Essays on the Role of Negative Externalities in
Mechanism and Market Design

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Business Administration
in the Graduate School of Duke University
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Abstract

This dissertation focuses on understanding how negative externalities affect managerial decisions, specifically, the resource allocation and pricing in competitive environments.

We first study optimal allocation and pricing on a network of competing buyers. Buyers have private information about the value of an item being sold (such as franchise contract, good, service). Furthermore, buyers place a premium on obtaining the item exclusively, i.e., if no competitors obtain the item at the same time. We show that the seller limited to offering posted price contracts should inflate the price in order to maximize revenues and capture the value buyers put on exclusivity. However, posted prices are not revenue-maximizing and there are theoretical barriers to discovering generic optimal allocation and pricing schemes, stemming both from mechanism design theory and computational complexity theory. We present an easy-to-implement hybrid auction-pricing procedure which revenue-dominates posted prices and is optimal in a full competition setting.

We next turn to a different type of negative externalities in which a buyer faces a possibility of losses, thus suffering a negative externality, due to scarce resource being allocated to competitors. We show that the existence of such negative externalities among market participants competing for a scarce resource allows for emergence of the no-allocation equilibrium with positive revenues for the seller. A monopolist selling $K$ indivisible items to a large number of competing unit-demand buyers who
face negative externalities whenever their rivals get the items, can exploit these negative externalities. If the number of buyers is large enough, the no-allocation equilibrium emerges: no items get allocated, yet buyers still pay the seller to avoid a potential exposure to negative externalities. We provide conditions on the magnitude of externalities and on the level of buyer competition that yield optimality of the no-allocation equilibrium.

Finally, we consider the setting where the scope of negative externalities is limited. A revenue-maximizing monopolist is selling a single indivisible good to buyers who face a loss if a rival buyer obtains it. The rivalry is modeled through a network, an arc between a pair of buyers indicates that a buyer considers another buyer its rival, and the magnitude of the loss is the private information of each buyer. First, given a network, we characterize the optimal mechanism. Second, we show that revenues depend on the network structure. Thus, in applications where it is possible, the monopolist might consider designing not only the mechanism but also the network (if not fully, at least partially). Third, we provide solutions to this joint network and mechanism design problem. Specifically, we determine revenue-maximizing rivalry networks (which in turn induce optimal mechanisms), and show that they are independent of distributional assumptions on buyers' independent private loss values, provided virtual values are bounded from zero. We achieve these results under different restrictions of how the monopolist can affect the network. When rivalry is symmetric, matchings are optimal (with at most one path on three vertices). However, asymmetric competitive relationships among buyers generate higher revenues than symmetric ones. The optimal asymmetric networks are characterized by (i) every buyer having at least one rival, and (ii) the existence of a buyer not considered a rival by anyone.
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Resource allocation and pricing in a competitive environment is a canonical problem in many managerial decision-making situations. For example, such questions arise naturally in allocation and pricing of procurement contracts, in franchising, in designing and structuring distribution or service agreements, in allocation and pricing of internet advertising, etc. A typical silent assumption when modeling such problems is that market participants do not care whether rivals obtain the resource or not. More generally, it is often assumed that market participants’ values are not affected by the impact of allocation decisions on the competitive structure of the market. However, this assumption, which is central to the analytical tractability of many models, might not always be realistic. A market participant may place a higher value on getting the resource exclusively, than on getting the resource along with competitors. Similarly, a market participant may face losses when competitors obtain the resource. For example, an Original Equipment Manufacturer (OEM)’s potential component suppliers, outsourcing plants, or distributors may put a premium on exclusively securing the procurement contracts, sourcing orders, or distribution agreements from the OEM. Similarly, if they get excluded from the allocation, they
may experience negative externalities due to loss of business opportunities.

We enrich standard models and study the effects of allocation-dependent valuations in asymmetric information settings. In addition, we model the potentially limited scope of dependence among competing buyers through a network in which an arc between a pair of buyers indicates that a buyer considers another buyer its competitor. Our approach allows for determining when it is profitable for the revenue-maximizing seller, or worthwhile for the social planner, to exploit these allocation-dependent valuations if they exist. Specifically, we develop optimal mechanisms to exploit such valuations. Figure 1.1 illustrates different types of privately held valuations that we study in this dissertation. We next discuss these in detail.

Let’s consider a setting of a monopolistic seller and two competing buyers ($i = 1, 2$), each of whom can get at most one item. In standard settings, a buyer is assumed to distinguish two outcomes, getting an item for which the buyer has the value $v_i^1$ and not getting an item for which the buyer has the value $v_i^0$. (The value of not getting the item is commonly assumed to be zero.) In contrast, in this work we consider the settings in which buyers distinguish among four different outcomes, depending on the realized allocations. Therefore, a buyer potentially has four different valuations: a valuation when the buyer obtains an item while the competitor does not, $v_i^{10}$, a valuation when both the buyer and the competitor obtain items, $v_i^{11}$, a valuation when neither the buyer nor the competitor obtains an item, $v_i^{00}$, and a valuation when the buyer does not obtain an item while the competitor does, $v_i^{01}$. Note that the first superscript indicates whether the buyer gets the item, while the second superscript indicates whether the competitor gets the item. In general, it is natural to assume that $v_i^{10} \geq v_i^{11} \geq v_i^{00} \geq v_i^{01}$. Standard models often assume that $v_i^{10} = v_i^{11} = v_i^{00} = v_i^{01}$, because there is no allocation-dependent possibility. To express allocation dependence, in this dissertation we will hence allow different values between $v_i^{10}$ and $v_i^{11}$, and (or) between $v_i^{00}$ and $v_i^{01}$. In particular, we will focus on two scenarios. One
The exclusivity is schematically described in the left side of Figure 1.1. The other scenario considered in this dissertation is that of negative externality, which occurs when the buyer’s valuation of not getting an item depends on whether the competitor

\[
v_i^{10} \geq v_i^{11} \geq v_i^{00} = v_i^{01}.
\]

The exclusivity is schematically described in the left side of Figure 1.1. The other scenario considered in this dissertation is that of negative externality, which occurs when the buyer’s valuation of not getting an item depends on whether the competitor

\[v_i^{00} = v_i^{01} = 0.\]
gets an item, i.e.,

\[ v_i^{10} = v_i^{11} \geq v_i^{00} \geq v_i^{01}. \]

The negative externality is schematically described in the right side of Figure 1.1.\(^2\)

Another important modeling aspect of our work is the limited scope of allocation-dependent valuations. We model this through the network structure. The underlying network structure representing competitive relationships plays a central role in the analysis. For example, a service supplier (e.g., call center, IT, and R&D) may care more about other service suppliers in the same region than those in a different region. A computer equipment manufacturer, when bidding for a display advertisement slot, cares about whether another computer equipment manufacturer’s ad is shown on the same page, but might not be negatively impacted if a financial institution’s ad were shown. Thus, a market participant’s valuation may be limited to a geographic area, a demographic segment, or a social network. Natural questions arise regarding allocation and pricing when valuations are limited to networks. Given a competitive network structure, how does a revenue-maximizing seller find the best way to exploit allocation-dependent valuations? If the seller could influence the competitive network structure (e.g., a retailer may design its network of suppliers; an internet advertisement seller may decide which bidders will be allowed to participate) or choose a structure among different competitive networks (e.g., Google may choose different communities to allocate an initial limited quantity of Google glasses), what is the revenue-maximizing network structure?

In this dissertation, we present three essays in which we develop analytical models to study these problems. The models also provide prescriptive recommendations for managerial decisions, and develop insights into understanding of complicated business settings.

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\(^2\) In this case, we can normalize the valuations when neither of the market participants obtains an item to be zero, i.e., \( v_i^{00} = 0. \)
In Chapter 2, we study how a monopolist could maximize revenues when exclusive allocations could be valuable to buyers. A monopolistic seller aims to allocate contracts to potential buyers, each of whom has a privately held value for the contract, and also places an additional private value on obtaining the contract exclusively. We first show that the revenue-maximizing seller, limited to posted prices, should inflate the price that would have been posted had there been no additional value for the exclusive allocation, and, thus, capture values that buyers place on exclusivity. Many luxury good products and products with the “coolness” factor, such as innovative electronic devices, often appear to be overpriced. Our result suggests that the overpricing is due to optimal price-setting, aimed at capturing an added value for exclusivity that buyers might associate with owning such products.

It is well-known that posted prices are revenue-maximizing when there is sufficient supply and when buyers do not have exclusivity valuations. However, we show that posted prices are not optimal in the presence of exclusivity valuations. Thus, the revenue-maximizing procedures are more complicated than posted prices and we use mechanism design techniques to study optimal allocation and pricing for exclusivity contracts. We show that privately held information and network structure could both impose insurmountable obstacles in the design of optimal procedures. Nonetheless, we introduce a hybrid auction-pricing procedure that revenue-dominates posted prices and that is optimal in some settings. The main idea of the hybrid auction-pricing procedure is to separate exclusive from non-exclusive allocations. The seller starts by running a standard optimal ascending auction with reserve for exclusive allocations only, with the caveat that there will be no exclusive allocations should the auction price raise to a predefined threshold value. (The procedure concludes by exclusive allocations if the market clears at any time during the exclusivity auction.) Only if the exclusivity is overdemanded at that threshold, the seller cancels

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3 This chapter is based on the recent working paper (Deng and Pekeč, 2013).
the exclusivity auction and offers items to all interested bidders at a predetermined posted price. This auction-pricing procedure not only demonstrates that in some settings it is possible to design sales procedures that overcome suboptimality of pricing and prohibitive complexity of optimal mechanisms in general, but our methodology also provides guidance on how to design other procedures that could be optimal for application-specific network and industry structures.

We next turn to a different type of negative externalities in which a buyer faces a possibility of losses, thus suffering a negative externality, due to scarce resources being allocated to competitors. In Chapter 3, we consider the information structure in which a buyer’s valuation for the resource is privately known, while the negative externality is her competitor’s private information.\footnote{This chapter is based on the published paper (Deng and Pekeč, 2011).} We show that the existence of even minuscule negative externalities allows for the emergence of the no-allocation equilibrium with positive revenues for the seller. In the no-allocation equilibrium, no items get allocated, yet buyers still pay the seller to avoid a potential exposure to negative externalities. The no-allocation equilibrium allows the seller of a limited number of items, for example, internet ad slots, to simultaneously (1) optimize revenues by collecting a small payment from each of the advertisers who are concerned with negative externality effects, and (2) ensure ad-free experience to its users. Our results indicate that it might be possible that ad-free user experiences can be supported not just by charging users, but could be subsidized by potential advertisers whose ads will not be shown.

In Chapter 4, we consider the information structure in which a buyer’s negative externality, when losing the resource to her competitors, is private and non-negligible.\footnote{This chapter is based on the recent working paper (Belloni et al., 2014).} The optimality of no-allocation equilibrium is shown in this information setting as well. We study the seller’s optimal network design problem. Somewhat
surprisingly, full competition is not optimal and the seller maximizes revenues when the impact of negative externalities for any buyer is limited to a single rival. In other words, with an undirected network representation of negative externalities, the optimal network is a matching. In a directed network setting, a network is optimal if and only if (1) each buyer could experience negative externalities, and (2) there exists at least one buyer who does not impose externalities on any other buyers. Any network can be restructured into a revenue-maximizing one with a small number of changes in pairwise relationships among buyers.
2

Optimal Allocation of Exclusivity Contracts

2.1 Introduction

Exclusivity rights are considered valuable in a variety of settings. For example, a contract securing rights to sell a product or offer a service is more valuable if no competitor secures the same contract. Consider a case of allocating a retail franchise, such as a car dealership, a chain restaurant, etc. If a buyer secures a franchise contract, the value of the business could be enhanced if competitors do not obtain a franchise: a portion of the business that a competitor could have served is likely to be captured by the sole franchise owner. Exclusivity contracts are commonplace for distribution agreements: a distributor often gets an exclusive contract to distribute a brand name product in a given geographic region. Exclusivity brings an additional value to contracts or purchases in many other settings, such as, advertising. Some examples of the exclusivity in advertising include having an ad shown exclusively on a webpage (i.e., without any competitors’ ads showing), or exclusivity in the form of sponsorship of events, celebrity or athlete product endorsements, building naming rights, or product placement and merchandizing agreements in mass media.
entertainment such as movies, TV shows, or video games. Exclusivity could be perceived as valuable from a consumer’s perspective as well. Luxury brands often try to create and exploit the perception of exclusivity that consumers might attach to purchasing their product. A consumer might value the product more highly if nobody else has the product, (e.g., designer clothing, a limited edition sports car, or obtaining a special feature or power-up in a game).

The notion of obtaining an exclusive contract inherently assumes that competitors are excluded from obtaining the same contract. Therefore, it is important to determine perceived competitors for each buyer, and, more generally, the structure of the competitor relationship. As is common with franchising and with exclusive sales, service and distribution agreements, the scope of exclusivity contracts is defined by the structure of the competitor relationship and could be restricted to a geographic area or a market segment. Geographical limitations are a natural limitation in a retail franchise example: unserved demand from a competitor who did not get the contract might only be captured by nearby locations. Hence, the notion of exclusivity in a franchise contract is typically limited to a geographic or demographic area (e.g., one Honda dealership in the city, no other McDonalds restaurants within 1km radius, etc.). In advertising, the scope of exclusivity might be limited to rivals who are perceived as direct competitors: for example, an online shoe retailer might perceive its ad allocation as exclusive, even if there are, say, financial institution ads shown on the same webpage, as long as no other shoe retailer ads are shown. In fact, multiple non-competing brands sponsorship or product placement is common (e.g., official credit card, official car, official drink, etc., of a sporting event). Finally, a social network could also define the scope of exclusivity. For example, designer clothing might not be considered as valuable if another person wears exactly the same item at an event; a limited edition sports car might not be perceived as valuable if an identical car is parked in the same parking lot; and even obtaining a special feature
or power in a game might not be worth any bragging rights if another friend already achieved the same. Thus, a social network could be the underlying topology for understanding the notion of exclusivity in product placement, and firms would have to take that into account when making decisions on targeting (groups of) consumers with exclusive or non-exclusive offers.

In this chapter, we study how to allocate and price exclusivity contracts, or any goods or services which could have added value if allocated exclusively. We consider the model in which a monopolistic seller aims to allocate items (contracts) to unit-demand buyers (one contract per buyer). Each buyer $i$ has a privately held value $v_i$ for the item and another privately held value $w_i$ for the item being allocated exclusively to them. For example, a car dealership has a private valuation for the franchise contract that would secure the right to sell a car brand, and another (higher) private valuation if such a franchise contract is exclusive making them the only dealer in the region selling that particular brand.

We start by analyzing how a revenue-maximizing seller should determine posted prices in order to capture potential value buyers put on exclusivity. In Section 2.2.1, we show that the revenue-maximizing seller, limited to posted prices, should inflate the price that would have been posted had there been no additional value for the exclusive allocation. The seller has to trade-off capturing higher revenues from high-valuation buyers with losing some low-value buyers due to the high price. High value buyers are also willing to buy at an inflated price since it increases the probability that no other buyers will get the item and thus unlock the exclusivity value for it. Thus, the inflated price signals the exclusivity potential to buyers who obtain the item. Many luxury good products and products with the “coolness” factor such as innovative electronic devices often appear to be overpriced. Our result suggests that

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1 The difference between these two values can be thought of as the exclusivity premium and, similarly, the difference between prices for non-exclusive and exclusive allocations determined by the optimal mechanisms can be thought of as the price of exclusivity.
the overpricing is due to optimal price setting aimed at capturing added value for exclusivity that buyers might associate with owning such products.

By quoting a single price to buyers who put a premium on exclusivity, the seller leaves it to buyers to evaluate risks associated with buying an item, without knowing whether they are paying for a non-exclusive or an exclusive allocation. Instead of quoting a single posted price, a seller could consider offering a two-price menu: a possibly different posted price for a non-exclusive allocation and another, presumably higher, posted price in the case that only one buyer obtains the item and thus gets it exclusively. Such a menu might not be easy to implement in everyday purchase settings, but it could potentially alleviate the uncertainty buyers face with respect to the exclusivity of their possible purchase and could, consequently, have a potential for higher seller’s revenues relative to single posted price schemes. Somewhat surprisingly, we show that the seller does not gain anything by quoting such a two-price menu. This makes the seller’s price-setting job easy as there are no benefits in complicated posted price schemes and the optimal strategy involves a single (inflated) price.

It is well-known that the posted prices are the optimal mechanism for the revenue-maximizing seller with sufficient supply if buyers do not put any additional value on exclusivity. In Section 2.2.2 we show that exclusivity valuations impose difficulties on the seller who seeks to maximize revenues. Posted prices in such settings are not revenue-maximizing, and the seller interested in maximizing revenues has to consider and implement more complex mechanisms, such as auctions. We use mechanism design techniques to study optimal allocation and pricing for exclusivity contracts in Section 2.2.2. The privately held two-dimensional information \((w_i, v_i)\) introduces well-known difficulties of the multi-dimensional mechanism design and makes the

\footnote{The optimal mechanism is a second-price auction with reserve. Since there is an item available for each buyer, purchases are made at the reserve price.}
problem analytically intractable. (We briefly discuss this in Section 2.2.2 as well.) We show how to overcome this analytical hurdle in some settings of special interest. For instance, we determine the optimal mechanism when the value for obtaining the item exclusively can be derived from non-exclusivity valuations. Two simple examples are additive exclusivity, in which a buyer’s exclusivity premium is a constant, and multiplicative exclusivity, in which a buyer’s exclusivity premium is a scaling of the buyer’s non-exclusivity valuation. In both cases, posted prices are revenue-deficient, and one has to resort to a design of ascending auctions with appropriately chosen reserve prices in order to optimize revenues.

Even in the settings in which the mechanism-design problem for selling items to buyers with exclusivity valuations can be solved analytically, a practical implementation of the optimal mechanism could be a hard problem for the seller to tackle. In Section 2.3, we illustrate such difficulties in a model that allows for a limited scope of the exclusivity. In this model, local exclusivity is determined by an underlying network topology. The network nodes correspond to buyers, while arcs define the set of perceived competitors for each buyer. A buyer considers a contract to be allocated exclusively to it if none of its neighbors (as defined by the network) obtains the item. Throughout, we assume that the network is publicly known.\footnote{For example, intermediaries or sellers of advertising space might know the social network of consumers being targeted; competitors for a given business are often known; the geographic or demographic area covered by an exclusive franchise agreement is also typically known and defined.} We introduce the notion of \textit{local linear exclusivity} (LLE) in which a buyer who obtains the item exclusively can capture some of the value of its competitors since, by virtue of exclusivity, none of them obtained the item. More formally, the exclusivity premium is a linear combination of the non-exclusive allocation values of all neighbors. For example, a car dealer $i$ who obtains the exclusive franchise contract covering some region, derives its exclusivity premium, $w_i - v_i$, from the values $v_j$, that its competitors, $j$, would not be able to realize (since they didn’t get the franchise contract). However, while
competitors are shut out of the market, buyer $i$, who got the contract exclusively, might only realize a fraction $\alpha_{ij}$ of $v_j$ (because it might not serve $j$’s customers as efficiently as $j$ could have, had they obtained the contract). Thus, buyer $i$ has the exclusivity valuation $w_i = v_i + \sum_j \alpha_{ij} v_j$. The model with LLE valuations is defined precisely in Section 2.3.

The range of difficulties the seller might face with implementing the optimal mechanism is illustrated within the LLE model. We first provide the optimal revenue-maximizing mechanism for LLE valuations in Section 2.3.1. The problem of determining which buyers should get the items according to the revenue-maximizing mechanism ranges from straightforward to computationally intractable. For example, it is straightforward to implement the optimal mechanism when all valuations are publicly known, with a natural condition that LLE valuations are bounded in the sense that competitors can never jointly benefit more than 100% of the value $v_j$ from buyer $j$ who didn’t obtain the item. (We call such a setting bounded local linear exclusivity, BLLE, and provide a formal definition in Section 2.3.) In such situations, the seller cannot benefit from any exclusive allocations since the premium buyers put on exclusivity does not compensate the loss of the values of competitors that didn’t obtain the item. Thus, the seller should simply sell non-exclusivity contracts to all buyers $i$ and charge them $v_i$. This observation suggests that in many settings exclusivity should not even be considered. For example, a monopolist manufacturer of a product, facing potential retailers with similar capabilities (e.g., the value of business at a given location is the same for anyone who sells the product in that particular location) and with knowledge of retailers’ pricing and costs of carrying the product, should not even consider exclusivity arrangements (such as making the product available exclusively at certain retailers) and should make the product widely available, i.e., at all retailers interested in carrying it.

In contrast, implementing the optimal mechanism within the LLE model could
be a non-trivial task for the seller. Diversely held private information also induces inefficiencies and could result in exclusive allocations, even when such exclusive allocations are impossible with publicly known information. Thus, it might be optimal for the seller who does not know the buyers’ values to allocate the item exclusively, even in the BLLE setting. Furthermore, private information also turns an allocation problem that is trivial to solve in the public information setting into a computationally hard problem in the private information setting. We also show a somewhat surprising non-monotonicity of revenues with respect to buyers’ valuations: if any of \( v_i \) or \( w_i \) increases or stays the same (the total value in the system increases), the seller’s revenues could decrease. The reason for this is that buyers’ information rents depend not only on the values but also on the underlying network structure and their locations in the network.

Even when buyers valuations are known to the seller, implementing the optimal mechanism might be an insurmountable challenge. We show in Section 2.3.1 that finding the optimal allocation is computationally hard, at least as hard as finding a maximum independent set in a network. This result indicates that in general, it will be impossible (unless P=NP) to define a procedure that would implement the optimal mechanism and, thus, one should not hope for developing any simplistic revenue-optimal pricing schemes or even optimal auctions (which would be guaranteed to end in a reasonable amount of time). While implementing optimal mechanism is hard in general, the problem could be manageable for some network structures.

We illustrate our findings in a supply chain contract setting in Section 2.3.2, where we show how a monopolistic revenue-maximizing supplier should allocate franchise contracts to competing retailers that have private information on the market share they could capture if they obtain the contract exclusively or non-exclusively.

In Section 2.4, we propose and analyze a simple ascending auction, that we name **Exclusivity Valuations Ascending (EVA) Auction**, for selling items in the presence
of exclusivity. The main idea is to separate exclusive from non-exclusive allocations. The seller starts by running a standard optimal ascending auction with reserve for exclusive allocations only, with the caveat that there will be no exclusive allocations should the auction price raise to a predefined threshold value. (The procedure concludes by exclusive allocations if the market clears at any time during the exclusivity auction.) Only if the exclusivity is overdemanded at that threshold, the seller cancels the exclusivity auction and offers items to all interested bidders at a predetermined posted price. The reasoning behind such an approach is that the seller could infer high enough non-exclusive valuations from those buyers demanding an exclusive allocation at a high price, and could instead charge a sufficiently high price for a non-exclusive contract to all buyers, potentially collecting revenues that are higher than the maximum possible revenues from limiting sales to exclusivity contracts. Given the complexity results from Section 2.3.1, this (or any other) straightforward auction cannot be the optimal mechanism for all possible settings and all network structures. However, we show that the EVA auction is the optimal mechanism in the LLE setting and with full competition among buyers (every buyer considers all other buyers to be its competitors). In order to establish the optimality of the EVA auction in Section 2.4, we first describe a revenue-maximizing direct mechanism in this setting, and then we show that the EVA auction implements the optimal allocation and pricing. The proof of the EVA auction optimality is twofold: we establish that the EVA auction is optimal under the assumption of truthful bidding behavior, and prove that truthful bidding is a perfect Bayesian equilibrium behavior. These results not only demonstrate that in some settings it is possible to design sales procedures that overcome suboptimality of pricing (established in Section 2.2.2) and prohibitive complexity of optimal mechanisms in general (that was presented in Section 2.3.1), but also provide guidance on how to design other procedures that could be optimal for application-specific network and industry structures.
We provide brief concluding remarks in Section 2.5. All proofs are relegated to the Appendix.

Related Literature

There is a large operations management literature that focuses on managing incentive conflicts in contracting, e.g., numerous supply chain models are reviewed in Cachon (2003). Specifically, pricing strategies are of practical importance due to the simplicity of the underlying contract form (i.e., posted prices) and have been widely studied (e.g., see Cachon and Feldman, 2011; Nasiry and Popescu, 2011). We show that there is a different set of issues that need to be addressed when exclusive allocations are a possibility. The structure of optimal mechanisms goes beyond pricing and we resort to the use of Myerson’s mechanism design techniques (Myerson, 1981). Mechanism design approach is standardly used in theoretical analyses involving privately held information in many contexts including operations management (e.g., see Gallien, 2006; Chen, 2007; Duenyas et al., 2013; Lobel and Xiao, 2013), although exclusivity has not been the focus of this operations management literature. From the abstract modeling perspective, exclusivity is a form of an allocation-dependent negative externality. Thus, our work is somewhat related to the literature that studies retailers’ stocking decisions, pricing decisions, and the supply chain performance by considering externalities among retailers in the same supply chain echelon (e.g., see Bernstein and Federgruen, 2004; Netessine and Zhang, 2005; Adida and DeMiguel, 2011).

The underlying network topology is crucial in modeling and understanding the local exclusivity, i.e., exclusivity with a limited scope. Several recent papers study allocation and pricing procedures on networks with positive and negative externalities. In Candogan et al. (2012), a monopolistic seller’s pricing strategies for a divisible good are examined in a public information setting with a local positive network ef-
fect, i.e., a buyer’s utility is increasing with the usage level of its peers. The work of Bhattacharya et al. (2011) provides allocation and pricing procedures on the network structure in a public information setting as well, and, in addition, focuses on algorithmic issues. In another paper focused on algorithmic issues, Haghpanah et al. (2011), positive externalities are modeled so that a buyer’s value is the product of a fixed private type and a known submodular function of the allocation of its peers, and the focus is on understanding algorithmic issues. In contrast, in this chapter, a buyer’s privately held value increases when none of its neighbors gets the item. This important distinction even impacts basic computational complexity findings. With exclusivity, the discrepancy of complexities of the revenue maximizing optimal solution does not stem out of the possible negative virtual valuations, but from the underlying network structure.

The allocation and pricing in the presence of different types of externalities, mostly motivated by problems arising in internet ad auctions, has been of interest to interdisciplinary research combining optimization, microeconomic theory, and algorithmic techniques and methodology. Since leveraging information on externality information may improve the efficiency or enhance the seller’s revenues, mechanisms that use externality valuation information have been explored in different formats, (e.g., Ghosh and Mahdian, 2008; Chen and Kempe, 2009; Ghosh and Sayedi, 2010; Constantin et al., 2010; Conitzer and Sandholm, 2012).

There is also a large amount of economics literature that addresses interdependent valuations (e.g., see survey Maskin, 2003). Unlike models of network externalities (e.g., Katz and Shapiro, 1985; Parker and Alstyne, 2005), in which buyers’ valuations are often assumed to depend on the (expected) size of their associated network, valuations in our model depend on the allocation in the neighborhood. Moreover, a variety of models for externalities have been studied in detail in, e.g., Jehiel et al. (1996), Jehiel and Moldovanu (2001), Aseff and Chade (2008), Figueroa and Skreta
(2011), and Brocas (2012). It is important to note that the notion of exclusivity, which we consider in this chapter, is fundamentally different from that of externalities: exclusivity valuation, unlike most models with externalities, imposes no externality on buyers who do not get the contract. Still, the techniques used to analyze mechanisms with interdependent valuations have a similar flavor to those one could use for dealing with exclusivity. In particular, externalities in Jehiel et al. (1996) are modeled as private information of the rivals which is similar to the information structure in our LLE setting. In our analysis of direct optimal mechanisms, we use classical Myerson’s methodology (Myerson, 1981), and in Section 2.2.2, we briefly discuss why it is difficult to handle information structures that cannot be embedded into Myerson’s framework.

2.2 Model

A monopolist seller has unlimited supply of identical items (e.g., contracts) that can be allocated among \( N = \{1, 2, \ldots, n\} \) unit-demand buyers. (Thus, we may assume there are \( K = n \) items.) Buyer \( i \)'s valuation for the item is \( v_i \) if obtaining the item non-exclusively, and \( w_i \) if obtaining the item exclusively (i.e., if none of the competitors obtains the item). Thus, buyer \( i \)'s type is represented by a vector \( v_i = (w_i, v_i) \). We assume

\[
    w_i \geq v_i \geq 0,
\]

where, without the loss of generality, we normalize buyer \( i \)'s value for not getting an item to zero. Note that the difference between the exclusive and non-exclusive valuation \( w_i - v_i \) can be thought of as the value of the exclusivity to buyer \( i \).

We consider the setting in which \( v_i \) is independent private information, while the number of buyers is publicly known. The seller’s valuation vector is assumed to be \((0, 0)\). Buyer \( i \)'s private information \( v_i = (w_i, v_i) \) is a realization of a continuous two-
dimensional random variable \((W_i, V_i)\) with joint cumulative distribution function \(F_i\) and with support \(\Omega = [\underline{w}, \overline{w}] \times [\underline{v}, \overline{v}]\). The corresponding density function is denoted by \(f_i\). Let \(F_i^{\lambda w + (1 - \lambda) v}\) denote the distribution of \(\lambda W_i + (1 - \lambda) V_i\) for \(\lambda \in [0, 1]\). (Thus, marginal distributions are \(F_i^w\) and \(F_i^v\).) To make analysis tractable, we make a standard regularity assumption, i.e., \(1 - F_i^{\lambda w + (1 - \lambda) v}\) is log-concave for \(\lambda \in [0, 1]\).

In what follows, we first derive the optimal posted price mechanism with exclusivity valuations. Then, we discuss a general mechanism design problem in the setting with exclusivity valuations and provide optimal mechanisms for two settings in which exclusivity valuations are perfectly correlated with non-exclusivity valuations.

2.2.1 Posted Price Mechanisms

The most prevalent way of facilitating trade is through posted prices. Here, we discuss how the seller should exploit the fact that buyers have higher valuations for an item should they obtain it exclusively.

A natural benchmark for assessing the effect of exclusivity valuations is the standard case in which there are no exclusivity premiums.

**Example 1.** Consider two buyers with independent private values \(v_i\) that are uniformly distributed on \([0, 1]\). If there are no exclusivity valuations, i.e., if \(w_i = v_i\), then the seller’s expected revenue is maximized by posting a price \(P = 0.5\). (The probability that a buyer buys the item at price \(P\) is the probability that \(v_i > P\), and, thus, the seller’s expected revenue per buyer is \(P(1 - P)\).) Therefore, the seller’s expected revenue from two buyers is \(2P(1 - P) = 0.5\).

If buyers have exclusivity valuations \(w_i = v_i + \varepsilon_i\), where \(\varepsilon_i\) is independent of \(v_i\) and also uniformly distributed on \([0, 1]\), the seller can set the price to \(P = 0.75 > 0.5\) and have the expected revenue of 0.75. Thus, the revenue-maximizing seller should exploit exclusivity valuations of buyers by inflating the posted price. We will show that \(P = 0.75\) is an optimal posted price in this case.
In order to determine the optimal posted price $P$, we first analyze a more complicated posted price mechanism. We study the seller posting a two-price menu \( \{P_{i}^{10}, P_{i}^{11}\}_{i=1}^{n} \) for each buyer. Each buyer $i$ can only accept or reject the entire menu. If buyer $i$ is the only buyer who accepts the price menu, it will get the item exclusively and pay $P_{i}^{10}$. If there is more than one buyer accepting the price menu offered to them, every such buyer $i$ gets the item non-exclusively and pays $P_{i}^{11}$. (The first digit in the superscript is the indicator of whether buyer $i$ gets the item or not and the second digit is the indicator of whether any other buyer gets the item or not. Thus, 10 in the superscript indicates the exclusive allocation to buyer $i$, and 11 indicates a non-exclusive allocation to buyer $i$.) Buyers are assumed to simultaneously accept or reject the two-price menu offered to them and the price they will pay will be determined only after all buyers’ responses are received by the seller.

Note that a (single) posted price mechanism is equivalent to a special case of a two-price menu where two prices are identical: \((P_{i}^{10}, P_{i}^{11}) = (P_{i}, P_{i})\). Somewhat surprisingly, single posted prices are sufficient to ensure optimality of expected revenues for the seller.

**Lemma 2.** A single posted price mechanism is revenue-optimal among two-price menus.

Let $P^{*}$ be the revenue-maximizing single posted price and let $R^{*}$ be the expected seller revenue in this case. Let $P^{0}$ be the revenue-maximizing posted price for buyers with no exclusivity value, i.e., $w_{i} = v_{i}$. Also, let $R^{0}$ be the expected seller revenue when $w_{i} = v_{i}$ and when the single price is $P^{0}$.

**Proposition 3.** Consider ex ante identical buyers. Then, $P^{*} \geq P^{0}$ and $R^{*} \geq R^{0}$.

Proposition 3 establishes that, in settings with buyers that value exclusivity, $w_{i} - v_{i} > 0$, the seller can increase revenues by exploiting these buyers’ exclusivity
values. Interestingly, the posted price \( P^* \) in such settings should be inflated relative to the optimal posted price \( P^0 \) when the value of exclusivity is ignored. When facing \( P^* > P^0 \), buyers have to trade-off the possibility of obtaining the item exclusively because competitors might be priced out of the market with an inflated price, with the possibility of themselves being priced out of the market due to an inflated price. The seller is facing the same trade-off: inflating the price will bring higher revenues from high-value buyers but will also leave low-value buyers priced out of the market. The proof of Lemma 2 demonstrates how to compute price \( P^* \) and, in particular, establishes that \( P^* = 0 \) in Example 1. Even if the distribution of buyers’ values are not known, Proposition 3 provides an easily implementable managerial guidance: a seller facing buyers that value exclusivity should inflate the posted price to capture some of these exclusivity values.

However, the posted price mechanism with an inflated price \( P^* \) is not optimal among all allocation and pricing procedures. Intuitively, the reason for the sub-optimality of the posted price mechanism is that buyers have multi-dimensional private valuations, but they can provide a one-dimensional response, i.e., to accept the posted price or not. Hence, we consider the structure of optimal allocation and pricing mechanisms in the next section.

2.2.2 Mechanism Design with Exclusivity

In this section, we study optimal mechanisms that exploit both the exclusivity and non-exclusivity valuations. By the Revelation Principle (Myerson, 1981), we consider direct mechanisms that allocate items based on buyers’ reports. Reports from all buyers are \( \hat{\mathbf{v}} = (\hat{v}_i, \hat{v}_{-i}) \in \Omega^n \). A direct mechanism specifies the allocation \( (p_i : \Omega^n \to \{0, 1\} \) is buyer \( i \)'s probability to get an item) and payments \( (m_i : \Omega^n \to \mathbb{R} \) is the payment from buyer \( i \) to the seller) for each \( \hat{\mathbf{v}} \in \Omega^n \). If buyer \( i \) does not
participate, it does not get any item.\(^4\)

Buyer \(i\)'s ex post utility when reporting its type as \(\hat{v}_i\), while its true type is \(v_i\), and when other buyers report \(v_{-i}\), is

\[
U_i(\hat{v}_i, v_i, v_{-i}) = w_ip_i(\hat{v}_i, v_{-i}) \prod_{j \neq i} (1 - p_j(\hat{v}_i, v_{-i})) \\
+ v_ip_i(\hat{v}_i, v_{-i}) \left( 1 - \prod_{j \neq i} (1 - p_j(\hat{v}_i, v_{-i})) \right) \\
- m_i(\hat{v}_i, v_{-i}).
\]

Let \(p\) denote \(\{p_i\}_{i=1}^n\) and \(m\) denote \(\{m_i\}_{i=1}^n\) and, thus, the LP relaxation of the seller’s Revenue Maximization Problem (General-RMP) is

\[
\max_{p,m} \sum_{i=1}^n \int m_i(v_i, v_{-i}) dF(v)
\]

subject to

- (EPIC) \(U_i(v_i, v_i, v_{-i}) \geq U_i(\hat{v}_i, v_i, v_{-i})\) for all \(i\) and all \(v_i, \hat{v}_i, v_{-i}\),
- (EPIR) \(U_i(v_i, v_i, v_{-i}) \geq 0\) for all \(i\) and all \(v_i, v_{-i}\),
- (Feasibility) \(0 \leq p_i(v) \leq 1\) for all \(i\),

where (EPIC) is the ex post incentive compatibility constraint to ensure truth-telling and (EPIR) is the ex post individual rationality constraint to ensure participation.

