Geometry of Bäcklund Transformations

by

Yuhao Hu

Department of Mathematics
Duke University

Date: ____________________________

Approved:

____
Robert Bryant, Supervisor

____
Hubert Bray

____
Lenhard L. Ng

____
Leslie Saper

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2018

Abstract

Geometry of Bäcklund Transformations

by

Yuhao Hu

Department of Mathematics
Duke University

Date: ______________________________

Approved:

____
Robert Bryant, Supervisor

____
Hubert Bray

____
Lenhard L. Ng

____
Leslie Saper

An abstract of a dissertation submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2018

Abstract

This thesis is a study of Bäcklund transformations using geometric methods. A Bäcklund transformation is a way to relate solutions of two PDE systems. If such a relation exists for a pair of PDE systems, then, using a given solution of one system, one can generate solutions of the other system by solving only ODEs.

My contribution through this thesis is in three aspects.

First, using Cartan's Method of Equivalence, I prove the generality result: a generic rank-1 Bäcklund transformation relating a pair of hyperbolic Monge-Ampère systems can be uniquely determined by specifying at most 6 functions of 3 variables. In my classification of a more restricted case, I obtain new examples of Bäcklund transformations, which satisfy various isotropy conditions.

Second, by formulating the existence problem of Bäcklund transformations as the integration problem of a Pfaffian system, I propose a method to study how a Bäcklund transformation relates the invariants of the underlying hyperbolic Monge-Ampère systems. This leads to several general results.

Third, I apply the method of equivalence to study rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems and partially classify those that are homogeneous. My classification so far suggests that those homogeneous Bäcklund transformations (relating two hyperbolic Monge-Ampère systems) that are genuinely rank-2 are quite few.

To my parents.

Contents

List of Abbreviations and Symbols

Symbols

$(M, \mathcal{I})$	An exterior differential system with manifold M and differential ideal $\mathcal{I}$.
$I^k \subset \Lambda^k(T^*M)$	The subbundle corresponding to the degree-k piece of $\mathcal{I}$, when $(M, \mathcal{I})$ is an exterior differential system.
$\langle \theta_1, ..., \theta_\ell \rangle$	The ideal of $\Omega^*(U)$ generated by differential forms $\theta_1, ..., \theta_\ell$ (defined on U) and their exterior derivatives.
$\langle \theta_1, ..., \theta_\ell \rangle_{\text{alg}}$	The ideal of $\Omega^*(U)$ algebraically generated by differential forms $\theta_1, ..., \theta_\ell$ defined on U.
$[\![\theta_1, ..., \theta_\ell]\!]$	The vector subbundle of $\Lambda^k(T^*U)$ generated by differential forms $\theta_1, ..., \theta_\ell$ (defined on U) of the same degree k.
$C(\mathcal{I})$	The Cartan system associated to the differential system $(M, \mathcal{I})$, i.e., the Frobenius system whose integral curves are precisely the Cauchy characteristics of $(M, \mathcal{I})$.
$X \lrcorner \omega$	The interior product of a smooth vector field X with a differential form ω.
$S^\perp \subset TM$	The distribution spanned by all vector fields X that satisfy $X \lrcorner \omega = 0$ for any $\omega \in S \subset \Omega^*(M)$.

Acknowledgements

I'd like to express my deepest gratitude to my advisor, Robert Bryant, for teaching me how to do mathematics and for his encouragement and guidance during my research. His influence on me goes beyond mathematics.

In the past years, I've benefited a great deal from attending lectures of and talking with professors in or outside Duke. Special thanks go to

Hubert Bray, Lenny Ng and Leslie Saper, for being on my committee.

Jeanne Clelland, for her inspiring work and her interest in my research.

Clark Bray, for all the training he has given me in mathematical teaching.

David Schaeffer. The memory of your first-year course I'll always cherish.

I'd like to acknowledge all my friends and fellow students who stood beside me over the years. Especially, I'd like to thank

Ma Luo and Zhiyong Zhao, for our brotherly 5+ years.

Mendel Nguyen, for mathematics, physics, Chopin and Driade.

Gavin Ball, Ryan Gunderson and Mike Bell, for kindly sharing their research.

Zhennan Zhou, for taking me on hikes and teaching me how to cook.

Sean Lawley, for all our conversations during 2012-14.

Rosa Zhou, for moon, star, and rose.

Thank you, my dear parents. Your thoughts are constantly on me, my well-being. How many times have you reminded me that I shall never give up. How deeply have you, by your love, passed on to me the value of a simple and unassuming life.

1

Introduction

In 1882, the Swedish mathematician A.V. Bäcklund proved the result (see [Bac83], [BGG03] or [CT80]): *Given a surface with a constant Gauss curvature* $K < 0$ *in* $\mathbb{E}^3$*, one can construct, by solving ODEs, a 1-parameter family of new surfaces in* $\mathbb{E}^3$ *with the Gauss curvature* K*.* This is the origin of the term "Bäcklund transformation".

Classically, a *Bäcklund transformation* is a PDE system $\mathcal{B}$ that relates solutions of two other PDE systems $\mathcal{E}_1$ and $\mathcal{E}_2$. More precisely, such a relation must satisfy the property: given a solution u of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$), substituting it in $\mathcal{B}$, one obtains a PDE system whose solutions can be found by ODE methods and produce solutions of $\mathcal{E}_2$ (resp. $\mathcal{E}_1$).

For example, the *Cauchy-Riemann system*

$$\begin{cases} u_x - v_y = 0, \\ u_y + v_x = 0 \end{cases} \tag{1.1}$$

is a Bäcklund transformation; it relates solutions of the Laplace equation $\Delta z = 0$ for $z(x, y)$ in the following way: If u satisfies $\Delta u = 0$, then, substituting it in (1.1), we obtain a compatible first order system for v, whose solutions can be found by ODE methods and satisfy $\Delta v = 0$, and *vice versa*.

As another example, consider the system of nonlinear equations

$$\begin{cases} z_x - \bar{z}_x = \lambda \sin(z + \bar{z}), \\ z_y + \bar{z}_y = \frac{1}{\lambda} \sin(z - \bar{z}), \end{cases} \tag{1.2}$$

where λ is a nonzero constant. One can show that (1.2) is a Bäcklund transformation relating solutions of the *sine-Gordon equation*

$$u_{xy} = \tfrac{1}{2} \sin(2u). \tag{1.3}$$

The system (1.2) is closely connected with the classical Bäcklund transformation relating surfaces in $\mathbb{E}^3$ with a negative constant Gauss curvature. For details, see [CT80].

In addition, a Bäcklund transformation may relate solutions of a parabolic equation (see [NC82]) or two equations that are nonequivalent (see [CI$^+$09]). Numerous other examples of Bäcklund transformations are discussed in [RS02]. Through these examples, Bäcklund transformations are found to have rich connection with surface theory in differential geometry and solitons in mathematical physics.

Among the examples discussed in [RS02], a Bäcklund transformation relating solutions of the *hyperbolic Tzitzeica equation* is particularly interesting. The *hyperbolic Tzitzeica equation* is the second-order equation for $h(x, y)$:

$$(\ln h)_{xy} = h - h^{-2}. \tag{1.4}$$

This equation was discovered by Tzitzeica in his study of *hyperbolic affine spheres* in the affine 3-space $\mathbb{A}^3$ (see [Tzi08] and [Tzi09]). He found that the system in α, β, and h

$$\begin{cases} \alpha_x = (h_x\alpha + \lambda\beta)h^{-1} - \alpha^2, \\ \alpha_y = \beta_x = h - \alpha\beta, \\ \beta_y = (h_y\beta + \lambda^{-1}\alpha)h^{-1} - \beta^2, \end{cases} \tag{1.5}$$

where λ is an arbitrary nonzero constant, is a Bäcklund transformation relating solutions of (1.4). More explicitly, if h solves the hyperbolic Tzitzeica equation (1.4), then, substituting it in the system (1.5), one obtains a compatible first-order PDE system for α and β, whose solutions can be found by solving ODEs; for each solution (α, β), the function

$$\bar{h} = -h + 2\alpha\beta$$

also satisfies the hyperbolic Tzitzeica equation (1.4). Furthermore, one can show that, unlike the systems (1.1) and (1.2), substituting a solution h of (1.4) into (1.5) yields a system whose solutions depend on 2 parameters instead of 1. Using our terminology (Definition 2.13 of Chapter 2), one can verify that the system (1.5) corresponds to a *rank*-2 Bäcklund transformation.

