Abstract. We introduce a non-partisan probability distribution on congres-
sional redistricting of North Carolina which emphasizes the equal partition
of the population and the compactness of districts. When random districts
are drawn and the results of the 2012 election were re-tabulated under the
drawn districtings, we find that an average of 7.6 democratic representatives
are elected. 95% of the randomly sampled redistrictings produced between
6 and 9 Democrats. Both of these facts are in stark contrast with the 4
Democrats elected in the 2012 elections with the same vote counts. This
brings into serious question the idea that such elections represent the “will of
the people.” It underlines the ability of redistricting to undermine the demo-
cratic process, while on the face allowing democracy to proceed.

1. Introduction

Democracy is typically equated with expressing the will of the people through
government. Perceived failures of democracy in representative governments are
typically attributed to the voice of the people being muted and obstructed by the
actions of special interests or the sheer size of government. The implication being
that the voice and will of the people exists as a clear well defined voice which only
needs to be better heard.

Yet the will of the people is not monolithic. It is not always so simple to obtain
a consensus or even a clear majority opinion. We rely on our elections as a proxy to
express our collective opinions and our political will. Of course, Arrow’s Impossibil-
ity Theorem guarantees that all electoral systems have paradoxes. However, there
is a more profound problem with this notion; the very concept of a clear majority
voice is problematic. In the United States, the federalist system, on the national
level, and district representation schemes, on the municipal level, acknowledge that
the people’s voice is geographically diverse and that we value the expression of that
diversity in our government. It is reasonable to ask how singular is the received
“will of the people” when it is filtered through geographically based districts. We
take election results to give the elected officials a mandate to act in the people’s
name. How sensitive are the election results to the choice of districts? By exten-
sion, how sensitive to the choice of redistricting is the received impression of the
“will of the people?” Our results show that the “will of the people” is not a single
election outcome but rather a distribution of possible outcomes. The exact same
vote counts can lead to drastically different outcomes depending on the choice of
districts.

With the increased insertion of politics into the congressional redistricting pro-
cess, exploring these questions in the context of house congressional districts seems
particularly important and timely. The 2010 redistricting of North Carolina is a
useful example and testing ground for this general line of inquiry. Most would agree that politics had a hand in the North Carolina redistricting process. The motivations were diverse. The twelfth district was drawn to create a majority black district. Others seemingly were drawn to split and pack different voting blocks to diminish their political power, particularly those of the democratic party. The question remains of how large was the effect of the redistricting on the outcome.

In the 2012 congressional elections, which were based on the 2010 districts, four out of the thirteen congressional seats were filled by Democrats. Yet in seeming contradiction, the majority of votes were cast for Democratic candidates on the state wide level. The election results hinged on the geographic positioning of congressional districts. While this outcome is clearly the result of politically drawn districts, perhaps it is not the result of excessive tampering. Our country has a long history of balancing the rights of urban areas with high population with those of more rural, less populated areas. Our federalist and electoral structures enshrined the idea that the one-person-one-vote ideal could be modified to support other objectives, particularly that of regionalism. It might be that in North Carolina the subversion of the results of the global vote count would happen in any redistricting which balances the representation of the urban with the rural or the beach, with the mountains, and each with the Piedmont. Maybe the vast majority of reasonable districts which one might draw would have these issues due to the geography of the population’s distribution. We are left asking the basic question: “How much does the outcome depend on the choice of districts?” This can be further refined by asking “what are the outcomes for a typical choice of districts?” or “When should a redistricting be considered outside the norm?” These last two refinements require some way of quantifying what the typical outcomes are for a given set of votes. This turns the usual election procedure on its head. We are interested in fixing the votes and then changing the redistricting and observing how the results change. Since we will explore these questions in the context of the American political system, we will assume that people vote for parties, not people, which is of course not true. However, in these polarized times it is not the worst approximation. We still find the results extremely illuminating.

Once one accepts that the expressed “people’s will” is not a single outcome but a distribution of outcomes depending on the redistricting, it expands the realm of possibility in evaluating redistrictings. The principle goal of this article is to construct an appropriate probability distribution on all redistrictings and explore its implications. Our probability distribution will be nonpartisan in that it will only make use of the distribution of the population and not any information about party affiliation, historical voting patterns, race or socioeconomic class.

We do not intend for this work to be the definitive answer in this direction. Rather, we have erred on the side of simplicity by constructing a probability distribution which only considers the compactness of districts and the degree to which the population is partitioned equally. More complicated procedures might value minority representation or traditional political boundaries more. We simply wish to show the utility of a simple probability distribution placed on the space of redistrictings to illuminate the degree to which the outcome of an election depends on the choice of districts and if a given redistricting produces representative results. Using such a distribution, one begins to obtain a feel for the degree to which the
perceived will of the people fluctuates and how typical, and perhaps “fair”, a given redistricting is.