(Throughout the rest of this chapter, we will simplify the notation by denoting \(U_i(v_i, v_i, v_{-i})\) by \(U_i(v_i, v_{-i})\).)

**Example 4.** Suppose no buyer puts a premium on exclusivity, i.e., \(w_i = v_i\) for all \(i\).

Here, given unit-demand buyers and sufficient supply to meet demand, the Problem

\(^4\) Although we focus on the deterministic mechanisms here, results in this section can be shown to hold for the non-deterministic mechanisms by redefining allocation probabilities as in Section 2.4.
(General-RMP) decomposes to \( n \) Myerson’s optimal auctions with one item and one bidder each. The optimal auction is a second price auction with reserve (Myerson, 1981), which, in the case of a single buyer, reduces to determining whether the reserve is met. Thus, the posted price mechanism is optimal: there exists a price \( a \) (auction reserve price) such that every buyer \( i \) willing to pay \( a \) gets the item.

This example demonstrates why posted prices are the most widespread way of facilitating trade when the seller has no capacity issues and can meet all demand. However, if exclusivity has value, not only is the use of posted prices suboptimal, but finding an optimal mechanism becomes a hard problem to tackle. The rest of this chapter is devoted towards this issue.

**Example 5.** We consider the same valuation setting as in Example 1, i.e., \( w_i = v_i + \varepsilon_i \), where \( v_i \) and \( \varepsilon_i \) are independently and uniformly distributed on \([0, 1]\). Problem (General-RMP) is a linear programming problem when the support \( \Omega \) is finite. We then find the seller’s optimal expected revenue by discretizing the type space. As shown in the Appendix, the seller’s optimal expected revenue stabilizes around 0.9. Recall the expected revenue from posted prices is 0.75. There is still a significant gap between the numerical result of the optimal mechanism and the expected revenue of posted price schemes.

Problem (General-RMP), as well as the corresponding social surplus maximization problem, is a multi-dimensional mechanism design problem and is extremely difficult to solve analytically. The core difficulty lies in defining information rents in the case of privately held multi-dimensional valuations (i.e., exclusive and non-exclusive valuations in this chapter). Incentive compatibility constraints in the multi-dimensional mechanism design problem can be characterized as monotonicity and integrability conditions, and, as pointed out in Jehiel et al. (1999), the integrability condition is the primary source of difficulties in generalizing the standard Myerson
approach. (We demonstrate this in the proof of Proposition 6.) Furthermore, it is well known that solutions to multi-dimensional mechanism design problems are sensitive to various details of the environment, e.g., the seller’s belief about the buyers’ types. Hence, there is little hope for finding closed-form solutions. This unappealing feature of multi-dimensional mechanism design problems has been demonstrated by Rochet and Choné (1998), Armstrong (1996), and Manelli and Vincent (2007).

However, this analytical hurdle can be overcome in some settings. For example, Figueroa and Skreta (2011) provides a rather general framework for identifying information structures, in which applying Myerson’s approach to solving the mechanism design problem will be successful.

In particular, the standard mechanism design techniques in Myerson (1981) are applicable provided that there exists a one-dimensional representation of the multi-dimensional privately held information, e.g., $w_i = \Theta_i (v_i)$, where $\Theta_i$ is a publicly known function is often a tractable case. We illustrate this with two valuation structures: additive exclusivity, $w_i = v_i + \theta^0_i$ (publicly known additive premium that buyer $i$ is ready to pay for obtaining the item exclusively), and multiplicative exclusivity, $w_i = \theta^1_i v_i$ (publicly known multiplier buyer $i$ is ready to pay for obtaining the item exclusively), where $\theta^0_i$ and $\theta^1_i$ are publicly known constants. (The latter is the information structure in the model of Aseff and Chade (2008).) Let $v$ denote $\{v_i\}_{i=1}^n$, and $\psi_i$ denote the virtual valuation, $\psi_i \triangleq v_i - (1 - F^v_i(v_i)) / f^v_i(v_i)$. Since $1 - F^v_i(v_i)$ is log-concave, $\psi_i$ is increasing in $v_i$.

**Proposition 6.** The revenue-maximizing mechanism $(p^*(v), m^*(v))$ when buyers have additive exclusivity valuation structure is given by (A.6) and (A.7). The revenue-maximizing mechanism $(p^*(v), m^*(v))$ when buyers have multiplicative exclusivity valuation structure is given by (A.8) and (A.9).

There are other restricted information structures for which the multi-dimensional
mechanism design problem can be solved, e.g., \( w_i = v_i + \Xi_i(v_{-i}) \), where \( \Xi_i \) is a publicly known function. It turns out that local exclusivity on a network can fit in this framework. We describe and analyze it in the next section.

2.3 Local Exclusivity on a Network

The scope of exclusivity might be limited to an area in a geographic or demographic network, to a market segment in a competition network, or to a group of people in a social network. We now formally define local exclusivity on a network.

Relationships among buyers are defined by a network \( (N,E) \) where \( E \) is the 0-1 adjacency matrix: \( e_{ij} = 1 \) if and only if buyer \( i \) considers buyer \( j, j \neq i \), to be related to it (e.g., \( i \) considers \( j \) as a competitor or \( i \) and \( j \) are geographical neighbors or directly connected in a social network). Let \( S(i) \subseteq N \setminus \{i\} \) denote the set of buyer \( i \)’s neighbors, i.e., the set of all other buyers that \( i \) considers to be related to it: \( S(i) = \{j \in N : e_{ij} = 1\} \).

Buyer \( i \) has exclusivity valuation \( w_i \) for the item if none of its neighbors \( j \in S(i) \) gets an item, and has non-exclusivity valuation \( v_i \) if there is a neighbor \( j \in S(i) \) who also obtains the item. We still consider the setting in which \( v_i \) is private information, while network \( (N,E) \) is publicly known.

Direct mechanisms are defined as in Section 2.2.2, and, thus, buyer \( i \)’s ex post utility when reporting its type as \( \hat{v}_i \), while its true type is \( v_i \), and when other buyers report \( v_{-i} \), is

\[
U_i(\hat{v}_i, v_i, v_{-i}) = w_ip_i(\hat{v}_i, v_{-i}) \prod_{j \in S(i)} (1 - p_j(\hat{v}_i, v_{-i})) + v_ip_i(\hat{v}_i, v_{-i}) \left( 1 - \prod_{j \in S(i)} (1 - p_j(\hat{v}_i, v_{-i})) \right) - m_i(\hat{v}_i, v_{-i}). \tag{2.2}
\]
Therefore, the LP relaxation of the seller’s Revenue Maximization Problem (General-RMP) is also similar to the one in Section 2.2.2 except for substituting $U_i(\hat{v}_i, v_i, v_{-i})$ with the formulation in (2.2).

Without imposing any structure on $w_i$, problem (General-RMP), as well as the corresponding social surplus maximization problem, is still a multi-dimensional mechanism design problem. Furthermore, a numerical approach to solve this problem also has limited potential, given that even simplistic instances exhibit computational complexity obstacles: for example, even if $v_i = (1, 0)$ for all $i$ (i.e., buyers only value exclusivity and this valuation is the same for all buyers and is publicly known, so there is no private information at all in this setting), the Problem (General-RMP) reduces to finding the maximum independent set on $(N, E)$.

To gain theoretical insights on the impact of exclusivity when allocating items on the network, we consider a simplified private information structure that exploits the exclusivity value on the underlying network $(N, E)$. In our model, the exclusivity valuation $w_i$ is derived from the privately held valuation $v_i$ for non-exclusive allocation, and the valuations $v_j, j \in S(i)$, of buyer $i$’s neighbors. In particular, we assume that buyer $i$’s exclusivity premium is a linear combination of non-exclusivity valuations of buyer $i$’s neighbors $j \in S(i)$:

$$w_i = v_i + \sum_{j \in S(i)} \alpha_{ij} v_j \quad (2.3)$$

with publicly known non-negative matrix $A = [\alpha_{ij}]$. If (2.3) holds, we say that valuations satisfy local linear exclusivity (LLE). Note that the publicly known network structure defines LLE (through neighborhoods $S(i)$ and weights $\alpha_{ij}$) and is of fundamental importance in our analysis.

If buyer $i$ gets the item exclusively, none of its neighbors $j$ gets the item and thus buyer $i$ can realize some of their unrealized values. For example, in an example of
buyers of advertising space (or potential buyers of a franchise), buyer \( j \) who does not get the ad space (franchise contract), will lose potential customers and buyer \( i \) might attract some fraction \( \alpha_{ij} \) of that lost value for \( j \), as \( j \)'s potential customers will be presented by \( i \)'s ad only (will be able to go to \( i \)'s franchise only). LLE valuations allow for one-dimensional representation of privately held information, even though buyer types are two-dimensional and, in fact, depend on diversely held private information in buyer \( i \)'s neighborhood (i.e., \( v_i = (w_i, v_i) \) is a function of \( v_i \) and \( v_j, j \in S(i) \)).

In what follows, it will be useful to distinguish a special case of LLE, where for every \( j \),

\[
\sum_{\{i: j \in S(i)\}} \alpha_{ij} \leq 1. \tag{2.4}
\]

In other words, if buyer \( j \) does not get the item, the most other buyers \( i \) for whom \( j \) is in their neighborhood, \( j \in S(i) \), can collectively benefit from buyer \( j \)'s unrealized value is bounded by \( v_j \), i.e., they cannot realize more than 100\% of the value \( j \) would have realized if allocated the item. Valuations that satisfy both (2.3) and (2.4) are said to satisfy bounded local linear exclusivity (BLLE).

2.3.1 Optimal Mechanisms for LLE Valuations

We study optimal mechanisms with local exclusivity on a network in this section. We first consider a complete information setting, i.e., the setting in which there is no privately held information and \( v_i \) are known to the seller. The ex post utility (2.2)

\[\text{5 Note that (BLLE) might not hold in situations in which buyer } i \text{ has better technology to profit from buyer } j \text{'s consumers than buyer } j \text{ might have, i.e., it could be that even single } \alpha_{ij} > 1. \text{ Still, BLLE conditions are intuitive when all buyers have similar technology that turns market share into profit/value.}\]
can be rewritten as

\[ U_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) + \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) p_i(v_i, v_{-i}) \prod_{j \in S(i)} (1 - p_j(v_i, v_{-i})) - m_i(v_i, v_{-i}) \]  

Without private information, there can be no misreporting, so the (EPIC) trivially holds, and the monopolistic seller can capture the entire social surplus by setting \( m_i \) to make the (EPIR) binding. Hence, the revenue maximization problem in the perfect information setting (FB-RMP) (also known as First Best (FB) solution as it gives an upper bound for what the seller can achieve in the optimal mechanism) is equivalent to the social surplus maximization problem, i.e.,

\[
\max \left\{ \sum_{i=1}^{n} \left( v_i + \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) \prod_{j \in S(i)} (1 - p_j(v)) \right) \right\} \]

subject to

(Feasibility) \( 0 \leq p_i(v) \leq 1 \) for all \( i \).

Note that the problem (FB-RMP) is a network optimization problem, and thus, the optimal solution fundamentally depends on the network structure.

**Proposition 7.** Suppose that buyer valuations are BLLE and publicly known. The seller maximizes revenues by allocating an item to every buyer.

Proposition 7 is straightforward but it provides an important benchmark for further analysis. It establishes that there are no exclusive allocations when the exclusivity premium is bounded by (2.4). Thus, solving the problem in the complete information setting is trivial and the seller should concentrate on providing sufficient supply and not exploit the (limited) potential of local exclusivity allocations. For example, if all buyers have similar capabilities (e.g., have similar business models),
then excluding any buyer will result in the loss of that buyers unrealized value which is at least as large as the additional value its neighbors in the network could have jointly realized due to his exclusion. (BLLE valuations have diseconomies of scale structure.)

In contrast, exploiting the exclusivity on the network may be necessary if exclusive valuations are not bounded (e.g., satisfying LLE). For example, if excluding a less capable buyer would allow its more capable neighbors in the network to jointly realize higher value from the buyer’s exclusion than the value the buyer would realize were he to have gotten the item.

**Proposition 8.** Suppose that buyer valuations are LLE and publicly known. Allocating exclusively to some buyers could be optimal. Furthermore, finding a deterministic optimal solution to the (FB-RMP) problem is at least as hard as finding the maximum independent set in \((N,E)\).

If the exclusivity premium is large compared to valuations of the players, the optimal solution will tend to allocate exclusively. Hence, as shown in the proof, one can construct large enough \(\alpha_{ij}\) such that solving (FB-RMP) finds the maximum independent set in \((N,E)\).

We now turn to the private information setting. We will show that, in contrast to Proposition 7, exclusive allocations are possible and that the mechanism design problem becomes computationally hard even for BLLE valuations.

Following the methodology in Jehiel et al. (1996), we can rewrite the (EPIC) as follows. By the Envelope Theorem,

\[
\frac{dU_i(v_i, v_{-i})}{dv_i} = \frac{\partial U_i(\hat{\nu}_i, v_i, v_{-i})}{\partial \nu_i} \Big|_{(\hat{\nu}_i) = (v_i)} = p_i(v_i, v_{-i}).
\]  

(2.6)

Obviously, \(U_i(v_i, v_{-i})\) is increasing in \(v_i\). Moreover, since \(U_i(v_i, v_{-i})\) is a convex function, it is equivalent to require \(dp_i(v_i, v_{-i})/dv_i \geq 0\), which means \(p_i(v_i, v_{-i})\) is
increasing in $v_i$. Hence, we can rewrite the interim utility function as

$$U_i(v_i, v_{-i}) = U_i(v_i, v_{-i}) + \int_{v_i}^{v_i} p_i(t, v_{-i}) dt. \quad (2.7)$$

We choose $v_i$ as the bottom type and make the bottom type binds

$$U_i(v_i, v_{-i}) = 0. \quad (2.8)$$

By (2.6) and $0 \leq p_i(v_i, v_{-i}) \leq 1$, we know that $U_i(v_i, v_{-i}) \geq 0$ for any $v_i$.

By (2.5), (2.7) and (2.8), we rewrite the ex post payment as

$$m_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) + \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) p_i(v_i, v_{-i}) \prod_{j \in S(i)} (1 - p_j(v_i, v_{-i}))$$

$$- \int_{v_i}^{v_i} p_i(t, v_{-i}) dt.$$

Thus, the seller’s expected revenue can now be expressed as

$$\sum_{i=1}^{N} \int m_i(v_i, v_{-i}) dF^v(v) = \int \sum_{i=1}^{n} \left( \psi_i + \gamma_i \prod_{j \in S(i)} (1 - p_j(v)) \right) p_i(v) dF^v(v), \quad (2.9)$$

where $\gamma_i$ is the exclusivity premium, i.e., $\gamma_i \triangleq \sum_{j \in S(i)} \alpha_{ij} v_j$. Note that $\gamma_i$ depends on the network structure, i.e., $S(i)$, fraction $\alpha_{ij}$ for $j \in S(i)$, and valuations of buyer $i$’s neighbors (and not virtual valuations).

For any set of realizations $\{v_i\}_{i \in N}$, the seller’s revenue maximization problem (SB-RMP) (also known as Second Best (SB) solution as the seller has to pay information rents due to information asymmetry) can be stated as a point-wise maximization problem, i.e.,

$$\max_{\{p_i\}_{i=1}^{n}} \sum_{i=1}^{n} \left( \psi_i + \gamma_i \prod_{j \in S(i)} (1 - p_j(v)) \right) p_i(v).$$
subject to

(Feasibility) \[ 0 \leq p_i(v) \leq 1 \text{ for all } i, \]

(Monotonicity) \[ p_i(v_i, v_{-i}) \text{ is increasing in } v_i. \]

Note that the last constraint is part of the (EPIC).

Also note that the existence of a negative virtual valuation \( \psi_i < 0 \) implies that there must be buyers who will not get an item. Looking at the objective function of the (SB-RMP) problem, buyer \( i \) with \( \psi_i < 0 \) will not be allocated an item non-exclusively, and if buyer \( i \) gets an exclusive allocation, then it must be that \( \gamma_i \) is large enough and consequently \( S(i) \neq \emptyset \), which means that none of buyers \( j, j \in S(i) \), will get the item.

The following proposition contrasts Proposition 7 and shows the computational hardness of the allocation problem in the private information environment. It is important to note that the hardness is not driven just by possibly negative virtual valuations \( \psi_i \), and could be due to the publicly known network structure.\(^\text{6}\)

**Proposition 9.** Suppose that buyer valuations \( v_i \) are privately held. Suppose that valuations are BLLE. Then allocating exclusively to some buyers could be optimal. Furthermore, finding a deterministic optimal solution to the (SB-RMP) problem is at least as hard as finding the maximum independent set in \((N, E)\), even if virtual valuations \( \psi_i \geq 0 \) for all \( i \).

Table 2.1 summarizes the results of Proposition 7, Proposition 8, and Proposition 9.

These results show how the complexity of making optimal allocation and pricing decisions varies even within LLE framework. Thus, the same conclusions extend

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\(^{\text{6}}\) Such a discrepancy is observed in many problems and typically stems out of the fact that virtual valuations computed using standard Myerson technique turn non-negative values into possibly negative ones, and the underlying optimization problem that allows for negative inputs has a different complexity than the problem restricted to non-negative inputs.
Table 2.1: Optimal Allocation and Complexity.

<table>
<thead>
<tr>
<th></th>
<th>BLLE</th>
<th>LLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Public Information</td>
<td>Non-Exclusive only</td>
<td>Exclusive or Non-Exclusive</td>
</tr>
<tr>
<td></td>
<td>(straightforward)</td>
<td>(could be hard, depends on network)</td>
</tr>
<tr>
<td></td>
<td>(Proposition 7)</td>
<td>(Proposition 8)</td>
</tr>
<tr>
<td>Private Information</td>
<td>Exclusive or Non-Exclusive</td>
<td>Exclusive or Non-Exclusive</td>
</tr>
<tr>
<td></td>
<td>(could be hard, depends on network)</td>
<td>(could be hard, depends on network)</td>
</tr>
<tr>
<td></td>
<td>(Proposition 9)</td>
<td>(Propositions 8,9)</td>
</tr>
</tbody>
</table>

to fully general information structures for which solving Problem (General-RMP) analytically is beyond reach.

The main reason behind studying complexity (and relating the hard instances to the maximum-independent set problem) is to demonstrate that there is little hope for creating a reasonable procedure for making optimal allocation and pricing decisions.\(^7\) Still, it is possible to have subclasses of network structures and LLE valuations for which implementing the optimal mechanism is possible with a simple procedure. The structure of results in this section indicates that both the underlying network structure and private information could be determining factors. In Section 2.4, we will discuss one such example.

We conclude this section by illustrating a non-monotonicity property of optimal mechanisms with exclusivity. This property could be considered as yet another indication that there is little hope for finding simple and intuitive procedures implementing optimal outcomes in general settings. When buyers have private information, they can get informational rents from the revenue-maximizing seller. These rents obviously depend on the value of privately held information. On the other

\(^7\) If such procedure were to exist and were guaranteed to end in reasonable time (e.g., so that the number of queries and information updates needed grows polynomially with the growth of the number of buyers), this would establish $P = NP$, and refute a central conjecture and decades-old open problem in theoretical computer science.
hand, it would be reasonable to expect that the seller’s optimal revenues increase as the total value in the system increases (i.e., $w_i, v_i$ increase or stay the same). However, this intuition does not hold because buyers’ information rents also depend on the network structure. We show that the non-monotonicity of revenues is possible, even for a network with two buyers.

**Example 10.** Consider a clique of size two, i.e., $S(1) = 2$ and $S(2) = 1$. Let $F_1 = F_2 = U[0,1]$. Let $v_1 = 0.01 + 2/3$, $v_2 = 2/3$, and $\alpha_{12} = \alpha_{21} = 0.4$. Consequently, $w_1 = 0.01 + 2.8/3$ and $w_2 = 0.004 + 2.8/3$. It turns out that the optimal mechanism is to allocate items (non-exclusively) to both buyers and the seller’s revenue is 1.25. However, if both $\alpha_{12}$ and $\alpha_{21}$ are increased to 0.7, the optimal mechanism is to allocate an item exclusively to buyer 1 and the seller’s revenue goes down to 1.133. The similar effect can occur when only $\alpha_{12}$ is increased to 0.9. Computational details are provided at the end of the Appendix.

### 2.3.2 LLE Valuations in Supply Chain Contracts

We now illustrate an optimal mechanism for LLE valuations in the supply chain contracting setting. Consider a two-period game between a monopolistic supplier and $n$ retailers. In period 1, the monopolistic supplier is selling identical buyback contracts $(\omega, b)$ to $n$ retailers in a market with stochastic demand $D$, where $\omega$ is the per unit price charged to the retailer and $b$ is the per unit payment given to the retailer for any remaining goods.\(^8\) Note that the supplier designs the allocation and pricing procedures to sell the contracts, while $(\omega, b)$ is pre-announced and fixed all through the two-period game. Let $G$ denote the cumulative distribution function and $\mu$ denote the mean value of $D$.

If retailer $i$ gets the contract non-exclusively, i.e., at least one of its neighbors

---

\(^8\) Revenue-sharing contracts are also included in this framework, since they are equivalent to buyback contracts.
gets the contract as well, the demand that retailer $i$ can seize in period 2 is $D_i = a_i^{11}D$; if retailer $i$ gets the contract exclusively, i.e., none of its neighbors gets the contract, the demand that retailer $i$ can seize in period 2 is $D_i = a_i^{10}D$; otherwise, the retailer gets zero. Furthermore, the market share vector $(a_i^{10}, a_i^{11})$ is retailer $i$’s private information, and satisfies $1 > a_i^{10} \geq a_i^{11} \geq 0$ and

$$\sum_{i=1}^{n} a_i^{11} \leq 1.$$ 

In addition, let $I^P$ denote the set of retailers getting the contracts and $a_i$ denote retailer $i$’s generic market share $(a_i = a_i^{10}, a_i^{11},$ or $0)$.

In period 2, retailer $i \in I^P$ decides the order quantity $q_i$ from the supplier. Let $c_s$ be the supplier’s per unit production cost and $c_r$ be the retailer’s per unit marginal cost. If the retailer does not satisfy the demand, there incurs a per unit goodwill penalty $g_r$ on the retailer and $g_s$ on the supplier. Also let the supplier’s salvage value be $v$ and the exogenous revenue rate of the product be $r$. Furthermore, the expected sales for the ordered quantity $q_i$ is defined as

$$ES_i(q_i) \triangleq E_D \left[ \min (q_i, a_iD) \right] = q_i - a_i \int_{0}^{q_i/a_i} G(D) \, dD,$$

the expected leftover inventory is defined as

$$EI_i(q_i) \triangleq E_D \left[ \max (q_i - a_iD, 0) \right] = q_i - ES_i(q_i),$$

and the expected lost-sales is defined as

$$EL_i(q_i) \triangleq E_D \left[ \max (a_iD - q_i, 0) \right] = a_i \mu - ES_i(q_i).$$

Then, retailer $i$’s expected profit is

$$\pi_i = \begin{cases} rES_i(q_i) + vEI_i(q_i) - g_rEL_i(q_i) - c_rq_i - T_i, & \text{if } i \in I^P; \\ 0, & \text{otherwise.} \end{cases}$$
where \( T_i \) is the payment from retailer \( i \) to the supplier. Meanwhile, the supplier’s profit is

\[
\pi_s = \sum_{i \in I^P} (T_i - g_s EL_i(q_i) - c_s q_i).
\]

Next, we study each retailer’s optimal order quantity and the corresponding expected profits of retailers and the supplier under the buyback contract \((\omega, b)\). Given the buyback contract \((\omega, b)\), the payment is \( T_i = \omega q_i - b EI_i(q_i) \), and, thus, the profit of retailer \( i \in I^P \) can be rewritten as

\[
\pi_i = (r + g_r - c_r - \omega) q_i - (r - v + g_r - b) a_i \int_{0}^{q_i/a_i} G(D) dD - a_i g_r \mu.
\]

First order conditions indicate that the optimal order quantity is

\[
q^*_i(\omega, b) = a_i G^{-1} \left( \frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right).
\]

Therefore, retailer \( i \)'s optimal profit is

\[
\pi^*_i(\omega, b) = a_i \Pi_r - a_i g_r \mu,
\]

where

\[
\Pi_r(\omega, b) = (r + g_r - c_r - \omega) G^{-1} \left( \frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right) - (r - v + g_r - b) \int_{0}^{G^{-1} \left( \frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right)} G(D) dD.
\]

Note that \( \Pi_r(\omega, b) \) can be regarded as the aggregate profit (after compensating the retailer’s goodwill penalty) of retailers. Furthermore, with \( q^*_i(\omega, b) \), the supplier’s profit is

\[
\pi_s(\omega, b) = \sum_{i \in I^P} (a_i \Pi_s(\omega, b) - a_i g_s \mu),
\]

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where
\[
\Pi_s(\omega, b) = (\omega + g_s - c_s) G^{-1}\left(\frac{r + g_r - c_r - \omega}{r - v + g_r - b}\right)
- (b + g_s) \int_0^{G^{-1}\left(\frac{r + g_r - c_r - \omega}{r - v + g_r - b}\right)} G(D) dD.
\]

Note that \(\Pi_s(\omega, b)\) can be regarded as the aggregate profit (after compensating the supplier’s goodwill penalty) of the supplier.

In order to focus on the allocation and pricing of local exclusivity in period 1, we consider the buyback contract \((\omega^*, b^*)\) that coordinates the supply chain and gives the supplier zero profit in period 2. In fact, how to split the period 2 profit in the coordinated supply chain depends on the bargaining powers between the supplier and retailers, and, thus, all splits are possible in real business scenarios. Theoretically speaking, in the First Best, i.e., \((a_{1i}^{10}, a_{1i}^{11})\) is publicly known, all buyback contracts that coordinate the supply chain, including \((\omega^*, b^*)\), can also give the supplier the maximal expected profit in the two-period game. However, in the Second Best, i.e., \((a_{1i}^{10}, a_{1i}^{11})\) is private information, the performance of the supplier in the two-period game depends on how to split the profit in the coordinated supply chain. On one extreme, the buyback contract that coordinates the supply chain and gives retailers zero profits can give the supplier the maximal expected profit in the two-period game and can also achieve the First Best. On the other extreme, the First Best can not be achieved, and the buyback contract \((\omega^*, b^*)\) provides a lower bound on the performance of the supplier in the two-period game among all possible buyback contracts that coordinate the supply chain. The reason is that \((\omega^*, b^*)\) maximizes the information rent that the supplier has to give to each retailer in order to induce truthful reporting in period 1. Therefore, we focus on this later extreme to study the allocation and pricing of local exclusivity in period 1. Note that the supplier will get non-zero profit in period 1 by selling the buyback contract to retailers, even though
the supplier gets zero in period 2 under \((\omega^*, b^*)\).

We then characterize \((\omega^*, b^*)\), which coordinates the supply chain and gives the supplier zero profit in period 2. Let \(\Pi_c\) denote the aggregate profit of the coordinated supply chain (after compensating all goodwill penalty). Following the standard results in Cachon (2003), this buyback contract \((\omega^*, b^*)\) must satisfy

\[
\begin{align*}
 r + g_r - c_r - \omega^* &= \lambda (r + g_s + g_r - c_s - c_r), \\
 r - v + g_r - b^* &= \lambda (r - v + g_s + g_r), \\
(1 - \lambda) \Pi_c &= g_s \mu.
\end{align*}
\]

Note that the simplest form of supply chain contracts, the wholesale-price contract with wholesale price set as \(c_s\), is an example of \((\omega^*, b^*)\), when \(g_r = g_s = v = c_r = 0\). Hence, with \((\omega^*, b^*)\), we have \(\Pi_r (\omega^*, b^*) = \lambda \Pi_c\), and the profit of retailer \(i \in I^P\) under \((\omega^*, b^*)\) is

\[
\pi^*_i (\omega^*, b^*; a_i) = a_i (\Pi_c - g_r \mu).
\]

Therefore, instead of focusing on privately held \((a_{i0}^*, a_{i1}^*)\), we can directly consider privately held profits \(\pi^*_i (\omega^*, b^*; a_{i0}^*)\) and \(\pi^*_i (\omega^*, b^*; a_{i1}^*)\) for each retailer and, moreover, the structure of \((a_{i0}^*, a_{i1}^*)\) can be carried over to \((\pi^*_i (\omega^*, b^*; a_{i0}^*), \pi^*_i (\omega^*, b^*; a_{i1}^*))\).

2.4 Exclusivity Valuation Ascending Auction

In general, exclusivity makes the design of revenue-maximizing sales procedures challenging for the seller. Posted prices are not optimal (Example 5) and implementable procedures for determining optimal allocation and pricing decisions are beyond reach (Table 2.1). In this section, we propose a simple ascending auction procedure, Exclusivity Valuations Ascending (EVA) Auction. EVA generalizes and outperforms posted pricing, and is in fact an optimal mechanism for one class of information and network structures.
The main idea behind EVA is to separate exclusive and non-exclusive allocation decisions. EVA first attempts to allocate exclusively by an ascending auction with reserve. The auction ends if the market clears before the ascending auction reaches a preannounced ceiling threshold price. If this ceiling price is achieved, the auction ends without any exclusive allocations, and items are offered to buyers at a fixed price. (Note that any posted price mechanism can be implemented by choosing not to run the exclusivity auction by setting a ceiling price at zero and by choosing the posted price at which items will be offered.)

We show that EVA implements the optimal mechanism in the setting with BLLE valuations and with full competition, i.e., every buyer considers all other buyers as competitors. Thus, EVA with appropriately chosen reserve, ceiling and posted prices is always at least as good as posted price mechanisms and it is optimal in at least some settings. However, given complexity results from Section 2.3, we know that EVA is not the optimal mechanism for all settings.

Instead of describing properties of the EVA auction directly, in this section, we show how the EVA auction arises naturally from studying the optimal mechanism design in the setting with BLLE valuations with the underlying network being a complete graph (i.e., \((N, E)\) a clique, so \(S(i) = N \setminus \{i\}\), and at most one item can be allocated exclusively). The reason for such an approach is to illustrate how one can try to design other procedures for allocating and pricing exclusivity that could turn out to be optimal in some other settings of interest.

We first describe in detail the allocation and pricing rules of the optimal mechanism in the setting with BLLE valuations and with full competition. Then we use this detailed description to design the EVA auction as the only candidate for the ascending auction implementation of the optimal mechanism, assuming that all buyers respond truthfully to announced prices. Finally, we show that truthful responding is a perfect Bayesian equilibrium.
2.4.1 Optimal mechanism with BLLE valuations and full competition

Without loss of generality, we can order buyers according to their virtual valuations $\psi_i$, i.e., $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$. Let $K^*$ be the cutoff for $\psi_i$, i.e. for $i \leq K^*$, $\psi_i \geq 0$ and for $i > K^*$, $\psi_i < 0$. If $K^* = 0$, let $\sum_{i=1}^{K^*} \psi_i = 0$. Let $j^* \triangleq \arg \max_{j \in N} (\gamma_j + \psi_j)$.

We use the subscripted $\psi_i$ to denote an arbitrary buyer $i$’s virtual valuation (not ordered). We also order other buyers $-i$ according to $\psi_{-i}$ excluding $\psi_i$, i.e., $\psi(1) \geq \psi(2) \geq \cdots \geq \psi(N-1)$. $K^*_{-i}$ is similarly defined.

Let
\[
Z_{i1}^{11} \triangleq \left\{ z_i : \psi_i(z_i) \geq 0 \text{ and } \sum_{h=1}^{K^*_{-i}} \psi(z_h) + \psi_i(z_i) > \gamma_j + \psi_j \text{ for all } j \right\}
\]
and
\[
Z_{i10}^{10} \triangleq \left\{ z_i : \gamma_i + \psi_i(z_i) \geq \gamma_j + \psi_j \text{ for all } j \text{ and } \gamma_i + \psi_i(z_i) \mathbb{1}_{\psi_i(z_i) < 0} \geq \sum_{h=1}^{K^*_{-i}} \psi(z_h) \right\}.
\]

Then we can define thresholds $y_{11}^{11}(v_{-i})$ for the non-exclusive allocation, and $y_{10}^{10}(v_{-i})$ for the exclusive allocation:
\[
y_{11}^{11}(v_{-i}) \triangleq \inf \{ z_i \in Z_{i1}^{11} \} \text{ and } y_{10}^{10}(v_{-i}) \triangleq \inf \{ z_i \in Z_{i10}^{10} \}.
\]

Let $Q_{i10}^{10}$ denote buyer $i$’s probability to get the item exclusively, let $Q_{i1k}^{1k}$ denote buyer $i$’s probability to get the item with $k$ items allocated to its neighbors, let $Q_{i00}^{00}$ denote buyer $i$’s probability not to get any item with no item allocated to its neighbors, and let $Q_{i0k}^{0k}$ denote buyer $i$’s probability not to get any item with $k$ items allocated to its neighbors.

Proposition 11. The optimal mechanism when buyers in full competition have BLLE valuations allocates as follows:
\[Q_i^{1\to K^*}(v) = \begin{cases} 1, & \text{for } i = 1, 2, \ldots, K^*, \\ \text{if } \sum_{h=1}^{K^*} \psi_h > \max\{\gamma_j^* + \psi_j^*, 0\} \text{ or } \sum_{h=1}^{K^*} \psi_h = 0 > \gamma_j^* + \psi_j^*; \\ Q_j^{10}(v) = 1, & \text{if } \gamma_j^* + \psi_j^* \geq \max\{\sum_{h=1}^{K^*} \psi_h, 0\}; \\ Q_i^{10}(v) = 1, & \text{for } i = 1, 2, \ldots, n, \\ \text{if } \max\{\sum_{h=1}^{K^*} \psi_h, \gamma_j^* + \psi_j^*\} < 0. \end{cases}\]

The payments are given by

\[m_i(v) = v_i p_i(v) + \gamma_i Q_i^{10}(v) - \int_{v_i}^{v_i} p_i(t, v_{-i}) dt. \tag{2.10}\]

In particular, if \(Q_i^{1\to K^*}(v) = 1\), the payment is \(m_i(v) = y_i^{11}(v_{-i})\); if \(Q_i^{10}(v) = 1\), the payment is \(m_i(v) = \gamma_i + y_i^{10}(v_{-i})\); otherwise, the payment is \(m_i(v) = 0\).

The proof of Proposition 11 requires that the proposed mechanism satisfy both (EPIR) and (EPIC) conditions, and there should be no incentive for the seller to deviate.

If ties need to be broken, we let the seller prefer exclusive allocation if \(\sum_{h=1}^{K^*} \psi_h = \gamma_j^* + \psi_j^*\). Moreover, \(Z_i^{11}\) and \(Z_i^{10}\) reflect the requirement of buyer \(i\)'s valuation given other buyers' valuations for the non-exclusive allocation and exclusive allocation, respectively. Since \(\sum_{\{i:j \in S(i)\}} \alpha_{ij} \leq 1\) and the virtual valuation \(\psi_j(v_{-i})\) is an increasing function, \(\psi_j(v_j) - \alpha_{ij} v_j\) for some \(j \in S(i)\) is also increasing in \(v_j\). By the definition of \(Z_i^{11}\) and \(Z_i^{10}\), we know that for any \(v_i \geq y_i^{11}(v_{-i})\) and \(Q_i^{1\to K^*}(v_i, v_{-i}) = 1\), \(v_i \in Z_i^{11}\); while for any \(v_i \geq y_i^{10}(v_i)\) and \(Q_i^{10}(v_i, v_{-i}) = 1\), \(v_i \in Z_i^{10}\). As a result, we have the specified formulation of the payments.

