Rational Points of Universal Curves in Positive Characteristics

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

For the moduli stack $\mathcal{M}_{g,n/\mathbb{F}_p}$ of smooth curves of type (g, n) over Spec $\mathbb{F}_p$ with the function field K, we show that if $g \geqslant 3$, then the only K-rational points of the generic curve over K are its n tautological points. Furthermore, we show that if $g \geqslant 3$ and $n = 0$, then Grothendieck's Section Conjecture holds for the generic curve over K. A primary tool used in this thesis is the theory of weighted completion developed by Richard Hain and Makoto Matsumoto.

My dissertation is dedicated to my late father, who taught me the meaning of not giving up and standing up again and again when I failed.

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Acknowledgements

First of all, I would like to thank my advisor, Richard Hain, for continuously spending time in discussing mathematics with me during his busy time with his unlimited patience, showing me his infinity enthusiasm for mathematics, and sharing his beautiful mathematical ideas. Without him guiding and encouraging me academically and personally, I would not have reached where I am today.

I also would like to thank Makoto Matsumoto for providing me essential knowledge to proceed in this project and his continuous support for my academic career.

I would like to thank Chad Schoen, Leslie Saper, Ezra Miller, William Pardon, for teaching me algebra and algebraic geometry, which has developed my understanding of the subjects significantly deeper.

I would like to thank my fellow graduate students and advanced students for years of intensive, fruitful, and fun mathematical and non-mathematical discussions in Office 025: Hangjun Xu, Kevin Kordek, Ioannis Sgouralis, Chris O'Neill, Andrew Goetz, and Benjamin Gaines.

I would like to thank Adrienne Piccono for always standing by me and giving me a lot of laughter whenever I was down.

Finally, I would like to thank my family for supporting me in many ways and encouraging me to continue my study especially last five years when my family were in a difficult time. They have given me the courage and power to stand up strong to finish what I started.

1

Introduction

Suppose that C is a geometrically irreducible smooth projective curve over a field k. Let G_k be the absolute Galois group of k. Associated to the curve C, there is a short exact sequence of algebraic fundamental groups:

$$1 \to \pi_1(C_{\bar{k}}, \bar{x}) \to \pi_1(C, \bar{x}) \to G_k \to 1,$$

where $\bar{k}$ is the separable closure of k and $C_{\bar{k}} = C \otimes_k \bar{k}$. Each k-rational point x of C induces a section s_x of $\pi_1(C, \bar{x}) \to G_k$, which is unique up to conjugation by elements of the geometric fundamental group $\pi_1(C_{\bar{k}}, \bar{x})$. Grothendieck's section conjecture states that when C is hyperbolic and k is a finitely generated infinite field, there is a bijection between the set of k-rational points and the set of conjugacy classes of sections of $\pi_1(C, \bar{x}) \to G_k$ via the association $x \mapsto [s_x]$. Hain proved in [20] that the sections conjecture holds for the restriction of the universal curve $\mathcal{C} \to \mathcal{M}_g$ to its generic point $\operatorname{Spec} k(\mathcal{M}_g)$ with $g \geqslant 5$ and char $k = 0$. In this paper, we will extend his results to positive characteristics. In order to make this paper self-contained, the majority of results needed are cited from Hain's original papers [18] and [20].

Before stating our main results, we need to introduce notations. A curve C/T of type (g,n) is a proper smooth family $C \to T$ of geometrically connected curves of genus g with distinct n sections $s_i : T \to C$. Suppose that $2g-2+n>0$. Let k be a field. Denote the moduli stack of curves of type (g,n) over $\mathrm{Spec}(k)$ by $\mathcal{M}_{g,n/k}$ and the universal curve over it by $\mathcal{C}_{g,n/k}$. Let K be the function field of $\mathcal{M}_{g,n/k}$. The generic curve of type (g,n) over K with $g \geqslant 3$ is the pullback of the universal curve $\mathcal{C}_{g,n/k}$ to the function field K. The key ingredient that allows us to use Hain's methods in positive characteristics is the comparison of algebraic fundamental groups of a certain finite étale cover of $\mathcal{M}_{g,n}$. For a prime number ℓ, there is a finite étale Galois cover $M^{\lambda}_{g,n}$ of $\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]} := \mathcal{M}_{g,n/\mathbb{Z}} \otimes \mathrm{Spec}(\mathbb{Z}[1/\ell])$ that is representable by a scheme and has a smooth compactification over $\mathbb{Z}[1/\ell]$ whose boundary is a relative normal crossing divisor over $\mathbb{Z}[1/\ell]$. Such covers were explicitly constructed by Boggi, de Jong, and Pikaart in [7], [26],and [40].

Denote the moduli stack of curves of type (g,n) over $\mathrm{Spec}(k)$ with an abelian level r by $\mathcal{M}_{g,n/k}[r]$. When the ground field k contains an rth root of unity $\mu_r(\bar{k})$, we always assume that $\mathcal{M}_{g,n/k}[r]$ is a geometrically connected, smooth stack over $\mathrm{Spec}(k)$.

Suppose that p is a prime number, ℓ is a prime number distinct from p, and m is a nonnegative integer. Let $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m] \to \mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$ be the universal curve over the stack $\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$.

Theorem 1. *Let K be the function field of $\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$. If $g \geqslant 3$, then the only K-rational points of $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$ are its n tautological points.*

The corresponding result in characteristic 0 follows from results in Teichmüller theory [12, 25] due to Hubbard, Earle and Kra. Our approach is to apply Hain's algebraic methods in positive characteristics.

Let $\mathbb{F}_q = \mathbb{F}_p[\zeta_{\ell^m}]$, where ζ_{ℓ^m} is a primitive ℓ^mth root of unity.

Theorem 2. *Let C/L be the restriction of the universal curve $\mathcal{C}_{g/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g/\mathbb{F}_q}[\ell^m]$ to the generic point* $\operatorname{Spec} L$ *of $\mathcal{M}_{g/\mathbb{F}_q}[\ell^m]$. Let $\bar{L}$ be the separable closure of L, and let $\bar{x}$ be a geometric point of $C_{\bar{L}}$. If $g \geqslant 3$, then the sequence*

$$1 \to \pi_1(C_{\bar{L}}, \bar{x}) \to \pi_1(C, \bar{x}) \to G_L \to 1$$

does not split.

Corollary 3. *The section conjecture holds for the generic curve C/L.*

The first key tool used in this paper is the theory of specialization homomorphism from [14, SGA 1, §X, XIII]. This allows us to compare the maximal pro-ℓ quotient of the fundamental groups of $\mathcal{M}^{\lambda}_{g,n/\bar{\mathbb{Q}}_p}$ and $\mathcal{M}^{\lambda}_{g,n/\bar{\mathbb{F}}_p}$ when $\ell \nmid p$. The essential tools used in Hain's original paper [20] and this paper are weighted completion and relative completion of profinite groups. The theory of weighted completion was developed by Hain and Matsumoto in [24]. For a curve C/T, let $\mathrm{GSp}(H_{\mathbb{Q}_\ell}) := \mathrm{GSp}(H^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Q}_\ell(1)))$ with ℓ a prime not in the residue characteristics $\mathrm{char}(T)$ of T. There are natural monodromy actions of $\pi_1(C, \bar{\eta})$ and $\pi_1(T, \bar{\eta})$ into $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$ with the Zariski closure R of their common images. One can take the weighted completion of $\pi_1(C, \bar{\eta}_C)$ and $\pi_1(T, \bar{\eta}_T)$ with respect to R to obtain $\mathbb{Q}_\ell$-proalgebraic groups $\mathcal{G}_C$ and $\mathcal{G}_T$. These are extensions of R by a prounipotent $\mathbb{Q}_\ell$-group. In this paper, R is equal to the whole group $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$. For the universal curve $\mathcal{C}_{g,n/k}[\ell^m] \to \mathcal{M}_{g,n/k}[\ell^m]$, the Zariski closure $\mathcal{G}^{\mathrm{geom}}_{\mathcal{M}_{g,n/\bar{k}}[\ell^m]}$ of the image in $\mathcal{G}_{\mathcal{M}_{g,n/k}[\ell^m]}(\mathbb{Q}_\ell)$ of the composite $\pi_1(\mathcal{M}_{g,n/\bar{k}}[\ell^m], \bar{\eta}) \to$
$\pi_1(\mathcal{M}_{g,n/k}[\ell^m], \bar{\eta}) \to$
$\mathcal{G}_{\mathcal{M}_{g,n/k}[\ell^m]}(\mathbb{Q}_\ell)$ is an extension of the reductive group $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$ by a prounipotent $\mathbb{Q}_\ell$-group and its Lie algebra $\mathfrak{g}^{\mathrm{geom}}_{g,n}$ is a pro-object of the category of the $\mathcal{G}_{\mathcal{M}_{g,n/k}[\ell^m]}$-modules. Each finite-dimensional $\mathcal{G}_{\mathcal{M}_{g,n/k}[\ell^m]}$-module V admits a natural weight filtration:

$$V = W_m V \supset W_{m-1} V \supset \cdots \supset W_n V$$

such that each weight graded quotient $\mathrm{Gr}^W_r V$ is a $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-module of weight r. Each natural weight filtration induced on $\mathfrak{g}^{\mathrm{geom}}_{g,n}$ satisfies the property that $\mathfrak{g}^{\mathrm{geom}}_{g,n} = W_0\mathfrak{g}^{\mathrm{geom}}_{g,n}$ and its pronilpotent radical $\mathfrak{u}^{\mathrm{geom}}_{g,n}$ is negatively weighted: $\mathfrak{u}^{\mathrm{geom}}_{g,n} = W_{-1}\mathfrak{u}^{\mathrm{geom}}_{g,n}$. Theorem 1 and 2 are proved by using the structure of the truncated Lie algebra $\mathrm{Gr}^W_\bullet(\mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3})$, which is defined in section 13.

2

Fundamental Groups

For a connected scheme X and a choice of a geometric point $\bar{\eta} : \operatorname{Spec} \Omega \to X$, we have the étale fundamental group of X denoted by $\pi_1(X, \bar{\eta})$, which is defined as the automorphism group of the fibre functor. More generally, for a Galois category $\mathcal{C}$ with a fundamental functor F, we have the fundamental group $\pi_1(\mathcal{C}, F)$ such that F is an equivalence of the category $\mathcal{C}$ and the category of finite sets on which $\pi_1(\mathcal{C}, F)$ acts continuously. When $\mathcal{C}$ is the category of finite étale covers E of X and $F = F_{\bar{\eta}} : E \mapsto E_{\bar{\eta}} := E \times_X \operatorname{Spec} \Omega$, we have $\pi_1(\mathcal{C}, F) = \pi_1(X, \bar{\eta})$. When X is a field k and $\bar{k}$ is an algebraic closure of k, we have $\pi_1(\operatorname{Spec} k, \operatorname{Spec} \bar{k}) = \operatorname{Gal}(k_{\text{sep}}/k)$, where k_{sep} is the separable closure of k in $\bar{k}$. In this paper, we will need the extension of this theory to the Deligne-Mumford stacks, which are constructed in [37].

2.1 A homotopy exact sequence

Let k be a field and fix an algebraic closure $\bar{k}$ of k. Let k_{sep} be the separable closure of k in $\bar{k}$. Suppose that X is a geometrically connected variety over k. Denote the base change to k_{sep} of X by $\bar{X}$. Let $\bar{x} : \operatorname{Spec}(\omega) \to \bar{X}$ be a geometric point of X. We may regard $\bar{x}$ as a geometric point of X by the base change morphism $\bar{X} \to X$. We have the following exact sequence of profinite groups:

$$1 \to \pi_1(\bar{X}, \bar{\eta}) \to \pi_1(X, \bar{\eta}) \to \mathrm{Gal}(k_{\mathrm{sep}}/k) \to 1$$

More generally, if X is a connected quasi-compact Deligne-Mumford stack over k, then the corresponding sequence of fundamental groups is exact [48, Cor. 6.6].

2.2 Comparison theorem

Suppose that k is a subfield of $\mathbb{C}$. Let $\bar{k}$ be the algebraic closure of k in $\mathbb{C}$. For a geometrically connected scheme X of finite type over k and a geometric point $\bar{\eta} : \operatorname{Spec} \mathbb{C} \to X$, there is a canonical isomorphism

$$\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \bar{\eta})^{\wedge} \cong \pi_1(X \otimes_k \bar{k}, \bar{\eta}),$$

where X^{an} denotes the complex analytic variety associated to X and $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \bar{\eta})^{\wedge}$ denotes the profinite completion of the topological fundamental group of X^{an} with the image of $\bar{\eta}$ as a base point. Furthermore, for a DM stack $\mathcal{X}$ over k, the corresponding analytic space denoted by $\mathcal{X}^{\mathrm{an}}$ is an orbifold (or a stack in the category of topological spaces) and we have the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{X}^{\mathrm{an}}, x)$ of $\mathcal{X}^{\mathrm{an}}$ with an appropriate base point $x \to \mathcal{X}^{\mathrm{an}}$. The above comparison theorem extends to DM stacks over k (see [38] for details): there is a canonical isomorphism

$$\pi_1^{\mathrm{orb}}(\mathcal{X}^{\mathrm{an}}, x)^{\wedge} \cong \pi_1(\mathcal{X} \otimes_k \bar{k}, x),$$

where $x : \operatorname{Spec} \mathbb{C} \to \mathcal{X}$ is a geometric point of $\mathcal{X}$.

2.3 Fundamental groups of curves

Let C be a smooth curve of genus g over an algebraically closed field k such that C is a complement of $n \geqslant 0$ closed points of its smooth compactification. Fix a geometric point $\bar{\eta}$ of C. The fundamental group of a smooth curve does not change under extensions of algebraically closed fields of characteristic zero [45, 5.6.7], and thus we may assume that k is a subfield of $\mathbb{C}$. Then by the comparison theorem the fundamental group $\pi_1(C, \bar{\eta})$ of C with base point $\bar{\eta}$ is isomorphic to the profinite completion of the group

$$\Pi_{g,n} := \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_n = 1 \rangle.$$

When char $k = p > 0$, Grothedieck proved in [14] that the maximal prime-to-p quotient [1]of $\pi_1(C, \bar{\eta})$, denoted by $\pi_1(C, \bar{\eta})^{(p')}$, is isomorphic to the maximal prime-to-p completion of the group $\Pi_{g,n}$.

2.4 Fundamental group of the generic point of a variety

Suppose that X is a smooth variety over a field k. Let $K = k(X)$ be the function field of X and $\bar{\eta} : \mathrm{Spec}(\bar{K}) \to X$ be a geometric point lying over the generic point of X. We may take this geometric point $\bar{\eta}$ as a base point for any open subvariety of X. By a divisor, we mean a finite union of closed integral subschemes of codimension one. For divisors $D \subset E$ of X defined over k, there is a canonical surjection

$$\pi_1(X - E, \bar{\eta}) \to \pi_1(X - D, \bar{\eta})$$

and thus there is a projective system of profinite groups:

$$\{\pi_1(X - D, \bar{\eta})\}_D,$$

where D is taken over the divisors of X defined over k. Fix an algebraic closure $\bar{K}$ of K. Let K_{sep} be the separable closure of K in $\bar{K}$. Then Zariski-Nagata purity [14, Theorem 3.1] implies

Proposition 2.4.1. *The canonical surjection*

$$\mathrm{Gal}(K_{\mathrm{sep}}/K) \to \varprojlim_D \pi_1(X - D, \bar{\eta})$$

[1] Here the maximal prime-to-p quotient $G^{(p')}$ of a profinite group G is the projective limit of its finite continuous quotients of order prime to p.

is an isomorphism.

Proof. The canonical surjection

$$\mathrm{Gal}(K_{\mathrm{sep}}/K) \cong \pi_1(\mathrm{Spec}(K), \mathrm{Spec}(\bar{K})) \to \pi_1(X - D, \bar{\eta})$$

for each divisor D of X induces a surjection

$$\phi : \mathrm{Gal}(K_{\mathrm{sep}}/K) \to \varprojlim_{D} \pi_1(X - D, \bar{\eta}).$$

To show that ϕ is injective, it will suffice to show that every open subgroup of $\mathrm{Gal}(K_{\mathrm{sep}}/K)$ contains the kernel of ϕ. Let H be an open subgroup of $\mathrm{Gal}(K_{\mathrm{sep}}/K)$. The corresponding fixed field of K^H in K_{sep} is a finite separable extension of K. The normalization Y of X with respect to K^H is finite over X. By Zariski-Nagata purity, Y is unramified over an open subscheme U of X this is a complement of a divisor D of X. Denote the restriction of Y to U by Y'. Then Y' is finite étale over U, and it corresponds to an open subgroup N of $\pi_1(U, \bar{\eta})$. Denote the canonical homomorphism $\mathrm{Gal}(K_{\mathrm{sep}}/K) \to \pi_1(U, \bar{\eta})$ by ϕ_U. Let W be the preimage $\phi_U^{-1}(N)$. It is an open subgroup of $\mathrm{Gal}(K_{\mathrm{sep}}/K)$, and it corresponds to a connected finite étale cover L of $\mathrm{Spec}(K)$. Pulling back Y' along the composition

$$L \to \mathrm{Spec}(K) \to U,$$

we obtain a section s of $L \times_U Y' \to L$, which follows from the fact that the image in $\pi_1(U, \bar{\eta})$ of $\pi_1(L, \bar{\eta})$ is contained in N. Since the diagram

$$\begin{array}{ccccc}
L \times_U Y' & \longrightarrow & \mathrm{Spec}(K^H) & \longrightarrow & Y' \\
{\scriptstyle s}\uparrow\downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & U
\end{array}$$

is a diagram of fiber products, we see that $W = \pi_1(L, \bar{\eta})$ is contained in H. This shows that H contains $\ker(\phi_U)$. Since $\ker(\phi) \subseteq \ker(\phi_U)$, we have shown that H contains $\ker(\phi)$. $\square$

3

Representations of $\mathrm{Sp}(H)$ and $\mathrm{GSp}(H)$

Suppose $g \geqslant 1$. Let A be a commutative ring and H_A be a free A-module of rank $2g$. Fix a nondegenerate, skew symmetric bilinear form $q : H_A \otimes H_A \to A$. For an A-algebra S, denote $H_A \otimes_A S$ by H_S. The general symplectic group $\mathrm{GSp}(H_S)$ is defined by

$$\mathrm{GSp}(H_S) = \{\phi \in \mathrm{GL}(H_S) | \phi^* q = \tau(\phi) q \quad \text{for some} \quad \tau(\phi) \in S^\times\}.$$

Associating $\tau(\phi)$ to ϕ is a surjective homomorphism $\mathrm{GSp}(H_S) \to \mathbb{G}_{m/S}$ and its kernel is the symplectic group $\mathrm{Sp}(H_S)$.

We regard $\mathrm{Sp}(H)$ and $\mathrm{GSp}(H)$ as group schemes defined over $\mathbb{Z}$ and their group of S-rational points are identified with the groups $\mathrm{Sp}(H_S)$ and $\mathrm{GSp}(H_S)$, respectively. There is an exact sequence of group schemes over $\mathbb{Z}$:

$$1 \to \mathrm{Sp}(H) \to \mathrm{GSp}(H) \xrightarrow{\tau} \mathbb{G}_m \to 1.$$

A $\mathrm{GSp}(H_A)$-module $A(n)$ is a free A-module of rank 1 with action of $\mathrm{GSp}(H_A)$ via the nth power of the homomorphism τ. Fixing an isomorphism $t : A \to A(1)$

mapping $1 \mapsto a_0$, we define a bilinear form $\theta := t \circ q$, which is a $\mathrm{GSp}(H_A)$-equivariant, nondegenerate, skew symmetric bilinear form. For a $\mathrm{GSp}(H_A)$-module V, we define $V(n)$ to be $V \otimes_A A(n)$. The dual pairing denoted by $\check{\theta}$ is the map

$$\check{\theta} : A(1) \to \Lambda^2 H_A,$$

which we view as an element of $\Lambda^2 H_A(-1)$ as well.

3.1 Weyl's construction

Here, we will briefly review the representation theory of $\mathrm{Sp}(H)$ and $\mathrm{GSp}(H)$. For example, see [13, §17]. We assume that $H = H_{\mathbb{Q}}$ with a $\mathrm{GSp}(H)$-equivariant, nondegenerate, skew symmetric, bilinear form $\theta : H^{\otimes 2} \to \mathbb{Q}(1)$. It follows from the fact that the irreducible representations of $\mathrm{Sp}(H)$ and $\mathrm{GSp}(H)$ are absolutely irreducible that for an extension $F \supset \mathbb{Q}$ of fields, the representations of $\mathrm{Sp}(H_F)$ and $\mathrm{GSp}(H_F)$ are obtained by extension of scalars from those of $\mathrm{Sp}(H)$ and $\mathrm{GSp}(H)$, respectively.

For each $n \geqslant 2$, consider the nth tensor power $H^{\otimes n}$ of H. The symmetric group S_n acts on $H^{\otimes n}$ by permuting its factors. For each $1 \leqslant i < j \leqslant n$, the ijth contraction map

$$\theta_{ij} : H^{\otimes n} \to H^{\otimes n-2}(1)$$

is defined by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n \mapsto \theta(u_i, u_j) u_1 \otimes \cdots \hat{u}_i \otimes \cdots \otimes \hat{u}_j \otimes \cdots \otimes u_n,$$

where $\hat{}$ indicates the omission of the corresponding components. The intersection of all the θ_{ij} is denoted by $H^{\langle n \rangle}$, which is both a $\mathrm{GSp}(H)$-subrepresentation and an S_n-subrepresentation of $H^{\otimes n}$. For each partition λ of n, a Young symmetrizer c_λ acts on $H^{\langle n \rangle}$ as an endomorphism and

$$H_{[\lambda]} := \mathrm{Im}\left(c_\lambda : H^{\langle n \rangle} \to H^{\langle n \rangle}\right)$$

is an irreducible GSp(H)-representation. The following result can be easily obtained from the basic results in [13].

Theorem 3.1.1. *Every irreducible representation of* GSp(H) *is isomorphic to the representation of the form* $H_{[\lambda]}(r)$*, where* $r \in \mathbb{Z}$ *and* λ *is a partition of an integer* $n \geqslant 0$ *into* $\leqslant g$ *parts. Moreover, each* $H_{[\lambda]}(r)$ *restricts to an irreducible* Sp(H)*-representation and every isomorphism class of irreducible* Sp(H)*-representations occurs in this way. Finally,* $H_{[\lambda]}(r) = H_{\langle\lambda'\rangle}(r')$ *as* Sp(H)*-representations if and only if* $\lambda = \lambda'$.

3.2 Key Sp(H) and GSp(H)-representations

Here we assume $H = H_{\mathbb{Q}_\ell}$. Define the central cocharacter

$$\omega : \mathbb{G}_m \to \mathrm{GSp}(H)$$

by mapping $z \mapsto z^{-1}\,\mathrm{id}$, which we call the standard cocharacter. Each irreducible GSp(H)-representation admits weight $\omega(V)$ as a $\mathbb{G}_m(\mathbb{Q})$-representation. In particular, H has weight -1. The composite

$$\mathbb{G}_m \xrightarrow{\omega} \mathrm{GSp}(H) \xrightarrow{\tau} \mathbb{G}_m$$

is given by $z \mapsto z^{-2}$, and thus the representation $\mathbb{Q}(r)$ has weight $-2r$. If an irreducible representation V has weight ω, then the irreducible representation $V(r)$ has weight $\omega - 2r$. As mentioned in the introduction, the proof of the main results uses a truncated graded Lie algebra whose graded quotients are GSp(H)-representations. We will introduce GSp(H)-representations appearing in the graded quotients in low degree. For a partition λ of a nonnegative integer n into $s \leqslant g$ nonnegative integers: $(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s \geqslant 0)$, denote the corresponding irreducible GSp(H)-representation $H_{[\lambda]}$ by $H_{[\lambda_1+\lambda_2+\ldots+\lambda_s]}$. The weight of $H_{[\lambda]}$ is given by $-(\lambda_1 + \cdots + \lambda_s)$.

The representations used in the proof of the main theorems are

$$H_{[0]}(1) = \mathbb{Q}_\ell(1), \quad H_{[1]} = H_{\mathbb{Q}_\ell}, \quad H_{[1^2]}, \quad \text{and} \quad H_{[1^3]}(-1).$$

We consider $H_{[1^2]}$ and $H_{[1^3]}(-1)$ as the quotient of $\Lambda^2 H$ and $\Lambda^3 H(-1)$, respectively. More explicitly, there are split exact sequences of $\mathrm{GSp}(H)$-representations:

$$0 \to H_{[1^2]} \to \Lambda^2 H \xrightarrow{\theta} \mathbb{Q}_\ell(1) \to 0,$$

and

$$0 \to H_{[1^3]}(-1) \to \Lambda^3 H(-1) \xrightarrow{\phi} H \to 0,$$

where ϕ is the twist of the map defined by

$$\phi : x \wedge y \wedge z \mapsto \theta(x,y)z + \theta(y,z)x + \theta(z,x)y.$$

It is easy to see that $\check{\theta}/g : \mathbb{Q}_\ell(1) \to \Lambda^2 H$ and $\_ \wedge \check{\theta}/(g-1) : u \mapsto \frac{u\wedge\check{\theta}}{g-1}$ are sections of θ and ϕ, respectively. For the rest of this paper, we denote $H_{[1^2]}$ by $\Lambda_0^2 H$ and $H_{[1^3]}(-1)$ by $\Lambda_0^3 H$. Also, using Hain's notation in [20], we denote $H_{[2^2]}$ by $H_{\boxplus}$.

The following computations are made by using computer program LiE and used in section 10. Kabanov's stability result [29] implies that the following decompositions are independent of g when $g \geqslant 6$.

Proposition 3.2.1 ([18, 10.2])**.** *If $g \geqslant 3$, we have:*

1.

$$\Lambda^2 H_{[1^3]} =$$

$$\begin{cases} H_{[1^6]} + H_{[1^4]} + H_{[1^2]} + H_{[2^2,1^2]} + H_{[2^2]} + H_{[0]} & : g \geqslant 6 \\ H_{[1^4]} + H_{[1^2]} + H_{[2^2,1^2]} + H_{[2^2]} + H_{[0]} & : g = 5 \\ H_{[1^2]} + H_{[2^2,1^2]} + H_{[2^2]} + H_{[0]} & : g = 4 \\ H_{[2^2]} + H_{[0]} & : g = 3 \end{cases}$$

2.

$$H_{[1]} \otimes H_{[1^3]} = \begin{cases} H_{[1^4]} + H_{[2,1^2]} + H_{[1^2]} & : g \geqslant 4 \\ H_{[2,1^2]} + H_{[1^2]} & : g = 3 \end{cases}$$

4

Monodromy Representation

4.1 Monodromy action in characteristic 0.

Suppose that T is a smooth geometrically connected variety over a field k of characteristic 0 and that $f : C \to T$ is a curve of type (g, n). Fix a geometric point $\bar{\eta} : \operatorname{Spec} \Omega \to T$ of T and denote the fiber of C over $\bar{\eta}$ by $C_{\bar{\eta}}$. For a prime number ℓ, denote $H^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1))$ by $H_{\mathbb{Z}_\ell}$. This is equipped with the cup product pairing $\theta : \Lambda^2 H_{\mathbb{Z}_\ell} \to \mathbb{Z}_\ell(1)$, which is skew symmetric and nondegenerate. The choice of a symplectic basis of $H_{\mathbb{Z}_\ell}$ gives an isomorphism $\mathrm{GSp}(H_{\mathbb{Z}_\ell}) \cong \mathrm{GSp}_g(\mathbb{Z}_\ell)$. Let $\bar{x}$ be a closed point of $C_{\bar{\eta}}$ that lies over $\bar{\eta}$.

Lemma 4.1.1. *If $g \geqslant 2$, then the homomorphism $\pi_1(C_{\bar{\eta}}, \bar{x}) \to \pi_1(C, \bar{x})$ induced by $i : C_{\bar{\eta}} \to C$ is injective.*

Proof. It is well known that there is an exact sequence of profinite groups

$$1 \to \Pi_g^\wedge \to \pi_1(\mathcal{M}_{g,1/k}, \bar{x}) \to \pi_1(\mathcal{M}_{g/k}, \bar{\eta}) \to 1.$$

The curve C is the pullback of the universal curve $\mathcal{M}_{g,1/k}$ along the morphism ϕ_f :

$T \to \mathcal{M}_{g/k}$ and we have the commutative diagram

$$\begin{array}{ccccccccc} & & \pi_1(C_{\bar\eta}, \bar x) & \longrightarrow & \pi_1(C, \bar x) & \longrightarrow & \pi_1(T, \bar\eta) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow \phi_{f*} & & \\ 1 & \to & \pi_1(C_{\bar\eta}, \bar x) & \to & \pi_1(\mathcal{M}_{g,1/k}, \bar x) & \to & \pi_1(\mathcal{M}_{g/k}, \bar\eta) & \to & 1, \end{array}$$

whose rows are exact. Therefore, the homomorphism $\pi_1(C_{\bar\eta}, \bar x) \to \pi_1(C, \bar x)$ is injective. □

Lemma 4.1.1 gives the exact sequence of algebraic fundamental groups

$$1 \to \pi_1(C_{\bar\eta}, \bar x) \to \pi_1(C, \bar x) \to \pi_1(T, \bar\eta) \to 1.$$

Thus the conjugation action of $\pi_1(C, \bar x)$ on $\pi_1(C_{\bar\eta}, \bar x)$ induces a natural monodromy representation

$$\rho_{\bar\eta} : \pi_1(T, \bar\eta) \to \mathrm{GSp}(H_{\mathbb{Z}_\ell})$$

such that the diagram

$$\begin{array}{ccc} \pi_1(T, \bar\eta) & \xrightarrow{\rho_{\bar\eta}} & \mathrm{GSp}(H_{\mathbb{Z}_\ell}) \\ \downarrow & & \downarrow \tau \\ G_k & \xrightarrow{\chi_\ell} & \mathbb{G}_m(\mathbb{Z}_\ell) \end{array}$$

commutes, where the left-hand vertical map is the canonical projection, the right-hand vertical map τ is the natural surjection, and where χ_ℓ is the ℓ-adic cyclotomic character.

Remark 4.1.2. Denote the smooth $\mathbb{Z}_\ell$-sheaf $R^1 f_* \mathbb{Z}_\ell(1)$ over T by $\mathbb{H}_{\mathbb{Z}_\ell}$. For a geometric point $\bar\eta$ of T, the monodromy action of $\pi_1(T, \bar\eta)$ on the stalk $H^1_{\text{ét}}(C_{\bar\eta}, \mathbb{Z}_\ell(1))$ of $\mathbb{H}_{\mathbb{Z}_\ell}$ at $\bar\eta$ coincides with $\rho_{\bar\eta}$.

4.2 Monodromy action in characteristic p.

Suppose that S is a connected scheme, and that $f : X \to S$ is a proper smooth morphism of schemes whose fibers are geometrically connected. Let $\bar{s} : \operatorname{Spec}\Omega \to S$ be a geometric point of S and $\bar{x}$ be a geometric point of the fiber $X_{\bar{s}}$ of X with a value in Ω. Let char (S) be the set of residue characteristics of S and let$\mathbb{L}$ be the set of prime numbers not in char(S). The following results are from [14, SGA 1, Exposé XIII, 4.3, 4.4]. Let K be the kernel of the canonical homomorphism $\pi_1(X,\bar{x}) \to \pi_1(S,\bar{s})$ and N be the kernel of the projection $K \to K^{\mathbb{L}}$ where $K^{\mathbb{L}}$ is the maximal pro-$\mathbb{L}$ quotient of K. Then N is a distinguished subgroup of $\pi_1(X,\bar{x})$ and we denote by $\pi_1'(X,\bar{x})$ the quotient of $\pi_1(X,\bar{x})$ by N. Also we denote by $\pi_1^{\mathbb{L}}(X_{\bar{s}},\bar{x})$ the maximal pro-$\mathbb{L}$ quotient of $\pi_1(X_{\bar{s}},\bar{x})$. In general, the sequence

$$\pi_1^{\mathbb{L}}(X_{\bar{s}},\bar{x}) \to \pi_1'(X,\bar{x}) \to \pi_1(S,\bar{s}) \to 1$$

is exact, but if the morphism $f : X \to S$ admits a section, it becomes also left exact:

$$1 \to \pi_1^{\mathbb{L}}(X_{\bar{s}},\bar{x}) \to \pi_1'(X,\bar{x}) \to \pi_1(S,\bar{s}) \to 1.$$

In this case, we obtain a monodromy action

$$\rho_{\bar{s}} : \pi_1(S,\bar{s}) \to \operatorname{Out}(\pi_1^{\mathbb{L}}(X_{\bar{s}},\bar{x})).$$

For the case where $f : X \to S$ has no sections, we have the following result provided that S is locally noetherian.

Proposition 4.2.1. *Suppose that S is a locally noetherian connected scheme, and that $f : X \to S$ is a proper smooth morphism with geometrically connected fibers. If $\bar{s} : \operatorname{Spec}\Omega \to S$ is a geometric point of S, and $\bar{x}$ a geometric point of the geometric fiber $X_{\bar{s}}$, then the sequence*

$$1 \to \pi_1^{\mathbb{L}}(X_{\bar{s}},\bar{x}) \to \pi_1'(X,\bar{x}) \to \pi_1(S,\bar{s}) \to 1$$

is exact.

Proof. First we note that the sequence

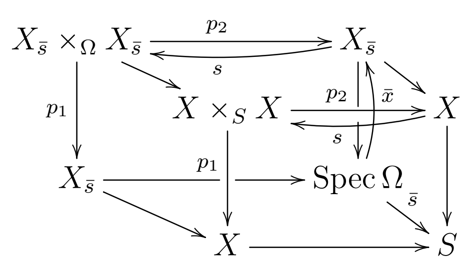
$$\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 1$$

is exact [14, SGA 1, Exposé X], so that $\pi_1(X_{\bar{s}}, \bar{x})$ maps onto the kernel K of the canonical projection $\pi_1(X, \bar{x}) \to \pi_1(S, \bar{s})$. There is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N' & \longrightarrow & \pi_1(X_{\bar{s}}, \bar{x}) & \longrightarrow & \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 1 & \longrightarrow & N & \longrightarrow & K & \longrightarrow & K^{\mathbb{L}} & \longrightarrow & 1, \end{array}$$

where the middle and right vertical maps are surjective, and N' is the kernel of the projection $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x})$. Since the middle map ϕ is surjective, we see that $\mathrm{Ker}(\phi'')$ maps onto $\mathrm{Coker}(\phi')$. Consequently, $\mathrm{Coker}(\phi')$ is a pro-$\mathbb{L}$ group. Thus, if $\mathrm{Coker}(\phi')$ is nontrivial, then N will admit a nontrivial finite $\mathbb{L}$-quotient, contradicting the maximality of $K^{\mathbb{L}}$. Hence ϕ' is surjective.

We claim now that the restriction to $\mathrm{Ker}(\phi)$ of the projection map $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x})$ is trivial. Consider the fiber product diagram



where p_1, p_2 denote the 1st and 2nd projections, respectively, and s is the diagonal section. This diagram induces the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) & \longrightarrow & \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) \times H & \longrightarrow & H & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) & \to & \pi_1'(X_{\bar{s}} \times_\Omega X_{\bar{s}}, s(\bar{x})) & \underset{\to}{\overset{s_*}{\leftarrow}} & \pi_1(X_{\bar{s}}, \bar{x}) & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) & \longrightarrow & \pi_1'(X \times_S X, s(\bar{x})) & \underset{\longrightarrow}{\overset{s_*}{\leftarrow}} & \pi_1(X, \bar{x}) & \to & 1 \end{array}$$

where H is the kernel of the canonical map $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x})$ and the first row is obtained by pulling back the middle exact sequence along the inclusion $H \to \pi_1(X_{\bar{s}}, \bar{x})$. The bottom two rows are exact by [14, SGA 1, Exposé XIII, 4.3, 4.4], and hence the right two squares are pullback squares. Note that H is equal to $\mathrm{Ker}(\phi)$. Denote also by s_* the map $H \to \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x}) \times H$ induced by the section s. By the commutativity of the diagram, $s_*(h) = (1, h)$ for all $h \in H$. Thus the composition

$$H \to \pi_1(X_{\bar{s}}, \bar{x}) \xrightarrow{s_*} \pi_1'(X_{\bar{s}} \times_{\Omega} X_{\bar{s}}, s(\bar{x})) \xrightarrow{p_{1*}} \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x})$$

is trivial. Since $p_{1*} \circ s_* : \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x})$ is equal to the canonical projection $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^{\mathbb{L}}(X_{\bar{s}}, \bar{x})$, our claim holds. Therefore, $\mathrm{Ker}(\phi'')$ is trivial by the Snake Lemma, and hence ϕ'' is an isomorphism. □

Suppose that T is a locally noetherian connected scheme, and that $C \to T$ is a curve. Fix a prime number ℓ different from $\mathrm{char}(T)$. Denote the maximal pro-ℓ quotient of $\pi_1(C_{\bar{\eta}}, \bar{x})$ by $\pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)}$. Then we have the exact sequence

$$1 \to \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} \to \pi_1'(C, \bar{x}) \to \pi_1(T, \bar{\eta}) \to 1,$$

from which we obtain a natural monodromy action of $\pi_1(T, \bar{\eta})$ on $\mathrm{Hom}(\pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)}, \mathbb{Z}_\ell(1)) \cong H^1_{et}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1))$. Denote $H^1_{et}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1))$ by $H_{\mathbb{Z}_\ell}$. The action of $\pi_1(T, \bar{\eta})$ respects the Weil pairing $\theta : \Lambda^2 H_{\mathbb{Z}_\ell} \to \mathbb{Z}_\ell(1)$. Hence we obtain a representation

$$\rho_{\bar{\eta}} : \pi_1(T, \bar{\eta}) \to \mathrm{GSp}(H_{\mathbb{Z}_\ell}).$$

In particular, when T is defined over a field k, we have the commutative diagram

$$\begin{array}{ccc} \pi_1(T, \bar{\eta}) & \xrightarrow{\rho_{\bar{\eta}}} & \mathrm{GSp}(H_{\mathbb{Z}_\ell}) \\ \downarrow & & \downarrow \tau \\ G_k & \xrightarrow{\chi_\ell} & \mathbb{G}_m(\mathbb{Z}_\ell) \end{array}$$

where the left-hand vertical map is the canonical projection, the right-hand vertical map τ is the natural surjection, and where χ_ℓ is the ℓ-adic cyclotomic character.

5

Moduli of Curves with A Teichmüller Level Structure

Suppose that C/T is a curve of type (g, n). Let $\mathbb{L}$ is the set of prime numbers distinct from char(T). Associated to the curve C/T, there exists a pro-object $\pi_1^{\mathbb{L}}(C'/T)$ of the category of locally constant sheaves of finite groups of order divisible by primes in $\mathbb{L}$, where C'/T is the curve obtained by removing the sections $s_1, \ldots, s_n$, see [11, §5]. This pro-object $\pi_1^{\mathbb{L}}(C'/T)$ is a locally constant étale sheaf over T such that each stalk $\pi_1^{\mathbb{L}}(C'/T)_{\bar{\eta}}$ is isomorphic to the maximal pro-$\mathbb{L}$ quotient of the fundamental group of the curve $C_{\bar{\eta}} - \{s_1(\bar{\eta}), \ldots, s_n(\bar{\eta})\}$. For a group G whose order is divisible by primes in $\mathbb{L}$, the sheaf of exterior homomorphisms

$$\mathcal{H}\mathrm{om}^{\mathrm{ext}}(\pi_1^{\mathbb{L}}(C'/T), G)$$

is defined to be the quotient of the locally constant sheaf

$$\mathcal{H}\mathrm{om}(\pi_1^{\mathbb{L}}(C'/T), G)$$

by conjugation action of the sheaf $\pi_1^{\mathbb{L}}(C'/T)$ on it. Then [11, 5.6] a Teichmüller structure α of level G on the curve C/T is a surjective exterior homomorphism

$$\alpha \in \Gamma(T, \mathcal{H}\mathrm{om}(\pi_1^{\mathbb{L}}(C'/T), G)).$$

5.1 Moduli stacks of curves with a non-abelian level structure

Suppose that $2g-2+n>0$. Denote the Deligne-Mumford compactification [11] of $\mathcal{M}_{g,n/\mathbb{Z}}$ by $\overline{\mathcal{M}}_{g,n/\mathbb{Z}}$. Fix a prime number ℓ. Finite étale coverings of $\mathcal{M}_{g,n}$ that are representable by a scheme and have a compactification that is smooth over $\operatorname{Spec}\mathbb{Z}[1/\ell]$ are essential to our comparison between characteristic zero and positive characteristic. The existence of such coverings was established by

1. de Jong and Pikaart for $n=0$ and all ℓ in [26],
2. Boggi and Pikaart for $n>0$ and odd ℓ in [7], and
3. Pikaart for $n>0$ and $\ell=2$ in [40].

Their results needed in this paper are summarized in the following statement:

Proposition 5.1.1. *For all prime numbers ℓ and all (g,n) satisfying $2g-2+n>0$, there is a finite étale Galois covering $M\to\mathcal{M}_{g,n}[1/\ell]:=\mathcal{M}_{g,n/\mathbb{Z}}\otimes\mathbb{Z}[1/\ell]$ over $\mathbb{Z}[1/\ell]$ that satisfies:*

1. *M is a separated scheme of finite type over $\mathbb{Z}[1/\ell]$;*
2. *the normalization $\overline{M}$ of $\overline{\mathcal{M}}_{g,n}[1/\ell]$ with respect to M is proper and smooth over $\mathbb{Z}[1/\ell]$;*
3. *the boundary $\overline{M}\backslash M$ is a relative normal crossing divisor over $\mathbb{Z}[1/\ell]$.*

In fact, M was taken to be the DM stack ${}_G\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$ of curves of type (g,n) with a Teichmüller structure of level G, where G was specifically taken to be:

1. the quotient of $\Pi_{g,0}$ by the normal subgroup generated by the third term of its lower central subgroup and all ℓ^mth powers when ℓ is odd and $n=0$;

2. the quotient of $\Pi_{g,0}$ by the normal subgroup generated by the fourth term of its lower central subgroup and all fourth powers when $\ell = 2$ and $n = 0$;

3. the quotient $\Pi_{g,n}/W^3\Pi_{g,n} \cdot \Pi_{g,n}^{\ell m}$, where W^3 denotes the third term of the weight filtration of $\Pi_{g,n}$ defined in [7] when ℓ is odd and $n > 0$;

4. the quotient $\Pi_{g,n}/W^4\Pi_{g,n} \cdot \Pi_{g,n}^{4}$, where W^4 denotes the third term of the weight filtration of $\Pi_{g,n}$ defined in [7] when $\ell = 2$ and $n > 0$,

where $\Pi_{g,n}^k$ is the subgroup of $\Pi_{g,n}$ generated by all kth powers. In [11], G is a finite quotient of $\Pi_{g,n}$ by a characteristic subgroup, but the same construction can be done when G is a finite quotient of $\Pi_{g,n}$ by an invariant subgroup, see §5.4. For $n \geqslant 2$, the subgroups $W^\bullet\Pi_{g,n} \cdot \Pi_{g,n}^k$ are not characteristic, but are invariant. For fixed prime numbers p and $\ell \neq p$, denote by $M_{g,n}^\lambda$ or simply M^λ the finite étale cover M of $\mathcal{M}_{g,n}[1/\ell]$ given by the above proposition.

5.2 Moduli stacks of curves with an abelian level

When G is a finite quotient by the subgroup $W^2\Pi_{g,n}\cdot\Pi_{g,n}^m$, we have $G \cong H_1(\Sigma_g, \mathbb{Z}/m\mathbb{Z})$, where Σ_g is a closed oriented genus g surface. In this case, we denote the moduli stack of n-pointed smooth projective curves with the Teichmüller structure of level $H_1(\Sigma_g, \mathbb{Z}/m/\mathbb{Z})$ by $\mathcal{M}_{g,n}[m]$. The stack $\mathcal{M}_{g,n}[m]$ is representable by a scheme for $m \geqslant 3$ (See [3, Chapter XVI, Theorem 2.11]. It is well known that the Deligne-Mumford compactification $\overline{M_{g,n}[m]}$ is never smooth if $g > 2$.

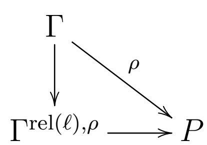
5.3 Relative Pro-ℓ Completion

The pro-ℓ completion of a group Γ with

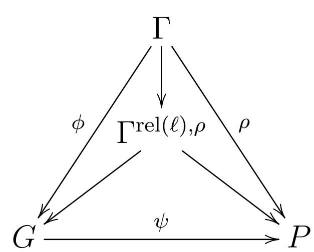
$H_1(\Gamma) \otimes_{\mathbb{Z}} \mathbb{F}_\ell = 0$ is trivial. Thus the pro-ℓ completions of the mapping class groups in genus at least 3 are trivial. On the other hand, their relative pro-ℓ completions are large enough to give us the information of their structure. Here we recall from [23] the definition of and some basic facts about relative pro-ℓ completion of a group. Suppose that:

1. Γ is a discrete group or profinite group;
2. P is a profinite group;
3. $\rho : \Gamma \to P$ is a continuous dense homomorphism.