An ultimate goal of studying Bäcklund transformations is solving the *Bäcklund problem*, which was considered by Goursat in [Gou25]. The statement of the *Bäcklund problem* is: *Find all pairs of systems of PDEs whose solutions are related by a Bäcklund transformation.* This problem remains unsolved.

However, recent works of Jeanne Clelland have shed new light on the classification of Bäcklund transformations. Her paper [Cle01], in particular, focuses on Bäcklund transformations relating solutions of two *hyperbolic Monge-Ampère systems*, which are second-order PDEs in the plane arising frequently in differential geometry.

Clelland's approach in [Cle01] to Bäcklund transformations involves several key steps. First, a Bäcklund transformation relating two hyperbolic Monge-Ampère systems is formulated as an *exterior differential system* ([BCG$^+$13]). This allows one to study a Bäcklund transformation geometrically as a manifold N with a structure $\mathcal{B}$. As a result, concepts such as equivalence and symmetry can be easily defined. Second, she applies *Cartan's method of equivalence* to derive local invariants of such a Bäcklund transformation. Third, by assuming all local invariants to be constants, she obtains a complete classification of homogeneous rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems, where *homogeneous* means that the symmetry group of a Bäcklund transformation acts transitively on the underlying manifold.

The aim of the present work is to address the following questions:

i. Up to equivalence, what is the generality of Bäcklund transformations relating two hyperbolic Monge-Ampère systems?

ii. Given two hyperbolic Monge-Ampère systems, how to tell whether they are related by a Bäcklund transformation?

iii. What can be concluded about the existence of Bäcklund transformations in higher ranks?

Of course, similar questions can be asked for Bäcklund transformations relating two PDE systems in broader classes (elliptic, hyperbolic, parabolic), but, for clarity, in this thesis, we focus on Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

This thesis is organized as follows.

In Chapter 2, we develop the basic concepts and techniques that are used in this work. These include the notion of an *exterior differential system*, in particular, that of a *hyperbolic Monge-Ampère system*; the notion of a *G-structure*, which is at the heart of an equivalence problem; and the notion of a *Bäcklund transformation*.

In the first two sections of Chapter 3, we prove a generality result:

Theorem 3.3. *A generic rank-*1 *Bäcklund transformation relating two hyperbolic Monge-Ampère systems can be specified uniquely, up to equivalence, by initial data consisting of at most 6 functions of 3 variables.*

This theorem implies the following

Corollary 3.4. *There exist hyperbolic Monge-Ampère systems that are not related to any other hyperbolic Monge-Ampère system by a generic rank-*1 *Bäcklund transformation.*

Before summarizing the main ideas of proving Theorem 3.3, we briefly explain what the term "generic" means. Given a rank-1 Bäcklund transformation $(N, \mathcal{B})$ relating two hyperbolic Monge-Ampère systems, there is an intrinsic way to define a tensor field T on N (Section 3.1). A rank-1 Bäcklund transformation is said to be *generic* if the tensor T takes generic values.

To prove Theorem 3.3, we first show that, given a generic rank-1 Bäcklund transformation $(N, \mathcal{B})$ relating two hyperbolic Monge-Ampère systems, where N is a 6-manifold, there is a canonical way to define a local coframing on N. One can show that two such coframings are locally equivalent up to diffeomorphism if and only if the corresponding Bäcklund transformations are locally equivalent. Then, we apply a theorem of Cartan ([Bry14]) to show that such local coframings, up to diffeomorphism, depend on at most 6 functions of 3 variables.

In the last section of Chapter 3, we focus on the case when two invariants of a generic rank-1 Bäcklund transformation are assumed to be specific constants. We can classify such Bäcklund transformations. In particular, we find new examples of Bäcklund transformations with cohomogeneity $1, 2$ and 3.

In Chapter 4, we study how obstructions to the existence of Bäcklund transformations may be expressed in terms of the invariants of the underlying hyperbolic Monge-Ampère systems. Such obstructions can be found by applying techniques of *exterior differential systems* ([BCG$^+$13]) to a rank-4 Pfaffian system. We obtain several results that inform us which pairs of hyperbolic Monge-Ampère systems may be related by a rank-1 Bäcklund transformation of a particular type.

In Chapter 5, motivated by the example (1.5), we study rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems, and partially classify those that are homogeneous[1]. The approach to classification is analogous to that in [Cle01]. The results of classification, as well as the cases that are work in progress, are summarized in Section 5.4.

Most calculations in Chapters 3,4, and 5 are performed using Maple™.

[1] It turns out that the rank-2 Bäcklund transformation corresponding to (1.5) is nonhomogeneous.

2

Background

2.1 Exterior Differential Systems

Definition 2.1. Let M be a smooth manifold, $\mathcal{I} \subset \Omega^*(M)$ a graded ideal that is closed under exterior differentiation. The pair $(M, \mathcal{I})$ is said to be an *exterior differential system* with space M and differential ideal $\mathcal{I}$.

Given an exterior differential system $(M, \mathcal{I})$, we use $\mathcal{I}^k$ to denote the degree-k piece of $\mathcal{I}$, namely, $\mathcal{I}^k = \mathcal{I} \cap \Omega^k(M)$, where $\Omega^k(M)$ stands for the $C^\infty(M)$-module of differential k-forms on M. If the rank of $\mathcal{I}^k$, restricted to each point, is locally a constant, then the elements of $\mathcal{I}^k$ are precisely smooth sections of a vector bundle denoted by I^k.

Definition 2.2. An *integral manifold* of an exterior differential system $(M, \mathcal{I})$ is a submanifold $i : N \hookrightarrow M$ satisfying $i^*\phi = 0$ for any $\phi \in \mathcal{I}$.

By this definition, the requirement of $\mathcal{I}$ being closed under exterior differentiation is natural, because of the identity $i^* \circ \mathrm{d} = \mathrm{d} \circ i^*$.

Intuitively, an exterior differential system is a coordinate-independent way to express a PDE system; an integral manifold, usually with a certain independence condition satisfied, corresponds to a solution of the PDE system. We illustrate this point by the following example.

Example 1. A single k-th order PDE in one unknown function u of n independent variables $x = (x^1, ..., x^n)$ can be expressed in the form

$$F(x, u, \partial_{\alpha_1} u(x), ..., \partial_{\alpha_k} u(x)) = 0, \tag{2.1}$$

where α_i ranges over all multiindices of length i from $1, 2, ..., n$. For simplicity, we regard two multiindices as equivalent if and only if they are reorderings of each other. Furthermore, if α is a multiindex of length i, then $j\alpha$, where $j \in \{1, ..., n\}$, is a multiindex of length $i + 1$.

The equation (2.1) corresponds to an exterior differential system $(M, \mathcal{I})$ induced from the canonical contact system on the k-jet bundle $J^k(\mathbb{R}^n, \mathbb{R})$. To be explicit, $J^k(\mathbb{R}^n, \mathbb{R})$ has the standard coordinates $(x, u, p_{\alpha_1}, ..., p_{\alpha_k})$, where α_i are multiindices in the sense above. The canonical contact system $\mathcal{C}$ on $J^k(\mathbb{R}^n, \mathbb{R})$ is algebraically generated by the 1-forms

$$\theta_0 = \mathrm{d}u - p_i \mathrm{d}x^i,$$

$$\theta_{\alpha_i} = \mathrm{d}p_{\alpha_i} - p_{j\alpha_i} \mathrm{d}x^j \quad (i = 1, 2, ..., k-1)$$

and their exterior derivatives.