Note that for the exclusive allocation, the buyer has to pay not only the minimum requirement of the allocation, but also the exclusivity premium \(\gamma_i\), which depends on the network structure and neighbors' valuations. However, the exclusivity premium \(\gamma_i\) is not affected by buyer \(i\)'s valuation. Therefore, when the non-exclusive allocation
prevails, the increase of buyer $i$’s valuation can not change the allocation into the exclusive allocation. Its own valuation will determine whether it will obtain the item or not rather than exclusivity or non-exclusivity. Therefore, buyer $i$’s payments are irrelevant to its valuation.

2.4.2 Ascending Auctions for Exclusivity Contracts

We now present an ascending price auction that implements the seller revenue-maximizing mechanism. For simplicity, we consider the case of two buyers. We first give a different presentation of the direct optimal mechanism, in which the following notation will be helpful in describing the optimal mechanism. Let $v_1 = x$ and $v_2 = y$, and then $\gamma_1(y) = \alpha_{12} y$ and $\gamma_2(x) = \alpha_{21} x$. Let $x_0(y)$ and $y_0(x)$ be functions implicitly defined by

$$
\psi_1(x_0(y)) = -\gamma_1(y) \quad \text{and} \quad \psi_2(y_0(x)) = -\gamma_2(x),
$$

respectively, and let $x_1(y)$ and $y_1(x)$ be functions implicitly defined by

$$
\psi_1(x_1(y)) - \psi_2(y) = \gamma_2(x_1(y)) - \gamma_1(y) \quad \text{and} \quad \psi_1(x) - \psi_2(y_1(x)) = \gamma_2(x) - \gamma_1(y_1(x)),
$$

respectively. Let $x_2$ and $y_2$ denote the roots of

$$
\psi_1(x_2) = \gamma_2(x_2) \quad \text{and} \quad \psi_2(y_2) = \gamma_1(y_2),
$$

respectively, and let $x_*$ and $y_*$ denote the roots of

$$
x_0(y_*) = x_*, \ y_0(x_*) = y_*, \ \text{and} \ x_1(y_*) = x_*, \ y_1(x_*) = y_*.
$$

The regularity assumption ensures that these functions are well defined. Let

$$
\alpha_{12}, \alpha_{21} \leq 1
$$

and, thus, $\psi_1(\cdot) - \gamma_2(\cdot)$ and $\psi_2(\cdot) - \gamma_1(\cdot)$ are increasing functions.

Using these notations, we describe direct optimal mechanisms.
Corollary 12. The optimal allocation is

\[
\begin{align*}
(p_1, p_2) &= (0, 0), \quad \text{if } x < x_0(y) \text{ and } y < y_0(x), \\
(p_1, p_2) &= (1, 1), \quad \text{if } x > x_2 \text{ and } y > y_2, \\
(p_1, p_2) &= (1, 0), \quad \text{if } x > \max\{x_0(y), x_1(y)\} \text{ and } y < y_2, \\
(p_1, p_2) &= (0, 1), \quad \text{if } y > \max\{y_0(x), y_1(x)\} \text{ and } x < x_2.
\end{align*}
\]

The optimal payments are

\[
(m_1, m_2) = \begin{cases} 
(0, 0), & \text{if } (p_1, p_2) = (0, 0), \\
(x_2, y_2), & \text{if } (p_1, p_2) = (1, 1), \\
\max\{x_1(y), x_0(y)\} + \alpha_{12} y, & \text{if } (p_1, p_2) = (1, 0), \\
0, \max\{y_1(x), y_0(x)\} + \alpha_{21} x, & \text{if } (p_1, p_2) = (0, 1).
\end{cases}
\]

Figure 2.1 illustrates the optimal mechanism for \(F_1 = F_2 = U[0, 1]\). Hence

\[
x_0(y) = (1 - \alpha_{12} y) / 2, \quad y_0(x) = (1 - \alpha_{21} x) / 2,
\]

\[
x_1(y) = y (2 - \alpha_{12}) / (2 - \alpha_{21}), \quad y_1(x) = x (2 - \alpha_{21}) / (2 - \alpha_{12}),
\]

\[
x_2 = 1 / (2 - \alpha_{21}), \quad y_2 = 1 / (2 - \alpha_{12}),
\]

and

\[
(x_*, y_*) = \left( (2 - \alpha_{12}) / (4 - \alpha_{12} \alpha_{21}), (2 - \alpha_{21}) / (4 - \alpha_{12} \alpha_{21}) \right).
\]

The region labeled (00) corresponds to no-allocation, \((p_1, p_2) = (0, 0)\), region (10) corresponds to the exclusive allocation to player 1, \((p_1, p_2) = (1, 0)\), region (01) corresponds to the exclusive allocation to player 2, \((p_1, p_2) = (0, 1)\), and region (11) corresponds to the non-exclusive allocation, \((p_1, p_2) = (1, 1)\).

Note that if \(\alpha_{12} = \alpha_{21} = 0\), then \(x_0 = y_0 = x_2 = y_2 = 1/2\), and all four regions are rectangular. If, instead, \(\alpha_{12} = 1\) or \(\alpha_{21} = 1\), the non-exclusive allocation region (11) disappears.

We next present an ascending price auction that implements the seller revenue-maximizing mechanism. In order to simplify the exposition, we assume \(F_1 = F_2 = U[0, 1]\).
Figure 2.1: Optimal Mechanism with $v$ Uniformly Distributed on $[0, 1]$.

Furthermore, without loss of generality, we focus on the non-degenerate case with $\alpha_{12} < 1$ and $\alpha_{21} < 1$. (Otherwise, a non-exclusive allocation is not possible and the problem reduces to selling the exclusive allocation to the highest bidder, i.e., the classical single item optimal auction Myerson (1981).)

Figure 2.1 provides guidance for designing an ascending auction. The auction has to start with the reservation prices so that buyer 1 with $v_1 = x \leq x^*$ and buyer 2 with $v_2 = y \leq y^*$ do not even participate in the auction. This shows that auction prices are not anonymous when buyers are not symmetric (which is the case for $\alpha_{12} \neq \alpha_{21}$). Throughout the auction, each buyer faces an increasing price for the exclusive allocation (regions 10 and 01) and clinches the exclusive allocation if the rival drops from the auction because its price for exclusive allocation becomes too high. If $v_1 = x \geq x_2$ and $v_2 = y \geq y_2$, the non-exclusive allocation (region 11) is optimal: this is achieved by simply stopping the auction when auction prices imply $v_1 = x \geq x_2$ and $v_2 = y \geq y_2$. If only one of the buyers accepts the offer at the beginning of the auction, the auction goes into a second stage, in which a new

---

$^9$ $x_*$ and $y_*$ are defined in (2.14).

$^{10}$ $x_2$ and $y_2$ are defined in (2.13).
take-it-or-leave-it offer will be presented to it.

**Exclusivity Valuation Ascending (EVA) Auction:**

Each of the two bidders is facing their own increasing price for exclusive allocation. Prices increase as time \( t \in [0, T] \) increases. At time \( t \), buyer 1 is quoted price

\[
P_{10}^1(t) = \left( \frac{\alpha_{12}(2 - \alpha_{21})}{2 - \alpha_{12}} + 1 \right) \left( x^* + \frac{x_2 - x^*}{T} t \right),
\]

for the exclusive allocation, while buyer 2 is quoted price

\[
P_{10}^2(t) = \left( \frac{\alpha_{21}(2 - \alpha_{12})}{2 - \alpha_{21}} + 1 \right) \left( y^* + \frac{y_2 - y^*}{T} t \right).
\]

for the exclusive allocation.

If neither buyer accepts the price quoted to them at time \( t = 0 \) (reserve price), the auction ends immediately with no allocation and no payments.

If only one buyer (say buyer 1) accepts the offer at time \( t = 0 \) (reserve price), the auction goes into a second stage. A take-it-or-leave-it offer is then presented to buyer 1: getting the item exclusively with price

\[
\tilde{P}_{10}^1(y) = \frac{1}{2} + \frac{1}{2} \alpha_{12} y,
\]

where buyer 2 reveals its type \( v_2 = y \) (to both the seller and buyer 1). If buyer 2 accepts the offer at time \( t = 0 \), the corresponding price is

\[
\tilde{P}_{10}^2(x) = \frac{1}{2} + \frac{1}{2} \alpha_{21} x,
\]

where buyer 1 reveals its type \( v_1 = x \) (to both the seller and buyer 2).

Otherwise, the auction continues until buyer \( i \) drops from the auction at time \( 0 < t^* < T \), which ends the auction with the rival obtaining the item exclusively at the price \( P_{-i}^{10}(t^*) \). If both buyers stay in the auction until time \( T \), the auction ends with the non-exclusive allocation that charges \( x_2 \) to buyer 1 and \( y_2 \) to buyer 2.\( \square \)
Note that if buyers are symmetric \textit{ex ante} (i.e., if $\alpha_{12} = \alpha_{21}$), then $P^{10}_1(t) = P^{10}_2(t)$.

**Proposition 13.** \textit{Bidding truthfully in EVA Auction is a perfect Bayesian equilibrium that implements the seller revenue-maximizing optimal mechanism.}

### 2.5 Concluding Remarks

In this chapter, we discuss how to allocate and price contracts, goods or services, when potential buyers have private valuations and put a premium on an exclusive allocation to them. The exclusivity is a key feature of our model since a buyer's value depends on the overall allocation, i.e., whether their competitors or peers are excluded from the allocation. The notion of exclusivity could have a limited scope and is naturally defined by proximity on a network in which buyers are represented by nodes and perceived relationships among buyers are represented by arcs. This underlying network structure is of critical importance when developing revenue-maximizing allocation and pricing procedures. In some cases, the network topology with natural limits on the value of exclusivity could provide insights that yield straightforward allocation and pricing procedures. In other cases, the network structure could be an unsurmountable obstacle to finding optimal solutions for the monopolistic seller.

We find that the revenue-maximizing seller facing buyers that put a premium on exclusivity can do better than ignoring this additional value buyers put on exclusive allocations. Even the seller who is committed to posted price mechanisms can increase revenues by inflating the price it would have charged if there were no additional value to buyers in the exclusive allocation. We show that optimal mechanisms go beyond posted prices, but determining optimal mechanism is a daunting task in a fully generic case with two-dimensional privately held buyer valuations. However, we present optimal mechanisms when underlying network topology defines the struc-
ture of exclusivity valuations (so that a value of exclusivity is a linear combination of non-exclusivity valuations of the neighbors in the network). If the non-exclusivity valuations in the network neighborhood limit the exclusivity valuation of a buyer, the revenue-maximizing seller who knows buyers’ valuations should allocate only non-exclusive contracts to all buyers. However, if buyers’ valuations are private, the seller would have to pay information rents and allocating exclusive contracts could be optimal. Interestingly, information rents depend on the network structure and create a somewhat surprising non-monotonicity effect: if buyers’ valuations increase, the seller revenue could decrease.

The fact that buyers have two-dimensional private information, a value for a non-exclusive allocation and a value for an exclusive allocation, introduces well-known fundamental difficulties of two-dimensional mechanism design problems. Our focus on information structures that happen to permit one-dimensional representation allows us to bypass this difficulty and gain insights on optimal mechanisms for a large class of settings that could be of practical relevance. Our approach is not limited to LLE valuations. We can derive results analogous to those presented in the chapter, as long as the buyers’ valuations \((w_i, v_i)\) can be parameterized by a well-behaved function \(\Gamma_i(v_1, ..., v_n)\) of one-dimensional information \(v_j\) that is privately held by each buyer \(j\). Specifically, we can extend our results from Section 2.3.1, Section 2.4, and Section 2.4.2, with either multiplicative exclusivity valuations or additive exclusivity valuations. While the approach is analogous, the structure of results does change, which is not surprising given that there is little hope of solving a generic two-dimensional mechanism design problem. For example, an ascending auction that implements the optimal mechanism for multiplicative exclusivity valuations has to simultaneously quote exclusivity and non-exclusivity prices, and possibly quote them non-anonymously, which is different from the EVA auction presented here.

The notion that buyers put a premium on exclusive allocation is closely related
to negative externalities, as the value to the buyer who gets the item decreases if any of the buyer’s competitors also get the item. However, exclusivity has an all-or-nothing feature in the sense that there is no incremental benefit as the number of local competitors decreases. One way to model negative externalities, and possibly exclusivity, would be to assume buyers demand more than one item and are interested in buying items designed for their competitors. In such a setting, an exclusive allocation corresponds to the buyer obtaining $k + 1$ items: an item designed for the buyer and all of the unique items designed for each of its $k$ neighbors. The difficulty with such an approach is that buyers have non-monotonic marginal values for items they demand: the first item has value $v_i$, the $k + 1$st has marginal value $w_i - v_i$, while all other items have marginal value zero. While there is a work on efficient mechanisms when buyers have decreasing marginal values (e.g., Ausubel, 2004), we are not aware of any results or general methodologies that could handle non-monotonic marginal values (and these are fundamental features of representing exclusivity) in the revenue-maximization setting. However, our approach with LLE valuations could be extended in a way that relaxes the notion of exclusivity. We could define a more general model for valuations so that every buyer that gets a contract (exclusively or not) has a value for it that is equal to $v_i + \sum_j \alpha_{ij} v_j$, where summation is over all $j \in S(i)$ that did not get the contract. In other words, buyer $i$ gets the value from the contract $v_i$ and a fraction of the value of the contract for each of its neighbors that were shut out of the market (capturing a fraction $\alpha_{ij}$ of the unrealized neighbor’s value $v_j$). Deriving an optimal mechanism in this case is analogous to our approach with LLE valuations.

Design of an ascending auction implementation of the optimal mechanism is possibly of interest for eventual practical use. Note that we have formally described the EVA auction for $n = 2$ buyers, although it should readily extend for any number of buyers. It has an appealing feature in that it separately auctions the two allocation
types in a straightforward manner: it first auctions off an exclusive contract using a standard ascending procedure, and then it allocates non-exclusive contracts only in the second stage. We leave questions on how to extend EVA on a general network and how well EVA or a similar procedure performs for general two-dimensional valuations for future research.

In conclusion, we provide initial theoretical insights on when the seller should consider allocating exclusively, who the seller should allocate exclusively to, and how exclusivity contracts should be priced. Our model is generic and simple, but the setup and findings provide possible directions for further research by extending the model parameters, generalizing the information structure, and adjusting the approach to fit applications where exclusivity plays a major role.
3 Money for Nothing: Exploiting Negative Externalities

3.1 Introduction

When market participants compete for a scarce resource, they face a possibility of losses due to resource being allocated to competitors. This is the case in many electronic commerce situations. A typical situation is that of internet advertising. The number of impressions per webpage is limited and could be considered a scarce resource when demand is high. Many advertisers rely on their internet presence and ad placement as the main channel for attracting customers. If an advertiser cannot secure a webpage impression to reach a potential customer who will be shown that webpage, the advertiser might prefer that no competitor’s ads are shown to the customer being exposed to competitors’ ads.\(^1\)

One should expect that design of optimal allocation and pricing mechanisms could

\(^1\) This effect is not limited to e-commerce: a telecom company that does not obtain a frequency license might lose potential future customers who will be captured by a competitor who might have secured that license; a drug developer who does not obtain a regulatory approval is better off if all competitors also fail to obtain an approval, than if any of competitors succeed and are first to the market.
be affected by the fact that market participants not only have private valuations for a scarce resource, but also face negative externalities when a competitor succeeds in obtaining the resource. One theoretical obstacle to finding an optimal mechanism is that information structure of market participants becomes two-dimensional. In addition, a potential usefulness of such optimal mechanism depends on whether such an optimal mechanism (a) allows for a manageable practical implementation, and (b) yields fundamentally different allocation and pricing decisions from those reached without exploiting negative externalities (e.g., if such externalities turn out to be negligible). In this chapter we show that a monopolistic seller (or a social planner) could exploit existence of even minuscule negative externalities, provided sufficiently large demand for a scarce resource.

Intuitively, even if negative externalities are small, not allocating any resources might be optimal when the number of buyers is large; a small number of winners would trigger a large total value of negative externalities for all losers. A recent school naming-rights example illustrates this point: thirteen donors gave combined $85 million, with minimum single donor gift of $5 million, to the Wisconsin School of Business at the University of Wisconsin-Madison to “preserve the Wisconsin name for at least 20 years. During that time, the school will not be named for a single donor or entity” (WSB, 2007). Not only did the Wisconsin School of Business raise $85 million for NOT naming the school (compared, e.g., with $55 million naming gift for the Tepper School of Business at Carnegie Mellon University (TSB, 2004)), but they can also try to sell the name again in 2027.

This example indicates a potential for implementing such no-allocation equilibrium in the context of electronic commerce, and internet advertising specifically. A monopolist seller of limited ad space could attempt to collect a small amount of money from each of the potential advertisers who are concerned with negative externality effects, and by doing so optimize revenues and ensure ad-free experience to
its users. Thus, our results show that ad-free user experience can be supported not just by a user-subscription fee model (used by several popular content providers), but could be subsidized by potential advertisers whose ads will not be shown.

In sponsored search, intense competition among internet advertisers for user attention and actions (measured by click-through rates and conversion rates), induces negative externalities whenever competing ads are displayed together. This point has been argued in Ghosh and Sayedi (2010). User attention and actions that an ad can attract in sponsored search depends on the total number of the ads shown simultaneously (Muthukrishnan, 2009; Reiley et al., 2010), as well as on the relative position of the ad (Craswell et al., 2008; Gomes et al., 2009). The externalities due to user attention and click through rates, also labeled quantity effects (Constantin et al., 2010), are analyzed in cascade models (e.g., Aggarwal et al., 2008; Kempe and Mahdian, 2008; Giotis and Karlin, 2008).

Since click through rates can be learned by a search engine as the data are observable, the quantity externalities are different from the value externalities, i.e., the value conditioned on receiving a click. These value externalities are private to advertisers (Constantin et al., 2010). The private information about value externalities may not be one dimensional, due to the dependence on the conversion probability (Ghosh and Sayedi, 2010). Furthermore, only the ad owner has information on conversions, so the value of negative externality imposed onto a competitor might actually be private information of the ad owner. Such information structure has been studied in economics literature (Jehiel et al., 1996) in the limited context of a single unit resource. We will build on this model and extend it to an arbitrary but finite number of resources and study relationships involving externality valuations and intensity of competition.

A reasonable conjecture would be that incorporating externality valuations in the ad auction design ought to improve expressiveness, and could therefore improve
both efficiency and the seller’s revenue (for general discussion of expressiveness, see, e.g., Parkes and Sandholm, 2005). Some recent papers explore mechanisms that use externality valuation information in different formats. In Ghosh and Sayedi (2010), extensions are designed and equilibria are analyzed for a Generalized Second Price auction in which an advertiser’s private value depends on whether the ad is allocated exclusively or not. In their model advertiser places two bids: one for exclusive placement and one for being placed with multiple other ads. (The difference in the valuations correspond to negative externality of competitors’ ads being placed.) The model in Ghosh and Mahdian (2008) assumes the value of an ad to an advertiser depends on the relative quality of the ad compared to the other ads that are shown simultaneously. In Constantin et al. (2010), a framework of the unit-bidder constraints for value externalities is considered: a bidder is allowed to specify a set of single competitor constraints, where each such constraint prohibits the bidder’s ad being shown together with the ad of the competitor identified by the constraint. Clearly, the focus of such modeling approach is to deal with externalities by limiting possible allocations. Given somewhat complicated valuation structure that accounts for all externalities, even computing optimal allocation in the complete information setting could be unmanageable. In Conitzer and Sandholm (2010), a general representation of settings with externalities is provided and then the efficient computation of optimal outcomes are studied. There is also a large body of literature on advertising, see Bagwell (2007).

In this chapter, we provide a theoretical analysis supporting the intuition behind optimality of the non-allocation. Our setting is that of a monopolistic seller of $K$ identical indivisible items with $n$ potential unit-demand buyers. The buyers have independent private values for the item, as well as independent private values for (negative) externality they would impose onto every rival if they get the item. This is the informational setting of the seminal work of Jehiel et al. (1996). They
manage to provide the optimal mechanism for the single item case ($K = 1$), despite two-dimensional valuation structure. The optimal mechanism with $K > 1$ items (Theorem 17) is a straightforward generalization of the methodology in Jehiel et al. (1996).

Our focus is on exploring properties of the optimal mechanism with multiple items and a large number of buyers. If there are sufficiently many buyers willing to pay a small amount in order to avoid negative externalities being imposed on them, the optimal mechanism will not allocate any items. We formalize this observation by specifying the joint effect of the number of buyers and the number of items to the emergence of an equilibrium with no allocation. Moreover, under the no-allocation equilibrium, the seller’s expected profit increases in the number of buyers and the number of items.

Since negative externalities are the driving force behind possibility of the no-allocation equilibrium, our analysis includes a variety of externality valuation structures. Specifically, we show conditions for emergence of the no-allocation equilibrium under the assumption of externalities being independent of item valuations (independent externalities), as well as under the assumption of externalities depending on item valuations (dependent externalities). We also allow for scaling the magnitude of externality valuations relative to the magnitude of item valuations. This case is important for settings in which negative externality imposed onto rivals is much smaller than the value of the items (e.g., the value of the ad shown is much higher than the negative externality if rival’s ad is shown.)

The multiple items setting of this chapter also provides a way for understanding the relationship between exclusivity pricing and pricing negative externalities. In our model, the exclusivity premium is set to the payoff difference between an exclusive allocation and a sharing allocation, and is equal to the magnitude of aggregate externalities.
Our results establish emergence of the no-allocation equilibrium when negative externalities are present (possibly small) and when competition is intense. Therefore, auction mechanisms that exploit negative externalities could, in contrast to standard mechanisms that ignore negative externality information, yield optimal pricing and (no-)allocation, which in turn makes them relevant and desirable for use in practice.

The chapter is organized as follows. In the next section, we describe our model. In Section 3.3, we show the optimal mechanism. Section 3.4 illustrates our findings regarding conditions for the no-allocation equilibrium, for uniformly distributed valuations, with both independent and dependent externalities. The generalization of our results for general distributions is demonstrated in section 3.5. Brief concluding remarks close the chapter.

3.2 Model

A risk-neutral seller is selling \( K \) identical indivisible items to \( n \) (\( n > K \)) potential risk-neutral buyers \( i, i = 1, \ldots, n \). Each buyer has unit demand. We adopt the information structure of Jehiel et al. (1996), and assume that buyer \( i \)’s private information is given by a two-dimensional type: \( t_i = (t_{1i}, t_{2i}) = (\pi_i, \alpha_i) \), where \( \pi_i \) is \( i \)'s valuation of the item, and \( \alpha_i \) is the externality imposed on buyer \( j \) (\( j \neq i \)) if buyer \( i \) obtains the item.\(^2\) Note that \(-\alpha_i\) can be viewed as a negative externality on buyer \( j \) when buyer \( i \) gets the item and buyer \( j \) does not, i.e., buyer \( i \) creates the negative (positive) externality on buyer \( j \) when \( \alpha_i > 0 \) (\( \alpha_i < 0 \)). The seller’s type is set to \( t_0 = (0, 0) \). Buyer types are independent across buyers, so the type space is \( T = T_1 \times T_2 \times \ldots \times T_n \) where types are drawn from \( T_i = [\pi_i, \pi_i] \times [\alpha_i, \alpha_i] \), according to the joint probability density function (hereafter, PDF) \( f_i \) and joint cumulative

\(^2\) Our analysis and all results readily extend to the environment with different externalities on different buyers, i.e., where \( \alpha_i \) is a length \( n-1 \) vector \( (\alpha_{i1}, \ldots, \alpha_{i(i-1)}, \alpha_{i(i+1)}, \ldots, \alpha_{in}) \). For simplicity of exposition, in this chapter we present the model with identical externalities imposed on different buyers, i.e., \( \alpha_{ij} = \alpha_i \) for all \( j \neq i \).
density function (hereafter, CDF) $F_i$.

If valuations and externalities are independent, $f_i(t^1_i, t^2_i) = f^1_i(t^1_i)f^2_i(t^2_i)$.

We also make the following standard assumption:

**Assumption 14.** $\frac{1 - F^1_i(t)}{F^1_i(t)}$ is a decreasing function for all $i$.

We next define the set of buyers with $K$ highest valuations and the set of buyer $i$’s rivals with $K$ highest externalities imposed on $i$.

**Definition 15.** $K^1\left(\{t^1_j\}_{j=1}^n\right) \triangleq \{j(n), j(n-1), ..., j(n-K+1)\}$, where $t^1_{j(n)} \geq t^1_{j(n-1)} \geq ... \geq t^1_{j(n-K+1)} \geq ... \geq t^1_{j(1)}$.

**Definition 16.** $K^2(\{t^2_j\}_{j=1}^n, i) \triangleq \{j(n), j(n-1), ..., j(n-K+1)\}$, where $t^2_{j(n)} \geq t^2_{j(n-1)} \geq ... \geq t^2_{j(n-K+1)} \geq ... \geq t^2_{j(2)}$ and $t^2_{j(1)} \triangleq t^2_i$.

A direct revelation mechanism is defined as $(x, p, \rho)$. $x$ is the payment vector from the buyer to the seller with $x_i : T \rightarrow R$. $p$ is the allocation (probability) vector with $p : T \rightarrow \{z \in R^n_+ | \sum z_i \leq K$ and $z_i \leq 1\}$. The probability $p_i(t_1, t_2, ..., t_n)$ is defined as the probability that buyer $i$ obtains one of the $K$ items, regardless of which one she occupies. $\rho$ is the trigger strategy vector when buyer $i$ refuses to participate with $\rho_i : T_{-i} \rightarrow \{z \in R^n_+ | \sum z_i \leq K$ and $z_i \leq 1\}$. We also define the interim payment as

$$y_i(t_i) = \int_{T_{-i}} x_i(t_1, ..., t_n)\phi_{-i}(t_{-i})dt_{-i}, \quad (3.1)$$

and interim allocation rule as

$$q_i(t_i) = \int_{T_{-i}} p_i(t_1, ..., t_n)\phi_{-i}(t_{-i})dt_{-i}, \quad (3.2)$$

where $\phi = f_1 \times f_2 \times ... \times f_n$. 
Note that when $K > 1$, externalities may be imposed on buyer $i$ even when she obtains the item since buyer $j$ could obtain the item as well. (Case $K = 1$ corresponds to the model in Jehiel et al. (1996).)

In addition, we assume that each buyer’s utility is additively separable. The interim utility of buyer $i$, when she reports $s_i$ with true type $t_i$ and her rivals truthfully report, is thus

$$U_i(s_i, t_i) = q_i(s_i) t_i^1 - \sum_{j \neq i} \int_{T_{-i}} p_j(s_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i} - y_i(s_i).$$

Obviously, the seller’s optimal trigger strategy for buyer $i$ is to sell the $K$ items to buyer $i$’s opponents who have the $K$ largest externalities. This strategy imposes the severest punishment on the buyer if she rejects to participate, i.e.,

$$\rho_{v(i, t_{-i})}(t_{-i}) = 1, \quad (3.3)$$

$$\rho_j(t_{-i}) = 0, \text{ for } j \neq v(i, t_{-i})$$

and

$$v(i, t_{-i}) \in K^2(\{t_j^2\}_{j=1}^n, i).$$

Under this optimal trigger strategy, the seller’s problem is

$$\max_{\{x, p\}} \sum_{i=1}^n \int_{T_i} y_i(t_i) f_i(t_i) dt_i \quad (3.4)$$

subject to

$$U_i(t_i, t_i) \geq U_i(s_i, t_i) \text{ for all } i \text{ and all } s_i, t_i \in T_i \quad (\text{ICC})$$

$$U_i(t_i, t_i) \geq A_i \text{ for all } i \text{ and all } t_i \in T_i \quad (\text{IRC})$$

where

$$A_i \triangleq -\int_{T_{-i}} (\sum_{j \in K^2(\{t_j^2\}_{h=1}^n, i)} t_j^2 \phi_{-i}(t_{-i}) dt_{-i}). \quad (3.5)$$
Inequality (ICC) is the Incentive Compatibility Constraint, which ensures that truthfully reporting is a Nash equilibrium. (IRC) is the Individual Rationality Constraint, under which there is no incentive for the buyers to reject participation.

In order to make analysis tractable, we will assume that distributions from which buyer types, $t_i = (\pi_i, \alpha_i)$, are drawn are i.i.d. across buyers. However, $\pi_i$ and $\alpha_i$ may or may not be correlated.

### 3.3 Optimal Mechanism

We follow approach of Jehiel et al. (1996) and their Proposition 2 that uses standard Myerson technique (Myerson, 1981) to obtain expression for the seller’s *ex ante* profit in the case $K = 1$, and thus we obtain

$$EP = -\sum_{i=1}^{n} A_i + \int \left( \sum_{i=1}^{n} [\pi_i - \frac{1 - F_i^1(\pi_i)}{f_i^1(\pi_i)}] - (n - 1)E_i \right) \cdot p_i(\pi_1, \ldots, \pi_n) \cdot f_1^1(\pi_1) \cdots f_i^1(\pi_n) d\pi_1 \cdots d\pi_n$$

where

$$E_i = \int_{\alpha}^{\bar{\alpha}} \tau f_i^2(\tau) d\tau. \tag{3.6}$$

when externalities are independent of item valuations, and

$$EP = -\sum_{i=1}^{n} A_i + \int \left( \sum_{i=1}^{n} [\pi_i - \frac{1 - F_i^1(\pi_i)}{f_i^1(\pi_i)}] - (n - 1)g_i(\pi_i) \right) \cdot p_i(\pi_1, \ldots, \pi_n) \cdot f_1^1(\pi_1) \cdots f_i^1(\pi_n) d\pi_1 \cdots d\pi_n$$

for externalities perfectly correlated with item valuations, i.e., for $\alpha_i = g_i(\pi_i)$.

Therefore, we have the following theorem illustrating the optimal allocation rules and the optimal interim payment rules.
Theorem 17. 1) If item valuations and externalities are independent, the optimal allocation rule is

\[
p^*_i(\pi_1, \ldots, \pi_n) = \begin{cases} 
1 & \text{if } i \in K^1(\{\pi_j - \frac{1-F_j^1(\pi_j)}{f_j^1(\pi_j)} - (n-1)E_j\}_{j=1}^n) \\
\quad \text{and } \pi_i - \frac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)E_i \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

2) If item valuations and externalities are perfectly correlated, i.e. \( \alpha_i = g_i(\pi_i) \), the optimal allocation rule is

\[
p^*_i(\pi_1, \ldots, \pi_n) = \begin{cases} 
1 & \text{if } i \in K^1(\{\pi_j - \frac{1-F_j^1(\pi_j)}{f_j^1(\pi_j)} - (n-1)g_j(\pi_j)\}_{j=1}^n) \\
\quad \text{and } \pi_i - \frac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)g_i(\pi_i) \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

3) The optimal interim payment is given by

\[
y^*_i(t_i) = -A_i + t_i^1 q^*_i(t_i) - \int_{t_i^1}^t q^*_i(v,t_i)dv - \sum_{j \neq i} \int_{T_{-i}} p^*_j(t_i,t_{-i})t_j^2 \phi_i(t_{-i})dt_{-i}, \quad (3.7)
\]

where \( q^*_i(t_i) \) is obtained by substituting \( p^*_i(\pi_1, \ldots, \pi_n) \) into the definition of the interim allocation rule (3.2).

3.4 Uniformly Distributed Types

Throughout this section we will assume that types are uniformly distributed. This will allow us to illustrate the conditions on the number of buyers \( n \) and the number of items \( K \) that yield the no-allocation equilibrium. We will also assume negative externalities, i.e. \( \alpha > 0 \).
We first consider the externality independence case with \( K \) items, i.e., \( f_i(t_1^i, t_2^i) = f_1^i(t_1^i)f_2^i(t_2^i) \). The distribution of externalities can be scaled up or scaled down relative to valuations by some positive number \( c \). We end this illustration by analyzing the externality dependence case.

### 3.4.1 Independent Externalities

We assume the valuation \( \pi_i \) is drawn from \([0, 1]\) uniform distribution, while the externalities \( \alpha_i \) are drawn from a scaled down (up) \([0, c]\) uniform distribution, where \( c < 1 \) (\( c > 1 \)). Then \( F^1(\pi_i) = \pi_i, f^1(\pi_i) = 1, F^2(\alpha_i) = \alpha_i/c, \) and \( f^2(\alpha_i) = 1/c \). By the definition of \( E \) (3.6), the externality is

\[
E = \int_0^c \frac{1}{c} \tau \, d\tau = \frac{c}{2}.
\]

By Theorem 17, the items will be sold to those buyers whose valuations \( \pi_i \) are among the first \( K \) largest, i.e. \( i \in K^1 \), and, satisfy

\[
\pi_i - \frac{1 - F^1(\pi_i)}{f^1(\pi_i)} - (n - 1)E \geq 0.
\]

With uniform distribution, the condition is

\[
\pi_i \geq \frac{1}{2}[1 + \frac{c}{2}(n - 1)]
\]

Therefore, the items will not be allocated if

\[
n > 1 + \frac{2}{c}.
\]

Since, for \( j = 0, ..., K - 1 \), the density of the \((n - 1 - j)^{th}\) largest out of \( n - 1 \) i.i.d. random variables with CDF \( F(x) \) and PDF \( f(x) \), is

\[
f_{x^{(n-1-j)}} = \frac{(n - 1)!}{(n - j - 2)!j!} F(x)^{n-j-2}(1 - F(x))^j f(x),
\]

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the threat $A$ can be rewritten as

$$A = -\frac{n-1}{n}c \left[ \left( \sum_{j=1}^{K-1} \frac{(n-2)!}{(n-j-2)!j!} \prod_{i=0}^{j-1} \frac{j-i}{n-(j-i)} \right) + 1 \right]. \quad (3.8)$$

The calculation also involves two integrals,

$$\int_{0}^{c} (n-1) \left( \frac{x}{c} \right)^{n-1} dx = \frac{n-1}{n}$$

and

$$\int_{0}^{c} \left( \frac{x}{c} \right)^{n-j-1} \left( 1 - \frac{x}{c} \right)^{j} dx = \frac{j}{n-j} \int_{0}^{c} \left( \frac{x}{c} \right)^{n-j} \left( 1 - \frac{x}{c} \right)^{j-1} dx \text{ for } j \geq 1.$$

The seller’s expected profit when the seller does not allocate any items is

$$EP = -nA$$

$$= (n-1)c \left[ \left( \sum_{j=1}^{K-1} \frac{(n-2)!}{(n-j-2)!j!} \prod_{i=0}^{j-1} \frac{j-i}{n-(j-i)} \right) + 1 \right]$$

$$= (n-1)c \left[ \left( \sum_{j=1}^{K-1} \frac{(n-2)!}{(n-j-2)!j!} \frac{j \cdot (j-1) \ldots 1}{(n-1) \cdot (n-2) \ldots (n-j)} \right) + 1 \right]$$

$$= (n-1)c \left[ \left( \sum_{j=1}^{K-1} \frac{(n-2)!}{(n-j-2)!j!} \frac{j!(n-j-1)!}{(n-1)!} \right) + 1 \right]$$

$$= c \left[ \sum_{j=1}^{K-1} (n-j-1) + (n-1) \right]$$

$$= nc \sum_{j=0}^{K-1} \frac{n-j-1}{n}$$

$$= c \left[ (n-1)K - \frac{K(K-1)}{2} \right] \quad (3.9)$$

where $\sum_{j=0}^{K-1} \frac{n-j-1}{n}$ is the sum of expected values of the $K$ largest order statistics for the uniform distribution.

In the next proposition we summarize the no-allocation result and describe how the number of buyers and the number of items affect the optimal expected profit.
Proposition 18. Suppose there are $K$ items and $n$ buyers with types $(\pi_i, \alpha_i)$ independently drawn from $U[0,1] \times U[0,c]$. If $n > 1 + 2/c$, the optimal mechanism will not allocate any of the items. Moreover, the seller’s expected profit is increasing in the number of buyers $n$, and in the number of items $K$.