Definition 5.3.1. The *relative pro-ℓ completion* of Γ with respect to ρ consists of a profinite group $\Gamma^{\mathrm{rel}(\ell),\rho}$ and the natural homomorphisms $\Gamma \to \Gamma^{\mathrm{rel}(\ell),\rho}$ and $\Gamma^{\mathrm{rel}(\ell),\rho} \to P$ that make the diagram



commute. It is characterized by the following universal mapping property: If G is a profinite group, $\psi : G \to P$ a continuous homomorphism with pro-ℓ kernel, and if $\phi : \Gamma \to G$ is a continuous homomorphism whose composition with ψ is ρ, then there is a unique continuous homomorphism $\Gamma^{\mathrm{rel}(\ell),\rho} \to G$ that makes the following diagram commute:



When the context is clear, we will omit ρ from the notation and denote $\Gamma^{\mathrm{rel}(\ell),\rho}$ by $\Gamma^{\mathrm{rel}(\ell)}$.

To construct the relative pro-ℓ completion of Γ with respect to ρ, consider all the commutative diagrams of the form

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho} & P \\ & {\scriptstyle\phi}\searrow \quad \nearrow{\scriptstyle\rho_\phi} & \\ & G_\phi & \end{array}$$

where G_ϕ is a profinite group, ϕ is a continuous dense homomorphism, and ρ_ϕ is continuous with $\ker \rho_\phi$ being a pro-ℓ group. Then the relative pro-ℓ completion $\Gamma^{\mathrm{rel}(\ell),\rho}$ of Γ with respect to ρ is canonically isomorphic to $\varprojlim G_\phi$, where the limit is taken over all the commutative diagrams of the form above. It is easy to see that relative pro-ℓ completion with respect to the trivial representation ρ of a discrete or profinite group is simply the classical pro-ℓ completion of the group denoted by $\Gamma^{(\ell)}$. The following propositions are the basic properties that are used in this paper.

Proposition 5.3.2 ([23, Prop. 2.3])**.** *(Naturality) Notations as in Definition 5.3.1. Suppose that $\rho_j : \Gamma_j \to P_j$ for $j = 1, 2$ are continuous dense homomorphisms. If the diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\rho_1} & P_1 \\ {\scriptstyle\phi_\Gamma}\downarrow & & \downarrow{\scriptstyle\phi_P} \\ \Gamma_2 & \xrightarrow{\rho_2} & P_2 \end{array} ,$$

where ϕ_Γ and ϕ_P are continuous homomorphisms, commutes, then there is a unique continuous homomorphism $\phi^{\mathrm{rel}(\ell)} : \Gamma_1^{\mathrm{rel}(\ell),\rho_1} \to \Gamma_2^{\mathrm{rel}(\ell),\rho_2}$ that makes the diagram

$$\begin{array}{ccccc} & & \overset{\rho_1}{\frown} & & \\ \Gamma_1 & \longrightarrow & \Gamma_1^{\mathrm{rel}(\ell),\rho_1} & \longrightarrow & P_1 \\ {\scriptstyle\phi_\Gamma}\downarrow & & \downarrow{\scriptstyle\phi^{\mathrm{rel}(\ell)}} & & \downarrow{\scriptstyle\phi_P} \\ \Gamma_2 & \longrightarrow & \Gamma_2^{\mathrm{rel}(\ell),\rho_2} & \longrightarrow & P_2 \\ & & \underset{\rho_2}{\smile} & & \end{array}$$

commute.

Proof. This follows from the universal mapping property. □

Proposition 5.3.3 ([23, Prop. 2.1])**.** *A homomorphism* $\rho : \Gamma \to P$ *from a discrete group to a profinite group induces a homomorphism* $\bar{\rho} : \Gamma^\wedge \to P$ *from the profinite completion of* Γ *to* P*. The natural homomorphism* $\Gamma \to \Gamma^\wedge$ *induces a natural isomorphism* $\Gamma^{\mathrm{rel}(\ell),\rho} \cong (\Gamma^\wedge)^{\mathrm{rel}(\ell),\bar{\rho}}$.

Proof. By naturality, there is a homomorphism $\phi : \Gamma^{\mathrm{rel}(\ell),\rho} \to (\Gamma^\wedge)^{\mathrm{rel}(\ell),\bar{\rho}}$. Since $\Gamma^{\mathrm{rel}(\ell),\rho}$ is a profinite group and the natural homomorphism $\Gamma \to \Gamma^{\mathrm{rel}(\ell),\rho}$ is continues, it factors through $\Gamma^\wedge$, and hence there is a homomorphism $\psi : (\Gamma^\wedge)^{\mathrm{rel}(\ell),\bar{\rho}} \to \Gamma^{\mathrm{rel}(\ell),\rho}$. The universal mapping property implies that ϕ and ψ are inverse to each other. □

Proposition 5.3.4 ([23, Prop. 2.4])**.** *(Right exactness) Suppose that* $\rho_j : \Gamma_j \to P_j$ *for* $j = 1, 2$ *and* 3 *are continuous dense homomorphisms as in the above definition. Suppose furthermore that the* Γ_j *are all discrete or all profinite groups. If the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_3 & \longrightarrow & 1\,, \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 1 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 1 \end{array}$$

where all the arrows are continuous and rows are exact, then the sequence

$$\Gamma^{\mathrm{rel}(\ell),\rho_1} \to \Gamma^{\mathrm{rel}(\ell),\rho_2} \to \Gamma^{\mathrm{rel}(\ell),\rho_3} \to 1$$

is exact.

Proof. See [23]. □

Proposition 5.3.5 ([23, Lemma 2.6])**.** *Suppose that*

$$1 \to K \to P \xrightarrow{\psi} \overline{P} \to 1$$

is a short exact sequence of profinite groups. Suppose that $\rho : \Gamma \to P$ *is a continuous dense homomorphism as in the above definition. Denote* $\psi \circ \rho$ *by* $\bar{\rho}$*. If* K *is a pro-*ℓ *group, then the natural homomorphism* $\Gamma^{\mathrm{rel}(\ell),\rho} \to \Gamma^{\mathrm{rel}(\ell),\bar{\rho}}$ *is an isomorphism.*

Proof. Since K is a pro-ℓ group and the kernel of the natural homomorphism $\Gamma^{\mathrm{rel}(\ell),\rho} \to P$ is pro-ℓ, the preimage N of K under the homomorphism $\Gamma^{\mathrm{rel}(\ell),\rho} \to P$ is also a pro-ℓ group, and hence, by the universal mapping property, there is a natural homomorphism $\Gamma^{\mathrm{rel}(\ell),\bar{\rho}} \to \Gamma^{\mathrm{rel}(\ell),\rho}$, which is an inverse of the natural homomorphism $\Gamma^{\mathrm{rel}(\ell),\rho} \to \Gamma^{\mathrm{rel}(\ell),\bar{\rho}}$. □

Example 5.3.6. Let Γ be a finite index normal subgroup of the mapping class group $\Gamma_{g,n}$. Denote by $\Gamma^\wedge_{g,n}$ and $\Gamma^\wedge$ the profinite completion of $\Gamma_{g,n}$ and Γ, respectively. Let $\rho : \Gamma^\wedge \to \mathrm{Sp}(\mathbb{Z}_\ell)$ be the homomorphism obtained by composing with the standard representation $\Gamma^\wedge_{g,n} \to \mathrm{Sp}(\mathbb{Z}_\ell)$. Suppose that $\psi : \mathrm{Sp}(\mathbb{Z}_\ell) \to \mathrm{Sp}(\mathbb{Z}/\ell\mathbb{Z})$ is reduction mod ℓ. If $\bar{\rho}$ is trivial, then since $\ker\psi$ is a pro-ℓ group, there are natural isomorphisms $\Gamma^{\mathrm{rel}(\ell),\rho} \cong \Gamma^{\mathrm{rel}(\ell),\bar{\rho}}$.

5.4 Fundamental Groups of Finite Étale Covers of Moduli Stacks of Curves

Suppose that g and n are non-negative integers satisfying $2g - 2 + n > 0$. Fix a closed oriented genus g surface Σ_g and a finite subset $P = \{p_1, p_2, \ldots, p_n\}$ of n distinct points in Σ_g. Denote the mapping class group of (Σ_g, P) by $\Gamma_{\Sigma_g,P}$. This is defined to be the group of isotopy classes of orientation preserving homeomorphisms which fix P pointwise. By the classification of surfaces, the homeomorphism class of (Σ_g, P) depends only on (g, n). Therefore, the group $\Gamma_{\Sigma_g,P}$ depends only on the pair (g, n), and thus it is denoted by $\Gamma_{g,n}$. Denote the complement $\Sigma_g - P$ of P in Σ_g by $\Sigma_{g,n}$. Denote the topological fundamental group $\pi_1^{\mathrm{top}}(\Sigma_{g,n}, *)$ of $\Sigma_{g,n}$ by $\Pi_{g,n}$. The standard presentation of $\Pi_{g,n}$ is

$$\Pi_{g,n} = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1\rangle.$$

Note that $\Pi_{g,0} = \Pi_{g,n}/\langle \gamma_1, \ldots, \gamma_n\rangle$. The geometric automorphisms of $\Pi_{g,n}$ are defined to be the ones that fix the conjugacy class of every γ_i and induce the identity on

$H_2(\Pi_{g,0}, \mathbb{Z})$. Denote the group of geometric automorphisms of $\Pi_{g,n}$ by $A_{g,n}$ and the group of the inner automorphisms of $\Pi_{g,n}$ by $I_{g,n}$. $I_{g,n}$ is clearly a normal subgroup of $A_{g,n}$. It is well known that there is a canonical isomorphism

$$\Gamma_{g,n} \cong A_{g,n}/I_{g,n}$$

(See [47, Theorem V.9]). The invariant subgroups of $\Pi_{g,n}$ are defined to be the ones that are stable under the action of $A_{g,n}$. For an invariant subgroup K of $\Pi_{g,n}$, there is a natural representation

$$\Gamma_{g,n} \to \mathrm{Out}(\Pi_{g,n}/K).$$

This representation is the key for the construction of M^λ.

Let k be a field of characteristic 0. For simplicity, assume that k is contained in $\mathbb{C}$ and denote the algebraic closure of k in $\mathbb{C}$ by $\bar{k}$. The moduli stack $\mathcal{M}_{g,n/\mathbb{C}}$ can be viewed as a complex analytic orbifold denoted by $\mathcal{M}^{\mathrm{an}}_{g,n/\mathbb{C}}$. Denote the orbifold fundamental group of $\mathcal{M}^{\mathrm{an}}_{g,n/\mathbb{C}}$ by $\pi_1^{\mathrm{orb}}(\mathcal{M}^{\mathrm{an}}_{g,n/\mathbb{C}}, \bar{\eta})$ with base point $\bar{\eta} \in \mathcal{M}_{g,n}(\mathbb{C})$. There is a natural isomorphism

$$\pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n/\mathbb{C}}, \bar{\eta}) \cong \Gamma_{g,n}.$$

Therefore, for each geometric point $\bar{\eta}$ of $\mathcal{M}_{g,n/\bar{k}}$, there is an isomorphism

$$\pi_1(\mathcal{M}_{g,n/\bar{k}}, \bar{\eta}) \cong \Gamma^\wedge_{g,n},$$

which is uniquely determined up to inner automorphisms, and there is an exact sequence

$$1 \to \Gamma^\wedge_{g,n} \to \pi_1(\mathcal{M}_{g,n/k}, \bar{\eta}) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

Let k be an algebraically closed field of characteristic $p > 0$. Denote the ring of p-adic Witt vectors over k by $W(k)$. When k is clear from context, we denote $W(k)$ by W. It is a characteristic zero complete discrete valuation ring with the residue field k. Fix an algebraic closure L of the fraction field of $W(k)$. There is an isomorphism $\Gamma_{g,n}^{\wedge} \cong \pi_1(\mathcal{M}_{g,n/L}, \bar{\eta})$ of the geometric fundamental group of $\mathcal{M}_{g,n/L}$ with the profinite completion of the mapping class group $\Gamma_{g,n}$. Fix a prime number $\ell \nmid p$. Let $G = \Pi_{g,n}/W^3\Pi_{g,n} \cdot \Pi_{g,n}^{\ell^m}$ for odd ℓ and $G = \Pi_{g,n}/W^4\Pi_{g,n} \cdot \Pi_{g,n}^4$ for $\ell = 2$, where the filtration $W^\bullet$ is defined in §5.1. Let M^λ be a finite étale cover of $\mathcal{M}_{g,n}[1/\ell]$ as in Proposition 5.1.1. Denote the kernel of the natural representation $\Gamma_{g,n} \to \mathrm{Out}(G)$ by $\Gamma_{g,n}^\lambda$. Denote the Teichüller space of the reference surface $\Sigma_{g,n}$ by $\mathcal{T}_{g,n}$. By construction, each connected component of the complex variety $M^\lambda \otimes \mathbb{C}$ is isomorphic to the analytic space $\mathcal{T}_{g,n}/\Gamma_{g,n}^\lambda$. Since $\Gamma_{g,n}^\lambda$ acts on $\mathcal{T}_{g,n}$ freely, we see that there is a natural conjugacy class of isomorphisms

$$\pi_1(M_{\mathbb{C}}^\lambda) \cong (\Gamma_{g,n}^\lambda)^\wedge,$$

where $M_{\mathbb{C}}^\lambda$ is a connected component of $M^\lambda \otimes \mathbb{C}$. Since ℓ is a unit in W, there is a natural morphism $\operatorname{Spec} W \to \operatorname{Spec} \mathbb{Z}[1/\ell]$. Choose a connected component of $M^\lambda \otimes_{\mathbb{Z}[1/\ell]} W$ and denote it by M_W^λ. Denote its base changes to L and k by M_L^λ and M_k^λ, respectively. Let $\bar{\eta}$ and $\bar{\xi}$ be a geometric point of M_L^λ and M_k^λ, respectively. The scheme M_L^λ is a connected finite étale cover of $\mathcal{M}_{g,n/L}$ and there is an isomorphism $\pi_1(M_L^\lambda, \bar{\eta}) \cong (\Gamma_{g,n}^\lambda)^\wedge$. Since the boundary of $\overline{M^\lambda}$ is a relative normal crossing divisor over $\mathbb{Z}[1/\ell]$, the boundary of the Zariski closure of M_W^λ in $\overline{M^\lambda} \otimes W$ is also a relative normal crossing divisor over W. This allows us to define a specialization homomorphism of tame fundamental groups [14, Exposé XIII]

$$sp : \pi_1^t(M_L^\lambda, \bar{\eta}) \to \pi_1^t(M_W^\lambda, \bar{\eta}) \cong \pi_1^t(M_W^\lambda, \bar{\xi}) \xleftarrow{\sim} \pi_1^t(M_k^\lambda, \bar{\xi}),$$

where the left-hand map is induced by base change to L, the map at middle is an isomorphism obtained by change of base points, and the right-hand map is the isomorphism induced by base change to k.

Theorem 5.4.1. *With notations as above, there is an isomorphism*

$$(\Gamma_{g,n}^\lambda)^{(\ell)} \cong \pi_1(M_k^\lambda, \bar{\xi})^{(\ell)},$$

which is uniquely determined up to inner automorphisms.

Proof. The smoothness of M^λ_W over W implies that the specialization morphism sp is surjective. This surjective homomorphism induces an isomorphism

$$sp^{(p')} : \pi_1(M^\lambda_L, \bar{\eta})^{(p')} \xrightarrow{\sim} \pi_1(M^\lambda_k, \bar{\xi})^{(p')}$$

upon taking maximal prime-to-p quotient. Hence we have an isomorphism

$$sp^{(\ell)} : \pi_1(M^\lambda_L, \bar{\eta})^{(\ell)} \xrightarrow{\sim} \pi_1(M^\lambda_k, \bar{\xi})^{(\ell)}$$

by taking maximal pro-ℓ quotient. □

Corollary 5.4.2. *With notations as above, there are natural conjugacy classes of isomorphisms*

$$(\Gamma_{g,n}[\ell^m])^{(\ell)} \cong \pi_1(\mathcal{M}_{g,n/k}[\ell^m])^{(\ell)}$$

and

$$\Gamma^{\mathrm{rel}(\ell)}_{g,n} \cong \pi_1(\mathcal{M}_{g,n/k})^{\mathrm{rel}(\ell)}.$$

Proof. For $A = L, W$, and k, denote $\mathcal{M}_{g,n/A}$ and $\mathcal{M}_{g,n/A}[\ell^m]$ by $\mathcal{M}_A$ and $\mathcal{M}_A[\ell^m]$, respectively. Let $\bar{\eta}$ and $\bar{\xi}$ be geometric points of M^λ_L and M^λ_k, respectively. Denote the images of $\bar{\eta}$ and $\bar{\xi}$ under morphisms by $\bar{\eta}$ and $\bar{\xi}$ also. The monodromy action $\pi_1(\mathcal{M}_A)^{\mathrm{rel}(\ell)} \to \mathrm{Sp}(\mathbb{Z}/\ell\mathbb{Z})$ factors through the finite group $\Gamma_{g,n}/\Gamma^\lambda_{g,n}$, which is the automorphism group of M^λ_A over $\mathcal{M}_A$. Denote this finite group by G. This implies that for $A = W$ and $A = k$, there is an exact sequence

$$1 \to \pi_1(M^\lambda_A, \bar{\xi})^{(\ell)} \to \pi_1(\mathcal{M}_A, \bar{\xi})^{\mathrm{rel}(\ell)} \to G \to 1.$$

Similarly, for $A = L$ and $A = W$, there is an exact sequence

$$1 \to \pi_1(M^\lambda_A, \bar{\eta})^{(\ell)} \to \pi_1(\mathcal{M}_A, \bar{\eta})^{\mathrm{rel}(\ell)} \to G \to 1.$$

Fix an isomorphism $\pi_1(M_W^\lambda, \bar{\xi}) \cong \pi_1(M_W^\lambda, \bar{\eta})$. These exact sequences fit into the commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & \pi_1(M_k^\lambda, \bar{\xi})^{(\ell)} & \to & \pi_1(\mathcal{M}_k, \bar{\xi})^{\mathrm{rel}(\ell)} & \to & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \pi_1(M_W^\lambda, \bar{\xi})^{(\ell)} & \to & \pi_1(\mathcal{M}_W, \bar{\xi})^{\mathrm{rel}(\ell)} & \to & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(M_W^\lambda, \bar{\eta})^{(\ell)} & \to & \pi_1(\mathcal{M}_W, \bar{\eta})^{\mathrm{rel}(\ell)} & \to & G & \to & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \to & \pi_1(M_L^\lambda, \bar{\eta})^{(\ell)} & \to & \pi_1(\mathcal{M}_L, \bar{\eta})^{\mathrm{rel}(\ell)} & \to & G & \to & 1,
\end{array}$$

where the left-hand vertical maps are all isomorphisms and the map $G \to G$ is an isomorphism induced by the fixed isomorphism $\pi_1(M_W^\lambda, \bar{\xi}) \cong \pi_1(M_W^\lambda, \bar{\eta})$. Therefore, the middle vertical maps are all isomorphisms and thus there are isomorphisms

$$\pi_1(\mathcal{M}_k, \bar{\xi})^{\mathrm{rel}(\ell)} \cong \pi_1(\mathcal{M}_L, \bar{\eta})^{\mathrm{rel}(\ell)} \cong \Gamma_{g,n}^{\mathrm{rel}(\ell)},$$

which are unique up to conjugation by elements of $\pi_1(\mathcal{M}_k, \bar{\xi})^{\mathrm{rel}(\ell)}$. Similarly, let G' be the quotient of $\pi_1(\mathcal{M}_A[\ell^m])$ by the finite index subgroup $\pi_1(M_A^\lambda)$. It is a finite ℓ-group. Using the exact sequences

$$1 \to \pi_1(M_A^\lambda, \bar{\xi})^{(\ell)} \to \pi_1(\mathcal{M}_A[\ell^m], \bar{\xi})^{(\ell)} \to G' \to 1,$$

where $A = W$ and $A = k$, and

$$1 \to \pi_1(M_A^\lambda, \bar{\eta})^{(\ell)} \to \pi_1(\mathcal{M}_A[\ell^m], \bar{\eta})^{(\ell)} \to G' \to 1,$$

where $A = L$ and W, we also have isomorphisms

$$\pi_1(\mathcal{M}_k[\ell^m], \bar{\xi})^{(\ell)} \cong \pi_1(\mathcal{M}_L[\ell^m], \bar{\eta})^{(\ell)} \cong \Gamma_{g,n}[\ell^m]^{(\ell)},$$

which are unique up to conjugation by elements of $\pi_1(\mathcal{M}_k[\ell^m], \bar{\xi})^{(\ell)}$. $\square$

6

Weighted Completion

In this chapter, we review the theory of weighted completion. The idea of weighted completion is to "linearize a profinite group" with weight data. The weighted completion is relatively computable, since it is controlled by cohomology. This is a generalization of the relative completion of a discrete group (which we will review in next chapter) due to Deligne, and was developed by Hain and Matsumoto in [22]. The main application of the completion in this paper is to the arithmetic mapping class groups and the profinite completion of the mapping class groups.

6.1 (Pro)algebraic groups

Suppose that F is a field of characteristic zero. In this paper, by an algebraic group over F we mean an affine group scheme of finite type over F and by a proalgebraic F-group $\mathcal{G}$ we mean a projective limit of a projective system consisting of surjective homomorphisms

$$\mathcal{G} = \varprojlim_{\alpha} G_\alpha,$$

where each G_α is an algebraic group over F. It is known that every affine group scheme arises as a proalgebraic group [32]. For an F-algebra R, the R-rational points $\mathcal{G}(R)$ of $\mathcal{G}$ is the projective limit of the projective system $\{G_\alpha(R)\}$,

$$\mathcal{G}(R) = \varprojlim G_\alpha(R).$$

The Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is the projective limit of the corresponding projective system of the Lie algebras $\mathfrak{g}_\alpha$

$$\mathfrak{g} = \varprojlim \mathfrak{g}_\alpha.$$

We consider the Lie algebra $\mathfrak{g}$ as a topological Lie algebra with the topology induced by the projective limit. Here we equip $\mathfrak{g}_\alpha$ the discrete topology. The neighborhoods of zero are the kernels of the natural surjections $\mathfrak{g}_\alpha$. The functor Lie taking $\mathcal{G}$ to its Lie algebra $\mathfrak{g}$ is exact; if the sequence

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$$

of proalgebraic F-groups is exact, then the sequence

$$0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}'' \to 0$$

of the corresponding Lie algebras is exact.

6.2 Prounipotent groups and pronilpotent Lie algebras

Suppose that F is a field of characteristic zero. For $n \geqslant 1$ and an F-algebra R the algebraic group $\mathbb{U}_n(R)$ is an algebraic subgroup of $\mathrm{GL}_n(R)$ consisting of upper triangular matrices whose diagonal entries are all 1. A unipotent F-group is an algebraic group over F that is isomorphic to an algebraic subgroup of $\mathbb{U}_n$ for some n. A *prounipotent F-group* $\mathcal{U}$ is the inverse limit of a surjective projective system of unipotent F-groups U_α:

$$\mathcal{U} = \varprojlim_\alpha U_\alpha.$$

The Lie algebra $\mathfrak{u}$ of the prounipotent F-group $\mathcal{U}$ is a *pronilpotent* Lie algebra that is the projective limit of the finite-dimensional nilpotent Lie algebras $\mathfrak{u}_\alpha$ of the U_α:

$$\mathfrak{u} = \varprojlim_\alpha \mathfrak{u}_\alpha.$$

For a unipotent group U and its Lie algebra $\mathfrak{u}$, the exponential map $\exp : \mathfrak{u} \to U$ is an isomorphism of algebraic varieties with the inverse given by the logarithm map $\log : U \to \mathfrak{u}$. The exponential and logarithm maps extend to prounipotent groups by taking a projective limit to give an isomorphism of proalgebraic varieties.

6.3 Continuous cohomology and homology of a Lie algebra

Let $\mathcal{G} = \varprojlim_\alpha G_\alpha$ be a proalgebraic group over F and $\mathfrak{g} = \varprojlim_\alpha \mathfrak{g}_\alpha$ be its Lie algebra. The continuous cohomology of $\mathfrak{g}$ is defined to be the inductive limit of the cohomology of its canonical finite-dimensional quotients $\mathfrak{g}_\alpha$:

$$H^\bullet(\mathfrak{g}) := \varinjlim_\alpha H^\bullet(\mathfrak{g}_\alpha)$$

and the continuous homology is defined to be the inverse limit:

$$H_\bullet(\mathfrak{g}) := \varprojlim_\alpha H_\bullet(\mathfrak{g}_\alpha)$$

These can be computed using the continuous Chevalley-Eilenberg complexes

$$\mathrm{Hom}_{\mathrm{cts}}(\Lambda^\bullet \mathfrak{g}, F) := \varinjlim_\alpha \mathrm{Hom}(\Lambda^\bullet \mathfrak{g}_\alpha, F)$$

and

$$\Lambda^\bullet \mathfrak{g} := \varprojlim_\alpha \Lambda^\bullet \mathfrak{g}_\alpha$$

There are natural isomorphisms

$$H^\bullet(\mathfrak{g}) = \mathrm{Hom}(H_\bullet(\mathfrak{g}), F) \text{ and } H_\bullet(\mathfrak{g}) = \mathrm{Hom}(H^\bullet(\mathfrak{g}), F).$$

6.4 A presentation of a pronilpotent Lie algebras

Pronilpotent Lie algebras are easy to deal with, since they can be easily expressed as a quotient of a free Lie algebra. Recall that for a F-vector space V, the free Lie algebra denoted by $\mathbb{L}(V)$ is the Lie subalgebra of the tensor algebra $T(V)$, where the bracket on $\mathbb{L}(V)$ is defined by $[u, v] := uv - vu$. It has a universal property; if there is a F-linear map $\phi : V \to \mathfrak{g}$ from V to a Lie algebra $\mathfrak{g}$, then the map ϕ extends to be a Lie algebra homomorphism $\mathbb{L}(V) \to \mathfrak{g}$. The free Lie algebra $\mathbb{L}(V)$ is graded by bracket length: there is an isomorphism

$$\mathbb{L}(V) \cong \bigoplus_{n \geqslant 1} \mathbb{L}_n(V).$$

Now, since a derivation on $\mathbb{L}(V)$ is determined by its effect on V, we see that $\operatorname{Der} \mathbb{L}(V) \cong \operatorname{Hom}_F(V, \mathbb{L}(V))$ and that the derivation Lie algebra $\operatorname{Der} \mathbb{L}(V)$ is graded:

$$\operatorname{Der} \mathbb{L}(V) \cong \bigoplus_{n \geqslant 1} \operatorname{Der}^n \mathbb{L}(V),$$

where $\operatorname{Der}^n \mathbb{L}(V) := \operatorname{Hom}_F(V, \mathbb{L}_{n+1}(V))$. The *free completed Lie algebra* $\mathbb{L}(V)^\wedge$ generated by V is defined to be

$$\mathbb{L}(V)^\wedge = \varprojlim_{W,n} \mathbb{L}(W)/L^n\mathbb{L}(W),$$

where W ranges over all finite-dimensional quotients of V and $L^\bullet\mathbb{L}(W)$ is the lower central series of $\mathbb{L}(W)$.

The following elementary fact is a key for finding a presentation for a pronilpotent Lie algebra.

Lemma 6.4.1. *If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism of nilpotent Lie algebras, then ϕ is surjective if and only if the induced linear map $H_1(\mathfrak{g}) \to H_1(\mathfrak{h})$ is surjective.*

□

Corollary 6.4.2. *If $\mathfrak{g}$ is a pronilpotent Lie algebra, then there exists a free completed Lie algebra $\mathfrak{f}$ and a continuous surjective Lie algebra homomorphism $\phi : \mathfrak{f} \to \mathfrak{g}$ such that the induced map $\tilde{\phi} : H_1(\mathfrak{f}) \to H_1(\mathfrak{g})$ is an isomorphism.*

Proof. Fix a continuous section of the natural projection $\mathfrak{g} \to H_1(\mathfrak{g})$. Let ϕ be the continuous Lie algebra homomorphism $\phi : \mathbb{L}(H_1(\mathfrak{g}))^\wedge \to \mathfrak{g}$ induced by this section. Then the induced map $\tilde{\phi} : H_1(\mathbb{L}(H_1(\mathfrak{g}))^\wedge) \to H_1(\mathfrak{g})$ is continuous and an isomorphism, and hence the above lemma implies that ϕ is a continuous surjection. □

Therefore, we obtain a presentation

$$\mathfrak{g} \cong \mathbb{L}(H_1(\mathfrak{g}))^\wedge/\mathfrak{r}.$$

Proposition 6.4.3. *If a closed Lie ideal $\mathfrak{r}$ of a free pronilpotent Lie algebra $\mathfrak{f}$ that is contained in $[\mathfrak{f}, \mathfrak{f}]$, then there is a continuous natural isomorphism*

$$H_2(\mathfrak{f}/\mathfrak{r}) \cong \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}].$$

Proof. Since $\mathfrak{r}$ is a subalgebra of of a free pronilpotent Lie algebra, it is free. Thus we have $H^k(\mathfrak{r}) = 0$ for all $k > 1$. Since $\mathfrak{r} \subset [\mathfrak{f}, \mathfrak{f}]$, together with the vanishing of cohomology for $k > 1$, the spectral sequence

$$E_2^{s,t} = H^s(\mathfrak{f}/\mathfrak{r}, H^t(\mathfrak{r})) \Rightarrow H^{s+t}(\mathfrak{f})$$

implies that $H^2(\mathfrak{f}/\mathfrak{r}) = H^0(\mathfrak{f}/\mathfrak{r}, H^1(\mathfrak{r}))$. Note that we have

$$H_0(\mathfrak{f}/\mathfrak{r}, H_1(\mathfrak{r})) = \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}].$$

Thus there is a natural isomorphism

$$H^2(\mathfrak{f}/\mathfrak{r}) = H^0(\mathfrak{f}/\mathfrak{r}, H^1(\mathfrak{r})) = \mathrm{Hom}_{\mathrm{cts}}(H_0(\mathfrak{f}/\mathfrak{r}, H_1(\mathfrak{r})), F) = \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{r}/[\mathfrak{r}, \mathfrak{f}], F).$$

Taking the dual, we get a natural isomorphism $H_2(\mathfrak{f}/\mathfrak{r}) \cong \mathfrak{r}/[\mathfrak{r}.\mathfrak{f}]$. □

Whether a Lie algebra homomorphism between pronilpotent Lie algebras is an isomorphism or not, we have a useful criterion. This is an analogue for pronilpotent Lie algebras of a classical result of Stallings [43].

Proposition 6.4.4. *Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism of pronilpotent Lie algebras. If ϕ induces an isomorphism $H_1(\mathfrak{g}) \to H_1(\mathfrak{h})$ and a surjection $H_2(\mathfrak{g}) \to H_2(\mathfrak{h})$, then ϕ is an isomorphism.*

Proof. Since ϕ induces an isomorphism on H_1, it is a surjection. Any continuous linear section of $\mathfrak{g} \to H_1(\mathfrak{g})$ induces Lie algebra surjections $\psi_1 : \mathbb{L}(H_1(\mathfrak{g})) \to \mathfrak{g}$ and $\psi_2 : \mathbb{L}(H_1(\mathfrak{h})) \to \mathfrak{h}$ by composing with ϕ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{r} & \longrightarrow & \mathbb{L}(H_1(\mathfrak{g})) & \xrightarrow{\psi_1} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathbb{L}(H_1(\mathfrak{h})) & \xrightarrow{\psi_2} & \mathfrak{h} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{r}$ and $\mathfrak{n}$ are the kernels of ψ_1 and ψ_2 and the middle map is the isomorphism induced by the isomorphism on H_1. Note that the right-hand vertical map is injective. By assumption, ϕ induces a surjection on H_2, and so Proposition 6.4.3 implies that the map $\mathfrak{r} \to \mathfrak{n}$ is also surjective. Hence, ϕ is an isomorphism. □

Corollary 6.4.5. *A pronilpotent Lie algebra $\mathfrak{g}$ is trivial if and only if $H_1(\mathfrak{g}) = 0$ and free if and only if $H_2(\mathfrak{g}) = 0$.*

6.5 Negatively weighted extensions

Suppose that F is a field of characteristic 0, that R is a reductive algebraic group defined over F, and that $w : \mathbb{G}_m \to R$ is a central cocharacter. Denote $\mathbb{G}_{m/F}$ by $\mathbb{G}_m$. Suppose that

$$1 \to U \to G \to R \to 1$$

is an extension of R by a unipotent group U in the category of algebraic F-groups. The abelianization $H_1(U)$ is an R-module, and therefore a $\mathbb{G}_m$-module via w. Thus we have the decomposition

$$H_1(U) = \bigoplus_{r \in \mathbb{Z}} H_1(U)_r,$$

where $\mathbb{G}_m$ acts on $H_1(U)_r$ via the rth power of its defining representation. We will say that this extension is *negatively weighted* with respect to w if $H_1(U)_r = 0$ for all $r \geqslant 0$ and that a proalgebraic group $\mathcal{G}$ which is an extension of R by a prounipotent group $\mathcal{U}$ is *negatively weighted* if it is an inverse limit of negatively weighted extensions of R by unipotent groups.

6.6 Weight filtrations

By the Levi decomposition [10, p. 158], the extension

$$1 \to U \to G \to R \to 1$$

splits and any two splittings differ by conjugation by an element of $U(F)$. Therefore there is a lift of the homomorphism $\mathbb{G}_m \to R$ to a homomorphism $\tilde{\omega} : \mathbb{G}_m \to \mathcal{G}$.

Lemma 6.6.1. *Any two lifts $\tilde{\omega} : \mathbb{G}_m \to G$ of ω are conjugate by an element of $U(F)$.*

Proof. Pulling back the sequence

$$1 \to U \to G \to R \to 1$$

along ω, we obtain the sequence

$$1 \to U \to \tilde{G} \to \mathbb{G}_m \to 1.$$

Any two sections of $\tilde{G} \to \mathbb{G}_m$ are conjugate by an element of $U(F)$ by the Levi decomposition. □

Fix a lift $\tilde{\omega} : \mathbb{G}_m \to G$. We can regard each finite-dimensional G-module V as a $\mathbb{G}_m$-module and thus have a decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where $\mathbb{G}_m$ acts on V_n via the nth power of the defining representation. Define a weight filtration of V by

$$W_n V := \bigoplus_{m \leqslant n} V_m.$$

The above weight decomposition respects Hom and tensor products; if U and V are finite-dimensional G-modules, then we have

$$\mathrm{Hom}(U, V)_n = \bigoplus_{l-k=n} \mathrm{Hom}(U_k, V_l)$$

and

$$(U \otimes V)_n = \bigoplus_{k+l=n} U_k \otimes V_l.$$

Therefore, it follows that weight filtrations are compatible with Hom and tensor products:

$$W_n \operatorname{Hom}(U, V) = \{\phi \in \operatorname{Hom}(U, V) | \phi(W_l U) \subseteq W_{l+n} V \text{ for all } l \in \mathbb{Z}\}$$

and

$$W_n(U \otimes V) = \sum_{k+l=n} W_k U \otimes W_l V$$

Let G be a negatively weighted extension of R by a unipotent group. Denote the Lie algebras of G, U, and R by $\mathfrak{g}$, $\mathfrak{u}$, and $\mathfrak{r}$, respectively. We can regard them as G-modules via the adjoint action of G. Fix a lift $\tilde{\omega}$ of ω.

Proposition 6.6.2. *The Lie algebras $\mathfrak{g}$, $\mathfrak{u}$, and $\mathfrak{r}$ are graded Lie algebras satisfying the properties:*

1. *if* $x \in \mathfrak{g}_k$ *and* $y \in \mathfrak{g}_l$, *then* $[x, y] \in \mathfrak{g}_{k+l}$.

2. $\mathfrak{u}_m \subseteq \mathfrak{g}_m$.

3. $\mathfrak{u}_m = 0$ *if* $m \geqslant 0$.

4. $\mathfrak{g} = W_0\mathfrak{g}$, $\mathfrak{u} = W_{-1}\mathfrak{g}$, *and* $\mathfrak{r} = \mathrm{Gr}_0^W \mathfrak{g}$.

Proof. The first assertion follows from the fact that the bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a G-module homomorphism. The second is clear. As to the third, define $\mathfrak{u}_{<0}$ to be the sub Lie algebra of $\mathfrak{u}$. It is a nilpotent Lie algebra. The inclusion $\mathfrak{u}_{<0} \to \mathfrak{u}$ induces an isomorphism $H_1(\mathfrak{u}_{<0}) \cong H_1(\mathfrak{u})$, since $H_1(\mathfrak{u})$ has only negative weights. By Lemma 6.4.1, the inclusion $\mathfrak{u}_{<0} \to \mathfrak{u}$ is indeed an isomorphism. For the last assertion, note that since ω is central, we have $\mathfrak{r} = \mathfrak{r}_0$, and that the sequence

$$0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{r} \to 0$$

is an exact sequence of graded Lie algebras. The result follows immediately. □

A key property of negatively weighted extensions used in this paper is the following.

Proposition 6.6.3. *For every finite-dimensional* G*-module* V, *each term* W_nV *of the weight filtration* $W_\bullet V$ *is a* G*-module and each graded quotient* $\mathrm{Gr}_n^W V := W_nV/W_{n-1}V$ *is an* R*-module of weight* n.

Proof. Every finite-dimensional G-module V is a $\mathfrak{g}$-module via the homomorphism $\mathfrak{g} \to \mathrm{End}_F(V)$. The map $\mathfrak{g} \otimes V \to V, x \otimes v \mapsto x(v)$ is G-equivariant, and hence if $x \in \mathfrak{g}_k$ and $v \in V_l$, then $x(v) \in V_{k+l}$. Since $\mathfrak{g} = W_0\mathfrak{g}$, the action of $\mathfrak{g}$ on V preserves the weight filtration of V. Since $\mathfrak{u} = W_{-1}\mathfrak{u}$, the image of W_lV under the action of $\mathfrak{u}$ is contained in $W_{l-1}V$. Thus $\mathfrak{u}$ acts trivially on the associated graded quotients $\mathrm{Gr}_l^W V$ for all $l \in \mathbb{Z}$, so does U. Write $G = U \rtimes s(R)$ with respect to the fixed splitting s of

$G \to R$. Since ω is central, the image of $s \circ \omega$ is central in $s(R)$. Thus the action of $s(R)$ on V preserves the weight filtration, so does the action of G. Together with the trivial action of U on $\mathrm{Gr}^W_\bullet V$, the induced action of G on $\mathrm{Gr}^W_\bullet V$ descends to the action of R. □

Although, by definition, the weight filtration depends on the choice of the lift $\tilde{\omega}$, it does not.

Proposition 6.6.4. *With notation as above, the weight filtration of V is independent of the choice of the lift $\tilde{\omega}$.*

Proof. Let $\tilde{\omega}$ and $\tilde{\omega}'$ be the lifts of $\omega : \mathbb{G}_m \to R$. There exists an element u of $U(F)$ such that $\tilde{\omega}' = u\tilde{\omega}u^{-1}$. Denote the weight decomposition with respect to $\tilde{\omega}$ and $\tilde{\omega}'$ by $V = \bigoplus V_n$ and $V = \bigoplus V_n'$, respectively. Then we see that $\tilde{\omega}'$ acts on uV_n with weight n, and so $uV_n \subseteq V_n'$. Similarly, $u^{-1}V_n' \subseteq V_n$ or $V_n' \subseteq uV_n$. Thus $uV_n = V_n'$. Since the action of G preserves the weight filtrations $W_\bullet$ and $W_\bullet'$, we have

$$\bigoplus_{m \leqslant n} V_m' = \bigoplus_{m \leqslant n} uV_m = u\left(\bigoplus_{m \leqslant n} V_m\right) \subseteq \bigoplus_{m \leqslant n} V_m.$$

A symmetric argument shows the other containment. □

The above results extend to the case where $\mathcal{G}$ is a proalgebraic F-group $\mathcal{G}$ that is a negatively weighted extension of G with respect to $\omega : \mathbb{G}_m \to R$. If V is a finite-dimensional $\mathcal{G}$-module, then the action of $\mathcal{G}$ factors through its algebraic quotient G_α. It is easy to see that G_α is an extension of R by a unipotent F-group U_α. Since the prounipotent radical $\mathcal{U}$ of $\mathcal{G}$ is negatively weighted, it follows that U_α is also negatively weighted, i.e., G_α is a negatively weighted extension of R. Thus the $\mathcal{G}$-module V admits a natural weight filtration. It is clear that this weight filtration does not depend on the quotient G_α.

6.6.5. Pro-and Ind-$\mathcal{G}$-modules: The construction of weight filtrations extend to projective limits and inductive limits of finite-dimensional $\mathcal{G}$-modules. If $\mathcal{V} = \varprojlim_\alpha V_\alpha$ is a projective limit of finite-dimensional $\mathcal{G}$-modules, then $W_n\mathcal{V}$ is given by $W_n\mathcal{V} = \varprojlim_\alpha W_n V_\alpha$ for each $n \in \mathbb{Z}$. similarly, if $\mathcal{V} = \varinjlim_\alpha V_\alpha$ is an inductive limit of finite-dimensional $\mathcal{G}$-modules, then $W_n\mathcal{V}$ is given by $W_n\mathcal{V} = \varinjlim_\alpha W_n V_\alpha$.

6.7 Category of weighted modules

An important property of weight filtrations is that morphisms of weight modules are strict and hence that taking associated graded quotients are exact. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are negatively weighted extensions of reductive groups R_1 and R_2, respectively, such that the diagram

$$\begin{array}{ccccc} \mathcal{G}_1 & \longrightarrow & R_1 & \xleftarrow{\omega_1} & \mathbb{G}_m \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} & & \| \\ \mathcal{G}_2 & \longrightarrow & R_2 & \xleftarrow{\omega_2} & \mathbb{G}_m \end{array}$$

commutes. We may regard G_2-modules as G_1-modules via the homomorphism Φ. We form a category as follows. Objects are the data $(V, \mathcal{G}, R, \omega)$, where $\omega : \mathbb{G}_m \to R$ is a central cocharacter, R is a reductive algebraic F-group, and $\mathcal{G}$ is a proalgebraic group that is a negatively weighted extension of R, and where V is a finite-dimensional $\mathcal{G}$-module. Morphisms of such modules are ones compatible with the diagram described above.

6.8 Strictness

Recall that a linear map $f : (V, W_\bullet) \to (V', W'_\bullet)$ of filtered vector spaces is strict if it satisfies the property

$$f(V) \cap W'_n V' = f(W_n V),$$

for all $n \in Z$. In general, the map f induces a map $\mathrm{Gr}_n(f) : \mathrm{Gr}_n^W V \to \mathrm{Gr}_n^{W'} V'$, but the map $\mathrm{Gr}_n(f)$ need not to be injective even when f is injective.

Example 6.8.1. Let V_1 and V_2 be finite-dimensional F-vector spaces. Let $V = V_1 \oplus V_2$. Define a filtration $W_\bullet V$ by setting

$$0 = W_0 V = W_1 V \subseteq W_2 V = V.$$

Define a filtration $W'_\bullet V$ by setting

$$0 = W'_0 V \subseteq W'_1 V = V_1 \subseteq W'_2 V = V.$$

Then the identity map $\mathrm{id} : V \to V$ preserves the weight filtrations, but $\mathrm{Gr}_2(\mathrm{id}) : \mathrm{Gr}_2^W V \to \mathrm{Gr}_2^{W'} V$ is not injective.

Subspaces and quotients of a filtered vector space $(V, W_\bullet)$ admit induced weight filtrations; if $A \hookrightarrow V$ is a subspace and $q : V \twoheadrightarrow B$ is a quotient, then

$$W_n A := A \cap W_n V \text{ and } W_n B := q(W_n V).$$

Proposition 6.8.2. *If a linear map $f : (V, W_\bullet) \to (V', W_\bullet)$ of filtered vector spaces is strict, then there are natural isomorphisms*

1. $\mathrm{im}\,\mathrm{Gr}_\bullet^W f \cong \mathrm{Gr}_\bullet^W \mathrm{im}\, f$
2. $\ker \mathrm{Gr}_\bullet^W f \cong \mathrm{Gr}_\bullet^W \ker f$ *and*

of graded vector spaces.

