as well as the average over the past 20 years. It also falls close to the middle of the range of defensible consumption rates of interest (2–7%).

Table 2  
Value today of \$100 in the future

Years in future	Discount rate model			Value relative to constant discounting	
	Constant 4% rate	Mean reverting	Random walk	Mean reverting	Random walk
0	\$100.00	\$100.00	\$100.00	1	1
20	45.64	46.17	46.24	1	1
40	20.83	21.90	22.88	1	1
60	9.51	10.61	12.54	1	1
80	4.34	5.23	7.63	1	2
100	1.98	2.61	5.09	1	3
120	0.90	1.33	3.64	1	4
140	0.41	0.68	2.77	2	7
160	0.19	0.36	2.20	2	12
180	0.09	0.19	1.81	2	21
200	0.04	0.10	1.54	3	39
220	0.02	0.06	1.33	3	75
240	0.01	0.03	1.18	4	145
260	0.00	0.02	1.06	5	285
280	0.00	0.01	0.97	7	568
300	0.00	0.01	0.89	11	1147
320	0.00	0.01	0.83	16	2336
340	0.00	0.00	0.78	26	4796
360	0.00	0.00	0.73	43	9915
380	0.00	0.00	0.69	74	20,618
400	0.00	0.00	0.66	131	43,102

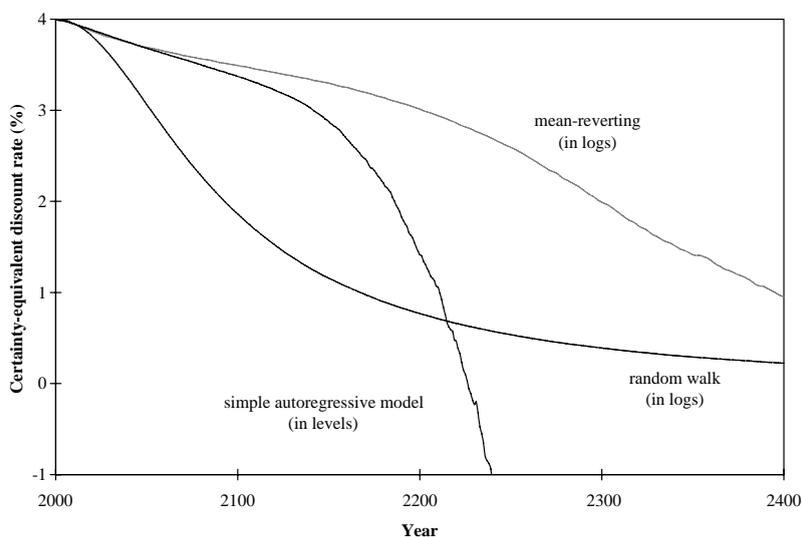


Fig. 2. Forecasts of certainty-equivalent discount rates.

#### 4.1. Certainty-equivalent discount rates and discount factors

The two important features to notice in Fig. 2 are the effect of using a model in logs (Eqs. (8) versus (1)) and the effect of a random walk versus mean-reverting model (Eqs. (10) versus (8)). The model in logs successfully avoids the possibility of negative interest rates in the future, eliminating negative certainty-equivalent rates. Eliminating the possibility of negative rates also slows the decline in future certainty-equivalent rates. Although the unlogged model generates a certainty-equivalent rate declining from 4% to 1% after 210 years, the comparable (mean-reverting) logged model requires almost 400 years to decline to 1%.

Because the exclusion of negative interest rates is relatively uncontroversial, the more interesting comparison is between the random walk and mean-reverting models. Both will decline toward zero because over many years simulated discount rates have more time to wander closer to zero. In the mean-reverting model, however, the chance that discount rates will be persistent enough to wander very far is much smaller than in the random walk model, which assumes such persistence with certainty.

The impact on certainty-equivalent discount rates is enormous. The random walk model implies rates that decline from 4% to 2% after 100 years, down to 1% after 200 years, and further declining to 0.5% after about 300 years. Meanwhile, the mean-reverting model indicates that certainty-equivalent rates stay above 3% for 200 years. Over the next 200 years, the decline is similar to the initial decline in the random walk model, with rates hitting 2% after 300 years and 1% after 400 years.<sup>28</sup>

These certainty-equivalent discount rates translate into dramatic differences in the valuation of future consequences as shown in Table 2. After only 100 years, conventional discounting at 4% undervalues the future by a factor of 3 based on the random walk model of interest rate behavior. After 200 years, that factor rises to about 40. That is, conventional discounting values \$100 in the year 2200 at 4 cents. The random walk model values the same \$100 at \$1.54—about 40 times higher. Going further into the future, conventional discounting is off by a factor of over 40,000 after 400 years. The same effects occur with the mean-reverting model, but lagged by 100–200 years (a factor of 3 after 200 years, and a factor of over 40 after 360 years).