Note that $n > 1 + 2/c$ is a sufficient condition to make the seller generate revenues without allocating any items. The result is intuitive: a small $c$ means low externalities, and thus it requires a large number of buyers to make the no-allocation equilibrium feasible. On the other hand, $c$ can be considered as a function of $n$. It follows from Proposition 18 that the no-allocation equilibrium emerges when $c(n) > 2/(n - 1)$. From (3.9), we can observe that, when $n$ is large enough, the seller can implement the no-allocation equilibrium and collect a small payment (close to $c$) from each buyer. This is achieved by threatening the buyers with allocating the items to other buyers, in particular, threatening to allocate to $K$ buyers with the largest negative externality values.

Since the optimal expected profit is concave in $K$, the seller could achieve the largest expected profit when

$$K^* = n - 1/2.$$

Therefore, the expected profit increases in the number of items, $K$, because $K \leq n - 1$.

Figure 3.1 shows how expected profits change with the number of buyers and the number of items, with $c = 1$, $1 + 2/c \leq n \leq 30$, and $1 \leq K \leq n - 1$. The seller’s expected profit increases linearly with the number of buyers $n$.

We further simplify the results above for a special case of the single item, $K = 1$, and for $c = 1$, which corresponds to the setting in Jehiel et al. (1996).

By the definition of $A_i$ (3.5), the expected utility of a nonparticipating buyer is
Figure 3.1: Expected Profit with n Buyers and K Items \((c = 1)\).

given by

\[
A = -(n - 1) \int_0^1 \tau^{n-1} d\tau = -\frac{n - 1}{n}.
\]

By the definition of \(E\) (3.6), the externality is still

\[
E = \int_0^1 \tau d\tau = \frac{1}{2}.
\]

According to Theorem 17, the single item will be assigned to the buyer \(i^* = \arg\max_i \{\pi_i - \frac{1-F^1(\pi_i)}{F^1(\pi_i)} - (n-1)E, 0\} = \arg\max_i \{2\pi_i - 1 - \frac{1}{2}(n-1), 0\}.\) This requires \(\pi_i \geq (n + 1)/4.\) Thus, the allocation rule is specified as

\[
p_{i^*}(t_1, ..., t_n) = 1 \quad \text{and} \quad p_{j \neq i^*}(t_1, ..., t_n) = 0.
\]

There are two cases.

Case 1: \((n + 1)/4 > 1.\) Since \(\pi_i \in [0, 1],\) \(p_i(t_1, ..., t_n) = 0\) for all \(i\) with the trigger strategy (3.3). Therefore, the interim payment for buyer \(i\) is \(y_i^*(\pi_i) = -A + 0 - 0 - 0 = (n - 1)/n.\) The expected profit of the seller at this case is thus \(EP = n - 1.\)
Case 2: \((n + 1)/4 \leq 1\). If \(\pi_i < (n + 1)/4\), buyer \(i\) will not get the single item, and by (3.7) the interim payment is

\[
y^*_i(\pi_i) = -A - 0 - 0 - (n - 1) \int_{\frac{n+1}{4}}^{1} \pi_i^{n-2} d\pi_i \int_{0}^{1} t_i^2 dt_i = \frac{n - 1}{n} - \frac{1}{2} [1 - \left(\frac{n + 1}{4}\right)^{n-1}].
\]

If \(\pi_i \geq (n + 1)/4\), buyer \(i\) may get the item. The optimal interim allocation rule is

\[
q^*_i(t_i) = \int_{t_i}^{T_{-i}} p^*_i(t_i, t_{-i}) dt_{-i} = \pi_i^{n-1},
\]

and the optimal interim payment (3.7) is specified as

\[
y^*_i(\pi_i) = -A + t_i^1 q^*_i(t_i) - \int_{0}^{\pi_i} q^*_i(v, t_i^2) dv - (n - 1) \int_{\pi_i}^{1} \pi_j^{n-2} d\pi_j \int_{0}^{1} t_j^2 dt_j
\]

\[
= \frac{n - 1}{n} + \pi_i^n - \frac{1}{n} \left[ \pi_i^n - \left(\frac{n + 1}{4}\right)^n \right] - \frac{1}{2} \left(1 - \pi_i^{n-1}\right)
\]

\[
= \frac{n - 1}{n} + \frac{n - 1}{n} \pi_i^n + \frac{1}{n} \left(\frac{n + 1}{4}\right)^n - \frac{1}{2} + \frac{1}{2} \pi_i^{n-1}.
\]

Therefore, the seller’s expected profit in this case is

\[
EP = \sum_{i=1}^{n} \int_{0}^{1} y^*_i(\pi_i) d\pi_i = n \int_{0}^{1} y^*_i(\pi_i) d\pi_i
\]

\[
= \frac{n(n + 1)}{4} \left[ \frac{n - 1}{n} - \frac{1}{2} \left[1 - \left(\frac{n + 1}{4}\right)^{n-1}\right]\right]
\]

\[
+ n \left(1 - \frac{n + 1}{4}\right) \left[ \frac{n - 1}{n} - \frac{1}{2} + \frac{1}{n} \left(\frac{n + 1}{4}\right)^n\right]
\]

\[
+ n \int_{\frac{n+1}{n}}^{1} \left(\frac{n - 1}{n} \pi_i^n + \frac{1}{2} \pi_i^{n-1}\right) d\pi_i.
\]

Numerically, when \(n = 1\), \(EP = 0.25\); when \(n = 2\), \(EP = 1.115\) (consistent with the results in Jehiel et al. (1996)); when \(n = 3\), \(EP = 2\); and when \(n = 4\), \(EP = 3\). For \(n \leq 3\), we are in Case 2. For \(n \geq 4\), we are in Case 1. Therefore, for sufficiently large \(n\), starting from \(n = 4\) in this example, the single item will not be allocated to
any buyer. However, the seller could raise revenues from the buyers by the trigger strategy. This is because each buyer would like to pay a small amount of money in order to avoid the impact of negative externalities. Moreover, the expected profit increases in the number of buyers since the expected profit increases in $n$ when $n = 1, 2, 3$ and $4$ and the expected profit is $n - 1$ when $n \geq 4$.

### 3.4.2 Dependent Externalities

Here we assume valuation $\pi_i$ is drawn from $[0, 1]$ uniform distribution, and consider a perfectly linear correlation case, $\alpha_i = c_i \pi_i$, where $c_i$ is publicly known, and $c_i > 0$. $c_i < 1$ means the externality is just a fraction of the valuation. Then, the allocation condition becomes

$$i \in K_1 \left( \pi_i - \frac{1 - F_1(\pi_i)}{f_1(\pi_i)} - (n - 1)c_i \pi_i \right)$$

and $\pi_i - \frac{1 - F_1(\pi_i)}{f_1(\pi_i)} - (n - 1)c_i \pi_i \geq 0$.

With uniform distribution, it is

$$[2 - (n - 1)c_i] \pi_i \geq 1.$$

Therefore, there will be no allocation if $2 - (n - 1)c_i \leq 0$ or $0 < 2 - (n - 1)c_i < 1$.

Since $\pi_i$ is drawn from $[0, 1]$ uniform distribution, $\alpha_i$ is uniform on $[0, c_i]$. By the definition of $A_i$ (3.5), we get the same expression as in the previous subsection, i.e.

(3.8)

$$A_i = -\frac{n - 1}{n} c_i \left[ \left( \sum_{j=1}^{K-1} \frac{(n - 2)!}{(n - j - 2)!j!} \prod_{i=0}^{j-1} \frac{j - i}{n - (j - i)} \right) + 1 \right].$$

In particular, if $K = 1$,

$$A_i = -(n - 1) \int_0^1 c_i \tau^{n-1} d\tau = -\frac{n - 1}{n} c_i.$$

Consequently, the expected profit under the no-allocation equilibrium is the same as in the externality independence setting.
The following proposition summarizes these results.

**Proposition 19.** Suppose there are \( K \) items and \( n \) buyers with types \((\pi_i, \alpha_i)\) where \( \pi_i \) is drawn from \( U[0, 1] \) and \( \alpha_i = c_i \pi_i \). If \( n > \max_i \{1 + 1/c_i\} \), the optimal mechanism will not allocate any of the items. Moreover, the seller’s expected profit is increasing in the number of buyers \( n \), and in the number of items \( K \).

The condition from the proposition is only sufficient for the existence of the no-allocation equilibrium. In fact, when externality valuations are perfectly correlated with item valuations, buyers with highest item valuations are simultaneously the buyers who would impose highest negative externalities on rivals. Thus, with the same scales, a smaller number of buyers than in the independent externalities case will make negative externalities dominate item valuations and yield the no-allocation equilibrium. On the other hand, buyers with larger downward scales, i.e., a smaller \( c_i \), impose smaller negative externalities on rivals. Thus, a larger number of buyers is required to yield the no-allocation equilibrium.

**Corollary 20.** Under conditions of either Proposition 18 or Proposition 19, the expected profits under the no-allocation equilibrium are

\[
EP = -\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{c_i}{n} \left[ (n-1)K - \frac{K(K-1)}{2} \right].
\]

### 3.5 No-allocation Equilibrium with Independent Externalities

We now show that the no-allocation equilibrium emerges for a large enough number of buyers, regardless of the distributional assumptions. We assume externality \( \alpha_i \) to be independent from the item valuations \( \pi_i \).

**Theorem 21.** Suppose there are \( K \) items for sale to \( n \) buyers with independent valuations and externalities. There exist \( N \) such that for \( n > N \), it is optimal for
The seller not to allocate any item. Moreover, the seller’s expected profit is positive under the no-allocation equilibrium, and is increasing in both \( K \) and \( n \).

The independence assumption is not a critical one and the same proof strategy would go through even when externalities depend on the item valuations. The exact conditions for the no-allocation result would be determined by the dependence structure of externalities and valuations.

3.6 Concluding Remarks

In this chapter we discuss an optimal mechanism that a monopolist seller could use to exploit possible (negative) externalities among large number of buyers interested in the limited number of items that are being sold. Specifically, we demonstrate that no allocation is an equilibrium even with modest negative externalities, provided sufficiently large demand. This provides an opportunity for the seller to generate revenues without having to sell any of the items. For example, a monopolistic seller of the ad space could generate revenues by charging potential advertisers for not showing any ads (thereby providing ads-free experience to the end users that are initial targets by the interested advertisers). For this strategy to be viable, (i) advertisers would have to be (ever so slightly) negatively impacted if rival’s ad is shown, and (ii) the number of interested advertisers has to be large relative to the number of available ad slots.

Our formal analysis of the properties of the optimal mechanism is first illustrated in the case of uniformly distributed valuations. We show that it is optimal for the seller not to allocate any item to any buyer when there are many buyers. Moreover, under the no-allocation equilibrium, the seller’s optimal expected profit is increasing in the number of buyers and the number of items. These results hold both in the case of independent externalities and in the case of dependent externalities. We then
proceed to show the robustness of the results in a general distribution case.

A critical piece of our model is the two-dimensional valuation structure that we adopt from Jehiel et al. (1996). In addition to the private valuation of the item, externalities imposed on rivals are also private information of the buyer who creates these externalities by getting the item. For example, only the ad owner knows the conversion rates and can have an estimate of the negative externalities imposed on competitors whose ads are not shown. We discuss a slight modification of the negative externalities in the Appendix B and show that our main result holds regardless if negative externality is imposed on all competitors or on auction losers only.

It would be interesting to consider the case in which the externalities imposed on a buyer who does not get the item is privately known to that buyer, and not to a rival who got the item. However, the problem of analytically finding an optimal mechanism in such setting remains intractable, except in special cases. (For example, certain information structures could be transformed into setting of this chapter.) This raises a question whether the emergence of the no-allocation equilibrium is a mere consequence of the valuation structure in our model or whether it holds for a variety of valuation structures. We conjecture that the intuitive reasoning (that we confirmed analytically within our model) ought to hold for any reasonable valuation structure: there should be no allocation when the sum of negative externalities imposed on a large number of buyers who do not get an item is larger than the total surplus of the small number of buyers who would get the items. In fact, special cases of our model such as commonly known value of negative externalities have direct special case analogues in other reasonable valuation structure models. Also, we can numerically confirm the results for a limited number of buyers with privately held values for the allocation in which negative externalities are imposed on them, but have to limit calculations to a small finite number of buyer types.

Our findings provide theoretical foundation for plausibility of existence of the no-
allocation equilibrium. Thus, such option should not be ignored whenever negative externalities are present and competition is intense. Hence, design of computationally manageable and implementable direct mechanisms that exploit privately held information about negative externalities could have implications in practice.

We leave for future research an investigation of optimal mechanisms for valuation structures in which each buyer has private information on its valuation for any allocation. Furthermore, investigating optimal mechanisms in such valuation model would likely allow for important insights on the relationship between exclusivity and negative externalities that go beyond basic observations in this chapter.
Mechanism and Network Design with Private Negative Externalities

4.1 Introduction

The potential for gains from winning in a competitive market often come hand in hand with the risk of losses if a rival wins instead. For example, a business seeking to carry a sought-after product or provide a specialized licensed service in a local area, could incur a loss if a rival secures the right to carry a product instead. Similarly, a company that is denied a regulatory approval (e.g., a drug developer seeking approval for a new drug, or a telecom company seeking a frequency spectrum license) is better off if all competitors also fail to obtain such approval, rather than if any of the competitors succeed and capture the market demand (e.g., for a drug or for bandwidth). Thus, comparisons with rivals’ successes or failures often matter as they are closely related to the market share, to capturing important customers or business opportunities, and to securing contracts, property rights or licenses.

This loss-exposure due to competitive considerations can be viewed as a negative externality: the value associated with a market outcome depends not only on one’s
allocation but also on the allocation to one’s rivals. If market participants are exposed to such negative externalities, i.e., if they value their relative competitive position and relationships, this information could potentially play a role in market design.

In this chapter, we show that the nature of competitive relationships dictates the format of the optimal mechanism, and thus, the underlying structure of competitive relationships among market participants is a relevant consideration in market design. Therefore, the monopolist should consider competitive relationships among market players when designing the market transaction rules.

Furthermore, once the optimal mechanism that maximizes the monopolist’s revenues is in place, impacting the structure of competitive relationships among market participants provides a new additional opportunity for improvement of the monopolist’s revenues. We show that the monopolist could be interested in investing in changes to these relationships, when possible. This is because different competitive structures not only have different optimal mechanisms, but might yield different revenues to the monopolist. We study this question of optimizing competitive relationships in the context of a revenue-maximizing monopolist who might have the ability to affect these competitive relationships among buyers. In particular, we characterize optimal structures from the monopolist’s perspective.

In many settings, it is possible for the monopolist to fully or partially impact the competitive relationships. For example, Facebook can choose their Preferred Marketing Developers (PMDs) in a given region. The competitive relationships among selected PMDs affect the best practices and choices for potential Facebook customers on that market. (Enterprise Resource Planing software providers have a similar ability in approving and licensing IT consulting companies for software implementation and integration.) When a manufacturer chooses a location for a production facility that depends on local suppliers (say, a food processing facility), it could take into account the competitive relationships among local suppliers for each of the locations...
considered and choose the location (and the optimal mechanism corresponding to
the competitive structure of local suppliers) that maximizes its expected revenues.
There are also many situations in which the regulatory environment impacts the
competitive relationships. For example, operating licenses for pharmacies are regu-
lated in many European countries in a way that limits the number of pharmacies in
a geographic and demographic area (e.g., distance between two pharmacies should
be at least 500 meters and there cannot be more than one pharmacy per 5,000 in-
habitants of any municipality.) Similarly, the number of taxi licenses is often capped
and, in addition, non-competition is mandated (e.g., while a Boston taxi can drive a
customer from Boston to Cambridge, MA, it is not allowed to pick up a customer in
Cambridge. The opposite restriction is in place for Cambridge taxis, thus eliminat-
ing any competition between Boston and Cambridge taxis.) These examples suggest
that understanding the impact of the competitive relationships among market par-
ticipants on market design is important for market-designer’s decision-making.

In this chapter, we develop a theoretical model that focuses on the privately held
negative externalities and their impact on the monopolist’s market design problem.
Thus, we attempt to capture the value of competitive relationships among buyers
with these externalities and abstract away from other potentially important aspects
of realistic market design considerations. In particular, we study a market design
problem for a risk-neutral revenue-maximizing monopolistic seller that has a single
item that could be allocated to any of the \( n \) buyers.\(^1\) The value of the item is publicly
known and is \( v \) to the buyer who obtains the item (e.g., price of the enterprise
resource software suite, frequency spectrum license, contract to supply a commodity
with a well-defined market price, etc.). However, each competitor \( i \) of the buyer who
obtained the item, suffers a loss of \( \alpha_i \) in that case. The loss-value \( \alpha_i \) is private to each

\(^1\) The case of the risk-neutral cost-minimizing monopsonist buyer of a single item facing \( n \) com-
peting sellers is analogous.
buyer $i$ and is the negative externality realized due to the allocation to a rival. If neither the buyer nor any of its rivals obtains the item (for example, if the item does not get allocated), then the value for such a buyer is zero. We formally introduce the model in Section 4.2.

In Section 4.3, we solve the mechanism design problem for a generic structure of competitive relationships, i.e., in the case where the set of rivals for every buyer is predetermined. The revenue-maximizing mechanism considers the aggregate negative externality that a possible allocation to buyer $i$ would create. The mechanism allocates the item if the benefit of the allocation is larger than the aggregate negative externality induced by it. There are two interesting features of the optimal mechanism. First, the optimal mechanism might not allocate the item.\(^2\) Second, even buyers who do not get the item and do not experience any negative externalities from the allocation, might have to pay. A recent school naming-rights example illustrates this point: thirteen donors gave a combined $85$ million, with a minimum single donor gift of $5$ million, to the Wisconsin School of Business at the University of Wisconsin-Madison to “preserve the Wisconsin name for at least 20 years. During that time, the school will not be named for a single donor or entity.” (The Wisconsin School of Business, 2007). Hence, each of these donors paid millions of dollars for not obtaining the item (and for ensuring that no other rival obtains it either). Not only did the Wisconsin School of Business raise $85$ million for not naming the school (compared, e.g., with the $55$ million naming gift for the Tepper School of Business at Carnegie Mellon University, 2004), but they can also try to raise funds again by (not) selling the name again in 2027.

The fact that the optimal mechanism, and thus the monopolistic seller’s revenues, depend on the structure of competitive relationships among buyers, indicates that

\(^2\) This is in contrast to the optimal mechanism with a publicly known value $v$ but without any negative externalities, i.e., without considering competitive relationships among buyers. The revenue-maximizing mechanism in this case simply allocates to any buyer and charges $v$. 72
the monopolist could have preferences over these structures, and, consequently, a revenue-driven interest in designing, changing, or influencing competitive relationships among buyers. Thus, in addition to finding and implementing an optimal mechanism, the market-designer has an additional important tool for revenue optimization: optimizing over structures of market participants’ competitive relationships. Ignoring the network design part of revenue-optimization is a choice that could result in leaving money on the table, even if the optimal mechanism is implemented.

In Section 4.4, we focus on this important aspect of the market designer’s problem and analyze expected revenues from optimal mechanisms for different structures of competitive relationships. In particular, we characterize competitive structures that are revenue-maximizing for the market designer. This requires jointly analyzing competitive structures and mechanisms, and cannot be decoupled: each structure has its own optimal mechanism and these mechanisms do change as the competitive structures change. Furthermore, it is not a priori clear that such optimal structures exist independently of problem parameters such as the value of the item $v$ and distributional assumptions on privately held negative externalities $\alpha_i$: one competitive structure might be optimal for one distributional assumption on negative externalities, while another might be optimal for a different distributional assumption. We show that, somewhat surprisingly, the optimality of competitive structures does not depend on the underlying distributional assumptions for externalities, and characterize these revenue-maximizing competitive structures and mechanisms.

A natural starting point for optimizing over competitive structures is the fully competitive setting in which every buyer experiences a negative externality whenever any other buyer obtains the item. In other words, every buyer considers all other buyers to be its rivals. In this setting, buyers are vulnerable and thus the revenue-

---
3 The numerical value of the expected revenues achieved by the optimal mechanism does depend on parameters.
maximizing mechanism should somehow exploit this by possibly capturing some of these negative externality valuations. The expected revenues in such fully competitive environments are higher than the expected revenues when there are no negative externalities. Thus, negative externalities can be exploited by the optimal mechanism in the fully competitive setting. We then investigate the extent to which negative externalities can be exploited in optimal mechanisms with limited competition. In particular, we analyze how the optimal mechanism performs when buyers are partitioned in disjoint competing blocks. Such fragmentation of competitive relationships is often due to geographical constraints. For example, as already noted, competition among taxis in the Boston area is fragmented into two competitive blocks on each side of the Charles River, while there is no competition among rivals from opposite sides of the river. Also, authorized retailers or licensed service providers (e.g., retailers of a luxury brand or authorized repair/service companies) compete for business with all rivals within the boundaries of a predefined region, such as a state or a country, and do not compete with rivals across these boundaries.\footnote{Fragmentation of competitive relationships could be due to other reasons, such as business type. For example, advertisers competing for an advertising slot perceive similar businesses as their competitors. An online shoe retailer is competing for potential customers with other shoe retailers, but does not necessarily lose potential business if it loses an ad to a financial institution. Thus, while shoe retailers and financial institutions compete for customers within their line of business, they are not in competitive relationships across lines of business.} In an attempt to capture the nature of competitive relationships in similar situations, we study settings in which buyers are fragmented into separate competing blocks, so that a buyer in a block considers all other buyers from that block only as its rivals (and thus it does not consider any buyer from a different block to be a rival). A fragmented competitive structure could have higher expected revenues than the fully competitive structure. Thus, the fully competitive setting is not revenue-maximizing among such market fragmentations. We show that the competitive structure fragmented into two-buyer competitive blocks is revenue-maximizing: each buyer considers only one other buyer
as its competitor (when there is an odd number of buyers, there is either exactly one block of three buyers or a single buyer that does not have rivals).

The reason why buyer competition fragmented into blocks of size two is revenue-optimal, is two-fold. First, it is important in terms of revenue considerations for each buyer to face a possibility of experiencing negative externality (and thus the willingness to pay to avoid such an outcome), but this can be achieved already with blocks of size two. Second, if a buyer who gets the item imposes negative externality on many other buyers (because many consider her a rival), payments will not be collected from those buyers since they will experience their worst possible outcome. Thus, having large blocks of buyers limits optimal mechanism’s revenue-potential from exploiting negative externalities from buyers in that block. Another way to explain why revenues in a market fragmented into buyer matchings dominates revenues in fully competitive markets, is through the value of privately held information on negative externalities. In the fully competitive market, each buyer’s information is relevant for \( n \) possible allocations \((n-1)\) rivals that could get an item, and no-allocation), while in the market fragmented in blocks of two buyers, this information is relevant for only two possible allocations (that to the unique rival or no-allocation). Thus, in a fully competitive market, privately held information is more valuable than in a fragmented market, so buyers can command higher information rents, which consequently lowers the expected revenue. Hence, while competition among buyers can be and should be exploited in order to maximize the monopolistic seller’s revenues, its effect on revenues is not monotone: too much competition (e.g., fully competitive structure) or lack of competition (no competitive relationships) are both revenue-deficient.

The model of localized competition through partitioning buyers into blocks generalizes to graphical representations. The buyers are modeled as nodes in the graph and an edge in the graph indicates that two buyers (nodes) are rivals. Thus, the graph neighborhood of a buyer (node) corresponds to the set of rivals. The fully
competitive setting corresponds to the complete graph, while the market fragmented in the blocks of size two corresponds to a matching. Graph representations allow for a much larger set of possible competitive structures, yet we establish that matchings remain optimal. (In the case of an odd number of users, the optimal graph is a matching with a path on three vertices, somewhat different than the case of partitioning in blocks.) In other words, the optimal mechanism on a matching has higher expected revenues than an optimal mechanism on any other graph. Note that the matchings are sparse among all graphs, i.e., a randomly selected or constructed graph representing competitive relationships is not likely to be revenue-maximizing.

The rivalry need not be mutual. A small local store might consider a multinational giant retailer a competitor and could be affected by the assortment and pricing of products the giant retailer carries. On the other hand, the multinational might not consider some small local store a rival.\(^5\) Such situations are better modeled with a directed network, rather than with an undirected graph. We show that breaking up the requirement for a competitive relationship to be mutual, fundamentally changes the structure of optimal networks. In order to describe the structure of optimal directed networks, we note two types of sources of revenue (i.e., buyer payments) for the monopolistic seller. First, the seller will collect \(v\) if they allocate the object. Second, the seller will collect some payment from all buyers who faced a possibility of experiencing negative externality, but the allocation did not impose it on them. We show that any directed network which ensures both sources of revenue is optimal. More precisely, the highest expected revenues are achieved with the optimal mechanism on a network in which (i) every buyer has at least one rival, and (ii) there exists a (benevolent) buyer that is not considered a rival by any other buyer. These

\(^5\) The distinction of mutual and one-directional relationships is important in social networks: Facebook friendships are bidirectional, while Twitter following is one-directional. This distinction does have potential implications for product placement and advertising decisions based on the underlying social network structure.
Table 4.1: Optimal Networks under Different Externality Structures and Objectives.

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<thead>
<tr>
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<th>Positive Externalities</th>
<th>Negative Externalities</th>
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<tbody>
<tr>
<td>Revenue Maximization</td>
<td>Complete Graph</td>
<td>This chapter</td>
</tr>
<tr>
<td>Social Surplus Maximization</td>
<td>Complete Graph</td>
<td>Benevolent Buyer</td>
</tr>
</tbody>
</table>

two properties can be simultaneously achieved by a directed network, but cannot be achieved by an undirected graph. In another contrast to the undirected case, optimal directed networks are dense among all networks in the sense that any network can be transformed into an optimal network by a small number of arc additions/deletions. Thus, even if changing competitive relationships among pairs of buyers is costly, one needs to make a small number of changes to reach an optimal network.

In Section 4.5, we discuss several important aspects of the presented results. We show that optimizing over network is an important tool at the market-designer’s disposal that cannot be ignored in the sense that the gap between revenues on the optimal network and revenues on a randomly chosen network can be arbitrarily large. We also provide conditions for the emergence of no-allocation in the optimal mechanism, just as in the case of Wisconsin School of Business naming rights fundraising.

We also discuss the robustness of our main findings to distributional assumptions on privately held information on negative externalities \( \alpha_i \) as well as choice of the objective. It turns out that the market design with the goal of maximizing social welfare (efficiency) is straightforward: any graph/network with a benevolent buyer (i.e., a buyer that does not impose negative externalities to any other buyer) is optimal and the mechanism allocates the item to this buyer. Thus, there are no benefits of exploiting competitive relationships if the welfare of all buyers is part of the objective. Similarly, the network optimization is straightforward in settings with
positive network externalities, that is, in settings where buyers experience positive externality when a neighbor obtains the item. In such settings, whether the objective is maximizing revenues or efficiency, the value of objective function is increasing with respect to edge/arc addition and thus, a fully connected graph is optimal. We summarize the optimal networks under different externality structures and objectives in Table 4.1. If one allows for simultaneous positive and negative externalities (or their virtual valuations), the structure of optimal networks becomes dependent on problem parameters such as $v$ and distributional assumptions about externalities. Hence, the case of revenue maximization on a network with negative externalities seems to be the only one in which there exists a robust characterization of non-trivial optimal structures.

Some concluding remarks are provided in Section 4.6. The proofs are relegated to the Appendix.

4.1.1 Related Literature

In this chapter, we consider the optimal network design when buyers have localized allocation dependent valuations. One important element of our framework is that buyers not only have valuations for getting the item, but they also have (negative) externality valuations (when losing the item to direct competitors). Therefore, our modeling of negative externalities is related to the interdependent valuations models in economics literature (e.g., see survey Maskin, 2003). In this literature, each buyer typically perceives all other buyers as direct competitors, which corresponds to a symmetric fully competitive setting that, in contrast to our setting, does not fully capture the underlying network structure. Models with externalities have been studied in, e.g., Jehiel et al. (1996), Jehiel et al. (1999), Jehiel and Moldovanu (2001), Aseff and Chade (2008), Figueroa and Skreta (2011), Deng and Pekeć (2011), and Brocas (2012). Furthermore, in some of the literature, such as Jehiel et al. (1996),
externalities are modeled as private information of the rivals. Such an information structure is critical in reducing the multi-dimensional mechanism design problem to a 1-dimensional problem, since a buyer has no incentive to misreport rivals’ externalities. Our work, however, considers a different private information structure in which negative externalities are buyers’ own private information. Such a setting enables us to study how the private information and network structure of negative externalities affects the market design. The work of Jehiel et al. (1999) identifies the difficulty of the multi-dimensional mechanism design problem in the setting where both valuations and externalities are buyers’ own private information. Moreover, by considering the symmetric case, they are able to establish the optimality of second-price auction formats. The symmetry assumption is critical: a consequence of the results we establish is that second-price auctions are not optimal for a generic network structure of externalities (even with publicly known item valuations). The reason is that buyers may have different competitor relationships, and the symmetry may not hold inherently in our (network) setting. In particular, under optimal undirected graphs, second-price auctions cannot implement the optimal mechanism, since losing buyers may pay differently (When the winning buyer is a direct competitor of a losing buyer, the losing buyer pays zero; however, when the winning buyer is not a direct competitor of a losing buyer, the losing buyer has to pay for not suffering from negative externalities.)

Another important element of our framework is network design. Network design problem has been considered mostly in the context of network formation games among buyers who build or maintain the network, in both economics literature (e.g., see Jackson, 2003; Epstein et al., 2009; Arcaute et al., 2013) and computer science.

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6 Under the information structure of Jehiel et al. (1996) with negative externalities, Deng and Pekeč (2011) provide a rationale for the no-allocation equilibrium. We show that no-allocation is a property of suboptimal network structures in our model, and that it could also emerge in the fully competitive setting and other sub-optimal network structures.
literature (e.g., see Anshelevich et al., 2003; Chen et al., 2008; Marden and Wierman, 2013). In these papers, the existence and quality of the equilibrium are studied from the perspective of independent buyers. There are also recent papers studying the impact of networks on economic systems. For example, Acemoglu et al. (2012), Acemoglu et al. (2013a), and Acemoglu et al. (2013b) analyze the role of networks as shock propagation and amplification mechanisms. Network design in our work has different motivation and goals: we design and evaluate network structures that maximize expected revenues for the associated optimal mechanism.

Furthermore, as externalities considered in this chapter are localized, the network structure is crucial to understanding these localized externalities. Thus, this chapter is also related to the economics literature of network externalities. In particular, buyers’ valuations are often assumed to depend on the (expected) size of their associated network (see, e.g., Katz and Shapiro, 1985; Parker and Alstyne, 2005) or the behavior of other buyers (see, e.g., Farrell and Saloner, 1985; Johari and Kumar, 2010), while valuations in our model depend on the allocation in the neighborhood. Several recent papers also study allocation and pricing procedures on networks with positive and negative externalities. In Candogan et al. (2012), a monopolistic seller’s pricing strategies for a divisible good are examined in a public information setting with a local positive network effect, i.e., a buyer’s utility increases with the usage level of its peers. In Haghpanah et al. (2013), positive externalities are modeled so that a buyer’s value is the product of a fixed private type and a known submodular function of the allocation of its peers. The work of Bhattacharya et al. (2011) examines algorithmic properties of some allocation and pricing procedures on a network in a setting where negative externalities are publicly known. This chapter focuses on the impact of privately held negative externalities on the structure of the optimal mechanism, and consequently, on the choice of the optimal network structure for the monopolistic seller (who is able to influence the structure of competitive relationships
among buyers).

The network structure in our framework is publicly known. While there is work that studies mechanisms with private information on certain proximity networks (see, e.g., Schummer and Vohra, 2002), allowing for private information on the perceived competitors would impose difficulties in finding the optimal mechanism as such private information on sets of competitors might not have sufficient structural properties for, e.g., establishing existence of belief-free optimal mechanisms. However, our results (with a publicly known network structure) provide a natural benchmark for any results in the setting with private information on the network structure.

The suboptimality of expected revenues in a fully competitive setting of our model (and optimality of fragmenting symmetric competitive relationships into pairwise matchings), could be viewed as an indication that some markets have the tendency to fragment. This has been argued in the context of some financial two-sided markets in a recent work of Peivandi and Vohra (2014), and has been observed in the context of labor markets (e.g., Roth and Xing, 1994).

4.2 Model

A monopolistic seller has an indivisible item that can be allocated among \( N = \{1, \ldots, n\} \) buyers. Buyer \( j \)’s valuation for obtaining the item is \( v \), but she faces a negative externality \(-\alpha_j\) if a competing buyer obtains the item instead. The competitive relationships among buyers are represented by a network \( \mathcal{N} = (N, E) \), with \( E \subset N^2 \). The nodes \( j \in N \) correspond to buyers, and an arc \((i,j) \in E\) represents that buyer \( i \) is perceived to be a competitor to buyer \( j \), i.e., that buyer \( j \) experiences negative externality \(-\alpha_j\) if buyer \( i \) obtains the item. The neighborhoods in such network capture information on competitive relationships: the set of buyer \( j \)’s competitors, i.e., those that could impose negative externality \(-\alpha_j\) to buyer \( j \) if they obtain the item, is denoted by \( \mathcal{N}^-(j) = \{i \in N : (i,j) \in E\} \), and the set of
buyers $i$ that perceive buyer $j$ to be their competitor, i.e., those that would suffer negative externality $-\alpha_i$ if buyer $j$ obtains the item, is denoted by $\mathcal{N}^+(j) = \{i \in N : (j, i) \in E\}$. We also use notation $d^+(j) = |\mathcal{N}^+(j)|$ and $d^-(j) = |\mathcal{N}^-(j)|$, and $\delta^+(\mathcal{N}) = \min_{j \in \mathcal{N}} d^+(j)$ and $\delta^-(\mathcal{N}) = \min_{j \in \mathcal{N}} d^-(j)$.

A special case is that of symmetric competitive relationships, $(i, j) \in E$ if and only if $(j, i) \in E$. Then, we use common terminology: $\mathcal{N}$ is a graph (i.e., an undirected network) and $\{i, j\}$ such that $(i, j), (j, i) \in E$ is an (undirected) edge. Note that, when $\mathcal{N}$ is a graph, $\mathcal{N}^+(j) = \mathcal{N}^-(j)$ and we denote it as $\mathcal{N}(j)$; consequently, $d^+(j) = d^-(j)$ and it is denoted by $d(j)$, and $\delta^+(\mathcal{N}) = \delta^-(\mathcal{N})$ is denoted by $\delta(\mathcal{N})$.

The information structure is as follows. The item valuation $v$ is publicly known and equal across buyers. The network $\mathcal{N}$ is publicly known (and potentially directed). The magnitude of the negative externality $\alpha_i$ is privately known by each buyer, and it is drawn independently from cumulative distribution function $F_i$. Its support is given by $\Omega = [\alpha, \alpha]$ and the corresponding density function is denoted by $f_i$.

By the Revelation Principle (Myerson, 1981), we consider direct mechanisms that allocate the item based on buyers’ reports. Reports from all the buyers are $\hat{\alpha} = (\hat{\alpha}_i, \hat{\alpha}_{-i}) \in \Omega$. A direct mechanism specifies the allocation probabilities ($p_i : \Omega^n \rightarrow [0, 1]$ is buyer $i$’s probability to get the item) and payments ($x_i : \Omega^n \rightarrow \mathbb{R}$ is the payment from buyer $i$ to the seller) for each $\hat{\alpha} \in \Omega^n$. If buyer $i$ does not participate, the trigger strategy is to allocate the item to one of buyer $i$’s competitors, i.e., $j \in \mathcal{N}^-(i)$, (see, e.g., Jehiel et al., 1996).

Buyer $i$’s ex post utility when she reports her type as $\hat{\alpha}_i$ while her true type is $\alpha_i$ and other buyers truthfully report is

$$U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i}) = v p_i(\hat{\alpha}_i, \alpha_{-i}) - \sum_{j \in \mathcal{N}^-(i)} \alpha_i p_j(\hat{\alpha}_i, \alpha_{-i}) - x_i(\hat{\alpha}_i, \alpha_{-i}), \quad (4.1)$$

7 Note that buyers need not to be ex ante identical, and a negative externality value $\alpha \in [\alpha, \alpha]$ can be held by any buyer. The buyer-specific support $\Omega_i = [\alpha_i, \alpha_i]$ is discussed in Section 4.5.3.
where we will write $U_i(\alpha_i, \alpha_j, \alpha_{-i})$ as $U_i(\alpha_{-i})$ for simplicity.