Proof. For each n, We have

$$\begin{aligned}
\operatorname{im} \operatorname{Gr}_n^W f &= \frac{f(W_n V) + W_n V'}{W_{n-1}V'} \\
&\cong \frac{f(W_n V)}{f(W_n V) \cap W_{n-1}V'} \\
&= \frac{\operatorname{im} f \cap W_n V'}{(\operatorname{im} f \cap W_n V') \cap W_{n-1}V'} \\
&= \frac{\operatorname{im} f \cap W_n V'}{\operatorname{im} f \cap W_{n-1}V'} \\
&= \frac{W_n \operatorname{im} f}{W_{n-1} \operatorname{im} f} \\
&= \operatorname{Gr}_n^W \operatorname{im} f
\end{aligned}$$

For the second, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_{n-1}V & \longrightarrow & W_n V & \longrightarrow & \operatorname{Gr}_n^W V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f(W_{n-1}V) & \longrightarrow & f(W_n V) & \longrightarrow & \operatorname{Gr}_n^W \operatorname{im} f & \longrightarrow & 0
\end{array}$$

The rows are exact and the vertical maps are all surjective since f is strict. Since $\operatorname{Gr}_n^W \operatorname{im} f \cong \operatorname{im} \operatorname{Gr}_n^W f$, it follows that

$$\begin{aligned}
\ker \operatorname{Gr}_n^W f &\cong \frac{\ker(f : W_n V \to f(W_n V))}{\ker(f : W_{n-1}V \to f(W_{n-1}V))} \\
&= \frac{\ker f \cap W_n V}{\ker f \cap W_{n-1}V} \\
&= \operatorname{Gr}_n^W \ker f
\end{aligned}$$

□

Proposition 6.8.3. *If $f : V_1 \to V_2$ is a morphism in the category of weighted modules, then it is strict. Consequently, the functor $\operatorname{Gr}_\bullet^W$ taking $V \rightsquigarrow \operatorname{Gr}_\bullet^W V$ is exact on the category of finite-dimensional $\mathcal{G}$-modules.*

Proof. First note that every object in the category of weighted modules has a natural weight filtration. Using the notation in the definition of the category 6.7, fix a lift $\tilde{\omega}_1 : \mathbb{G}_m \to \mathcal{G}_1$ of ω_1. Then $\Phi \circ \tilde{\omega}_1$ is a lift of ω_2. Since f is $\mathcal{G}$-equivariant, it becomes a $\mathbb{G}_m$-homomorphism. Hence f preserves the weight filtrations. With respect to these lifts of ω_1 and ω_2, we may regard f as a map of graded vector spaces and so it is strict. Suppose now that $0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$ is an exact sequence in the category of weighted modules. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{n-1}V' & \longrightarrow & W_{n-1}V & \longrightarrow & W_{n-1}V'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}$$

By strictness, the first row is exact. This diagram induces the exact sequence

$$0 \to V'/W_{n-1} \to V/W_{n-1} \to V''/W_{n-1} \to 0$$

of weight modules. The strictness again induces the exact sequence

$$0 \to \operatorname{Gr}_n^W V' \to \operatorname{Gr}_n^W V \to \operatorname{Gr}_n^W V'' \to 0$$

for each $n \in \mathbb{Z}$. □

Main properties of negatively weighted extensions used in this paper are summarized as the following.

Proposition 6.8.4 ([24, Thms. 3.9 & 3.12])**.** *Suppose that R is a reductive F-group and that $w : \mathbb{G}_m \to R$ is a central cocharacter. If $\mathcal{G}$ is a proalgebraic group that is a negatively weighted extension of R with respect to w by a prounipotent group, then every finite dimensional $\mathcal{G}$-module V has a natural weight filtration $W_\bullet$:*

$$0 = W_nV \subset \cdots \subset W_{r-1}V \subset W_rV \subset \cdots W_mV = V.$$

It is characterized by the property that the action of $\mathcal{G}$ on the rth weight graded quotient

$$\operatorname{Gr}^W_r V := W_r V / W_{r-1} V$$

factors through $\mathcal{G} \to R$ and is an R-module of weight r. The weight filtration is preserved by $\mathcal{G}$-module homomorphisms and the functor $\operatorname{Gr}^W_\bullet$ on the category of finite-dimensional $\mathcal{G}$-modules is exact. $\square$

6.9 Presentations of $\mathcal{G}$

Each splitting of the extension of R by a prounipotent group

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

induces an action of R on $\mathcal{U}$ and an isomorphism

$$\mathcal{G} \cong \mathcal{U} \rtimes R$$

that is compatible with the projection to R. Hence in order to give a presentation of $\mathcal{G}$, it will suffice to give a presentation of $\mathcal{U}$ with its R-action, but the latter is determined by a presentation of the Lie algebra $\mathfrak{u}$ as an R-module. Fixing a splitting of $\mathcal{G} \to R$ gives $\mathfrak{u}$ an R-module structure. Then the natural projection $\mathfrak{u} \to H_1(\mathfrak{u})$ is an R-homomorphism, and so it induces a continuous R-section $s : H_1(\mathfrak{u}) \to \mathfrak{u}$. Since $\mathfrak{u}$ is pronilpotent, the section s induces an R-isomorphism

$$\mathbb{L}(H_1(\mathfrak{u}))^\wedge / \mathfrak{r} \cong \mathfrak{u}.$$

6.10 Weighted completion of a profinite group

Weighted completion of a profinite group Γ is similar to continuous relative completion. It plays an essential role in [20]. A key property of weighted completion is that it induces weight filtrations with strong exactness properties on the Γ-representations that factor through its weighted completion. Here we take F to be $\mathbb{Q}_\ell$, where ℓ is a prime number. Denote $\mathbb{G}_{m/\mathbb{Q}_\ell}$ by $\mathbb{G}_m$. Suppose that:

1. Γ is a profinite group;

2. R is a reductive algebraic group defined over $\mathbb{Q}_\ell$;

3. $w : \mathbb{G}_m \to R$ is a central cocharacter;

4. $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ is a continuous homomorphism with Zariski dense image.

Definition 6.10.1. The *weighted completion* of Γ with respect to ρ and w consists of a proalgebraic $\mathbb{Q}_\ell$-group $\mathcal{G}$, that is a negatively weighted extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

where $\mathcal{U}$ is a prounipotent $\mathbb{Q}_\ell$-group and a continuous Zariski dense homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ whose composition with $\mathcal{G}(\mathbb{Q}_\ell) \to R(\mathbb{Q}_\ell)$ is ρ. It is characterized by the following universal mapping property: If G is an affine (pro)algebraic $\mathbb{Q}_\ell$-group that is a negatively weighted extension

$$1 \to U \to G \to R \to 1$$

of R (with respect to w) by a (pro)unipotent group U, and if $\phi : \Gamma \to G(\mathbb{Q}_\ell)$ is a continuous homomorphism whose composition with $G(\mathbb{Q}_\ell) \to R(\mathbb{Q}_\ell)$ is ρ, then there is a unique homomorphism of proalgebraic $\mathbb{Q}_\ell$-groups $\Phi : \mathcal{G} \to G$ that commutes with the projections to R and such that $\phi = \Phi \circ \tilde{\rho}$:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\rho}} & \mathcal{G} \\ {\scriptstyle \phi}\downarrow & \overset{\Phi}{\swarrow} & \downarrow \\ G & \longrightarrow & R \end{array}$$

Proposition 6.10.2. *The weighted completion of* Γ *with respect to* ρ *and* w *always exists.*

Proof. Consider the category whose objects are pairs $(p : G \to R, \phi : \Gamma \to G(\mathbb{Q}_\ell))$, where p is a negatively weighted extension of R and ϕ is a continuous homomorphism with Zariski dense image such that $p \circ \phi = \rho$. A morphism between $(p_1 : G_1 \to R, \phi_1 : \Gamma \to G_1(\mathbb{Q}_\ell))$ and $(p_2 : G_2 \to R, \phi_2 : \Gamma \to G_2(\mathbb{Q}_\ell))$ is a homomorphism $f : G_1 \to G_2$ such that $\phi_2 = f \circ \phi_1$. Note that the Zariski density of ϕ_1 and ϕ_2 implies that such a homomorphism f is unique. A fiber product in this category is given as follows. Since $p_1 \circ \phi_1 = \rho = p_2 \circ \phi_2$, there exists an induced map $\phi : \Gamma \to (G_1 \times_R G_2)(\mathbb{Q}_\ell)$. Let G be the Zariski closure of the image of ϕ in $G_1 \times_R G_2$. That G is a negatively weighted extension of R follows from that $G_1 \times_R G_2$ is a negatively weighted extension by the unipotent $\mathbb{Q}_\ell$-group $U_1 \times U_2$, where U_i is the unipotent radical of G_i, $i = 1, 2$. Thus $(p : G \to R, \phi : \Gamma \to G(\mathbb{Q}_\ell))$ is a fiber product in this category. The weighted completion $(\mathcal{G} \to R, \tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell))$ of $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ with respect to ω is the projective limit of all objects of this category. □

6.11 Naturality

The naturality of weighted completions is a useful technical property needed in this thesis. With the notation in the definition 6.10 of weighted completions, suppose that the following diagram

$$\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\psi} & \Gamma_2 \\
\downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\
R_1(\mathbb{Q}_\ell) & \xrightarrow{\xi} & R_2(\mathbb{Q}_\ell) \\
\uparrow{\scriptstyle \omega_1} & & \uparrow{\scriptstyle \omega_2} \\
\mathbb{G}_m(\mathbb{Q}_\ell) & = & \mathbb{G}_m(\mathbb{Q}_\ell)
\end{array}$$

commutes, where the map $\psi : \Gamma_1 \to \Gamma_2$ is a continuous homomorphism of profinite groups and $\xi : R_1 \to R_2$ is a homomorphism of algebraic $\mathbb{Q}_\ell$-groups. For $i = 1, 2$, let $(\mathcal{G}_i, \tilde{\rho}_i : \Gamma_i \to \mathcal{G}_i(\mathbb{Q}_\ell)$ be the weighted completion of ρ_i with respect to ω_i.

Proposition 6.11.1. *With the notation as above, there exists a unique homomorphism $\Psi : \mathcal{G}_1 \to \mathcal{G}_2$ of proalgebraic $\mathbb{Q}_\ell$-groups such that the diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\psi} & \Gamma_2 \\ \downarrow{\scriptstyle \tilde{\rho}_1} & & \downarrow{\scriptstyle \tilde{\rho}_2} \\ \mathcal{G}_1(\mathbb{Q}_\ell) & \xrightarrow{\Psi} & \mathcal{G}_2(\mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ R_1(\mathbb{Q}_\ell) & \xrightarrow{\xi} & R_2(\mathbb{Q}_\ell) \end{array}$$

commutes.

Proof. Denote the pullback of $\mathcal{G}_2$ along the homomorphism $\xi : R_1 \to R_2$ by $\tilde{\mathcal{G}}_2$. We have a unique induced homomorphism $\hat{\rho} : \Gamma_1 \to \tilde{\mathcal{G}}_2(\mathbb{Q}_\ell)$ such that the composition with the projection to R_1 is ρ_1. Let $\mathcal{G}$ be the Zariski closure of the image of $\hat{\rho}$ in $\tilde{\mathcal{G}}_2$. Then $\mathcal{G}$ is a negatively weighted extension of R_1, and hence by the universal property of weighted completion, we obtain a map $\mathcal{G}_1 \to \mathcal{G}$. Composing with the map $\tilde{\mathcal{G}}_2 \to \mathcal{G}_2$, we get a desired map $\Psi : \mathcal{G}_1 \to \mathcal{G}_2$ that makes the above diagram commute. □

Remark 6.11.2. The map Ψ induces a Lie algebra map $\tilde{\Psi} : \mathfrak{g}_1 \to \mathfrak{g}_2$ that preserves the weight filtrations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and is strict with respect to them.

It is natural to ask whether weighted completion is an exact functor. The answer is that it is, in general, only a right exact functor. Suppose that

$$1 \to \pi \to \Gamma_1 \to \Gamma_2 \to 1$$

is a sequence of profinite groups such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \\ & & & & R(\mathbb{Q}_\ell) & = & R(\mathbb{Q}_\ell) & & \\ & & & & \uparrow{\scriptstyle \omega} & & \uparrow{\scriptstyle \omega} & & \\ & & & & \mathbb{G}_m(\mathbb{Q}_\ell) & = & \mathbb{G}_m(\mathbb{Q}_\ell) & & \end{array}$$

commutes, where ρ_1 and ρ_2 are continuous homomorphisms with Zariski dense images. Denote the weighted completions of ρ_1 and ρ_2 with respect to ω by $(\mathcal{G}_1, \tilde{\rho}_1 : \Gamma_1 \to \mathcal{G}_1(\mathbb{Q}_\ell))$ and $(\mathcal{G}_2, \tilde{\rho}_2 : \Gamma_2 \to \mathcal{G}_2(\mathbb{Q}_\ell))$, respectively. Denote the ℓ-adic unipotent completion of π over $\mathbb{Q}_\ell$ by $\pi^{\mathrm{un}}_{\mathbb{Q}_\ell}$.

Proposition 6.11.3. *If the sequence*

$$1 \to \pi \to \Gamma_1 \xrightarrow{\phi} \Gamma_2 \to 1$$

is exact, then the sequence of proalgebraic $\mathbb{Q}_\ell$-groups

$$\pi^{\mathrm{un}}_{\mathbb{Q}_\ell} \to \mathcal{G}_1 \xrightarrow{\Phi} \mathcal{G}_2 \to 1$$

is exact.

Proof. The Zariski density of ρ_1 and ρ_2 implies that the induced map $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ is surjective. Let $\mathcal{K}$ be the kernel of Φ. Denote the prounipotent radicals of $\mathcal{G}_1$ and $\mathcal{G}_2$ by $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Notice that $\mathcal{K}$ is equal to the kernel of $\mathcal{U}_1 \to \mathcal{U}_2$, which is the restriction of Φ. Thus $\mathcal{K}$ is a prounipotent $\mathbb{Q}_\ell$-group. By commutativity, π maps to $\mathcal{K}(\mathbb{Q}_\ell)$ and by the universal property of ℓ-adic completion, there is a unique map $\pi^{\mathrm{un}}_{\mathbb{Q}_\ell} \to \mathcal{K}$ such that the map $\pi \to \mathcal{K}(\mathbb{Q}_\ell)$ factors through $\pi^{\mathrm{un}}_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \to \mathcal{K}(\mathbb{Q}_\ell)$. Let $\mathcal{N}$ be the Zariski closure in $\mathcal{K}$ of the image of $\pi \to \mathcal{K}(\mathbb{Q}_\ell)$. We claim that $\mathcal{N} = \mathcal{K}$. Since $\mathcal{N} \subseteq \mathcal{K}$, we have an induced map $\hat{\Phi} : \mathcal{G}_1/\mathcal{N} \to \mathcal{G}_2$. The homomorphism

$\tilde{\rho}_1 : \Gamma_1 \to \mathcal{G}_1(\mathbb{Q}_\ell)$ induces a map $\hat{\rho}_1 : \Gamma_1/\pi \cong \Gamma_2 \to (\mathcal{G}_1/\mathcal{N})(\mathbb{Q}_\ell)$ that makes the diagram

$$\begin{array}{ccccc} & & \Gamma_2 & & \\ & \swarrow^{\hat{\rho}_1} & \downarrow{\scriptstyle \tilde{\rho}_2} & \searrow^{\rho_2} & \\ (\mathcal{G}_1/\mathcal{N})(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}_2(\mathbb{Q}_\ell) & \longrightarrow & R(\mathbb{Q}_\ell) \end{array}$$

commute. Since $\mathcal{G}_1/\mathcal{N}$ is an negatively weighted extension of R by a prounipotent $\mathbb{Q}_\ell$-group $\mathcal{K}/\mathcal{N}$, by the universal property of $\mathcal{G}_2$, we get a map $\Psi : \mathcal{G}_2 \to \mathcal{G}_1/\mathcal{N}$. By universal property of weighted completion, we have $\tilde{\Phi} \circ \Psi = \mathrm{id}$, and so Ψ is injective. By Zariski density, Ψ is surjective also, and hence $\mathcal{G}_1/\mathcal{N} \cong \mathcal{G}_2$. This implies that $\mathcal{N} = \mathcal{K}$. □

Our main application of weighted completion is on families of curves. The following result gives us a criterion when a sequence of completions is exact.

Proposition 6.11.4. *With the notation as in Proposition 6.11.3, suppose that $H_1(\pi^{\mathrm{un}}_{\mathbb{Q}_\ell})$ is finite-dimensional, that the action of Γ_2 on $H_1(\pi)$ induces a $\mathcal{G}_2$-action on $H_1(\pi^{\mathrm{un}}_{\mathbb{Q}_\ell})$, and that the weight filtration induced on $H_1(\pi^{\mathrm{un}}_{\mathbb{Q}_\ell})$ has finite dimensional graded quotients which vanish in weights $r \geqslant 0$. If $\pi^{\mathrm{un}}_{\mathbb{Q}_\ell}$ has trivial center, then the sequence of completions*

$$1 \to \pi^{\mathrm{un}}_{\mathbb{Q}_\ell} \to \mathcal{G}_1 \to \mathcal{G}_2 \to 1$$

is exact.

Proof. Since weighted completion is right exact, the sequence

$$\pi^{\mathrm{un}}_{\mathbb{Q}_\ell} \to \mathcal{G}_1 \to \mathcal{G}_2 \to 1$$

is exact. Denote the Lie algebra of $\pi^{\mathrm{un}}_{\mathbb{Q}_\ell}$ by $\mathfrak{p}$. Note that the $\mathcal{G}_2$-action on $H_1(\pi^{\mathrm{un}}_{\mathbb{Q}_\ell})$ is induced by the conjugation action of Γ_1 on π. Hence this conjugation action of Γ_1 induces the action of $\mathcal{G}_1$ on $H_1(\pi^{\mathrm{un}}_{\mathbb{Q}_\ell}) = H_1(\mathfrak{p})$, which is negatively weighted. The weight filtration of $H_1(\mathfrak{p})$ induces a weight filtration on the completed free Lie algebra $\mathbb{L}(H_1(\mathfrak{p}))^\wedge$, which in turn induces one on $\mathfrak{p}$ via a surjection $\mathbb{L}(H_1(\mathfrak{p}))^\wedge \to \mathfrak{p}$. Since $H_1(\mathfrak{p}()$ is negatively weighted, it follows that $\mathfrak{p}$ is negatively weighted. Since $H_1(\mathfrak{p})$ has finite dimensional graded quotients, it follows that each weight quotient $\mathfrak{p}/W_r\mathfrak{p}$ of $\mathfrak{p}$ is finite dimensional. Denote the group of automorphisms $\mathfrak{p}/W_r\mathfrak{p}$ preserving the weight filtration by $\mathrm{Aut}_W(\mathfrak{p}/W_r)$. The finite-dimensionality of $\mathfrak{p}/W_r\mathfrak{p}$ implies that $\mathrm{Aut}_W(\mathfrak{p}/W_r)$ is an algebraic group over $\mathbb{Q}_\ell$. Thus the group of automorphisms of $\mathfrak{p}$ preserving the weight filtration, denoted by $\mathrm{Aut}_W(\mathfrak{p})$, is a proalgebraic $\mathbb{Q}_\ell$- group, since it is the inverse limit of the $\mathrm{Aut}_W(\mathfrak{p}/W_r)$. The Zariski closure of the image of $\Gamma_1 \to \mathrm{Aut}_W(\mathfrak{p})(\mathbb{Q}_\ell)$ is a negatively weighted extension of R. Thus by the universality of weighted completion, we obtain a map $\mathcal{G}_1 \to \mathrm{Aut}_W(\mathfrak{p})$ that makes the diagram

$$\begin{array}{ccccc}
\pi & \longrightarrow & \Gamma_1 & \searrow & \\
\downarrow & & \downarrow & & \\
\mathcal{P}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}_1(\mathbb{Q}_\ell) & \longrightarrow & (\mathrm{Aut}_W(\mathfrak{p}))(\mathbb{Q}_\ell)
\end{array}$$

commute. The center-freeness of $\mathcal{P}$ implies that the adjoint action of $\mathcal{P}$ on $\mathfrak{p}$ is injective, and hence that the map $\mathcal{P} \to \mathcal{G}_1$ is injective. □

6.12 Structure of the pronilpotent Lie algebra $\mathfrak{u}$

Recall the data for the weighted completion of a profinite group Γ. Suppose that V is a finite-dimensional R-representation. V can be decomposed as $V = \bigoplus_{n\in\mathbb{Z}} V_n$ under the $\mathbb{G}_m$-action through ω. Since ω is central in R, each V_n is an R-submodule. We say that V is *pure* of weight n if $V = V_n$, and that V is *negatively weighted* if $V_n = 0$ for all $n \geqslant 0$. V can be considered as a continuous Γ-module via the homomorphism $\rho : \Gamma \to R(\mathbb{Q}_\ell)$. Denote by $H^\bullet(\Gamma, V)$ the continuous cohomology of Γ with coefficients in V.

Let $(\mathcal{G}, \tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell))$ be the weighted completion of $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ with respect to ω. Then the prounipotent $\mathbb{Q}_\ell$-group $\mathcal{U}$ of $\mathcal{G}$ is the projective limit of the unipotent groups U_α, where U_α is the unipotent radical of the negatively weighted extension $\rho_\alpha : \Gamma \to G_\alpha(\mathbb{Q}_\ell)$. The Lie algebra $\mathfrak{u}$ of $\mathcal{U}$ is the projective limit of the finite-dimensional nilpotent Lie algebras $\mathfrak{u}_\alpha$:

$$\mathfrak{u} = \varprojlim_\alpha \mathfrak{u}_\alpha$$

Similarly, we have

$$H_1(\mathfrak{u}) = \varprojlim_\alpha H_1(\mathfrak{u}_\alpha) \text{ and } H_2(\mathfrak{u}) = \varprojlim_\alpha H_2(\mathfrak{u}_\alpha).$$

Note that there is an R-module isomorphism $H_1(\mathfrak{u}) \cong H_1(\mathcal{U})$.

Theorem 6.12.1 ([24, Thms. 4.6])**.** *For all finite-dimensional pure R-modules V of weight r, there are natural isomorphisms*

$$\operatorname{Hom}_R(H_1(\mathcal{U}), V) \cong \operatorname{Hom}_R(\operatorname{Gr}^W_r H_1(\mathcal{U}), V) \cong \begin{cases} H^1(\Gamma, V) & r < 0 \\ 0 & r \geqslant 0 \end{cases}$$

Proof. Since $H_1(\mathcal{U})$ is negatively weighted, if $r \geqslant 0$, then $\operatorname{Hom}(H_1(\mathcal{U}), V) = 0$. Assume that $r < 0$. Let $[f]$ be the class of a continuous cocycle $f : \Gamma \to V$. Then define a map $\hat{\rho} : \Gamma \to V \rtimes R$ by setting $\gamma \mapsto (f(\gamma), \rho(\gamma))$. Here we consider the R-module V as a unipotent $\mathbb{Q}_\ell$-group, and then $V \rtimes R$ is a negatively weighted extension of R. The Zariski closure of the image of $\hat{\rho}$ in $V \rtimes R$ is also a negatively weighted extension of R. Thus by the universal property of weighted completion, we obtain a map $\Phi_f : \mathcal{G} \to V \rtimes R$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{G} & \longrightarrow & R & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & V & \longrightarrow & V \rtimes R & \longrightarrow & R & \longrightarrow & 1 \end{array}$$

commutes. Hence we obtain an R-module homomorphism $f_* : H_1(\mathcal{U}) \to V$. Recall that V acts on the cocycles $\Gamma \to V$ by conjugation; if $a \in V$ and $g : \Gamma \to V$ is a cocycle, then $(aga^{-1})(\gamma) = a + g(\gamma) - \gamma a$ (V is a Γ-module via ρ). Now, since the ℓ-adic unipotent completion $(\ker\rho)^{\mathrm{un}}_{/\mathbb{Q}_\ell}$ of the kernel of ρ surjects onto $\mathcal{U}$, the image of $\ker\rho$ in $\mathcal{U}(\mathbb{Q}_\ell)$ is Zariski dense. For any $a \in V$, the restrictions to $\ker\rho$ of the cocycles f and $g := afa^{-1}$ agree. Thus the induced morphisms $\Phi_f : \mathcal{G} \to V \rtimes R$ and $\Phi_g : \mathcal{G} \to V \rtimes R$ agree on $\mathcal{U}$. This shows $f_* = g_*$ and so we have obtained a map $\Psi : H^1(\Gamma, V) \to \mathrm{Hom}_R(H_1(\mathcal{U}), V), [f] \mapsto [f]_*$. Conversely, let $\phi : H_1(\mathcal{U}) \to V$ be a continuous R-module homomorphism. Pushing out the extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1,$$

along the canonical projection $\mathcal{U} \to H_1(\mathcal{U})$, we obtain the exact sequence

$$1 \to H_1(\mathcal{U}) \to \mathcal{G}' \to R \to 1.$$

Further, pushing out this sequence along the map ϕ, we get an exact sequence

$$1 \to V \to \mathcal{G}_\phi \to R \to 1.$$

Composing $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ with the induced map $\mathcal{G} \to \mathcal{G}_\phi$, we obtain a map $\tilde{\phi} : \Gamma \to \mathcal{G}_\phi(\mathbb{Q}_\ell)$ that lifts ρ. Since $\mathcal{G}_\phi$ is an extension of a reductive group by a unipotent group, it uniquely splits up to conjugation by an element of V. Hence $\tilde{\phi}$ gives a class of a continuous cocycle $\Gamma \to V$. Therefore, we obtain a map $\Omega : \mathrm{Hom}_R(H_1(\mathcal{U}), V) \to H^1(\Gamma, V)$. It can be easily checked that Ψ and Ω are inverse to each other. □

Let $\{V_\alpha\}$ be a set of representatives of isomorphism classes of irreducible representations of R. Denote the weight of V_α by $w(\alpha)$. The following result allows us to compute the generators of $\mathfrak{u}$ via the cohomology of Γ.

Theorem 6.12.2 ([24, Thm. 4.8]). *If $H^1(\Gamma, V_\alpha)$ is finite-dimensional for all α with $w(\alpha) < 0$, then*

$$H^1(\mathfrak{u}) \cong \bigoplus_{\{\alpha : w(\alpha) \geqslant 1\}} H^1(\Gamma, V_\alpha^*) \otimes V_\alpha$$

and

$$H_1(\mathfrak{u}) \cong \varprojlim_F \bigoplus_{\alpha \in F} H^1(\Gamma, V_\alpha)^* \otimes V_\alpha,$$

where F ranges over the finite subset of $\{\alpha : w(\alpha) < 0\}$.

Furthermore, we can bound the relations in $\mathfrak{u}$.

Theorem 6.12.3 ([24, Thm. 4.9])**.** *There is a natural R-homomorphism*

$$\Phi : H^2(\mathfrak{u}) \hookrightarrow \bigoplus_{\{\alpha : w(\alpha) \geqslant 2\}} H^2(\Gamma, V_\alpha^*) \otimes V_\alpha.$$

7

Relative Completion

If Γ is a discrete group with $H_1(\Gamma) \otimes \mathbb{Q} = 0$, then the unipotent completion of Γ is trivial. For example, this is the case for the mapping class groups Γ_g for $g \geqslant 3$. The relative completion of a discrete group Γ with respect to a reductive representation is a proalgebraic $\mathbb{Q}$-group that is an extension of a reductive algebraic $\mathbb{Q}$-group by a prounipotent $\mathbb{Q}$-group. This is a generalization of the (Malcev) unipotent completion of a discrete group. The definition of the relative completion is due to Deligne. Hain further developed and generalized it in [15, 17, 18].

7.1 Relative completion of a discrete group.

We recall the definition of relative completion. Suppose that:

1. Γ is a discrete group;
2. R is a reductive algebraic group defined over F, where F is a field of characteristic zero;
3. $\rho : \Gamma \to R(F)$ is a homomorphism with Zariski dense image.

Definition 7.1.1. The *relative completion* of Γ with respect to ρ consists of a proalgebraic F-group $\mathcal{G}$, that is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

where $\mathcal{U}$ is a prounipotent F-group and a Zariski dense homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(F)$ whose composition with $\mathcal{G}(F) \to R(F)$ is ρ. It is characterized by the following universal mapping property: If G is an affine (pro)algebraic F-group that is an extension

$$1 \to U \to G \to R \to 1$$

of R by a (pro)unipotent group U, and if $\phi : \Gamma \to G(F)$ is a homomorphism whose composition with $G(F) \to R(F)$ is ρ, then there is a unique homomorphism of proalgebraic F-groups $\Phi : \mathcal{G} \to G$ that commutes with the projections to R and such that $\phi = \Phi(F) \circ \tilde{\rho}$:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\rho}} & \mathcal{G}(F) \\ {\scriptstyle \phi}\downarrow & \swarrow{\scriptstyle \Phi} & \downarrow \\ G(F) & \longrightarrow & R(F) \end{array}$$

To construct the relative completion of Γ with respect to ρ, consider all the commutative diagrams of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(F) & \longrightarrow & E(F) & \longrightarrow & R(F) & \longrightarrow & 1 \\ & & & & {\scriptstyle \tilde{\rho}}\uparrow & \nearrow{\scriptstyle \rho} & & & \\ & & & & \Gamma & & & & \end{array}$$

where E is a linear algebraic F-group which is an extension of R by a unipotent F-group U, and where $\tilde{\rho}$ is a Zariski dense homomorphism which lifts $\rho : \Gamma \to R(F)$. All homomorphisms in the top row are algebraic group homomorphisms. Morphisms of such diagrams are defined in the obvious way. Moreover, the collection of such diagrams forms an inverse system [15, Prop. 2.1]. Then the completion $\mathcal{G}$ with respect to ρ is defined to be the inverse limit

$$\mathcal{G} = \varprojlim E$$

over all the above commutative diagrams.

An important property of relative completion is that it behaves well under base change. Suppose that E is an extension of F. By extending scalars to E, every proalgebraic group G over F yields a proalgebraic group $G \otimes_F E$. Assume that the image of $\Gamma \to R(E) = (R \otimes_F E)(E)$ is Zariski dense in $R \otimes_F E$. By the universal mapping property of the relative completion $\mathcal{G}_E$ of Γ with respect to $\rho : \Gamma \to R(E)$, we obtain a natural homomorphism $\mathcal{G}_E \to \mathcal{G} \otimes_F E$.

Theorem 7.1.2. *The natural homomorphism $\mathcal{G}_E \to \mathcal{G} \otimes_F E$ is an isomorphism.*

Proof. See [21, §3.2]. □

When R is the trivial group, one has $\mathcal{G} = \mathcal{U}$, which is a prounipotent group, and the pair $(\mathcal{U}, \Gamma \to \mathcal{U})$ is called the *unipotent completion* of Γ over F. It will be denoted by $\Gamma^{\mathrm{un}}_{/F}$.

7.2 Relative completion of $\Gamma^{\lambda}_{g,n}$

Suppose $2g - 2 + n > 0$. Let $H_A = H_1(\Sigma_g, A)$, $\Gamma = \Gamma^{\lambda}_{g,n}$, and $R = \mathrm{Sp}(H_{\mathbb{Q}})$, where $H_A = H_1(\Sigma_g, A)$ is the first homology group of the compact reference surface Σ_g. Let $\rho : \Gamma_{g,n} \to \mathrm{Sp}(H_{\mathbb{Q}})$ be the representation of the mapping class group on the first homology of the surface. Since the image of ρ is $\mathrm{Sp}(H_{\mathbb{Z}})$, ρ is a Zariski dense representation. Denote by $\mathcal{G}^{\mathrm{geom}}_{g,n}$ the relative completion of $\Gamma_{g,n}$ with respect to ρ and by $\mathcal{U}^{\mathrm{geom}}_{g,n}$ its prounipotent radical. Note that the base change theorem above implies that the relative completion of $\Gamma_{g,n}$ with respect to $\rho : \Gamma_{g,n} \to \mathrm{Sp}(H_{\mathbb{Q}_\ell})$ is $\mathcal{G}^{\mathrm{geom}}_{g,n} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Recall that the Torelli group $T_{g,n}$ is the kernel of the natural homomorphism $\Gamma_{g,n} \to \mathrm{Sp}(\mathbb{Z})$. Let $K_{g,n}$ be the subgroup of the Torelli group $T_{g,n}$ generated by Dehn twists along the separating simple closed curves on $\Sigma_{g,n}$. The following result is a special case of a result [39, Theorem B] of Putman.

Theorem 7.2.1. *Suppose that $g \geqslant 3$ and that $n \geqslant 0$. If Γ is a finite index subgroup of $T_{g,n}$ that contains $K_{g,n}$, then the inclusion induces an isomorphism $H_1(\Gamma, \mathbb{Q}) \to H_1(T_{g,n}, \mathbb{Q})$.*

Corollary 7.2.2 ([16, Cor. 6.7])**.** *Suppose that $g \geqslant 3$ and $n \geqslant 0$. If Γ is a finite index subgroup of $\Gamma_{g,n}$ that contains $K_{g,n}$, then the group $\mathcal{G}^{\mathrm{geom}}_{g,n}$ and the homomorphism $\Gamma \hookrightarrow \Gamma_{g,n} \to \mathcal{G}^{\mathrm{geom}}_{g,n}$ is the completion of Γ relative to the restriction of the representation $\rho : \Gamma_{g,n} \to \mathrm{Sp}(H_{\mathbb{Q}})$.*

7.3 Continuous relative completion of a profinite group.

We will need the profinite analogue of relative completion, since our main objects are profinite groups. Here we take the coefficient field F to be the field $\mathbb{Q}_\ell$ for some prime number ℓ.

Definition 7.3.1. Suppose that:

1. Γ is a profinite group;
2. R is a reductive algebraic group defined over $\mathbb{Q}_\ell$;
3. $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ is a continuous homomorphism with Zariski dense image.

The continuous relative completion of Γ with respect to ρ is a proalgebraic $\mathbb{Q}_\ell$-group $\mathcal{G}$ that is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

of R by a prounipotent $\mathbb{Q}_\ell$-group $\mathcal{U}$ and a continuous Zariski dense homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ which lifts ρ to $\mathcal{G}(\mathbb{Q}_\ell)$.

Like the relative completion of a discrete group, the continuous completion of a profinite group is also characterized by a universal mapping property that is the same as one for the discrete case except that all homomorphisms are required to be continuous in the ℓ-adic profinite case.

Denote the profinite completion of a discrete group Γ by $\Gamma^\wedge$. Consider Γ as a topological group whose neighborhoods of the identity are defined to be the finite index normal subgroups. We have the following theorem:

Theorem 7.3.2 ([20, Thm. 6.3])**.** *Suppose that* Γ *is a discrete group,* R *is a reductive* $\mathbb{Q}_\ell$*-group, and that* $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ *is a continuous, Zariski dense representation. Let* $\rho_\ell : \Gamma^\wedge \to R(\mathbb{Q}_\ell)$ *be the continuous extension of* ρ *to* $\Gamma^\wedge$*. If* $\mathcal{G}$ *and* $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ *is the completion of* Γ *with respect to* ρ*, then:*

1. $\tilde{\rho}$ *is continuous and thus induces a continuous homomorphism* $\hat{\rho}_\ell : \Gamma^\wedge \to \mathcal{G}(\mathbb{Q}_\ell)$*;*

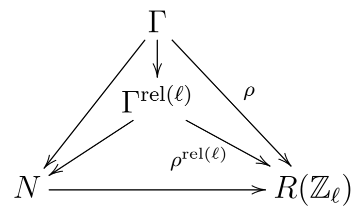
2. $\mathcal{G}$ *and* $\hat{\rho}_\ell$ *is the continuous relative completion of* $\Gamma^\wedge$ *with respect to* ρ_ℓ*.*

Suppose that Γ is a profinite group and that $\rho : \Gamma \to R(\mathbb{Z}_\ell)$ is a continuous homomorphism such that the composition with the inclusion $R(\mathbb{Z}_\ell) \to R(\mathbb{Q}_\ell)$ has Zariski dense image. Recall that $\rho^{\mathrm{rel}(\ell)} : \Gamma^{\mathrm{rel}(\ell),\rho} \to R(\mathbb{Z}_\ell)$ is the relative pro-ℓ completion of Γ with respect to ρ. Since $\Gamma \to \Gamma^{\mathrm{rel}(\ell),\rho}$ is surjective, $\rho^{\mathrm{rel}(\ell)} : \Gamma^{\mathrm{rel}(\ell),\rho} \to R(\mathbb{Q}_\ell)$ has Zariski dense image.

Proposition 7.3.3. *The continuous relative completion of* $\Gamma^{\mathrm{rel}(\ell),\rho}$ *with respect to the homomorphism* $\rho^{\mathrm{rel}(\ell)} : \Gamma^{\mathrm{rel}(\ell),\rho} \to R(\mathbb{Q}_\ell)$ *is isomorphic to the continuous relative completion* $\mathcal{G}$ *of* Γ *with respect to* ρ*.*

Proof. Denote $\Gamma^{\mathrm{rel}(\ell),\rho}$ by $\Gamma^{\mathrm{rel}(\ell)}$. Denote the relative completion of $\Gamma^{\mathrm{rel}(\ell)}$ with respect to the homomorphism $\rho^{\mathrm{rel}(\ell)} : \Gamma^{\mathrm{rel}(\ell)} \to R(\mathbb{Q}_\ell)$ by $\mathcal{G}^{\mathrm{rel}(\ell)}$. By functoriality, there exists a map $\phi : \mathcal{G} \to \mathcal{G}^{\mathrm{rel}(\ell)}$. Denote the image of Γ in $\mathcal{G}(\mathbb{Q}_\ell)$ by N. The kernel K of

$N \to R(\mathbb{Z}_\ell)$ lies in $\mathcal{U}(\mathbb{Q}_\ell)$ and is pro-ℓ. Thus, by the universal mapping property of $\Gamma^{\mathrm{rel}(\ell)}$, there exists a unique continuous homomorphism $\Gamma^{\mathrm{rel}(\ell)} \to N$ such that the diagram



commutes. Now, the universal mapping property of relative completion gives a map $\psi : \mathcal{G}^{\mathrm{rel}(\ell)} \to \mathcal{G}$, and it ensures that the maps ϕ and ψ are inverse to each other. □

8

Weighted Completion and Families of Curves

Suppose that k is a field, that T is a locally noetherian geometrically connected scheme over k, and that $C \to T$ is a curve of genus $g \geqslant 2$. Fix an algebraic closure $\bar{k}$ of k. Denote the base change to $\bar{k}$ of C and T by $C \otimes_k \bar{k}$ and $T \otimes_k \bar{k}$, respectively. Let $\bar{\eta} : \operatorname{Spec} \Omega \to T \otimes_k \bar{k}$ be a geometric point of $T \otimes_k \bar{k}$. By abuse of notation, $\bar{\eta}$ also denotes the image of $\bar{\eta}$ in T. Denote the geometric fiber of $C \otimes_k \bar{k}$ over $\bar{\eta}$ by $C_{\bar{\eta}}$. Let $\bar{x}$ be a geometric point of the fiber $C_{\bar{\eta}}$. The images of $\bar{x}$ in $C \otimes_k \bar{k}$ and C are also denoted by $\bar{x}$. Fix a prime number ℓ that is different from $\operatorname{char}(k)$. In this section, $H_{\mathbb{Z}_\ell} = H^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1))$ and $H_{\mathbb{Q}_\ell} = H_{\mathbb{Z}_\ell} \otimes \mathbb{Q}_\ell$. Let R be the Zariski closure of the image of the natural monodromy representation

$$\rho_{T,\bar{\eta}} : \pi_1(T, \bar{\eta}) \to \operatorname{GSp}(H_{\mathbb{Q}_\ell}).$$

Assuming that R contains the homotheties[1], we have the central cocharacter defined by

$$\omega : \mathbb{G}_m \to R \qquad z \mapsto z^{-1}\mathrm{id}_H,$$

[1] A theorem of Bogomolov [9] implies that this is always true when k is a number field.

which we call the standard cocharacter.

Recall that $\mathbb{L}_T$ is the set of prime numbers distinct from $\mathrm{char}(T)$. When there is no risk of confusion, we simply denote it by $\mathbb{L}$. Recall also that the morphism $C \to T$ induces an exact sequence of profinite groups

$$1 \to \pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x}) \to \pi_1'(C, \bar{x}) \to \pi_1(T, \bar{\eta}) \to 1,$$

where the middle group is the quotient by the image of the kernel of $\pi_1(C_{\bar{\eta}}) \to \pi_1^{\mathbb{L}}(C_{\bar{\eta}})$.

Proposition 8.0.4. *Suppose that T is a locally noetherian connected scheme. Let $f : C \to T$ be a proper smooth family of curves of genus $g \geqslant 2$. Then the monodromy action $\pi_1(C, \bar{x}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$ factors through the quotient $\pi_1'(C, \bar{x})$.*

Proof. Consider the fiber product diagram.

$$\begin{array}{ccccc}
C_{\bar{\eta}} & \longrightarrow & C \times_T C & \xrightarrow{p_2} & C \\
\downarrow & & {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle s} & & \downarrow{\scriptstyle f} \\
\mathrm{Spec}(\Omega) & \xrightarrow{\bar{x}} & C & \xrightarrow{f} & T
\end{array}$$

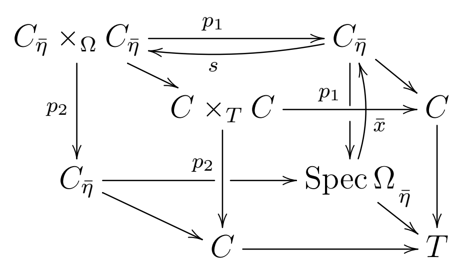
where $\bar{\eta} = f \circ \bar{x}$, p_1 and p_2 are canonical projections onto the first and second components, respectively, and s is the diagonal section of p_1. This diagram induces the following commutative diagram of algebraic fundamental groups:

$$\begin{array}{ccccccc}
\pi_1(C_{\bar{\eta}}, \bar{x}) & \longrightarrow & \pi_1(C \times_T C, \bar{x}) & \underset{s_*}{\overset{p_{1*}}{\rightleftarrows}} & \pi_1(C, \bar{x}) & \longrightarrow & 1 \\
\| & & \downarrow{\scriptstyle \mathfrak{p}_{2*}} & & \downarrow & & \\
\pi_1(C_{\bar{\eta}}, \bar{x}) & \longrightarrow & \pi_1(C, \bar{x}) & \longrightarrow & \pi_1(T, \bar{\eta}) & \longrightarrow & 1
\end{array}$$

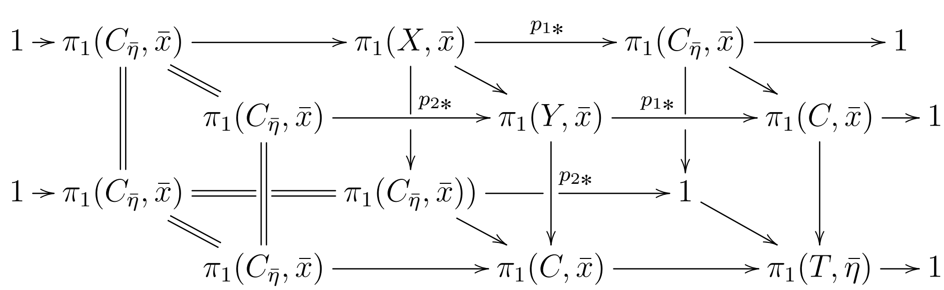
where the rows are exact. Pushing out along the surjection $\pi_1(C_{\bar{\eta}}, \bar{x}) \to \pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x})$, we obtain a diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x}) & \longrightarrow & \pi_1'(C \times_T C, \bar{x}) & \underset{s'_*}{\overset{p'_{1*}}{\rightleftarrows}} & \pi_1(C, \bar{x}) & \longrightarrow & 1 \\
& & \| & & \downarrow{\scriptstyle p'_{2*}} & & \downarrow & & \\
1 & \longrightarrow & \pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x}) & \longrightarrow & \pi_1'(C, \bar{x}) & \longrightarrow & \pi_1(T, \bar{\eta}) & \longrightarrow & 1
\end{array}$$

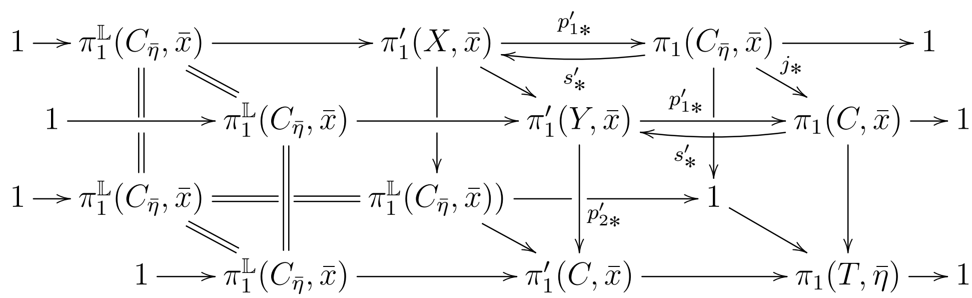
where the maps p'_{1*}, p'_{2*}, and s'_* are induced by p_{1*}, p_{2*}, and s_*, respectively. The left exactness of the first row follows since $p_1 : C \times_T C \to C$ admits a section s. We see that $\pi_1(C, \bar{x})$ acts on $\pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x})$ by conjugation via s'_*. We claim that this action factors through $\pi'_1(C, \bar{x})$ via p'_{2*}, but this is equivalent to saying that the kernel N of the projection $\pi_1(C, \bar{x}) \to \pi'_1(C, \bar{x})$ acts trivially on $\pi_1^{\mathbb{L}}(C_{\bar{\eta}}, \bar{x})$ through this action. Consider the following fiber product diagram:



By abuse of notation, p_1 and p_2 also denote the projection of $C_{\bar{\eta}} \times_\Omega C_{\bar{\eta}}$ onto the first and second components, respectively, and similarly for the diagonal section s of $p_1 : C_{\bar{\eta}} \times_\Omega C_{\bar{\eta}} \to C_{\bar{\eta}}$. This diagram induces the following commutative diagram of algebraic fundamental groups:



where, for simplicity, X and Y denote $C_{\bar{\eta}} \times_\Omega C_{\bar{\eta}}$ and $C \times_T C$, respectively. Pushing out along the surjection $\pi_1(C_{\bar\eta},\bar x)\to\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$, we obtain the diagram:



Note that $\pi_1(C_{\bar\eta},\bar x)$ acts on $\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$ as inner automorphisms via the section s'_*. Since the center of $\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$ is trivial, the inner automorphism group $\mathrm{Inn}(\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x))$ is also a pro-$\mathbb{L}$ group, and thus this action of $\pi_1(C_{\bar\eta},\bar x)$ factors through $\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$. That is, the kernel N' of the projection $\pi_1(C_{\bar\eta},\bar x)\to\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$ acts trivially as inner automorphisms. We have seen that N' maps onto the kernel of $K\to K^{\mathbb{L}}$ via the natural map $j_*:\pi_1(C_{\bar\eta},\bar x)\to\pi_1(C,\bar x)$, where K is the kernel of $\pi_1(C)\to\pi_1(T)$. By the property of fiber product diagram, N acts trivially on $\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$ via the section s'_* of the map $p'_{1*}:\pi_1'(Y,\bar x)\to\pi_1(C,\bar x)$. Hence the action of $\pi_1(C,\bar x)$ on $\pi_1^{\mathbb{L}}(C_{\bar\eta},\bar x)$ descends to the action of $\pi_1'(C,\bar x)$ via the projection p'_{2*}. $\square$

Lemma 8.0.5. *The monodromy representation $\pi_1(C,\bar x)\to\mathrm{GSp}(H_{\mathbb{Q}_\ell})$ factors through $\pi_1(T,\bar\eta)$.*

Proof. This follows immediately from the existence of the commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_1(C_{\bar\eta},\bar x) & \to & \pi_1(C,\bar x) & \to & \pi_1(T,\bar\eta) & \to & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathrm{Inn}(\Pi^{(\ell)}) & \to & \mathrm{Aut}(\Pi^{(\ell)}) & \to & \mathrm{Out}(\Pi^{(\ell)}) & \to & 1,
\end{array}$$

where $\Pi^{(\ell)}$ denotes the maximal pro-ℓ quotient $\pi_1(C_{\bar\eta},\bar x)^{(\ell)}$ of $\pi_1(C_{\bar\eta},\bar x)$ and rows are exact. $\square$

Since the canonical map $\pi_1(C,\bar x)\to\pi_1(T,\bar\eta)$ is surjective, it follows that the monodromy representation $\pi_1(C,\bar x)\to R(\mathbb{Q}_\ell)$ is also Zariski dense. Denote by $\mathcal{G}_C$

and $\mathcal{G}_T$ the weighted completions of $\pi_1(C,\bar{x})$ and $\pi_1(T,\bar{\eta})$ with respect to ω and their monodromy representations to R, respectively, and denote their prounipotent radicals by $\mathcal{U}_C$ and $\mathcal{U}_T$. Since the canonical map $\pi_1(C\otimes_k \bar{k},\bar{x}) \to \pi_1(T\otimes_k \bar{k},\bar{\eta})$ is surjective, their images in $R(\mathbb{Q}_\ell)$ are equal. Denote their common Zariski closure by R^{geom}, which is a reductive subgroup of R. Denote by $\mathcal{G}_C^{\mathrm{geom}}$ and $\mathcal{G}_T^{\mathrm{geom}}$ the continuous relative completion of $\pi_1(\bar{C},\bar{x})$ and $\pi_1(\bar{T},\bar{\eta})$ with respect to their monodromy representations to $R^{\mathrm{geom}}(\mathbb{Q}_\ell)$, respectively, and denote their prounipotent radicals by $\mathcal{U}_C^{\mathrm{geom}}$ and $\mathcal{U}_T^{\mathrm{geom}}$.