In Section 2 we describe our main results. In Section 3 we detail the construction of a probability measure on possible redistrictings. In Section 4 we explain a number of illustrative examples designed for those not familiar with the types of Gibbs probability measures constructed in Section 3. In Section 5 we discuss the algorithm used to draw example redistrictings from our probability measure. In Section 7 we discuss how the model is calibrated to produce acceptable results. In Section 8 we make a small digression to discuss the racial make up of districts produced by our measure. Finally, in Section 10 we make some concluding remarks and discuss some future directions.

2. Summary of Results

To examine the effect of the choices of districts on North Carolina Congressional elections, we will place a probability distribution on the space of all possible redistricting plans of the state into thirteen federal Congressional districts. We will then choose a redistricting according to this probability and then rerun the North Carolina congressional elections from 2012 using the actual votes recorded in each voting tabulation district (VTD). This will produce a winner from each of the thirteen congressional districts. We will record the number of democrats (or equivalently republicans) elected and then repeat the procedure drawing a new, fresh redistricting from our probability distribution and again recording the outcome of the election with these new districts. After many such draws, we obtain a histogram showing the distribution of outcomes. It shows the sensitivity of the results to the districts chosen. The results of this procedure for our principle model are given by the histogram given in Figure 2 which shows fraction of different numbers of Democratic representatives obtained by sampling about 100 random redistrictings from our random distribution. This distribution gives a quantitative measure of the “will of the people” in a given election. Those in search of a single number might well take the mean (7.6 Democratic representatives) or the median (7 Democratic representatives). However, the entirety of the distribution gives more information. Over 50% of the samples produce either 7 or 8 Democratic representative. All of the samples produce between 6 and 9 Democratic representatives. These results should be compared with the current North Carolina house delegation which has only 4 Democratic representatives.

In light of these results, it might be reasonable to accept redistrictings which produce outcomes within one standard deviation of the mean to be truly representative while those outside to be suspect and not representative of the will of the people. Not once in our run were 4 or less Democratic seats produced. While it is possible, our results show that it is extremely unlikely that a random redistricting, chosen according to our nonpartisan probability distribution, would produce 5 or less Democratic seats if the actual vote counts from the 2012 election are used.

In building our probability measure on redistrictings, we will only include the three main legal requirements of a redistricting plan. First, the districts should be connected. Second, they should come as close as possible to having equal number of people. Lastly, they should be as compact as possible. Our procedure will tacitly include the fact that redistricting should have some relationship to historical districts and communities; however, we postpone that discussion to later. (See
We emphasize that no information about party affiliation or the vote counts from any election were used in generating the probability distribution. Furthermore, the results in Figure 2 used the actual vote tally from the 2012 congressional election.

Given the suppressing nature of the results, it is natural to ask if the districts produced are reasonable. Figures 10–10 give example redistrictings chosen randomly from the collection we produced. The law mandates that each district should have as close to \( \frac{1}{13} \) of the total population as possible. All of the districts considered in making Figure 2 had a relative deviation of less than 0.7% from the ideal of \( \frac{1}{13} \). 50% percent had a relative deviation less than 0.09%. This compares favorably with the districting currently under use which has relative deviations less than 0.7%.

Our goal is neither to provide a method for producing usable redistricting nor a definitive model for the likelihood of a given redistricting, but rather to provide a simple model which can be understood and which can shed light on the “the people’s will.” Nonetheless, our model did contain a tunable parameter, denoted \( \lambda \), which varies between zero and one and measures the relative weight given to the constraints of equal division of population versus the compactness of districts. When \( \lambda \) is close to zero, all of the weight is given to the compactness, while when \( \lambda \) is close to one, all of the weight is given to the division of population. Since the two effects are not necessarily on the same scale, \( \lambda \) equal one-half does not necessarily represent the equal balancing of the effects. One must look at the sample redistrictings to calibrate the parameters. The data quoted above and shown in Figure 2 used \( \lambda = 0.3 \) which was chosen because the districts produced with this value were overwhelmingly better than the currently used districts at splitting the population evenly between the thirteen districts and at being more compact.

We further discuss selecting a value of \( \lambda \) in Section 7. Nonetheless, in Figure 2 we summarize the results with four different values of \( \lambda \) as well as two different versions of the method used to sample the probability distribution: “Long Period” and “Short Period” methods. The “Long Period” method is preferred and was used in Figure 2 and the data quoted above. The “Short Period” method allows for more data to be collected but is less effective at drawing from the desired probability distribution.
distribution. A discussion of the issues involved is given in Section 6 and Section 7.

![Figure 2](image_url)

**Figure 2.** Plotted are the summary of the elections for four different values of the parameter $\lambda$ used in defining our probability distribution of redistrictings. Each of the four different values of $\lambda$ is under a shorter and longer heating/cooling cycles. The solid green dot is the median. The dark blue box is centered on the median to contain 50% of the points. The lighter blue box is centered to contain 90% of the data. The hollow squares give the max and min values.