Let $w : \mathbb{R}^n \to \mathbb{R}$ be a C^k-function. Then $\hat{w} : \mathbb{R}^n \to J^k(\mathbb{R}^n, \mathbb{R})$, defined by

$$\hat{w}(x) = (x, w(x), \partial_{\alpha_1} w(x), ..., \partial_{\alpha_k} w(x))$$

is an n-dimensional integral manifold of the exterior differential system $(J^k(\mathbb{R}^n, \mathbb{R}), \mathcal{C})$. This $\hat{w}$ is called the *lifting* of w to $J^k(\mathbb{R}^n, \mathbb{R})$. Conversely, suppose that $\hat{v} : \mathbb{R}^n \to J^k(\mathbb{R}^n, \mathbb{R})$ is an n-dimensional integral manifold of $(J^k(\mathbb{R}^n, \mathbb{R}), \mathcal{C})$ on which $x^1, x^2, ..., x^n$ are independent functions, in other words, on which $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$ everywhere. We then have

$$p_i \circ \hat{v} = \frac{\partial}{\partial x_i}(u \circ \hat{v}), \quad p_{ij} \circ \hat{v} = \frac{\partial^2}{\partial x_i \partial x_j}(u \circ \hat{v}), ...$$

Therefore, locally, a C^k-function $w : \mathbb{R}^n \to \mathbb{R}$ is in one-to-one correspondence with an integral manifold of $(J^k(\mathbb{R}^n, \mathbb{R}), \mathcal{C})$ on which $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$.

Now let $M \subset J^k(\mathbb{R}^n, \mathbb{R})$ be defined by the equation

$$F(x, u, p_{\alpha_1}, ..., p_{\alpha_k}) = 0,$$

where F is as in (2.1). If ∇F is nonzero at a point, then, by shrinking to a neighborhood of that point, we can assume M to be a smooth manifold. Under this assmuption, let $\mathcal{C}_M$ be the restriction of the contact system $\mathcal{C}$ to M. Then, by an argument similar to the above, one can show that an integral manifold of the exterior differential system $(M, \mathcal{C}_M)$ on which $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$ is in one-to-one correspondence with a solution of (2.1) whose lifting to $J^k(\mathbb{R}^n, \mathbb{R})$ is contained in M.

Regarding two exterior differential systems, the following notion of equivalence is natural.

Definition 2.3. Two exterior differential systems $(M, \mathcal{I})$ and $(N, \mathcal{J})$ are said to be *equivalent up to diffeomorphism*, or *equivalent*, for brevity, if there exists a diffeomorphism $\phi : M \to N$ such that $\phi^*\mathcal{J} = \mathcal{I}$. Such a ϕ is called an *equivalence* between both systems. An equivalence between $(M, \mathcal{I})$ and itself is called a *symmetry* of $(M, \mathcal{I})$.

Example 2. When $J^1(\mathbb{R}^n, \mathbb{R})$ with coordinates (x^i, u, p_i) is regarded as a contact manifold with the contact form

$$\theta = \mathrm{d}u - p_i \mathrm{d}x^i,$$

a symmetry of the exterior differential system $(J^1(\mathbb{R}^n, \mathbb{R}), \mathcal{C})$ is just a contact transformation of the space $J^1(\mathbb{R}^n, \mathbb{R})$ to itself.

We end this section by presenting the *Frobenius Theorem*, formulated in terms of exterior differential systems. This theorem is at the heart of our characterization of a Bäcklund transformation. For a proof of the Frobenius theorem, see Chapter II of [BCG$^+$13].

Theorem 2.1. (Frobenius) *Let $(M^n, \mathcal{I})$ be an exterior differential system. If, on an open $U \subset M$, the differential ideal $\mathcal{I}$ is generated by k linearly independent 1-forms $\theta_1, ..., \theta_k$ satisfying*

$$\mathrm{d}\theta_i \equiv 0 \mod \theta_1, \theta_2, ..., \theta_k,$$

then any $p \in U$ has an open neighborhood $V \subset U$ on which there exists a coordinate system $(x^1, ..., x^{n-k}, y^1, ..., y^k)$ such that the ideal $\mathcal{I}$ is generated by $\mathrm{d}y^1, \mathrm{d}y^2, ..., \mathrm{d}y^k$.

Remark 1. One can show ([BCG$^+$13]) that a coordinate system $(x^1, ..., x^{n-k}, y^1, ..., y^k)$ in the conclusion of Theorem 2.1 can be found by solving systems of ODEs. Once such a coordinate system is obtained, setting the y^i $(i = 1, ..., k)$ to be constants defines an $(n-k)$-dimensional integral manifold of $(M, \mathcal{I})$. Each such integral manifold is called a *leaf* associated to the system $(M, \mathcal{I})$.

Definition 2.4. Given an exterior differential system $(M, \mathcal{I})$. If the assumption in Theorem 2.1 holds on a neighborhood of every $p \in M$ for a constant k, then $(M, \mathcal{I})$ is called a *rank-k Frobenius system.*

2.2 Hyperbolic Monge-Ampère Systems

Among second order PDEs for 1 unknown function of 2 independent variables, the *Monge-Ampère equations* are of special interest, as they frequently arise in differential geometry (see [Bry02]) and the calculus of variations (see [BGG03]).

The general form of a *Monge-Ampère equation* for $z(x, y)$ is

$$A(z_{xx}z_{yy} - z_{xy}^2) + Bz_{xx} + 2Cz_{xy} + Dz_{yy} + E = 0, \tag{2.2}$$

where A, B, C, D, E are functions of x, y, z, z_x, z_y. A Monge-Ampère equation (2.2) is said to be *elliptic* (resp., *hyperbolic*, *parabolic*) if $AE - BD + C^2$ is negative (resp., positive, zero).

For example, the sine-Gordon equation (1.3) and the Tziteica equation (1.4) are hyperbolic Monge-Ampère equations; the Cauchy-Riemann equation (1.1) and the equation[1] $z_{xx}z_{yy} - z_{xy}^2 = 1$ are elliptic Monge-Ampère equations; in the classical calculus of variations, the *Euler-Lagrange equation* for a first-order functional

$$\int_\Omega L(x, z(x), \nabla z(x))\mathrm{d}x, \quad \Omega \subset \mathbb{R}^2,\ L : J^1(\Omega, \mathbb{R}) \to \mathbb{R}$$

is always Monge-Ampère (see [BGG03][2]).

A Monge-Ampère equation can be formulated as an exterior differential system on a contact manifold. In the hyperbolic case, we follow [BGH95] to give the

Definition 2.5. A *hyperbolic Monge-Ampère system* $(M, \mathcal{I})$ is an exterior differential system, where M is a 5-manifold, $\mathcal{I}$ being locally algebraically generated by $\theta \in \mathcal{I}^1$ and $d\theta, \Omega \in \mathcal{I}^2$ satisfying

(1) $\theta \wedge (\mathrm{d}\theta)^2 \neq 0$;

(2) $[\![\mathrm{d}\theta, \Omega]\!]$, modulo θ, has rank 2;

(3) $(\lambda \mathrm{d}\theta + \mu\Omega)^2 \equiv 0 \mod \theta$ has two distinct solutions $[\lambda_i : \mu_i] \in \mathbb{RP}^1$ $(i = 1, 2)$.

In these three conditions, the first says that θ is contact; the second says that the corresponding PDE system is nonempty; the third characterizes hyperbolicity, that is, each integral surface of $(M, \mathcal{I})$ is foliated by two distinct families of characteristics.

Example 3. A PDE of the form

$$z_{xy} = F(x, y, z, z_x, z_y)$$

[1] cf. Jørgen's Theorem in [Bry02].

[2] In [BGG03], a Monge-Ampère equation is a second order PDE for 1 unknown function of n independent variables, and it is shown that an Euler-Lagrange equation can be formulated as an *Euler-Lagrange exterior differential system*, which can be intrinsically characterized. In this thesis, we only consider the case when $n = 2$.

corresponds to a hyperbolic Monge-Ampère system $(M, \mathcal{I})$, where $M = J^1(\mathbb{R}^2, \mathbb{R})$ with the standard coordinates (x, y, z, p, q); $\mathcal{I}$ is algebraically generated by

$$\theta = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,$$

$$\mathrm{d}\theta = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q,$$

$$\Omega = (\mathrm{d}p - F(x, y, z, p, q)\mathrm{d}y) \wedge \mathrm{d}x.$$

In particular, Ω and $\mathrm{d}\theta + \Omega$ are decomposable 2-forms.