Throughout the chapter we will focus on ex-post constraints. Therefore, the seller’s revenue maximization problem is

$$\max_{p, x} \sum_{i=1}^{n} \int x_i (\alpha_i, \alpha_{-i}) dF(\alpha)$$

subject to

\begin{align*}
(\text{EPIC}) & \quad U_i(\alpha_i, \alpha_{-i}) \geq U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i}) \text{ for all } i \text{ and all } \alpha_i, \hat{\alpha}_i, \alpha_{-i}, \\
(\text{EPIR}) & \quad U_i(\alpha_i, \alpha_{-i}) \geq -\alpha_i 1_{N^-(i) \neq \emptyset} \text{ for all } i \text{ and all } \alpha_i, \alpha_{-i}, \\
(\text{Feasibility}) & \quad \sum_{i=1}^{n} p_i (\alpha_i, \alpha_{-i}) \leq 1 \text{ and } p_i (\alpha_i, \alpha_{-i}) \geq 0, \text{ for all } i.
\end{align*}

4.3 Optimal Mechanism for a Given Network

This section solves the optimal mechanism design problem under the negative externality structure discussed in the previous section when the network $\mathcal{N}$ is fixed. This characterization is of interest on its own, but it also plays a crucial role in the following sections in which we consider the designer of the mechanism who can further optimize over the network itself.

To describe the optimal mechanism, define the virtual negative externality

$$\pi_i(\alpha_i) := \alpha_i - \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)}$$

and for a set of nodes $A \subseteq N$ we let $\pi_A(\alpha) = \sum_{j \in A} \pi_j(\alpha_j)$ and $p_A(\alpha) = \sum_{j \in A} p_j(\alpha)$. By the Envelope Theorem, the seller’s expected revenue can be expressed through
the virtual negative externalities as

\[
\sum_{i=1}^{n} \int x_i(\alpha_i, \alpha_{-i}) \, dF(\alpha) = \sum_{i=1}^{n} \alpha 1_{\{\mathcal{N}^{-}(i) \neq \emptyset\}} + \int \sum_{i=1}^{n} \left\{ p_i(\alpha) \left( v - \pi_{\mathcal{N}^{+}(i)}(\alpha) \right) \right\} \, dF(\alpha).
\]

(4.2)

The first term in (4.2) represents the revenue from individual negative externalities, while the second term is the revenue from payments for selling the good, discounted by the information rent for the externalities a buyer imposes on the system. The proof of the equality in (4.2) is in the Appendix.

This allows us to restate the seller’s revenue maximization problem as a function of the allocation variable

\[
\Pi(\mathcal{N}) := \max_p \left\{ \Pi(p; \mathcal{N}) := \sum_{i=1}^{n} \alpha 1_{\{\mathcal{N}^{-}(i) \neq \emptyset\}} + \int \sum_{i=1}^{n} \left\{ p_i(\alpha) \left( v - \pi_{\mathcal{N}^{+}(i)}(\alpha) \right) \right\} \, dF(\alpha) \right\}
\]

subject to

(Feasibility) \quad \sum_{i=1}^{n} p_i(\alpha_i, \alpha_{-i}) \leq 1 \text{ and } p_i(\alpha_i, \alpha_{-i}) \geq 0, \text{ for all } i,

(Monotonicity) \quad \sum_{j \in \mathcal{N}^{-}(i)} p_j(\alpha_i, \alpha_{-i}) \text{ is decreasing in } \alpha_i.

Note that the last constraint is part of the (EPIC). To solve the problem above, we apply the standard argument of ignoring monotonicity, maximizing point-wise for each \( \alpha \), and then verifying that the solution satisfies monotonicity under regularity conditions. The following result summarizes the optimal allocation and payment rules under standard monotonicity assumption on the virtual negative externalities (i.e., \( \pi_i \) is non-decreasing in \( \alpha_i \)).
Theorem 22. Suppose $\pi_i(\cdot)$ is non-decreasing for each $i \in N$ and the network $\mathcal{N}$ is given. Then it is optimal to allocate to buyer $i$ at evaluations $\alpha$ if and only if

$$
\pi_{\mathcal{N}^+(i)}(\alpha) = \min_{j \in \mathcal{N}} \pi_{\mathcal{N}^+(j)}(\alpha) \text{ and } \pi_{\mathcal{N}^+(i)}(\alpha) \leq v,
$$

in which case $p^*_i(\alpha) = 1$, and $p^*_j(\alpha) = 0$ for $j \in N \setminus \{i\}$, is optimal. Moreover, the optimal revenue equals to

$$
\Pi(\mathcal{N}) = \sum_{i=1}^{n} \alpha 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} + \int_{\alpha} \max_{i \in \mathcal{N}} (v - \pi_{\mathcal{N}^+(i)}(\alpha))^+ dF(\alpha), \quad (4.3)
$$

and for each $i \in N$ the associated optimal payment rule is given by

$$
x^*_i(\alpha) = vp^*_i(\alpha) + \alpha_i \left\{ 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} - p^*_i(\mathcal{N}^-(i)) \right\}
- \int_{\alpha} \left\{ 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} - p^*_i(t, \alpha_{-i}) \right\} dt. \quad (4.4)
$$

Theorem 22 characterizes the optimal mechanism for any given network $\mathcal{N}$. The allocation is determined by the impact of a buyer on others, $\mathcal{N}^+$, while the payment of a buyer is determined by how the buyer is affected, $\mathcal{N}^-$. Therefore, the network structure $\mathcal{N}$ plays a critical role in the optimal mechanism.

Theorem 22 also establishes that the payments of the optimal mechanism are between $vp^*_i(\alpha)$ and $vp^*_i(\alpha) + \alpha_i$. The first two terms in $x^*_i(\alpha)$ can be seen as the “cost” for the agent of the allocation and the last term in $x^*_i(\alpha)$ is negative and is the information rent which can be interpreted as a discount for the buyer.

The optimal payment (4.4) indicates that, unlike the symmetric setting of Jeljel et al. (1999), second-price auctions may not be able to implement the optimal mechanism for a generic network $\mathcal{N}$, since losing buyers’ payments may differ. In particular, if the winning buyer is a direct competitor of a losing buyer, the losing

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8 We use the notation $(t)_+ = \max\{0, t\}$. 85
buyer pays zero; but if the winning buyer is not a direct competitor of a losing buyer, such losing buyer has to pay for not being exposed to negative externalities.

4.4 Optimal Network Design

The previous section characterizes the revenue maximizing mechanism for a given network. In order to further improve revenues, the seller needs to resort to changing the network itself. In this section we study the problem of the revenue-maximizing seller who does not only design the optimal mechanism but also optimizes over the set of feasible networks. We will consider three different cases: (i) a feasible network must have a group structure; (ii) a feasible network is symmetric, i.e., an undirected graph; and (iii) any network is feasible.

The characterization of the optimal mechanism under a fixed network $N$ established in Section 4.3 points to two potential difficulties with this joint network and mechanism optimization problem. First, the optimal mechanism could change as the underlying network $N$ changes. Second, there is no monotonicity of expected revenues with respect to network arc additions or deletions.

Formally, in this section, we are addressing the following problem:

$$\max_{N \subset \mathbb{N}} \Pi(N) = \max_{N \subset \mathbb{N}} \max_{p \in P} \Pi(p; N)$$

(4.5)

where $\mathbb{N}$ denotes the set of feasible network structures (e.g., group structure, undirected graph, or a generic network), and where $P$ denotes the set of feasible allocation rules. More precisely, $P$ is the set of allocation rules satisfying (EPIC), (EPIR), and (Feasibility) constraints, and, thus, $\Pi(N)$ is given by equation (4.3) in Theorem 22. We call the optimal solution to problem (4.5) an optimal network (e.g., optimal group structure, optimal undirected graph, or optimal directed network).\footnote{Note that optimal network $N^*$ defines the optimal mechanism (by Theorem 22) and, consequently, the allocation rule $p^*$ that jointly with $N$ maximizes the righthand side of (4.5).}

The fo-
cus of our analysis is on the impact of negative externalities on the solution to (4.5). In order to isolate the impact of negative externalities, throughout this section, the virtual negative externalities are assumed to be non-negative, namely we assume that \( \pi_i(\alpha_i) \geq 0 \) for all \( \alpha_i \in \Omega \). This condition implies that negative externalities are non-zero, i.e. \( \alpha > 0 \). Several commonly used distributions imply this condition. For example, \( \alpha_i \sim U[\alpha, \bar{\alpha}] \) for \( 0 < \alpha \leq \bar{\alpha} \leq 2\alpha \); \( \alpha_i - \bar{\alpha} \sim \exp(\lambda) \) where \( \lambda \alpha \geq 1 \). (In Section 4.5.5, we briefly discuss the consequences of relaxing this condition.)

We first consider problem (4.5) with restrictive choice of \( \mathcal{N} \): we only consider network structures corresponding to partitioning players into groups, and describe the structure of optimal group structures. We show in Section 4.4.1 that the optimal group structures are largely independent of the distributional assumption on negative externalities. However, the distributional assumptions do matter for a complete description of optimal group structures when the number of players is odd. Interestingly, once the set of feasible structures \( \mathcal{N} \) is extended to include all undirected graphs, in Section 4.4.2 we characterize optimal graphs and establish that their optimality is independent of distributional assumptions. The optimal undirected graphs are unique (up to a graph homomorphism). In contrast, in Section 4.4.3 we demonstrate that there is a large number of optimal directed networks, all independent of distributional assumptions.

### 4.4.1 Optimal Group Design

A group structure is defined by a partition of buyers, i.e.,

\[
\mathcal{G} = \left\{ \{N_h\}_{h=1}^H : \bigcup_{h=1}^HN_h = N \right\},
\]

where \( H \) is the number of sets in the partition. Buyers in the set \( N_h \) are said to be in the same group \( h \), and have mutual competition relationships, i.e., buyer \( i \in N_h \) when getting the item imposes negative externalities on all other buyers.
\( j \in N_h \setminus \{i\} \). However, a buyer does not perceive any other buyers in a different group as competitors and vice versa. Therefore, under a group structure \( \mathcal{G} \), we have \( \mathcal{G}(i) = N_h \setminus \{i\} \) for every buyer \( i \in N_h \).

Group structures that partition buyers in as equal-sized sets \( N_h \) as possible are important in our analysis. A partition \( \{N_h : h \in L\} \) of \( A \) is said to be \( k \)-equipartition if \( |N_h| = k \) for all \( h \in L = \{1, \ldots, l\} \); to be an almost \( k \)-equipartition if \( ||N_h| - k| \leq 1 \) with equality holding for at most one \( h \).

In order to optimize over group structures we need to account that the optimal mechanism will also change as we search over different groups as characterized in Theorem 22. With ex ante identical buyers (who have the same externality distribution and item value), there are optimal allocation rules that are symmetric among buyers in the same group. This means that any permutation on valuations of buyers in \( \mathcal{G}(i) \) will not change the optimal allocation rule. Therefore we can focus on the set of symmetric allocation rules to search the optimal allocation rule.

**Theorem 23.** Consider the optimization problem (4.5) where \( \mathbb{R} \) is the set of group structures and externality valuations are identically distributed. Almost \( 2 \)-equipartitions are optimal solutions to problem (4.5) for any non-negative and non-decreasing \( \pi_i \). Furthermore, if \( v > \alpha - 1/f(\alpha) > 0 \), every optimal solution to problem (4.5) is an almost \( 2 \)-equipartition.

When the number of buyers \( n \) is even, Theorem 23 implies that the optimal group design is \( 2 \)-equipartition, i.e., the revenues are maximized when buyers are fragmented and partitioned into competitive pairs (\( |\mathcal{G}(i)| = 1 \) for all \( i \in N \)). Note that this optimal group structure is independent of distributional assumptions on \( \alpha_i \) (as long as \( \pi_i(\alpha_i) > 0 \)).

When \( n \) is an odd number, the optimal group structure is still fragmented and consists of at least \( (n - 3)/2 \) pairs of competing buyers. The remaining three buyers
are either grouped in a clique of three competing buyers, or are fragmented into a pair of competing buyers with the last buyer not having any competitors. The competitive relationship among these three buyers in the optimal group design depends on the item value and the externality distribution. If the externality is sufficiently large, it is optimal to have these three buyers in a clique. However, if the item value is sufficiently large, it is optimal to have a single buyer without any competitors. (A numerical example is provided in Example 1 of the Appendix.)

The following corollary of Theorem 23 demonstrates that, under mild assumptions on the value $v$, the optimal group structure induces allocation.

**Corollary 24.** Under the conditions of Theorem 23 and provided $v > \alpha$, the optimal mechanism induced by an optimal group structure with $n > 3$ always allocates the item.

Note that optimal mechanisms for arbitrary group structures do not necessarily allocate the item. In fact, there are group structures for which no-allocation is optimal for a range of $v$ that intersects with $v > \alpha$, see Section 4.5.1.

### 4.4.2 Optimal Undirected Graphs

In this section we consider undirected graphs. Using $\mathcal{N}(i) = \mathcal{N}^-(i) = \mathcal{N}^+(i)$ to denote the neighbors of buyer $i$, the optimal mechanisms under a network design $\mathcal{N}$ are described in Theorem 22. Furthermore, the intuition behind Theorem 23 extends to the case of undirected graphs, with calling a graph $k$-regular if $d_i = k$ for all $i$, and almost $k$-regular if $|d_i - k| \leq 1$ with equality holding for at most one $i \in N$.

**Theorem 25.** Consider the optimization problem (4.5) where $\mathcal{R}$ is the set of undirected graphs. Almost $1$-regular graphs with $\delta = 1$ are optimal solutions to problem (4.5) for any non-negative and non-decreasing $\pi_i$. Furthermore, if $v > \alpha - 1/f_i(\alpha) >
0 for all \(i \in N\), every optimal solution to problem (4.5) is an almost 1-regular graph with \(\delta = 1\).

Theorem 25 demonstrates that optimal undirected graphs are unique up to a graph homomorphism (since buyers are ex ante symmetric): for even \(n\), the optimal graph is a matching, and for odd \(n\) the optimal graph is a matching on \(n - 3\) vertices and a path on the remaining three vertices.\(^{10}\) The almost 1-regularity of optimal undirected graphs corresponds to the almost 2-equipartition structure of optimal group structures. However, unlike optimal group structures and their natural graph representations, optimal graphs do not have a clique of size three or three independent vertices of degree zero. Moreover, unlike optimal group structures, the optimality of the described graphs is independent of the distributional assumption (provided virtual valuations are positive, as stated in the theorem).

It also follows from Theorem 25 that the optimal network among all undirected graphs always allocates under mild conditions on the value \(v\).

**Corollary 26.** Under the conditions of Theorem 25 and provided \(v > \bar{\alpha}\), the optimal mechanism induced by an optimal undirected graph always allocates the item.

### 4.4.3 Optimal Directed Networks

Next we fully extend the set of feasible structures and optimize (4.5) over all (directed) networks. As in preceding subsections, we build upon the characterization of Theorem 22. The flexibility of asymmetric relations allows for emergence of new optimal structures (and additional revenue gains), as characterized by the following theorem.

**Theorem 27.** Consider the optimization problem (4.5) where \(\mathcal{N}\) is the set of directed graphs. Let \(\pi_i\) be non-decreasing and non-negative for \(i \in N\). A network \(N\) is

\(^{10}\) This is the unique almost 1-regular graph with \(\delta = 1\) when \(n\) is odd, up to a graph homomorphism.
optimal if and only if (i) $\delta^- \geq 1$, and (ii) $\delta^+ = 0$. Moreover, the optimal revenue associated with those networks is $v + \alpha n$. Furthermore, any directed network can be transformed into an optimal network by $O(n)$ arc additions/deletions.

Results in Theorem 27 describe two characterizing properties of the optimal networks: (i) each buyer has loss-exposure, i.e., could experience negative externalities, and (ii) there exists a buyer who does not impose negative externalities on any other buyers. These two conditions describe the best-case scenario from revenue-maximization perspective: (i) there is a potential to exploit loss-exposure of each buyer, and (ii) the revenues from allocating the item do not need to be offset by negative externalities imposed by such allocation. Note that these two conditions cannot be simultaneously achieved by an undirected graph (even with ex-ante symmetric buyers), hence any optimal network must have asymmetric competitive relationships among buyers. These conditions do not depend on distributional assumptions, and consequently, optimality is independent of these assumptions (analogous to the undirected graph case). Furthermore, the implementation of the optimal mechanism on any optimal network is straightforward: the item is allocated to one of the “benevolent” buyers (i.e., to any buyer $i$ with $d^+(i) = 0$) who is charged $v + \alpha$, while all other buyers are charged $\alpha$.

**Corollary 28.** Under the conditions of Theorem 25 and provided $v > 0$, the optimal mechanism induced by an optimal directed network always allocates the item.

We illustrate optimal networks by presenting two extremal structures: the $S_{ij}$-network which has the minimum number of arcs among all optimal networks (there needs to be at least $n$ arcs due to condition (i)), and $K_{\{i\}}$-network which has the maximum number of arcs among all optimal networks (there has to be at least $n - 1$ arcs missing due to condition (ii)). The $S_{ij}$-network is a variant of a star network defined by $n$ arcs: $n - 1$ arcs $(i, k)$, $k \in N \setminus \{i\}$ and the arc $(j, i)$. (Figure 4.1a shows
Figure 4.1: Extremal Optimal Networks with Four Buyers.

$S_{12}$-network for four buyers.) $K_{\{i\}}$-network has all possible arcs except for $n - 1$ arcs $(k, i), k \in N \setminus \{i\}$. (Figure 4.1b shows $K_{\{3\}}$-network for four buyers.) There are many more optimal directed networks: Theorem 27 also establishes that, in sharp contrast to undirected graphs, optimal (directed) network structures are not unique and are “near” any network. More precisely, an optimal network can be obtained from any given network with at most $2(n - 1)$ arc additions/deletions. Such number of additions/deletions is very small compared with the average distance between two (uniformly drawn) random networks which is $n(n - 1)/2 = O(n^2)$. Thus, if influencing competitive relationships among pairs of buyers is not prohibitively costly, the mechanism designer can improve expected revenues by investing in influencing a small number of competitive relationships among pairs of buyers.

4.5 Additional Results and Extensions

In this section, we examine the robustness of our findings in several ways.

We first investigate the property that the item is always allocated by the optimal mechanism on the optimal structures (Corollaries 24, 26 and 28). In Section 4.5.1, we show that such property does not hold for many suboptimal network structures: no-allocation equilibrium can emerge, i.e., it may be optimal for the seller not to
allocate the item. Even when the item is always allocated by the optimal mechanism for a suboptimal network structure, in Section 4.5.2, we indicate that the revenue loss could be substantial. More precisely we compare revenues between random and optimal undirected graphs when allocation always occurs.

In Section 4.5.3, we discuss the impact of the assumption that buyers are ex ante symmetric and point to some results that generalize to settings with heterogeneous buyers.

We also discuss the solution of the joint mechanism and network design problem (4.5) when buyers’ valuations are publicly known in Section 4.5.4.

Finally, in Section 4.5.5, we relax the requirement for virtual valuations of negative externalities to be bounded from zero. When negative externalities are non-substantial in the sense that the virtual values can be both positive and negative, the solution to the joint mechanism and network design problem (4.5) depends on the distributional details of negative externalities and is sensitive to other problem parameters, such as the item value $v$. Hence, with such a relaxation, there are no belief-free optimal solutions to problem (4.5).

### 4.5.1 No-Allocation Equilibrium

Corollaries 24, 26 and 28 provide sufficient conditions under which the optimal network structure induces an optimal mechanism which always allocates the item. For non-optimal network structures, however, the corresponding optimal mechanism might not allocate. The purpose of this subsection is to investigate when no-allocation equilibrium arises from the optimal mechanism for a given network. We next formally state conditions on the value $v$ for no-allocation equilibrium to exist.

**Proposition 29.** Given a network $\mathcal{N}$, let $\delta^+ := \min_{i \in N} |\mathcal{N}^+(i)|$. Suppose $\pi_i$ is non-decreasing for $i \in N$. Provided that $\delta^+ \min_{i \in N} (\alpha - 1/f_i(\alpha)) > v$, the optimal
mechanism induced by $\mathcal{N}$ has $p^*_i(\alpha) = 0$ for all $i \in N$.

The proof of Proposition 29 is a direct consequence of Theorem 22. If the value $v$ does not grow with the number of buyers and virtual negative externality is bounded away from zero, $\pi_i(\alpha) = \alpha - 1/f_i(\alpha) > 0$ for all $i$, networks in which every node is highly connected are prone to no-allocation. Therefore, even small negative externalities may still be worth exploiting with the no-allocation equilibrium. Under relaxed assumptions, the following related proposition provides a lower bound for the probability of no allocation.

**Proposition 30.** Given a network $\mathcal{N}$, let $\delta^+ := \min_{i \in N} |\mathcal{N}^+(i)|$. Suppose $\pi_i$ is non-decreasing for $i \in N$. Then, the probability of no-allocation arising as the outcome of the optimal mechanism induced by $\mathcal{N}$ is at least

$$1 - n \exp\left(-\frac{\alpha \delta^+ - v}{2\alpha^2}\right).$$

The intuition is similar to one behind Proposition 29. If the value of allocating the item is not too large, it is optimal not to allocate if all agents are connected and subject to negative externality valuations. Finally, a necessary condition for the no-allocation equilibrium is that isolated nodes cannot exist.

**Proposition 31.** Given a network $\mathcal{N}$, let $\delta^+ := \min_{i \in N} |\mathcal{N}^+(i)|$. Suppose $\pi_i$ is non-decreasing for $i \in N$ and $v > 0$. If no allocation is optimal for some realization of the types, there are no isolated nodes, i.e. $\delta^+ = \min_{i \in N} |\mathcal{N}^+(i)| \geq 1$.

Results form this subsection suggest that no allocation emerges in large networks where every buyer has a large number of perceived competitors. Thus, large highly competitive network structures cannot be optimal due to results of Corollaries 24, 26 and 28.

4.5.2 Revenue Comparison between Random and Optimal Undirected Graphs

Not allocating the item forfeits an opportunity to get $v$ from allocating the item and, as shown in Corollaries 24, 26 and 28, the optimal mechanism allocates the
item on the optimal structure. However, there are many other structures whose corresponding optimal mechanisms always allocate the item. In this subsection, we briefly investigate the difference between revenues from the optimal structure and a random structure which always allocates the item (and hence capitalizes on the item value $v$).

Specifically, we focus on undirected graphs. As shown in Section 4.4.2, the almost 1-regular graph is optimal and yields $\alpha(n - 1) + v$ in revenue. (Note that by ignoring negative externalities, a suboptimal mechanism can easily achieve $v$ by allocating the item to any buyer.)

In order to simplify analysis, in this subsection, we further assume $v \geq n\bar{\alpha}$ in addition to $\pi_i(\alpha_i) \geq 0$. To compare revenues between optimal undirected graphs (almost 1-regular graph) and random graphs, we consider a random graph $R$ with $n$ nodes where each edge is added independently with probability $q$. Therefore, with high probability $1 - \gamma$, we have

$$| R^+(i) | - (n - 1)q \leq \sqrt{nq(1 - q) \log(2/\gamma)}.$$  \hspace{1cm} (4.6)

In turn this allows us to bound the revenue of the optimal mechanism given the

$^{11}$ The same conclusion would hold provided $\max_{i \leq n} E[\alpha_i^2] < \infty$ and $v \geq \bar{v}_\xi = \min \{ \tilde{v} : E[\pi_N(\alpha) \wedge \tilde{v}] \geq \xi E[\pi_N(\alpha)] \}$ for $\xi \in (0, 1]$. Note that if $\bar{v}_\xi \leq n\bar{\alpha}$ but $\bar{v}_\xi$ allows for unbounded support of $\alpha_i$. 
network $\mathcal{R}$ with high probability as

$$
\Pi(p^*; \mathcal{R}) = \sum_{i=1}^{n} \alpha 1_{(\mathcal{R}^-(i)=\emptyset)} + \int \sum_{i=1}^{n} \{p^*_i(\alpha) (v - \pi_{\mathcal{R}^+(i)})\} \ dF(\alpha)
$$

$$
= (1) \ \alpha n + v - E[\sum_{i=1}^{n} p^*_i(\alpha) \pi_{\mathcal{R}^+(i)}]
$$

$$
\leq (2) \ \alpha n + v - E[\min_{i \in N} \pi_{\mathcal{R}^+(i)}]
$$

$$
\leq \alpha n + v - \alpha \min_{i \in N} |\mathcal{R}^+(i)| - E \left[ \min_{i \in N} \pi_{\mathcal{R}^+(i)} - \alpha |\mathcal{R}^+(i)| \right]
$$

$$
\leq \alpha n + v - \alpha \min_{i \in N} |\mathcal{R}^+(i)| + E \left[ \max_{i \in N} |\pi_{\mathcal{R}^+(i)} - \alpha |\mathcal{R}^+(i)| | \right]
$$

where (1) holds by $v \geq n\bar{\alpha} \geq n \max_{i \in N} \pi_i(\alpha_i) \geq \pi_{\mathcal{R}^+(i)}(\alpha)$, (2) holds by allocations adding up to one pointwise. By (4.6), $\min_{i \in N} |\mathcal{R}^+(i)| \geq nq - \sqrt{nq(1-q) \log(2/\gamma)}$ with probability $1 - \gamma$. Next, because $\pi_j(\alpha_j) - \alpha$ is independent across $j$ and has zero mean, we have

$$
E \left[ \max_{i \in N} |\sum_{j \in N(i)} \{\pi_j(\alpha_j) - \alpha\}| \right]
$$

$$
\leq \{E[\max_{i \in N} |\sum_{j \in N(i)} \{\pi_j(\alpha_j) - \alpha\}|^2]\}^{1/2}
$$

$$
\leq (1) \ \sqrt{2} \{1 + 4 \log(2n)\}^{1/2} \{E[\max_{i \in N} \sum_{j \in N(i)} \pi_j^2(\alpha_j)]\}^{1/2}
$$

$$
\leq (2) \ \sqrt{2} \{1 + 4 \log(2n)\}^{1/2} \{E[\sum_{i \in N} \pi_i^2(\alpha_i)]\}^{1/2}
$$

$$
\leq (3) \ \sqrt{2} \{1 + 4 \log(2n)\}^{1/2} \{n \max_{i \in N} E[\alpha_i^2]\}^{1/2},
$$

where (1) follows from Nemirovski’s inequality (e.g., see Exercise 11.8 in Boucheron et al., 2013), (2) from $\mathcal{N}(i) \subset N$, and (3) from $0 \leq \pi_i(\alpha) \leq \alpha$ by assumption and its
definition. Then, collecting terms we have

$$\Pi(p^*; R) \leq n\alpha(1 - q) + v + \alpha\sqrt{n}\{q(1 - q)\log(2/\gamma)\}^{1/2} + \sqrt{n}\{2 + 8\log(2n)\}^{1/2}\alpha$$

where the last two terms are $o(n)$. Since the optimal matching yields \(\alpha(n - 1) + v\), we have effective gains of the order of \(n\alpha q\).

Therefore, as \(n\) grows large, the revenue gains from choosing an optimal structure rather than a random one that guarantees allocation, are unbounded. This motivates a revenue-maximizing seller to consider the aspect of network optimization.

### 4.5.3 Heterogenous Buyers

Throughout the analysis, we allowed for heterogenous distributions \(F_i\) describing privately held negative externalities \(\alpha_i\). However, we assumed that support \(\Omega = [\alpha, \alpha]\) is identical for all \(F_i\). Our methods and most results extend naturally when allowing for heterogeneous support \(\Omega_i = [\alpha_i, \bar{\alpha_i}]\).

For instance, the characterization of the optimal mechanism for a given network shown in Theorem 22 readily extends to the heterogeneous support setting (with \(\alpha\) replaced by \(\alpha_i\) in (4.4) and (4.3).

Similarly, Theorem 27 directly generalizes to the setting with heterogeneous support \(\Omega_i = [\alpha_i, \bar{\alpha_i}]\) (now with the expected revenue from an optimal directed network expressed as \(v + \sum_i \alpha_i\)).

Allowing for heterogenous support does not fully generalize Theorem 25.

**Theorem 32.** Assume \(\Omega_i = [\alpha_i, \bar{\alpha_i}]\) and consider the optimization problem (4.5) where \(\mathcal{R}\) is the set of undirected graphs. An optimal undirected graph is 1-regular. Furthermore, if there exists \(\alpha\) such that \(\alpha_i = \alpha\) for all \(i \in N\) and if \(v > \alpha - 1/f_i(\alpha) > 0\) for all \(i \in N\), then the undirected graph is optimal if and only if it is almost 1-regular with \(\delta = 1\).
Thus, results and the proof arguments of Theorem 25 directly extend only to the setting with buyers that have heterogeneous support for externality valuations, provided that the lowest possible externality is the same for all buyers. (With private information, $\alpha_i$ are critical parameters for determining the seller’s expected revenues.) When $\alpha_i$ are different, the seller could exploit these differences and focus on some particular buyers. The optimal way to exploit such differences is distribution-dependent, and in the Appendix we provide an example (see Example 2) with three buyers where the optimal graph could either have one or two edges, depending on the problem parameters. Also, with heterogeneous $\alpha_i$, while almost 1-regular graphs are optimal, there could be other optimal graphs.\textsuperscript{12}

4.5.4 Efficiency Objective

Competitive relationships among buyers have potential to negatively impact their values, and the optimal network and mechanism design is exploiting these negative externalities. However, if the designer’s objective is efficiency, i.e., maximizing social welfare (surplus), then the negative externalities could only have a negative impact on the objective. Thus, efficient mechanisms will avoid negatively impacting buyers’ values, whenever possible.

More formally, the efficiency objective is to maximize the sum of buyers’ values\textsuperscript{13}

$$T(N) = \max_{p,x} \sum_{i=1}^{n} \left[ v p_i - \sum_{j \in N^{-}(i)} \alpha_i p_j \right]$$

\textsuperscript{12} The optimal mechanism will avoid imposing negative externalities to any node with a sufficiently large value of $\alpha_i$ (and “sufficiently large” is distribution-dependent, as it depends on the relative comparisons of $v$ and virtual valuations). Thus, an optimality is not affected by any edge additions/deletions among such nodes with “sufficiently large” $\alpha_i$ (provided that these nodes are not isolated).

\textsuperscript{13} The payments $x_i$ cancel out in the objective as they are transfers from buyers’ to the seller.
subject to

\[(EPIC) \quad U_i(\alpha_i, \alpha_{-i}) \geq U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i}) \text{ for all } i \text{ and all } \alpha_i, \hat{\alpha}_i, \alpha_{-i},\]

\[(EPIR) \quad U_i(\alpha_i, \alpha_{-i}) \geq -\alpha_i 1_{\{N^-(i) \neq \emptyset\}} \text{ for all } i \text{ and all } \alpha_i, \alpha_{-i},\]

\[(Feasibility) \quad \sum_{i=1}^{n} p_i(\alpha_i, \alpha_{-i}) \leq 1 \text{ and } p_i(\alpha_i, \alpha_{-i}) \geq 0, \text{ for all } i.\]

Thus, given the feasibility constraint and negative externalities, \(v\) is an upper bound on the value of the objective function (4.7) for any network structure \(N\). However, for a network structure with \(\delta^+ = 0\) (i.e., with a buyer that is not perceived a competitor by anyone), a mechanism that allocates to a buyer \(i\) with \(d^+(i) = 0\) achieves the total surplus of \(v\).

**Proposition 33.** A network structure \(N\) has \(T(N) = v\) if and only if \(\delta^+(N) = 0\).

Thus, the total welfare is maximized for network structures that have at least one “benevolent” buyer who will get the item. This observation aligns with simple intuition that eliminating competitive relationships should limit the impact of negative externalities and thus weakly improve total welfare. However, as demonstrated in the chapter, when the objective is revenue-maximization, managing the structure of competitive relationships has a delicate impact on revenues: one one hand, eliminating competitive relationships eliminates possibilities for revenue generation, while on the other hand, introducing too many competitive relationships among buyers also negatively impacts revenues.

### 4.5.5 Generalized Externality Values

The main focus of this chapter are negative externalities and their impact on mechanism and network design. In this subsection, however, we discuss the implications of possibly relaxing the condition \(\pi_i(\alpha) \geq 0\) for all \(\alpha \in \Omega_i\).
Table 4.2: Optimal Networks with $\min \sum_{i \in N} (|N^+(i)| + |N^-(i)|)$ (Different Distributions of $\alpha_i$ and Different values of $v$).

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_i \sim U [-1, 0]$</th>
<th>$\alpha_i \sim U [0, 1]$</th>
<th>$\alpha_i \sim U [1, 2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = 0$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>$v = 10$</td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
</tbody>
</table>

A natural setting contrasting negative externalities is that of positive externalities. Specifically, requiring that in our setup virtual valuations always be negative, $\pi_i(\alpha) \leq 0$ for all $\alpha \in \Omega_i$. (Since the impact on buyer’s utility is $-\alpha_i$, buyers are experiencing positive externalities.)

With positive externalities, it is intuitive and straightforward to establish that adding pairwise relationships among buyers can only increase revenues (as all neighbors of the buyer who obtains the item benefit from such allocation and there are no negative consequences of any allocation), and consequently, the revenue-maximizing network structure is a complete graph.

Thus, in both cases with the uniform impact of externalities, corresponding to all positive externalities or all sufficiently large negative externalities (i.e., independently drawn externalities with $\max_i \pi_i(\overline{\alpha}_i) \leq 0$ or $\min_i \pi_i(\alpha_i) > 0$, respectively), the optimal network structure does not depend on any further distributional details. In the case of positive externalities, the revenues are maximized on a complete graph, while determining the existence and form of the optimal structures in presence of negative externalities has been the main focus of this chapter.

However, when the impact of externalities is not uniform, (i.e., if $\min_i \pi_i(\alpha_i) < 0 < \max_i \pi_i(\overline{\alpha}_i)$), the optimal network structure does depend on distributional details
and problem parameters. We illustrate this with a concrete example of four buyers. We present optimal networks for two different item values, $v = 0$ and $v = 10$, and for externalities independently and uniformly distributed on $[-1, 0]$ (positive externalities), on $[0, 1]$ (both positive and negative impact possible since $\pi_i(\alpha) = -1 < 0 < 1 = \pi_i(\pi)$), and on $[1, 2]$ (negative externalities). The optimal network structure with the smallest number of arcs for each case is presented in Table 4.2.

Specifically, the optimal network structure changes with changes in externality valuations. As externalities change from uniformly positive to uniformly significantly negative, the optimal network changes from the complete graph to optimal networks identified in this chapter. While the optimal network structures in these uniform cases are belief-free (i.e., they do not depend on details of distributional assumptions) and as such can provide general insights in optimal competitive market structures, the optimal networks in intermediate cases that simultaneously allow for both positive and negative externalities depend on the problem parameters (e.g., the case of optimal networks with $\alpha_i \sim U[0, 1]$ for different values of $v$ in Table 4.2) and consequently a general parameter-free guidance on optimal market structure cannot be provided.

4.6 Conclusions

We study the impact of rivalry among competing buyers on the revenues of a monopolist seller of a single indivisible object. We show that the structure of competitive relationships among buyers dictates the design of the optimal mechanism and, consequently, the seller’s expected revenues. Furthermore, we establish the existence of optimal rivalry structures for the revenue-maximizing seller. While the seller can exploit and benefit from buyers’ loss exposure due to rivalry, we show that the seller’s expected revenue is maximized for structures that limit the rivalry. Thus, fierce competitive relationships among buyers undermine revenue potential of the monopolist
Competitive relationships among buyers are modeled as negative externalities: a buyer suffers a loss (which is private information, independent across buyers) if any of its rivals obtains the item. We show that the revenue-maximizing mechanism for the seller depends on the network describing the structure of the competitive relationships (that are common knowledge). In the optimal mechanism for a given network, buyers might pay the seller even if they don’t obtain the item: they pay to avoid the loss that would be induced by an allocation to a rival. When such payments are large in aggregate, it might be optimal for the seller not to allocate the item.