We can explore the implications of an alternative underlying interest rate (e.g., based on equity or after-tax rates) by repeating the simulations with alternative initial rates. We do this for the alternatives of 2% and 7%—what one might think of as upper and lower bounds on the consumer rate of interest. Using the parameter estimates in Column 1 of Table 1, but initializing the random walk to either 2% or 7% in 2000, we can again compute expected discount factors and compare them to constant discounting at the initial rate. We find that the *relative* effect of interest rate uncertainty (measured by the ratio of discount factors with and without the random walk disturbances) rises as the initial rate rises. At a horizon of 400 years and an initial rate of 7%, valuation under the assumption of a random walk is 530 million times higher than a constant 7% rate. Meanwhile, the comparable factor based on a 2% rate is just a little over 100. Intuitively, the effect must be smaller for low discount rates (e.g., 2%) because the range of possible lower rates (0–2%) is narrower.

<sup>28</sup> Recall that the certainty-equivalent forward rate is the period-to-period rate at some time  $t$  in the future—not an average rate for discounting between period  $t$  and the present. To get a better sense of the effect of discount rate uncertainty on present values, we must look at expected discount factors, as we do below.

These results provide a precise answer to the question posed by the title of this paper: “How much do uncertain rates increase valuations?” At horizons of several hundred years, the answer is, “Quite a lot.” After 400 years the random walk model, which we consider more convincing, increases the value by a factor of over 40,000 relative to conventional discounting at 4%. The mean-reverting model increases the value by a factor of 130. Because the numbers in these tables represent expected values, one can alternatively combine the results by weighting the values based on the probability given to each model. For example, placing an equal probability on both models yields an increase in value of roughly 21,000 after 400 years relative to a constant rate. A weight of 25% on the random walk model yields a factor of roughly 11,000. These large increases in intertemporal prices can substantially alter the evaluation of policy consequences over long horizons, as we now investigate.

#### 4.2. Relevance for climate change policy evaluation

An important application of discounting the distant future is valuation of the consequences of climate change due to human activities, namely the burning of fossil fuels and emission of carbon dioxide that remain in the atmosphere for hundreds of years. Conventional analyses, using constant rates of 4–5%, tend to produce low estimates of climate change damages (see, e.g., [37]), that recommend moderate if not marginal mitigation action. Other analyses, based on lower discount rates, produce significantly higher climate damage estimates and recommend aggressive action [16].

How does our approach weigh in? We first simulate Nordhaus’ model to provide an estimate of the marginal damages in future years from an extra ton of emissions in the year 2000. This time path of the flow of climate damages depicts a sensible pattern of damages having a delayed, then an increasing, and finally a declining effect as emissions decay.<sup>29</sup> We then apply alternative discount rate paths to estimate the present value of damages,  $PV$ , from a marginal ton of emissions today—and hence the marginal benefit of avoided emissions:

$$PV = \sum_{t=2000}^{2400} MD_t E[P_t] \quad (11)$$

where  $MD_t$  is the flow of marginal damages computed from Nordhaus’ model, and  $E[P_t]$  are the various expected discount factors we constructed both in Table 2 and our sensitivity analyses.

The results of these calculations are shown in Table 3. We first observe that the value associated with the constant discount rate of 4% is reasonably close to the value of \$5.29 reported in Table 5.7 of Nordhaus [37].<sup>30</sup> Next, note that the effect of discount rate uncertainty based on the random walk model is quite large—increasing the estimated benefits of mitigation by over 80% to more than \$10 using the benchmark 4% rate.<sup>31</sup> The mean-reverting model yields a more modest

<sup>29</sup> See [36] for details on the estimated time profile of climate damages, which we generated by simulating the model described in Nordhaus [37].

<sup>30</sup> A constant discount rate of 4.15% delivers a net present value of exactly \$5.29. Note that Nordhaus does not use a constant rate based on his assumption of a growth slowdown.

<sup>31</sup> At first glance, one might think based on Table 2 that the effect should have been even greater. However, even though benefit flows in the distant future may be severely undercounted by discounting that ignores uncertainty, the error is not as great for near-term benefit flows, and near-term flows still carry greater weight than distant flows in the overall calculation.

Table 3

Expected discounted value of climate mitigation benefits (per ton carbon)

		Benefits from 1 ton of carbon mitigation (1989 dollars)	Relative to constant rate
Government bond rate (4%)	Constant 4% rate	\$5.74	—
	Random walk model	\$10.44	+ 82%
	Mean-reverting model	\$6.52	+ 14%
2% rate	Constant 2% rate	\$21.73	—
	Random walk model	\$33.84	+ 56%
	Mean-reverting model	\$23.32	+ 7%
7% rate	Constant 7% rate	\$1.48	—
	Random walk model	\$2.88	+ 95%
	Mean-reverting model	\$1.79	+ 21%

increase of only 14%, to about \$7. As suggested above, one could combine these estimates by assigning probabilities to each of the two models and taking expectations. Equal weighting, for example, would yield an estimate of \$8.50.