By pushing out the exact sequence

$$\pi_1(C_{\bar{\eta}},\bar{x}) \to \pi_1(C,\bar{x}) \to \pi_1(T,\bar{\eta}) \to 1$$

along the surjection $\pi_1(C_{\bar{\eta}},\bar{x}) \to \pi_1(C_{\bar{\eta}},\bar{x})^{(\ell)}$, we obtain the exact sequence

$$1 \to \pi_1(C_{\bar{\eta}},\bar{x})^{(\ell)} \to \pi_1'(C,\bar{x}) \to \pi_1(T,\bar{\eta}) \to 1$$

that fits in the commutative diagram

$$\begin{array}{ccccccccc}
 & & \pi_1(C_{\bar{\eta}},\bar{x}) & \longrightarrow & \pi_1(C,\bar{x}) & \longrightarrow & \pi_1(T,\bar{\eta}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \pi_1(C_{\bar{\eta}},\bar{x})^{(\ell)} & \longrightarrow & \pi_1'(C,\bar{x}) & \longrightarrow & \pi_1(T,\bar{\eta}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi^{(\ell)}) & \longrightarrow & \mathrm{Aut}(\Pi^{(\ell)}) & \to & \mathrm{Out}(\Pi^{(\ell)}) & \to & 1.
\end{array}$$

Denote by $\mathcal{G}_C'$ the weighted completion of $\pi_1'(C,\bar{x})$ with respect to ω and its monodromy representation $\pi_1'(C,\bar{x}) \to R(\mathbb{Q}_\ell)$.

Lemma 8.0.6. *With the notations above, there is a canonical isomorphism*

$$\mathcal{G}_C \cong \mathcal{G}_C'.$$

Similarly, there is a canonical isomorphism

$$\mathcal{G}_C^{\mathrm{geom}} \cong \mathcal{G}_C'^{\mathrm{geom}}.$$

Proof. By the functoriality of weighted completion, there is a unique map $\phi : \mathcal{G}_C \to \mathcal{G}'_C$. Denote the kernel of $\pi_1(C, \bar{x}) \to \pi'_1(C, \bar{x})$ by N. Recall that N is the kernel of the maximal pro-ℓ quotient $K \to K^{(\ell)}$, where K is the kernel of the canonical projection $\pi_1(C, \bar{x}) \to \pi_1(T, \bar{\eta})$. We have the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & \pi_1(C,\bar{x}) & \to & \pi'_1(C,\bar{x}) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{U}_C(\mathbb{Q}_\ell) & \to & \mathcal{G}_C(\mathbb{Q}_\ell) & \longrightarrow & R(\mathbb{Q}_\ell) & \longrightarrow & 1 \end{array}$$

Since compact subgroups of $\mathcal{U}(\mathbb{Q}_\ell)$ are pro-ℓ groups, the left vertical map must be trivial. Hence the canonical map $\pi_1(C, \bar{x}) \to \mathcal{G}(\mathbb{Q}_\ell)$ factors through $\pi'_1(C, \bar{x})$. By the universal property of weighted completion, there exists a unique map $\psi : \mathcal{G}'_C \to \mathcal{G}_C$. It is easy to see that ϕ and ψ are inverse to each other. □

Denote the continuous ℓ-adic unipotent completion of $\pi_1(C_{\overline{\eta}}, \bar{x})$ by $\mathcal{P}$. It is a prounipotent $\mathbb{Q}_\ell$-group. Since compact subgroups of $\mathbb{Q}_\ell$-points of a prounipotent group is pro-ℓ, the canonical map $\pi_1(C_{\overline{\eta}}, \bar{x}) \to \mathcal{P}$ factors through $\pi_1(C_{\overline{\eta}}, \bar{x})^{(\ell)}$, and furthermore there is a unique isomorphism $\mathcal{P} \cong \pi_1(C_{\overline{\eta}}, \bar{x})^{(\ell),\mathrm{un}}_{/\mathbb{Q}_\ell}$ of $\mathcal{P}$ and the unipotent completion of the maximal pro-ℓ quotient of $\pi_1(C_{\overline{\eta}}, \bar{x})$, since both completions admit the same universal property.

Proposition 8.0.7. *With the notation as above:*

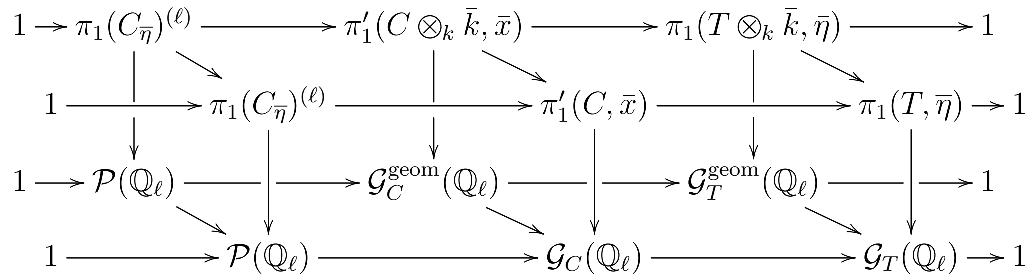
1. *There are exact sequences*

$$1 \to \mathcal{P} \to \mathcal{G}_C \to \mathcal{G}_T \to 1$$

and

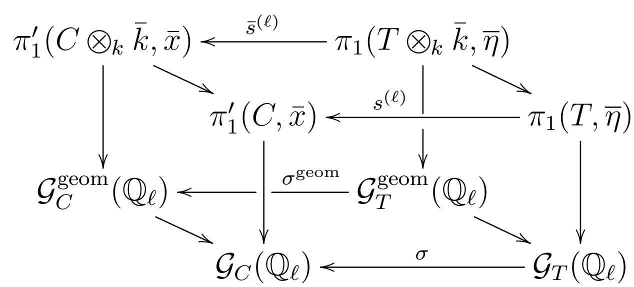
$$1 \to \mathcal{P} \to \mathcal{G}_C^{\mathrm{geom}} \to \mathcal{G}_T^{\mathrm{geom}} \to 1$$

of proalgebraic $\mathbb{Q}_\ell$-groups such that the diagram



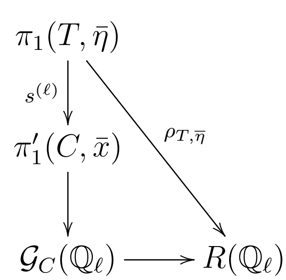
commutes.

2. *Every section s of $\pi_1(C, \bar{x}) \to \pi_1(T, \bar{\eta})$ induces sections $s^{(\ell)}$ and $\bar{s}^{(\ell)}$ of $\pi_1'(C, \bar{x}) \to \pi_1(T, \bar{\eta})$ and $\pi_1'(C \otimes_k \bar{k}, \bar{x})) \to \pi_1(T \otimes_k \bar{k}, \bar{\eta})$, respectively, and sections σ and σ^{geom} of $\mathcal{G}_C \to \mathcal{G}_T$ and $\mathcal{G}_C^{\mathrm{geom}} \to \mathcal{G}_T^{\mathrm{geom}}$, respectively, such that the diagram*



commutes.

Proof. The first part of the proposition follows from Proposition 6.11.4 with the right exactness of relative and weighted completion and the fact [35] that $\mathcal{P}$ has trivial center and $H_1(\mathcal{P})$ is pure of weight -1. For the second part of the proposition, the sections $s^{(\ell)}$ and $\bar{s}^{(\ell)}$ are induced by base change to $\bar{k}$ and by pushout. Since the diagram



commutes and since $\mathcal{G}_C$ is a negatively weighted extension of R, it follows from the universal mapping property of $\mathcal{G}_T$ that there exists a unique map $\sigma : \mathcal{G}_T \to \mathcal{G}_C$, which is a section of $\mathcal{G}_C \to \mathcal{G}_T$. A similar argument applies for the section σ^{geom}. $\square$

Denote the Lie algebras of R, $\mathcal{G}_C$, $\mathcal{G}_T$, $\mathcal{U}_C$, $\mathcal{U}_T$, $\mathcal{P}$ by $\mathfrak{r}$, $\mathfrak{g}_C$, $\mathfrak{g}_T$, $\mathfrak{u}_C$, $\mathfrak{u}_T$, $\mathfrak{p}$, respectively. These admit natural weight filtrations as objects of the category of $\mathcal{G}_C$-modules. By Proposition 6.8.4, their rth graded quotient is an R-module of weight r. Since $H_1(\mathcal{P}) = H_1(\mathfrak{p})$ is pure of weight -1, it follows that $\mathfrak{p} \cong W_{-1}\mathfrak{p}$, and by Proposition 6.8.4, we have

$$\mathfrak{g}_A = W_0\mathfrak{g}_A,\ W_{-1}\mathfrak{g}_A = \mathfrak{u}_A,\ \text{and}\ \mathrm{Gr}^W_0 \mathfrak{g}_A \cong \mathfrak{r},$$

where $A = C$ and $A = T$. The following corollary follows immediately from the fact that the functor $\mathrm{Gr}^W_\bullet$ is exact on the category of $\mathcal{G}_C$-modules.

Corollary 8.0.8. *With the notation above: There is an exact sequence*

$$0 \to \mathrm{Gr}^W_\bullet \mathfrak{p} \to \mathrm{Gr}^W_\bullet \mathfrak{g}_C \to \mathrm{Gr}^W_\bullet \mathfrak{g}_T \to 0$$

of graded Lie algebras in the category of R-modules.

9

Weighted Completion of Arithmetic Mapping Class Groups

In this section, we summarize and extend the results of [20, §8]. Suppose that g and n are integers satisfying $2g-2+n>0$. Fix prime numbers p and $\ell \neq p$. Denote the finite Galois cover of the moduli stack $\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$ given by Proposition 5.1.1 by $M^{\lambda}_{g,n}$. Choose a connected component of the base change to $\mathbb{Z}_p^{\mathrm{ur}}$ of $M^{\lambda}_{g,n}$ and denote it by $M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}$, where $\mathbb{Z}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Z}_p$. For $R=\bar{\mathbb{Q}}_p$ and $R=\bar{\mathbb{F}}_p$, the base change M^{λ}_R of $M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}$ is a connected smooth variety over R. Since $M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}$ is of finite type over $\mathbb{Z}_p^{\mathrm{ur}}$, it can be defined over some finite unramified extension S of $\mathbb{Z}_p$. Denote the fraction field and residue field of S by L and k, respectively. Denote the absolute Galois group of L and k by G_L and G_k, respectively.

Fix a geometric point $\bar{\eta}$ of $M^{\lambda}_{\bar{\mathbb{Q}}_p}$ and $\bar{\xi}$ of $M^{\lambda}_{\bar{\mathbb{F}}_p}$. Let $C_{\bar{y}}$ be the fiber of the universal curve over $\bar{y}$, where $\bar{y}=\bar{\eta}$ and $\bar{y}=\bar{\xi}$. Recall that for a $\mathbb{Z}_\ell$-module A,

$$H_A := H^1_{\text{ét}}(C_{\bar{y}}, A(1)).$$

Since the image of the ℓ-adic cyclotomic character $\chi_\ell : G_L \to \mathbb{Z}_\ell{}^\times$ is infinite, the image of $\chi_\ell : G_L \to \mathbb{G}_m(\mathbb{Q}_\ell)$ is Zariski dense. The image of the monodromy representation

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{Q}}_p,\bar{\eta}} : \pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$$

is of finite index in $\mathrm{Sp}(H_{\mathbb{Z}_\ell})$, and hence it is Zariski dense in $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \longrightarrow & \pi_1(M^\lambda_L, \bar{\eta}) & \longrightarrow & G_L & \longrightarrow & 1 \\ & & \downarrow \rho^{\mathrm{geom}}_{\bar{\mathbb{Q}}_p} & & \downarrow \rho_L & & \downarrow \chi_\ell & & \\ 1 & \longrightarrow & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathrm{GSp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}_\ell) & \longrightarrow & 1 \end{array}$$

implies that the image of the monodromy representation

$$\rho_{L,\bar{\eta}} : \pi_1(M^\lambda_L, \bar{\eta}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$$

is also Zariski dense. Denote the weighted completion of $\pi_1(M^\lambda_L, \bar{\eta})$ with respect to $\rho_{L,\bar{\eta}}$ and the standard cocharacter ω by

$$\mathcal{G}_{M^\lambda_L} \text{ and } \tilde{\rho}_{L,\bar{\eta}} : \pi_1(M^\lambda_L, \bar{\eta}) \to \mathcal{G}_{M^\lambda_L}(\mathbb{Q}_\ell).$$

Denote the pullback to $M^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}}$ of the universal curve $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ by $f : \mathcal{C}^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}} \to M^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}}$. Let $\pi : M^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}} \to \mathbb{Z}_p^{\mathrm{ur}}$ be the structure morphism of $M^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}}$ over $\mathbb{Z}_p^{\mathrm{ur}}$.

Proposition 9.0.9. *The image of the monodromy representation*

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}} : \pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$$

is pro-ℓ.

Proof. Since the kernel of the reduction map $\mathrm{Sp}(H_{\mathbb{Z}_\ell}) \to \mathrm{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$ is a pro-ℓ group, the statement then will follow, if the composition

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p} : \pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \overset{\rho^{\mathrm{geom}}}{\to} \mathrm{Sp}(H_{\mathbb{Z}_\ell}) \to \mathrm{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$$

is trivial. By the proper-smooth base change theorem [30, Ch.6 §4], the sheaf $R^1f_*\mu_\ell$ is a constructible locally constant étale sheaf on $M^\lambda_{\mathbb{Z}_p^{\mathrm{ur}}}$. Its fiber over a geometric point $\bar{y}$ is isomorphic to $H^1_{\text{ét}}(C_{\bar{y}}, \mu_{\ell^m}) = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g}$. Denote $R^1f_*\mu_{\ell^m}$ by $\mathcal{F}$. Let $\bar{s}_1$ be a geometric point lying over the generic point and $\bar{s}_2$ be the closed point of $\mathbb{Z}_p^{\mathrm{ur}}$. By a generalization of the proper-smooth base change theorem [14, SGA 1 Exposé XIII, 2.9], the specialization morphism, induced by the specialization $\bar{s}_1 \to \bar{s}_2$,

$$H^0_{\text{ét}}(M^\lambda_{\bar{\mathbb{F}}_p}, \mathcal{F}) = (\pi_*\mathcal{F})_{\bar{s}_2} \to (\pi_*\mathcal{F})_{\bar{s}_1} = H^0_{\text{ét}}(M^\lambda_{\bar{\mathbb{Q}}_p}, \mathcal{F})$$

is an isomorphism. Note that $H^0_{\text{ét}}(M^\lambda_{\bar{\mathbb{Q}}_p}, \mathcal{F}) = (\mathcal{F}_{\bar{\eta}})^{\pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta})}$. Since the standard representation $\Gamma^\lambda_{g,n} \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$ factors through the level ℓ^m subgroup $\Gamma_{g,n}[\ell^m]$, the composition with the reduction mod-ℓ^m map

$$\Gamma^\lambda_{g,n} \to \mathrm{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$$

is trivial, and so is the monodromy representation

$$\pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta}) \to \mathrm{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}}).$$

Thus we have

$$(\mathcal{F}_{\bar{\eta}})^{\pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta})} = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g},$$

which implies that

$$(\mathcal{F}_{\bar{\xi}})^{\pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi})} = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g}.$$

Therefore, the monodromy $\rho^{\mathrm{geom}} : \pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$ is trivial. □

Corollary 9.0.10. *The image of the monodromy representation*

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p, \bar{\xi}} : \pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$$

is pro-ℓ.

Proof. We use the same notation as in the proof of the above proposition. Denote the automorphism group of the étale cover $M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}} \to \mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m]$ by G. Note that $H^0_{\text{ét}}(M^{\lambda}_{\bar{\mathbb{F}}_p}, \mathcal{F})^G = H^0_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathcal{F})$ and that G acts trivially on $H^0_{\text{ét}}(M^{\lambda}_{\bar{\mathbb{F}}_p}, \mathcal{F})$ as it acts trivially on $H^0_{\text{ét}}(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \mathcal{F})$. Thus it follows that

$$H^0_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathcal{F}) = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g},$$

which implies that the monodromy $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell], \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$ has a pro-ℓ image. $\square$

Proposition 9.0.11. *The monodromy representation*

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}} : \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$$

has a finite index image in $\mathrm{Sp}(H_{\mathbb{Z}_\ell})$*, and so does the monodromy representation*

$$\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}} : \pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell}).$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1'(\mathcal{C}^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{x}') & \longrightarrow & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1'(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{x}') & \longrightarrow & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \phi & & \downarrow \phi' & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1'(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{x}) & \longrightarrow & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta}) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1'(\mathcal{C}^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{x}) & \longrightarrow & \pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \longrightarrow & 1,
\end{array}$$

whose rows are exact and the vertical maps between the second and third rows are isomorphisms. This diagram commutes once we fix an isomorphism $\phi : \pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{x}') \cong \pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{x})$, which determines an isomorphism $\phi' : \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) \cong \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta})$. Fix such an isomorphism. The proof of Proposition 9.0.9 also shows that the monodromy representation $\rho_{\mathbb{Z}_p^{\mathrm{ur}},\bar{\xi}} : \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$ also has a pro-ℓ image, since

$H^0_{\text{ét}}(M^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \mathcal{F}) \cong H^0_{\text{ét}}(M^{\lambda}_{\bar{\mathbb{F}}_p}, \mathcal{F})$ by the generalization of proper-smooth base change theorem. Thus it follows that the image of the monodromy representation $\pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x}') \to \text{Sp}(H_{\mathbb{Z}_\ell})$ is also pro-ℓ. This implies that the image of $\pi_1'(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x}')$ in $\text{Aut}(\pi_1(C_{\bar{\xi}})^{(\ell)})$ under its conjugation action on $\pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)}$ is also pro-ℓ, and hence this conjugation action factors through $\pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x}')^{(\ell)}$. Since the center of $\pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)}$ is trivial, it follows that the composition

$$\pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)} \to \pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x}')^{(\ell)} \to \text{Aut}(\pi_1(C_{\bar{\xi}})^{(\ell)})$$

is injective. Thus by taking maximal pro-ℓ quotients of the above diagram, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1(\mathcal{C}^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi})^{(\ell)} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{\xi})^{(\ell)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1(\mathcal{C}^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\text{ur}}}, \bar{\eta})^{(\ell)} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1(\mathcal{C}^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{x})^{(\ell)} & \longrightarrow & \pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta})^{(\ell)} & \longrightarrow & 1,
\end{array}$$

whose rows are exact and vertical maps are all isomorphisms. From this diagram, we see that the diagram

$$\begin{array}{ccc}
\pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi}) \\
\downarrow & & \downarrow \\
\pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta})^{(\ell)} & \xrightarrow{\cong} & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi})^{(\ell)} \\
\downarrow & & \downarrow \\
\text{Sp}(H_{\mathbb{Z}_\ell}) & \xrightarrow{\cong} & \text{Sp}(H_{\mathbb{Z}_\ell})
\end{array}$$

commutes, where the bottom isomorphism is induced by ϕ. Since the composition of the two left-hand vertical maps is the standard representation $(\Gamma^{\lambda}_{g,n})^{\wedge} \to \text{Sp}(H_{\mathbb{Z}_\ell})$, it has finite-index image, and so does the monodromy representation $\rho^{\text{geom}}_{\bar{\mathbb{F}}_p, \bar{\xi}}$:

$\pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$. The density of the monodromy $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$ follows since its image in $\mathrm{Sp}(H_{\mathbb{Z}_\ell})$ contains the image of $\pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$. $\square$

By Proposition 9.0.11, the image of $\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}} : \pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \mathrm{Sp}(H_{\mathbb{Q}_\ell})$ is Zariski dense. Since the image of the ℓ-adic cyclotomic character $\chi : G_k \to \mathbb{Z}_\ell^\times$ is infinity, the image of $\chi_\ell : G_k \to \mathbb{G}_m(\mathbb{Q}_\ell)$ is Zariski dense. The commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(M^\lambda_{\bar{\mathbb{F}}_p}, \bar{\xi}) & \longrightarrow & \pi_1(M^\lambda_k, \bar{\xi}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p}} & & \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \chi_\ell} & & \\
1 & \longrightarrow & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathrm{GSp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}_\ell) & \longrightarrow & 1.
\end{array}$$

implies that the monodromy representation

$$\rho_{k,\bar{\xi}} : \pi_1(M^\lambda_k, \bar{\xi}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$$

has Zariski dense image. Denote the weighted completion of $\pi_1(M^\lambda_k, \bar{\xi})$ with respect to $\rho_{k,\bar{\xi}}$ and the standard cocharacter ω by

$$\mathcal{G}_{M^\lambda_k} \quad \text{and} \quad \tilde{\rho}_{k,\bar{\xi}} : \pi_1(M^\lambda_k, \bar{\eta}) \to \mathcal{G}_{M^\lambda_k}(\mathbb{Q}_\ell).$$

Let $\bar{y}$ denote $\bar{\eta}$ and $\bar{\xi}$. Similarly, we have the weighted completion of $\pi_1(M^\lambda_S, \bar{y})$ with respect to $\rho_{S,\bar{y}} : \pi_1(M^\lambda_S, \bar{y}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$ and the cocharacter ω, denoted by

$$\mathcal{G}_{M^\lambda_S}, \quad \text{and} \quad \rho_{S,\bar{y}} : \pi_1(M^\lambda_S, \bar{y}) \to \mathcal{G}_{M^\lambda_S}(\mathbb{Q}_\ell).$$

Recall that $\mathcal{G}^{\mathrm{geom}}_{g,n/\mathbb{Q}_\ell}$ and $(\Gamma_{g,n})^\wedge \to \mathcal{G}^{\mathrm{geom}}_{g,n/\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$ is the relative completion of $(\Gamma_{g,n})^\wedge$ with respect to the standard representation $(\Gamma_{g,n})^\wedge \to \mathrm{Sp}(H_{\mathbb{Q}_\ell})$. For $g \geqslant 3$, the continuous relative completion of $\pi_1(M^\lambda_{\bar{\mathbb{Q}}_p}, \bar{\eta})$ with respect to its standard representation $\rho^{\mathrm{geom}}_{\bar{\mathbb{Q}}_p,\bar{\eta}}$ is isomorphic to $\mathcal{G}^{\mathrm{geom}}_{g,n/\mathbb{Q}_\ell}$ by Corollary 7.2.2 and Theorem 7.3.2. Similarly, for

$g \geqslant 3$, the continuous relative completion of $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta})$ with respect to its standard representation to $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$ is isomorphic to $\mathcal{G}^{\mathrm{geom}}_{g,n/\bar{\mathbb{Q}}_p}$ [18, Prop. 3.3]. When the field F is clear from context, we will denote $\mathcal{G}^{\mathrm{geom}}_{g,n/F}$ by $\mathcal{G}^{\mathrm{geom}}_{g,n}$.

Proposition 9.0.12. *The continuous relative completion of* $\pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi})$ *with respect to* $\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}}$ *is isomorphic to* $\mathcal{G}^{\mathrm{geom}}_{g,n}$*. Similarly, the continuous relative completion of* $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi})$ *with respect to* $\rho^{\mathrm{geom}}_{\bar{\mathbb{F}}_p,\bar{\xi}}$ *is isomorphic to* $\mathcal{G}^{\mathrm{geom}}_{g,n}$*.*

Proof. Fix an isomorphism $\phi : \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta}) \cong \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi})$. We have the following commutative diagram

$$\begin{array}{ccccccc}
\pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \longrightarrow & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta}) & \xrightarrow{\cong} & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) & \longleftarrow & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1(M^{\lambda}_{\bar{\mathbb{Q}}_p}, \bar{\eta})^{(\ell)} & \xrightarrow{\cong} & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta})^{(\ell)} & \xrightarrow{\cong} & \pi_1(M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi})^{(\ell)} & \xleftarrow{\cong} & \pi_1(M^{\lambda}_{\bar{\mathbb{F}}_p}, \bar{\xi})^{(\ell)} \\
& \searrow & \downarrow & & \downarrow & \swarrow & \\
& & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \xrightarrow{\cong} & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & & ,
\end{array}$$

where the isomorphism $\mathrm{Sp}(H_{\mathbb{Q}_\ell}) \cong \mathrm{Sp}(H_{\mathbb{Q}_\ell})$ is induced by the isomorphism ϕ and the isomorphisms on the second row are ones in the proof of Theorem 5.4.1. By taking the relative completion of each of the profinite groups with respect to its corresponding monodromy representation, we obtain the commutative diagram of proalgebraic $\mathbb{Q}_\ell$-groups

$$\begin{array}{ccccccc}
\mathcal{G}^{\mathrm{geom}}_{g,n} & \longrightarrow & \mathcal{G}^{\mathrm{geom}}_{M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \xrightarrow{\cong} & \mathcal{G}^{\mathrm{geom}}_{M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \longleftarrow & \mathcal{G}^{\mathrm{geom}}_{M^{\lambda}_{\bar{\mathbb{F}}_p}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{G}^{\mathrm{geom},(\ell)}_{g,n} & \xrightarrow{\cong} & \mathcal{G}^{\mathrm{geom},(\ell)}_{M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \xrightarrow{\cong} & \mathcal{G}^{\mathrm{geom},(\ell)}_{M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \xleftarrow{\cong} & \mathcal{G}^{\mathrm{geom},(\ell)}_{M^{\lambda}_{\bar{\mathbb{F}}_p}} \\
& \searrow & \downarrow & & \downarrow & \swarrow & \\
& & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \xrightarrow{\cong} & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & & .
\end{array}$$

Since the vertical maps between the first and second rows are isomorphism by Proposition 7.3.3, it follows that

$$\mathcal{G}^{\mathrm{geom}}_{M^{\lambda}_{\bar{\mathbb{F}}_p}} \cong \mathcal{G}^{\mathrm{geom}}_{M^{\lambda}_{\mathbb{Z}_p^{\mathrm{ur}}}} \cong \mathcal{G}^{\mathrm{geom}}_{g,n}.$$

A similar argument applies to the relative completion of $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi})$. $\square$

For a field F whose ℓ-adic cyclotomic character has an infinite image, denote the weighted completion of G_F with respect to the ℓ-adic cyclotomic character $\chi_\ell : G_F \to \mathbb{G}_m(\mathbb{Q}_\ell)$ and $\omega : z \mapsto z^{-2}$ by $\mathcal{A}_F$.

Throughout the rest of this section, for a prime ℓ, let M denote the étale covers $M^\lambda_{g,n}$ and $\mathcal{M}_{g,n}[\ell^m]$ of $\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$. As in above, we fix a connected component of the base change to S of M and denote it by M_S, where S is some finite unramified extension of $\mathbb{Z}_p$ over which M decomposes as a finite disjoint union of geometrically connected components. Recall that L and k are the fraction field and the residue field of S, respectively.

Proposition 9.0.13 ([20, 8.1]). *Applying weighted completion to the two right-hand columns and relative completion to the left-hand column of diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(M_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \longrightarrow & \pi_1(M_L, \bar{\eta}) & \longrightarrow & G_L & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho^{\mathrm{geom}}_{\bar{\mathbb{Q}}_p}} & & \downarrow{\scriptstyle \rho_L} & & \downarrow{\scriptstyle \chi_\ell} & & \\ 1 & \longrightarrow & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathrm{GSp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}_\ell) & \longrightarrow & 1 \end{array}$$

gives a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{G}^{\mathrm{geom}}_{g,n} & \longrightarrow & \mathcal{G}_{M_L} & \longrightarrow & \mathcal{A}_L & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathrm{GSp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}_\ell) & \longrightarrow & 1 \end{array}$$

whose rows are exact. Similar results hold if we replace the sequence

$$1 \to \pi_1(M_{\bar{\mathbb{Q}}_p}, \bar{\eta}) \to \pi_1(M_L, \bar{\eta}) \to G_L \to 1$$

with the exact sequence

$$1 \to \pi_1(M_{\bar{\mathbb{F}}_p}, \bar{\xi}) \to \pi_1(M_k, \bar{\xi}) \to G_k \to 1$$

and

$$1 \to \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{y}) \to \pi_1(M_S, \bar{y}) \to \pi_1(S, \bar{y}) \to 1,$$

where $\bar{y} = \bar{\eta}$ *and* $\bar{y} = \bar{\xi}$. □

Denote the prounipotent radicals of $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathcal{G}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, and $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$ by $\mathcal{U}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathcal{U}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, and $\mathcal{U}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$, respectively. Denote the Lie algebras of $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathcal{G}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$, $\mathcal{U}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathcal{U}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, and $\mathcal{U}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$ by $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathfrak{g}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$, $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$, $\mathfrak{u}^{\mathrm{geom}}_{M_{\mathbb{Z}_p^{\mathrm{ur}}}}$, and $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$, respectively.

Proposition 9.0.14. *Let* $F = L$ *and* k *and* $\bar{y} = \bar{\eta}$ *and* $\bar{\xi}$*, respectively. If* $2g-2+n > 0$ *and* $g \geqslant 3$*, then the natural action of* $\pi_1(M_F, \bar{y})$ *on* $\pi_1(M_{\bar{F}}, \bar{y})$ *induces an action of* $\mathcal{G}_{M_F}$ *on* $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$. *Therefore,* $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$ *and* $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ *are pro-objects of the category of* $\mathcal{G}_{M_F}$*-modules, and thus admit natural weight filtrations.*

Proof. The conjugation action of $\pi_1(M_F, \bar{y})$ on $\pi_1(M_{\bar{F}}, \bar{y})$ induces a homomorphism

$$\pi_1(M_F, \bar{y}) \to \mathrm{Aut}(\mathcal{G}^{\mathrm{geom}}_{M_{\bar{F}}})$$

and thus a homomorphism

$$\psi : \pi_1(M_F, \bar{y}) \to \mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}).$$

Define the filtration $D_\bullet \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$ by

$$D_0 \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}} = \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}, \quad D_{-i} \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}} = L^i \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}, \quad \text{for } i \geqslant 1$$

where $L^i \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ is the ith term of the lower central series of $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ with $L^1 \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}} = \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$. Proposition 9.0.12 implies that we have an isomorphism $\mathfrak{u}^{\mathrm{geom}}_{g,n} \cong \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ and it is shown in [18] that each graded quotient of $\mathfrak{u}^{\mathrm{geom}}_{g,n}$ associated to the lower central series of $\mathfrak{u}^{\mathrm{geom}}_{g,n}$ is finite dimensional. Since every automorphism of $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$ preserves the filtration, we have $\mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}) \cong \varprojlim_i \mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}/D_{-i})$, and hence $\mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}})$ is proalgebraic. Since $\mathfrak{p}$ is a $\mathcal{G}_{M_F}$-module, it has a natural weight filtration, and the exactness of the functor

$\mathrm{Gr}^W_\bullet$ and the fact that $H_1(\mathfrak{p})$ has weight -1 imply that the natural weight filtration on $\mathfrak{p}$ agrees with its lower central filtration. Similarly, $\mathrm{Der}\,\mathfrak{p}$ has a natural weight filtration as a $\mathcal{G}_{M_F}$-module. Johnson's computation of the abelianization of the Torelli group [28] implies that $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,1}) \to \mathrm{Gr}^W_{-1}\mathrm{Der}\,\mathfrak{p}$ is an isomorphism. Thus $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,1})$ is pure of weight -1. For $n > 1$, the morphism $\mathcal{M}_{g,n/\bar{F}} \to (\mathcal{M}_{g,1/\bar{F}})^n$ induces an injective map $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,n}) \to H_1(\mathfrak{u}^{\mathrm{geom}}_{g,1})^{\oplus n}$, and the morphism $\mathcal{M}_{g,1/\bar{F}} \to \mathcal{M}_{g/\bar{F}}$ induces a surjective map $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,1}) \to H_1(\mathfrak{u}^{\mathrm{geom}}_{g})$. Therefore, $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,n})$ is pure of weight -1 for all $n \geqslant 0$ and so is $H_1(\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}})$ for all $n \geqslant 0$. This implies that each graded quotient $\mathrm{Gr}^D_{-i}\,\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ is pure of weight $-i$ as a $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-module. Now, the monodromy representation $\rho_{F,\bar{y}} : \pi_1(M^\lambda_{\bar{F}}, \bar{y}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$ factors through $\mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_M)(\mathbb{Q}_\ell)$. Since the image of $\rho_{F,\bar{y}}$ is Zariski dense in $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$, the Zariski closure in $\mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_M)$ of the image of $\pi_1(M_F, \bar{y})$ in $\mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_M)(\mathbb{Q}_\ell)$ maps onto $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$ with prounipotent kernel K. Since K is negatively weighted, the universal mapping property of $\mathcal{G}_{M_F}$ gives a map $\mathcal{G}_{M_F} \to \mathrm{Aut}(\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}})$, which makes $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$ and so $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ as $\mathcal{G}_{M_F}$-modules. □

Remark 9.0.15. The induced natural weight filtration on $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}}$ indeed agrees with the filtration defined in this proof. The weight filtration clearly satisfies

$$\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}} = W_0\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}} \quad \text{and} \quad W_{-1}\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{F}}} = \mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}.$$

Again, the exactness of the functor $\mathrm{Gr}^W_\bullet$ and the fact that $H_1(\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}})$ has weight -1 imply that $W_{-r}\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$ is the rth term of the lower central series of $\mathfrak{u}^{\mathrm{geom}}_{M_{\bar{F}}}$. This coincidence allows us to apply the results of [18] in this paper.

Proposition 9.0.16. *The isomorphisms*

$$\mathfrak{g}^{\mathrm{geom}}_{g,n} \cong \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}} \cong \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$$

are morphisms in the category of $\mathcal{G}_{M_L}$-modules.

Proof. First consider the diagram

$$\begin{array}{ccccccc}
1 \longrightarrow & \pi_1(M_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \longrightarrow & \pi_1(M_L, \bar{\eta}) & \longrightarrow & G_L & \to 1 \\
& \downarrow & & \downarrow & & \| & \\
1 \to & \pi_1(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}, \bar{\eta})) & \to & \pi_1(\mathcal{M}_{g,n/L}, \bar{\eta}) & \to & G_L & \to 1,
\end{array}$$

whose rows are exact. $\pi_1(M_L, \bar{\eta})$ acts on $\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}, \bar{\eta})$ by conjugation via the homomorphism $\pi_1(M_L, \bar{\eta}) \to \pi_1(\mathcal{M}_{g,n/L}, \bar{\eta})$. This conjugation action induces an action of $\mathcal{G}_{M_L}$ on $\mathfrak{g}_{g,n}^{\mathrm{geom}}$ and hence the isomorphism $\mathfrak{g}_{M_{\bar{\mathbb{Q}}_p}}^{\mathrm{geom}} \to \mathfrak{g}_{g,n}^{\mathrm{geom}}$ is a $\mathcal{G}_{M_L}$-module homomorphism. Secondly, consider the diagram

$$\begin{array}{ccccccc}
1 \to & \pi_1(M_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \to & \pi_1(M_L, \bar{\eta}) & \longrightarrow & G_L & \longrightarrow 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta}) & \to & \pi_1(M_S, \bar{\eta}) & \to & \pi_1(S, \bar{\eta}) & \to 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) & \to & \pi_1(M_S, \bar{\xi}) & \to & \pi_1(S, \bar{\xi}) & \to 1 \\
& \uparrow & & \uparrow & & \uparrow & \\
1 \to & \pi_1(M_{\bar{\mathbb{F}}_p}, \bar{\xi}) & \to & \pi_1(M_k, \bar{\eta}) & \longrightarrow & G_k & \longrightarrow 1.
\end{array}$$

A choice of an isomorphism $\phi : \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta}) \cong \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi})$ determines isomorphisms $\pi_1(S, \bar{\eta}) \cong \pi_1(S, \bar{\xi})$ and $\pi_1(M_S, \bar{\eta}) \cong \pi_1(M_S, \bar{\xi})$, which makes the above diagram commute. Pushing out this diagram along the surjection $\pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi}) \to \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi})^{(\ell)}$ induces the commutative diagram

$$\begin{array}{ccccccc}
1 \longrightarrow & \pi_1(M_{\bar{\mathbb{Q}}_p}, \bar{\eta})^{(\ell)} & \longrightarrow & \pi_1'(M_L, \bar{\eta}) & \longrightarrow & G_L & \longrightarrow 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \longrightarrow & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\eta})^{(\ell)} & \longrightarrow & \pi_1'(M_S, \bar{\eta}) & \longrightarrow & \pi_1(S, \bar{\eta}) & \longrightarrow 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \longrightarrow & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}}, \bar{\xi})^{(\ell)} & \longrightarrow & \pi_1'(M_S, \bar{\xi}) & \longrightarrow & \pi_1(S, \bar{\xi}) & \longrightarrow 1 \\
& \uparrow & & \uparrow & & \uparrow & \\
1 \longrightarrow & \pi_1(M_{\bar{\mathbb{F}}_p}, \bar{\xi})^{(\ell)} & \longrightarrow & \pi_1'(M_k, \bar{\xi}) & \longrightarrow & G_k & \longrightarrow 1,
\end{array}$$

where rows are exact and all the left-hand vertical maps and the vertical maps between the third and fourth rows are isomorphisms. Thus $\pi_1(M_L, \bar{\eta})$ acts on

$\pi_1(M_{\bar{\mathbb{F}}_p},\bar{\xi})^{(\ell)}$ through the conjugation action of $\pi_1'(M_k,\bar{\xi})$ on $\pi_1(M_{\bar{\mathbb{F}}_p},\bar{\xi})^{(\ell)}$. Hence the induced isomorphism $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}} \cong \mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$ is a $\mathcal{G}_{M_L}$-module homomorphism. □

Recall that for a prime number ℓ, the corresponding finite étale cover $M^{\lambda}_{g,n}$ of $\mathcal{M}_{g,n}$ is defined over $\mathbb{Z}[1/\ell]$. Suppose that F is a field of characteristic zero such that the image of the ℓ-adic cyclotomic character $\chi_\ell : G_F \to \mathbb{G}_m(\mathbb{Z}_\ell)$ is infinity and such that a connected component M^{λ}_F of the base change to F of $M^{\lambda}_{g,n}$ is geometrically connected.

Proposition 9.0.17 ([20, 8.2])**.** *If $g \geqslant 3$, then for all $m \geqslant 1$ the natural homomorphism*

$$\mathcal{G}_{\mathcal{M}_{g,n/F}[m]} \to \mathcal{G}_{\mathcal{M}_{g,n/F}}$$

is an isomorphism, and furthermore for all prime numbers $\ell \geqslant 3$ the natural homomorphisms

$$\mathcal{G}_{M^{\lambda}_F} \to \mathcal{G}_{\mathcal{M}_{g,n/F}[\ell^m]} \to \mathcal{G}_{\mathcal{M}_{g,n/F}}$$

are isomorphisms.

From this point, we will denote the weighted completions $\mathcal{G}_{M^{\lambda}_F}$, $\mathcal{G}_{\mathcal{M}_{g,n/F}[m]}$, and $\mathcal{G}_{\mathcal{M}_{g,n/F}}$ by simply $\mathcal{G}_{g,n/F}$ and omit F when F is clear from the context. Similarly, we will denote the Lie algebras $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$ and $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$ by $\mathfrak{g}^{\mathrm{geom}}_{g,n}$. They are pro-objects in the category of $\mathcal{G}_{g,n}$-modules.

9.1 Variants

The comparison of the relative completions $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}$ and $\mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$ can be extended to the relative completion of the universal curve over M. Denote the pullback to $M_{\mathbb{Z}_p^{\mathrm{ur}}}$ of the universal curve $\mathcal{C}_{g,n}$ by $\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}$. The diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \to & \pi_1(C_{\bar\xi},\bar x')^{(\ell)} & \to & \pi_1'(\mathcal{C}_{\bar{\mathbb{F}}_p},\bar x') & \to & \pi_1(M_{\bar{\mathbb{F}}_p},\bar\xi) & \to & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(C_{\bar\xi},\bar x')^{(\ell)} & \to & \pi_1'(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}},\bar x') & \to & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}},\bar\xi) & \to & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(C_{\bar\eta},\bar x)^{(\ell)} & \to & \pi_1'(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}},\bar x) & \to & \pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}},\bar\eta) & \to & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \to & \pi_1(C_{\bar\eta},\bar x)^{(\ell)} & \to & \pi_1'(\mathcal{C}_{\bar{\mathbb{Q}}_p},\bar x) & \to & \pi_1(M_{\bar{\mathbb{Q}}_p},\bar\eta) & \to & 1
\end{array}$$

commutes, where rows are exact, after fixing an isomorphism $\pi_1(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}},\bar x')\cong\pi_1(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}},\bar x)$, which determines isomorphisms $\pi_1(C_{\bar\xi},\bar x')\cong\pi_1(C_{\bar\eta},\bar x)$ and $\pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}},\bar\xi)\cong\pi_1(M_{\mathbb{Z}_p^{\mathrm{ur}}},\bar\eta)$. Applying continuous relative completion to this diagram with respect to their natural monodromy representation to $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$ and taking Lie algebras, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{F}}_p}} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{g,n} & \longrightarrow & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{g,n} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{g,n} & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{Q}}_p}} & \longrightarrow & \mathfrak{g}^{\mathrm{geom}}_{g,n} & \longrightarrow & 1,
\end{array}$$

where rows are exact and all the left and right-hand vertical maps are isomorphisms. Proposition 8.0.7 implies that the map $\mathfrak{p}\to\mathfrak{g}^{\mathrm{geom}}$ is injective, since the composition $\mathfrak{p}\to\mathfrak{g}^{\mathrm{geom}}\to\mathfrak{g}$ is injective. Thus there is an isomorphism

$$\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{Q}}_p}}\cong\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{F}}_p}}.$$

As there is an isomorphism $\mathfrak{g}^{\mathrm{geom}}_{M_{\bar{\mathbb{Q}}_p}}\cong\mathfrak{g}^{\mathrm{geom}}_{g,n}$, there is an isomorphism $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{Q}}_p}}\cong\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$, and these isomorphisms are morphisms in the category of $\mathcal{G}_{\mathcal{C}_{g,n}}$-modules. Hence we will denote the Lie algebras $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{Q}}_p}}$ and $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{F}}_p}}$ by $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$. The canonical morphism $\mathcal{G}_{\mathcal{C}_{g,n}}\to\mathcal{G}_{g,n}$ makes $\mathfrak{g}^{\mathrm{geom}}_{g,n}$ a $\mathcal{G}_{\mathcal{C}_{g,n}}$-module.