What is striking about Figure 2 is that the basic conclusions are relatively insensitive to the choice of $\lambda$ or the details of how we sample the system. All of the distributions we produced have a mean around seven. All of the distributions are fairly concentrated around the mean with 50% of the values concentrated on only two values, one always being seven. It is also worth noting that all but one of the distributions never produced only four democratic representatives which was the number of democratic representatives elected in the 2012 elections. The one that did, did so in less than 5% of the samples and was the distribution possessing fewest desirable properties of all those considered. Hence a randomly chosen redistricting from any of the probability distribution considered is all but guaranteed to produce drastically different results from the 2012 elections while using the exact same votes as the 2012 election.

It is clear that the current situation of only four democratic representatives is not representative of the will of the people. The results of the election seem to have very little to do with the will of the people. It is worth recalling that our probability distribution on all possible redistrictings was agnostic relative to the different political parties. It uses no information about the number of registered democrats or republicans in a district nor its racial or socioeconomic make up. We are simply drawing, in an unbiased way, redistrictings, favoring those which are compact and those which have approximately 1/13th of the state’s population. The result left the authors wondering, “Is this democracy?”
3. A probability distribution on reasonable redistricting

We will represent the state of North Carolina as a graph \( G \) with edges \( E \) and vertices \( V \). Each vertex represents a Voting Tabulation District (VTD) and an edge between two vertices exists if the two VTDs are adjacent on the map. This graph representing the North Carolina voting landscape has over 2500 vertices and over 8000 edges.

A redistricting plan is a function from the set of vertices to the integers from one to thirteen (since North Carolina has thirteen seats in the house of representatives). More formally, recalling that \( V \) was the set of vertices, we will represent a redistricting plan by a function \( \xi : V \to \{1, 2, \ldots, 13\} \). We let \( \mathcal{R} \) denote the space of all redistricting plans. If \( \xi(v) = i \) for some \( v \in V \) then the VTD represented by vertex \( v \) is in district \( i \). Similarly for \( i \in \{1, 2, \ldots, 13\} \) and a plan \( \xi \), the \( i \)th district which we will denote by \( D_i(\xi) \) is given by \( \{v \in V : \xi(v) = i\} \). We wish to consider redistricting plans \( \xi \) such that each district \( D_i(\xi) \) is a single connected component. We will denote the collection of such redistricting plans by \( \mathcal{R}_{\text{connected}} \subset \mathcal{R} \).

As already described in Section 2, our goal is to produce a probability distribution on the space of redistrictings. To define our probability measure on the space of redistrictings, we will first assign a score to each redistricting plan \( \xi \) according to how well it satisfies our ideals, with lower scores preferred. To describe the score function on the space of redistricting plans, we need to attach to our graph \( G = (V, E) \) some data which gives relevant features of each VTD. We define the positive functions \( \text{pop}(v) \) and \( \text{area}(v) \) for a vertex \( v \in V \) as respectively the population and geographic area of the VTD associated with the vertex \( v \). We extend these functions to a collection of vertices \( A \subset V \) by

\[
\text{pop}(A) = \sum_{v \in A} \text{pop}(v) \quad \text{and} \quad \text{area}(A) = \sum_{v \in A} \text{area}(v).
\]

(1)

We will think of the boundary of a district \( D_i(\xi) \) to be the subset of the edges \( E \) which connect vertices inside of \( D_i(\xi) \) to vertices outside of \( D_i(\xi) \). We will write \( D_i(\xi) \) for the boundary of the district \( D_i(\xi) \). Since we want to include the exterior boundary of each district (the section bordering an adjacent state or the ocean), we add to \( V \) the vertex \( o \) which represents the “outside” and connect it with an edge to each vertex representing a VTD which is on the boundary of the state. We will always assume that any redistricting \( \xi \) always satisfies \( \xi(v) = 0 \) if and only \( v = o \). Since \( \xi \) always satisfies \( \xi(o) = 0 \) and hence \( o \not\in D_i(\xi) \) for \( i \geq 1 \), it does not matter that we have not defined \( \text{area}(o) \) or \( \text{pop}(o) \) as \( o \) is never included in the districts.