Expected revenues depend both on the distributional assumptions the seller uses to model buyers’ independent private information on their losses, and on the network structure describing competitive relationships. However, we show that the optimal network structure is independent of distributional assumptions (provided that anticipated losses are sufficiently large). Thus, the revenue-maximizing seller who has ability to shape the network structure of competitive relationships among buyers, would choose the same optimal network structure regardless of its beliefs on buyers’ loss values.

Specifically, we show that for symmetric relationships, matchings are optimal structures (with a single path on three vertices in the case of an odd number of buyers). Thus, the seller’s revenues are maximized when facing maximally fragmented rival buyers grouped in pairs of mutually perceived competitors. This result indicates that the effect of negative externalities could be (at least in our setting) one of the drivers for market fragmentation to emerge naturally.

Symmetry of competitive relationships limits revenue potential for the seller. We show that there are many other asymmetric structures that revenue-dominate matchings. There are many optimal networks, none of which are symmetric, and are
characterized by (i) every buyer having a possibility of experiencing a loss, and (ii) the existence of a buyer that is not perceived as a rival by anyone.

It is interesting to note that, unlike efficiency objective and/or positive externalities settings, non-trivial network structures emerge as optimal ones, yet the optimality does not depend on the distributional assumptions (provided sufficiently large negative externalities). If some buyers could experience a gain rather than a loss when a rival obtains the item (positive externality), or if the virtual valuation of a loss changes sign, we show that the structure of the optimal competitive structure depends on the distributional assumption.

Our results are established in a model that deliberately focuses on the effects of negative externalities, and could be viewed as a starting point of the analysis in richer settings. For example, even if the item value is buyers’ private information, it is likely that the optimal structures are not far from those we identify in this chapter (e.g., if uncertainty about this privately held information is not large). Similarly, if multiple items are to be allocated to unit-demand buyers, it is straightforward to extend some of our results to that setting (as one needs the existence of multiple buyers not considered rivals by anyone). The main insight from our work, that the monopolist’s revenues depend on the network structure and that non-trivial networks are optimal in our model, indicates that potentially influencing competitive relationships among buyers is a relevant strategy that a revenue-maximizing monopolist should consider in richer and potentially more realistic settings.
Appendix A

Chapter 2 Results

Proof of Lemma 2

We first study the Bayesian Nash equilibrium among buyers for any given price menu. Let \( A_i \in [0, 1] \) denote buyer \( i \)'s equilibrium probability to accept the price menu. Buyer \( i \) accepts the price menu if

\[
(w_i - P_{i0}^{10}) \prod_{j \neq i} (1 - A_j) + (v_i - P_{i1}^{11}) \left( 1 - \prod_{j \neq i} (1 - A_j) \right) \geq 0. \tag{A.1}
\]

We define a function \( H_i (\eta^1, \eta^2; \{ A_j \}_{j \neq i}) \triangleq \eta^1 \prod_{j \neq i} (1 - A_j) + \eta^2 \left( 1 - \prod_{j \neq i} (1 - A_j) \right) \), and condition (A.1) is equivalent to \( H_i (w_i, v_i; \{ A_j \}_{j \neq i}) \geq H_i \left( P_{i0}^{10}, P_{i1}^{11}; \{ A_j \}_{j \neq i} \right) \).

Let the distribution of \( H_i (W_i, V_i; \{ A_j \}_{j \neq i}) \) be \( F^H_i (\cdot; \{ A_j \}_{j \neq i}) \). Then the equilibrium condition is

\[
A_i = 1 - F^H_i \left( H_i \left( P_{i0}^{10}, P_{i1}^{11}; \{ A_j \}_{j \neq i} \right); \{ A_j \}_{j \neq i} \right). \tag{A.2}
\]

Since there are unlimited supply of items, \( \sum_{i=1}^n A_i \leq n \). Let \( A \triangleq \{ A_i \}_{i=1}^n \) and
it is clear that $A$ is non-empty, compact and convex. Note that, for any given 
\[ \{P_{10}^i, P_{11}^i\}_{i=1}^n, H_i \left( P_{10}^i, P_{11}^i; \{A_j\}_{j\neq i} \right) \] 
is continuous in $\{A_j\}_{j\neq i}$ and, thus,
\[ F_i^H \left( H_i \left( P_{10}^i, P_{11}^i; \{A_j\}_{j\neq i} \right) \right) \]
is continuous in $\{A_j\}_{j\neq i}$. Therefore, by Kakutani fixed-point theorem, the equilibrium exists. Moreover, we can rewrite condition (A.2) as
\[ H_i \left( P_{10}^i, P_{11}^i; \{A_j\}_{j\neq i} \right) = (F_i^H)^{-1} \left( 1 - A_i; \{A_j\}_{j\neq i} \right). \quad (A.3) \]

Next, we study the optimal price menu. The seller’s expected revenue is
\[
\sum_{i=1}^n \left( P_{10}^i A_i \prod_{j \neq i} (1 - A_j) + P_{11}^i A_i \left( 1 - \prod_{j \neq i} (1 - A_j) \right) \right) \\
= \sum_{i=1}^n A_i H_i \left( P_{10}^i, P_{11}^i; \{A_j\}_{j\neq i} \right) \\
= \sum_{i=1}^n A_i (F_i^H)^{-1} \left( 1 - A_i; \{A_j\}_{j\neq i} \right).
\]

Note that the last equality comes from condition (A.3). Let $A^*$ be the optimal solution for this problem. Then any price menu $\{P_{10}^i, P_{11}^i\}$ is optimal if
\[ H_i \left( P_{10}^{10*}, P_{11}^{11*}; \{A_j^*\}_{j\neq i} \right) = (F_i^H)^{-1} \left( 1 - A_i^*; \{A_j^*\}_{j\neq i} \right). \]

If a single price is quoted for each buyer, i.e., $P_{10}^i = P_{11}^i = P_i$, then we construct
\[ P_i^* = H_i \left( P_i^*, P_i^*; \{A_j^*\}_{j\neq i} \right) = (F_i^H)^{-1} \left( 1 - A_i^*; \{A_j^*\}_{j\neq i} \right) \] 
and $P_i^*$ is optimal. Note that $P_i^*$ is the average price of the price menu $\{P_{10}^{10*}, P_{11}^{11*}\}$, satisfying
\[ P_i^* = P_{10}^{10*} \prod_{j \neq i} (1 - A_j^*) + P_{11}^{11*} \left( 1 - \prod_{j \neq i} (1 - A_j^*) \right). \quad (A.4) \]
Remarks: Some parts of the technical proof of Lemma 2 have an intuitive explanation as follows. Consider a two-price mechanism \((P^{10}_j; P^{11}_j)\) for \(j = 1, \ldots, n\). Pick a buyer \(i\) with \(P^{10}_i \neq P^{11}_i\) and denote the probability that no other buyer among \(-i\) wins a contract as \(a_i\). Hence, if buyer \(i\) accepts the two-price menu \((P^{10}_i; P^{11}_i)\), her expected payment does not depend on her type and equals \(a_i P^{10}_i + (1 - a_i) P^{11}_i\). Let us now consider a mechanism that proposes the same two-price menu \((P^{10}_j; P^{11}_j)\) to all other buyers \(j \neq i\) and a single price menu \(P_i = a_i P^{10}_i + (1 - a_i) P^{11}_i\) to buyer \(i\). Given the same decisions of \(-i\) buyers, buyer \(i\) faces the same expected payment \(a_i P^{10}_i + (1 - a_i) P^{11}_i\). Since buyer \(i\) faces the same expected payment, the acceptance decisions of each buyer \(i\)'s type does not change. Neither does change the expected payment of buyer \(i\) to the seller. Hence, the two-price menu can be replaced with one-price menu for one buyer without affecting the decisions of all buyers and the expected revenue of the seller. With this procedure, we can proceed sequentially.

However, this intuition does not address the existence of the equilibrium. □

Proof of Proposition 3

Consider ex ante identical buyers and then the seller quotes the same price for each buyer. Let \(A\) denote a buyer’s equilibrium probability to accept the price menu.

If the seller ignores the exclusivity, it believes that buyers accept the posted price based on their non-exclusivity valuations. Hence, the seller posts a price \(\widetilde{P}\) for each buyer to maximize its expected revenue \(n\widetilde{P} \left(1 - F^v \left(\widetilde{P}\right)\right)\), where \(1 - F^v \left(\widetilde{P}\right)\) is the probability that a buyer will accepts the price menu. The log transformation of the objective is \(\log n + \log \widetilde{P} + \log \left(1 - F^v \left(\widetilde{P}\right)\right)\). Since \(1 - F^v (\cdot)\) is log-concave, the
first order condition is valid and the optimal $P^0$ is determined by

$$P^0 - \frac{1 - F^v(P^0)}{f^v(P^0)} = 0.$$

If the seller considers the exclusivity, by Lemma 2, it is sufficient to consider single posted price. The seller’s expected revenue under price $P$ is $nP(1 - F^H(P; A))$, where

$$A = 1 - F^H(P; A). \quad (A.5)$$

The log transformation of the objective is $\log n + \log P + \log (1 - F^H(P; A))$. Since $1 - F^H(P; A(P))$ is log-concave in $P$, the first order condition is valid and the optimal $P^*$ is determined by

$$\frac{1}{P^*} + \frac{-f^H(P^*; A(P^*)) - \frac{\partial F^H(P^*; A(P^*))}{\partial A} dA(P^*)}{1 - F^H(P^*; A(P^*))} = 0.$$

By condition (A.5),

$$\frac{dA}{dP} = -f^H(P; A) - \frac{\partial F^H(P; A)}{\partial A} \frac{dA}{dP},$$

and, thus,

$$\frac{dA}{dP} = \frac{-f^H(P; A)}{1 + \frac{\partial F^H(P; A)}{\partial A}}.$$

Therefore, the first order condition can be rewritten as

$$P^* - \frac{1 - F^H(P^*; A(P^*))}{f^H(P^*; A)} \left(1 + \frac{\partial F^H(P^*; A(P^*))}{\partial A}\right) = 0.$$

We next prove that $(1 - F^H(\hat{P}; A)) / f^H(\hat{P}; A) \geq (1 - F^v(\hat{P})) / f^v(\hat{P})$ for any $A$ and any $\hat{P}$. We first derive the distribution of $H(W_i, V_i; A)$ for given $A$. Since $v_i$ is private information and $w_i \geq v_i \geq 0$, there always exist a constant $\beta' \geq 0$
and a random variable $\varepsilon'_i \geq 0$ independent of $V_i$ such that $W_i = V_i + \beta' V_i + \varepsilon'_i$. Hence, $H(W_i, V_i; A)$ can be rewritten as $H(W_i, V_i; A) = V_i (1 + \beta) + \varepsilon_i$, where $\beta = \beta' (1 - A)^{n-1}$ and $\varepsilon_i = \varepsilon'_i (1 - A)^{n-1}$.

Let the distribution of $\varepsilon_i$ be $F^\varepsilon$ and the density be $f^\varepsilon$. Since buyers are ex ante identical, the distribution of $\varepsilon_i$ is identical across buyers. Then

$$F^H \left( \hat{P}; A \right) = \Pr \left( V_i (1 + \beta) + \varepsilon_i \leq \hat{P} \right) = \int_0^{\hat{P}} F^v \left( \frac{\hat{P} - \varepsilon_i}{1 + \beta} \right) f^\varepsilon (\varepsilon_i) d\varepsilon_i,$$

and

$$f^H \left( \hat{P}; A \right) = \frac{1}{1 + \beta} \int_0^{\hat{P}} f^v \left( \frac{\hat{P} - \varepsilon_i}{1 + \beta} \right) f^\varepsilon (\varepsilon_i) d\varepsilon_i.$$

Let $LHS \triangleq \left( 1 - F^H \left( \hat{P}; A \right) \right) / f^H \left( \hat{P}; A \right)$, and then

$$\int_0^{\hat{P}} \left( F^v \left( \frac{\hat{P} - \varepsilon_i}{1 + \beta} \right) + \frac{LHS}{1 + \beta} f^v \left( \frac{\hat{P} - \varepsilon_i}{1 + \beta} \right) \right) f^\varepsilon (\varepsilon_i) d\varepsilon_i = 1.$$

By the first mean value theorem of integration, there exists $\xi \in \left[ 0, \hat{P} \right]$ such that

$$\left( F^v \left( \frac{\hat{P} - \xi}{1 + \beta} \right) + \frac{LHS}{1 + \beta} f^v \left( \frac{\hat{P} - \xi}{1 + \beta} \right) \right) F^\varepsilon \left( \hat{P} \right) = 1.$$

Therefore,

$$LHS = \frac{1}{F^\varepsilon \left( \hat{P} \right)} - \frac{F^v \left( \frac{\hat{P} - \xi}{1 + \beta} \right)}{f^v \left( \frac{\hat{P} - \xi}{1 + \beta} \right)} (1 + \beta).$$

Since $1 - F^v (\cdot)$ is log concave, $\left( \hat{P} - \xi \right) / (1 + \beta) \leq \hat{P}$, $1 / F^\varepsilon \left( \hat{P} \right) \geq 1$, and $1 + \beta \geq 1$, we must have $LHS \geq \left( 1 - F^v \left( \hat{P} \right) \right) / f^v \left( \hat{P} \right)$.
We also prove that \( \partial F^H \left( \hat{P}; A \right) / \partial A \geq 0 \). Let the distribution of \( \varepsilon'_i \) be \( F^{\varepsilon'} \) and the density be \( f^{\varepsilon'} \). Note that \( F^{\varepsilon'} \) and \( f^{\varepsilon'} \) are independent of \( A \). Since \( \varepsilon_i = \varepsilon'_i (1 - A)^{n-1} \)

\[
f^{\varepsilon} (\varepsilon_i) = \frac{1}{(1 - A)^{n-1}} f^{\varepsilon'} \left( \frac{\varepsilon_i}{(1 - A)^{n-1}} \right).
\]

Hence, \( F^H \left( \hat{P}; A \right) \) can be rewritten as

\[
F^H \left( \hat{P}; A \right) = \frac{1}{(1 - A)^{n-1}} \int_0^{\hat{P}} F^{v} \left( \frac{\hat{P} - \varepsilon_i}{1 + \beta' (1 - A)^{n-1}} \right) f^{\varepsilon'} \left( \frac{\varepsilon_i}{(1 - A)^{n-1}} \right) d\varepsilon_i.
\]

Using the first mean value theorem of integration, there exists \( \zeta \in [0, \hat{P}] \) such that

\[
F^H \left( \hat{P}; A \right) = \frac{1}{(1 - A)^{n-1}} F^{v} \left( \frac{\hat{P} - \zeta}{1 + \beta' (1 - A)^{n-1}} \right) F^{\varepsilon'} \left( \frac{\hat{P}}{(1 - A)^{n-1}} \right).
\]

It is clear that \( \partial F^H \left( \hat{P}; A \right) / \partial A \geq 0 \).

Finally, we prove \( P^* \geq P^0 \). Since

\[
1 - F^H \left( P^0; A \left( P^0 \right) \right) \geq 1 - F^{v} \left( P^0 \right) \quad \text{and} \quad \frac{F^H \left( P^0; A \left( P^0 \right) \right)}{\partial A} \geq 0,
\]

the first order condition indicates that

\[
P^0 - \frac{1 - F^H \left( P^0; A \left( P^0 \right) \right)}{f^H \left( P^0; A \right)} \left( 1 + \frac{\partial F^H \left( P^0; A \left( P^0 \right) \right)}{\partial A} \right) \leq P^0 - \frac{1 - F^{v} \left( P^0 \right)}{f^{v} \left( P^0 \right)} = 0.
\]

Note that \( 1 - F^H \left( P; A \left( P \right) \right) \) is log-concave in \( P \), we must have \( P^* \geq P^0 \).

Furthermore, the optimal expected profit with exclusivity consideration is

\[
R^* = nP^* \left( 1 - F^H \left( P^*; A \left( P^* \right) \right) \right)
\]

\[
\geq nP^0 \left( 1 - F^H \left( P^0; A \left( P^0 \right) \right) \right)
\]

\[
\geq nP^0 \left( 1 - F^{v} \left( P^0 \right) \right) = R^0,
\]

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where the first inequality comes from the fact $P^*$ is optimal in the posted price mechanism with exclusivity consideration, while the second inequality comes from the fact that $F^H(\hat{P}^*; A) \leq F^v(\hat{P})$ for any $A$ and $\hat{P}$. In addition, when the seller ignores the exclusivity valuations, $nP_0(1 - F^v(P_0))$ is the expected revenue from the seller’s perspective, and $nP_0(1 - F^H(P_0; A(P_0)))$ is the expected revenue from the researcher’s perspective.

Remarks: Some parts of the technical proof of Proposition 3 have an intuitive explanation as follows. Consider an optimal pricing mechanism when there is no exclusivity and denote its price as $P^0$. Also consider the environment when buyers have exclusivity: $w_i > v_i$ with positive probability. Let us assume that buyers use the “old” strategies (the same as when there is no exclusivity; these strategies might not be optimal), and, thus, have the same probability of contract acceptance. Since, buyer $i$ has a premium of exclusivity (this happens if other buyers do not get the items with positive probability) more types $v_i$ of the agent are ready to accept price $P^0$. Let us increase the price for buyer $i$ until $P_i = P' \leq P^0$ such that the probability of that buyer accepts the contract remains as for the case when there is no exclusivity. For price $P'$ buyer $i$ is playing a best response to other buyers’ (not yet optimal) strategies. We repeat this procedure with each buyer. Since buyers are identical, this procedure results in the same price $P'$ for each buyer. Note that in the new mechanism each buyer plays a best response and accepts the contract with the same probability as for $P^0$ and the environment with no exclusivity. The new mechanism might not be optimal, but it has a higher price and a higher revenue. Hence, the optimal mechanism will have a higher revenue.

However, this intuitive reasoning cannot provide the exact optimal price and expected revenue.
Proof of Example 5

Problem (General-RMP) is a linear programming problem when the support Ω is finite. We discretize the type space Ω as follows,

\[ \Omega(M) = \{(i/(M-1) + j/(M-1), i/(M-1)) : i, j = 0, 1, \ldots, M - 1\}, \]

where \( M \) is the number of possible outcomes of \( v_i \) or \( \varepsilon_i \). Moreover, we consider discrete uniform distributions on both \( v_i \) and \( \varepsilon_i \). The following Table A.1 shows the numerical solutions of Problem (General-RMP).

| Grid Index \( M \) | \( |\Omega| \) | Seller’s Expected Revenue |
|-------------------|-----------|--------------------------|
| 2                 | 4         | 1.188                    |
| 3                 | 9         | 1.056                    |
| 4                 | 16        | 0.995                    |
| 5                 | 25        | 0.961                    |
| 6                 | 36        | 0.938                    |
| 7                 | 49        | 0.924                    |
| 8                 | 64        | 0.915                    |
| 9                 | 81        | 0.907                    |
| 10                | 100       | 0.901                    |

The number of variables is \( 4|\Omega|^2 = 4M^2 \) and the number of (EPIC) constraints is \( 2|\Omega|^2(|\Omega| - 1) = 2(M^6 - M^4) \). Hence, it rather quickly becomes computationally unmanageable. □

Proof of Proposition 6

1) We first prove the result for additive exclusivity. Following the methodology in Myerson (1981), we can rewrite the (EPIC) as follows. By Envelope Theorem,

\[ \frac{dU_i(v_i, v_{-i})}{dv_i} = \left. \frac{\partial U_i(\tilde{v}_i, v_i, v_{-i})}{\partial v_i} \right|_{(\tilde{v}_i) = (v_i)} = p_i(v_i, v_{-i}). \]
Obviously, $U_i(v_i, v_{-i})$ is increasing in $v_i$. Moreover, since $U_i(v_i, v_{-i})$ is a convex function, it is equivalent to require $dp_i(v_i, v_{-i})/dv_i \geq 0$, which means $p_i(v_i, v_{-i})$ is increasing in $v_i$. Hence, we can rewrite the interim utility function as

$$U_i(v_i, v_{-i}) = U_i(v_i, v_{-i}) + \int_{v_i}^{v_i} p_i(t, v_{-i}) \, dt.$$  

Note that the integral on the righthand side is a line integral when exclusive and non-exclusive valuations have one-dimensional representation. However, with a generic relationship between exclusive and non-exclusive valuations, this integral may not be a line integral and may be path dependent. We choose $v_i$ as the bottom type and make the bottom type binds $U_i(v_i, v_{-i}) = 0$. Since $0 \leq p_i(v_i, v_{-i}) \leq 1$, we know that $U_i(v_i, v_{-i}) \geq 0$ for any $v_i$.

With additively exclusivity valuation, the ex post utility (2.1) can be rewritten as

$$U_i(\hat{v}_i, v_i, v_{-i}) = v_i p_i(\hat{v}_i, v_{-i}) + \theta_0 p_i(\hat{v}_i, v_{-i}) \prod_{j \neq i} (1 - p_j(\hat{v}_i, v_{-i})) - m_i(\hat{v}_i, v_{-i}).$$

We can then rewrite the ex post payment as

$$m_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) + \theta_0 p_i(v_i, v_{-i}) \prod_{j \neq i} (1 - p_j(v_i, v_{-i}))$$

$$- \int_{v_i}^{v_i} p_i(t, v_{-i}) \, dt.$$  

Thus, the seller’s expected revenue can now be expressed as

$$\sum_{i=1}^{n} \int m_i(v) \, dF^v(v) = \int \sum_{i=1}^{n} \left( \psi_i + \theta_0 \prod_{j \neq i} (1 - p_j(v)) \right) p_i(v) \, dF^v(v).$$

We introduce the following notation of the set of allocation,

$$\mathcal{P}_{add} = \{ p : 0 \leq p_i(v) \leq 1 \text{ and } p_i(v) \text{ is increasing in } v_i \text{ for all } i \}. $$
Therefore, the optimal mechanism, i.e., functions \((p^*(v), m^*(v))\), satisfies

\[
p^*(v) \in \arg \max_{p \in \mathcal{P}_{\text{add}}} \left\{ \sum_{i=1}^{n} \left( \psi_i + \theta_i^0 \prod_{j \neq i} (1 - p_j(v)) \right) p_i(v) \right\}, \tag{A.6}
\]

and

\[
m^*_i(v) = v_i p^*_i(v) + \theta_i^0 p^*_i(v) \prod_{j \neq i} (1 - p^*_j(v)) - \int_{v_i}^{v_i} p^*_i(t, v_{-i}) dt. \tag{A.7}
\]

2) We then prove the result for multiplicative exclusivity. By Envelope Theorem,

\[
\frac{dU_i(v_i, v_{-i})}{dv_i} = \left. \frac{\partial U_i(\hat{v}_i, v_i, v_{-i})}{\partial v_i} \right|_{(\hat{v}_i) = (v_i)} = p_i(v_i, v_{-i}) \left( 1 + (\theta^1_i - 1) \prod_{j \neq i} (1 - p_j(v_i, v_{-i})) \right).
\]

Obviously, \(U_i(v_i, v_{-i})\) is increasing in \(v_i\). Moreover, since \(U_i(v_i, v_{-i})\) is a convex function, it is equivalent to require \(dp_i(v_i, v_{-i})/dv_i \geq 0\), which means \(p_i(v_i, v_{-i})\) is increasing in \(v_i\). Hence, we can rewrite the interim utility function as

\[
U_i(v_i, v_{-i}) = U_i(v_i, v_{-i}) + \int_{v_i}^{v_i} p_i(t, v_{-i}) \left( 1 + (\theta^1_i - 1) \prod_{j \neq i} (1 - p_j(t, v_{-i})) \right) dt.
\]

We choose \(\underline{v}_i\) as the bottom type and make the bottom type binds \(U_i(\underline{v}_i, v_{-i}) = 0\). Since \(0 \leq p_i(v_i, v_{-i}) \leq 1\), we know that \(U_i(v_i, v_{-i}) \geq 0\) for any \(v_i\).

With multiplicative exclusivity valuation, the ex post utility (2.1) can be rewritten as

\[
U_i(\hat{v}_i, v_i, v_{-i}) = v_i p_i(\hat{v}_i, v_{-i}) \left( 1 + (\theta^1_i - 1) \prod_{j \neq i} (1 - p_j(\hat{v}_i, v_{-i})) \right) - m_i(\hat{v}_i, v_{-i}).
\]
We can then rewrite the ex post payment as
\[ m_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j(v_i, v_{-i})) \right) \]
\[ - \int_{v_i}^{v_i} p_i(t, v_{-i}) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j(t, v_{-i})) \right) dt \]

Thus, the seller’s expected revenue can now be expressed as
\[ \sum_{i=1}^{n} \int m_i(v) dF^v(v) = \int \sum_{i=1}^{n} \psi_i p_i(v) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j(v)) \right) dF^v(v) \]

We introduce the following notation of the set of allocation,
\[ \mathcal{P}_{mul} = \{ p : 0 \leq p_i(v) \leq 1 \}
\] and \[ p_i(v) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j(v)) \right) \] is increasing in \( v_i \) for all \( i \).

Therefore, the optimal mechanism, i.e., functions \((p^*(v), m^*(v))\), satisfies
\[ p^*(v) \in \arg \max_{p \in \mathcal{P}_{mul}} \left\{ \sum_{i=1}^{n} \psi_i \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j(v)) \right) p_i(v) \right\} \], \hspace{1cm} (A.8)

and
\[ m_i^*(v) = v_i p_i^*(v) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j^*(v)) \right) \]
\[ - \int_{v_i}^{v_i} p_i^*(t, v_{-i}) \left( 1 + \left( \theta_i^1 - 1 \right) \prod_{j \neq i} (1 - p_j^*(t, v_{-i})) \right) dt \]. \hspace{1cm} (A.9)
Proof of Proposition 7

Let $N^1$ be the set of buyers who get items and $N^0$ be the set of buyers who do not get items.

If an item is allocated to buyer $i$, the seller can charge buyer $i$ at most $v_i + \sum_{j \in S(i)} \alpha_{ij} v_j 1_{\{S(i) \subseteq N^0\}}$. To see this upper bound, we consider two different cases.

Case 1: $S(i) \cap N^0 = S(i)$. This indicates that buyer $i$ gets an item exclusively and thus the seller can raise $v_i + \sum_{j \in S(i)} \alpha_{ij} v_j$ from buyer $i$.

Case 2: $S(i) \cap N^0 \neq S(i)$. This indicates that at least one of buyer $i$'s neighbors gets an item and thus the seller can raise $v_i$ from buyer $i$.

If an item is not allocated to buyer $i$, the seller can charge buyer $i$ at most 0.

Let $S^{-1}(j)$ denote the set of buyers for whom buyer $j$ is in its neighborhood, i.e., $S^{-1}(j) = \{i : j \in S(i)\}$. Hence, the total payments from the system is bounded above by

$$\sum_{i \in N^1} \left( v_i + \sum_{j \in S(i)} \alpha_{ij} v_j 1_{\{S(i) \subseteq N^0\}} \right)$$

$$= \sum_{i \in N^1} v_i + \sum_{i \in N} \sum_{j \in N} \alpha_{ij} v_j 1_{\{i \in N^1, S(i) \subseteq N^0, j \in S(i)\}}$$

$$\leq \sum_{i \in N^1} v_i + \sum_{j \in N} \sum_{i \in N} \alpha_{ij} v_j 1_{\{j \in N^0, S^{-1}(j) \nsubseteq N^0, i \in S^{-1}(j)\}}$$

$$= \sum_{i \in N^1} v_i + \sum_{j \in N^0} v_j 1_{\{S^{-1}(j) \nsubseteq N^0\}} \sum_{i \in S^{-1}(j)} \alpha_{ij}, \tag{A.10}$$

where the inequality comes from the fact that

$$\{(i, j) : i \in N^1, S(i) \subseteq N^0, j \in S(i)\} \subseteq \{(i, j) : j \in N^0, S^{-1}(j) \nsubseteq N^0, i \in S^{-1}(j)\}.$$

With BLLE valuations, equation (A.10) is smaller than

$$\sum_{i \in N^1} v_i + \sum_{j \in N^0} v_j 1_{\{S^{-1}(j) \nsubseteq N^0\}} \leq \sum_{i \in N^1} v_i + \sum_{i \in N^0} v_i = \sum_{i \in N} v_i,$$
which is the total payment when the seller allocates an item to every buyer. Hence, the optimal allocation is to allocate an item to every buyer and the optimal payment is to charge \( v_i \) to buyer \( i \). □

Proof of Proposition 8

Let everybody have the same exclusivity premium, i.e., \( \gamma = \sum_{j \in S(i)} \alpha_{ij} v_j \) for all \( i \). Note that such instance occurs when \( v_i = \bar{v} \) for all \( i \) and \( \alpha_{ij} = \frac{c}{|S(i)|} \), where \( c \) is a constant. Suppose there are exactly \( k - 1 \) exclusive allocations. Then the seller can get at most \( (k - 1)(\bar{v} + \gamma) \) from these \( k - 1 \) exclusive allocations, plus at most \( (n - k + 1)\bar{v} \) from \( n - k + 1 \) non-exclusive allocations.

Next, suppose there are exactly \( k \) exclusive allocations. Then the seller can get at least \( k(\bar{v} + \gamma) \). Therefore, the revenue for \( k \) exclusive allocations dominate that for \( k - 1 \) exclusive allocations if

\[
\gamma \geq (n - k + 1)\bar{v} + (k - 1)\bar{v} \quad \text{Merlin 11.11}
\]

Note that \( \gamma \) is increasing in \( \alpha_{ij} \). Hence, if \( \alpha_{ij} \) is sufficiently large such that condition (A.11) holds, then finding the optimal allocation also results in determining the size of the largest independent set. □

Proof of Proposition 9

By the definition of virtual valuation, \( v_i \geq \psi_i (v_i) \) holds. Furthermore, there always exists \( \underline{\psi}, \overline{\psi} \), and \( v_i \) such that \( 0 \leq \underline{\psi} \leq \psi_i (v_i) \leq \overline{\psi} < v_i \), where \( \overline{\psi} (\underline{\psi}) \) is an upper (lower) bound on the virtual valuation. Since \( \overline{\psi} < v_i \), there also exists \( \{v_i\}_{i \in N} \) such that

\[
0 \leq \overline{\psi} \leq \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) / n,
\]
even when (BLLE) holds. Note that such instance occurs when \( v_i = v \) and \( \psi = v \left( \sum_{j \in S(i)} \alpha_{ij} \right) / n \).

Consider the realization \( \{v_i\}_{i \in N} \) such that \( v_i > 0, \ \psi_i(v_i) \leq \psi \) and \( 0 \leq \underline{\psi} \leq \bar{\psi} \leq \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) / n \) for all \( i \). Also let everybody have the same exclusivity premium, i.e., \( \gamma = \sum_{j \in S(i)} \alpha_{ij} v_j \). By equation (2.9) and condition (A.11) in the proof of Proposition 8, the revenue for \( k \) exclusive allocations dominate that for \( k - 1 \) exclusive allocations if

\[
\gamma \geq (n - k + 1)\bar{\psi} + (k - 1)\bar{\psi} - k\underline{\psi}.
\]

It is clear that this condition holds, since \( 0 \leq \underline{\psi} \leq \bar{\psi} \leq \left( \sum_{j \in S(i)} \alpha_{ij} v_j \right) / n \). Then finding a deterministic optimal solution to the (SB-RMP) problem also results in determining the size of the largest independent set. \( \square \)

Proof of Proposition 11

We prove the proposition in the following steps.

**Step 1: The proposed mechanism satisfies the (EPIR).**

By the construction of the payment rule,

\[
m_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) + \gamma_i Q_{i}^{10}(v_i, v_{-i}).
\]

Therefore,

\[
U_i(v_i, v_{-i}) = v_i p_i(v_i, v_{-i}) + \gamma_i Q_{i}^{10}(v_i, v_{-i}) - m_i(v_i, v_{-i}) = 0.
\]

**Step 2: The proposed mechanism satisfies the (EPIC).**

To prove this claim, we need only check the \( p_i(v) \triangleq \sum_{k=1}^{n-1} Q_i^{1k}(v) + Q_i^{10}(v) \) is non-decreasing in \( v_i \).
If $\max\{\sum_{h=1}^{K^*} \psi_h, \gamma_{j^*} + \psi_{j^*}\} < 0$, $Q_i^{00} = 1$ for $i = 1, 2, \ldots, n$. We have $p_i(v) = 0$. This case means $\gamma_j + \psi_j < 0$. It is a trivial case to verify the monotonicity. In the following proof, we only consider the case $\max\{\sum_{h=1}^{K^*} \psi_h, \gamma_{j^*} + \psi_{j^*}\} \geq 0$.

Case 1: $j \in \{1, \ldots, K^*\}$.

If $\sum_{h=1}^{K^*} \psi_h > \gamma_{j^*} + \psi_{j^*}$, then $Q_j^{1K^*} = 1$. Hence, we have $p_j(v) = 1$.

If $\gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_h$ and $j = j^* (\gamma_j + \psi_j \geq \gamma_i + \psi_i, \text{ for } i = 1, \ldots, n)$, then $Q_j^{10} = 1$. Hence, we have $p_j(v) = 1$.

If $\gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_h$ and $j \neq j^* (\gamma_{j^*} + \psi_{j^*} > \gamma_j + \psi_j)$, then $Q_j^{10} = 1$. Hence, we have $p_j(v) = 0$.

From $p_j(v) = 0$ to $p_j(v) = 1$, we have two possibilities.

1) If $j = j^*$ when $p_j(v) = 1$, then $\psi_j$ changes from $\psi_j \leq \gamma_{j^*} + \psi_{j^*} - (\psi_1 + \cdots + \psi_{j-1} + \psi_{j+1} + \cdots + \psi_{K^*})$ and $\psi_j < \gamma_{j^*} + \psi_{j^*} - \gamma_j$ to $\psi_j \geq \gamma_i + \psi_i - \gamma_j$ for any $i = 1, \ldots, n$.

2) If $j \neq j^*$ when $p_j(v) = 1$, then $\psi_j$ changes from $\psi_j \leq \gamma_{j^*} + \psi_{j^*} - (\psi_1 + \cdots + \psi_{j-1} + \psi_{j+1} + \cdots + \psi_{K^*})$ and $\psi_j < \gamma_{j^*} + \psi_{j^*} - \gamma_j$ to $\psi_j \geq \gamma_{j^*} + \psi_{j^*} - (\psi_1 + \cdots + \psi_{j-1} + \psi_{j+1} + \cdots + \psi_{K^*})$ and $\psi_j < \gamma_{j^*} + \psi_{j^*} - \gamma_j$.

In both cases, $\psi_j$ is increasing. As $\psi_j$ itself is increasing in $v_j$, we see that $p_j(v)$ is non-decreasing in $v_i$.

Case 2: $j \in \{K^* + 1, \ldots, n\}$.

If $\sum_{h=1}^{K^*} \psi_h > \gamma_{j^*} + \psi_{j^*}$, then $Q_i^{1K^*} = 1$ for $i = 1, 2, \ldots, K^*$. Hence, we have $p_j(v) = 0$.

If $\gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_h$ and $j \neq j^* (\gamma_{j^*} + \psi_{j^*} > \gamma_j + \psi_j)$, then $Q_j^{10} = 1$. Hence, we have $p_j(v) = 0$.

If $\gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_h$ and $j = j^* (\gamma_j + \psi_j \geq \gamma_i + \psi_i, \text{ for } i = 1, \ldots, n)$, then $Q_j^{10} = 1$. Hence, we have $p_j(v) = 1$.

From $p_j(v) = 1$ to $p_j(v) = 0$, we also have two possibilities.

1) If $j = j^*$ when $p_j(v) = 0$, then $\psi_j$ changes from $\psi_j \geq \psi_1 + \psi_2 + \cdots + \psi_{K^*} - \gamma_j$
and $\psi_j \geq \gamma_i + \psi_i - \gamma_j$, for any $i = 1, ..., n$ to $\psi_j < \psi_1 + \psi_2 + \cdots + \psi_{K^*} - \gamma_j$ and $\psi_j \geq \gamma_i + \psi_i - \gamma_j$, for any $i = 1, ..., n$.