Proposition 9.1.1. *Each section x of the universal curve $f : \mathcal{C}_k \to M_k$ induces a well-defined* $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$*-equivariant section of* $\mathrm{Gr}^W_\bullet f_* : \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}} \to \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{g,n}$.

Proof. By Proposition 8.0.7, each section x induces a section σ^{geom} of $f_* : \mathcal{G}^{\mathrm{geom}}_{\mathcal{C}_{\bar{\mathbb{F}}_p}} \to \mathcal{G}^{\mathrm{geom}}_{M_{\bar{\mathbb{F}}_p}}$, which is well defined up to conjugation by an element of $\mathcal{P}$. Thus the induced section $d\sigma^{\mathrm{geom}}$ of $df_* : \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}} \to \mathfrak{g}^{\mathrm{geom}}_{g,n}$ is a morphism of $\mathcal{G}_{\mathcal{C}_{g,n}}$-modules and is well defined up to addition of a section of the form $\mathrm{ad}(u) \circ d\sigma^{\mathrm{geom}}$ with u an element of $\mathfrak{p}$. Since $\mathrm{ad}(u) \in W_{-1} \operatorname{Der} \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$, the sections $d\sigma^{\mathrm{geom}}$ and $d\sigma^{\mathrm{geom}} + \mathrm{ad}(u) \circ d\sigma^{\mathrm{geom}}$ induce the same section of $\mathrm{Gr}^W_\bullet df_* : \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}} \to \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{g,n}$. Denote this section by $\mathrm{Gr}^W_\bullet d\sigma^{\mathrm{geom}}$. Since the action of $\mathcal{U}_{\mathcal{C}_{g,n}}$ on $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$ and $\mathfrak{g}^{\mathrm{geom}}_{g,n}$ is negatively weighted, the graded Lie algebras $\mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$ and $\mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{g,n}$ are $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-modules and $\mathrm{Gr}^W_\bullet d\sigma^{\mathrm{geom}}$ is $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant. □

10

Generators and Relations

In [20], Hain notices that the structure of $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}$ as a graded Lie algebra in the category of $S_n \times \mathrm{GSp}(H_{\mathbb{Q}_\ell})$-modules is an essential factor to understanding the rational points of the universal curve. In order to study the structure of $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}$, Hain uses the computations in [18] where a weight filtration is constructed by using Hodge theory. In [20] and this paper, the weight filtration on $\mathfrak{g}^{\mathrm{geom}}_{g,n}$ is constructed by using weighted completion. However, the weight filtration given by weighted completion agrees with the filtration produced by the lower central series of $\mathfrak{u}^{\mathrm{geom}}_{g,n}$, and so does the Hodge theoretical weight filtration on $\mathfrak{u}^{\mathrm{geom}}_{g,n}$. The agreement of the two construction follows from that in both construction $H_1(\mathfrak{u}^{\mathrm{geom}}_{g,n})$ is pure of weight -1 and the exactness of the functor $\mathrm{Gr}^W_\bullet$.

In this section, we summarize Hain's computation of the presentations of the Lie algebras $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}$ and computations in [18]. The details of this section can be found in [20, §8, §9] and [18].

10.1 S_n action on $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}$

We observe that $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}$ has an S_n-module structure as follows. Let F be a field of characteristic zero such that the ℓ-adic cyclotomic character $\chi_\ell : G_F \to \mathbb{G}_m(\mathbb{Q}_\ell)$ has Zariski dense image. Fix an algebraic closure $\bar{F}$ of F. The projection morphisms $\mathcal{M}_{g,n/F} \to \mathcal{C}_{g/F}$, $(C, x_1, \ldots, x_n) \mapsto (C, x_j)$ for $j = 1, \ldots, n$ induce an inclusion $\mathcal{M}_{g,n/F} \to \mathcal{C}^n_{g/F}$, where $\mathcal{C}^n_{g/F}$ denotes the nth power of the universal curve over $\mathcal{M}_{g/F}$. There is a natural monodromy representation $\rho : \pi_1(\mathcal{C}^n_{g/F}, \bar{\eta}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$, which has Zariski dense image. Denote the weighted completion of $\pi_1(\mathcal{C}^n_{g/F}, \bar{\eta})$ with respect to ρ and the standard cocharacter ω by $\widehat{\mathcal{G}}_{g,n}$. Denote the continuous relative completion of $\pi_1(\mathcal{C}^n_{g/\bar{F}}, \bar{\eta})$ with respect to $\bar{\rho} : \pi_1(\mathcal{C}^n_{g/\bar{F}}, \bar{\eta}) \to \mathrm{Sp}(H_{\mathbb{Q}_\ell})$ by $\widehat{\mathcal{G}}^{\mathrm{geom}}_{g,n}$. Denote the prounipotent radical of $\widehat{\mathcal{G}}_{g,n}$ and $\widehat{\mathcal{G}}^{\mathrm{geom}}_{g,n}$ by $\widehat{\mathcal{U}}_{g,n}$ and $\widehat{\mathcal{U}}^{\mathrm{geom}}_{g,n}$, respectively, and denote the Lie algebras of $\widehat{\mathcal{G}}_{g,n}$, $\widehat{\mathcal{G}}^{\mathrm{geom}}_{g,n}$, $\widehat{\mathcal{U}}_{g,n}$, and $\widehat{\mathcal{U}}^{\mathrm{geom}}_{g,n}$ by $\hat{\mathfrak{g}}_{g,n}$, $\hat{\mathfrak{g}}^{\mathrm{geom}}_{g,n}$, $\hat{\mathfrak{u}}_{g,n}$, and $\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}$, respectively. The inclusion $\mathcal{M}_{g,n/F} \to \mathcal{C}^n_{g/F}$ induces an $\mathcal{G}_{g,n/F}$-module homomorphism $\mathfrak{g}^{\mathrm{geom}}_{g,n} \to \hat{\mathfrak{g}}^{\mathrm{geom}}_{g,n}$ and thus an $S_n \times \mathrm{GSp}(H)$-module homomorphism $\mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{g,n} \to \mathrm{Gr}^W_\bullet \hat{\mathfrak{g}}^{\mathrm{geom}}_{g,n}$, where the action of S_n is induced by the action of S_n on $\mathcal{C}^n_{g/F}$ by permuting the n marked points.

Proposition 10.1.1 ([20, 8.7]). *If $g \geqslant 3$, then*

$$\mathrm{Gr}^W_j \mathfrak{u}^{\mathrm{geom}}_{g,n} \to \mathrm{Gr}^W_j \hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}$$

is an isomorphism for $j = -1$ and a surjective for $j = -2$ with kernel isomorphic to the $S_n \times \mathrm{GSp}(H_{\mathbb{Q}_\ell})$-module $\bigoplus_{i<j} \mathbb{Q}_\ell(1)$, where S_n acts by permuting the factors.

10.2 Presentation of $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1}/W_{-3}$

The action of $\pi_1(\mathcal{M}_{g,1/F}, \bar{\eta})$ on $\mathfrak{p}$ induces an action of $\mathcal{G}_{g,1/F}$ on $\mathfrak{p}$. Thus $\mathfrak{p}$ has a natural weight filtration. This $\mathcal{G}_{g,1/F}$-action on $\mathfrak{p}$ induces a Lie algebra homomorphism $\mathfrak{g}_{g,1} \to$

$\operatorname{Der}\mathfrak{p}$. Composing with the Lie algebra homomorphism $\mathfrak{g}_{g,1}^{\mathrm{geom}} \to \mathfrak{g}_{g,1}$ induced by the natural map $\pi_1(\mathcal{M}_{g,1/\bar{F}}, \bar{\eta}) \to \pi_1(\mathcal{M}_{g,1/F}, \bar{\eta})$, we obtain a $\mathcal{G}_{g,1}$-equivariant Lie algebra homomorphism $\mathfrak{g}_{g,1}^{\mathrm{geom}} \to \operatorname{Der}\mathfrak{p}$ and hence a weight graded Lie algebra homomorphism $\mathrm{Gr}^W_\bullet \mathfrak{g}_{g,1}^{\mathrm{geom}} \to \mathrm{Gr}^W_\bullet \operatorname{Der}\mathfrak{p} \cong \operatorname{Der}\mathrm{Gr}^W_\bullet \mathfrak{p}$.

Proposition 10.2.1 ([18, §9, §10]). *If $g \geqslant 3$, then the homomorphism*

$$\mathrm{Gr}^W_j \mathfrak{u}_{g,1}^{\mathrm{geom}} \to \mathrm{Gr}^W_j \operatorname{Der}\mathfrak{p}$$

is an isomorphism for $j = -1$ and an injection for $j = -2$.

Therefore, in order to compute the presentation of the Lie algebra $\mathrm{Gr}^W_\bullet \mathfrak{u}_{g,1}^{\mathrm{geom}}/W_{-3}$, we consider the action of $\mathrm{Gr}^W_\bullet \mathfrak{u}_{g,1}^{\mathrm{geom}}/W_{-3}$ on $\mathrm{Gr}^W_\bullet \mathfrak{p}$. We have seen that the natural weight filtration on $\mathfrak{p}$ induced by the $\mathcal{G}_{g,1}$ action agrees with the lower central series of $\mathfrak{p}$. Recall that $\check{\theta}$ is a map $\mathbb{Q}(1) \to \Lambda^2 H$, viewed as a map $\check{\theta} : \mathbb{Q}(1) \to \mathbb{L}_2(H)$ via the canonical isomorphism $\Lambda^2 H \cong \mathbb{L}_2(H)$.

Proposition 10.2.2 ([33, 10.3]). *There is a natural $\mathrm{GSp}(H)$-equivariant Lie algebra isomorphism*

$$\mathrm{Gr}^W_\bullet \mathfrak{p} \cong \mathbb{L}(H)/(\operatorname{im}\check{\theta}).$$

Remark 10.2.3. Since inner automorphisms map to the identity element in $\mathrm{GSp}(H)$, they act trivially on this isomorphism and consequently this isomorphism does not depend on the choice of a base point.

The free Lie algebra $\mathbb{L}(V)$ generated by a F-vector space V is graded by bracket length: there is an isomorphism

$$\mathbb{L}(V) \cong \bigoplus_{n \geqslant 1} \mathbb{L}_n(V).$$

Now, since a derivation on $\mathbb{L}(H)$ is determined by its effect on H, we see that $\operatorname{Der}\mathbb{L}(H) \cong \mathrm{Hom}_F(H, \mathbb{L}(H))$ and that the derivation Lie algebra $\operatorname{Der}\mathbb{L}(H)$ is graded:

$$\operatorname{Der}\mathbb{L}(H) \cong \bigoplus_{n \geqslant 1} \operatorname{Der}^n \mathbb{L}(H),$$

where $\operatorname{Der}^n \mathbb{L}(H) := \operatorname{Hom}_F(H, \mathbb{L}_{n+1}(H))$.

We recall the following well known fact.

Proposition 10.2.4. *If $g \geqslant 2$, then there is a natural $\operatorname{GSp}(H)$-equivariant graded Lie algebra homomorphism*

$$\delta : \mathbb{L}\big((\Lambda^3 H)(-1)\big) \to \operatorname{Der} \mathbb{L}(H)$$

satisfying the properties:

1. *$\delta(u)$ kills the image of $\check{\theta}$ for all $u \in \mathbb{L}((\Lambda^3 H)(-1))$.*
2. *$\delta(x \wedge \check{\theta}) = \operatorname{ad}(x) - \theta(x, \ )\check{\theta}$ for all $x \in H$.*
3. *$\operatorname{im}\big\{\mathbb{L}_2\big((\Lambda^3 H)(-1)\big) \to \operatorname{Der}^2 \mathbb{L}(H)\big\} \cong H_{\boxplus} \oplus \Lambda_0^2 H \oplus F(1)$*

□

In fact, the homomorphism δ is obtained by twisting the $\operatorname{GSp}(H)$-equivariant linear mapping

$$\Lambda^3 H \to \big[\operatorname{Der}^1 \mathbb{L}(H)\big](1)$$

defined by

$$x \wedge y \wedge z \mapsto \big(u \mapsto \theta(u,x)[y,z] + \theta(u,y)[z,x] + \theta(u,z)[x,y]\big).$$

By the property (i), δ induces a $\operatorname{GSp}(H)$-equivariant graded Lie algebra homomorphism

$$\mathbb{L}\big((\Lambda^3 H)(-1)\big) \to \operatorname{Der} \operatorname{Gr}^W_\bullet \mathfrak{p},$$

which we denote by δ also.

Theorem 10.2.5. *[18, Cor. 5.7 and §9,11] If $g \geqslant 3$, then there is a Lie algebra surjection*

$$q : \mathbb{L}\big((\Lambda^3 H)(-1)\big) \to \operatorname{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1}$$

that makes the diagram

$$\begin{array}{ccc} \mathbb{L}\big((\Lambda^3 H)(-1)\big) & \xrightarrow{q} & \mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1} \\ & {\scriptstyle\delta}\searrow \quad \swarrow & \\ & \mathrm{Gr}^W_\bullet \mathrm{Der}\,\mathfrak{p} & \end{array}$$

commute. Consequently, there are isomorphisms

$$\mathrm{Gr}^W_j \mathfrak{u}^{\mathrm{geom}}_{g,1} \cong \mathrm{Gr}^W_j \mathrm{Der}\,\mathfrak{p} \cong \begin{cases} (\Lambda^3 H)(-1) = \Lambda^3_0 H \oplus H & : j = -1, \\ H_{\boxplus} \oplus \Lambda^2_0 H & : j = -2. \end{cases}$$

□

The exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{u}^{\mathrm{geom}}_{g,1} \to \mathfrak{u}^{\mathrm{geom}}_{g} \to 0$$

induces the exact sequence of weight graded Lie algebras

$$0 \to \mathrm{Gr}^W_\bullet \mathfrak{p} \to \mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1} \to \mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g} \to 0.$$

The copy of H in $\mathrm{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,1}$ can be identified as the image of the composition map

$$H \xrightarrow{i} \big(\Lambda^3 H(-1)\big) \xrightarrow{q} \mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1},$$

where the inclusion $i : H \to \big(\Lambda^3 H(-1)\big)$ is defined by $x \to x \wedge \check{\theta}$. This copy of H in $\mathrm{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,1}$ corresponds to the inner derivations $\mathrm{ad}(x)$ in $\mathrm{Gr}^W_{-1} \mathrm{Der}\,\mathfrak{p}$. Similarly, identify the copy of $\Lambda^2_0 H$ in $\mathrm{Gr}^W_{-2} \mathfrak{u}^{\mathrm{geom}}_{g,1}$ as the image of the composition map

$$\Lambda^2 H \xrightarrow{\Lambda^2 i} \Lambda^2\big((\Lambda^3 H)(-1)\big) \xrightarrow{[\ ,\ ]} \mathrm{Gr}^W_{-2} \mathfrak{u}^{\mathrm{geom}}_{g,1}.$$

In order to determine the presentation of $\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,1}/W_{-3}$, we need to determine the bracket

$$\Lambda^2 \mathrm{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,1} = \Lambda^2\big((\Lambda^3 H)(-1)\big) \to \mathrm{Gr}^W_{-2} \mathfrak{u}^{\mathrm{geom}}_{g,1}.$$

Using the decomposition $(\Lambda^3 H)(-1) = \Lambda_0^3 H \oplus H$, this bracket is decomposed into three $\mathrm{GSp}(H)$-equivariant mappings:

$$\Lambda^2 H \to \mathrm{Gr}^W_{-2}\, \mathfrak{u}^{\mathrm{geom}}_{g,1}, \quad H \otimes \Lambda_0^3 H \to \mathrm{Gr}^W_{-2}\, \mathfrak{u}^{\mathrm{geom}}_{g,1}, \quad \Lambda^2(\Lambda_0^3 H) \to \mathrm{Gr}^W_{-2}\, \mathfrak{u}^{\mathrm{geom}}_{g,1}.$$

These brackets can be computed in $\mathrm{Gr}^W_\bullet \mathrm{Der}\, \mathfrak{p}$.

Proposition 10.2.6. *[18, §12] If $g \geqslant 3$, then the three brackets are determined as:*

$$\mathrm{im}\left\{\Lambda^2 H \to \mathrm{Gr}^W_{-2} \mathrm{Der}\, \mathfrak{p}\right\} = \Lambda_0^2 H.$$

$$\mathrm{im}\left\{H \otimes \Lambda_0^3 H \to \mathrm{Gr}^W_{-2} \mathrm{Der}\, \mathfrak{p}\right\} = \Lambda_0^2 H.$$

$$\mathrm{im}\left\{\Lambda^2(\Lambda_0^3 H) \to \mathrm{Gr}^W_{-2} \mathrm{Der}\, \mathfrak{p}\right\} = \begin{cases} H_{\boxplus} & g = 3, \\ H_{\boxplus} \oplus \Lambda_0^2 H & g \geqslant 4. \end{cases}$$

□

10.3 Presentations of $\mathrm{Gr}^W_\bullet\, \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}$

Hain introduces an $S_n \times \mathrm{GSp}(H)$-module denoted by $\Lambda_n^3 H$. For $u \in \Lambda^3 H(-1)$, denote the image of u in $\Lambda_0^3 H$ by $\bar{u}$. For each positive integer n, we define

$$\Lambda_n^3 H := \{(u_1, \ldots, u_n) \in \left(\Lambda^3 H(-1)\right)^n | \bar{u}_1 = \cdots = \bar{u}_n\}.$$

This $S_n \times \mathrm{GSp}(H)$ submodule of $\left(\Lambda^3 H(-1)\right)^n$ is isomorphic to

$$\Lambda_0^3 H \oplus H^{\oplus n}$$

as an $S_n \times \mathrm{GSp}(H)$-module.

Theorem 10.3.1 ([20, 9.11])**.** *If $g \geqslant 3$ and $n \geqslant 0$, then there are natural $S_n \times \mathrm{GSp}(H)$-equivariant isomorphisms*

$$H_1(\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}) \cong H_1(\mathfrak{u}^{\mathrm{geom}}_{g,n}) \cong \mathrm{Gr}^W_{-1}\, \mathfrak{u}^{\mathrm{geom}}_{g,n} \cong \Lambda_n^3 H.$$

There is an exact sequence

$$0 \to \mathbb{Q}_\ell(1)^{\binom{n}{2}} \to \mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \to \mathrm{Gr}^W_{-2}\,\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n} \to 0$$

of $S_n \times \mathrm{GSp}(H)$*-modules and an* $S_n \times \mathrm{GSp}(H)$*-equivariant isomorphism*

$$\mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \cong H_{\boxplus} \oplus \left(\Lambda^2_0 H\right)^n \oplus \mathbb{Q}_\ell(1)^{\binom{n}{2}}.$$

□

In order to determine the presentations of $\mathrm{Gr}^W_\bullet\,\mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}$, we need to determine the bracket $[\ ,\ ] : \Lambda^2\,\mathrm{Gr}^W_{-1}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \to \mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n}$. For this purpose, we write

$$\Lambda^3_n H = \Lambda^3_0 H \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n,$$

where H_j corresponds to the jth marked point. By Theorem 10.3.1, $\Lambda^2\,\mathrm{Gr}^W_{-1}\,\mathfrak{u}^{\mathrm{geom}}_{g,n}$ can be decomposed as

$$\Lambda^2\,\mathrm{Gr}^W_{-1}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \cong \Lambda^2\Lambda^3_0 H \oplus \bigoplus_{j=1}^n (H_j \otimes \Lambda^3_0 H) \oplus \bigoplus_{j=1}^n \Lambda^2 H_j \oplus \bigoplus_{i<j} H_i \otimes H_j,$$

and we also have

$$\mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \cong \bigoplus_{j=1}^n \Lambda^2_0 H_j \oplus \bigoplus_{i<j} \mathbb{Q}_\ell(1)_{ij} \oplus H_{\boxplus}.$$

By [18, §13], we may choose this decomposition so that the bracket

$$[\ ,\ ] : \Lambda^2 H_j \to \mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \stackrel{\mathrm{proj}}{\to} \Lambda^2_0 H_j$$

is the quotient map and so that the bracket

$$[\ ,\ ] : H \otimes H \cong H_i \otimes H_j \to \mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \stackrel{\mathrm{proj}}{\to} \mathbb{Q}_\ell(1)_{ij}$$

is the cup product pairing θ. Fix $\mathrm{GSp}(H)$-equivariant projections

$$\mathbf{c} : \Lambda^2\Lambda^3_0 H \to \Lambda^2_0 H \quad \mathbf{d} : H \otimes \Lambda^3_0 H \to \Lambda^2_0 H$$

$$\mathbf{e} : \Lambda^2 H \to \Lambda^2_0 H, \quad \psi : \Lambda^2\Lambda^3_0 H \to \mathbb{Q}_\ell(1).$$

By Proposition 3.2.1, each of these projections are unique up to a scalar multiplication. Denote the GSp(H)-equivariant projections

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to \Lambda^2\Lambda^3_0 H \xrightarrow{\mathbf{c}} \Lambda^2_0 H$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to H_i \otimes \Lambda^3_0 H \xrightarrow{\mathbf{d}} \Lambda^2_0 H$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to \Lambda^2 H_j \xrightarrow{\mathbf{e}} \Lambda^2_0 H$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to H_i \otimes H_j \xrightarrow{\mathbf{e}} \Lambda^2_0 H$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to \Lambda^2 H_i \xrightarrow{\theta} \mathbb{Q}_\ell(1)$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to H_i \otimes H_j \xrightarrow{\theta} \mathbb{Q}_\ell(1)$$

$$\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} \to \Lambda^2\Lambda^3_0 H \xrightarrow{\psi} \mathbb{Q}_\ell(1)$$

by $\mathbf{c}$, $\mathbf{d}_j$, $\mathbf{e}_j$, $\mathbf{e}_{ij}$, θ_i, θ_{ij}, and ψ, respectively. By Proposition 3.2.1 and the above decomposition of $\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n}$, we have

Proposition 10.3.2 ([20, 9.12])**.** *If $g \geqslant 3$ and $n \geqslant 0$, then*

$$\operatorname{Hom}_{\mathrm{GSp}(H)}(\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n}, \Lambda^2_0 H)$$

has a basis

$$\{\mathbf{d}_1, \dots, \mathbf{d}_n, \mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{ij} : 1 \leqslant i < j \leqslant n\} : g = 3$$

$$\{\mathbf{c}, \mathbf{d}_1, \dots, \mathbf{d}_n, \mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{ij} : 1 \leqslant i < j \leqslant n\} : g \geqslant 4$$

and

$$\operatorname{Hom}_{\mathrm{GSp}(H)}(\Lambda^2 \operatorname{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n}, \mathbb{Q}_\ell(1))$$

has a basis

$$\{\psi, \theta_1, \dots, \theta_n, \theta_{ij} : 1 \leqslant i < j \leqslant n\}$$

for all $g \geqslant 3$. □

Denote the projections

$$\mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \to \Lambda^2_0 H_j$$

$$\mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \to \mathbb{Q}_\ell(1)_{ij}$$

$$\mathrm{Gr}^W_{-2}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} \to H_{\boxplus}$$

by p_j, q_{ij}, and $p_{\boxplus}$, respectively.

Proposition 10.3.3 ([20, 9.13],[18, §12]). *If $g \geqslant 3$, then after rescaling $\mathbf{c}$, $\mathbf{d}$, and ψ by nonzero constants if necessary,*

$$\begin{aligned} p_j \circ [\ ,\ ] &= \mathbf{d}_j + \mathbf{e}_j && : g = 3 \\ p_j \circ [\ ,\ ] &= \mathbf{c} + \mathbf{d}_j + \mathbf{e}_j && : g \geqslant 4 \\ q_{ij} \circ [\ ,\ ] &= \psi + \theta_{ij} - \frac{1}{g}(\theta_i + \theta_j) && : g \geqslant 3 \end{aligned}$$

Furthermore, $p_{\boxplus} \circ [\ ,\ ]$ is non-zero and restricts to zero on each $H_j \otimes \Lambda^3_0 H$ and each $H_i \otimes H_j$ for $i \leqslant j$. □

Remark 10.3.4. The author of [20] seems to omit the factor $-\frac{1}{g}(\theta_i + \theta_j)$ by mistake.

10.4 The Lie Algebras $\mathfrak{d}_{g,n}$

In this section, we associate a graded two-step Lie algebra $\mathfrak{d}_{g,n}$ to the graded Lie algebra $\mathrm{Gr}^W_\bullet\,\mathfrak{u}^{\mathrm{geom}}_{g,n}$. When $g \geqslant 4$, the n tautological sections of the universal curve determine the $\mathrm{GSp}(H)$-equivariant Lie algebra sections of the projection $\mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}$.

For $g \geqslant 3$ and $n \geqslant 0$, define

$$\mathfrak{d}_{g,n} = \left(\mathrm{Gr}^W_\bullet\,\mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3}\right) / \left(\Lambda^2_0 H\right)^\perp,$$

where $(\Lambda^2_0 H)^\perp$ denotes the $\mathrm{GSp}(H)$-invariant complement of its $\Lambda^2_0 H$-isotypical component. This is a graded Lie algebra in the category of $S_n \times \mathrm{GSp}(H)$-modules and each jth graded quotient is given by

$$(\mathfrak{d}_{g,n})_j = \begin{cases} \Lambda^3_n H & : j = -1 \\ (\Lambda^2_0 H)^n & : j = -2 \\ 0 & : j \leqslant -3 \end{cases}$$

Let F be a number field, a finite extension of $\mathbb{Q}_p$, or a finite field of characteristic p. Recall that M denotes the étale covers $M^\lambda_{g,n}$ and $\mathcal{M}_{g,n}[\ell^m]$. Assume that $M \otimes F$ is decomposed into finitely many geometrically connected components. Fix a component of $M \otimes F$ and denote it by M_F.

Proposition 10.4.1. *Each section x of the universal curve $f : \mathcal{C}_F \to M_F$ induces a well-defined* $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$*-equivariant section of* $\mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}$.

Proof. By Proposition 9.1.1, each section x of the family $\mathcal{C}_F \to M_F$ induces a well-defined $\mathrm{GSp}(H)$-equivariant section $\mathrm{Gr}^W_\bullet \, d\sigma^{\mathrm{geom}}_x$ of the projection

$$\mathrm{Gr}^W_\bullet f_* : \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}} \to \mathrm{Gr}^W_\bullet \mathfrak{g}^{\mathrm{geom}}_{g,n}.$$

The open immersion $\mathcal{M}_{g,n+1/\bar{\mathbb{Q}}_p} \to \mathcal{C}_{g,n/\bar{\mathbb{Q}}_p}$ induces a surjection $\pi_1(\mathcal{M}_{g,n+1/\bar{\mathbb{Q}}_p}, \overline{\eta}) \to \pi_1(\mathcal{C}_{g,n/\bar{\mathbb{Q}}_p}, \overline{\eta})$, which induces a Lie algebra surjection $\mathfrak{g}^{\mathrm{geom}}_{g,n+1} \to \mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$. This Lie algebra surjection is a $\mathcal{G}_{g,n+1}$-module homomorphism and hence it preserves the natural weight filtrations on $\mathfrak{g}^{\mathrm{geom}}_{g,n+1}$ and $\mathfrak{g}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}$. Moreover its kernel is isomorphic to $\mathbb{Q}_\ell(1)^n$. Thus it follows that

$$\left(\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}/W_{-3}\right)/(\Lambda^2_0 H)^\perp \cong \left(\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n+1}/W_{-3}\right)/(\Lambda^2_0 H)^\perp = \mathfrak{d}_{g,n+1},$$

and hence that the section $\mathrm{Gr}^W_\bullet \, d\sigma^{\mathrm{geom}}_x$ induces a section s_x of $\mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}$. □

Using the decomposition $\Lambda^3_n H \cong \Lambda^3_0 H \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n$, we denote the elements of $\Lambda^3_n H$ by $(v; u_1 \ldots, u_n)$, where v is an element of $\Lambda^3_0 H$ and u_i is an element of H_j for each $j = 1, \ldots, n$. The surjectivity of the map $H \otimes \Lambda^3_0 H \to \Lambda^2_0 H$ implies that the linear projection

$$\Lambda^3_{n+1}H \to \Lambda^3_n H, \quad (v; u_0, u_1, \ldots, u_n) \mapsto (v; u_1, \ldots, u_n)$$

induces a $\mathrm{GSp}(H)$-equivariant Lie algebra projection

$$\epsilon_n : \mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}.$$

The following result is a key for understanding the rational points of the universal curve and follows from Schur's lemma.

Proposition 10.4.2. *If $g \geqslant 4$, then there are exactly n $\mathrm{GSp}(H)$-equivariant sections of ϵ_n:*

$$s_j : (v; u_1, \ldots, u_n) \mapsto (v; u_j, u_1, \ldots, u_n)$$

for each $j = 1, \ldots, n$.

For $g = 3$, the sections of ϵ_n are $s_1, \ldots, s_n$ and the section

$$\zeta_n : (v; u_1, \ldots, u_n) \mapsto (v; 0, u_1, \ldots, u_n).$$

Proof. Let s be a section of ϵ_n. Then the Schur's lemma implies that $\mathrm{Gr}^W_{-1} s$ is given by

$$\mathrm{Gr}^W_{-1} s : (v; u_1, \ldots, u_n) \mapsto (v; \sum_{j=1}^{n} a_j u_j, u_1, \ldots, u_n),$$

where $v \in \Lambda^3_0 H$, $u_j \in H_j$ for each j, and the a_j are some constants in $\mathbb{Q}_\ell$. Thus it then follows that we have

$$\mathrm{Gr}^W_{-1} s : (w_1, \ldots, w_n) \mapsto (\sum_{j=1}^{n} a_j^2 w_j, w_1, \ldots, w_n),$$

where $w_j \in \Lambda^2_0 H_j$ for each j. On the other hand, the surjectivity of the bracket $H \otimes \Lambda^3_0 H \to \Lambda^2_0 H$ implies that $\mathrm{Gr}^W_{-2} s$ is given by

$$\mathrm{Gr}^W_{-1} s : (w_1, \ldots, w_n) \mapsto (\sum_{j=1}^{n} a_j w_j, w_1, \ldots, w_n).$$

Hence $a_j^2 = a_j$ for $j = 1, \ldots, n$. When $g \geqslant 4$, the bracket $[\ ,\ ] : \Lambda^2\Lambda_0^3 H \to \Lambda_0^2 H$ is nontrivial and maps diagonally to $\bigoplus_{j=1}^m \Lambda_0^2 H_j \subset \mathrm{Gr}^W_{-2}\, \mathfrak{d}_{g,m}$ for $m \geqslant 1$. Since s is a Lie algebra section, this gives the relation $\sum_{j=1}^n a_j^2 = 1$. Together with the relation $a_j^2 = a_j$ for $j = 1, \ldots, n$, we conclude that $a_j = 1$ for some j and $a_i = 0$ for $i \neq j$. When $g = 3$, we do not have the relation $\sum_{j=1}^n a_j^2 = 1$, but the commutativity $\mathrm{Gr}^W_{-2}\, s \circ [\ ,\ ] = [\ ,\ ] \circ \mathrm{Gr}^W_{-1}\, s$ and the independence of the projections d_j and e_j give us the relation $\sum_{i \neq j} a_i a_j = 0$. Thus when $g = 3$, we can conclude that at most one $a_j = 1$ and $a_i = 0$ for $i \neq j$. □

Remark 10.4.3. For $n = 1$, the section s_1 is induced by the tautological section of the universal curve $\mathcal{C}_{g,1} \to \mathcal{M}_{g,1}$. For $n > 1$, considering the jth projection $\mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$, we can see that the jth tautological section induces the section s_j. In deed, there is the commutative diagram

$$\begin{array}{ccc} \mathfrak{d}_{g,n+1} & \longrightarrow & \mathfrak{d}_{g,2} \\ {\scriptstyle s_j'}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle s_1} \\ \mathfrak{d}_{g,n} & \xrightarrow{pr_j} & \mathfrak{d}_{g,1}, \end{array}$$

where pr_j is the map induced by the jth projection $\mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$ and s_j' is the map induced by the jth tautological section. From this diagram, it is easy to see that the section s_j' agrees with the section s_j.

When $g = 3$, there exists an extra section ζ_n of ϵ_n. This is due to the fact that $\Lambda^2\Lambda_0^3 H$ does not contain $\Lambda_0^2 H$. We explain briefly here that the section ζ_n cannot be induced by a rational point. The point is that the section induced by a rational point respects a natural integral structure on $\mathrm{Gr}^W_{-1}\, \mathfrak{u}^{\mathrm{geom}}_{g,n}$ that comes from the image of the Torelli group in $\mathcal{U}^{\mathrm{geom}}_{g,n}$, see [18]. Suppose that A is an integral domain with fraction field F. For a positive integer n, define a lattice in $\Lambda_n^3 H_F$ by

$$\Lambda_n^3 H_A = \{(u_1, \ldots, u_n) \in (\Lambda^3 H_A(-1))^n : \bar{u}_1 = \cdots \bar{u}_n\}(-1),$$

where $\bar{u}$ is the image of u in $\Lambda^3_0 H_A$.

Proposition 10.4.4 ([20, 10.7])**.** *Suppose that $g \geqslant 3$ and $n \geqslant 0$. If A is an integral domain and $g-1 \notin A^\times$, then ζ_n does not restrict to a section of the projection*

$$\Lambda^3_{n+1} H_A \to \Lambda^3_n H_A, \quad (u_0, u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n)$$

.

Proof. This follows easily from the fact that the sequence

$$0 \to H \xrightarrow{\wedge\check{\theta}} \Lambda^3 H(-1) \to \Lambda^3_0 H \to 0$$

splits if and only if $g - 1 \in A^\times$. □

As a summary of this section, we have

Corollary 10.4.5. *Suppose that $n \geqslant 1$. If $g \geqslant 4$, or if $g = 3$ and $\ell = 2$, then the only $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant sections of $\epsilon_n : \mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}$ that respect the integral structure on Gr^W_{-1} are the ones induced by the tautological sections of the universal curve. In particular, there are no $\mathrm{GSp}(H)$-equivariant sections of $\mathfrak{d}_{g,1} \to \mathfrak{d}_{g,0}$.* □

In Appendix A, we introduce the hyperelliptic mapping class groups and their weighted completions to improve this result. Theorem A.4.17 implies that there are elements $\xi_1, \tilde{\xi}_1 \in H_{\boxplus} \subset \mathrm{Gr}^W_{-2}\, \mathfrak{u}^{\mathrm{geom}}_{g,1}$ such that the bracket $[\xi_1, \tilde{\xi}_1]$ is nonzero and lies in $\mathrm{Gr}^W_{-4}\, \mathfrak{p} \subset \mathrm{Gr}^W_{-4}\, \mathfrak{u}^{\mathrm{geom}}_{g,1}$. We lift the elements $\xi_1, \tilde{\xi}_1$ to $\xi_n, \tilde{\xi}_n \in H_{\boxplus} \subset \mathrm{Gr}^W_{-4}\, \hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}$ via the section $\mathfrak{u}^{\mathrm{geom}}_{g,1} \to \hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}$ induced by the diagonal section of the projection $\mathcal{C}^n_g \to \mathcal{M}_{g,1}$ onto any component. Then the bracket $[\xi_n, \tilde{\xi}_n]$ is nonzero and maps diagonally to $\bigoplus_{j=1}^n \mathrm{Gr}^W_{-4}\, \mathfrak{p}_j$, where $\mathfrak{p}_j$ corresponds to the projection $\mathcal{C}^n_g \to \mathcal{M}_{g,1}$ onto the j-th component for each $j = 1, \ldots, n$.

Proposition 10.4.6. *Let F be an algebraic number field or a finite field of characterisc p. If $g \geqslant 3$ and $\ell \neq p$, then the sections $s_1, \ldots, s_n$ are the only* $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-*equivariant sections that come from the sections of $\pi_1(\mathcal{C}_{g,n/F}, \bar{x}) \to \pi_1(\mathcal{M}_{g,n/F}, \bar{\eta})$. When $n = 0$, there are no sections of ϵ_0 that come from a section of*

$$\mathrm{Gr}^W_\bullet \,\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,1}/W_{-5} \to \mathrm{Gr}^W_\bullet \,\hat{\mathfrak{u}}^{\mathrm{geom}}_{g}/W_{-5}.$$

Proof. Each section s of the projection $\pi_1(\mathcal{C}_{g,n/F}, \bar{x}) \to \pi_1(\mathcal{M}_{g,n/F}, \bar{\eta})$ induces a graded Lie algebra section of $\mathrm{Gr}^W_\bullet \,\mathfrak{u}^{\mathrm{geom}}_{\mathcal{C}_{g,n}} \to \mathrm{Gr}^W_\bullet \,\mathfrak{u}^{\mathrm{geom}}_{g,n}$, and thus a section of

$$\mathrm{Gr}^W_\bullet \,\mathfrak{u}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}/W_{-5} \to \mathrm{Gr}^W_\bullet \,\mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-5}.$$

Results from Chapter 12 imply that this section descends to a section $\hat{s}$ of

$$\mathrm{Gr}^W_\bullet \,\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n+1}/W_{-5} \to \mathrm{Gr}^W_\bullet \,\hat{\mathfrak{u}}^{\mathrm{geom}}_{g,n}/W_{-5}.$$

By Schur's lemma, $\mathrm{Gr}^W_{-1} \hat{s}$ is given by

$$\mathrm{Gr}^W_{-1} \hat{s} : (v; u_1, \ldots, u_n) \mapsto (v; \sum_{j=1}^n a_j u_j, u_1, \ldots, u_n),$$

where $v \in \Lambda^3_0 H$, $u_j \in H_j$ for each j, and the a_j are some constants in $\mathbb{Q}_\ell$. It then follows that the restriction of $\mathrm{Gr}^W_{-4} \hat{s}$ to $\bigoplus_{j=1}^n \mathrm{Gr}^W_{-4} \mathfrak{p}_j$ is given by

$$\mathrm{Gr}^W_{-1} \hat{s} : (t_1, \ldots, t_n) \mapsto (\sum_{j=1}^n a_j^4 t_j, t_1, \ldots, t_n) \in \mathrm{Gr}^W_{-4} \mathfrak{p}_0 \oplus \bigoplus_{j=1}^n \mathrm{Gr}^W_{-4} \mathfrak{p}_j,$$

where t_j is in $\mathrm{Gr}^W_{-4} \mathfrak{p}_j$ for each j. Since $(\mathrm{Gr}^W_{-2} \hat{s})(\xi_n) = \xi_{n+1}$ and $(\mathrm{Gr}^W_{-2} \hat{s})(\tilde{\xi}_n) = \tilde{\xi}_{n+1}$ and $\hat{s}$ is a Lie algebra section, we have

$$\mathrm{Gr}^W_{-4} \hat{s} : (\gamma, \ldots, \gamma) \mapsto (\gamma, \gamma, \ldots, \gamma),$$

where $\gamma := [\xi_1, \tilde{\xi}_1]$. Thus we have the relation

$$\sum_{j=1}^n a_j^4 = 1.$$

Together with the relation $a_j^2 = a_j$ for each j, we see that $a_j = 1$ for some j and $a_i = 0$ for $i \neq j$. Thus the section s induces s_j for some j. $\square$

11

The Characteristic Class of A Rational Point

In [20], Hain defined a characteristic class κ_x for a T-rational point x of the curve $C \to T$, where T is a smooth variety over a field k with char $(k) = 0$. For our comparison purpose, we need to redefine this characteristic class for curves $C \to T$, where T is defined over a more general base ring, e.g., $\mathbb{Z}_p$. In this section, we will explain how this can be done and extend the results used in [20] to positive characteristics. Let B be a connected scheme. Suppose that T is a geometrically connected smooth scheme over B and that $f : C \to T$ is a curve of genus g. In this section, we associate a cohomology class κ_x in $H^1_{\text{ét}}(T, R^1 f_* \mathbb{Q}_\ell(1))$ to a rational point $x \in C(T)$. Denote the relative Jacobian of $f : C \to T$ by $\pi : J_{C/T} \to T$. $J_{C/T}$ is a family of jacobians and is an abelian scheme over T. Note that $J_{C/T}$ has a zero section $s_0 : T \to J_{C/T}$. Let $\overline{\eta} : \operatorname{Spec} \Omega \to T$ be a geometric point of T. Denote the fiber of f over $\overline{\eta}$ by $C_{\overline{\eta}}$ and the fiber of $J_{C/T} \to T$ over $\overline{\eta}$ by $(J_{C/T})_{\overline{\eta}}$. Let $\bar{x}$ be a geometric point of $C_{\overline{\eta}}$. Note that $(J_{C/T})_{\overline{\eta}}$ is the jacobian variety of the curve $C_{\overline{\eta}}$. When ℓ is not in char(T), there are natural isomorphisms

$$\pi_1((J_{C/T})_{\overline{\eta}}, \bar{x})^{(\ell)} \cong \pi_1(C_{\overline{\eta}}, \bar{x})^{(\ell),\text{ab}} \cong H^1_{\text{ét}}(C_{\overline{\eta}}, \mathbb{Z}_\ell(1)),$$

where ab denotes maximal abelian quotient. Denote the lisse sheaf $R^1 f_* A(1)$ over T by $\mathbb{H}_A$, where $A = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. Then we have

$$H_A := H^1_{\text{ét}}(C_{\overline{\eta}}, A(1)) = \left(\mathbb{H}_A\right)_{\overline{\eta}}.$$

By Proposition 4.2.1, there is an exact sequence

$$1 \to \pi_1((J_{C/T})_{\overline{\eta}}, \bar{x})^{(\ell)} \to \pi_1'(J_{C/T}, \bar{x}) \to \pi_1(T, \overline{\eta}) \to 1.$$

Thus the zero section s_0 determines a splitting

$$\pi_1'(J_{C/T}, \bar{x}) \cong \pi_1((J_{C/T})_{\overline{\eta}}, \bar{x})^{(\ell)} \rtimes \pi_1(T, \overline{\eta}) \cong H_{\mathbb{Z}_\ell} \rtimes \pi_1(T, \overline{\eta}),$$

which is well-defined up to conjugation action of $H_{\mathbb{Z}_\ell}$. To each rational point $x \in C(T)$, we associate the divisor $D_x := (2g-2)x - \omega_{C/T}$, where $\omega_{C/T}$ is the relative canonical divisor of the family $C \to T$. The divisor D_x is homologically trivial on each geometric fiber, and hence gives a section of $J_{C/T} \to T$, which determines a κ_x in

$$H^1_{\text{cts}}(\pi_1(T, \overline{\eta}), H_{\mathbb{Z}_\ell}) \cong H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Z}_\ell}).$$

Tensoring with $\mathbb{Q}_\ell$, we obtain a class in $H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Q}_\ell})$, which we denote also by κ_x.

Remark 11.0.7. This class behaves well under base change.

11.1 Classes of the universal curve over $\mathcal{M}_{g,n}$

Let F be a field of characteristic zero. Suppose that T is a noetherian geometrically connected scheme over F. Denote the class in $H^1_{\text{ét}}(\mathcal{M}_{g,1/F}, \mathbb{H}_{\mathbb{Q}_\ell})$ of the tautological section of the universal curve $\mathcal{C}_{g,1/F} \to \mathcal{M}_{g,1/F}$ by κ. This class is universal in the sense that for each rational point $x \in C(T)$, the class $\kappa_x \in H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Q}_\ell})$ is the pullback of κ, i.e., $\kappa_x = \phi^*\kappa$, where $\phi : T \to \mathcal{M}_{g,1/F}$ is the morphism induced by x. Denote the class of the jth tautological section of the universal curve $\mathcal{C}_{g,n/F} \to \mathcal{M}_{g,n/F}$ by κ_j.

Proposition 11.1.1 ([20, 12.1])**.** *If $g \geqslant 3$, $n \geqslant 0$, and $m \geqslant 1$, then for all fields F of characteristic zero,*

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/F}[m], \mathbb{H}_{\mathbb{Q}_\ell}) = \mathbb{Q}_\ell\kappa_1 \oplus \mathbb{Q}_\ell\kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell\kappa_n.$$

□

Suppose that p is a prime number, and that ℓ is a prime number distinct from p and m is a positive integer such that $\ell^m \geqslant 3$. Denote a connected component of the base change to $\mathbb{Z}_p^{\text{ur}}$ of $\mathcal{M}_{g,n}[\ell^m]$ by $\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$. Denote the universal curve over $\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ by $\mathcal{C}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$, and denote the relative Jacobian of $\mathcal{C}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ over $\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ by $J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$. For $A = \bar{\mathbb{Q}}_p$ and $\bar{\mathbb{F}}_p$, the base change to A of $J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ and $\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ are denoted by $J_A[\ell^m]$ and $\mathcal{M}_A[\ell^m]$, respectively. Let $\bar{\xi}$ and $\bar{\eta}$ be geometric points of $\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m]$ and $\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m]$, respectively. We consider $\bar{\xi}$ and $\bar{\eta}$ as geometric points of $\mathcal{M}_{g,n/\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ via canonical morphisms induced by base change. Denote the fiber over $\bar{\xi}$ and $\bar{\eta}$ of $\mathcal{C}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m] \to \mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m]$ by $C_{\bar{\xi}}$ and$C_{\bar{\eta}}$. Let $\bar{x}'$ and $\bar{x}$ be a geometric point of $C_{\bar{\xi}}$ and $C_{\bar{\eta}}$, respectively. By Proposition 4.2.1, we have the diagram (∗∗)

$$\begin{array}{ccccccccc}
1 \to & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell),\text{ab}} & \to & \pi_1'(J_{\bar{\mathbb{F}}_p}[\ell^m], \bar{x}') & \to & \pi_1(\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi}) & \to 1 \\
& \| & & \downarrow & & \downarrow & \\
1 \to & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell),\text{ab}} & \to & \pi_1'(J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{x}') & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{\xi}) & \to 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell),\text{ab}} & \to & \pi_1'(J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{x}) & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{\eta}) & \to 1 \\
& \| & & \uparrow & & \uparrow & \\
1 \to & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell),\text{ab}} & \to & \pi_1'(J_{\bar{\mathbb{Q}}_p}[\ell^m], \bar{x}) & \to & \pi_1(\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta}) & \to 1,
\end{array}$$

that commutes after fixing an isomorphism $\pi_1(J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{x}') \cong \pi_1(J_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{x})$, which determines an isomorphism $\pi_1(\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{\xi}) \cong \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\text{ur}}}[\ell^m], \bar{\eta})$. The rows of the diagram are exact and the vertical maps between the second and third row are isomorphisms.