Example 4. The oriented orthonormal frame bundle $\mathcal{O}$ over the Euclidean space $\mathbb{E}^3$ consists of elements of the form $(\mathbf{x}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{x} \in \mathbb{R}^3$, and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an oriented orthonormal frame at $\mathbf{x}$. On $\mathcal{O}$, we have the canonical structure equations

$$\mathrm{d}\omega^i = -\omega^i_j \wedge \omega^j,$$

$$\mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j,$$

where $\omega^i_j = -\omega^j_i$. Consider the exterior differential system $(\mathcal{O}, \mathcal{I})$, where

$$\mathcal{I} = \langle \omega^3, \mathrm{d}\omega^3, \mathrm{d}\omega^1_2 - K\omega^1 \wedge \omega^2 \rangle_{\mathrm{alg}}, \qquad K\, const.$$

An integral surface of $(\mathcal{O}, \mathcal{I})$ on which $\omega^1 \wedge \omega^2 \neq 0$ corresponds to a surface S in $\mathbb{E}^3$ with constant Gauss curvature K and with an orthonormal frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ attached to it, $\mathbf{e}_3$ being normal to S.

Let X be a nonzero vector field on $\mathcal{O}$ that annihilates $\omega^1, \omega^2, \omega^3, \omega^1_3$ and ω^2_3. For each ϕ in the three algebraic generators of $\mathcal{I}$ above, we have

$$X \lrcorner\, \phi = 0, \qquad \mathcal{L}_X \phi = 0.$$

This implies that on the quotient space M^5 of $\mathcal{O}$ by the flow of X, letting $\pi : \mathcal{O} \to M$ be the quotient map, there exists a well-defined differential form α such that $\pi^*\alpha = \phi$. It follows that $\mathcal{I}$ descends to M to be an exterior differential system $\mathcal{J}$, in the sense that $\mathcal{I}$ is algebraically generated by the elements of $\pi^*\mathcal{J}$. Moreover, $(M, \mathcal{J})$ is a hyperbolic Monge-Ampère system if and only if the constant $K < 0$. This, in part, follows from the equality

$$(\lambda \mathrm{d}\omega^3 + \mu(\mathrm{d}\omega^1_2 - K\omega^1 \wedge \omega^2))^2 \equiv (\lambda^2 + K\mu^2)(\mathrm{d}\omega^3)^2 \mod \omega^3.$$

By Definition 2.5, on each hyperbolic Monge-Ampère system $(M, \mathcal{I})$, locally there exist 1-forms $\theta, \omega^1, \omega^2, \omega^3, \omega^4$, linearly independent everywhere, such that

$$\mathcal{I} = \langle \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle.$$

One can show that the pair of Pfaffian systems $\mathcal{I}_{10} = \langle \theta, \omega^1, \omega^2 \rangle$ and $\mathcal{I}_{01} = \langle \theta, \omega^3, \omega^4 \rangle$ are well-defined up to ordering. Restricted to an integral surface of $(M, \mathcal{I})$, each of $\mathcal{I}_{10}$ and $\mathcal{I}_{01}$ becomes a rank-1 Frobenius system whose integral curves are the characteristics of $(M, \mathcal{I})$ in the usual sense of hyperbolic PDEs. This motivates the

Definition 2.6. Given a hyperbolic Monge-Ampère system[3] $(M, \mathcal{I})$, where $\mathcal{I} = \langle \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4 \rangle$, the systems $\mathcal{I}_{10} = \langle \theta, \omega^1, \omega^2 \rangle$ and $\mathcal{I}_{01} = \langle \theta, \omega^3, \omega^4 \rangle$ are called the *characteristic systems* associated to $(M, \mathcal{I})$.

2.3 Equivalence of G-structures

Definition 2.7. Let M be an n-dimensional manifold. A *coframe* at $p \in M$ is a linear isomorphism $u_p : T_pM \to \mathbb{R}^n$. The set of all coframes on M forms a principal right $\mathrm{GL}(n, \mathbb{R})$-bundle with the group action $u \cdot g := g^{-1}u$. This is called the *coframe bundle* over M, denoted as $\mathcal{F}(M)$. A local section of $\mathcal{F}(M)$ defined on $U \subset M$ is said to be a *coframing* on U.

Many (local) differential geometric structures on a smooth manifold M^n can be equivalently expressed by a notion of 'admissible' coframings, with the property that

[3] The notion of a *characteristic system* can apply to hyperbolic exterior differential systems in general. See [BGH95].

any two such coframings differ pointwise by the action of a Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$. For example, a Riemannian metric g on M^n can be viewed as all local coframings $(\omega^1, \omega^2, ..., \omega^n)$ satisfying $g = (\omega^1)^2 + \cdots + (\omega^n)^2$, any two such coframings (defined on the same domain) differing by a pointwise action of the orthogonal group $O(n)$; a symplectic form Ω on M^{2n} can be viewed as all local coframings $(\omega^1, \omega^2, ..., \omega^{2n})$ satisfying $\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2n-1} \wedge \omega^{2n}$, any two such coframings differing by a pointwise action of the symplectic group $\mathrm{Sp}(2n)$; an almost complex structure J on M^{2n} is characterized by any basis of $(1,0)$-forms on M, whose real and imaginary parts comprising a local coframing, any two such coframings differing by a pointwise action of $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$; etc. This motivates the following definition.

Definition 2.8. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup. A *G-structure* on a smooth manifold M^n is a principal G-subbundle of the coframe bundle $\mathcal{F}(M)$.

When dealing with equivalence between two local geometric structures defined in terms of coframes, it is necessary to take into account the ambiguity in the choice of admissible coframings. For example, Let (M_i, g_i) $(i = 1, 2)$ be two Riemannian n-manifolds, where $g_i = (\omega^1_{(i)})^2 + \cdots + (\omega^n_{(i)})^2$. These Riemannian structures are locally equivalent if and only if there exists a (local) diffeomorphism $\phi : U_1 \to U_2$ $(U_i \subset M_i$ open) such that $\phi^* \omega_{(2)} = \gamma \omega_{(1)}$ for some $\gamma : U_1 \to O(n)$, where $\omega_{(i)} = (\omega^1_{(i)}, \omega^2_{(i)}, ..., \omega^n_{(i)})$ $(i = 1, 2)$. However, the ambiguity represented by γ can be removed by passing to G-structures, as we will describe below.

Definition 2.9. Let $\mathcal{G}$ be a G-structure on M, with $\pi : \mathcal{G} \to M$ being the submersion. The *tautological* 1*-form* τ on $\mathcal{G}$ is the $\mathbb{R}^n$-valued 1-form determined by the equation $\tau(\mathbf{v}) = u(\pi_*(\mathbf{v}))$, for any $u \in \mathcal{G}$ and $\mathbf{v} \in T_u\mathcal{G}$.

Definition 2.10. Two G-structures, $\mathcal{G}_1$ and $\mathcal{G}_2$, with tautological 1-forms τ_1, τ_2, respectively, are said to be *equivalent* if there exists a diffeomorphism $\hat{\phi} : \mathcal{G}_1 \to \mathcal{G}_2$, such that $\hat{\phi}^* \tau_2 = \tau_1$.

The following proposition is proved in [Gar89].