Given an edge \( e \in E \) which connects the two vertices \( v, \tilde{v} \in V \), we define \( \text{boundary}(e) \) to be the length of common border of the VTDs associated with the vertex \( v \) and \( \tilde{v} \). As before, we extend the definition to the boundary of a set of edges \( B \subset E \) by

\[
\text{boundary}(B) = \sum_{e \in B} \text{boundary}(e).
\]

(2)

With these preliminaries out of the way, we return to defining the score functions used to assess the goodness of a redistricting. We will construct our total score function as a convex combination of two terms: a population score \( J_{\text{pop}} \) and a

\[\text{area}(A) = \sum_{v \in A} \text{area}(v)\]
compactness score $J_{\text{compact}}$. We define the population score by

$$J_{\text{pop}}(\xi) = c_{\text{pop}} \sum_{i=1}^{13} \left( \text{pop}(D_i(\xi)) - \frac{N_{\text{pop}}}{13} \right)^2$$

where $N_{\text{pop}}$ is the total population of North Carolina, $\text{pop}(D_i(\xi))$ is the population of the district $D_i(\xi)$ as defined in (1), and $c_{\text{pop}}$ is a positive constant which is used to make the size of the two score terms comparable.

We define the compactness score $J_{\text{compact}}$ as a ratio of the sum of the perimeter to the total area of each district. This ratio, often referred to as the “isoperimetric constant” of a region, is minimized for a circle which is the most compact shape. Hence we define

$$J_{\text{compact}}(\xi) = c_{\text{compact}} \sum_{i=1}^{13} \frac{[\text{boundary}(D_i(\xi))]^2}{\text{area}(D_i(\xi))}.$$

where $D_i(\xi)$ is the set of edges which define the boundary, $\text{boundary}(D_i(\xi))$ is the length of the boundary of district $D_i(\xi)$ and $\text{area}(D_i(\xi))$ is its area. As before $c_{\text{compact}}$ is used to make the size of $J_{\text{compact}}$ and $J_{\text{pop}}$ comparable. After some experimentation, we found that $c_{\text{pop}} = 1/5000$ and $c_{\text{compact}} = 2000$ bring $J_{\text{compact}}$ and $J_{\text{pop}}$ to about the same scale. This compactness measure is one of two measures often used in the legal literature where it is referred to as the parameter score $[4, 5]$.

A second measure often used is the ratio of the area of the district to that of the smallest circle which contains the district. This second measure, usually referred to as the dispersion score, is more sensitive to overly elongated districts though the parameter score also penalizes them. We will not use the dispersion score since the spatial location of each VTD was not readily available.

Recall that $R_{\text{connected}}$ was the collection of redistricting policies in which all districts are connected. For any $\lambda \in [0, 1]$ and $\xi \in R_{\text{connected}}$, we define $J_{\lambda}$ by

$$J_{\lambda}(\xi) = \lambda J_{\text{pop}}(\xi) + (1 - \lambda) J_{\text{compact}}(\xi).$$

The parameter $\lambda$ dictates the balance between the two energies. The lower $J_{\lambda}(\xi)$ is, the more evenly distributed the population is and the more compact the districts $D_i(\xi)$ are.

So far we have defined $J_{\lambda}(\xi)$ for $\xi \in R_{\text{connected}}$. We now extended the definition by

$$J_{\lambda} = \begin{cases} 
\lambda J_{\text{pop}}(\xi) + (1 - \lambda) J_{\text{compact}}(\xi) & \xi \in R_{\text{connected}} \\
\infty & \xi \notin R_{\text{connected}} 
\end{cases}$$

We will see that if $J_{\lambda}(\xi) = \infty$, then the probability that the redistricting $\xi$ is considered will be zero.

Next for all $\beta > 0$ and $\lambda \in [0, 1]$, we define the probability measure $P_{\lambda, \beta}$ on the space of redistrictings $R$ by

$$P_{\lambda, \beta}(\xi) = \frac{e^{-\beta J_{\lambda}(\xi)}}{Z_{\lambda, \beta}}$$

where $Z_{\lambda, \beta}$ is simply the normalization constant defined so that $P_{\lambda, \beta}(R) = 1$. In other words,

$$Z_{\lambda, \beta} = \sum_{\xi \in R} e^{-\beta J_{\lambda}(\xi)}.$$
Notice that as promised if $\xi \notin R_{\text{connected}}$ then $e^{-\beta J_\lambda(\xi)} = e^{-\infty} = 0$ and we see that $\mathcal{P}_{\lambda,\beta}(R_{\text{connected}}) = 1$. Hence, all of the probability is concentrated on redistrictings which have simply connected districts. The positive constant $\beta$ is often called the “inverse temperature” in analogy with statistical mechanics and gas dynamics. When $\beta$ is very small (the high temperature regime), different elements of $R_{\text{connected}}$ have close to equal probability. As $\beta$ increases (the “temperature decreases”), the measure concentrates the probability on the redistrictings $\xi \in R$ which minimize $J_\lambda(\xi)$. After some experimentation we found that $\beta = 0.01$ produced districts with reasonably well balanced populations and compact footprints when compared to the districts currently used.