Lemma 11.1.2. *Suppose that $n \geqslant 1$. If $\bar{*}$ is a geometric point of $\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m]$ and $\bar{y}$ is a geometric point of the fiber $C_{\bar{*}}$, then the sequence of the maximal pro-ℓ quotients*

$$1 \to \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} \to \pi_1(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)} \to \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*})^{(\ell)} \to 1$$

of the exact sequence

$$1 \to \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} \to \pi_1'(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y}) \to \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*}) \to 1$$

is exact.

Proof. A tautological section induces the closed immersion $\psi : \mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m] \to J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m]$ that makes the diagram

$$(*) \qquad \begin{array}{ccccccccc} 1 & \to & \pi_1(C_{\bar{*}}, \bar{y})^{(\ell)} & \to & \pi_1'(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y}) & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*}) & \to & 1 \\ & & \downarrow & & \downarrow \psi_* & & \| & & \\ 1 & \to & \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} & \to & \pi_1'(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y}) & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*}) & \to & 1 \end{array}$$

commute, where the left-hand vertical map is the canonical projection. Denote the kernel of the projection $\pi_1(C_{\bar{*}}, \bar{y})^{(\ell)} \to \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}}$ by N. Then ψ_* induces an isomorphism

$$\pi_1'(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})/N \cong \pi_1'(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y}).$$

Since $\pi_1(C_{\bar{*}}, \bar{y})^{(\ell)} \to \pi_1(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)}$ is injective, there is an isomorphism

$$\pi_1(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)}/N \cong \left(\pi_1'(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})/N\right)^{(\ell)}.$$

Taking maximal pro-ℓ quotient of the diagram $(*)$ and pushing out along the surjection $\pi_1(C_{\bar{*}}, \bar{y})^{(\ell)} \to \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}}$, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} & \to & \pi_1(\mathcal{C}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)}/N & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*})^{(\ell)} & \to & 1 \\ & & \| & & \downarrow & & \| & & \\ & & \pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} & \to & \pi_1(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)} & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{*})^{(\ell)} & \to & 1, \end{array}$$

where the middle vertical map is an isomorphism. Thus it follows that the map $\pi_1(C_{\bar{*}}, \bar{y})^{(\ell),\mathrm{ab}} \to \pi_1(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{y})^{(\ell)}$ is injective. □

Proposition 11.1.3. *Assume the notations above. If $g \geqslant 3$ and $n \geqslant 1$, then*

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) = \mathbb{Q}_\ell\kappa_1 \oplus \mathbb{Q}_\ell\kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell\kappa_n.$$

Moreover, we have

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) = \mathbb{Q}_\ell\kappa_1 \oplus \mathbb{Q}_\ell\kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell\kappa_n,$$

where $\mathbb{F}_q = \mathbb{F}_p[\zeta_{\ell^m}]$ and ζ_{ℓ^m} is a primitive ℓ^mth root of unity.

Proof. By Lemma 11.1.2, taking pro-ℓ completion of the diagram $(**)$ gives the commutative diagram

$$\begin{array}{ccccccccc}
1 \to & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell),\mathrm{ab}} & \to & \pi_1(J_{\bar{\mathbb{F}}_p}[\ell^m], \bar{x}')^{(\ell)} & \to & \pi_1(\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi})^{(\ell)} & \to 1 \\
& \| & & \downarrow & & \downarrow & \\
1 \to & \pi_1(C_{\bar{\xi}}, \bar{x}')^{(\ell),\mathrm{ab}} & \to & \pi_1(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{x}')^{(\ell)} & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\xi})^{(\ell)} & \to 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell),\mathrm{ab}} & \to & \pi_1(J_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{x})^{(\ell)} & \to & \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\eta})^{(\ell)} & \to 1 \\
& \| & & \uparrow & & \uparrow & \\
1 \to & \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell),\mathrm{ab}} & \to & \pi_1(J_{\bar{\mathbb{Q}}_p}[\ell^m], \bar{x})^{(\ell)} & \to & \pi_1(\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta})^{(\ell)} & \to 1,
\end{array}$$

whose rows are exact and the vertical maps between the second and third row are isomorphisms induced by change of base points. Furthermore, the maps

$$\pi_1(\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta})^{(\ell)} \to \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\eta})^{(\ell)} \leftarrow \pi_1(\mathcal{M}_{\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\xi})^{(\ell)} \leftarrow \pi_1(\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m], \bar{\eta})^{(\ell)}$$

are isomorphisms, and hence by exactness all the vertical maps are isomorphisms. This implies that there is an isomorphism

$$H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta})^{(\ell)}, (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\eta}}\right) \cong H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi})^{(\ell)}, (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\xi}}\right).$$

For $A = \bar{\mathbb{Q}}_p$, $\bar{\mathbb{F}}_p$, $\bar{\gamma} = \bar{\eta}$, $\bar{\xi}$, and $\bar{y} = \bar{x}$, $\bar{x}'$, respectively, the diagram

$$\begin{array}{ccccccc}
1 \to & \pi_1(C_{\bar{\gamma}}, \bar{y})^{(\ell),\mathrm{ab}} & \to & \pi_1'(J_A[\ell^m], \bar{y}) & \to & \pi_1(\mathcal{M}_A[\ell^m], \bar{\gamma}) & \to 1 \\
& \| & & \downarrow & & \downarrow & \\
1 \to & \pi_1(C_{\bar{\gamma}}, \bar{y})^{(\ell),\mathrm{ab}} & \to & \pi_1(J_A[\ell^m], \bar{y})^{(\ell)} & \to & \pi_1(\mathcal{M}_A[\ell^m], \bar{\gamma})^{(\ell)} & \to 1
\end{array}$$

is the pullback diagram along the surjection $\pi_1(\mathcal{M}_A[\ell^m], \bar{\gamma}) \to \pi_1(\mathcal{M}_A[\ell^m], \bar{\gamma})^{(\ell)}$. Thus there is a canonical isomorphism

$$H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/A}[\ell^m], \bar{\gamma}), (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\gamma}}\right) \cong H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/A}[\ell^m], \bar{\gamma})^{(\ell)}, (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\gamma}}\right).$$

Therefore, we have isomorphisms

$$\begin{aligned} H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}) &\cong H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \bar{\eta}), (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\eta}}\right) \\ &\cong H^1_{\mathrm{cts}}\left(\pi_1(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\gamma}), (\mathbb{H}_{\mathbb{Z}_\ell})_{\bar{\xi}}\right) \\ &\cong H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}). \end{aligned}$$

Under this isomorphism, the classes κ_j of the jth tautological section correspond in $H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})$ and $H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})$. Hence our claim follows from Proposition 11.1.1. As to the second claim, the spectral sequence

$$H^s(G_{\mathbb{F}_q}, H^t_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})) \Rightarrow H^{s+t}_{\text{ét}}(\mathcal{M}_{\mathbb{F}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})$$

and the fact that $H^0_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}) = 0$ imply that we have

$$H^1_{\text{ét}}(\mathcal{M}_{\mathbb{F}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}) = H^1_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})^{G_{\mathbb{F}_q}} \subset H^1_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}).$$

Since the tautological sections are defined over $\mathbb{Z}$ and hence defined over $\mathbb{F}_q$ by base change, the corresponding classes κ_j's lie in $H^1_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})^{G_{\mathbb{F}_q}}$. Tensoring with $\mathbb{Q}_\ell$, we have

$$H^1_{\text{ét}}(\mathcal{M}_{\mathbb{F}_q}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) = H^1_{\text{ét}}(\mathcal{M}_{\bar{\mathbb{F}}_q}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) = \mathbb{Q}_\ell\kappa_1 \oplus \mathbb{Q}_\ell\kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell\kappa_n.$$

□

11.2 The ℓ-adic Abel-Jacobi map

Suppose that $\pi : A \to T$ is an abelian scheme over a smooth scheme over a field F whose fibers are polarized abelian varieties. For a prime number ℓ not equal to char(F), the ℓ-adic Abel-Jacobi map agrees with the association

$$A(T) \to H^1_{\text{ét}}(T, R^1\pi_*\mathbb{Z}_\ell(1)), \qquad x \mapsto \kappa_x.$$

Lemma 11.2.1 ([20, 12.2]). *If $\pi : A \to T$ is a family of polarized abelian varieties over a noetherian scheme T, then the kernel of the ℓ-adic Abel-Jacobi map*

$$A(T) \to H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Z}_\ell})$$

is the subgroup $\bigcap_n \ell^n A(T)$ of ℓ^∞-divisible points, where ℓ is not in char (T).

Corollary 11.2.2 ([20, 12.3]). *With notations as above, if the group $A(T)$ of sections of $\pi : A \to T$ is finitely generated, then the kernel of*

$$A(T) \to H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Q}_\ell})$$

is finite.

Remark 11.2.3. By a generalization of the Mordell-Weil Theorem [36] by Néron, when T is a geometrically connected smooth variety over a field that is finitely generated over its prime subfield, $A(T)$ is finitely generated. This is the case, for example, for the universal curve $\mathcal{C}_{g,n/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m]$.

Applying this result to the relative Jacobian $\pi : J_{C/T} \to T$ associated to the family of curves $f : C \to T$, where T is a geometrically connected smooth variety over a field F.

Corollary 11.2.4 ([20, 12.4]). *Assume that the group of sections $J_{C/T}(T)$ of $\pi : J_{C/T} \to T$ is finitely generated. If x and y are sections of $f : C \to T$ and $\kappa_x = \kappa_y$, then $x - y$ is torsion in $J_{C/T}(T)$.*

Proof. Recall that the classes κ_x and κ_y are the images of $(2g-2)x - \omega_{C/T}$ and $(2g-2)y - \omega_{C/T}$, respectively. That $\kappa_x = \kappa_y$ implies that $(2g-2)(x-y)$ lies in the kernel of the map $J_{C/T}(T) \to H^1_{\text{ét}}(T, \mathbb{H}_{\mathbb{Q}_\ell})$. By Corollary 11.2.2, the divisor $(2g-2)(x-y)$ is torsion. □

11.3 The image of κ_j in $\mathrm{Hom}_{\mathrm{GSp}(H)}(H_1(\mathfrak{u}_{g,n}^{\mathrm{geom}}), H)$

Proposition 6.12.1 implies that there is a natural isomorphism

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) \cong \mathrm{Hom}_{\mathrm{GSp}(H)}(H_1(\mathfrak{u}_{g,n}^{\mathrm{geom}}), H_{\mathbb{Q}_\ell}).$$

We can explicitly describe the image of the class κ_x in $\mathrm{Hom}_{\mathrm{GSp}(H)}(H_1(\mathfrak{u}_{g,n}^{\mathrm{geom}}), H_{\mathbb{Q}_\ell})$.

Denote the $\mathrm{GSp}(H)$-equivariant projection $\Lambda^3_1 H \to H$ by h. This projection is induced by twisting the projection $\Lambda^3 H \to H(1)$:

$$x \wedge y \wedge z \mapsto \theta(x, y)z + \theta(y, z)x + \theta(z, x)y.$$

Denote the $\mathrm{GSp}(H)$-equivariant homomorphism $\Lambda^3_n H \to H$

$$\Lambda^3_n H \to \left(\Lambda^3_1 H\right)^n \xrightarrow{pr_j} \Lambda^3_1 H \xrightarrow{h} H$$

by h_j.

Proposition 11.3.1 ([20, 12.5 & 12.6],[22, 6.5])**.** *If $g \geqslant 3$ and $n \geqslant 1$, for each $j = 1, \ldots, n$, the* $\mathrm{GSp}(H)$*-equivariant homomorphism*

$$H_1(\mathfrak{u}_{g,n}^{\mathrm{geom}}) \cong \mathrm{Gr}^W_{-1} \mathfrak{u}_{g,n}^{\mathrm{geom}} \cong \Lambda^3_n H \xrightarrow{2h_j} H$$

corresponds to the class κ_j under the isomorphism

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{Q}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) \cong \mathrm{Hom}_{\mathrm{GSp}(H)}(H_1(\mathfrak{u}_{g,n}^{\mathrm{geom}}), H).$$

Fixing an isomorphism $\pi_1(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\eta}) \cong \pi_1(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}[\ell^m], \bar{\xi})$ determines the isomorphisms $(\mathbb{H}_{\mathbb{Q}_\ell})_{\bar{\eta}} \cong (\mathbb{H}_{\mathbb{Q}_\ell})_{\bar{\xi}}$ and $\mathfrak{u}^{\mathrm{geom}}_{\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m]} \cong \mathfrak{u}^{\mathrm{geom}}_{\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m]}$ that make the diagram

$$\begin{array}{ccc} H^1_{\text{ét}}\left(\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}\right) & \to & \mathrm{Hom}_{\mathrm{GSp}(H)}\left(H_1\left(\mathfrak{u}^{\mathrm{geom}}_{\mathcal{M}_{\bar{\mathbb{Q}}_p}[\ell^m]}\right), (\mathbb{H}_{\mathbb{Q}_\ell})_{\bar{\eta}}\right) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}\left(\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}\right) & \to & \mathrm{Hom}_{\mathrm{GSp}(H)}\left(H_1\left(\mathfrak{u}^{\mathrm{geom}}_{\mathcal{M}_{\bar{\mathbb{F}}_p}[\ell^m]}\right), (\mathbb{H}_{\mathbb{Q}_\ell})_{\bar{\xi}}\right) \end{array}$$

commute. Hence we have

Corollary 11.3.2. *If $g \geqslant 3$ and $n \geqslant 1$, for each $j = 1, \dots, n$, the* $\mathrm{GSp}(H)$*-equivariant homomorphism $2h_j$ corresponds to the class κ_j under the isomorphism*

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \mathbb{H}_{\mathbb{Q}_\ell}) \cong \mathrm{Hom}_{\mathrm{GSp}(H)}(H_1(\mathfrak{u}^{\mathrm{geom}}_{g,n}), H).$$

Remark 11.3.3. The $\mathrm{GSp}(H)$-equivariant projection

$$\Lambda^3_n H = \Lambda^3_0 H \oplus H_1 \oplus \cdots \oplus H_n \to H_j$$

onto the jth copy of H is equal to $h_j/(g-1)$ and corresponds to the class $\kappa_j/(2g-2)$ under this isomorphism.

12

Generic Sections of Fundamental Groups

The content of this section should be well known to experts. However, because of its key role in the proof of Theorem 2, we will give a brief introduction of the results needed in the proof.

Suppose that S is the spectrum of an excellent henselian discrete valuation ring R whose residue field k is a perfect field of characteristic $p \geqslant 0$. Denote the fraction field of R by K. Fix an algebraic closure $\bar{K}$ of K. Suppose that $\pi : X \to S$ is a proper smooth morphism with geometrically connected fibers. Let $\bar{x}$ and $\bar{x}'$ be geometric points of the fibers $X_{\bar{K}}$ and $X_{\bar{k}}$, respectively. We also consider $\bar{x}$ and $\bar{x}'$ as geometric points of X via the morphisms $j : X_{\bar{K}} \to X$ and $i : X_{\bar{k}} \to X$ induced by base change. Fixing an isomorphism $\pi_1(X, \bar{x}) \cong \pi_1(X, \bar{x}')$ gives the commutative diagram $(*)$

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(X_{\bar{K}}, \bar{x}) & \to & \pi_1(X_K, \bar{x}) & \to & G_K & \to & 1 \\ & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow & & \\ 1 & \to & \pi_1(X_{\bar{k}}, \bar{x}') & \to & \pi_1(X_k, \bar{x}') & \to & G_k & \to & 1 \end{array}$$

whose rows are exact and vertical maps are surjective. The surjective maps

$$\pi_1(X_{\bar{K}}, \bar{x}') \to \pi_1(X_{\bar{k}}, \bar{x}'), \qquad \pi_1(X_K, \bar{x}) \to \pi_1(X_k, \bar{x}')$$

in the diagram are the specialization homomorphism defined in [14, SGA 1, X].

Denote the kernel of the natural map $G_K \to G_k$ by I_k. It is the Galois group of the maximal unramified subextension K^{ur} in $\bar{K}$ of K. For a section s of $\pi_1(X_K, \bar{x}) \to G_K$, we define the *ramification* of s to be the map

$$\mathrm{ram}_s = \mathrm{sp} \circ s|_{I_k} : I_k \to \pi_1(X_{\bar{k}}, \bar{x}').$$

This sits in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_k & \longrightarrow & G_K & \longrightarrow & G_k & \to & 1 \\ & & \downarrow{\scriptstyle ram_s} & & \downarrow{\scriptstyle \mathrm{sp}\circ s} & & \| & & \\ 1 & \to & \pi_1(X_{\bar{k}}, \bar{x}') & \to & \pi_1(X_k, \bar{x}') & \to & G_k & \to & 1. \end{array}$$

From this, we see that $\mathrm{ram}_s^{\mathrm{ab}} : I_k^{\mathrm{ab}} \to \pi_1(X_{\bar{k}}, \bar{x}')^{\mathrm{ab}}$ is a G_k-equivariant map and that when ram_s is trivial, the section s induces a section s_0 of $\pi_1(X_k, \bar{x}') \to G_k$. A section s with trivial ram_s is called *unramified.* A section of $\pi_1(X_K) \to G_K$ induced by a rational point in $X_K(K)$ is unramified.

Now, suppose that ℓ is a prime number distinct from $\mathrm{char}(k) = p$. Pushing out the diagram $(*)$ along the surjection $\pi_1(X_{\bar{k}}) \to \pi_1(X_{\bar{k}})^{(\ell)}$, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\bar{K}}, \bar{x})^{(\ell)} & \longrightarrow & \pi'_1(X_K, \bar{x}) & \overset{s'}{\rightleftarrows} & G_K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{sp}^{(\ell)}} & & \downarrow{\scriptstyle \mathrm{sp}'} & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X_{\bar{k}}, \bar{x}')^{(\ell)} & \longrightarrow & \pi'_1(X_k, \bar{x}') & \longrightarrow & G_k & \longrightarrow & 1. \end{array}$$

The restriction of the composite $\mathrm{sp}' \circ s'$ to I_k induces the map

$$\mathrm{ram}_s^{(\ell)} : \mathbb{Z}_\ell(1) \to \pi_1(X_{\bar{k}})^{(\ell)}.$$

Proposition 12.0.4 ([44, Prop. 91])**.** *With the same notation as in above, suppose that the fibers of $\pi : X \to S$ are curves and that the residue field k of S is finitely generated over its prime subfield. Then $\mathrm{ram}_s(I_k)$ is a free pro-p group. In particular, $\mathrm{ram}_s^{(\ell)}$ is trivial and each section of $\pi_1(X_k) \to G_K$ induces a section of $\pi'_1(X_k) \to G_k$.*

Proof. If $\mathrm{ram}_s(I_k)$ is nontrivial, then we may find an open subgroup N of $\pi_1(X_{\bar{k}})$ such that

$$\mathrm{ram}_s(I_k) \subset N, \quad \text{and} \quad \mathrm{ram}_s(I_k) \not\subset \overline{[N,N]},$$

where $\overline{[N,N]}$ denotes the closure of $[N,N]$ in $\pi_1(X_{\bar{k}})$. The subgroup N corresponds to a finite étale cover of $X_{\bar{k}}$. Since N is the intersection of an open subgroup of $\pi_1(X)$ with $\pi_1(X_{\bar{k}})$, we may replace $X_{\bar{k}}$ and X with appropriate finite étale covers to reduce to the case where $N = \pi_1(X_{\bar{k}})$. Suppose $\mathrm{ram}_s(I_k)$ has a finite quotient that is an ℓ group. Then ram_s induces a nontrivial G_k-equivariant map

$$ram_s^{(\ell),\mathrm{ab}} : \mathbb{Z}_\ell(1) \to \pi_1(X_{\bar{k}})^{(\ell),\mathrm{ab}}.$$

This contradicts the Frobenius weights in étale cohomology. Therefore, if $p = 0$, ram_s is unramified, and otherwise $\mathrm{ram}_s(I_k)$ is a pro-p group. Furthermore, the p-cohomological dimension of $\pi_1(X_{\bar{k}})$ is $\leqslant 1$, which implies that $cd_p(\mathrm{ram}_s(I_k)) \leqslant 1$ and hence $\mathrm{ram}_s(I_k)$ is a free pro-p group. □

Suppose that $f : C \to T$ is a family of curves over an irreducible regular scheme T of finite type over a field F. Let L be the function field of T and ℓ a prime number distinct from $\mathrm{char}(F)$. Let $\bar{\eta}$ be a geometric generic point of C. The image of $\bar{\eta}$ in T is a geometric generic point of T. In the following, fundamental groups are defined by using this choice of base points.

Proposition 12.0.5. *Each section of $\pi_1(C_L) \to G_L$ induces a pro-ℓ section of $\pi_1(C) \to \pi_1(T)$. Consequently, there is a bijection between the set of conjugacy classes of pro-ℓ sections of $\pi_1(C_L) \to G_L$ and that of $\pi_1(C) \to \pi_1(T)$.*

Proof. Each section s of $\pi_1(C_L) \to G_L$ comes from the projective system of sections $\varprojlim_i U_i \to \pi_1(C_{U_i})$, where each U_i is a complement of finitely many prime divisors of T. Thus it will be enough to show that for each open subscheme U of T that is a complement of a prime divisor, each section s of $\pi_1(C_U) \to \pi_1(U)$ induces a pro-ℓ section of $\pi_1(C) \to \pi_1(T)$. Let Y be a prime divisor of T and U its complement in T. Let R be the henselization of the local ring of T at Y. Then R is an excellent henselian discrete valuation ring. Denote the fraction field of R by K and the residue field by k. We see that the fiber product $\operatorname{Spec} R \times_T U$ is isomorphic to $\operatorname{Spec} K$. We claim that the image of the inertia group I_k in $\pi_1(U)$ is equal to the kernel of the canonical surjection $\pi_1(U) \to \pi_1(T)$. Clearly, the image of I_k is contained in the kernel. Let H be an open subgroup of $\pi_1(U)$ containing the image of I_k. Then the preimage N in G_K of H under the homomorphism $G_K \to \pi_1(U)$ is an open subgroup of G_K containing I_k. Since I_k is the kernel of the canonical surjection $G_K \to \pi_1(\operatorname{Spec} R)$, the image N' of N in $\pi_1(\operatorname{Spec} R)$ is an open subgroup of $\pi_1(\operatorname{Spec} R)$ whose preimage in G_K is N. Its corresponding finite étale cover of $\operatorname{Spec} R$ pulls back to the finite étale cover of K that corresponds to the subgroup N. Let X be the finite étale cover of U corresponding to H. Let T' be the normalization of T with respect to X. Then $\nu : T' \to T$ is finite and the pullback of ν to U is just $X \to U$. Pulling back ν along the canonical morphism $\operatorname{Spec} R \to T$, we obtain a finite morphism $\nu' : T'' \to \operatorname{Spec} R$, where T'' is a normal scheme. We may choose the component Q of T'' whose fundamental group has the image in $\pi_1(\operatorname{Spec} R)$ equal to N'. Let W be the finite étale cover of $\operatorname{Spec} R$ corresponding to N'. Then Q and W are isomorphic to each other over the generic point of $\operatorname{Spec} R$. Pulling back W along the morphism $Q \to \operatorname{Spec} R$, we obtain a finite étale cover Q' of Q that admits a section j. The composite

$$Q \xrightarrow{j} Q' \to W$$

is a finite birational morphism of integral normal schemes and thus is an isomorphism. Therefore, Q is unramified over the closed point of $\operatorname{Spec} R$, which implies that T' is unramified over the generic point of Y. By Zariski-Nagata [14, SGA 1 X Thm. 3.1],

T' is unramified over whole T. Since T' pulls back to X, it follows that H contains the kernel of $\pi_1(U) \to \pi_1(T)$, whence our claim holds.

Now, each section s of $\pi_1(C_U) \to \pi(U)$ induces a section s_K of $\pi_1(C_K) \to K$. The above claim and the fact that $ram^{(\ell)}_{s_K}$ is unramified imply that the section s descends to a pro-ℓ section of $\pi_1(C) \to \pi_1(T)$. $\square$

13

The Proof of Theorem 1 and 2

Lemma 13.0.6. *If $g \geqslant 3$ and $n \geqslant 1$, there is no* $\mathrm{GSp}(H)$*-equivariant homomorphism*

$$\mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-3} \to \mathrm{Gr}^W_\bullet \mathfrak{u}^{\mathrm{geom}}_{g,2}/W_{-3}$$

that induces the map $(v; u_1, \ldots, u_n) \mapsto (v; u_1, u_1)$ *on* Gr^W_{-1}.

Proof. If $u \in H$, denote by $u^{(j)}$ the corresponding element in the jth copy of H in

$$\mathrm{Gr}^W_{-1} \mathfrak{u}^{\mathrm{geom}}_{g,n} = \Lambda^3_0 H \oplus H_1 \oplus \cdots \oplus H_n.$$

Fix a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ for H and $(u \cdot v)$ denotes the intersection number of u and v Suppose that such a homomorphism ϕ exists. Recall that, for a positive integer m, we have

$$\mathrm{Gr}^W_{-2} \mathfrak{u}^{\mathrm{geom}}_{g,m} = H_{\boxplus} \oplus \bigoplus_{1 \leqslant i < j \leqslant m} \mathbb{Q}_\ell(1)_{ij} \oplus \bigoplus_{j=1}^{m} \Lambda^2_0 H_j,$$

where $\mathbb{Q}_\ell(1)_{ij}$ is spanned by $\sum_{k=1}^g [a_k^{(i)}, b_k^{(j)}]$. Denote the element $\sum_{k=1}^g [a_k^{(i)}, b_k^{(j)}]$ in $\mathrm{Gr}^W_{-2} \mathfrak{u}^{\mathrm{geom}}_{g,m}$ by Θ_{ij}. We claim first that ϕ vanishes on the $\mathbb{Q}_\ell(1)$ component. For any

$i < j$ and $u, v \in H$, the bracket $[u^{(i)}, v^{(j)}]$ is computed in [18, §12] and is given by

$$[u^{(i)}, v^{(j)}] = \frac{(u \cdot v)}{g} \sum_{k=1}^{g} [a_k^{(i)}, b_k^{(j)}] \text{ in } \operatorname{Gr}_{-2}^W \mathfrak{u}_{g,m}^{\text{geom}}.$$

For $1 < j$, we have $\phi(v^{(j)}) = (0, 0)$ in $\operatorname{Gr}_{-1}^W \mathfrak{u}_{g,2}^{\text{geom}}$, and hence $[\phi(u^{(i)}), \phi(v^{(j)})] = 0$. Since ϕ is a homomorphism, it follows that $\phi(\Theta_{ij}) = 0$, and therefore ϕ vanishes on $\mathbb{Q}_\ell(1)_{ij}$ for all $1 \leqslant i < j \leqslant n$.

Next, we will compute $\sum_{k=1}^{g} [\phi(a_k^{(1)}), \phi(b_k^{(1)})]$ in $\operatorname{Gr}_{-2}^W \mathfrak{u}_{g,2}^{\text{geom}}$. Denote the element $\sum_{k=1}^{g} [a_k^{(i)}, b_k^{(i)}]$ in $\operatorname{Gr}_{-2}^W \mathfrak{u}_{g,m}^{\text{geom}}$ by Θ_i. Theorem 12.6 in [18] implies that we have

$$\begin{aligned}
\phi(\Theta_1) = \sum_{k=1}^{g} [\phi(a_k^{(1)}), \phi(b_k^{(1)})] &= \sum_{k=1}^{g} [a_k^{(1)} + a_k^{(2)}, b_k^{(1)} + b_k^{(2)}] \\
&= \sum_{k=1}^{g} \left([a_k^{(1)}, b_k^{(1)}] + [a_k^{(2)}, b_k^{(2)}] + \frac{2}{g} \Theta_{12} \right) \\
&= (\Theta_1 + \Theta_2 + 2\Theta_{12}) \text{ in } \operatorname{Gr}_{-2}^W \mathfrak{u}_{g,2}^{\text{geom}}.
\end{aligned}$$

But one has the relation [18, Thm. 12.6]

$$\Theta_i + \frac{1}{g} \sum_{j \neq i} \Theta_{ij} = 0, \qquad \text{for } 1 \leqslant i \leqslant m,$$

in $\operatorname{Gr}_{-2}^W \mathfrak{u}_{g,m}^{\text{geom}}$, so in $\operatorname{Gr}_{-2}^W \mathfrak{u}_{g,2}^{\text{geom}}$

$$\phi(\Theta_1) = \Theta_1 + \Theta_2 + 2\Theta_{12} = \frac{-1}{g}\Theta_{12} + \frac{-1}{g}\Theta_{12} + 2\Theta_{12} = \frac{2g-2}{g}\Theta_{12} \neq 0.$$

Therefore we have reach a contradiction. □

Recall that p is a prime number, ℓ is a prime number distinct from p, and m is a nonnegative integer.

Proposition 13.0.7. *Suppose that $g \geqslant 3$, $n \geqslant 1$, and $\ell^m \geqslant 3$. If x is a section of the universal curve $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m] \to \mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$ and $\kappa_x = \kappa_j$, then x is the jth tautological point x_j.*

Proof. Without loss of generality, we may assume that $j = 1$. The section x is defined over some finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$, which we may assume to contain a ℓ^mth root of unity $\mu_{\ell^m}(\bar{\mathbb{F}}_p)$. Thus we consider x as a section of $\mathcal{C}_{g,n/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m]$. Denote the relative Jacobian of $\mathcal{C}_{g,n/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m]$ by J. By Corollary 11.2.4, $[x - x_1]$ is a torsion in $J(\mathcal{M}_{\mathbb{F}_q}[\ell^m])$. Denote this torsion by t. If $t = 0$, then, since $g \geqslant 3$, $x = x_1$. If $t \neq 0$, then the sections x and x_1 are disjoint, and hence they induce the morphism

$$\mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g,2/\mathbb{F}_q}[\ell^m] \quad y \mapsto (C_y; x_1(y), x(y)),$$

where C_y is the fiber at y of $\mathcal{C}_{g,n/\mathbb{F}_q}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{F}_q}[\ell^m]$. By Corollary 11.3.2, $\kappa_x = \kappa_1$ implies that the induced $\mathrm{GSp}(H)$-equivariant homomorphism

$$\phi : \mathrm{Gr}^W_{-1}\,\mathfrak{u}^{\mathrm{geom}}_{g,n} = \Lambda^3_0 H \oplus H_1 \oplus \cdots \oplus H_n \to \mathrm{Gr}^W_{-1}\,\mathfrak{u}^{\mathrm{geom}}_{g,2} = \Lambda^3_0 H \oplus H_1 \oplus H_2$$

is given by

$$(v; u_1, \ldots, u_n) \mapsto (v; u_1, u_1).$$

This is impossible by Lemma 13.0.6. Thus $t = 0$, and we are done. □

Proof of Theorem 1. It is enough to show for the case $\ell^m \geqslant 3$. The valuative criterion of properness and the normality of $\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$ implies that each K-rational point of $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$ gives a unique section of the universal curve. Hence we have $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m](K) = \mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m](\mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m])$. Let x be a section of $\mathcal{C}_{g,n/\bar{\mathbb{F}}_p}[\ell^m] \to \mathcal{M}_{g,n/\bar{\mathbb{F}}_p}[\ell^m]$. The section x induces a section s_x of $\epsilon_n : \mathfrak{d}_{g,n+1} \to \mathfrak{d}_{g,n}$. By Corollary 10.4.5, we have $s_x = s_j$ for some $j \in \{1, \ldots, n\}$. Recall that s_j is the section of ϵ_n induced by the jth tautological point. Corollary 11.3.2 implies that $\kappa_x = \kappa_j$ and thus we have $x = x_j$ by Proposition 13.0.7. □

Proof of Theorem 2. Suppose that there is a section s of $\pi_1(C, \bar{x}) \to G_L$. By Corollary 12.0.5, the section s induces a pro-ℓ section $s^{(\ell)}$ of

$$1 \to \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} \to \pi_1'(\mathcal{C}_{\mathbb{F}_q}[\ell^m], \bar{x}) \to \pi_1(\mathcal{M}_{\mathbb{F}_q}[\ell^m], \bar{\eta}) \to 1,$$

which induces a GSp(H)-equivariant section of $\epsilon_0 : \mathfrak{d}_{g,1} \to \mathfrak{d}_{g,0}$. By Proposition 10.4.6, there is no GSp(H)-equivariant section of ϵ_0 that come from the projection $\pi_1'(\mathcal{C}_{\mathbb{F}_q}[\ell^m], \bar{x}) \to \pi_1(\mathcal{M}_{\mathbb{F}_q}[\ell^m], \bar{\eta})$. Therefore, there is no section of $\pi_1(C, \bar{x}) \to G_L$.

□

Appendix A

Weighted Completion of The Hyperelliptic Mapping Class Groups

In this chapter, we introduce the hyperelliptic mapping class groups and their weighted completions with respect to their natural monodromy representations. We will introduce the 3-step nilpotent Lie algebra $\mathfrak{b}_{g,n}$ associated to the completion. This is the hyperelliptic analogue of the 2-step nilpotent Lie algebra $\mathfrak{d}_{g,n}$ described in 10.4. In this chapter, we will show that the geometric sections of $\beta_n : \mathfrak{b}_{g,n+1} \to \mathfrak{b}_{g,n}$ are exactly ones that come from the tautological sections of the universal hyperelliptic curve and their hyperelliptic conjugates. Also, the technical result used here can be used to improve the result for the sections of $\epsilon_n : \mathfrak{d}_{3,n+1} \to \mathfrak{d}_{3,n}$. More precisely, the zero section ζ_n of ϵ_n cannot come from a section of $\mathrm{Gr}^w_\bullet\, \mathfrak{u}^{\mathrm{geom}}_{\mathcal{C}_{g,n}}/W_{-5} \to \mathrm{Gr}^W_\bullet\, \mathfrak{u}^{\mathrm{geom}}_{g,n}/W_{-5}$.

A.1 Hyperelliptic mapping class groups

Suppose that σ_g is a compact oriented surface of genus $g \geqslant 2$. Let Γ_g be the mapping class group of the surface Σ_g. Recall that it is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. Fix a hyperelliptic involution σ :

$\Sigma_g \to \Sigma_g$, which is an orientation-preserving diffeomorphism of S of order 2 with $2g+2$ fixed points. Such a diffeomorphism is unique up to isotopy. The hyperelliptic mapping class group Δ_g is defined to be the centralizer of σ in Γ_g:

$$\Delta_g := \{\phi \in \Gamma_g | \phi\sigma\phi^{-1} = \sigma\}.$$

Define $\Delta_{g,n}$ to be the fiber product

$$\Delta_{g,n} := \Gamma_{g,n} \times_{\Gamma_g} \Delta_g,$$

where $\Gamma_{g,n}$ is the mapping class group of type (g,n) and $\Gamma_{g,n} \to \Gamma_g$ is the natural projection obtained by forgetting the n marked points.

Define also $\Delta_{g,1^w}$ to be the subgroup of elements of Δ_g that fixes a specific Weierstrass point. More precisely, define $\Delta_{g,1^w}$ as follows. Denote the set of the Weierstrass points of Σ_g by W. Then there is an exact sequence of groups

$$1 \to \Delta_g[2] \to \Delta_g \to \mathrm{Aut}(W) \to 1,$$

where $\Delta_g[2]$ is the level 2 subgroup of Δ_g and the map $\Delta_g \to \mathrm{Aut}(W)$ is obtained by the action of Δ_g on W. Fixing a Weierstrass point p, we have the stabilizer subgroup $\mathrm{Aut}(W)_p$ of $\mathrm{Aut}(W)$. Define $\Delta_{g,1^w}$ to be the pullback of $\mathrm{Aut}(W)_p$ along the map $\Delta_g \to \mathrm{Aut}(W)$: there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_g[2] & \longrightarrow & \Delta_{g,1^w} & \longrightarrow & \mathrm{Aut}(W)_p & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_g[2] & \longrightarrow & \Delta_g & \longrightarrow & \mathrm{Aut}(W) & \longrightarrow & 1,
\end{array}$$

where rows are exact.

A.2 Moduli stacks of smooth hyperelliptic curves

Assume that k is a field of characteristic zero. Denote the moduli functor of smooth hyperelliptic curves over k by $\mathcal{H}_g$. It is a contravariant functor from the category of

k-schemes to the category of sets:

$$\mathcal{H}_{g/k} : Sch_{/k} \to Set$$

which assigns to each k-scheme T the set of isomorphism classes of families of smooth hyperelliptic curves of genus g:

$$\mathcal{H}_{g/k}(T) = \{C \to T \text{ family of smooth hyperelliptic curves of genus } g\}/\cong .$$

It was shown by Arsie and Vistoli [4] that $\mathcal{H}_{g/k}$ is a Deligne-Mumford stack that is isomorphic to a quotient stack:

$$\mathcal{H}_{g/k} \cong [\mathbb{A}_{sm}(2, 2g+2)/(\mathrm{GL}_2/\mu_{g+1})],$$

where $\mathbb{A}_{sm}$ is the space of homogeneous smooth forms of degree $2g+2$ in 2 indeterminates and GL_2 acts by $A \cdot f(x) = f(A^{-1}x)$. It is an irreducible smooth algebraic stack of finite type over $\operatorname{Spec} k$ of dimension $2g-1$. It is a closed substack of the moduli stack $\mathcal{M}_{g/k}$ of smooth curves of genus g over k. Define $\mathcal{H}_{g,n/k}$ to be the fiber product

$$\mathcal{H}_{g,n/k} = \mathcal{M}_{g,n/k} \times_{\mathcal{M}_{g/k}} \mathcal{H}_{g/k}.$$

Denote the universal curve over $\mathcal{H}_{g,n}$ by $\mathcal{C}_{\mathcal{H},g,n} \to \mathcal{H}_{g,n}$. It is the pullback of $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ to $\mathcal{H}_{g,n}$.

Variant A.2.1. Define $\mathcal{H}_{g,1^w/\bar{k}}$ to be the finite cover of the stack $\mathcal{H}_{g/\bar{k}}$ that corresponds to the finite quotient $\Delta_g/\Delta_{g,1^w}$. The moduli stack $\mathcal{H}_{g/\bar{k}}[2]$ of smooth hyperelliptic curves with abelian level 2 factors through the cover $\mathcal{H}_{g,1^w/\bar{k}}$:

$$\begin{array}{c} \mathcal{H}_{g/\bar{k}}[2] \\ \downarrow \\ \mathcal{H}_{g,1^w/\bar{k}} \\ \downarrow \\ \mathcal{H}_{g/\bar{k}} \end{array}$$

Variant A.2.2. Denote the n-th power of the universal curve $\mathcal{M}_{g/k}$ by $\mathcal{C}^n_{g/k}$ and the n-th power of the universal hyperelliptic curve over $\mathcal{H}_{g/k}$ by $\mathcal{C}^n_{\mathcal{H},g/k}$. By convention, we set $\mathcal{C}^0_{g/k} = \mathcal{M}_{g/k}$ and $\mathcal{C}^1_{g/k} = \mathcal{M}_{g,1/k}$. Note that $\mathcal{M}_{g,n/k}$ is an open substack of $\mathcal{C}^n_{g/k}$ and that $\mathcal{H}_{g,n/k}$ is an open substack of $\mathcal{C}^n_{\mathcal{H},g/k}$. We have the following fiber product diagram

$$\begin{array}{ccccc} \mathcal{H}_{g,n/k} & \longrightarrow & \mathcal{C}^n_{\mathcal{H},g/k} & \longrightarrow & \mathcal{H}_{g/k} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{g,n/k} & \longrightarrow & \mathcal{C}^n_{g/k} & \longrightarrow & \mathcal{M}_{g/k}. \end{array}$$

Forgetting the first component of $\mathcal{C}^{n+1}_{g/k}$ gives a curve $\pi : \mathcal{C}^{n+1}_{g/k} \to \mathcal{C}^n_{g/k}$. Pulling back π to $\mathcal{M}_{g,n/k}$, we obtain the universal curve $\mathcal{C}_{g,n/k}$ of type (g,n) over $\mathcal{M}_{g,n/k}$, which pulls back to the universal hyperelliptic curve $\mathcal{C}_{\mathcal{H},g,n/k}$ of type (g,n) over $\mathcal{H}_{g,n/k}$.

A.3 Fundamental groups $\pi_1(\mathcal{H}_{g,n/k})$ and their natural monodromy representations

In this section, assume that k is an algebraic number field or a finite extension of the p-adic number $\mathbb{Q}_p$ with some prime number p. By the comparison theorem [38], we have a unique conjugacy class of isomorphisms

$$\pi_1(\mathcal{H}_{g,n/\bar{k}}) \cong \pi_1^{\mathrm{orb}}(\mathcal{H}^{\mathrm{an}}_{g,n})^{\wedge} \cong (\Delta_{g,n})^{\wedge},$$

where $\mathcal{H}^{\mathrm{an}}_{g,n}$ denotes the associated analytic stack and $^\wedge$ denotes profinite completion. Let $\bar{\eta} : \operatorname{Spec}\Omega \to \mathcal{H}_{g,n/\bar{k}}$ be a geometric point of $\mathcal{H}_{g,n/\bar{k}}$. We regard $\bar{x}$ as a geometric point of $\mathcal{H}_{g,n/k}$ via the natural morphism $\mathcal{H}_{g,n/\bar{k}} \to \mathcal{H}_{g,n/k}$. The algebraic fundamental group $\pi_1(\mathcal{H}_{g,n/k}, \bar{\eta})$ sits in the exact sequence of algebraic fundamental groups

$$1 \to \pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta}) \to \pi_1(\mathcal{H}_{g,n/k}, \bar{\eta}) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

Let $\ell \neq p$. For $A = \mathbb{Z}_\ell$ or $A = \mathbb{Q}_\ell$, set $H_A = H^1_{\text{ét}}(C_{\bar{\eta}}, A(1))$, where $C_{\bar{\eta}}$ is the fiber of the universal curve over the geometric point $\bar{\eta}$. The étale cohomology group H_A is equipped with a nondegenerate alternating form $\theta : H_A \otimes H_A \to A(1)$. There is a natural monodromy representation $\rho : \pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta}) \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$ that comes from the action of Δ_g on the surface Σ_g. It follows from the result of A'Campo [1] that the image of ρ^{geom} in $\mathrm{Sp}(H_{\mathbb{Z}_\ell})$ contains a finite-index subgroup and hence that ρ^{geom} has a Zariski-dense image in $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$. The natural monodromy action of $\pi_1(\mathcal{H}_{g,n/k}, \bar{\eta})$ on $H_{\mathbb{Q}_\ell}$ gives a homomorphism

$$\rho : \pi_1(\mathcal{H}_{g,n/k}, \bar{\eta}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$$

that fits in the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta}) & \longrightarrow & \pi_1(\mathcal{H}_{g,n/k}, \bar{\eta}) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \rho^{\mathrm{geom}}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \chi_\ell} & & \\
1 & \longrightarrow & \mathrm{Sp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathrm{GSp}(H_{\mathbb{Q}_\ell}) & \longrightarrow & \mathbb{G}_{m/\mathbb{Q}_\ell} & \longrightarrow & 1,
\end{array}$$

where χ_ℓ is the ℓ-adic cyclotomic character. The Zariski density of ρ^{geom} and χ_ℓ implies that ρ also has a Zariski-dense image.