Proposition 2.1. *Let M_i $(i = 1, 2)$ be two smooth n-manifolds, each with a coframing ω_i defined on some open neighborhood $U_i \subset M_i$. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup. Let $\mathcal{G}_i$ be the G-structure on U_i defined by $\mathcal{G}_i = \{\omega_i(p) \cdot h | \; p \in U_i, \; h \in G\}$. There exists a diffeomorphism $\phi : U_1 \to U_2$ and a map $g : U_1 \to G$ such that $\phi^*\omega_2 = g\omega_1$ if and only if there exists an equivalence of G-structures $\hat{\phi} : \mathcal{G}_1 \to \mathcal{G}_2$.*

Example 5. Consider a hyperbolic Monge-Ampère system $(M, \mathcal{I})$. A local coframing $\Theta = (\theta^0, \theta^1, ..., \theta^4)$ defined on an open neighborhood $U \subset M$ is said to be 0-*adapted* if, on U, the differential ideal $\mathcal{I}$ can be expressed as

$$\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle.$$

This is a pointwise condition on Θ. Moreover, given any two 0-adapted coframings, on a common domain, they must relate by a pointwise action of the subgroup $G_0 \subset \mathrm{GL}(5, \mathbb{R})$ generated by matrices of the form

$$g = \begin{pmatrix} a & \mathbf{0} & \mathbf{0} \\ \mathbf{b}_1 & A & 0 \\ \mathbf{b}_2 & 0 & B \end{pmatrix}, \qquad a \neq 0; \; A, B \in \mathrm{GL}(2, \mathbb{R}); \; \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2,$$

and

$$J = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & I_2 \\ \mathbf{0} & I_2 & 0 \end{pmatrix}.$$

Consequently, two hyperbolic Monge-Ampère systems are equivalent if and only if the corresponding G_0-structures are equivalent. This treatment of a differential system as a geometric structure is what we mean by "geometry of differential systems".

To close this section, let us mention a useful property of the tautological 1-form of a G-structure on M: the so-called *reproducing property*.

Proposition 2.2. *Let $\mathcal{G}$ be a G-structure on M. Let τ be the tautological* 1*-form on $\mathcal{G}$. For any local section $\sigma : U \to \mathcal{G}$ ($U \subset M$ open), we have*

$$\sigma^*\tau = \sigma,$$

where, on the left-hand-side of the equality, σ is regarded as a differentiable map, whereas, on the right-hand-side, it is regarded as a coframing on M.

In particular, if certain differential conditions are satisfied by admissible coframings, then the reproducing property leads to corresponding restrictions on the equations satisfied by the tautological 1-form on a G-structure.

2.4 Bäcklund Transformations

We follow [AF15] to define Bäcklund transformations, though we will, in later chapters, mostly be concerned with those relating hyperbolic Monge-Ampère systems. For the latter, a definition can be found in Chapter 4 of [BGG03].

Definition 2.11. Let $(M, \mathcal{I})$ be an exterior differential system. A *rank-k integrable extension* of $(M, \mathcal{I})$ is an exterior differential system $(N, \mathcal{J})$ with a submersion $\pi : N \to M$ that satisfies the condition: for each $p \in N$, there exists an open neighborhood $U \subset N$ ($p \in U$) such that

(1) on U, the differential ideal $\mathcal{J}$ is algebraically generated by elements of $\pi^*\mathcal{I}$ together with 1-forms $\theta_1, ..., \theta_k \in \Omega^1(U)$, where $k = \dim N - \dim M$;

(2) for any $p \in U$, let F_p denote the fiber $\pi^{-1}(\pi(p))$; the 1-forms $\theta_1, ..., \theta_k$ restrict to T_pF_p to be linearly independent.

In Definition 2.11, roughly speaking, $\pi : N \to M$ is a bundle and $\mathcal{J}$ defines a 'connection' on this bundle that is flat over the integral manifolds of $\mathcal{I}$. In more detail, condition (1) implies that, if $S \subset M$ is an integral manifold of $(M, \mathcal{I})$, then $\mathcal{J}$ restricts to $\pi^{-1}(S)$ to be a Frobenius system; hence, $\pi^{-1}(S)$ is foliated by integral manifolds of $(N, \mathcal{J})$. Condition (2) implies that, restricting to any integral manifold of $(N, \mathcal{J})$, π is an immersion, whose image is an integral manifold of $(M, \mathcal{I})$.

Example 6. Let $(M, \mathcal{I})$ be an exterior differential system. The obvious submersion $\pi : M \times \mathbb{R} \to M$ induces an integrable extension $(M \times \mathbb{R}, \langle \pi^{-1}\mathcal{I}, dt \rangle)$ of $(M, \mathcal{I})$, where t is a coordinate on the $\mathbb{R}$-factor.

Proposition 2.3. *The composition of two integrable extensions is an integrable extension.*

Proof. Let $\pi_1 : (M_1, \mathcal{I}_1) \to (M_2, \mathcal{I}_2)$ and $\pi_2 : (M_2, \mathcal{I}_2) \to (M_3, \mathcal{I}_3)$ be integrable extensions. By definition, $\mathcal{I}_2$ is algebraically generated by $\pi_2^*\mathcal{I}_3$ and certain 1-forms $\alpha_1, ..., \alpha_p$; $\mathcal{I}_1$ is thus algebraically generated by $(\pi_2 \circ \pi_1)^*\mathcal{I}_3$, $\pi_1^*\alpha_1, ..., \pi_1^*\alpha_p$ and q 1-forms $\beta_1, ..., \beta_q$. Clearly, for $\pi_2 \circ \pi_1$, the first condition in the definition of integrable extensions is satisfied. To check the second condition, suppose that there exist constants c_i, f_j, $x \in M_1$ such that $\left(\sum_{i=1}^p c_i \pi_1^* \alpha_i + \sum_{j=1}^q f_j \beta_j\right)(\mathbf{v}) = 0$ for any $\mathbf{v} \in T_x M_1$ satisfying $\pi_{2*}(\pi_{1*}(\mathbf{v})) = \mathbf{0}$. Since π_1 is an integrable extension, each $T_x M_1$ is a direct sum of $V_1 := \ker_x(\pi_1)$ and $V_2 := \ker_x(\beta_1, ..., \beta_q)$. For the previous equality to be satisfied on V_1, all f_j must be equal to zero. Since π_{1*} restrict to V_2 to be an linear isomorphism, for the equality to hold on $V_2 \cap \ker((\pi_2 \circ \pi_1)_*)$, c_i must all be zero. □

Definition 2.12. A *Bäcklund transformation* relating two exterior differential systems, $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$, is a quadruple $(N, \mathcal{B}; \pi_1, \pi_2)$ where, for each $i \in \{1, 2\}$, $\pi_i : N \to M_i$ makes $(N, \mathcal{B})$ an integrable extension of $(M_i, \mathcal{I}_i)$. Such a Bäcklund transformation is represented by the diagram

$$\begin{array}{ccc} & (N, \mathcal{B}) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (M_1, \mathcal{I}_1) & & (M_2, \mathcal{I}_2) \end{array}$$

Definition 2.13. In Definition 2.12, if M_1, M_2 have the same dimension, which is not required in general, then the *rank* of $(N, \mathcal{B}; \pi_1, \pi_2)$ is the fiber dimension of either π_1 or π_2. If $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$ are equivalent exterior differential systems, then $(N, \mathcal{B}; \pi_1, \pi_2)$ is called an *auto-Bäcklund transformation* of either $(M_i, \mathcal{I}_i)$.

By Definitions 2.11 and 2.12, it is clear that, given a Bäcklund transformation $(N, \mathcal{B}; \pi_1, \pi_2)$ relating $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$, we can start with an integral manifold S of $(M_1, \mathcal{I}_1)$; restrict $\mathcal{B}$ to $\pi_1^{-1}S$, which becomes a Frobenius system; then solve this Frobenius system and project, by π_2, each leaf associated to it into M_2. As a result, one produces a family of integral manifolds of $(M_2, \mathcal{I}_2)$. By Remark 1, it is easy to see that only ODE methods are used in this process. This is what we mean by "a Bäcklund transformation allows one to use a known solution of a PDE system and ODE methods to obtain solutions of a second PDE system."

Example 7. Let $\pi : (N, \mathcal{J}) \to (M, \mathcal{I})$ be an integrable extension. The quadruple $(N, \mathcal{J}; \pi, \pi)$ is an auto-Bäcklund transformation of $(M, \mathcal{I})$. However, such a Bäcklund transformation does not produce new integral manifolds of $(M, \mathcal{I})$ from a given one.