3.1. Estimating Boundary Sizes. Unfortunately, not all of the data we need to define $J_\lambda$ is readily available. The information to define $\text{pop}(v)$ and $\text{area}(v)$ is publicly available for all of the VTDs in North Carolina [2] [1], while the value for $\text{boundary}(e)$ is not. In principle it could be obtained from the map showing the North Carolina VTDs. However, we had no means to efficiently automate this task. Since each of the VTDs is relatively small and many make up a given congressional district, we opted to employ a simple approximation.

Since the area of a VTD and the number of neighboring VTDs are readily available, we approximated the length of the shared boundary between two VTDs as follows. By assuming each VTD is a circle, we expressed the circumference as a function of the area by

$$ \text{Circumference} = 2\pi \frac{3}{2} \sqrt{\text{Area}}. $$

Then if we further make the approximation that the circumference is equally shared with each neighboring VTD, we can estimate the shared boundary between two adjacent VTDs as

$$ \text{Shared Boundary} = 2\pi \frac{3}{2} \frac{\sqrt{\text{Area}}}{\text{Number of Neighbors}} $$

Since this approximation could be centered on either of the two vertices which make up the edge, we average the two answers. Denoting the degree of a vertex by $\deg(v)$ and noting that constants will not change the relative size of the term we define

$$ \text{boundary}(e) = \frac{1}{2} \left( \frac{\sqrt{\text{area}(v)}}{\deg(v)} + \frac{\sqrt{\text{area}(v')}}{\deg(v')} \right) \quad (6) $$

if $e$ is an edge connecting vertices $v$ and $v'$ with neither edge being the “outside” vertex $o$. If the edge $e$ connects an interior vertex $v$ with the outside vertex $o$, we set

$$ \text{boundary}(e) = \frac{\sqrt{\text{area}(v)}}{\deg(v)} \quad (7) $$

since the estimate centered at $o$ does not make sense.

4. Motivating and Explanatory Examples

To clarify concepts used to define a probability measure on the space of redistrictings, we now give some elementary, illustrative examples. Those familiar with the idea of a Gibbs measure can skip ahead to the next section.
To better understand the role of the inverse temperature $\beta$, we begin by constructing a series of measures on the integers $\{1, 2, \ldots, 20\}$. As in the redistrictings setting, we begin by defining a score function which we denote by $\mathcal{H}$. For any $i \in \{1, 2, \ldots, 20\}$ we define

$$\mathcal{H}(i) = (i - 5)^2 \cdot (i - 15)^2 + 1$$

and the probability measure $\mathcal{P}_\beta$ by

$$\mathcal{P}_\beta(x) = \frac{e^{-\beta \mathcal{H}(x)}}{Z_\beta}$$

where $Z_\beta = \sum_{i=1}^{20} e^{-\beta \mathcal{H}(i)}$.

Figure 3 shows a plot of $\mathcal{P}_\beta$ for $\beta$ equal to .001, .005, and .01. Since $\mathcal{H}(x)$ is smallest when $x$ is 5 or 15, the measure tends to concentrate around these values. However, the degree it does so is governed by the inverse temperature $\beta$. When $\beta$ is very small, the probability is more evenly distributed. When $\beta$ is larger, the probability is highly concentrated around the minimum of $\mathcal{H}$.

![Figure 3](image)

**Figure 3.** Plot of the probability of the point $\{1, 2, \ldots, 20\}$ for $\mathcal{P}_\beta$ with $\beta$ equal to 0.001, 0.005, and 0.01.

Now let's consider a second example to explore the roll of the relative weighting parameter $\lambda$ in (3). Again we will construct an artificial example in lower dimensions where the effects are easier to visualize. We will place a probability measure $\mathcal{P}_{\lambda, \beta}$ on $\{1, \ldots, 20\}^2$, the space of all pairs $(i, k)$ where $i, k \in \{1, \ldots, 20\}$. As before we will define

$$\mathcal{P}_{\lambda, \beta}(i, k) = \frac{e^{-\beta \mathcal{H}_{\lambda}(i, k)}}{Z_{\beta, \lambda}}$$

where the score function $\mathcal{H}_\lambda$ is defined for $\lambda \in [0, 1]$ by

$$\mathcal{H}_\lambda(i, k) = \lambda \mathcal{H}_{\text{single}}(i, k) + (1 - \lambda) \mathcal{H}_{\text{pair}}$$

with $\mathcal{H}_{\text{single}}(i, k) = \mathcal{H}(i) + \mathcal{H}(k)$ where $\mathcal{H}$ was defined in (8), $\mathcal{H}_{\text{pair}}$ is defined by $\mathcal{H}_{\text{pair}}(i, j) = (i - k)^2$, and as before the normalizing constant is given by

$$Z_{\beta, \lambda} = \sum_{i, k=1}^{20} e^{-\beta \mathcal{H}_\lambda(i, k)}.$$
Figure 4 shows the probability in a two-dimensional “heat map” for different values of $\lambda$ and $\beta = 0.001$. Red points denote relatively high probability, white points denote probability in the middle of the range, and blue points denote relatively low probabilities. The value of $\lambda$ starts high in the upper right plot and decreases clockwise to a lowest value in the upper left corner.