A.4 The hyperelliptic Johnson homomorphism and Dehn twists

In this section, we briefly introduce the hyperelliptic analogue of the Johnson homomorphism. The detail of the construction of the hyperelliptic Johnson homomorphism can be found in Spivey's thesis [42]. First we recall the Johnson homomorphism. Let Π be the topological fundamental group $\pi_1^{\mathrm{top}}(\Sigma_g, *)$ of the surface Σ_g. Denote the lower central series of Π by $L^\bullet\Pi$:

$$L^1\Pi = \Pi \text{ and } L^k\Pi = [L^{k-1}\Pi, \Pi] \text{ for } k \geqslant 2.$$

Let $\mathrm{Gr}_L^m \Pi := L^m\Pi/L^{m+1}\Pi$ for each $m \geqslant 1$. Each $\mathrm{Gr}_L^m \Pi$ is a torsion-free $\mathrm{Sp}(H)$-module. Let $H = H_1(\Sigma_g, \mathbb{Z})$ equipped with the standard symplectic basis,

$$a_1, b_1, \ldots, a_g, b_g$$

and the intersection paring $\theta : H \otimes H \to \mathbb{Z}$. The form θ is a nondegenerate skew-symmetric bilinear form. We may regard θ as an element of $\Lambda^2 H$ via the isomorphism $H \cong H^*$ induced by θ. We have $\theta = \sum_{i=1}^{g} a_i \wedge b_i$. Then there is an isomorphism $\mathrm{Gr}_L^2 \Pi \cong \Lambda^2 H / \langle\theta\rangle$. Note that the mapping class group $\Gamma_{g,1}$ acts on Π and there is an exact sequence of $\Gamma_{g,1}$-groups

$$1 \to \mathrm{Gr}_L^2 \Pi \to \Pi/L^3\Pi \to \mathrm{Gr}_L^1 \Pi \to 1$$

Identifying $\mathrm{Gr}_L^1 \Pi \cong H$ and $\mathrm{Gr}_L^2 \Pi \cong \Lambda^2 H/\langle\theta\rangle$, this exact sequence can be rewritten as

$$1 \to \Lambda^2 H/\langle\theta\rangle \to \Pi/L^3\Pi \to H \to 1.$$

Recall that the Torelli group $T_{g,1}$ is the kernel of the natural representation $\Gamma_{g,1} \to \mathrm{Sp}(H)$. Then the Johnson homomorphism

$$\tau : T_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H/\langle\theta\rangle)$$

is defined as follows. For $u \in H$, let $\tilde{u}$ be a lift of u in $\Pi/L^3\Pi$. For an element $\phi \in T_{g,1}$, the element $\phi(\tilde{u})\tilde{u}^{-1}$ lies in $\Lambda^2 H/\langle\theta\rangle$, since ϕ acts trivially on H by definition. The Johnson homomorphism τ is defined to be the map $u \mapsto \phi(\tilde{u})\tilde{u}^{-1}$. It can be easily checked that $\tau(\phi)$ is independent of the choice of the life $\tilde{u}$ of u. Johnson showed in [27, 28] that $\operatorname{im}\tau = \Lambda^3 H$ for $g \geqslant 2$ and $H_1(T_{g,1}, \mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}] = \Lambda^3 H \otimes \mathbb{Z}[\frac{1}{2}]$ for $g \geqslant 3$.

The hyperelliptic Torelli group $T\Delta_g$ is defined to be the intersection of T_g and Δ_g in Γ_g, or equivalently the kernel of the natural representation $\Delta_g \to \mathrm{Sp}(H)$. Since $T\Delta_g$ fixes each Weierstrass point, $T\Delta_g$ is a subgroup of $\Delta_{g,1^w}$. Furthermore, it is easy to see that $T\Delta_g$ coincides with the kernel of the natural representation $\Delta_{g,1^w} \to \mathrm{Sp}(H)$. A construction similar to one for the Johnson homomorphism gives a $\Delta_{g,1^w}$-equivariant homomorphism

$$\omega : T\Delta_g \to \mathrm{Hom}(H, \Lambda^2 H/\langle\theta\rangle).$$

However, this homomorphism is trivial, since the hyperelliptic involution σ acts trivially on $T\Delta_g$, while it acts as $-\operatorname{id}$ on $\operatorname{Hom}(H, \Lambda^2 H/\langle\theta\rangle)$. Thus this suggests that we consider the next graded quotient $\operatorname{Gr}_L^3 \Pi$ and the exact sequence of $\Delta_{g,1^w}$-groups

$$1 \to \operatorname{Gr}_L^3 \Pi \to \Pi/L^4\Pi \to \Pi/L^3\Pi \to 1.$$

The triviality of the homomorphism ω implies that $T\Delta_g$ acts trivially on $\Pi/L^3\Pi$. Thus the same construction for the Johnson homomorphism gives a $\Delta_{g,1^w}$-equivariant homomorphism

$$\tau^{\text{hyp}} : T\Delta_g \to \operatorname{Hom}(\Pi/L^3\Pi, \operatorname{Gr}_L^3 \Pi).$$

Since $\operatorname{Gr}_L^3 \Pi$ is abelian and the abelianization of $\Pi/L^3\Pi$ is $\operatorname{Gr}_L^1 \Pi = H$, we have

$$\operatorname{Hom}(\Pi/L^3\Pi, \operatorname{Gr}_L^3 \Pi) = \operatorname{Hom}(H, \operatorname{Gr}_L^3 \Pi).$$

This $\Delta_{g,1^w}$-equivariant homomorphism

$$\tau^{\text{hyp}} : T\Delta_g \to \operatorname{Hom}(H, \operatorname{Gr}_L^3 \Pi)$$

is defined to be the hyperelliptic Johnson homomorphism. Note that the restriction of τ^{hyp} to the kernel N of τ^{hyp} induces a homomorphism

$$\tau_2^{\text{hyp}} : N \to \operatorname{Hom}(H, \operatorname{Gr}_L^5 \Pi).$$

More generally, if N_j is the kernel of the homomorphism $\Delta_g \to \operatorname{Aut}(\Pi/L^{2j}\Pi)$, then there is a homomorphism

$$\tau_j^{\text{hyp}} : N_j \to \operatorname{Hom}(H, \operatorname{Gr}_L^{2j+1} \Pi).$$

Note that $N_1 = T\Delta_g$.

A.4.1 The image of a Dehn twist under τ^{hyp}

In this section, set $H = H_1(\Sigma_g, \mathbb{Q})$. For each $j = 1, \ldots, g-1$, let C_j be the separating simple closed curve in Σ_g shown in the figure below. Then $\Sigma - C_i$ is the disjoint

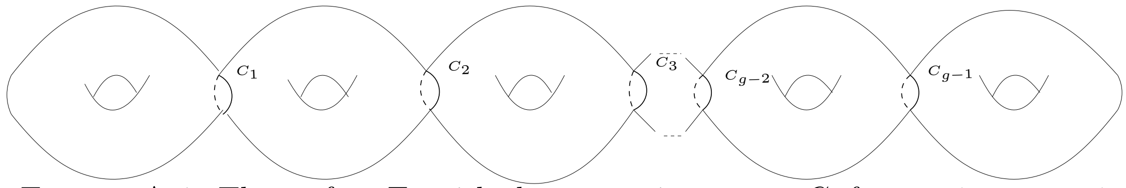


FIGURE A.1: The surface Σ_g with the separating curves C_j for $j = 1, \ldots, g-1$

union of two components S'_j and S''_j with boundary curve C_j. Let $\theta'_j = \sum_{i=1}^{j} a_i \wedge b_i$ and $\theta''_j = \sum_{i=j+1}^{g} a_i \wedge b_i$. Note that $\theta = \theta'_j + \theta''_j$. Let $\mathbb{L}(H)$ be the free Lie algebra generated by H. Denote the lower central series of $\mathbb{L}(H)$ by $L^\bullet\mathbb{L}(H)$. It is graded by bracket length:

$$\mathbb{L}(H) = \bigoplus_n \mathrm{Gr}_L^n \mathbb{L}(H),$$

where $\mathrm{Gr}_L^n \mathbb{L}(H) = L^n\mathbb{L}(H)/L^{n+1}\mathbb{L}(H)$. There is an isomorphism of graded Lie algebras

$$\mathrm{Gr}_L^\bullet \Pi \otimes \mathbb{Q} \cong \mathbb{L}(H)/\langle\theta\rangle.$$

In low degree, we have

$$\mathrm{Gr}_L^1 \Pi \otimes \mathbb{Q} = \mathbb{L}_1(H) = H, \ \ \mathrm{Gr}_L^2 \Pi \otimes \mathbb{Q} = \mathbb{L}_2(H)/\langle\theta\rangle, \text{ and } \mathrm{Gr}_L^3 \Pi \otimes \mathbb{Q} = \mathbb{L}_3(H)/[\theta, H].$$

Denote the Lie algebra $\mathbb{L}/\langle\theta\rangle$ by $\mathfrak{p}$. Define $\phi : \mathrm{Sym}^2\Lambda^2 H \to \mathrm{Hom}(H, \mathbb{L}_3(H)/[\theta, H])$ by

$$(u_1 \wedge v_1) \cdot (u_2 \wedge v_2) \mapsto \Big(x \mapsto \frac{1}{2}\{\theta(u_1, x)[v_1, [u_2, v_2]] - \theta(v_1, x)[u_1.[u_2, v_2]]$$
$$+ \theta(u_2, x)[v_2, [u_1, v_1]] - \theta(v_2, x)[u_2, [u_1, v_1]]\}\Big)$$

It is easy to see that ϕ is a $\mathrm{Sp}(H)$-equivariant homomorphism.

Lemma A.4.1. *For $I \subset \{1, \ldots, g\}$, set $\theta_I = \sum_{i \in I} a_i \wedge b_i$, $H_I = \mathrm{Span}\langle a_i, b_i | i \in I\rangle$, and $H_I^c = \mathrm{Span}\langle a_i, b_i | i \notin I\rangle$. Then we have*

$$\phi(\theta_I^2) : x \mapsto \begin{cases} 0 & x \in H_I^c \\ [x, \theta_I] & x \in H_I \end{cases}$$

Proof. A simple computation suffices. □

Let ω_j be the isotopy class of the Dehn twist around C_j. For simplicity, fix a Weierstrass point p in S'_1. Note that each ω_j is an element in $T\Delta_g$. In order to compute $\tau^{\text{hyp}}(\omega_j)$, we need to compute the action of ω on $\pi_1(\Sigma_g, p)$, first. We do this by computing the action of ω_j on the standard generating set of $\pi_1(\Sigma_g, x)$ consisting of the classes γ_{2i-1}, γ_{2i} based at x for $i = 1, \ldots, g$:

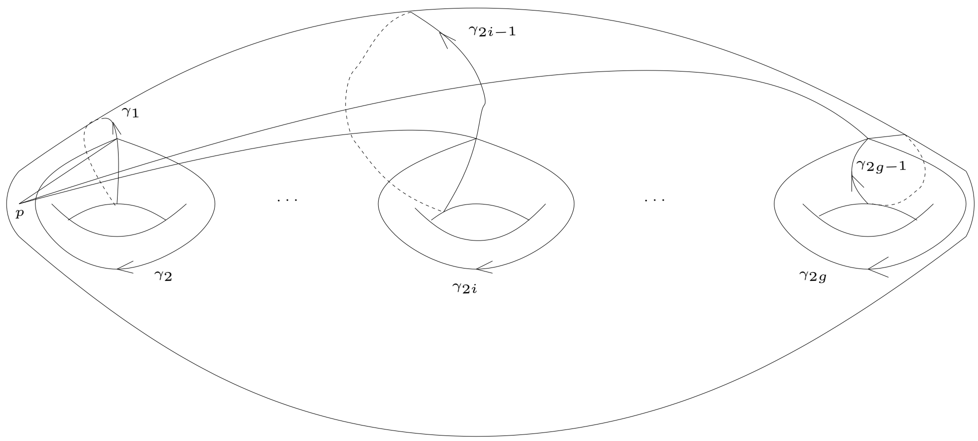


FIGURE A.2: The surface Σ_g with the standard generators γ_{2i-1} and γ_{2i} for $i = 1, \ldots, g$

Proposition A.4.2. *With notation as above, we have*

$$\tau^{\text{hyp}}(\omega_j) = \phi((\theta''_j)^2) : x \mapsto \begin{cases} 0 & x \in H_I^c \\ [x, \theta''_j] & x \in H_I \end{cases}$$

where $I = \{j+1, \ldots, g\}$.

Proof. Since the classes γ_i for $i \leqslant 2j$ do not intersect with C_j, we have $\omega_j(\gamma_i) = \gamma_i$ for $i \leqslant 2j$. Now, fix a point y_j on C_j and a path ρ_j from x to y_j. Considering C_j as a loop based at y_j, the composition $\rho_j C_j \rho_j^{-1}$ is a loop based at x. Note that the homotopy class of $\rho_j C_j \rho_j^{-1}$ is equal to the class

$$([\gamma_1, \gamma_2] \cdots [\gamma_{2j-1}, \gamma_{2j}])^{-1} \in \pi_1(\Sigma_g, x).$$

For $i > 2j$, we have

$$\omega_j(\gamma_i) = ([\gamma_1, \gamma_2] \cdots [\gamma_{2j-1}, \gamma_{2j}])\, \gamma_i\, ([\gamma_1, \gamma_2] \cdots [\gamma_{2j-1}, \gamma_{2j}])^{-1}$$
$$= \left[\prod_{k=1}^{j} [\gamma_{2k-1}, \gamma_{2k}], \gamma_i \right] \gamma_i,$$

which shows that

$$\omega_j(\gamma_i)\gamma_i^{-1} \in L^3\Pi.$$

Reducing mod $L^4\Pi$, we obtain the class

$$\left[\sum_{k=1}^{j} [u_{2k-1}, u_{2k}], u_i \right] \in \mathrm{Gr}_L^3 \Pi = \mathbb{L}_3(H)/[\theta, H],$$

where $u_{2k-1} = a_k$ and $u_{2k} = b_k$ for each $k = 1, \ldots, g$. Since $\sum_{k=1}^{j}[u_{2k-1}, u_{2k}] = \theta'_j$ and $\theta'_j + \theta''_j = \theta$, we can express

$$\left[\sum_{k=1}^{j} [u_{2k-1}, u_{2k}], u_i \right] = [\theta'_j, u_i] = [u_i, \theta''_j] \text{ in } \mathbb{L}_3(H)/[\theta, H].$$

The u_i with $i > 2j$ form a basis for H_I, and so we have

$$\tau^{\mathrm{hyp}}(\omega_j)(x) = [x, \theta''_j] \quad \text{if } x \in H_I$$

. It is clear that we have

$$\tau^{\mathrm{hyp}}(\omega_j)(x) = 0 \quad \text{if } x \in H_I^c.$$

□

A.4.2 The derivation Lie algebras $\mathrm{Der}\,\mathbb{L}(H)$ *and* $\mathrm{Der}\,\mathfrak{p}$

Note that the derivation Lie algebra $\mathrm{Der}\,\mathbb{L}(H) = \mathrm{Hom}(H, \mathbb{L}(H))$. Since $\mathbb{L}(H) = \mathrm{Gr}_L^{\bullet}\,\mathbb{L}(H)$, we have

$$\mathrm{Der}\,\mathbb{L}(H) = \bigoplus_n \mathrm{Der}^n\,\mathbb{L}(H),$$

where $\mathrm{Der}^n \mathbb{L}(H) = \mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathbb{L}(H))$ for each n. Similarly, note that $\mathrm{Der}\,\mathfrak{p} = \mathrm{Der}\,\mathrm{Gr}_L^\bullet \mathfrak{p}$ and that $\mathrm{Gr}_L^\bullet \mathrm{Der}\,\mathfrak{p} = \mathrm{Der}\,\mathrm{Gr}_L^\bullet \mathfrak{p}$. It is easy to see that $\mathrm{Der}^n \mathfrak{p} := \mathrm{Gr}_L^n \mathrm{Der}\,\mathfrak{p}$ is an $\mathrm{Sp}(H)$-submodule of $\mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathfrak{p})$.

Proposition A.4.3. *For each $n \geqslant 1$, there is an exact sequence of* $\mathrm{Sp}(H)$*-modules*

$$0 \to \mathrm{Der}^n \mathfrak{p} \to \mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathfrak{p}) \to \mathrm{Gr}_L^{n+2} \mathfrak{p} \to 0.$$

Proof. Each homomorphism $\delta \in \mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathfrak{p})$ induces a derivation $\tilde{\delta}$ of the free Lie algebra $\mathbb{L}(H)$ by regarding $\mathrm{Gr}_L^{n+1} \mathfrak{p}$ as an $\mathrm{Sp}(H)$-submodule of $\mathrm{Gr}_L^{n+1} \mathbb{L}(H)$. We have a decomposition

$$\mathrm{Gr}_L^2 \mathbb{L}(H) = \mathrm{Gr}_L^2 \mathfrak{p} \oplus \mathbb{Q}\theta.$$

Denote by $D(\delta)$ the image in $\mathrm{Gr}_L^{n+2} \mathfrak{p}$ of the element $\tilde{\delta}(\theta)$ under the surjection $\mathrm{Gr}_L^\bullet \mathbb{L}(H) \to \mathrm{Gr}_L^\bullet \mathfrak{p}$. It is clear that δ induces a derivation of $\mathfrak{p}$ if and only if $D(\delta) = 0$. The association $\delta \mapsto D(\delta)$ is easily seen to be a homomorphism and so we have

$$\mathrm{Der}^n \mathfrak{p} = \ker(D : \mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathfrak{p}) \to \mathrm{Gr}_L^{n+2} \mathfrak{p})).$$

It remains to show that D is surjective. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(H, \mathrm{Gr}_L^{n+1} \mathfrak{p}) & \xrightarrow{D} & \mathrm{Gr}_L^{n+2} \mathfrak{p} \\ \downarrow & & \| \\ H \otimes \mathrm{Gr}_L^{n+1} \mathfrak{p} & \xrightarrow{[\ ,\ ]} & \mathrm{Gr}_L^{n+2} \mathfrak{p}, \end{array}$$

where the left vertical map is induced by θ. Since $a_i^* = -\theta(b_i, \ )$ and $b_i^* = \theta(a_i, \ )$, via the isomorphism $H \cong H^*$ induced by θ, δ maps to the element

$$\sum_{i=1}^{g} a_i \otimes \delta(b_i) - b_i \otimes \delta(a_i) \text{ in } H \otimes \mathrm{Gr}_L^{n+1} \mathfrak{p}.$$

Then this element maps to the element via the bracket

$$\sum_{i=1}^{g} [a_i, \delta(b_i)] - [b_i, \delta(a_i)] = \sum_{i=1}^{g} [\delta(a_i), b_i] + [a_i, \delta(b_i)],$$

which is equal to $D(\delta)$. Thus it follows that D is surjective. □

Corollary A.4.4. *In the representation ring $R(\mathrm{Sp}(H))$, we have*

$$\mathrm{Der}^n \mathfrak{p} = H \otimes \mathrm{Gr}_L^{n+1} \mathfrak{p} - \mathrm{Gr}_L^{n+2} \mathfrak{p}.$$

Let H_λ denote the isomorphism class of the irreducible $\mathrm{Sp}(H)$-representation that corresponds to a partition λ of a nonnegative integer n of $1 \leqslant s \leqslant g$ part: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$. The Kabanov's stability [29] gives the following result.

Proposition A.4.5. *For all $g \geqslant 3$, the irreducible decomposition of $\mathrm{Gr}_L^m \mathfrak{p}$ when $1 \leqslant m \leqslant 4$ is given by*

$$\mathrm{Gr}_L^1 \mathfrak{p} = H_{[1]};$$

$$\mathrm{Gr}_L^2 \mathfrak{p} = H_{[1^2]};$$

$$\mathrm{Gr}_L^3 \mathfrak{p} = H_{[2+1]};$$

$$\mathrm{Gr}_L^4 \mathfrak{p} = H_{[2+1^2]} + H_{[2]} + H_{[3+1]}.$$

Remark A.4.6. The above computation was done by the compute program LiE.

Together with the decomposition for each $\mathrm{Gr}_L^m \mathfrak{p}$, the following result can be easily computed.

Corollary A.4.7. *For all $g \geqslant 3$, we have*

$$\mathrm{Der}^1 \mathfrak{p} = H_{[1^3]} + H_{[1]};$$

$$\mathrm{Der}^2 \mathfrak{p} = H_{[2^2]} + H_{[1^2]};$$

$$\mathrm{Der}^3 \mathfrak{p} = H_{[3+1^2]} + H_{[2+1]} + H_{[3]}.$$

It was shown in [6] that the graded Lie algebra $\mathfrak{p}$ has the trivial center, and hence we may regard $\mathfrak{p}$ as a graded Lie ideal of $\mathrm{Der}\,\mathfrak{p}$ via the adjoint action. Denote the quotient $\mathrm{Der}\,\mathfrak{p}/\mathfrak{p}$ by $\mathrm{Out\,Der}\,\mathfrak{p}$.

Proposition A.4.8. *For all $g \geqslant 3$, we have*

$$\operatorname{Out}\operatorname{Der}^1 \mathfrak{p} = H_{1^3]};$$

$$\operatorname{Out}\operatorname{Der}^2 \mathfrak{p} = H_{[2^2]};$$

$$\operatorname{Out}\operatorname{Der}^3 \mathfrak{p} = H_{[3+1^2]} + H_{[3]}.$$

The following lemma shows that $\tau^{\mathrm{hyp}}(\omega_j)$ induces a derivation of $\mathfrak{p}$.

Lemma A.4.9. *For $I \subset \{1, \ldots, g\}$, $\phi(\theta_I^2)$ lies in $\operatorname{Der}^2 \mathfrak{p}$, where $\theta_I = \sum_{i \in I} a_i \wedge b_i$.* □

A.4.3 The outer action of a commuting pair of Dehn twists

In this section, we will show that there is a pair of commuting pair of Dehn twists such that the bracket of the outer part of the image of each of the Dehn twist under the hyperelliptic Johnson homomorphism is a nontrivial inner derivation in $\operatorname{Der}^4 \mathfrak{p}$.

Proposition A.4.10. *Assume $g \geqslant 3$. The map $\phi : \operatorname{Sym}^2 \Lambda^2 H \to \operatorname{Hom}(H, \operatorname{Gr}_L^3 \mathfrak{p})$ defined in A.4.1 induces a homomorphism*

$$\tilde{\phi} : \operatorname{Sym}^2 \Lambda^2 H / (\Lambda^4 H \oplus \mathbb{Q}) \to \operatorname{Hom}(H, \operatorname{Gr}_L^3 \mathfrak{p}).$$

Proof. First note that the representation $\Lambda^4 H$ sits inside $\operatorname{Sym}^2 \Lambda^2 H$ via the map

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 \mapsto (v_1 \wedge v_2)(v_3 \wedge v_4) + (v_1 \wedge v_3)(v_4 \wedge v_2) + (v_1 \wedge v_4)(v_2 \wedge v_3).$$

The corresponding projection is given by

$$(u_1 \wedge u_2)(u_3 \wedge u_4) \mapsto u_1 \wedge u_2 \wedge u_3 \wedge u_4.$$

A copy of the trivial representation $\mathbb{Q}$ sits inside $\operatorname{Sym}^2 \Lambda^2 H$ via the map

$$1 \mapsto \theta^2.$$

Simple computations show that $\phi(\Lambda^4 H) = 0$ and $\phi(\theta^2) = 0$. Thus ϕ induces an $\operatorname{Sp}(H)$-homomorphism

$$\tilde{\phi} : \operatorname{Sym}^2 \Lambda^2 H / (\Lambda^4 H \oplus \mathbb{Q}) \to \operatorname{Hom}(H, \operatorname{Gr}_L^3 \mathfrak{p}).$$

□

Remark A.4.11. In fact, the map $\tilde{\phi}$ is an isomorphism onto $\operatorname{Der}\mathfrak{p}$.

In order to separate the $H_{[2^2]}$ component from $\operatorname{Sym}^2\Lambda^2 H$, we need a projection onto a copy of $H_{[1^2]}$ that is not contained in the submodule $\Lambda^4 H \subset \operatorname{Sym}^2\Lambda^2 H$. Note that a copy of $\Lambda^2 H$ sits inside $\Lambda^4 H$ via the map

$$u \wedge v \mapsto u \wedge v \wedge \theta.$$

The other copy of $\Lambda^2 H$ sits inside $\operatorname{Sym}^2\Lambda^2 H$ via the map

$$u \wedge v \mapsto (u \wedge v) \cdot \theta.$$

Define a map $\pi : \operatorname{Sym}^2\Lambda^2 H \to \Lambda^2 H$ by

$$\begin{aligned}(u_1 \wedge v_1)(u_2 \wedge v_2) \mapsto\ & \theta(u_1, v_1) v_2 \wedge u_2 + \theta(v_2, u_2) u_1 \wedge v_1 \\ & + \frac{1}{2}\{\theta(u_1, v_2) v_1 \wedge u_2 + \theta(v_1, u_2) u_1 \wedge v_2 + \theta(u_1, u_2) v_2 \wedge v_1 + \theta(v_2, v_1) u_1 \wedge u_2\}.\end{aligned}$$

One can easily check that π is an $\operatorname{Sp}(H)$-homomorphism and vanishes on $\Lambda^4 H$ as a submodule of $\operatorname{Sym}^2\Lambda^2 H$.

Lemma A.4.12. *Consider the irreducible* $\operatorname{Sp}(H)$*-module* $H_{[1^2]}$ *as the submodule of* $\Lambda^2 H$ *given by the kernel of the map* $u \wedge v \mapsto \theta(u, v)$*. Then the composition*

$$H_{[1^2]} \xrightarrow{\cdot\theta} \operatorname{Sym}^2\Lambda^2 H \xrightarrow{\pi} \Lambda^2 H$$

is given by multiplication by $-g-1$*. Also the composition*

$$\mathbb{Q}\theta \xrightarrow{\cdot\theta} \operatorname{Sym}^2\Lambda^2 H \xrightarrow{\pi} \Lambda^2 H$$

is given by multiplication by $-2g-1$*.* □

Define $\operatorname{Sp}(H)$-equivariant projections $p_1 : \Lambda^2 H \to \mathbb{Q}\theta$ by $u \wedge v \mapsto \frac{\theta(u,v)}{g}\theta$ and $p_2 : \Lambda^2 H \to H_{[1^2]}$ by $u \wedge v \mapsto u \wedge v - \frac{\theta(u,v)}{g}\theta$. define $p : \Lambda^2 H \to \operatorname{Sym}^2\Lambda^2 H$ by

$$u \wedge v \mapsto \left(\frac{1}{-2g-1}(\cdot\theta \circ p_1) + \frac{1}{-g-1}(\cdot\theta \circ p_2)\right)(u \wedge v).$$

Corollary A.4.13. *We have* $\pi \circ p = \mathrm{id}$ *on* $\Lambda^2 H$. □

Corollary A.4.14. *We have*

$$\mathrm{Sym}^2\Lambda^2 H = \Lambda^4 H \oplus H_{[2^2]} \oplus \Lambda^2 H.$$

Proof. Since π is surjective, we have $\mathrm{Sym}^2\Lambda^2 H = \ker\pi \oplus \Lambda^2 H$. We have seen that π vanishes on $\Lambda^4 H$, and so $\ker\pi$ contains $\Lambda^4 H$. Since $\Lambda^2 H$ does not contain a copy of the irreducible representation $H_{[2^2]}$, $H_{[2^2]}$ is contained in $\ker\pi$. We also observe that the submodule $H_{[2^2]}$ is not contained in $\Lambda^4 H$. Therefore, together with the irreducible decomposition of $\mathrm{Sym}^2\Lambda^2 H$, we have

$$\ker\pi = \Lambda^4 H \oplus H_{[2^2]}.$$

□

For each vector $(u_1 \wedge v_1)(u_2 \wedge v_2) \in \mathrm{Sym}^2\Lambda^2 H$, we can express uniquely

$$(u_1 \wedge v_1)(u_2 \wedge v_2) = \delta_1 + \delta_2 + \delta_3,$$

where δ_1, δ_2, and δ_3 are vectors of $\Lambda^4 H$, $H_{[2^2]}$, and $\Lambda^2 H$, respectively.

Lemma A.4.15. *We have*

$$\delta_1 + \delta_2 = (u_1 \wedge v_1)(u_2 \wedge v_2) - \{\theta(u_1, v_1)p(v_2 \wedge u_2) + \theta(v_2, u_2)p(u_1 \wedge v_1)\}$$

$$-\frac{1}{2}\{\theta(u_1, v_2)p(v_1 \wedge u_2) + \theta(v_1, u_2)p(u_1 \wedge v_2) + \theta(u_1, u_2)p(v_2 \wedge v_1) + \theta(v_2, v_1)p(u_1, u_2)\}.$$

□

Now, we consider a specific pair of commuting Dehn twists on Σ_g. Consider the separating simple closed curves C_1 and C_{g-1}. Fix a Weierstrass point p in $S_1'' \cap S_{g-1}'$: Recall that the corresponding isotopy classes of the Dehn twists around C_1 and C_{g-1}

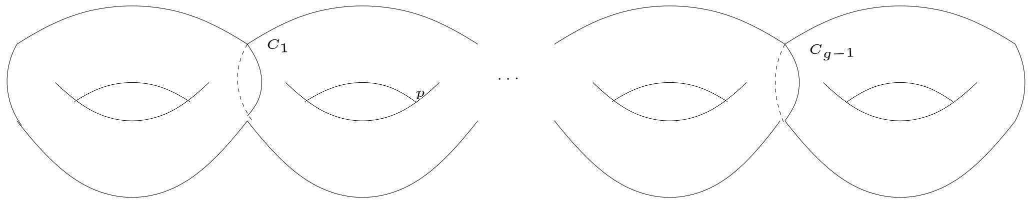


FIGURE A.3: The surface Σ_g with the separating curves C_1 and C_{g-1} and a fixed Weierstrass point p.

are denoted by ω_1 and ω_{g-1}, respectively. By Proposition A.4.2, we have

$$\tau^{\text{hyp}}(\omega_1) = \phi((a_1 \wedge b_1)^2),$$

and

$$\tau^{\text{hyp}}(\omega_{g-1}) = \phi((a_g \wedge b_g)^2).$$

Remark A.4.16. In order to apply Proposition A.4.2, one needs to adjust the result according to the position of the fixed Weierstrass point on Σ_g.

Denote $\tau^{\text{hyp}}(\omega_1)$ and $\tau^{\text{hyp}}(\omega_{g-1})$ by ω and $\tilde{\omega}$. By Lemma A.4.9, ω and $\tilde{\omega}$ lies in $\operatorname{Der}^2 \mathfrak{p}$. Since $\operatorname{Der}^2 \mathfrak{p} = H_{[2^2]} + H_{[1^2]}$, we can express ω and $\tilde{\omega}$ as

$$\omega = \xi_{[2^2]} + \xi_{[1^2]},$$

and

$$\tilde{\omega} = \tilde{\xi}_{[2^2]} + \tilde{\xi}_{[1^2]},$$

where $\xi_{[2^2]}$, $\tilde{\xi}_{[2^2]}$ are vectors in $H_{[2^2]}$ and $\xi_{[1^2]}$, $\tilde{\xi}_{[1^2]}$ are vectors in $H_{[1^2]}$.

Theorem A.4.17. *With notation as above, if $g \geqslant 3$, then the vector $[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}]$ is a nontrivial inner derivation in* $\operatorname{Der}^4 \mathfrak{p}$.

Proof. The following diagram

$$\begin{array}{ccc} T\Delta_g & \xrightarrow{\tau_1^{\text{hyp}}} & \operatorname{Der}^2 \mathfrak{p} \\ {\scriptstyle [\ ,\ ]}\downarrow & & \downarrow{\scriptstyle [\ ,\ ]} \\ N_2 & \xrightarrow{\tau_2^{\text{hyp}}} & \operatorname{Der}^4 \mathfrak{p} \end{array}$$

commutes, where the left-hand vertical map is the map taking the commutator of two elements and the right-hand vertical map is the bracket of the derivation Lie algebra $\operatorname{Der}\mathfrak{p}$. Since the Dehn twists ω_1 and ω_{g-1} commute in $T\Delta_g$ being disjoint from each other, it then follows that $[\omega, \tilde{\omega}] = 0$. Thus we have

$$\begin{aligned} 0 &= [\omega, \tilde{\omega}] \\ &= [\xi_{[2^2]} + \xi_{[1^2]}, \tilde{\xi}_{[2^2]} + \tilde{\xi}_{[1^2]}] \\ &= [\xi_{[2^2]}, \tilde{\xi}_{[2^2]}] + [\xi_{[2^2]}, \tilde{\xi}_{[1^2]}] + [\xi_{[1^2]}, \tilde{\xi}_{[2^2]}] + [\xi_{[1^2]}, \tilde{\xi}_{[1^2]}], \end{aligned}$$

which shows that the vector $[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}]$ is inner in $\operatorname{Der}\mathfrak{p}$, since $\mathfrak{p}$ is a Lie ideal in $\operatorname{Der}\mathfrak{p}$ via the adjoint action. It remains to show that this vector is nontrivial. By Corollary A.4.14, we can express $(a_1 \wedge b_1)^2$ and $(a_g \wedge b_g)^2$ as

$$(a_1 \wedge b_1)^2 = \delta_1 + \delta_2 + \delta_3 \text{ in } \Lambda^4 H \oplus H_{[2^2]} \oplus \Lambda^2 H$$

and

$$(a_g \wedge b_g)^2 = \tilde{\delta}_1 + \tilde{\delta}_2 + \tilde{\delta}_3 \text{ in } \Lambda^4 H \oplus H_{[2^2]} \oplus \Lambda^2 H.$$

Recall that the homomorphism $\phi : \operatorname{Sym}^2\Lambda^2 H \to \operatorname{Hom}(H, \mathbb{L}_3(H)/[\theta, H])$ vanishes on $\Lambda^4 H$. Thus we have

$$\phi(\delta_2) = \xi_{[2^2]} \text{ and } \phi(\delta_3) = \xi_{[1^2]}$$

and

$$\phi(\tilde{\delta}_2) = \tilde{\xi}_{[2^2]} \text{ and } \phi(\tilde{\delta}_3) = \tilde{\xi}_{[1^2]}.$$

Using Lemma A.4.15, we obtain

$$\delta_1 + \delta_2 = (a_1 \wedge b_1)^2 - \frac{3}{g+1}(a_1 \wedge b_1) \cdot \theta + c\theta^2$$

and

$$\tilde{\delta}_1 + \tilde{\delta}_2 = (a_g \wedge b_g)^2 - \frac{3}{g+1}(a_g \wedge b_g) \cdot \theta + \tilde{c}\theta^2,$$

where c and $\tilde{c}$ are some constants in $\mathbb{Q}$. Since ϕ vanishes on $\Lambda^4 H$ and $\mathbb{Q}\theta^2$, we then have

$$[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}] = \left[\phi\left((a_1 \wedge b_1)^2 - \frac{3}{g+1}(a_1 \wedge b_1)\cdot\theta\right), \phi\left((a_g \wedge b_g)^2 - \frac{3}{g+1}(a_g \wedge b_g)\cdot\theta\right)\right].$$

We evaluate this bracket on the vector $a_2 \in H$ by using the diagrammatic description for of $\operatorname{Der}\mathbb{L}(H)$ (see [34] for the description). Then we have

$$[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}](a_2) = \frac{9}{(g+1)^2}\Big[a_2, \big[[a_1, b_1], [a_g, b_g]\big]\Big] \in \operatorname{Gr}^5_L \mathfrak{p}.$$

This vector can be mapped to the highest vector of the irreducible representation $H_{[3+1^2]}$ appearing in $\operatorname{Gr}^5_L \mathfrak{p}$ via the action of $\mathfrak{sp}$, where $\mathfrak{sp}$ is the Lie algebra of $\operatorname{Sp}(H)$, and hence it is nonzero in $\mathfrak{p}$. Thus the vector $[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}]$ is a nonzero inner derivation in $\operatorname{Der}\mathfrak{p}$. □

A.5 Relative and weighted completions of hyperelliptic mapping class groups

Let k be an algebraic number field or a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\bar{k}$ of k. Assume $g \geqslant 3$. Let $\bar{\eta} : \operatorname{Spec}\Omega \to \mathcal{H}_{g,n/\bar{k}}$ be a geometric point of the stack $\mathcal{H}_{g,n/\bar{k}}$. Let $C_{\bar{\eta}}$ be the fiber of the universal curve over $\bar{\eta}$. Denote by $\mathcal{D}^{\mathrm{geom}}_{g,n}$ the relative completion of the algebraic fundamental group $\pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta})$ with respect to the natural monodromy representation

$$\rho^{\mathrm{geom}} : \pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta}) \to \pi_1(\mathcal{H}_{g/\bar{k}}, \bar{\eta}) \to \operatorname{Sp}(H_{\mathbb{Q}_\ell}),$$

where $H_{\mathbb{Q}_\ell} = H^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Q}_\ell(1))$.

Denote the prounipotent radical of $\mathcal{D}^{\mathrm{geom}}_{g,n}$ by $\mathcal{V}^{\mathrm{geom}}_{g,n}$ and its Lie algebra by $\mathfrak{v}^{\mathrm{geom}}_{g,n}$. Denote the ℓ-adic unipotent completion (see Appendix B for the definition) of $\pi_1(C_{\bar{\eta}}, \bar{x})$ by $\mathcal{P}$ and its Lie algebra by $\mathfrak{p}$, where $\bar{x}$ is a geometric point of $C_{\bar{\eta}}$. When $n = 1$, we have the exact sequence of pronilpotent Lie algebras

$$0 \to \mathfrak{p} \to \mathfrak{v}_{g,1}^{\mathrm{geom}} \to \mathfrak{v}_{g}^{\mathrm{geom}} \to 0.$$

Variant A.5.1. The universal curve over $\mathcal{H}_{g,1^w/\bar{k}}$ induces a modular map

$$\Psi : \mathcal{H}_{g,1^w/\bar{k}} \to \mathcal{H}_{g,1/\bar{k}}$$

that makes the diagram

$$\begin{array}{ccc} \mathcal{H}_{g,1^w/\bar{k}} & & \\ {\scriptstyle\Psi}\downarrow & \searrow & \\ \mathcal{H}_{g,1/\bar{k}} & \longrightarrow & \mathcal{H}_{g/\bar{k}} \end{array}$$

commute. Let $\bar{\eta} : \operatorname{Spec} \Omega \to \mathcal{H}_{g,1^w/\bar{k}}$ be a geometric point of $\mathcal{H}_{g,1^w/\bar{k}}$. We regard $\bar{\eta}$ as geometric points of $\mathcal{H}_{g,1/\bar{k}}$ and $\mathcal{H}_{g/\bar{k}}$ as well. Denote by $\mathcal{D}_{g,1^w}^{\mathrm{geom}}$ the relative completion of $\pi_1(\mathcal{H}_{g,1^w/\bar{k}}, \bar{\eta})$ with respect to the natural monodromy representation

$$\rho_w : \pi_1(\mathcal{H}_{g,1^w/\bar{k}}, \bar{\eta}) \to \pi_1(\mathcal{H}_{g/\bar{k}}, \bar{\eta}) \to \mathrm{Sp}(H_{\mathbb{Q}_\ell}).$$

Denote the prounipotent radical of $\mathcal{D}_{g,1^w}^{\mathrm{geom}}$ by $\mathcal{V}_{g,1^w}^{\mathrm{geom}}$ and its Lie algebra by $\mathfrak{v}_{g,1^w}^{\mathrm{geom}}$. We have the following diagram

$$\begin{array}{ccccccccc} & & & & \mathfrak{v}_{g,1^w}^{\mathrm{geom}} & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{v}_{g,1}^{\mathrm{geom}} & \longrightarrow & \mathfrak{v}_{g}^{\mathrm{geom}} & \longrightarrow & 0. \end{array}$$

Remark A.5.2. The monodromy representation ρ_w agrees with the one induced from the profinite completion of the homomorphism $\Delta_{g,1^w} \to \mathrm{Sp}(H_1(\Sigma_g, \mathbb{Z}_\ell))$. Thus the map ρ_w has a Zariski-dense image in $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$.

Denote by $\mathcal{D}_{g,n}$ and $\widehat{\mathcal{D}}_{g,n}$, respectively, the weighted completions of $\pi_1(\mathcal{H}_{g,n/k}, \bar{\eta})$ and $\pi_1(\mathcal{C}_{\mathcal{H},g/k}^n, \bar{\eta})$ with respect to the monodromy representation

$$\rho : \pi_1(\mathcal{H}_{g,n/\bar{k}}, \bar{\eta}) \to \pi_1(\mathcal{C}_{\mathcal{H},g/k}^n, \bar{\eta}) \to \pi_1(\mathcal{H}_{g/\bar{k}}, \bar{\eta}) \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$$

and the central cocharacter $\omega : \mathbb{G}_m \to \mathrm{GSp}(H_{\mathbb{Q}_\ell})$ defined by $z \mapsto z^{-1}\,\mathrm{id}$. Denote the pronilpotent radicals of $\mathcal{D}_{g,n}$ and $\widehat{\mathcal{D}}_{g,n}$ by $\mathcal{V}_{g,n}$ and $\widehat{\mathcal{V}}_{g,n}$, respectively, and their Lie algebras by $\mathfrak{v}_{g,n}$ and $\widehat{\mathfrak{v}}_{g,n}$. When $n = 1$, the sequence of $\mathcal{D}_{g,1}$-modules

$$0 \to \mathfrak{p} \to \mathfrak{v}_{g,1} \to \mathfrak{v}_g \to 0$$

is exact and the diagram

$$\begin{array}{ccccccccc} & & & & \mathfrak{v}^{\mathrm{geom}}_{g,1^w} & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{v}^{\mathrm{geom}}_{g,1} & \longrightarrow & \mathfrak{v}^{\mathrm{geom}}_{g} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{v}_{g,1} & \longrightarrow & \mathfrak{v}_g & \longrightarrow & 0 \end{array}$$

commutes. As a $\mathcal{D}_{g,1}$-module, the Lie algebra $\mathfrak{v}_g$ admits a natural weight filtration $W_\bullet\mathfrak{v}_g$ that satisfies the properties: $W_{-1}\mathfrak{v}_g = \mathfrak{v}_g$ and the action of $\mathcal{D}_{g,1}$ on each graded quotient $\mathrm{Gr}^W_{-m}\,\mathfrak{v}_g := W_{-m}\mathfrak{v}_g/W_{-m-1}\mathfrak{v}_g$ factors through $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$.

Proposition A.5.3. *For $g \geqslant 2$, each graded quotient $\mathrm{Gr}^W_{-2m-1}\,\mathfrak{v}_g = 0$ for $m \geqslant 0$.*

Proof. Recall that there is an exact sequence

$$1 \to \pi_1(\mathcal{H}_{g/\bar{k}}, \bar{\eta}) \to \pi_1(\mathcal{H}_{g/k}, \bar{\eta}) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

Denote the groups $\pi_1(\mathcal{H}_{g/\bar{k}}, \bar{\eta})$, $\pi_1(\mathcal{H}_{g/k}, \bar{\eta})$, and $\mathrm{Gal}(\bar{k}/k)$ by Δ^{geom}, Δ, and G_k, respectively. Let V be a geometrically nontrivial, irreducible representation of $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$ (recall that an irreducible $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$ is geometrically irreducible if it is nontrivial when restricted to $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$). The above exact sequence gives rise to a spectral sequence

$$E_2^{s,t} = H^s(G_k, H^t(\Delta^{\mathrm{geom}}, V)) \Rightarrow H^{s+t}(\Delta, V).$$

Thus there is an exact sequence

$$0 \to H^1(G_k, H^0(\Delta^{\mathrm{geom}}, V)) \to H^1(\Delta, V)$$

$$\to H^0(G_k, H^1(\Delta^{\text{geom}}, V)) \to H^2(G_k, H^0(\Delta^{\text{geom}}, V)).$$

Since $\rho^{\text{geom}} : \Delta^{\text{geom}} \to \mathrm{Sp}(H_{\mathbb{Q}_\ell})$ has a Zariski-dense image, we have $H^0(\Delta^{\text{geom}}, V) = 0$, and hence we have an isomorphism $H^1(\Delta, V) \cong H^1(\Delta^{\text{geom}}, V)^{G_k}$. Denote the relative completion of Δ_g with respect to the natural map $\Delta_g \to \mathrm{Sp}_g(\mathbb{Q}) := \mathrm{Sp}(H_1(\Sigma_g, \mathbb{Q}))$ by $\mathcal{D}_{g/\mathbb{Q}}$. It is a proalgebraic $\mathbb{Q}$-group that is an extension of $\mathrm{Sp}_g(\mathbb{Q})$ by a prounipotent $\mathbb{Q}$-group $\mathcal{V}_{g/\mathbb{Q}}$. Note that we also view $\mathrm{Sp}_g(\mathbb{Q})$ as an algebraic $\mathbb{Q}$-group. Since the hyperelliptic involution σ acts trivially on Δ_g by conjugation, the induced action of σ on $\mathcal{D}_{g/\mathbb{Q}}$ is trivial, ans so is the action on $\mathcal{V}_{g/\mathbb{Q}}^{\text{geom}}$. Now, if V is a finite dimensional $\mathrm{Sp}_g(\mathbb{Q})$-representation, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Sp}_g(\mathbb{Q})}(H_1(\mathcal{V}_{g/\mathbb{Q}}^{\text{geom}}), V) \cong H^1(\Delta_g, V).$$

Therefore, if σ acts on V as $-\mathrm{id}$, then we see that $H^1(\Delta_g, V) = 0$. By Base Change Theorem [21, Thm. 3.4], there is an isomorphism $\mathcal{V}_g^{\text{geom}} \cong \mathcal{V}_{g/\mathbb{Q}}^{\text{geom}} \otimes \mathbb{Q}_\ell$. This implies that there is an isomorphism

$$H^1(\Delta^{\text{geom}}, V \otimes \mathbb{Q}_\ell) \cong H^1(\Delta_g, V) \otimes \mathbb{Q}_\ell.$$

Suppose now that V is a finite-dimensional $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-representation of weight $-2m-1$. Then when restricted to $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$, it is a quotient of $H_{\mathbb{Q}_\ell}^{\otimes l}$ with l odd, and thus the hyperelliptic involution acts on V as $-\mathrm{id}$. Hence we have $H^1(\Delta_g, V) = 0$, and so $H^1(\Delta^{\text{geom}}, V) = 0$. Therefore, $H^1(\Delta, V) = 0$. Since there is an isomorphism

$$\mathrm{Hom}_{\mathrm{GSp}(H_{\mathbb{Q}_\ell})}(\mathrm{Gr}^W_{-2m-1} H_1(\mathfrak{v}_g), V) \cong H^1(\Delta, V),$$

we have $\mathrm{Gr}^W_{-2m-1} H_1(\mathfrak{v}_g) = 0$. Since a pronilpotent Lie algebra $\mathfrak{n}$ is generated by $H_1(\mathfrak{n})$, it then follows that $\mathrm{Gr}_{-2m-1} \mathfrak{v}_g = 0$ for $m \geqslant 0$. □

The following result is an immediate consequence of Tanaka's computation [46].