Example 8. In Example 4, the orthonormal frame bundle $\mathcal{O}$ can be viewed as a subgroup of $\mathrm{GL}(4, \mathbb{R})$; an element $(\mathbf{x}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \in \mathcal{O}$ corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}).$$

Let K be a negative constant. Let θ, r be constants satisfying $K = -\sin^2\theta / r^2$. The element

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \in \mathcal{O}$$

induces a map $\psi_g : \mathcal{O} \to \mathcal{O}$ defined by

$$\psi_g(u) = ug, \quad \forall u \in \mathcal{O} \subset \mathrm{GL}(4, \mathbb{R}).$$

Let $\hat{\mathcal{I}} \subset \Omega^*(\mathcal{O})$ be the differential ideal generated by elements of $\mathcal{I}$ and $\psi_g^*\mathcal{I}$. One can verify that the quadraple $(\mathcal{O}, \hat{\mathcal{I}}; \pi, \pi \circ \psi_g)$ defines an auto-Bäcklund transformation of the system $(M, \mathcal{J})$. This is the classical Bäcklund transformation relating surfaces in $\mathbb{E}^3$ with a negative constant Gauss curvature ([Bac83]).

Definition 2.14. Given a fiber bundle $\pi : E \to B$, for any $p \in E$, the *vertical tangent space* of E at p is the kernel of $\pi_* : T_pE \to T_{\pi(p)}B$.

Definition 2.15. A Bäcklund transformation $(N, \mathcal{B}; \pi_1, \pi_2)$ is called *nontrivial* if the two fibrations π_1, π_2 have distinct vertical tangent spaces at each point $p \in N$.

If $(N, \mathcal{B}; \pi_1, \pi_2)$ is a nontrivial Bäcklund transformation of rank k, then one can show that, given any p-dimensional integral manifold S of $(M_1, \mathcal{I}_1)$, $\pi_2(\pi_1^{-1}S)$ has a dimension strictly greater than p; thus, $\pi_2(\pi_1^{-1}S)$ must be foliated by a family of p-dimensional integral manifolds of $(M_2, \mathcal{I}_2)$.

Example 9. Let $(N, \mathcal{B}; \pi, \bar{\pi})$ be a rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$. On some open subsets $U \subset M$ and $\bar{U} \subset \bar{M}$, we have

$$\mathcal{I} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle, \qquad \bar{\mathcal{I}} = \langle \bar{\eta}^0, \bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4 \rangle.$$

Let $V = \pi^{-1}U \cap \bar{\pi}^{-1}\bar{U}$. It is easy to see that the Cauchy characteristics of the system $\langle \pi^*\eta^0 \rangle$ (defined on V) are precisely the fibers of $\pi|_V$; similarly for $\bar{\pi}|_V$. Therefore, if $(N, \mathcal{B}; \pi, \bar{\pi})$ is nontrivial, then $\pi^*\eta^0$ and $\bar{\pi}^*\bar{\eta}^0$ must be linearly independent when restricted to each tangent space of N. In particular, it follows that, on V, the differential ideal $\mathcal{B}$ is algebraically generated by elements of $\pi^*\mathcal{I}$ and $\bar{\pi}^*\bar{\eta}^0$, which needs to be the same as the system algebraically generated by elements of $\bar{\pi}^*\bar{\mathcal{I}}$ and $\pi^*\eta^0$.

Definition 2.16. A nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems is said to be *normal* if, in the notations of Example 9, on N, the system $[\![\mathrm{d}\eta^0, \mathrm{d}\bar{\eta}^0]\!]$ has rank 2 modulo $\eta^0, \bar{\eta}^0$.

Consider two Bäcklund transformations, one relating $(M_1, \mathcal{I}_1)$ and $(M_2, \mathcal{I}_2)$, the other relating $(M_2, \mathcal{I}_2)$ and $(M_3, \mathcal{I}_3)$, as the following diagram shows.

$$\begin{array}{ccccccc} & & (N_1, \mathcal{B}_1) & & & (N_2, \mathcal{B}_2) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \swarrow^{\pi_3} & & \searrow^{\pi_4} & \\ (M_1, \mathcal{I}_1) & & & (M_2, \mathcal{I}_2) & & & (M_3, \mathcal{I}_3) \end{array}$$

The Whitney sum of the fiber bundles $\pi_2 : N_1 \to M_2$ and $\pi_3 : N_2 \to M_2$, denoted as $N_1 \oplus N_2$, admits two submersions

$$p_1 : N_1 \oplus N_2 \to N_1, \qquad p_2 : N_1 \oplus N_2 \to N_2$$

such that $\pi_2 \circ p_1 = \pi_3 \circ p_2$. Let $\mathcal{B}$ denote the differential ideal on $N_1 \oplus N_2$ algebraically generated by $p_1^* \mathcal{B}_1$ and $p_2^* \mathcal{B}_2$.

Proposition 2.4. $(N_1 \oplus N_2, \mathcal{B}; \pi_1 \circ p_1, \pi_4 \circ p_2)$ *is a Bäcklund transformation relating* $(M_1, \mathcal{I}_1)$ *and* $(M_3, \mathcal{I}_3)$.

Proof. By taking into account Proposition 2.3, it suffices to show that p_1 and p_2 are integrable extensions. Let $\mathcal{B}_1$ be algebraically generated by $\alpha_1, ..., \alpha_k$ and $\pi_2^* \mathcal{I}_2$, $\mathcal{B}_2$ by $\beta_1, ..., \beta_l$ and $\pi_3^* \mathcal{I}_2$. Hence, $\mathcal{B}$ is algebraically generated by $p_1^* \alpha_1, ..., p_1^* \alpha_k$, $p_2^* \beta_1, ..., p_2^* \beta_l$ and $p_1^* \pi_2^* \mathcal{I}_2$ (the latter being the same as $p_2^* \pi_3^* \mathcal{I}_2$). Now suppose that $\mathbf{v}$ is tangent to a fiber of p_1. By construction, $p_{2*}\mathbf{v}$ is tangent to a fiber of π_3. If $p_2^*(\beta_i)(\mathbf{v}) = 0$ for all $i = 1, ..., l$, it is necessary that $p_{2*}\mathbf{v} = 0$, because π_3 is an integrable extension. Since p_2, restricting to each fiber of p_1, is an immersion, we have $\mathbf{v} = \mathbf{0}$. This prove that p_1 is an integrable extension. The case for p_2 is similar. □

The above discussion suggests that, for exterior differential systems, being Bäcklund-related is an equivalence relation. However, it is unknown whether this remains true if one restricts to the notion of being Bäcklund-related *at a particular rank*.

We close this section with two more definitions, which will be useful later.

Definition 2.17. Let $\pi : N \to M$ be a submersion. A p-form on N that takes value in $\pi^*(\Lambda^p(T^*M))$ is said to be a π*-semi-basic* p-form.

Definition 2.18. Let M be a smooth manifold. Let $E \subset \Lambda^k(T^*M)$ be a vector subbundle, and X a smooth vector field defined on M. We say that E is *invariant under the flow of* X if, for any (smooth) local section $\omega : U \to E$, where $U \subset M$ is open, the Lie derivative $\mathcal{L}_X\omega$ remains a section of E defined on U.

Remark 2. It is easy to see that, to verify the condition in Definition 2.18, it suffices to make a choice of basis sections $\sigma_1, ..., \sigma_k : U \to E$ (assuming that E has rank k), and verify that, for each i, $\mathcal{L}_X\sigma_i$ is a linear combination of $\sigma_1, ..., \sigma_k$.

3

The Problem of Generality

In this chapter, the objects of study are nontrivial rank-1 Bäcklund transformations relating a pair of hyperbolic Monge-Ampère systems. Since many classical examples belong to this category, it is highly desirable to have a complete classification of Bäcklund transformations of this kind. In [Cle01], by establishing a G-structure associated to a Bäcklund transformation, Clelland approached the classification problem using Cartan's Method of Equivalence, restricting to the case when all local invariants of the structure are constants (a.k.a the homogeneous case). Her classification found 15 types, among which 11 are analogues of the classical Bäcklund transformation between surfaces in $\mathbb{E}^3$ with a negative constant Gauss curvature.