As one would expect based on the earlier discussion, the relative effect of uncertainty on the present value of expected mitigation benefits is larger when the comparison involves a higher initial discount rate. This reflects the greater opportunity for uncertainty to lower rates when the initial rate is higher (versus a low initial rate where the rate simply cannot go much lower). Here, the effect of uncertainty is a 95% increase in discounted mitigation benefits relative to constant discounting with a 7% rate and a 56% increase with a 2% rate, both based on the random walk model. The mean reverting model again yields a more modest 21% increase using the 7% rate, and a 7% increase using the 2% rate.

Note that while the dollar value of discounted climate benefits is sensitive to the magnitude of the benefits profile we have chosen for illustration, the proportional increase due to incorporation of the effect of discount rate uncertainty depends only on the general shape of the benefits profile. In addition, because we focus on a 400-year horizon, our results are in some sense conservative; extending the horizon further introduces damages that are counted more heavily in the presence of uncertainty.<sup>32</sup> Applying the uncertainty-adjusted discount factors to other greenhouse gases with longer atmospheric residence (e.g., perfluorocarbons and sulfur hexafluoride), or to climate damage profiles that include catastrophic events or other permanent consequences (e.g., species loss), would also generate larger increases in discounted climate damages because the consequences would be more heavily concentrated in the future. In general, the greater the proportion of benefit flows occurring in the distant future, the greater the error introduced through discounting that ignores uncertainty in the discount rate itself. Other applications can be

<sup>32</sup>We have estimated that this understatement might be about 4% based on an extrapolation of the rate of decline in discounted carbon benefits at the end of our 400-year horizon.

explored by applying the random walk and mean-reverting discount factors (or a weighted combination thereof) from Table 2.

## 5. Conclusions

Properly discounting over long horizons requires that one consider the uncertainty surrounding future discount rates. Uncertainty coupled with persistence implies that appropriate future certainty-equivalent rates will decline toward the lower bound of possible future discount rates. Quantifying this effect requires not only a forecast of future rates but also estimates of their variance and autocorrelation.

Using 200 years of historical data on high-grade, long-term government bonds in the United States (primarily US Treasury bonds), we estimated several models of interest rate behavior. We found that the sampling error associated with our estimates of interest rate autocorrelation makes it difficult to draw conclusions based on our simple analytical model, which treats the autocorrelation as known. In fact, this sampling error is large enough that we are unable to reject the hypothesis that interest rates in either levels or logs follow a random walk.

Turning to simulations that both treat the autocorrelation as uncertain and allow a log-normal model for interest rates (which avoids negative values), we find significant effects associated with uncertainty in the long run, regardless of the underlying model. Although the average long-term real rate of return on government bonds is around 4%, the appropriate rate to discount the distant future (more than 400 years) is around 0.5% based on a random walk model, and 1% based on a mean-reverting model. Over horizons of less than 400 years, the random walk model suggests declines to much lower rates: 2% after 100 years, 1% after 200 years, and 0.5% after 300 years. Certainty-equivalent rates for the mean-reverting model, on the other hand, remain above 3% for next 200 years, declining to 2% only after 300 years and 1% after 400 years.

Over 100 of years, these certainty equivalent rates translate into considerable increases in the valuation of future consequences. After 400 years, the random walk model (which we consider more compelling) increases discounted values by a factor of more than 40,000; the mean-reverting model increases the value by a factor of 130. Although the 200 years of data used in our analysis does not give us statistical evidence to unambiguously favor either the mean-reverting or the random walk models, our opinion is that the latter makes more sense. Over long horizons, such as millennia, we see persistent changes in observed interest rates. Moreover, after a long period of relatively low rates, we think that such rates are more likely to persist rather than return to a higher long-run average. These propositions—that changes in rates are persistent over hundreds if not thousands of years, and that low rates for many recent years suggest low rates in the future—are consistent with the random walk model. At a practical level, the random walk model generates prediction intervals with better coverage in split samples than the mean-reverting model.

Applied to the consequences of greenhouse gas emissions, these results indicate that the expected marginal benefits from climate change mitigation could be understated by a factor of 2 in analyses that ignore uncertainty in the discount rate itself. Furthermore, we would expect similar understatements in the analysis of other problems with very long time horizons, including long-lived infrastructure projects, hazardous and radioactive waste disposal, and biodiversity preservation.

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