Lemma A.5.4. *If $g \geqslant 2$, then $\mathrm{Gr}^W_{-2} \mathfrak{v}_g$ does not contain a copy of $H_{[1^2]}$.*

Proof. We use the same notation as in the proof of Proposition A.5.3. Note that there is an isomorphism

$$\mathrm{Gr}^W_{-2}\,\mathfrak{v}_g = \mathrm{Gr}^w_{-2}\,H_1(\mathfrak{v}_g),$$

since $\mathfrak{v}_g$ is generated by H_1 and $\mathrm{Gr}^W_{-1}\,H_1(\mathfrak{v}_g) = 0$. Tanaka's computation [46] implies that $H^1(\Delta^{\mathrm{geom}}, H_{[1^2]})$ vanishes. Therefore, together with the isomorphism

$$\mathrm{Hom}_{\mathrm{GSp}(H_{\mathbb{Q}_\ell})}(\mathrm{Gr}^W_{-2}\,H_1(\mathfrak{v}_g), H_{[1^2]}) \cong H^1(\Delta, H_{[1^2]}),$$

the result follows. □

Lemma A.5.5. *For $g \geqslant 2$, $\mathrm{Gr}^W_{-2}\,\mathfrak{v}_g$ contains at least one copy of $H_{[2^2]}$.*

Proof. This follows from the fact that the hyperelliptic Johnson homomorphism τ^{hyp} factors through the composition of $\mathrm{Sp}(H_{\mathbb{Q}_\ell})$-equivariant homomorphisms

$$H_1(\mathfrak{v}_g^{\mathrm{geom}}) \to \mathrm{Gr}^W_{-2}\,H_1(\mathfrak{v}_g) \to \mathrm{Der}_{-2}\,\mathfrak{p},$$

and the fact that the $H_{[2^2]}$-component of the image of τ^{hyp} is nontrivial. □

Remark A.5.6. The Lie algebra $\mathfrak{p}$ admits a natural weight filtration, and so does $\mathrm{Der}\,\mathfrak{p}$. Since the weight filtration of $\mathfrak{p}$ coincides with its lower central series, we have $\mathrm{Gr}^W_{-m}\,\mathfrak{p} = \mathrm{Gr}^m_L\,\mathfrak{p}$ for $m \geqslant 1$. Thus, we have $\mathrm{Der}_{-m}\,\mathfrak{p} = \mathrm{Der}^m\,\mathfrak{p}$.

A.6 The Lie algebras $\mathfrak{b}_{\mathfrak{g},\mathfrak{n}}$

The projection $\mathcal{C}^n_{\mathcal{H},g/k} \to \mathcal{H}_{g/k}$ induces the exact sequence

$$1 \to \prod_{j=1}^{n} \pi_1(C_{\bar\eta}, \bar x_j) \to \pi_1(\mathcal{C}^n_{\mathcal{H},g/k}, \bar\eta) \to \pi_1(\mathcal{H}_{g/k}, \bar\eta) \to 1,$$

where $\bar x_j$ is the image of $\bar\eta$ in $\mathcal{H}_{g,1/k}$ under the j-th projection map $\mathcal{C}^n_{\mathcal{H},g/k} \to \mathcal{H}_{g,1/k}$. Denote the ℓ-adic unipotent completion of $\pi_1(C_{\bar\eta}, \bar x_j)$ by $\mathfrak{p}_j$. The $\mathfrak{p}_j$ are all naturally isomorphic and hence we denote each $\mathfrak{p}_j$ by $\mathfrak{p}$ for simplicity. Taking the weighted completions of $\pi_1(\mathcal{C}^n_{\mathcal{H},g/k}, \bar{\eta})$ and $\pi_1(\mathcal{H}_{g/k}, \bar{\eta})$ induces the exact sequence of $\widehat{\mathcal{V}}_{g,n}$-modules

$$0 \to \mathfrak{p}^n \to \widehat{\mathfrak{v}}_{g,n} \to \widehat{\mathfrak{v}}_g \to 0.$$

Therefore, we have

Proposition A.6.1. *For $g \geqslant 2$ and $n \geqslant 0$, we have*

$$\begin{aligned}
\mathrm{Gr}^W_{-1}\, \widehat{\mathfrak{v}}_{g,n} &= (\mathrm{Gr}^W_{-1}\, \mathfrak{p})^n;\\
\mathrm{Gr}^W_{-2}\, \widehat{\mathfrak{v}}_{g,n} &= (\mathrm{Gr}^W_{-2}\, \mathfrak{p})^n \oplus \mathrm{Gr}^W_{-2}\, \mathfrak{v}_g;\\
\mathrm{Gr}^W_{-3}\, \widehat{\mathfrak{v}}_{g,n} &= (\mathrm{Gr}^W_{-3}\, \mathfrak{p})^n;\\
\mathrm{Gr}^W_{-4}\, \widehat{\mathfrak{v}}_{g,n} &= (\mathrm{Gr}^W_{-4}\, \mathfrak{p})^n \oplus \mathrm{Gr}^W_{-4}\, \mathfrak{v}_g.
\end{aligned}$$

Assume that $g \geqslant 2$ and $n \geqslant 0$. We define the Lie algebra $\mathfrak{b}_{\mathfrak{g},\mathfrak{n}}$ to be

$$\mathfrak{b}_{g,n} = \widehat{\mathfrak{v}}_{g,n}/W_{-4}\widehat{\mathfrak{v}}_{g,n}$$

Proposition A.6.2. *Suppose that $g \geqslant 2$. Each section of the universal hyperelliptic curve $\mathcal{C}_{\mathcal{H},g,n/k} \to \mathcal{H}_{g,n/k}$ induces a well-defined $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant Lie algebra section of $\beta_n : \mathfrak{b}_{g,n+1} \to \mathfrak{b}_{g,n}$.*

Proof. The fiber product diagram

$$\begin{array}{ccc}
\mathcal{C}_{\mathcal{H},g,n/k} & \longrightarrow & \mathcal{C}^{n+1}_{\mathcal{H},g/k}\\
\downarrow & & \downarrow\\
\mathcal{H}_{g,n/k} & \longrightarrow & \mathcal{C}^n_{\mathcal{H},g/k}
\end{array}$$

induces the following commutative diagram of $\mathcal{D}_{\mathcal{H},g,n}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \widehat{\mathfrak{v}}_{g,n+1} & \longrightarrow & \widehat{\mathfrak{v}}_{g,n} & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{v}_{\mathcal{H},g,n} & \longrightarrow & \mathfrak{v}_{g,n} & \longrightarrow & 0,
\end{array}$$

where the rows are exact and the middle and right vertical maps are surjective. Each section s of $\mathcal{C}_{\mathcal{H},g,n/k} \to \mathcal{H}_{g,n/k}$ induces a $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant graded Lie algebra section $\mathrm{Gr}^W_\bullet ds_*$ of $\mathrm{Gr}^W_\bullet \mathfrak{v}_{\mathcal{H},g,n} \to \mathrm{Gr}^W_\bullet \mathfrak{v}_{g,n}$. Hence it induces a section $\mathrm{Gr}^W_\bullet ds_*/W_{-5}$ of

$$\mathrm{Gr}^W_\bullet(\mathfrak{v}_{\mathcal{H},g,n}/W_{-5}) \to \mathrm{Gr}^W_\bullet(\mathfrak{v}_{g,n}/W_{-5}).$$

Denote the kernel of $\mathfrak{v}_{g,n} \to \widehat{\mathfrak{v}}_{g,n}$ by $\mathfrak{k}$. The restriction of $\mathrm{Gr}^W_\bullet ds_*/W_{-5}$ to $\mathfrak{k}/W_{-5}$ maps into $\mathfrak{p}/W_{-5}$, and this is trivial. Thus the section $\mathrm{Gr}^W_\bullet ds_*/W_{-5}$ descends to a graded section of $\mathrm{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n+1}/W_{-5}) \to \mathrm{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n}/W_{-5})$, which induces a section of $\beta_n : \mathfrak{b}_{g,n+1} \to \mathfrak{b}_{g,n}$. □

Proposition A.6.3. *If $g \geqslant 3$ and $n \geqslant 1$, then there are nonzero elements $\delta_n, \tilde{\delta}_n \in \mathrm{Gr}^W_{-2} \mathfrak{v}_g \subset \mathrm{Gr}^W_{-2} \widehat{\mathfrak{v}}_{g,n}$ such that the bracket $[\delta_n, \tilde{\delta}_n]$ is nonzero and maps diagonally into $(\mathrm{Gr}^W_{-4} \mathfrak{p})^n \subset \mathrm{Gr}^W_{-4} \widehat{\mathfrak{v}}_{g,n}$.*

Proof. Recall notation from A.4.3. Note that the hyperelliptic Torelli group $T\Delta_g$ maps to $\mathcal{V}^{\mathrm{geom}}_{g,1^w}$. Composing with the logarithm map, $T\Delta_g$ maps to the pronilpotent Lie algebra $\mathfrak{v}^{\mathrm{geom}}_{g,1^w}$. The adjoint action of $\mathfrak{v}^{\mathrm{geom}}_{g,1^w}$ on $\mathfrak{p}$ factors through the adjoint action of $\mathfrak{v}_{g,1}$ on $\mathfrak{p}$. Denote the image of the commuting Dehn twists ω_1 and ω_{g-1} in $\mathfrak{v}_{g,1}$ by ω' and $\tilde{\omega}'$, respectively. The fact that the homomorphism $T\Delta_g \to \mathrm{Hom}(H, \mathrm{Gr}^2_L \mathfrak{p})$ with $H = H_1(\Sigma_g, \mathbb{Q})$ is trivial implies that $\mathrm{Gr}^W_{-1} \omega' = \mathrm{Gr}^W_{-1} \tilde{\omega}' = 0$. Since $\mathrm{Gr}^W_{-2} \mathfrak{v}_{g,1} = \mathrm{Gr}^W_{-2} \mathfrak{p} \oplus \mathrm{Gr}^W_{-2} \mathfrak{v}_g$, we can express

$$\mathrm{Gr}^W_{-2} \omega' = \delta_{\mathfrak{p}} + \delta_{\mathfrak{v}}, \text{ and } \mathrm{Gr}^W_{-2} \tilde{\omega}' = \tilde{\delta}_{\mathfrak{p}} + \tilde{\delta}_{\mathfrak{v}}.$$

Since $\mathrm{Gr}^W_{-2} \mathfrak{v}_g$ does not contain a copy of $H_{[1^2]}$ by Lemma A.5.4, we can see that the elements $\delta_{\mathfrak{v}}$ and $\tilde{\delta}_{\mathfrak{v}}$ map to $\xi_{[2^2]}$ and $\tilde{\xi}_{[2^2]}$, respectively (recall that $\xi_{[2^2]}$ and $\tilde{\xi}_{[2^2]}$ are defined in A.4.3). Since the Dehn twists ω and ω_{g-1} commutes, the bracket $[\omega', \tilde{\omega}']$ in the Lie algebra $\mathfrak{v}_{g,1}$ is trivial. Since $\mathrm{Gr}^W_{-1} \omega' = \mathrm{Gr}^W_{-1} \tilde{\omega}' = 0$, we have

$$\mathrm{Gr}^W_{-4}[\omega', \tilde{\omega}'] = [\delta_{\mathfrak{p}} + \delta_{\mathfrak{v}}, \tilde{\delta}_{\mathfrak{p}} + \tilde{\delta}_{\mathfrak{v}}] = 0.$$

Therefore, the bracket $[\delta_{\mathfrak{v}}, \tilde{\delta}_{\mathfrak{v}}]$ lies in $\mathrm{Gr}^W_{-4}\,\mathfrak{p}$. By Theorem A.4.17, the bracket $[\xi_{[2^2]}, \tilde{\xi}_{[2^2]}]$ is nontrivial, and so is the bracket $[\delta_{\mathfrak{v}}, \tilde{\delta}_{\mathfrak{v}}]$ in $\mathrm{Gr}^W_{-4}\,\mathfrak{p}$. This completes the case for $n = 1$. For $n > 1$, the diagonal section $\mathcal{C}^1_{\mathcal{H},g} \to \mathcal{C}^n_{\mathcal{H},g/k}$ induces a graded section $\mathrm{Gr}^W_\bullet\, d\Delta$ of the projection $\mathrm{Gr}^W_\bullet\, \widehat{\mathfrak{v}}_{g,n} \to \mathrm{Gr}^W_\bullet\, \widehat{\mathfrak{v}}_{g,1}$. Note that the restriction of $\mathrm{Gr}^W_{-2}\, d\Delta$ to $\mathrm{Gr}^W_{-2}\,\mathfrak{v}_g$ is the identity. Let $\delta_n = (\mathrm{Gr}^W_{-2}\, d\Delta)(\delta_{\mathfrak{v}})$ and $\tilde{\delta}_n = (\mathrm{Gr}^W_{-2}\, d\Delta)(\tilde{\delta}_{\mathfrak{v}})$. Then the bracket $[\delta_n, \tilde{\delta}_n]$ maps to $(\mathrm{Gr}^W_{-4}\,\mathfrak{p})^n$ diagonally, since $\mathrm{Gr}^W_\bullet\, d\Delta$ is a graded Lie algebra section. The bracket $[\delta_n, \tilde{\delta}_n]$ maps to $[\delta_{\mathfrak{v}}, \tilde{\delta}_{\mathfrak{v}}]$, which is nonzero, and so is $[\delta_n, \tilde{\delta}_n]$. □

A.7 The geometric sections of $\beta_n : \mathfrak{b}_{g,n+1} \to \mathfrak{b}_{g,n}$

In this section, we will compute $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant Lie algebra sections of β_n that are induced by the sections of the universal hyperelliptic curve $\mathcal{C}_{\mathcal{H},g,n/k} \to \mathcal{H}_{g,n/k}$. We call them *geometric* sections of β_n. We first study all possible sections of β_n. Remember that $\mathfrak{p}^{n+1} = \oplus_{j=0}^n \mathfrak{p}_j \subset \widehat{\mathfrak{v}}_{g,n+1}$, where the j-th component $\mathfrak{p}_j$ corresponds to the projection of $\mathcal{C}^{n+1}_{\mathcal{H},g/k}$ onto its j-th component.

Lemma A.7.1. *Suppose $g \geqslant 2$. Each $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-equivariant Lie algebra section ζ of β_n is determined by its effect on Gr^W_{-1}:*

$$\mathrm{Gr}^W_{-1}\,\zeta : \mathrm{Gr}^W_{-1}\,\mathfrak{p}_1 \oplus \cdots \oplus \mathrm{Gr}^W_{-1}\,\mathfrak{p}_n \to \mathrm{Gr}^W_{-1}\,\mathfrak{p}_0 \oplus \mathrm{Gr}^W_{-1}\,\mathfrak{p}_1 \cdots \oplus \mathrm{Gr}^W_{-1}\,\mathfrak{p}_n.$$

Proof. Note that we have $\mathrm{Gr}^W_{-1}\,\mathfrak{p}_j = H_{\mathbb{Q}_\ell}$ for each j. By Schur's lemma, we have

$$(\mathrm{Gr}^W_{-1}\,\zeta)(u_1, \ldots, u_n) = (\sum_{j=1}^n a_j u_j, u_1, \ldots, u_n),$$

where the a_j are some constants in $\mathbb{Q}_\ell$. Since ζ is a Lie algebra section, we see that $\mathrm{Gr}^W_{-2}\,\zeta$ is given by

$$\mathrm{Gr}^W_{-2}\,\zeta : (v_1, \ldots, v_n, r) \mapsto (\sum_{j=1}^n a_j^2 v_j, v_1, \ldots, v_n, r) \in \mathrm{Gr}^W_{-2}\,\mathfrak{p}_0 \oplus \bigoplus_{j=1}^n \mathrm{Gr}^W_{-2}\,\mathfrak{p}_j \oplus \mathrm{Gr}^W_{-2}\,\mathfrak{v}_g,$$

where each v_j is in $\operatorname{Gr}^w_{-2}\mathfrak{p}_j$ and r in $\operatorname{Gr}^w_{-2}\mathfrak{v}_g$, and that $\operatorname{Gr}^W_{-3}\zeta$ is given by

$$\operatorname{Gr}^W_{-3}\zeta : (w_1, \ldots, w_n) \mapsto (\sum_{j=1}^n a_j^3 w_j, w_1, \ldots, w_n) \in \operatorname{Gr}^W_{-3}\mathfrak{p}_0 \oplus \bigoplus_{j=1}^n \operatorname{Gr}^W_{-3}\mathfrak{p}_j,$$

where each w_j is in $\operatorname{Gr}^w_{-3}\mathfrak{p}_j$. □

Lemma A.7.2. *With notation as above. Let ζ be a section of β_n. Then we have $a_j = 0, 1,$ or -1 for each $j = 1, \ldots, n$.*

Proof. Recall that we have $\operatorname{Der}^W_{-1}\mathfrak{p} = H_{[1^3]} + H_{[1]}$ and $\operatorname{Der}^W_{-2}\mathfrak{p} = H_{[2^2]} + H_{[1^2]}$. The bracket $[\ ,\ ] : H_{[1]} \otimes H_{[2^2]} \to H_{[2+1]} \subset \operatorname{Der}^W_{-3}\mathfrak{p}$ is not trivial and so surjective. This implies that the bracket

$$[\ ,\ ] : \operatorname{Gr}^W_{-1}\mathfrak{p}_j \otimes \operatorname{Gr}^W_{-2}\mathfrak{v}_g \to \operatorname{Gr}^W_{-3}\mathfrak{p}_j$$

is also surjective. Therefore, we also have

$$\operatorname{Gr}^W_{-3}\zeta : (w_1, \ldots, w_n) \mapsto (\sum_{j=1}^n a_j w_j, w_1, \ldots, w_n) \in \operatorname{Gr}^W_{-3}\mathfrak{p}_0 \oplus \bigoplus_{j=1}^n \operatorname{Gr}^W_{-3}\mathfrak{p}_j.$$

Thus we obtain the relations $a_j^3 = a_j$ or $a_j(a_j^2 - 1) = 0$ for each $j = 1, \ldots, n$. □

By the above lemmas, we conclude that there are 3^n many sections of β_n. However, most of them do not come from the sections of the universal hyperelliptic curve. For each $j = 1, \ldots, n$, denote the section of β_n that corresponds to $a_j = 1$ and $a_i = 0$ for $i \neq j$ by ζ_j.

Theorem A.7.3. *If $g \geqslant 3$, then there are exactly $2n$ geometric sections of β_n given by $\zeta_1, \ldots, \zeta_n$ and $-\zeta_1, \ldots, -\zeta_n$.*

Proof. Each geometric section of β_n comes from a section of the Lie algebra surjection $\operatorname{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n+1}/W_{-5}) \to \operatorname{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n}/W_{-5})$. By Proposition A.6.3, there are elements

$\delta_n, \tilde{\delta}_n \in \mathrm{Gr}^W_{-2}\,\mathfrak{v}_g \subset \mathrm{Gr}^W_{-2}\,\widehat{\mathfrak{v}}_{g,n}$ such that the bracket $[\delta_n, \tilde{\delta}_n]$ is not zero and maps diagonally to $\bigoplus_{j=1}^n \mathrm{Gr}^W_{-4}\,\mathfrak{p}_j$. Denote the irreducible $\mathrm{GSp}(H_{\mathbb{Q}_\ell})$-submodule containing $[\delta_n, \tilde{\delta}_n]$ of $\mathrm{Gr}^W_{-4}\,\mathfrak{p}$ by V. Let $\hat{\zeta}$ be a section of $\mathrm{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n+1}/W_{-5}) \to \mathrm{Gr}^W_\bullet(\widehat{\mathfrak{v}}_{g,n}/W_{-5})$. Then the restriction of $\mathrm{Gr}^W_{-4}\,\hat{\zeta}$ to $\bigoplus_{j=1}^n \mathrm{Gr}^W_{-4}\,\mathfrak{p}_j \subset \mathrm{Gr}^W_{-4}\,\widehat{\mathfrak{v}}_{g,n}$ is given by

$$(\mathrm{Gr}^W_{-4}\,\hat{\zeta})(t_1, \ldots, t_n) = (\sum_{j=1}^n a_j^4 t_j, t_1, \ldots, t_n).$$

Since $\hat{\zeta}(\delta_n) = \delta_{n+1}$ and $\hat{\zeta}(\tilde{\delta}_n) = \tilde{\delta}_{n+1}$, we have

$$(\mathrm{Gr}^W_{-4}\,\hat{\zeta})(x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_n),$$

where $x_j = [\delta_n, \tilde{\delta}_n]$ for each $j = 0, \ldots, n$. It then follows that we have the relation $\sum_{j=1}^n a_j^4 = 1$. This implies that $a_j^2 = 1$ for some j and $a_i = 0$ for $i \neq j$. Therefore, $\hat{\zeta}$ induces either ζ_j or $-\zeta_j$. Now, it is clear that the j-th tautological section and its hyperelliptic conjugate of the universal curve $\mathcal{C}_{\mathcal{H},g,n/k} \to \mathcal{H}_{g,n/k}$ induce the sections ζ_j and $-\zeta_j$, respectively. $\square$

Appendix B

Unipotent Completion

We will review Quillen's Malcev completion [41] and its explicit construction. Our main application of the completion is to complete the topological fundamental group of a compact orientable surface.

B.1 Construction of Malcev completion

Definition B.1.1. Let F be a field of characteristic zero. Suppose Γ is a group. The Malcev completion over F of Γ is a prounipotent F-group Γ^{un} with a Zariski-dense homomorphism $\rho : \Gamma \to \Gamma^{\mathrm{un}}(F)$ that satisfies the following universal property: if there is a Zariski-dense homomorphism $\hat{\rho} : \Gamma \to G(F)$ to the F-rational points of a prounipotent F-group G, then there exists a unique algebraic homomorphism $\phi : \Gamma^{\mathrm{un}} \to G$ such that $\hat{\rho} = \phi(F) \circ \rho$.

Let $R\Gamma$ be the group algebra $R[\Gamma]$ over a commutative ring R. The group algebra $R\Gamma$ is equipped with the augmentation homomorphism $\epsilon : R\Gamma \to R$ taking $\gamma \in \Gamma$ to 1. The kernel of ϵ is denoted by J_R, which we call the augmentation ideal. In this chapter, we simply write J instead. Together with the augmentation, the comultiplication $\Delta : R\Gamma \to R\Gamma \otimes R\Gamma$ defined by $\Delta(\gamma) = \gamma \otimes \gamma$ and the antipode $S : R\Gamma \to R\Gamma$ defined by $S(\gamma) = \gamma^{-1}$ make $R\Gamma$ into a cocommutative Hopf algebra. Completing with respect to the powers of J,which is called the J-adic completion, we obtain a complete Hopf algebra

$$R\Gamma\hat{} = \varprojlim_{n} R\Gamma/J^n$$

equipped with the completed comultiplication

$$\Delta : R\Gamma\hat{} \to R\Gamma \widehat{\otimes} R\Gamma$$

where $\widehat{\otimes}$ denotes the complete tensor product:

$$R\Gamma \widehat{\otimes} R\Gamma := \varprojlim_{n,m} R\Gamma/J^n \otimes R\Gamma/J^m$$

Proposition B.1.2. *Suppose that Γ is a group and that R is a ring. Then the map $\phi : \Gamma \to J_R$ taking γ to $\gamma - 1$ induces an R-module isomorphism*

$$H_1(\Gamma, R) \cong J_R/J_R^2.$$

Proof. It is enough to show for $R = \mathbb{Z}$. First note that we have

$$\gamma_1\gamma_2 - (\gamma_1 - 1) - (\gamma_2 - 1) - 1 = (\gamma_1 - 1)(\gamma_2 - 1) \equiv 0 \bmod J_{\mathbb{Z}}^2,$$

and so

$$\gamma_1\gamma_2 - 1 \equiv (\gamma_1 - 1) + (\gamma_2 - 1) \bmod J_{\mathbb{Z}}^2.$$

This shows that the map $\gamma \mapsto \gamma - 1$ induces a homomorphism $\tilde{\phi} : H_1(\Gamma, \mathbb{Z}) = \Gamma^{\text{ab}} \to J_{\mathbb{Z}}/J_{\mathbb{Z}}^2$. For $a = \sum n_i\gamma_i \in J_{\mathbb{Z}}$, define a map $\psi : J_{\mathbb{Z}} \to \Gamma^{\text{ab}}$ by setting $\psi(a) = \sum_i n_i[\gamma_i]$, where $[,]$ denotes classes in Γ^{ab}. It is easy to see that ψ induces a map $\tilde{\psi} : J_{\mathbb{Z}}/J_{\mathbb{Z}}^2 \to \Gamma^{\text{an}}$ and that $\tilde{\phi}$ and $\tilde{\psi}$ are inverse to each other. □

A vector space is said to be linearly compact if V is the inverse limit of a projective system consisting of its finite-dimensional quotients: $V \cong \varprojlim_\alpha V/W_\alpha$

Proposition B.1.3. *Suppose that R is a field. If $H_1(\Gamma, R)$ is finite-dimensional, then $R\Gamma^\wedge$ is a linearly compact R-vector space.*

Proof. Since $R\Gamma^\wedge = \varprojlim_n R\Gamma/J^n$, it will suffice to show that $R\Gamma/J^n$ is finite-dimensional for all n. This is clearly true for $n = 1$. Inductively, this is true if each J^n/J^{n+1} is finite-dimensional, but the multiplication induces a surjection

$$\left(J/J^2\right)^{\otimes n} \to J^n/J^{n+1},$$

and so we are done since $H_1(\Gamma, R) \cong J/J^2$. □

For the rest of this chapter, assume that F is a field of characterisc zero. Denote the completion of J by $\widehat{J}$.

The set of *primitive* elements of $F\Gamma^\wedge$ denoted by $\mathfrak{p}$ is defined by

$$\mathfrak{p} := \{X \in F\Gamma^\wedge | \Delta(X) = X \otimes 1 + 1 \otimes X\}.$$

Then $\mathfrak{p}$ admits a Lie algebra structure with the bracket defined by $[X, Y] = XY - YX$(it can be easily checked that $\Delta([X, Y]) = [X, Y] \otimes 1 + 1 \otimes [X, Y]$).

The set of *group-like* elements of $F\Gamma^\wedge$ denoted by $\mathcal{P}$ is defined by

$$\mathcal{P} := \{x \in R\Gamma^\wedge | \Delta(X) = X \otimes X \text{ and } \epsilon(X) = 1\}.$$

Proposition B.1.4. *The set $\mathcal{P}$ is a subgroup of the group of units of $F\Gamma^\wedge$:*

$$\mathcal{P} = \{x \in (F\Gamma^\wedge)^\times | \Delta(X) = X \otimes X\}.$$

More precisely, if $X \in \mathcal{P}$, then $S(X) \in \mathcal{P}$ and $S(X)X = 1 = XS(X)$.

Proof. Denote the multiplication of $F\Gamma^\wedge$ by ∇ and the unit by u. Suppose that $X \neq 0$ and $\Delta(X) = X \otimes X$. Then by a property of Hopf algebra, we have $X = (\nabla \circ (\epsilon \otimes \mathrm{id}) \circ \Delta)(X) = \epsilon(X)X$, and so $\epsilon(X) = 1$. Also, we have $\nabla \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon = \nabla \circ (\mathrm{id} \otimes S) \circ \Delta$, and so $S(X)X = 1 = XS(X)$. Thus X is a unit. Finally, we have $(S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S$, and so $\Delta(S(X)) = S(X) \otimes S(X)$. □

The J-adic topology of $F\Gamma^\wedge$ induces topologies on $\mathfrak{p}$ and $\mathcal{P}$ as follows. The filtration of $F\Gamma^\wedge$ by the powers of $\widehat{J}$ induces filtrations on $\mathfrak{p}$ and $\mathcal{P}$ by setting

$$\mathfrak{p}^n := \mathfrak{p} \cap \widehat{J}^n \text{ and } \mathcal{P}^n := \mathcal{P} \cap (1 + \widehat{J}^n).$$

Note that $\mathfrak{p} \subset \widehat{J}$ because $X = (\nabla \circ (\epsilon \otimes \mathrm{id}) \circ \Delta)(X) = \epsilon(X) + X$ and hence $\epsilon(X) = 0$, and that $\mathcal{P} \subset 1 + \widehat{J}$. Therefore, these filtrations satisfy

$$\mathfrak{p} = \mathfrak{p}^1 \supseteq \mathfrak{p}^2 \supseteq \mathfrak{p}^3 \cdots$$

and

$$\mathcal{P} = \mathcal{P}^1 \supseteq \mathcal{P}^2 \supseteq \mathcal{P}^3 \cdots ,$$

and give topologies on $\mathfrak{p}$ and $\mathcal{P}$ with respect to which $\mathfrak{p}$ and $\mathcal{P}$ are complete:

$$\mathfrak{p} \cong \varprojlim_n \mathfrak{p}/\mathfrak{p}^{n+1} \text{ and } \mathcal{P} \cong \varprojlim_n \mathcal{P}/\mathcal{P}^{n+1}$$

For each n, denote $\mathfrak{p}/\mathfrak{p}^{n+1}$ and $\mathcal{P}/\mathcal{P}^{n+1}$ by $\mathfrak{p}_n$ and $\mathcal{P}_n$, respectively. The logarithm and exponential mappings are homeomorphisms (with respect to the J-adic topology) of $\widehat{J}$ and $1 + \widehat{J}$ mutually inverse to each other:

$$\log : 1 + \widehat{J} \to \widehat{J} \text{ and } \exp : \widehat{J} \to 1 + \widehat{J}$$

By restricting, these maps induce filtration-preserving homeomorphisms

$$\log : \mathcal{P} \to \mathfrak{p} \text{ and } \exp : \mathfrak{p} \to \mathcal{P}.$$

Denote the lower central series of a Lie algebra $\mathfrak{g}$ by $L^\bullet\mathfrak{g}$ with $L^1\mathfrak{g} = \mathfrak{g}$. Note that $L^m\mathfrak{p} \subset \mathfrak{p}^m$ for each m. Hence, if $H_1(\Gamma, F)$ is finite-dimensional, then each $\mathfrak{p}_n$ is a finite-dimensional nilpotent Lie algebra over F and the logarithm and exponential mappings are polynomial bijections between $\mathfrak{p}_n$ and $\mathcal{P}_n$. Thus, each $\mathcal{P}_n$ is a unipotent F-group, and so $\mathcal{P}$ is a prounipotent F-group with its pronilpotent Lie algebra $\mathfrak{p}$.

For simplicity, here we are not distinguishing between an algebraic group over F and the F-rational points of the algebraic group.

Now consider the induced filtration $\mathfrak{p}_n^\bullet$ defined by $\mathfrak{p}_n^m := \mathfrak{p}^m/\mathfrak{p}^{n+1}$ for each m.

Proposition B.1.5. *The filtration $\mathfrak{p}_n^\bullet$ agrees with the lower central series $L^\bullet\mathfrak{p}_n$ for each $n \geqslant 1$.*

Proof. First, we prove that each graded quotient $\mathrm{Gr}_m^J\,\mathfrak{p}_n := \mathfrak{p}_n^m/\mathfrak{p}_n^{m+1}$ is isomorphic to J^m/J^{m+1}. Note that we have an injection $\mathrm{Gr}_1^J\,\mathfrak{p}_n = \mathfrak{p}^1/\mathfrak{p}^2 \hookrightarrow J/J^2$. Since the isomorphism $H_1(\Gamma, F) \to J/J^2$ factors through $\mathfrak{p}^1/\mathfrak{p}^2$, we see that $\mathrm{Gr}_1^J\,\mathfrak{p}_n \cong J/J^2$. Consider the commutative diagram:

$$\begin{array}{ccc} (\mathrm{Gr}_1^J\,\mathfrak{p}_n)^{\otimes m} & \xrightarrow{\theta} & (J/J^2)^{\otimes m} \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ \mathrm{Gr}_m^J\,\mathfrak{p}_n & \xrightarrow{j} & J^m/J^{m+1} \end{array}$$

Since the composition $\psi \circ \theta$ is surjective, it follows that the map j is surjective, but j is the natural inclusion, so it is an isomorphism. This shows that each graded quotient $\mathrm{Gr}_m^J\,\mathfrak{p}_n$ is generated by $\mathrm{Gr}_1^J\,\mathfrak{p}_n$. Now, note that $L^m\mathfrak{p}_n \subseteq \mathfrak{p}_n^m$ for each m, and hence that, in particular, there is a canonical projection $p : \mathrm{Gr}_L^1\,\mathfrak{p}_n \to \mathrm{Gr}_1^J\,\mathfrak{p}_n$. Now, consider the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{Gr}_L^1\,\mathfrak{p}_n)^{\otimes m} & \xrightarrow{\otimes p} & (\mathrm{Gr}_1^J\,\mathfrak{p}_n)^{\otimes m} \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ \mathrm{Gr}_L^m\,\mathfrak{p}_n & \longrightarrow & \mathrm{Gr}_m^J\,\mathfrak{p}_n \end{array}$$

where $\mathrm{Gr}_L^m\,\mathfrak{p}_n := L^m\mathfrak{p}_n/L^{m+1}\mathfrak{p}_n$. Since the composition $\phi \circ \otimes p$ is surjective, we see that the bottom horizontal map induced by the inclusion $L^m\mathfrak{p}_n \subseteq \mathfrak{p}_n^m$ is surjective. This implies that the quotient $\mathfrak{p}_n^m/(L^m\mathfrak{p}_n + \mathfrak{p}_n^{m+1}) = 0$, that is, $\mathfrak{p}_n^m = L^m\mathfrak{p}_n + \mathfrak{p}_n^{m+1}$. Thus we have $\mathfrak{p}_n^n = L^n\mathfrak{p}_n$ since $\mathfrak{p}_n^{n+1} = 0$, and inductively we have $\mathfrak{p}_n^m = L^m\mathfrak{p}_n$ for each m and n. $\square$

The image of the canonical inclusion $\Gamma \to F\Gamma^{\wedge}$ is contained in $\mathcal{P}$. Thus we have a canonical homomorphism $\widehat{\rho} : \Gamma \to \mathcal{P}$. Composing with the natural projection $\mathcal{P} \to \mathcal{P}_n$, we obtain a homomorphism $\rho_n : \Gamma \to \mathcal{P}_n$.

Proposition B.1.6. *If $H_1(\Gamma, F)$ is finite-dimensional, then the image of ρ_n is Zariski-dense for each n.*

Proof. Denote the Zariski closure of the image of ρ_n in $\mathcal{P}_n$ by $\mathcal{Z}_n$. Then the homomorphisms $\Gamma \to \mathcal{Z}_n \to \mathcal{P}_n$ induce the homomorphisms

$$H_1(\Gamma, F) \to H_1(\mathcal{Z}_n) \to H_1(\mathcal{P}_n),$$

whose composition is an isomorphism. To see that this composition is an isomorphism, note that the composition

$$H_1(\mathcal{P}_n) \cong H_1(\mathfrak{p}_n) = \operatorname{Gr}^1_L \mathfrak{p}_n = \operatorname{Gr}^J_1 \mathfrak{p}_n = \mathfrak{p}/\mathfrak{p}^2 = J/J^2$$

is induced by the logarithm map denoted by $\tilde{\log}$, and that the diagram

$$\begin{array}{ccc} H_1(\Gamma, F) & \longrightarrow & H_1(\mathcal{P}_n) \\ & {\scriptstyle \gamma \mapsto \gamma - 1} \searrow & \downarrow {\scriptstyle \tilde{\log}} \\ & & J/J^2 \end{array}$$

commutes. Thus the induced map $H_1(\mathcal{Z}_n) \to H_1(\mathcal{P}_n)$ is surjective. Since $\mathcal{Z}_n$ and $\mathcal{P}_n$ are unipotent, it follows that the natural inclusion $\mathcal{Z}_n \hookrightarrow \mathcal{P}_n$ is surjective as well. □

Proposition B.1.7. *If $H_1(\Gamma, F)$ is finite-dimensional, then the canonical homomorphism $\rho : \Gamma \to \mathcal{P}$ is the Malcev completion of Γ over F.*

Proof. Let $\rho : \Gamma \to \Gamma^{\mathrm{un}}(F)$ be the Malcev completion over F. Since each homomorphism ρ_n is Zariski-dense, the natural homomorphism $\widehat{\rho} : \Gamma \to \mathcal{P}(F)$ has Zariski-dense image. Thus by the universal property of the Malcev completion, there exists a unique algebraic homomorphism $\phi : \Gamma^{\mathrm{un}} \to \mathcal{P}$ such that $\phi(F) \circ \rho = \widehat{\rho}$. The

Zariski-density implies that ϕ is surjective. Let $\rho_U : \Gamma \to U(F)$ be a Zariski-dense homomorphism from Γ to the F-rational points of a unipotent F-group U. We may regard U as an algebraic subgroup of the group $\mathbb{U}_n \subset \mathrm{GL}_{n/F}$ of the upper triangular matrices with one's on the diagonal for some n. The representation ρ_U induces a homomorphism $\widehat{\rho_U} : F\Gamma \to gl_n(F)$. Note that $\widehat{\rho_U}(J)$ is contained in the subgroup of nilpotent matrices in $gl_n(F)$. Thus $\widehat{\rho_U}$ induces a map $\tilde{\rho_U} : F\Gamma/J^n \to gl_n(F)$. Note that the diagram

$$\begin{array}{ccc} \Gamma & & \\ \downarrow & \searrow^{\rho_U} & \\ F\Gamma/J^n & \xrightarrow[\tilde{\rho_U}]{} & gl_n(F) \end{array}$$

commutes, and that the image of Γ in $F\Gamma/J^n$ is contained in $\mathcal{P}_{n-1}$. By Zariski-density, there is a surjective algebraic homomorphism $\psi_U : \mathcal{P}_{n-1} \to U$ such that the diagram

$$\begin{array}{ccc} \Gamma & & \\ \downarrow & \searrow^{\rho_U} & \\ \mathcal{P}_{n-1}(F) & \xrightarrow[\psi_U(F)]{} & U(F) \end{array}$$

commutes. This implies that there exists a unique homomorphism $\psi : \mathcal{P} \to \Gamma^{\mathrm{un}}$ such that $\phi \circ \psi = \mathrm{id}$. It is clear that ϕ and ψ are inverse to each other. $\square$

Corollary B.1.8. *If $\Gamma^{\mathrm{un}}_{/F}$ is the Malcev completion of Γ over F, then the natural homomorphism*

$$\Gamma^{\mathrm{un}}_{/F} \to \Gamma^{\mathrm{un}}_{/\mathbb{Q}} \otimes F$$

is an isomorphism.

Proof. The F-rational points $\Gamma^{\mathrm{un}}(F)$ of $\Gamma^{\mathrm{un}}_{/F}$ is the set of group-like elements of $\widehat{F\Gamma}$ and the F-rational points $(\Gamma^{\mathrm{un}}_{/\mathbb{Q}} \otimes F)(F)$ of $\Gamma^{\mathrm{un}}_{/\mathbb{Q}} \otimes F$ is the set of group-like elements of $\widehat{\mathbb{Q}\Gamma} \otimes F = \widehat{F\Gamma}$. $\square$

B.2 Continuous ℓ-adic completion

In this section, we take F to be $\mathbb{Q}_\ell$. Our main target groups are topologically finitely generated profinite groups. The topology of $\mathbb{Q}_\ell$ induces a topology on the group of $\mathbb{Q}_\ell$-rational points $U(\mathbb{Q}_\ell)$ of a unipotent $\mathbb{Q}_\ell$-group U.

Definition B.2.1. Suppose that Γ is a profinite group. The continuous ℓ-adic unipotent completion of Γ is a prounipotent $\mathbb{Q}_\ell$-group $\Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell}$ with a continuous Zariski-dense homomorphism $\rho : \Gamma \to \Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$ that satisfies the following universal property: if there is a continuous Zariski-dense homomorphism $\hat{\rho} : \Gamma \to U(\mathbb{Q}_\ell)$ to the $\mathbb{Q}_\ell$-rational points of a prounipotent $\mathbb{Q}_\ell$-group U, then there exists a unique algebraic homomorphism $\phi : \Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell} \to U$ such that $\hat{\rho} = \phi(\mathbb{Q}_\ell) \circ \rho$.

Remark B.2.2. Since profinite groups are a compact topological groups, the image of ρ is compact in $U(\mathbb{Q}_\ell)$. Since compact subgroups of $U(\mathbb{Q}_\ell)$ are pro-ℓ groups, the map ρ factors through the maximal pro-ℓ quotient $\Gamma^{(\ell)}$ of Γ.

Denote the continuous ℓ-adic unipotent completion of $\Gamma^{(\ell)}$ by $\Gamma^{\mathrm{un}(\ell)}_{/\mathbb{Q}_\ell}$.

Proposition B.2.3. *The natural homomorphism* $\Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell} \to \Gamma^{\mathrm{un}(\ell)}_{/\mathbb{Q}_\ell}$ *is an isomorphism.*

Proof. The inverse of the natural homomorphism can be easily obtained by the universal property of $\Gamma^{\mathrm{un}(\ell)}_{/\mathbb{Q}_\ell}$. □

The following result shows a relation between the Malcev completion of a finitely generated group and the ℓ-adic completion of its profinite completion.

Proposition B.2.4. *Suppose that* Γ *is a group and that* $\widehat{\Gamma}$ *is its profinite completion. Let* $\Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell}$ *and* $\Gamma^{\mathrm{un}\wedge}_{/\mathbb{Q}_\ell}$ *be the Malcev completion and the* ℓ*-adic completion of* Γ *and* $\widehat{\Gamma}$*, respectively. If* Γ *is finitely generated, then the natural homomorphism* $\Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell} \to \Gamma^{\mathrm{un}\wedge}_{/\mathbb{Q}_\ell}$ *is an isomorphism. In particular, there are natural isomorphisms*

$$\Gamma^{\mathrm{un}}_{/\mathbb{Q}_\ell} \cong \Gamma^{\mathrm{un}\wedge}_{/\mathbb{Q}_\ell} \cong \Gamma^{\mathrm{un}(\ell)}_{/\mathbb{Q}_\ell}.$$

Proof. First, we will show that the natural homomorphism $\Gamma \to \Gamma^{\mathrm{un}}_{\mathbb{Q}_\ell}$ is continuous with respect to the pro-ℓ topology of Γ. Since $\Gamma^{\mathrm{un}}_{\mathbb{Q}_\ell}$ is the inverse limit of unipotent $\mathbb{Q}_\ell$-groups, it will suffice to show that a homomorphism $\Gamma \to U(\mathbb{Q}_\ell)$ of Γ to the $\mathbb{Q}_\ell$-rational points of a unipotent $\mathbb{Q}_\ell$-group is continuous. We may assume that U is the upper triangular unipotent algebraic subgroup of GL_n for some n. For each $m \in \mathbb{Z}$, denote by $U(\ell^m\mathbb{Z}_\ell)$ the matrices whose ij-th entry is in $\ell^{(j-i)m}\mathbb{Z}_\ell$ when $i < j$, is 1 when $i = j$, and is 0 when $i > j$. Since Γ is finitely generated, the image of Γ lies in $U(\ell^m\mathbb{Z}_\ell)$ for some m. Note that the $U(\ell^m\mathbb{Z}_\ell)$'s form a basic open set of neighborhoods of the identity in $U(\mathbb{Q}_\ell)$ and the filtration

$$\cdots \supset U(\ell^{n-1}\mathbb{Z}_\ell) \supset U(\ell^n\mathbb{Z}_\ell) \supset U(\ell^{n+1}\mathbb{Z}_\ell) \cdots \supset$$

has the property that each quotient is a group of ℓ-power order. Since the image of Γ is contained in $U(\ell^m\mathbb{Z}_\ell)$, the preimages of $U(\ell^n\mathbb{Z}_\ell)$ are all finite-index subgroups of ℓ-powers of Γ. Therefore, the natural homomorphism $\Gamma \to \Gamma^{\mathrm{un}}_{\mathbb{Q}_\ell}$ factors through the pro-ℓ completion $\Gamma^{(\ell)}$ of Γ. In particular, it factors through $\widehat{\Gamma}$, and hence this induces a homomorphism $\Gamma^{\mathrm{un}\wedge}_{\mathbb{Q}_\ell} \to \Gamma^{\mathrm{un}}_{\mathbb{Q}_\ell}$. Clearly, this map is the inverse of the natural homomorphism $\Gamma^{\mathrm{un}}_{\mathbb{Q}_\ell} \to \Gamma^{\mathrm{un}\wedge}_{\mathbb{Q}_\ell}$. □

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Tatsunari Watanabe was born on July 27, 1983, in Ichikawa Chiba, Japan. After receiving his associate degree in general science from Foothill College in June, 2007, he transferred to the University of California, Berkeley, where he majored in pure mathematics and received his B.A. in mathematics with highest honors and Phi Beta Kappa. In 2009, He was awarded Dorothea Klumpke Roberts Prize as one of the top graduating mathematics majors in UCB. In Fall 2009, he entered the graduate program in mathematics at Duke University. He is expected to obtain a Ph.D. under the supervision of Richard Hain in May, 2015.

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