2) If $j \neq j^*$ when $p_j (v) = 0$, then $\psi_j$ changes from $\psi_j \geq \psi_1 + \psi_2 + \cdots + \psi_{K^*} - \gamma_j$ and $\psi_j \geq \gamma_i + \psi_i - \gamma_j$, for any $i = 1, ..., n$ to $\psi_j < \gamma_j^* + \psi_j^* - \gamma_j$.

In both cases, $\psi_j$ is decreasing. As $\psi_j$ itself is increasing in $v_j$, we see that $p_j (v)$ is non-decreasing in $v_i$.

**Step 3:** The proposed mechanism is the equilibrium, i.e. there is no strict incentive for the seller to deviate.

If $\max \{\sum_{h=1}^{K^*} \psi_h, \gamma_{j^*} + \psi_{j^*}\} < 0$, $Q_i^{00} = 1$ for $i = 1, 2, ..., n$. Obviously, there is no incentive to deviate, or else, the seller would obtain a negative profit. Then we consider the case $\max \{\sum_{h=1}^{K^*} \psi_h, \gamma_{j^*} + \psi_{j^*}\} \geq 0$ as follows.

Case 1: If $\sum_{h=1}^{K^*} \psi_h > \gamma_{j^*} + \psi_{j^*}$, then $Q_i^{1K^*-1} = 1$ for $i = 1, 2, ..., K^*$.

Consider a deviation as $Q_{i'}^{1K^*-1} = 1 - \epsilon$ for $i' \in \{1, 2, ..., K^*\}$, where $\epsilon$ is a small positive number.

Subcase 1: $Q_{i'}^{0K^*} = \epsilon$, $Q_{i'}^{1K^*-1} = 1$ for $i' \neq i$ and $i \in \{1, 2, ..., K^*\}$, and $Q_{i'}^{1K^*-1} = \epsilon$ for $i'' \in \{K^* + 1, ..., n\}$. By the definition of $K^*$, if $n \geq K^*$, there is no incentive because $\psi_{i''} < 0$.

Subcase 2: $Q_{i'}^{0k} = \epsilon$ for $k < K^*$, $Q_{i'}^{1k-1} = \epsilon$ for $i' \in \{1, 2, ..., k\}$ if $i' > k$ or $i \in \{1, 2, ..., k + 1\} \setminus \{i'\}$ if $i' \leq k$ (this is the best deviation for allocating only $k$ items), and $Q_{i'}^{0k} = \epsilon$ for $i'' \in \{k + 1 \text{ (or } k + 2), ..., n\}$. Obviously, there is no strict incentive to this since $\psi_{i''} \geq 0$ for $i'' \leq K^*$.

Subcase 3: $Q_{i'}^{0j} = \epsilon$, $Q_{j'}^{10} = \epsilon$ for $j \neq i'$, and $Q_{i'}^{1K^*-1} = 1 - \epsilon$ for $i' \neq i'$ and $i \in \{1, 2, ..., K^*\}$. Then the expected revenue is $(1 - \epsilon)(\sum_{h=1}^{K^*} \psi_h) + \epsilon(\gamma_j + \psi_j)$. By the condition $\sum_{h=1}^{K^*} \psi_h > \gamma_{j^*} + \psi_{j^*}$ and the definition of $j^*$, $(1 - \epsilon)(\sum_{h=1}^{K^*} \psi_h) + \epsilon(\gamma_j + \psi_j) < \sum_{h=1}^{K^*} \psi_h$. Therefore, there is no incentive to deviate.

Subcase 4: $Q_{i'}^{10} = \epsilon$ and $Q_{i'}^{1K^*-1} = 1 - \epsilon$. By the similar argument in subcase 2, we
know there is no incentive to deviate.

Subcase 5: \( Q_{i}^{1k} = \epsilon \) for \( k < K^* \), and \( Q_{i}^{1k} = \epsilon \) for \( i \in \{1, 2, ..., k\} \) if \( i' > k \) or \( i \in \{1, 2, ..., k+1\} \setminus \{i'\} \) if \( i' \leq k \). Obviously, there is no strict incentive as in Subcase 2.

Subcase 6: \( Q_{i}^{00} = \epsilon \). Obviously, there is no strict incentive because \( v_{0i}^{00} = 0 \).

Case 2: If \( \gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_{h} \), then \( Q_{i}^{10} = 1 \). Consider a deviation as \( Q_{i}^{10} = 1 - \epsilon \), where \( \epsilon \) is a small positive number.

Subcase 1: if \( j^* \in \{1, 2, ..., k\} \) for \( k \leq K^* \), \( Q_{i}^{1k-1} = \epsilon \) for \( i \in \{1, 2, ..., k\} \). Then the expected revenue is \((1 - \epsilon)(\gamma_{j^*} + \psi_{j^*}) + \epsilon(\sum_{h=1}^{K^*} \psi_{h})\). By the condition \( \gamma_{j^*} + \psi_{j^*} \geq \sum_{h=1}^{K^*} \psi_{h} \), there is no strict incentive to deviate.

Subcase 2: if \( j^* \notin \{1, 2, ..., k\} \) for \( k \leq K^* \), \( Q_{i}^{0k} = \epsilon \), and \( Q_{i}^{1k-1} = \epsilon \) for \( i \in \{1, 2, ..., k\} \). By the similar argument in subcase 1, there is no strict incentive to deviate.

Subcase 3: \( Q_{i}^{01} = \epsilon \), and \( Q_{i}^{10} = \epsilon \). The expected revenue is \((1 - \epsilon)(\gamma_{j^*} + \psi_{j^*}) + \epsilon(\gamma_{j^*} + \psi_{j^*})\). By the definition of \( j^* \), there is no strict incentive to deviate.

Subcase 4: \( Q_{i}^{00} = \epsilon \). Obviously, there is no strict incentive.

Therefore, there is no strict incentive for the seller to deviate from the proposed allocation. The payment rule is just a direct result from condition (2.10).

Proof of Corollary 12

By Proposition 11, a non-exclusive allocation occurs iff \( \psi_{1}(x) + \psi_{2}(y) > \psi_{1}(x) + \gamma_{1}, \psi_{1}(x) + \psi_{2}(y) > \psi_{2}(y) + \gamma_{2} \), and \( \psi_{1}(x) + \psi_{2}(y) \geq 0 \). An exclusive allocation to buyer 1 occurs iff \( \psi_{1}(x) + \gamma_{1} \geq \psi_{1}(x) + \psi_{2}(y), \psi_{1}(x) + \gamma_{1} \geq \psi_{2}(y) + \gamma_{2} \), and \( \psi_{1}(x) + \gamma_{1} \geq 0 \). Similarly, an exclusive allocation to buyer 2 occurs iff \( \psi_{2}(y) + \gamma_{2} \geq \psi_{1}(x) + \psi_{2}(y), \psi_{2}(y) + \gamma_{2} \geq \psi_{1}(x) + \gamma_{1} \), and \( \psi_{2}(y) + \gamma_{2} \geq 0 \). Otherwise, no allocation occurs.

Since \( \gamma_{1} \geq 0 \) and \( \gamma_{2} \geq 0 \), the above condition can be simplified as follows. A non-exclusive allocation occurs iff \( \psi_{2}(y) > \gamma_{1} = \alpha_{12}y \) and \( \psi_{1}(x) > \gamma_{2} = \alpha_{21}x \). An
exclusive allocation to buyer 1 occurs iff \( \gamma_1 \geq \psi_2(y) \), \( \psi_1(x) - \psi_2(y) \geq \gamma_2 - \gamma_1 \), and \( \psi_1(x) \geq -\gamma_1 \). An exclusive allocation to buyer 2 occurs iff \( \gamma_2 \geq \psi_1(x) \), \( \gamma_2 - \gamma_1 \geq \psi_1(x) - \psi_2(y) \), and \( \psi_2(y) \geq -\gamma_2 \).

Moreover, when \( \psi_2(y) = \gamma_1 \) and \( \psi_1(x) = \gamma_2 \), \( \psi_1(x) - \psi_2(y) + \gamma_1 - \gamma_2 = 0 \) holds. When \( \psi_1(x) = -\gamma_1 \) and \( \psi_2(y) = -\gamma_2 \), \( \psi_1(x) - \psi_2(y) + \gamma_1 - \gamma_2 = 0 \) also holds. Therefore, the valuation space is divided into four regions.

The proof is completed by noting that \( \psi_1(\cdot), \psi_2(\cdot), \gamma_1(\cdot), \gamma_2(\cdot), \psi_1(\cdot) - \gamma_2(\cdot) \), and \( \psi_2(\cdot) - \gamma_1(\cdot) \) are increasing functions. \( \Box \)

Proof of Proposition 13

In order to simplify the proof, we rewrite the price system as follows.

\[
P_{10}^1(0) = \frac{2 + \alpha_{12} - \alpha_{12} \alpha_{21}}{4 - \alpha_{12} \alpha_{21}} \quad \text{and} \quad P_{10}^2(0) = \frac{2 + \alpha_{21} - \alpha_{12} \alpha_{21}}{4 - \alpha_{12} \alpha_{21}},
\]

\[
P_{10}^1(t) = \alpha_{12} y(t) + \frac{2 - \alpha_{12}}{2 - \alpha_{21}} y(t) \quad \text{and} \quad P_{10}^2(t) = \alpha_{21} x(t) + \frac{2 - \alpha_{21}}{2 - \alpha_{12}} x(t),
\]

where

\[
y(t) = \frac{2 - \alpha_{21}}{2 - \alpha_{12}}, \quad y(0) = \frac{2 - \alpha_{21}}{4 - \alpha_{12} \alpha_{21}},
\]

\[
y(T) = \frac{1}{2 - \alpha_{12}}, \quad \text{and} \quad y(T) = y(0) + \frac{y(T) - y(0)}{T} t,
\]

and the final prices are

\[
P_{1}^{11} = \frac{1}{2 - \alpha_{21}} \quad \text{and} \quad P_{2}^{11} = \frac{1}{2 - \alpha_{12}}.
\]

Without the loss of generality, we consider that only buyer 1 accepts the offer at \( t = 0 \). Since buyer 2 always has zero utility, it must truthfully report its valuation. Buyer 1 prefers the exclusive allocation iff

\[
x + \alpha_{12} y \geq \frac{1}{2} + \frac{1}{2} \alpha_{12} y,
\]
which is $x \geq 1/2 - \alpha_{12}y/2 = x_0(y)$. Moreover, when $x(t) = 1/2 - \alpha_{12}y(t)/2$, asking for exclusivity only brings zero utility. A similar argument works on the case where only buyer 2 accepts the offer at $t = 0$.

If the auction does not end at time $t = 0$, we will prove the following equilibrium: Buyer 1 does not quit until $x(t) = x$ and buyer 2 does not quit until $y(t) = y$.

We first consider the case when prices reach $\{P_{10}^1(T), P_{20}^2(T)\}$. This indicates that $x \geq x(T)$ and $y \geq y(T)$. Note that buyer 1 does not have incentive to deviate from the proposed equilibrium. Buyer 1 gets positive utility by asking for exclusivity if $x \geq x(T)$. Therefore, buyer 1 with $x \geq x(T)$ will not quit before time $T$, which only gives it zero utility. A similar argument works on buyer 2.

Consider $x(0) \leq x \leq x(T)$ and $y(0) \leq y \leq y(T)$. Without the loss of generality, we show that buyer 1 does not have incentive to deviate from the proposed equilibrium.

1) Buyer 1 with $x(t)$ will not postpone quitting. If buyer 1 asks for exclusivity until $x(\tau)$ for $\tau \geq t$, the expected utility is

$$
\int_{y(\tau)}^{y(T)} \frac{1}{1 - y(t)} \left( x(t) + \alpha_{12}y(\tilde{t}) - P_{10}^1(\tilde{t}) \right) dy(\tilde{t}) + \int_{y(\tau)}^{y(T)} 0 dy(\tilde{t}) + \int_{y(T)}^{1} 0 dy(\tilde{t})
$$

$$
= \int_{y(t)}^{y(\tau)} \frac{1}{1 - y(t)} \left( x(t) + \alpha_{12}y(\tilde{t}) - \alpha_{12}y(\tilde{t}) - \frac{2 - \alpha_{12}}{2 - \alpha_{21}}y(\tilde{t}) \right) dy(\tilde{t}).
$$

If buyer 1 quits at time $t$, the expected utility is 0. Then we need to verify

$$
x(t) \leq \frac{2 - \alpha_{12}}{2 - \alpha_{21}} (y(t) + y(\tau)).
$$

Since $x(t) = y(t)(2 - \alpha_{12}) / (2 - \alpha_{21})$ and $y(\tau) \geq y(t)$, the above condition holds. Hence, buyer 1 will quit at time $t$ if $x = x(t)$.

2) Buyer 1 with $x(\tau)$ for $\tau > t$ will ask for exclusivity at $t$. By the above proof,
buyer 1 asks for exclusivity until \( x(\tau) \). By asking for exclusivity, it gets

\[
\int_{y(t)}^{y(\tau)} \frac{1}{1 - y(t)} \left( x(\tau) + \alpha_{12} y(\tilde{t}) - P_{1}^{10}(\tilde{t}) \right) dy(\tilde{t})
\]

\[
= \int_{y(t)}^{y(\tau)} \frac{1}{1 - y(t)} \left( x(\tau) + \alpha_{12} y(\tilde{t}) - \alpha_{12} y(\tilde{t}) - 2 \frac{\alpha_{12}}{2 - \alpha_{21}} y(\tilde{t}) \right) dy(\tilde{t}).
\]

If buyer 1 quits, the expected utility is 0. Then we need to verify

\[
x(\tau) \geq 1 - \frac{\alpha_{12}}{2 - \alpha_{21}} (y(t) + y(\tau)).
\]

Since \( x(\tau) = y(\tau) (2 - \alpha_{12}) / (2 - \alpha_{21}) \) and \( y(\tau) > y(t) \), the above condition holds.

Hence, buyer 1 with \( x(\tau) \) for \( \tau > t \) will ask for exclusivity at \( t \).

The argument works on buyer 2 as well.

At time 0, buyer 1 with \( x < P_{1}^{10}(0) \) does not have incentive to deviate from quitting. Buyer 2 with \( y < P_{2}^{10}(0) \) does not have incentive to deviate, either.

Since the price system is consistent with the payment rule in the optimal mechanism and the construction of the above auction is consistent with the allocation rule, the auction implements the optimal mechanism. \( \square \)

Proof of the Non-monotonicity Example

Consider \( F_{1} = F_{2} = U[0,1] \). Let \( x = 0.01 + 2/3, y = 2/3 \), and \( \alpha_{12} = \alpha_{21} = 0.4 \). Then, in an optimal solution, items are allocated to both buyers, since \( x = 0.01 + 2/3 > 1 / (2 - \alpha_{21}) = 1/1.6 \) and \( y = 2/3 > 1 / (2 - \alpha_{21}) = 1/1.6 \). Buyer 1 pays \( m_{1} = 1/1.6 \) and buyer 2 also pays \( m_{2} = 1/1.6 \).

Next, consider an increase in \( \alpha_{12} \) to \( \tilde{\alpha}_{12} = 0.9 \). An item is allocated exclusively to buyer 1, since \( y = 2/3 < 1 / (2 - \tilde{\alpha}_{12}) = 1/1.1, \) \( x = 0.01 + 2/3 > y(2 - \tilde{\alpha}_{12}) / (2 - \alpha_{21}) = \frac{2}{3}/1.1, \) and \( 2x + \tilde{\alpha}_{12} y > 1 \). Then the payment is \( \tilde{m}_{1} = \tilde{\alpha}_{12} y + \)
\[ y (2 - \hat{\alpha}_{12}) / (2 - \alpha_{21}) = 0.6 + \frac{2.14}{3.16} \simeq 1.058 \text{ (since } y = \frac{2}{3} \geq (2 - \alpha_{21}) / (4 - \hat{\alpha}_{12}\alpha_{21}) = 1.6/3.64) \text{ and } \hat{m}_2 = 0. \text{ Therefore, } m_1 + m_2 > \hat{m}_1 + \hat{m}_2. \]

Also, consider an increase in both \( \alpha_{12} \) and \( \alpha_{21} \) to 0.7. An item is still allocated exclusively to buyer 1, since \( y < 1 / (2 - \hat{\alpha}_{12}) \), \( x = 0.01 + 2/3 > y = 2/3 \), and \( 2x + \hat{\alpha}_{12}y > 1 \). Then the payment is \( \hat{m}_1 = \hat{\alpha}_{12}y + y (2 - \hat{\alpha}_{12}) / (2 - \hat{\alpha}_{21}) = 1.72 \simeq 1.133 \text{ (since } y = 2/3 \geq (2 - \hat{\alpha}_{21}) / (4 - \hat{\alpha}_{12}\hat{\alpha}_{21}) = 1.3/3.51) \text{ and } \hat{m}_2 = 0. \text{ Therefore, } m_1 + m_2 > \hat{m}_1 + \hat{m}_2. \]
Appendix B

Chapter 3 Results

Proof of Theorem 21

By the Assumption 1, the condition

\[
\pi_i - \frac{1 - F^1(\pi_i)}{f^1(\pi_i)} - (n - 1)E \geq 0
\]

can be reduced to the condition

\[
\pi_i \geq \tilde{\pi}(n),
\]

where \(\tilde{\pi}(n)\) increases with \(n\). (Note that this argument may not hold in the externality dependence case for an arbitrary \(f^1(\cdot)\) function and an arbitrary number \(c\).)

As a result, we may have \(\tilde{\pi}(N_1) > \pi\) for some large enough \(N_1\). This is analogous to the no-allocation result from the uniform distribution case.

In order to study the expected profit under the no-allocation equilibrium, we consider \(K = 1\) first. The interim payment is

\[
y_i(\pi_i) = (n - 1) \int_{\alpha}^{\pi} \tau (F^2(\tau))^{n-2} f^2(\tau) d\tau = \alpha - \int_{\alpha}^{\pi} (F^2(\tau))^{n-1} d\tau.
\]
Obviously,
\[
\frac{\partial y_i(\pi_i)}{\partial n} = - \int_\alpha^\pi (F^2(\tau))^{n-1} \ln(F^2(\tau)) d\tau \geq 0,
\]
since \( F^2(\tau) \in [0, 1] \).

Since we are considering negative externalities, i.e. \( \pi \geq \alpha \geq 0 \), we have
\[
\tau(F^2(\tau))^{n-2} f^2(\tau) \geq 0
\]
for \( \alpha \leq \tau \leq \pi \). This indicates \( y_i(\pi_i) > 0 \) since \( n > 1 \).

Hence,
\[
EP = ny_i(\pi_i)
\]
is increasing in \( n \) for \( n > N_1 \).

Now we consider \( K \geq 1 \) under the no-allocation equilibrium. For \( j = 0, \ldots, K-1 \), the density of the \((n-1-j)\)th largest out of \( n-1 \) i.i.d. random variables with CDF \( F(x) \) and PDF \( f(x) \), is
\[
f_{x^{(n-1-j)}} = \frac{(n-1)!}{(n-j-2)!j!} F(x)^{n-j-2}(1-F(x))^j f(x).
\]
Therefore, the seller’s expected profit is
\[
EP = n \sum_{j=0}^{K-1} \int_\alpha^\pi \frac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau) \tau d\tau.
\]

With negative externalities, for each \( j \),
\[
h(n, j) \triangleq \int_\alpha^\pi \frac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau) \tau d\tau \geq 0.
\]
This indicates \( EP \) is increasing in \( K \) by adding more nonnegative terms.
For each \( j > 0 \), we do the integration by parts,

\[
\int_{\alpha}^{\pi} F^2(\tau)^{n-j-2}(1 - F^2(\tau))^j f^2(\tau) \tau d\tau \\
= \int_{\alpha}^{\pi} \frac{1}{n-j-1}(1 - F^2(\tau))^j \tau dF^2(\tau)^{n-j-1} \\
= \frac{1}{n-j-1}(1 - F^2(\tau))^j F^2(\tau)^{n-j-1} \tau \bigg|_{\alpha}^{\pi} \\
- \frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1} \left( (1 - F^2(\tau))^j d\tau - j (1 - F^2(\tau))^{j-1} \tau dF^2(\tau) \right) \\
= \frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1} (1 - F^2(\tau))^j d\tau \\
- \frac{1}{n-j-1} \int_{\alpha}^{\pi} (F^2(\tau))^{n-j-1} (1 - F^2(\tau))^j d\tau \\
= \frac{1}{n-j-1} \frac{1}{n-j} j(1 - F^2(\tau))^{j-1} F^2(\tau)^{n-j} \tau \bigg|_{\alpha}^{\pi} \\
- \frac{1}{n-j-1} \frac{1}{n-j}. \\
\int_{\alpha}^{\pi} j F^2(\tau)^{n-j-1} [(1 - F^2(\tau))^{j-1} d\tau - (j-1)(1 - F^2(\tau))^{j-2} \tau dF^2(\tau)] \\
- \frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau.
\]

If \( j > 1 \), the first term is zero and we can also continue the iteration of integration.
by parts,

\[
\int_{\alpha}^{\pi} F^2(\tau)^{n-j-2}(1 - F^2(\tau))^j f^2(\tau) \tau d\tau
\]

\[= 0 + \frac{1}{n-j-1} \int_{\alpha}^{\pi} (j-1) \left(1 - F^2(\tau)\right)^{j-2} F^2(\tau)^{n-j} \tau dF^2(\tau)\]

\[-\frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j} \left(1 - F^2(\tau)\right)^{j-1} d\tau\]

\[-\frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1} \left(1 - F^2(\tau)\right)^j d\tau.\]

For \( j > 0 \), we thus have the general form

\[
\int_{\alpha}^{\pi} F^2(\tau)^{n-j-2}(1 - F^2(\tau))^j f^2(\tau) \tau d\tau
\]

\[= \left(\frac{1}{n-j-1} \Pi_{l=0}^{j-1} \frac{1}{n-j+l(j-l)}\right) \frac{1}{\alpha}\]

\[-\frac{1}{n-j-1} \sum_{l=0}^{j-1} \left(\Pi_{m=0}^{l} \frac{j-m}{n-j+m}\right) \int_{\alpha}^{\pi} F^2(\tau)^{n-j+l} \left(1 - F^2(\tau)\right)^{j-l-1} d\tau\]

\[-\frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1} \left(1 - F^2(\tau)\right)^j d\tau\]

\[= \frac{(n-j-2)!}{(n-1)!} \frac{1}{j!} \frac{1}{\alpha}\]

\[-\frac{1}{n-j-1} \sum_{l=0}^{j-1} \frac{j!(n-j-1)!}{(j-l-1)! (n-j+l)!} \int_{\alpha}^{\pi} F^2(\tau)^{n-j+l} \left(1 - F^2(\tau)\right)^{j-l-1} d\tau\]

\[-\frac{1}{n-j-1} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1} \left(1 - F^2(\tau)\right)^j d\tau.\]
Therefore, for each $j > 0$

$$h(n, j) = \int_\alpha^\pi \frac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2}(1 - F^2(\tau))^j f^2(\tau) \tau \, d\tau$$

$$= -\sum_{t=0}^{j-1} \frac{(n-1)!}{(j-l-1)!(n-j+l)!} \int_\alpha^\pi F^2(\tau)^{n-j+l}(1 - F^2(\tau))^{j-l-1} \, d\tau$$

$$+ a - \frac{(n-1)!}{(n-j-1)!j!} \int_\alpha^\pi F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j \, d\tau. \quad (B.1)$$

This formulation also yields the case $j = 0$.

Since for $j > 0$

$$(n-1) - (n-j-1) = j,$$

and for $0 \leq l \leq j - 1$

$$(n-1) - (n-j+l) = j - l - 1.$$

Hence,

$$\frac{(n-1)!}{(n-j-1)!j!}$$

has finite $n$ terms in the numerator, and for each $l$

$$\frac{(n-1)!}{(j-l-1)!(n-j+l)!}$$

also has finite $n$ terms in the numerator. Therefore,

$$- \frac{(n-1)!}{(n-j-1)!j!} \int_\alpha^\pi F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j \, d\tau$$

and

$$- \sum_{t=0}^{j-1} \frac{(n-1)!}{(j-l-1)!(n-j+l)!} \int_\alpha^\pi F^2(\tau)^{n-j+l}(1 - F^2(\tau))^{j-l-1} \, d\tau$$

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both increase in \(n\) for \(n > N_2\), where \(N_2\) is a sufficiently large number. To see this, take the first one for example. The derivative is

\[
\frac{\partial}{\partial n} \left( -\frac{(n-1)!}{(n-j-1)!j!} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau \right) = -\int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau \frac{\partial}{\partial n} \left( -\frac{(n-1)!}{(n-j-1)!j!} \right)
\]

\[
-\frac{(n-1)!}{(n-j-1)!j!} \frac{\partial}{\partial n} \left( \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau \right).
\]

By Leibniz’s rule, we have

\[
-\int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau \frac{\partial}{\partial n} \left( -\frac{(n-1)!}{(n-j-1)!j!} \right)
\]

\[
-\frac{(n-1)!}{(n-j-1)!j!} \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j \ln (F^2(\tau)) d\tau
\]

\[
= \int_{\alpha}^{\pi} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j \left( -\frac{(n-1)!}{(n-j-1)!j!} \ln (F^2(\tau)) - \frac{\partial}{\partial n} \left( -\frac{(n-1)!}{(n-j-1)!j!} \right) \right) d\tau.
\]

Since \(F^2(\tau) \in [0, 1]\), the derivative is positive for sufficiently large \(n\).

As a result, for arbitrary \(K\),

\[
EP = n \sum_{j=0}^{K-1} h(n, j)
\]

is increasing in \(n\) for \(n > N \triangleq \max\{N_1, N_2\}\). \(\square\)

**An Alternative Model**

We discuss the scenario where negative externalities are imposed only if the buyer does not obtain any of the item. Hence, the interim utility function is revised as

\[
U(s_i, t_i) = q_i(s_i)t_i^1 - y_i(s_i) - (1 - q_i(s_i)) \sum_{j \neq i} \int_{\tau_i} p_j(s_i, \tau_i)t_j^2 \phi_{-i}(\tau_i)dt_{-i}.
\]
As a result, the envelope theorem gives potentials similar to the standard case, i.e.,

\[
\frac{dU(t_i, t_i)}{dt_i^1} = q_i(t_i)
\]

and

\[
\frac{dU(t_i, t_i)}{dt_i^2} = 0.
\]

The interim payment is thus

\[
y_i(t_i) = -A_i + t_i^1 q_i(t_i) - \int_{\pi_i}^{t_i^1} q_i(v, t_i^2)dv
\]

\[
- (1 - q_i(t_i)) \sum_{j \neq i} p_j(t_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i}
\]

where \(A_i\) has the same definition.

Hence, with independent externalities, the seller’s expected profit is

\[
EP = -\sum_{i=1}^{n} A + \int (\sum_{i=1}^{n} [\pi_i - \frac{1 - F^i(\pi_i)}{f^i(\pi_i)}] - (n - 1) E_i) \cdot
\]

\[
p_i(\pi_1, \ldots, \pi_n) f^1(\pi_1) \ldots f^1(\pi_n) d\pi_1 \ldots d\pi_n
\]

\[
+ \sum_{i=1}^{n} \int q_i(t_i) (\sum_{j \neq i} \int_{T_{-i}} p_j(t_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i}) f^1(t_i^1) f^2(t_i^2) dt_i
\]

where

\[
E_i = \int_{\alpha}^{\pi} \tau f_i^2(\tau)d\tau.
\]

We focus on the last term, which is

\[
Ext \triangleq \sum_{i=1}^{n} \int q_i(t_i) (\sum_{j \neq i} \int_{T_{-i}} p_j(t_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i}) f^1(t_i^1) f^2(t_i^2) dt_i
\]

\[
= \sum_{i=1}^{n} \int p_i(t_i, t_{-i}) (\sum_{j \neq i} E \int_{T_{-i}} p_j(t_i, t_{-i}) \phi_{-i}(t_{-i}) dt_{-i}) f^1(t_i^1) \phi_{-i}(t_{-i}) dt_i^1
\]

\[
= \int (\sum_{i=1}^{n} \sum_{j \neq i} p_i(t_i, t_{-i}) E \int_{T_{-i}} p_j(t_i, t_{-i}) \phi_{-i}(t_{-i}) dt_{-i}) f^1(t_i^1) \phi_{-i}(t_{-i}) dt_i^1
\]

\[
f^1(\pi_1) \ldots f^1(\pi_n) d\pi_1 \ldots d\pi_n \geq 0.
\]
Since the allocation probabilities should satisfy
\[ 0 \leq \sum_{i=1}^{n} p_i(t_i, t_{-i}) \leq K, \]
the $Ext$ must be bounded above by some finite positive number $\tilde{Ext}$.

Moreover, with the i.i.d. assumption about the distributions and from (B.1), we obtain

\[
-nA = n \sum_{j=0}^{K-1} h(n, j)
\]
\[
= n \sum_{j=0}^{K-1} (\alpha - \frac{(n-1)!}{(n-j-1)!!}) j! \int_{\alpha}^{\infty} F^2(\tau)^{n-j-1}(1 - F^2(\tau))^j d\tau
\]
\[
- \sum_{l=0}^{j-1} \frac{{(n-1)!}}{((j-l-1)!!(n-j+l)!)} \int_{\alpha}^{\infty} F^2(\tau)^{n-j+l}(1 - F^2(\tau))^{j-l-1} d\tau
\]
\[
\rightarrow \infty \text{ as } n \rightarrow \infty
\]

because the last two terms converge to zero as $n \rightarrow \infty$.

In addition, $- (n-1) E_i$ goes to $-\infty$ as $n \rightarrow \infty$. Therefore, when $n$ is sufficiently large, it is still optimal to implement the no-allocation equilibrium and the seller’s expected profit is
\[ EP^* = -nA, \]
which is positive and increasing in $n$ and $K$ as illustrated in Theorem 21.
Appendix C

Chapter 4 Results

A. Proofs of Section 3.

Proof of Equation (4.2).

We rewrite the (EPIC) as follows. By the Envelope Theorem,

\[
\frac{dU_i(\alpha_i, \alpha_{-i})}{d\alpha_i} = \left. \frac{\partial U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i})}{\partial \alpha_i} \right|_{\hat{\alpha}_i = \alpha_i} = -\sum_{j \in N^-(i)} p_j(\alpha_i, \alpha_{-i}). \quad (C.1)
\]

By (4.1), (C.1), and (EPIR), we rewrite the ex post payment as

\[
x_i(\alpha_i, \alpha_{-i}) = v p_i(\alpha_i, \alpha_{-i}) + \alpha_1 1_{N^-(i) \neq \emptyset} - \sum_{j \in N^-(i)} \alpha_j p_j(\alpha_i, \alpha_{-i})
\]

\[
+ \int_{\Delta} \sum_{j \in N^-(i)} p_j(t, \alpha_{-i}) dt. \quad (C.2)
\]
The seller’s expected revenue is

\[ \sum_{i \in N} \int x_i(\alpha_i, \alpha_{-i}) \, dF(\alpha) \]

\[ = \sum_{i \in N} \{ \alpha 1_{N^- (i) \neq \emptyset} + \int \left( v p_i(\alpha) - \sum_{j \in N^- (i)} \alpha_i p_j(\alpha) \right) \, dF(\alpha) \]

\[ + \int \int_{\alpha} \sum_{j \in N^- (i)} p_j(t, \alpha_{-i}) \, dt \, dF(\alpha) \} \]

\[ = \sum_{i \in N} \{ \alpha 1_{N^- (i) \neq \emptyset} + \int v p_i(\alpha) \, dF(\alpha) \} - \sum_{i \in N} \int \sum_{j \in N^- (i)} \alpha_i p_j(\alpha) \, dF(\alpha) \]

\[ + \sum_{i \in N} \int_{\alpha} \sum_{j \in N^- (i)} p_j(t, \alpha_{-i}) \, dF_i(\alpha_i) \, dF_{-i}(\alpha_{-i}) \]

\[ = \sum_{i \in N} \{ \alpha 1_{N^- (i) \neq \emptyset} + \int v p_i(\alpha) \, dF(\alpha) \} - \sum_{i \in N} \int \sum_{j \in N^- (i)} \alpha_i p_j(\alpha) \, dF(\alpha) \]

\[ + \sum_{i \in N} \int_{\alpha} \sum_{j \in N^- (i)} p_j(t, \alpha_{-i}) \, dF_i(\alpha_i) \, dF_{-i}(\alpha_{-i}) \]

\[ = \sum_{i \in N} \{ \alpha 1_{N^- (i) \neq \emptyset} + \int v p_i(\alpha) \, dF(\alpha) \} - \sum_{i \in N} \int \sum_{j \in N^- (i)} \alpha_i p_j(\alpha) \, dF(\alpha) \]

\[ + \sum_{i \in N} \int_{\alpha} \sum_{j \in N^- (i)} p_j(t, \alpha_{-i}) \frac{1 - F_i(t)}{f_i(t)} \, dF_i(t) \, dF_{-i}(\alpha_{-i}) \]

\[ = \sum_{i \in N} \{ \alpha 1_{N^- (i) \neq \emptyset} + \int v p_i(\alpha) \, dF(\alpha) \} \]

\[ + \sum_{i \in N} \int \left( \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)} - \alpha_i \right) \left( \sum_{j \in N^- (i)} p_j(\alpha) \right) \, dF(\alpha). \]

Note that the second equality comes from changing the order of integration.
We can further rewrite the revenue as

\[
\sum_{i \in N} \int x_i (\alpha_i, \alpha_{-i}) \, dF(\alpha)
\]

\[
= \sum_{i \in N} \left( \alpha 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} + \int v p_i (\alpha) \, dF(\alpha) \right)
\]

\[
+ \int \sum_{i \in N} \sum_{j \in \mathcal{N}^-(i)} \left( \left( \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)} - \alpha_i \right) p_j (\alpha_i) \right) \, dF(\alpha)
\]

\[
= \sum_{i \in N} \left( \alpha 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} + \int v p_i (\alpha) \, dF(\alpha) \right)
\]

\[
+ \int \sum_{j \in N} \sum_{i \in \mathcal{N}^+(j)} \left( \left( \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} - \alpha_j \right) p_i (\alpha) \right) \, dF(\alpha)
\]

\[
= \sum_{i \in N} \alpha 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} + \int \sum_{i \in N} \left( p_i (\alpha) \left( v - \sum_{j \in \mathcal{N}^+(i)} \left( \alpha_j - \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} \right) \right) \right) \, dF(\alpha).
\]

Note that the third equality comes from changing the order of summation. The fourth equality is changing the notation. ∎

**Proof of Theorem 22.**

Ignoring (Monotonicity) the seller’s revenue maximization problem decouples for each \( \alpha \in \Omega^n \). Since the first term is constant in \( p \), the solution of this relaxed problem is setting \( p_i^*(\alpha) = 1 \) for any index \( i^* \in \arg \max_{i \in N} v - \pi_{\mathcal{N}^-(i)}(\alpha) \), and \( p_j(\alpha) = 0 \) for \( j \in N \setminus \{i^*\} \). This choice satisfies (Feasibility). Next we show this also satisfies (Monotonicity).
Fix \( k \in N \) and we will show that \( \sum_{j \in N^-(k)} p_j(\alpha_k, \alpha_{-k}) \) is decreasing in \( \alpha_k \). By increasing \( \alpha_k \) we have that \( \pi_k(\alpha_k) \) is non-decreasing. Since \( j \in N^-(k) \) implies that \( k \in \mathcal{N}^+(j) \) we have that \( \pi_{\mathcal{N}^+(j)}(\alpha) \) is non-decreasing in \( \alpha_k \). Then the choice of \( p \) above yields that \( \sum_{j \in N^-(k)} p_j(\alpha_k, \alpha_{-k}) \) is decreasing in \( \alpha_k \). \( \square \)

B. Proofs of Section 4.

Proof of Theorem 23.