Since homogeneous structures, up to equivalence, only depend on constants, the following question remains to be answered: *What kind of initial data do we need to specify in order to determine a Bäcklund transformation?* In Section 3.2, in the generic case, we prove an upper bound for the magnitude of such initial data. In Section 3.3, we provide several examples of Bäcklund transformations with higher cohomogeneity, which we found by specifying only two structure invariants.

3.1 G-structure Equations for Bäcklund Transformations

According to Definition 2.12, it may appear that $(N, \mathcal{B}; \pi_1, \pi_2)$ being a Bäcklund transformation imposes conditions on all components in this quadruple. However, when it is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems, one only needs to impose conditions on the exterior differential system $(N, \mathcal{B})$, as the following proposition shows.

Proposition 3.1. ([Cle01]) *An exterior differential system $(N^6, \mathcal{B})$ is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems if and only if, for each $p \in N$, there exists an open neighborhood $V \subset N$ $(p \in V)$ and a coframing $(\theta^0, \bar{\theta}^0, \theta^1, ..., \theta^4)$, defined on V, satisfying the conditions:*

(1) *the differential ideal $\mathcal{B} = \langle \theta^0, \bar{\theta}^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle_{\text{alg}}$;*

(2) *the vector bundles $E_0 = [\![\theta^0]\!]$, $E_1 = [\![\theta^0, \theta^1, \theta^2]\!]$ and $E_2 = [\![\theta^0, \theta^3, \theta^4]\!]$ are invariant along the flow of X (see Definition 2.18), where X is a nonvanishing vector field on V that annihilates $\theta^0, \theta^1, ..., \theta^4$;*

(2)′ *the vector bundles $\bar{E}_0 = [\![\bar{\theta}^0]\!]$, $\bar{E}_1 = [\![\bar{\theta}^0, \theta^1, \theta^2]\!]$ and $\bar{E}_2 = [\![\bar{\theta}^0, \theta^3, \theta^4]\!]$ are invariant along the flow of $\bar{X}$, where $\bar{X}$ is a nonvanishing vector field on V that annihilates $\bar{\theta}^0, \theta^1, ..., \theta^4$;*

(3) *for some nonvanishing functions $A_1, ..., A_4$ defined on V,*

$$\mathrm{d}\theta^0 \equiv A_1\theta^1 \wedge \theta^2 + A_2\theta^3 \wedge \theta^4 \mod \theta^0,$$

$$\mathrm{d}\bar{\theta}^0 \equiv A_3\theta^1 \wedge \theta^2 + A_4\theta^3 \wedge \theta^4 \mod \bar{\theta}^0.$$

Proof. In one direction, assume that $(N, \mathcal{B}; \pi, \bar{\pi})$ is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$. Maintaining the notation in Example 9 of Chapter 2, and dropping the pull-back symbol when there is no confusion, we have that each $p \in N$ has an open neighborhood V such that, on V,

$$\mathcal{B} = \langle \eta^0, \bar{\eta}^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\text{alg}} = \langle \eta^0, \bar{\eta}^0, \bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4 \rangle_{\text{alg}}.$$

Thus, there exist nonvanishing functions A_i $(i = 1, ..., 4)$ defined on V such that

$$\begin{aligned} \mathrm{d}\eta^0 &\equiv A_1 \eta^1 \wedge \eta^2 + A_2 \eta^3 \wedge \eta^4 \quad \mod \eta^0, \\ \mathrm{d}\bar{\eta}^0 &\equiv A_3 \eta^1 \wedge \eta^2 + A_4 \eta^3 \wedge \eta^4 \quad \mod \eta^0, \bar{\eta}^0. \end{aligned} \tag{3.1}$$

By adding appropriate multiples of η^0 to $\eta^1, ..., \eta^4$, we can put (3.1) in the form

$$\mathrm{d}\bar{\eta}^0 \equiv A_3 \eta^1 \wedge \eta^2 + A_4 \eta^3 \wedge \eta^4 \quad \mod \bar{\eta}^0. \tag{3.2}$$

It is easy to see that the resulting coframing $(\eta^0, \bar{\eta}^0, \eta^1, ..., \eta^4)$ satisfies the conditions (1) and (3). Next, we show that it also satisfies the conditions (2) and (2)$'$.

Using the congruence (3.2), it is easy to see that the *Cartan system*[1]

$$C(\langle \bar{\eta}^0 \rangle) = \langle \bar{\eta}^0, \eta^1, ..., \eta^4 \rangle \subset \Omega^*(V).$$

As a result, $\bar{\pi}^* \bar{\mathcal{I}}$ is included in the intersection

$$C(\langle \bar{\eta}^0 \rangle) \cap \mathcal{B} = \langle \bar{\eta}^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\text{alg}},$$

Consequently, by switching $(\bar{\eta}^1, \bar{\eta}^2)$ and $(\bar{\eta}^3, \bar{\eta}^4)$ if needed, we have the following relations of rank-3 vector bundles over V:

$$[\![\bar{\eta}^0, \eta^1, \eta^2]\!] = [\![\bar{\eta}^0, \bar{\eta}^1, \bar{\eta}^2]\!], \tag{3.3}$$

$$[\![\bar{\eta}^0, \eta^3, \eta^4]\!] = [\![\bar{\eta}^0, \bar{\eta}^3, \bar{\eta}^4]\!]. \tag{3.4}$$

Let $\bar{X}$ be a vector field on V annihilated by $\bar{\eta}^0, \bar{\eta}^1, ..., \bar{\eta}^4$. By construction, the right-hand-sides of (3.3) and (3.4) are invariant under the flow of $\bar{X}$; it follows that the same holds for $[\![\bar{\eta}^0, \eta^1, \eta^2]\!]$ and $[\![\bar{\eta}^0, \eta^3, \eta^4]\!]$. This proves that $(\eta^0, \bar{\eta}^0, \eta^1, ..., \eta^4)$ satisfies Condition (2)$'$. The verification of Condition (2) is similar.

[1] Let $(M, \mathcal{I})$ be an exterior differential system, the *Cartan system* of $(M, \mathcal{I})$ is the Frobenius system that annihilates the Cauchy characteristic distribution of $(M, \mathcal{I})$. See [BCG+13] for details.

For the other direction, assume that $(\theta^0, \bar{\theta}^0, \theta^1, ..., \theta^4)$ is a coframing defined on an open set $V \subset N$ and satisfying the conditions (1)-(3). The quotient of V by the integral curves of $\langle \theta^0, \theta^1, ..., \theta^4 \rangle$ is a 5-manifold U. Let $f : V \to U$ be the projection. By Condition (2), there exists a system $\mathcal{I} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle$ defined on U satisfying

$$[\![f^*\eta^0]\!] = [\![\theta^0]\!], \quad [\![f^*\eta^0, f^*\eta^1, f^*\eta^2]\!] = [\![\theta^0, \theta^1, \theta^2]\!], \quad [\![f^*\eta^0, f^*\eta^3, f^*\eta^4]\!] = [\![\theta^0, \theta^3, \theta^4]\!].$$

It is then easy to show, using Conditions (1) and (3), that $\mathcal{I} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\text{alg}}$ is a hyperbolic Monge-Ampère ideal. A similar argument applies to the quotient of V by the integral curves of $\langle \bar{\theta}^0, \theta^1, ..., \theta^4 \rangle$. It follows that $(N, \mathcal{B})$ is a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. □

Corollary 3.1. *Let* $(N^6, \mathcal{B}; \pi_1, \pi_2)$ *be a nontrivial rank*-1 *Bäcklund transformation relating two hyperbolic Monge-Ampère systems. A coframing defined on an open subset* $V \subset N$ *that satisfies Conditions* (1)-(3) *in Proposition 3.1 can always be arranged to satisfy the extra condition:* $A_2 = A_3 = 1$.

Proof. This is obtained by scaling θ^0 and $\bar{\theta}^0$. □

Definition 3.1. A coframing as concluded in Corollary 3.1 is said to be 0-*adapted* to the Bäcklund transformation $(N, \mathcal{B})$.