![Heat map images](image_url)

**Figure 4.** Heat maps of $P_{\lambda,\beta}(i,k)$ with red denoting relatively high probability and blue relatively low. All plots use $\beta = 0.001$. The values of $\lambda$ are 0.95 (upper right), 0.75 (lower right), 0.25 (lower left), and 0.05 (upper left).

Notice how as $\lambda$ shifts, the qualitative features of the measure shift. When $\lambda$ is small, the value of the score is dominated by $H_{\text{pair}}(i,k)$ which favors pairs where $i$ and $k$ are close. When $\lambda$ is large, $H_{\text{single}}(i,k)$ dominates and the most probability is placed on pairs near the points $\{(5,5), (5,15), (15,5)(15,15)\}$. For intermediate values of $\lambda$, a balance is struck between the two goals.

5. **Rerunning elections with the 2012 votes**

To any redistricting $\xi \in \mathcal{R}$ we will associate an election outcome using the actual 2012 votes in each VTD. We let $\text{votes}_D(v)$ and $\text{votes}_R(v)$ denote respectively the number of democratic and republican votes in the VTD associated with the vertex $v$ in the 2012 congressional elections as obtained from the North Carolina Board of Elections [3]. Then as before, for a set of vertices $A \subset V$ we define

\[
\text{votes}_D(A) = \sum_{v \in A} \text{votes}_D(v) \quad \text{and} \quad \text{votes}_R(A) = \sum_{v \in A} \text{votes}_R(v).
\]
Then for any redistricting $\xi$ we define
\[
\text{dem}(\xi) = \# \{ i : \text{votes}_D(D_i(\xi)) > \text{votes}_R(D_i(\xi)) \}
\]
\[
\text{repub}(\xi) = \# \{ i : \text{votes}_D(D_i(\xi)) < \text{votes}_R(D_i(\xi)) \}
\]
where $\# C$ is the number of elements in the set $C$. Hence $\text{dem}(\xi)$ and $\text{repub}(\xi)$ give the number of democratic and republican congressional seats associated to a redistricting $\xi$ if we assume that each voter votes for the party they did in the 2012 election.

We are then primarily interested in the distribution of $\text{dem}(\Xi)$ (or equivalently $\text{repub}(\Xi)$) when $\Xi$ is chosen randomly according to the probability distribution $P_{\lambda,\beta}$. This will give some understanding of the make up of the congressional delegation for typical redistricting. It is also useful to record mean of $\text{dem}(\Xi)$ which is defined by
\[
\mu_{\text{dem}} = \sum_{\xi \in \mathcal{R}} \text{dem}(\xi) \lambda_{\lambda,\beta}(\xi).
\]

6. Sampling the Probability Measure $P_{\lambda,\beta}$

There are more than $13^{2500} \approx 7.2 \times 10^{2784}$ different redistrictings in $\mathcal{R}$. Current estimates on the number of atoms in the universe range from $10^{78}$ to $10^{82}$ and the number of seconds since the big bang at the creation of the universe is estimated to be $4.3 \times 10^{17}$ seconds. While there are significantly less redistricting in $\mathcal{R}_{\text{connected}}$ (the set of simply connected redistrictings), it is certainly not practical to enumerate the redistrictings to find those with the lowest values of $J_\lambda$ and hence the largest probability.

The standard and very effective way to escape this curse of dimensionality is to use a Markov chain Monte Carlo (MCMC) algorithm to sample from the probability distribution $P_{\lambda,\beta}$. The basic idea is to define a random walk on $\mathcal{R}_{\text{connected}}$ which has $P_{\lambda,\beta}$ as its globally attracting stationary measure. We do this using the standard Metropolis-Hastings algorithm which we now briefly explain.

The Metropolis-Hastings algorithm is designed to use one Markov transition kernel $Q$ (the proposal chain) to sample from another Markov transition kernel which has a unique stationary distribution $\mu$ (the target distribution). $Q(\xi,\xi')$ gives the probability of moving from the redistricting $\xi$ to the redistricting $\xi'$ in the proposal Markov chain and is assumed to be readily computable. We wish to use $Q$ to draw a sample distributed according to $\mu$. The algorithm proceeds as follows:

1. Choose some initial state $\xi \in \mathcal{R}$.
2. Propose a new state $\xi'$ with transition probabilities given by $Q(\xi,\xi')$.
3. Accept the proposed state with probability $p = \min\left(1, \frac{\mu(\xi')Q(\xi',\xi)}{\mu(\xi)Q(\xi,\xi')}\right)$.
4. Repeat steps 2 and 3.