First rewrite the seller’s expected revenue (4.2) for a given allocation rule \( p \) and group structure \( \mathcal{G} \) as

\[
\Pi(p; \mathcal{G}) := \sum_{i \in N} \alpha \mathbb{1}_{\{\mathcal{G}(i) \neq \emptyset\}} + \int \sum_{i \in N} p_i(\alpha) v dF(\alpha) - \sum_{i \in N} \sum_{j \in \mathcal{G}(i)} \int p_i(\alpha) \pi_j(\alpha) dF(\alpha). \tag{C.3}
\]

Let \( \mathcal{G} \) denote all possible group structures and let \( \mathcal{P}(\mathcal{G}) \) denote all feasible allocation rules. The seller’s group design problem is stated as follows

\[
\max_{\mathcal{G} \in \mathcal{G}} \max_{p \in \mathcal{P}(\mathcal{G})} \Pi(p; \mathcal{G}).
\]

Next, for any \( j \in N \) define the linear functional \( H_j \) as

\[
H_j(p_i) \triangleq \int p_i(\alpha; \mathcal{G}) \pi_j(\alpha_j) dF(\alpha).
\]

Note that, since \( \pi_i \geq 0 \) (indicating \( \underline{\alpha} > 0 \)) for \( i \in N \) and \( 0 \leq p_A(\alpha) \leq 1 \) for \( A \subseteq N \),
we have \( H_j(p_i) \geq 0 \) and

\[
\sum_{i \in A} H_j(p_i) = H_j(p_A)
\]

\[
= \int p_A(\alpha) \left( \alpha_j - \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} \right) dF(\alpha)
\]

\[
\leq \int \left( \alpha_j - \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} \right) dF(\alpha)
\]

\[
= \int \alpha_j dF_j(\alpha_j) - \int (1 - F_j(\alpha_j)) d\alpha_j
\]

\[
= \int \alpha_j dF_j(\alpha_j) + \alpha - \int \alpha_j dF_j(\alpha_j) = \alpha.
\]

Because of the group structure and ex-ante symmetry, note that for \( p \in \mathcal{P}(\mathcal{G}) \), \( H_j(p_i) = \overline{H}(p_i) \) for any \( j \in \mathcal{G}(i) \). The third term of equation (C.3) can be restated as

\[
\sum_{i \in N} \sum_{j \in \mathcal{G}(i)} \int p_i(\alpha) \pi_j(\alpha_j)dF(\alpha) = \sum_{i \in N} \sum_{j \in \mathcal{G}(i)} H_j(p_i) = \sum_{i \in N} \overline{H}(p_i) d(i),
\]

where \( d(i) := |\mathcal{G}(i)| \) is the degree of buyer \( i \), and \( F_i = F_j \) for all indices. With these notations, \( \Pi(p; \mathcal{G}) \) can be restated as

\[
\sum_{i \in N} \alpha 1_{\{\mathcal{G}(i) \neq \emptyset\}} + v \int \sum_{i \in N} p_i dF(\alpha) - \sum_{i \in N} \overline{H}(p_i) d(i).
\]

Given an arbitrary group design \( \mathcal{G} \) and the associated optimal allocation rule \( p^*(\cdot; \mathcal{G}) \), recall that

\[
\Pi(p^*(\alpha; \mathcal{G}); \mathcal{G}) = \sum_{i \in N} \alpha 1_{\{d(i) \geq 1\}} + v \int \sum_{i \in N} p_i^* (\alpha; \mathcal{G}) dF(\alpha)
\]

\[
- \sum_{i \in N} \overline{H}(p_i^*(\alpha; \mathcal{G})) d(i).
\]
Let \( M \) denote an almost 2-equipartition, \( d(i) = 1 \) for all buyers with the exception of one group (let buyer \( h \) in this group) with \( d(h) = 2 \).

We will first prove that

\[
\Pi(p^* (\alpha; G) ; M) \geq \Pi(p^* (\alpha; G) ; G).
\]

Step 1. Consider the group structure with \( d(i) < 1 \) for some \( i \). Suppose there are two isolated nodes, \( i \) and \( j \). Connecting \( i \) and \( j \) yields an additional revenue of

\[
2\alpha - H(p_j) - H(p_i) \geq 0.
\]

Therefore, the seller can increase its revenue by setting the optimal network to contain at most one isolated node.

Step 2. Consider the group structure with \( d(i) > 1 \) for some \( i \). Since \( \overline{H}(p_i) \geq 0 \), the seller’s revenue can be increased by decreasing the degree of buyer \( i \) while maintaining \( d(i) \geq 1 \) for buyer \( i \). Also note that when connecting an isolated buyer with one of other buyers, the seller’s expected revenue from this isolated buyer will not decrease since \( \overline{H}(p_i) \leq \alpha \). These operations will result in the group structure of almost 2-equipartition. When \( n \) is an even number, it is possible to set \( d(i) = 1 \) for all \( i \). When \( n \) is an odd number, the group structure is set to be either size-two groups plus an isolated node or size-two groups plus a size-three group.

Hence, \( \Pi(p^* (\alpha; G) ; M) \geq \Pi(p^* (\alpha; G) ; G) \).

Note that \( p^* (\cdot; G) \) may not be in the set of \( \mathcal{P}(M) \), which includes all symmetric allocation rules among buyers in the same group of \( M \). However, with ex ante identical buyers, there are optimal allocation rules that are symmetric among buyers in the same group. Hence, there exists \( p^* (\cdot; M) \in \mathcal{P}(M) \) such that

\[
\Pi(p^* (\alpha; M) ; M) \geq \Pi(p^* (\alpha; G) ; M).
\]

Therefore, when \( n \) is an even number, 2-equipartitions are optimal. When \( n \) is an odd number, the seller should set as many size-two groups as possible until there are
three buyers left. Analogous to Step 1, isolating each of the three buyers is dominated by grouping two of the buyers and leaving the third buyer independent. Therefore, the seller faces two possible options: either fully connecting all three buyers (option 1) or grouping two of the buyers and leaving the third buyer independent (option 2). Which one is optimal depends on the parameters.

To demonstrate this, let’s consider an example of three buyers. With option 1, the seller’s optimal revenue from the three buyers \( i = 1, 2, 3 \), is

\[
3\alpha + v \int \sum_{i=1}^{3} p_i^* dF(\alpha) - 2 \sum_{i=1}^{3} H(p_i^*) ,
\]

where \( p^* \) is the optimal allocation rule under option 1. With option 2, the seller’s optimal revenue is

\[
2\alpha + v.
\]

Suppose the item value \( v \) is sufficiently large such that allocating the item is always optimal in option 1. Then the seller’s optimal revenue in option 1 is \( 3\alpha + v - 2\alpha \), which is smaller than the seller’s optimal revenue in option 2, i.e., \( 2\alpha + v \). Hence, option 2 dominates option 1.

Suppose the item value \( v \) is so small such that \( \int p_i^* dF(\alpha) = \varepsilon \) for some \( \varepsilon > 0 \). Then there exists \( \tilde{\alpha}_i \in [\underline{\alpha}, \overline{\alpha}] \) for buyer \( i \) such that the seller’s optimal revenue in option 1 is

\[
3\alpha + 3\varepsilon v - 2\varepsilon \sum_{i=1}^{3} \pi_i(\tilde{\alpha}_i) .
\]

Thus, for small enough \( \varepsilon > 0 \), option 1 dominates option 2 when

\[
v < \frac{\alpha}{1 - 3\varepsilon} - \frac{2\varepsilon \sum_{i=1}^{3} \pi_i(\tilde{\alpha}_i)}{1 - 3\varepsilon} .
\]

This condition is feasible if

\[
\frac{\alpha}{1 - 3\varepsilon} - \frac{2\varepsilon \sum_{i=1}^{3} \pi_i(\tilde{\alpha}_i)}{1 - 3\varepsilon} > 2 (\overline{\alpha} - 1/f(\overline{\alpha})).
\]
There exist parameter values such that this condition holds.

We next prove that almost 2-equipartitions are uniquely optimal if $v > \alpha - 1/f(\alpha) > 0$.

Note that in Step 1, $p^*_i + p^*_j \leq 1$. By definition of $H(p^*_i)$ and $H(p^*_j)$, at least one of them is strictly smaller than $\alpha$. Hence, connecting $i$ and $j$ yields an additional revenue of

$$2\alpha - H(p^*_i) - H(p^*_j) > 0,$$

and, the seller can strictly increase its revenue by setting the optimal network to contain at most one isolated node. Thus, we will start from the group structure with at most one isolated node and consider two cases, following the argument in Step 2 earlier in the proof.

By assumption, we have $\pi_i > 0$ for all $i$ and $v > \alpha - 1/f(\alpha)$. The latter condition indicates that buyer $i$ with $d(i) = 1$ gets the item with positive probability for any $\mathcal{G}$ with $\mathcal{G}(j) \neq \emptyset$ for all $j$.

Case 1: $n$ is an even number.

If there exists $i$ such that $\mathcal{G}(i) = \emptyset$, then $p^*_i(\alpha; \mathcal{G}) = 1$ for all $\alpha$. Note that $p^*_i(\alpha; \mathcal{G}) = 1$ for all $\alpha$ is not the optimal allocation rule under $\mathcal{M}$. Hence,

$$\Pi(p^*(\alpha; \mathcal{M}); \mathcal{M}) > \Pi(p^*(\alpha; \mathcal{G}); \mathcal{M}) \geq \Pi(p^*(\alpha; \mathcal{G}); \mathcal{G}).$$

If $\mathcal{G}(i) \neq \emptyset$ for all $i$, there are two possibilities.

(i) There exists some $i$ with $p^*_i(\alpha; \mathcal{G}) = 0$ for all $\alpha$, such $p^*(\alpha; \mathcal{G})$ is not the optimal allocation rule under $\mathcal{M}$. Thus, $\Pi(p^*(\alpha; \mathcal{M}); \mathcal{M}) > \Pi(p^*(\alpha; \mathcal{G}); \mathcal{M}) \geq \Pi(p^*(\alpha; \mathcal{G}); \mathcal{G})$.

(ii) $p^*_i(\alpha; \mathcal{G}) > 0$ for some $\alpha$ and all $i$, $H(p^*_i) > 0$ by the definition of $H$. Thus, the seller’s revenue can be strictly increased by decreasing $d(i)$ while maintaining $d(i) \geq 1$ for buyer $i$.

Hence, setting $d(i) = 1$ for all $i$ is strictly optimal.
Case 2: \( n \) is an odd number.

We focus on the externality distribution such that size-two groups plus a size-three group (also denoted by \( \mathcal{M} \)) maximize the seller’s revenue. Using argument analogous to Case 1, it can be shown that the group structure with size-two groups plus a size-three group is strictly optimal when there exists \( i \) such that \( \mathcal{G}(i) = \emptyset \).

When \( \mathcal{G}(i) \neq \emptyset \) for all \( i \), \( n = 3 \) is straightforward, and thus we consider \( n \geq 5 \). There are also two possibilities.

(i) There exists at least four buyers with \( p_i^*(\alpha; \mathcal{G}) = 0 \) for all \( \alpha \), such \( p^*(\alpha; \mathcal{G}) \) is not an optimal allocation rule under \( \mathcal{M} \). Thus,

\[
\Pi(p^*(\alpha; \mathcal{M}) ; \mathcal{M}) > \Pi(p^*(\alpha; \mathcal{G}) ; \mathcal{M}) \geq \Pi(p^*(\alpha; \mathcal{G}) ; \mathcal{G}).
\]

(ii) There exist at most three buyers with \( p_i^*(\alpha; \mathcal{G}) = 0 \) for all \( \alpha \) (and for the rest of other buyers \( p_j^*(\alpha; \mathcal{G}) > 0 \) for some \( \alpha \)), \( H(p_j^*) > 0 \). Thus, the seller’s revenue can be strictly increased by decreasing the degree of buyer \( j \) while maintaining \( d(j) \geq 1 \) for buyer \( j \).

Therefore, if \( v > \alpha - 1/f(\alpha) > 0 \), every optimal solution to problem (4.5) is an almost 2-equipartition. □

Remark: Note that if the value of the item is smaller than the smallest virtual externalities, \( v \leq \alpha - 1/f(\alpha) \), then the item is not allocated in the optimal mechanism, \( p_i^*(\alpha; \mathcal{G}) = 0 \) with \( \mathcal{G}(i) \neq \emptyset \) for all \( i \), and, thus, \( H(p_i^*) = 0 \). Hence, in this case, almost 2-equipartitions cannot make the seller strictly better off in terms of the expected revenue. In particular, when \( n \) is an even number, any group structure with \( d(i) \geq 1 \) achieves the same expected revenue as almost 2-equipartitions. When \( n \) is an odd number, an isolated node plus groups of any size (strictly greater than two) are always optimal.
Proof of Corollary 24.

Let $d(i)$ be the degree of buyer $i$ in the optimal group structure. Then by Theorem 22, the seller always allocates the item in the optimal mechanism if $v > \bar{v} \min_{i \in N} d(i)$. Note that when $n$ is an odd number, $n = 3$, and the optimal group structure is a size-3 group, the condition is $v > 2\bar{v}$. In all other cases, $v > \bar{v}$ is sufficient for the seller’s always allocating the item. □

Proof of Theorem 25.

Let $\mathcal{N}$ denote an arbitrary (undirected) graph design so that $\mathcal{N}(i) = \mathcal{N}^-(i) = \mathcal{N}^+(i)$. By Theorem 22 we have

$$
\max_p \Pi(p; \mathcal{N}) = \sum_{i=1}^n \alpha 1_{\{i \neq \emptyset\}} + \int \max_{i \in \mathcal{N}} \left(v - \pi_{\mathcal{N}(i)}(\alpha)\right)_+ dF(\alpha)
$$

First we show that the optimal structure is connected, namely

$$
\delta := \min_{i \in \mathcal{N}} |\mathcal{N}(i)| \geq 1.
$$

Suppose $d(i^*) = 0$ for some $i^*$. Let $\tilde{\mathcal{N}} = \mathcal{N} \cup \{(i^*, j^*)\}$ denote the graph with an additional edge connecting $i^*$ to $j^* \in N$. Then, since $\max_{i \in \mathcal{N}} \left(v - \pi_{\mathcal{N}(i)}(\alpha)\right)_+ = v$ by $d(i^*) = 0$, we have

$$
\max_p \Pi(p; \mathcal{N}) = \sum_{i=1}^n \alpha 1_{\{i \neq \emptyset\}} + v = \sum_{i=1}^n \alpha 1_{\{i \neq \emptyset\}} + \alpha + \int (v - \alpha) dF(\alpha)
$$

$$
= \sum_{i=1}^n \alpha 1_{\{i \neq \emptyset\}} + \int \{v - \pi_{j^*}(\alpha)\} dF(\alpha)
$$

$$
\leq \sum_{i=1}^n \alpha 1_{\{i \neq \emptyset\}} + \int \max_{i \in \mathcal{N}} \left(v - \pi_{\mathcal{N}(i)}(\alpha)\right)_+ dF(\alpha)
$$

$$
= \max_p \Pi(p; \tilde{\mathcal{N}}),
$$

where we used that $\int \pi_i(\alpha) dF(\alpha) = \alpha$. The inequality comes from the assumption that $\pi_i \geq 0$ is for all $i \in \mathcal{N}$. Thus an optimal network is connected, i.e., $\delta \geq 1$. We divide the rest of the proof in two cases.
Case 1: $n$ is an even number. Let $\mathcal{M}$ denote a matching and again by Theorem 22

$$
\max_p \Pi(p; \mathcal{M}) = n\alpha + \int \sum_{i=1}^{n} \max_{i \in N} \left( v - \pi_{\mathcal{M}(i)}(\alpha) \right) + dF(\alpha).
$$

Since $\pi_i$ is non-negative for all $i \in N$ and each realization of $\alpha \in \Omega$,

$$
\max_{i \in N} \left( v - \pi_{\mathcal{N}(i)}(\alpha) \right) + \leq \max_{i \in N} \left( v - \pi_i(\alpha_i) \right) + \leq \max_{i \in N} \left( v - \pi_{\mathcal{M}(i)}(\alpha) \right) + .
$$

The equality comes from the fact that each buyer is connected to another buyer once and only once in $\mathcal{M}$. Together with $\pi_i = \sum_{i=1}^{n} \alpha 1_{\{\mathcal{M}(i) \neq \emptyset\}} = \sum_{i=1}^{n} \alpha 1_{\{\mathcal{N}(i) \neq \emptyset\}}$, we have that $\mathcal{M}$ dominates $\mathcal{N}$, i.e.,

$$
\max_p \Pi(p; \mathcal{M}) \geq \max_p \Pi(p; \mathcal{N}).
$$

Case 2: $n$ is an odd number. Note that $n = 1$ is straightforward. Assume $n \geq 3$.

(Step 1. Reduction to union of disjoint Star Graphs.) Let $\bar{\mathcal{N}}$ be the graph obtained by removing an edge between buyer $i$ and $j \in \mathcal{N}(i)$ where $|\mathcal{N}(i)| \geq 2$ and $|\mathcal{N}(j)| \geq 2$. Since $\pi_i(\alpha_i) \geq 0$ for all $i \in N$, and $\bar{\mathcal{N}}(i) \neq \emptyset$ and $\bar{\mathcal{N}}(j) \neq \emptyset$, we have $\pi_{\bar{\mathcal{N}}(i)}(\alpha) \leq \pi_{\mathcal{N}(i)}(\alpha)$. It follows that $\max_p \Pi(p; \bar{\mathcal{N}}) \geq \max_p \Pi(p; \mathcal{N})$. Repeating this operation, the final network (denoted as $\tilde{\mathcal{N}}$) is a union of disjoint star graphs.

(Step 2. Reduction to union of disjoint edges and paths over three vertices.) Let the degree of a central buyer $i^c$ be $d(i^c)$. We next prove that, if $d(i^c) \geq 3$, it is possible to improve revenues by the following operation: For $j, k \in \bar{\mathcal{N}}(i^c)$ remove the edges $\{(j, i^c)\}$ and $\{(k, i^c)\}$, and add the edge $\{(j, k)\}$. Let $\tilde{\mathcal{N}}$ denote the newly constructed network. The seller’s optimal revenue under network $\tilde{\mathcal{N}}$ is

$$
n\alpha + \int \max \left\{ 0, v - \pi_{\bar{\mathcal{N}}(i^c)}(\alpha), v - \pi_{i^c}(\alpha_{i^c}), \max_{i \in \mathcal{N}(\bar{\mathcal{N}}(i^c) \cup \{i^c\})} v - \pi_{\bar{\mathcal{N}}(i)}(\alpha) \right\} dF(\alpha).
$$
However, the seller’s optimal revenue under network $\tilde{N}$ is

$$n\alpha + \int \max \{ 0, \ v - \pi_{\tilde{N}(i^c)}(\alpha), v - \pi_j(\alpha_j), v - \pi_k(\alpha_k), v - \pi_{i^c}(\alpha_{i^c}), \max_{i \in N \setminus (\tilde{N}(i^c) \cup \{i^c\})} v - \pi_{\tilde{N}(i)}(\alpha) \} dF(\alpha).$$

Since $\pi_i(\alpha_i) \geq 0$ for all $i \in N$ and $\tilde{N}(i^c) \subset \tilde{N}(i^c)$, $v - \pi_{\tilde{N}(i)}(\alpha) \leq v - \pi_{\tilde{N}(i)}(\alpha)$. Hence,

$$\max_p \Pi(p; N) \leq \max_p \Pi(p; \tilde{N}).$$

For a star graph with buyer $i^c$, this operation can be applied on buyers in $\tilde{N}(i^c)$ until $d(i^c) = 1$ or $d(i^c) = 2$. The above argument works for all star graphs in the network, and, thus, the network consisting of a union of disjoint edges and paths over three vertices improves the seller’s expected revenue.

(Step 3. Reduction to at most one path over three vertices.) Finally, we show that the seller’s expected revenue improves by by converting two paths over three vertices into three disjoint edges, while keeping the rest of the graph unchanged. Label buyer 1, 2, and 3 in the first path over three vertices with edges $\{(1, 2)\}$ and $\{(1, 3)\}$, and label buyer 4, 5, and 6 in the second path over three vertices with edges $\{(4, 5)\}$ and $\{(4, 6)\}$. The seller’s optimal revenue under a network consisting of two paths over three vertices, i.e., $N_{2V}$, is

$$\max_p \Pi(p; N_{2V}) = n\alpha + \int \max \{ 0, \ v - \pi_2(\alpha_2) - \pi_3(\alpha_3), v - \pi_5(\alpha_5) - \pi_6(\alpha_6), \max_{i \in N \setminus \{1, 4\}} (v - \pi_{N_{2V}(i)}(\alpha)) \} dF(\alpha).$$

Note that $v - \pi_2(\alpha_2) - \pi_3(\alpha_3) \leq \max \{ v - \pi_2(\alpha_2), v - \pi_3(\alpha_3) \}$ and $v - \pi_5(\alpha_5) - \pi_6(\alpha_6) \leq \max \{ v - \pi_5(\alpha_5), v - \pi_6(\alpha_6) \}$. If the seller converts the two paths over three vertices into three disjoint edges, the seller’s optimal revenue under this converted
network, i.e., $\mathcal{N}_{3,M}$ is

$$
\max_p \Pi(p; \mathcal{N}_{3,M}) = n_\alpha + \max \left\{ 0, \max_{i \in \{1, \ldots, 6\}} (v - \pi_i(\alpha_i)), \max_{i \in \mathcal{N} \setminus \{1, \ldots, 6\}} (v - \pi_{\mathcal{N}_V(i)}(\alpha)) \right\} dF(\alpha).
$$

Hence, $\max_p \Pi(p; \mathcal{N}_{2V}) \leq \max_p \Pi(p; \mathcal{N}_{3,M})$. Therefore, the graph with matchings and only one path over three vertices is optimal.

We next prove that almost 1-regular graphs with $\delta = 1$ are uniquely optimal if $v > \alpha - 1/f_i(\alpha) > 0$ for all $i \in N$.

Note that $\pi_i > 0$ and $v > \alpha - 1/f_i(\alpha)$ for all $i \in N$. The latter condition indicates that buyer $i$ with $d(i) = 1$ gets the item with positive probability for any $\mathcal{N}$ with $\mathcal{N}(j) \neq \emptyset$ for all $j$.

Since buyers have the same item value and the same support for externality valuation, together with the fact that $\pi_i(\alpha) = \alpha > \alpha$, there always exists some realization of $\alpha$ such that for every $i$ and all $j \neq i$

$$
v - \pi_i(\alpha_i) > \max (0, v - \pi_j(\alpha_j)).
$$

Hence, with $\pi_i > 0$ for all $i$, there exist some realization of $\alpha$ such that

$$
\max_{i \in N} (v - \pi_i(\alpha_i))_+ > \max_{i \in N} (v - \pi_{\mathcal{N}(i)}(\alpha))_+,
$$

where $\mathcal{N} \neq \mathcal{N}^*$ with $\mathcal{N}(i) \neq \emptyset$ for all $i$ and $\mathcal{N}^*$ denotes almost 1-regular graphs.

Thus, almost 1-regular graphs strictly dominate any $\mathcal{N} \neq \mathcal{N}^*$ with $\mathcal{N}(i) \neq \emptyset$ for all $i$. In particular, when $n$ is an even number, for some realization of $\alpha$,

$$
\max_{i \in \mathcal{N}} (v - \pi_{\mathcal{N}(i)}(\alpha))_+ < \max_{i \in \mathcal{N}} (v - \pi_i(\alpha))_+,
$$

and it follows that

$$
\max_p \Pi(p; \mathcal{M}) > \max_p \Pi(p; \mathcal{N}).
$$
Similarly, when \( n \) is an odd number, this strict inequality on the expected revenues holds, i.e., \( \max_p \Pi (p; \tilde{N}) > \max_p \Pi (p; N) \), \( \max_p \Pi (p; \tilde{N}) > \max_p \Pi (p; \tilde{N}) \), and \( \max_p \Pi (p; N_{3M}) > \max_p \Pi (p; N_{2V}) \).

We next prove that expected revenues for almost 1-regular graphs strictly dominate that from \( N(i) \) with \( N(i^*) = \emptyset \) for some \( i^* \). As in the first part of the proof, let \( \tilde{N} = N \cup \{(i^*, j^*)\} \) denote the graph with an additional edge from \( i^* \) to \( j^* \in N \). (Without loss of generality, we consider that \( \tilde{N}(i) \neq \emptyset \) for all \( i \).) There are two possibilities.

(i) \( v < \bar{\alpha} \). There exist some realization of \( \alpha \) such that

\[
v - \pi_{j^*}(\alpha_{j^*}) < 0 \leq \max_{i \in N} \left( v - \pi_{\tilde{N}(i)}(\alpha) \right)_+.
\]

(ii) \( v \geq \bar{\alpha} \). Here, \( v - \pi_{j^*}(\alpha_{j^*}) \geq 0 \) for any \( \alpha \). There are two subcases. When \( \tilde{N} = N^* \), there exist some realization of \( \alpha \) such that \( v - \pi_{j^*}(\alpha_{j^*}) < \max_{i \in N} \left( v - \pi_{\tilde{N}(i)}(\alpha) \right)_+ \) still holds (since buyers have the same item value and the same support for externality distribution). When \( \tilde{N} \neq N^* \), we have shown above, as in (C.5), that there exists some realization of \( \alpha \) such that \( \max_{i \in N} \left( v - \pi_{\tilde{N}(i)}(\alpha) \right)_+ < \max_{i \in N} \left( v - \pi_{N^* (i)}(\alpha) \right)_+ \).

Hence, we have

\[
\int \left\{ v - \pi_{j^*}(\alpha_{j^*}) \right\} dF(\alpha) < \int \max_{i \in N} \left( v - \pi_{N^* (i)}(\alpha) \right)_+ dF(\alpha).
\]

Therefore, if \( v > \bar{\alpha} - 1/f_i(\alpha) > 0 \) for all \( i \in N \), every optimal solution to problem (4.5) is an almost 1-regular graph with \( \delta = 1 \).

This proof approach could have also been used to establish the distribution-independent part of the optimality of group structures, i.e, for the case of even \( n \). We still include a separate proof for optimal group structures as it exploits a natural symmetry in that setting and also exposes why, in general, asymmetric structures need to be considered. \( \Box \)
Remark: Note that if $v \leq \alpha - 1/f_i(\alpha)$ for all $i$, there is no allocation on almost 1-regular graphs for any realization of $\alpha$. Hence, there exist other undirected graphs that achieve the same expected revenue as almost 1-regular graphs. In particular, any undirected graph $\mathcal{N}$ with $\mathcal{N}(i) \neq \emptyset$ is optimal.

Proof of Theorem 27.

Recall that the seller’s expected revenue under any network $\mathcal{N}$ is

$$\Pi(\mathcal{P}, \mathcal{N}) = \sum_{i=1}^{n} \alpha 1_{\{\mathcal{N}^{-}(i) \neq \emptyset\}} + \int \sum_{i=1}^{n} \left\{ p_i(\alpha) \left( v - \pi_{\mathcal{N}^{+}(i)} \right) \right\} dF(\alpha).$$

Note that since $\pi_i \geq 0$ (which indicates that $\alpha > 0$), $\max_{i \in \mathcal{N}} \left( v - \pi_{\mathcal{N}^{+}(i)} \right) \leq v$. Consider any network with the following two properties: (i) each buyer experiences externalities, i.e., $\delta^- \geq 1$, and (ii) there exists at least one buyer who does not impose any externality on any other buyers, i.e., $\delta^+ = 0$. Thus, the seller’s optimal revenue under any of these networks is

$$n\alpha + v,$$

which is an upper bound on $\Pi(\mathcal{P}, \mathcal{N})$. Therefore, the constructed network is optimal for the seller among all networks.

We next prove that for any network $\mathcal{N}$, at most $2(n-1) = O(n)$ arc additions/deletions are needed to transform $\mathcal{N}$ to an optimal network.

If $\delta^+ \geq 1$, each buyer if getting the item imposes negative externalities on at least one of other buyers. Since $\sum_{i=1}^{n} d(i)^+ = \sum_{i=1}^{n} d(i)^-$, there exists a buyer $j$ who experiences negative externalities, i.e., $d(j)^- \geq 1$. We first remove all edges starting from buyer $j$. This can be achieved by at most $n-1$ arc deletions. We then add arcs (not starting from buyer $j$) pointing to buyers who do not experience negative externalities. This can be achieved by at most $n-1$ arc additions, since $d(j)^- \geq 1$. Thus, at most $2(n-1)$ arc additions/deletions are needed.
If $\delta^+ = 0$, there exists a buyer $h$ who does not impose negative externalities on any buyer if getting the item. In order to complete the transformation to an optimal network, we add edges (not starting from buyer $h$) pointing to buyers who do not experience negative externalities. This can be achieved by at most $n$ arc additions. Since $n \geq 2$, then $2(n - 1) \geq n$.

Hence, at most $2(n - 1)$ arc additions/deletions are needed in order to achieve the optimal network.\[\square\]

C. Proofs of Section 5.

Proof of Proposition 29.

By Theorem 22, for any $\alpha \in \Omega^n$, no allocation occurs if $\min_{i \in N} \pi_{N^+(i)}(\alpha) > v$. Since $\pi_i$ is non-decreasing we have $\pi_i(\alpha_i) \geq \pi_i(\alpha) = \alpha - 1/f_i(\alpha)$. Therefore $\pi_{N^+(i)}(\alpha) \geq \delta^+ \min_{i \in N} (\alpha - 1/f_i(\alpha)) > v$ by the assumed condition.\[\square\]

Proof of Proposition 30.

No allocation is optimal in the event $\min_{i \in N} \pi_{N^+(i)}(\alpha) > v$. Recall that $(\alpha_i)_{i=1}^n$ are independent across $i$ and $E[\pi_i(\alpha_i)] = \alpha$. Thus,

$$\Pr \left( \min_{i \in N} \pi_{N^+(i)}(\alpha) \leq v \right) \leq n \max_{i \in N} \Pr \left( \pi_{N^+(i)}(\alpha) \leq v \right) = n \max_{i \in N} \Pr \left( \pi_{N^+(i)}(\alpha) - \alpha |N^+(i)| \leq v - \alpha |N^+(i)| \right) \leq n \exp(-\{\alpha \delta^+ - v\}^2/(2\pi^2)).$$

where we used that $0 \leq \pi_i(\alpha_i) \leq \alpha_i$ and Hoeffding inequality.\[\square\]

Proof of Proposition 31.

If there exists a buyer $i$ such that $N^+(i) = \emptyset$, then $\pi_{N^+(i)} = 0 < v$. Thus, it is always profitable for the seller to allocate the item to buyer $i$. This is a contradiction to the optimality of no allocation.\[\square\]
Proof of Theorem 32.

Since buyers have different externality distributions with $\alpha_i = \alpha$ for all $i$, we need $\pi_i > 0$ for all $i$ and $v > \max_{i \in N} (\alpha - 1/f_i(\alpha)) > 0$ to guarantee that almost 1-regular graphs are uniquely optimal. With these conditions and the fact that $\pi_i(\alpha) = \alpha > \alpha$, there always exists some realization of $\alpha$ such that for every $i$ and all $j \neq i$

$$v - \pi_i(\alpha_i) > \max (0, v - \pi_j(\alpha_j)).$$

The rest of the proof is similar to Theorem 25.\Box

Remark: Note that if $v \leq \max_{i \in N} (\alpha - 1/f_i(\alpha))$, there may exist other networks that achieve the same expected revenue as almost 1-regular graphs. We illustrate this with the following numerical example.

Consider four buyers with buyer $i$'s externality uniformly distributed on $[5, 5 + i]$, for $i = 1, ..., 4$. Thus, $\max_{i \in N} (\alpha - 1/f_i(\alpha)) = 4$. Buyers have the item valuation $v = 3$.

Almost 1-regular graphs are also optimal. Paths over four vertices, in particular, $1 - 4 - 3 - 2$ and $1 - 3 - 4 - 2$, achieve the same expected revenue as almost 1-regular graphs. The seller's expected revenue is 20.31.
Proof of Proposition 33.

The objective function (4.7) can be expressed as

\[
\sum_{i=1}^{n} \left[ v p_i - \sum_{j \in \mathcal{N}^- (i)} \alpha_i p_j \right] = \sum_{i=1}^{n} v p_i - \sum_{j=1}^{n} \sum_{i \in \mathcal{N}^+(j)} \alpha_i p_j
\]

\[
= \sum_{i=1}^{n} v p_i - \sum_{i=1}^{n} \sum_{j \in \mathcal{N}^+(i)} \alpha_j p_i
\]

\[
= \sum_{i=1}^{n} v p_i - \sum_{\{i: d^+(i) \geq 1\}} \sum_{j \in \mathcal{N}^+(i)} \alpha_j p_i
\]

\[
= \sum_{\{i: d^+(i) = 0\}} v p_i + \sum_{\{i: d^+(i) \geq 1\}} \left[ v - \sum_{j \in \mathcal{N}^+(i)} \alpha_j \right] p_i
\]

\[
\leq v,
\]

where the first equality is due to changing the order of summation and the second equality is due to changing the notation of \( i \) and \( j \). The third and fourth equality separate contribution of nodes with \( d^+(i) = 0 \) and those with \( d^+(i) \geq 1 \). Finally, the inequality follows from (Feasibility) and from \( \alpha_j > 0 \) for all \( j \). The equality is achieved throughout (C.7) if and only if

\[
\sum_{\{i: d^+(i) = 0\}} p_i = 1
\]

(and, consequently, \( p_i = 0 \) for all \( i \) with \( d^+(i) \geq 1 \)). Note that such allocation also satisfies (EPIC) and (EPIR) constraints. Thus, \( T(\mathcal{N}) = v \) if and only if \( \{i: d^+(i) = 0\} \) is non-empty, i.e. \( \delta^+(\mathcal{N}) = 0. \)
D. Illustrative Theoretical Examples.

Example 1: Two Cases for Optimal Groups Design with Odd $n$.

Consider $n = 3$ buyers. Each buyer’s externality $\alpha_i$ is identically, independently, and uniformly distributed on $[1.55, 3]$. If $v = 1.6$, the seller’s expected revenue by putting all three buyers in a single group is 4.8, while the revenue when leaving a single buyer out of the group is 4.7. Thus, putting all three buyers in a group dominates leaving a single buyer out. However, when $v = 2$, the seller’s expected revenue by putting all three buyers in a group is 4.9, while the one by leaving a single buyer out is 5.1. In fact, if $v \leq 1.72$, putting all three buyers in a group dominates leaving a single buyer out. Otherwise, leaving a single buyer out dominates putting all three buyers in a group. □

Example 2: Non-Identical Buyers.

Consider $n = 3$ buyers. There are four possible graph structures. We know the isolated graph and the complete graph are not optimal. Therefore, without loss of generality, it is sufficient to only consider $\mathcal{N}_{1-2,3} = (1, 2, 3, \{\{(1, 2)\}, \{3\})$ and $\mathcal{N}_{2-1-3} = (1, 2, 3, \{\{(1, 2)\}, \{(1, 3)\})$. With $\mathcal{N}_{1-2,3}$, the expected revenue for the seller is

$$\alpha_1 + \alpha_2 + v,$$

while, with $\mathcal{N}_{2-1-3}$, the expected revenue is

$$\alpha_1 + \alpha_2 + \alpha_3 + \int \max \{0, v - \pi_2 - \pi_3, v - \pi_1\} dF(\alpha).$$

Note that if $v \leq \min \{\pi_2 + \pi_3, \pi_1, \pi_3\}$ for all $\alpha$, then $\mathcal{N}_{2-1-3}$ is optimal. Similarly, if $v \geq \max \{\pi_1, \pi_2 + \pi_3\}$ for all $\alpha$ and $\alpha_1 \geq \alpha_3$, then $\mathcal{N}_{1-2,3}$ is optimal. □


Biography

Changrong Deng was born on December 16, 1984 in Guiyang, China. He joined the Fuqua School of Business, Duke University as a Ph.D. student in Decision Sciences in 2009. Prior to joining Duke, he received his B.S. in Biological Sciences from Peking University, China, his B.A. in Economics from Peking University, China, and his M.A. in Economics from Peking University, China.