After an initial burn-in period, the stationary distribution of this Markov chain matches the stationary measure $\mu$. Thus, the states can be treated as samples from the desired distribution.

The stationary measure we would like to sample is $P_{\lambda,\beta}$. Our initial state is the districting that was used for the 2012 US House of Representatives election. We define the proposal chain used for proposing new redistricting in the following way:

1. Uniformly pick a conflicted edge at random. An edge, $e = (u, v)$ is a conflicted edge if $\xi(u) \neq \xi(v)$, $\xi(u) \neq 0$, $\xi(v) \neq 0$. 
(2) For chosen edge \( e = (u, v) \), with probability \( \frac{1}{2} \), either:

\[
\xi'(w) = \begin{cases} 
\xi(w) & w \neq u \\
\xi(v) & u
\end{cases}
\]

or

\[
\xi'(w) = \begin{cases} 
\xi(w) & w \neq v \\
\xi(u) & v
\end{cases}
\]

Let \( \text{con}(\xi) \) be the number of conflicted edges for districting \( \xi \). Then we have

\[
q(\xi, \xi') = \frac{1}{2 \text{con}(\xi)}
\]

The acceptance probability is given by:

\[
p = \min\left(1, \frac{\text{con}(\xi)}{\text{con}(\xi')} e^{-\beta (J_{\lambda}(\xi') - J_{\lambda}(\xi))}\right)
\]

Recall that if a districting \( \xi' \) is not connected, then \( J_{\lambda}(\xi') = \infty \). Thus, proposed redistrictings that are not connected are never accepted.

After a burn in period, every \( m \)-th districting can be taken as a sample from \( \mathcal{P}_{\lambda, \beta} \) for some \( m \). If \( m \) is long enough, the samples will be essentially independent. In our test, we used \( m = 40,000 \) and \( m = 100,000 \). The principle results quoted in Section 2 used the larger value. The fact that the results for the two values were similar lends credence to conclusion that \( m \) was taken sufficiently large.

The time the system takes to equilibrate and explore the state space depends on the inverse temperature parameter \( \beta \). The smaller the \( \beta \), the longer it takes. When \( \beta \) is large, it is harder to accept steps, so the Markov chain will get trapped in a valley where the energy \( J_{\lambda} \) has a local minimum. Alternatively, when \( \beta \) is small, the Markov chain easily explores the sample space without settling into a valley. Since we wish to heavily favor the valleys, we will use a \( \beta = 0.01 \) which we found to be a “low temperature” value for the system with most of the probability concentrated at relatively good minimizers.

To make sure that the Markov chain explores the sample space while still producing sample districtings of low score, we use a heating and cooling process. The Markov chain alternates between using a lower value of \( \beta = 0.001 \) (higher temperature) and the higher value of \( \beta = 0.01 \) (lower temperature). If samples are going to be drawn every \( m \)-th step, then the value of \( \beta \) will switch every \( \frac{m}{2} \) steps, namely \( n = 20,000 \) when \( m = 40,000 \) or \( n = 50,000 \) when \( m = 100,000 \). The sample is taken at the end of each cooling period.

Since the space of redistrictings is enormous, we are only in reality sampling from a hopefully large region around our initial condition. In this way all of our samples are related to the current configuration. The samples produced will be closer on average to the current districting than would be redistrictings chosen randomly according to \( \mathcal{P}_{\lambda, \beta} \). In this way we are tacitly honoring the requirement that the redistricting have a relation to historical districts if possible.

7. Calibrating the parameter \( \lambda \)

To calibrate the parameter \( \lambda \), we tested \( \lambda = 0.1, 0.2, 0.3, \) and \( 0.4 \). We tune \( \lambda \) so that the values of \( J_{\text{pop}} \) and \( J_{\text{compact}} \) obtained with \( \mathcal{P}_{\lambda, \beta} \) are comparable to those obtained with the current districts. We also compared the two different frequencies of cycling between \( \beta = 0.01 \) and \( \beta = 0.001 \): the “short period” of \( n = 20,000 \) and the “long period” of \( n = 50,000 \). The results are given in Figure 6. Since the results are comparable for the two choices of \( n \), we conclude that \( n = 50,000 \) is sufficiently large to obtain good samples. We would like to choose \( \lambda \) so that \( J_{\text{pop}} \) and \( J_{\text{compact}} \) are almost always below the current values. This is true for \( \lambda = .3 \) or
Figure 5. Population and Compactness scores vs 4 values of $\lambda$ each under the shorter and longer heating/cooling cycles. Solid green dot is the median. Dark blue box is centered on the median to contain 50% of the points. Lighter blue box is centered to contain 90% of the data. Hollow squares give the max and min values. Solid black horizontal lines give the value of each score for the current redistricting.

$\lambda = .4$. Since our current districting is not very compact but does a good job of evenly dividing the population, we have selected $\lambda = .3$ as our preferred value.

8. Racial Make Up

In 2010, about 21% of the state’s population was African American. In the 2010 and previous redistrictings, an effort was made to produce a district which could elect African American representatives. Two of the current districts have 50% or more African Americans. Since African Americans tend to vote Democratic, including any term in the score function to produce such a district could be perceived as partisan gerrymandering which we want to avoid in our study.

Nonetheless it is interesting to note the racial makeup of the districts created by our model and compare them to the current figures. Figure 7 gives the results for the African American and Hispanic populations. Our samples never produced a majority African American district. The two districts with largest African American representation had on average around 36% and 32% African American population which compares favorably to the state wide percentage of 22%, but not to the current districts. The Hispanic population also saw a drop in their percentage in their
most populous district from 14% in the current districts to on average 11% in the redistricting produced by our model.

It is of course possible to add additional score functions to prefer the creation of minority districts. The current results should be seen as the results if there is no intervention on the policy side to produce a particular minority district.

9. Technical Notes

There are a few technical notes that we would like to address in this section. In section 3, we define a districting as a function $\xi : V \rightarrow \{1, 2, \ldots, 13\}$. To refer to the current congressional districts, we need each vertex of our graph to belong to only one congressional district. We initially defined each vertex as a VTD. However, there are 65 VTDs which lie within two congressional districts. We split each of these VTDs to create two vertices, one for each of the two congressional districts. As before, vertices are connected when they are adjacent on the map. The congressional district boundaries are used to determine the boundary between the split VTDs, allowing us to determine which vertices are adjacent. Since population data is only available at the VTD level, we approximate the population for a split VTD as half of the population of the original VTD. Similarly, we approximate the
area of a split VTD as half of the area of the original VTD. For brevity, we refer to each vertex as a VTD, even though some of the vertices are a split VTD.

Another technical note is that there are some votes that cannot be attributed to a specific VTD. For example, absentee voting allows votes to be cast outside of an individuals home VTD. The number of such votes is negligible to those votes that can be attributed to a VTD and we simply neglect them.

A small error in the MCMC code lead to mildly nonsymmetric transition probabilities in some of the earlier data collected (in contrast to what was explained in Section 6). Some of that data is included in the analysis in this preliminary draft. New runs which will replace all of this data are under way but only partially complete. Comparison with new data generated shows no qualitative change from what is shown here. All of the nonsymmetric data will be replaced in the final version.

10. Conclusions

We have provided a prototype probability measure on the space of congressional redistrictings of North Carolina. The measure was non-partisan in that it considers no information beyond the total population and shape of the districts. The probability model was then calibrated to produce redistrictings which are comparable to the current district to the extent they partition the population equally and produce compact districts. Then effectively independent draws were made from this probability distribution using the Metropolis-Hastings variant of Markov chain Monte Carlo. For each redistricting drawn, the 2012 U.S. House of representatives election was retabulated using the actual vote counts to determine the party affiliation of the winner in each district. The statistics of the number of democratic winners give a portrait of the range of outcomes possible for the given set of votes cast. This distribution could be viewed as the true will of the people.

Redistrictings producing outcomes which are significantly different than the typical results obtained from random sampled redistrictings are arguable at odds with the will of the people expressed in the record of their votes. The fact that the election outcomes are so dependent on the choice of redistrictings demonstrates the need for checks and balances to ensure that democracy is served when redistrictings are drawn and the election outcome is representative of the votes casted.

It seems unreasonable to expect that politics would not enter into the process of redistricting. Since the legislators represent the people and presumably express their will, restricting their ability to express that will seems contrary to the very idea of democracy. This seems to be the opinion of a number of the current Supreme Court Justices. Yet the work in this note could likely be developed into a criteria to decide when a redistricting fails to be sufficiently democratic. It would perhaps be reasonable to only allow redistrictings which yield the more typical results, eschewing the most atypical as a subversion of the peoples will. This would still leave plenty of room for politics but add a counter-weight to balance that role of partisanship when it acts against the democratic ideals of a republic govened by the people.
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References


Jonathan Mattingly, Departments of Mathematics and Statistical Science, Duke University, Durham NC 27708
E-mail address: jonm@math.duke.edu

Christy Vaughn, Department of Mathematics, Duke University, Durham NC 27708
Figure 7. Current districting of NC in 2014
Figure 8. A sample redistricting from $\mathcal{P}_{\lambda,\beta}$ with $\lambda = .3$ and the long heating/cooling cycle.
Figure 9. A sample redistricting from $\mathcal{P}_{\lambda,\beta}$ with $\lambda = .3$ and the long heating/cooling cycle.
Figure 10. A sample redistricting from $\mathcal{P}_{\lambda, \beta}$ with $\lambda = .3$ and the long heating/cooling cycle.