Given a nontrivial rank-1 Bäcklund transformation $(N, \mathcal{B}; \pi_1, \pi_2)$, one may wonder whether its 0-adapted coframings are precisely the local sections of a G-structure over N. However, this is not true. For example, consider a 0-adapted coframing $(\theta^0, \bar{\theta}^0, \theta^1, ..., \theta^4)$ defined on an open subset $U \subset N$ with corresponding functions A_1, A_4. Let $T : U \to \mathrm{GL}(6, \mathbb{R})$ be the transformation

$$T : (\theta^0, \bar{\theta}^0, \theta^1, \theta^2, \theta^3, \theta^4) \mapsto (A_1^{-1}\theta^0, A_4^{-1}\bar{\theta}^0, \theta^3, \theta^4, \theta^1, \theta^2). \tag{3.5}$$

The coframing on the right-hand-side is clearly 0-adapted. However, the same transformation, when applied to a 0-adapted coframing with corresponding functions

A_1', A_4' that are different from A_1, A_4, may not result in a 0-adapted coframing.

One simple strategy, as taken by [Cle01], to avoid this imperfection is by, in addition to understanding the subbundles $[\![\theta^0]\!]$ and $[\![\bar{\theta}^0]\!]$ as an ordered pair, fixing an order for the pair of subbundles $[\![\theta^0, \bar{\theta}^0, \theta^1, \theta^2]\!]$ and $[\![\theta^0, \bar{\theta}^0, \theta^3, \theta^4]\!]$. Once this is considered, all local 0-adapted coframings respecting such an ordering are precisely the local sections of a G-structure, where $G \subset \mathrm{GL}(6, \mathbb{R})$ is the Lie subgroup consisting of matrices of the form

$$g = \begin{pmatrix} \det(B) & 0 & 0 & 0 \\ 0 & \det(A) & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \quad A = (a_{ij}), B = (b_{ij}) \in \mathrm{GL}(2, \mathbb{R}). \tag{3.6}$$

Now let $\mathcal{G}$ denote this G-structure on N. Let $\omega = (\omega^1, \omega^2, ..., \omega^6)$ be the tautological 1-form on $\mathcal{G}$. Let $\mathfrak{g}$ be the Lie algebra of G. Using the conditions in Proposition 3.1 and the reproducing property, one can show that ω satisfies the following structure equations, recorded from [Cle01] with a slight change of notation:

$$\mathrm{d}\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} = -\begin{pmatrix} \beta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_0 - \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 0 & \beta_3 & \beta_0 - \beta_1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} \tag{3.7}$$

$$+ \begin{pmatrix} A_1(\omega^3 - C_1\omega^1) \wedge (\omega^4 - C_2\omega^1) + \omega^5 \wedge \omega^6 \\ \omega^3 \wedge \omega^4 + A_4(\omega^5 - C_3\omega^2) \wedge (\omega^6 - C_4\omega^2) \\ B_1\omega^1 \wedge \omega^2 + C_1\omega^5 \wedge \omega^6 \\ B_2\omega^1 \wedge \omega^2 + C_2\omega^5 \wedge \omega^6 \\ B_3\omega^1 \wedge \omega^2 + C_3\omega^3 \wedge \omega^4 \\ B_4\omega^1 \wedge \omega^2 + C_4\omega^3 \wedge \omega^4 \end{pmatrix},$$

where the matrix in α and β is a $\mathfrak{g}$-valued 1-form, called a *pseudo-connection* of $\mathcal{G}$; the second term on the right-hand-side is called the *intrinsic torsion* of $\mathcal{G}$.

It is easy to see that the intrinsic torsion above, as a map defined on $\mathcal{G}$, takes value in a 10-dimensional representation of G and is G-equivariant. It is proved in

[Cle01] that this representation decomposes into 6 irreducible components, as shown by the following equations, where $u \in \mathcal{G}$, and g is as in (3.6):

$$A_1(u \cdot g) = \frac{\det(A)}{\det(B)} A_1(u), \quad A_4(u \cdot g) = \frac{\det(B)}{\det(A)} A_4(u), \tag{3.8}$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u \cdot g) = \det(AB) A^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u), \quad \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u \cdot g) = \det(AB) B^{-1} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u),$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(u \cdot g) = \det(B) A^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(u), \quad \begin{pmatrix} C_3 \\ C_4 \end{pmatrix}(u \cdot g) = \det(A) B^{-1} \begin{pmatrix} C_3 \\ C_4 \end{pmatrix}(u).$$

Definition 3.2. Let G and $\mathcal{G}$ be as above. The Bäcklund transformation[2] corresponding to $\mathcal{G}$ is said to be *generic* if, at each point $u \in \mathcal{G}$, the intrinsic torsion takes value in a G-orbit with the largest possible dimension.

3.2 An Estimate of Generality

In this section, we address the problem of generality for *generic* rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. The main recipe are a *G-structure reduction procedure* described in [Gar89] and a *theorem of Cartan* described in [Bry14].

Lemma 3.2. *Let $(N, \mathcal{B})$ be a nontrivial rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. Let $\mathcal{G}$ be the associated G-structure. If $(N, \mathcal{B})$ is generic, then, at each $u \in \mathcal{G}$, the intrinsic torsion takes value in an 8-dimensional G-orbit.*

Proof. Let $W_1 := \mathrm{span}((B_1, B_2), (C_1, C_2))$ and $W_2 := \mathrm{span}((B_3, B_4), (C_3, C_4))$ at each point $u \in \mathcal{G}$. By (3.8), the function $A_1 A_4$ and the dimensions of W_1 and W_2 are all invariant under the G-action. Let T denote the intrinsic torsion of $\mathcal{G}$. We claim that, for each $u \in \mathcal{G}$, the G-orbit of $T(u)$ is at most 8-dimensional and that

[2] To be precise, this is a Bäcklund transformation with an ordered pair of characteristic systems.

this occurs precisely when W_1 and W_2 are both 2-dimensional. To see why this is true, first note that if one of W_i $(i = 1, 2)$ has dimension less than 2 at $u \in \mathcal{G}$, then the dimension of $u \cdot G$ is at most 7-dimensional. If both W_1, W_2 have dimension 2 at $u \in \mathcal{G}$, then it is easy to show that there exists a unique $g \in G$ such that, at $u' = u \cdot g$,

$$\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} B_3 & C_3 \\ B_4 & C_4 \end{pmatrix} = \begin{pmatrix} \epsilon_2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

where $\epsilon_i = \pm 1$ $(i = 1, 2)$. This completes the proof. □

Let $(N, \mathcal{B})$ be a generic rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems. Following from Lemma 3.2, each point $p \in N$ has a connected open neighborhood $U \subset N$ on which a canonical coframing $(\omega^1, \omega^2, ..., \omega^6)$ can be determined. Such a coframing satisfies the equation (3.7), where all differential forms are defined on U instead of $\mathcal{G}$, and the equations (3.9), where the sign of each ϵ_i is determined. This motivates the following

Definition 3.3. Let N be a 6-manifold. A coframing $(\omega^1, \omega^2, ..., \omega^6)$ defined on an open subset $U \subset N$ is said to be 1-*adapted* (to a generic rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems) if there exist 1-forms α_i, β_i $(i = 0, ..., 3)$ and two functions A_1, A_4 defined on U such that the equations (3.7) and (3.9) are satisfied.

Theorem 3.3. *Let N be a 6-manifold. For each $p \in N$, a 1-adapted coframing (Definition 3.3) defined on a small open neighborhood $U \subset N$ of p can be uniquely determined, up to diffeomorphism, by specifying at most 6 functions of 3 variables.*

Proof. Let $U \subset N^6$ be a sufficiently small connected open subset. Suppose that $\omega = (\omega^1, \omega^2, ..., \omega^6)$ is a 1-adapted coframing on U in the sense of Definition 3.3. It follows that there exist functions P_{ij} $(i = 0, ..., 7;\ j = 1, ..., 6)$ defined on U such that ω satisfies (3.7) and (3.9) with

$$\alpha_i = P_{ij}\omega^j, \quad \beta_i = P_{i+4,j}\omega^j \qquad (i = 0, ..., 3;\ j = 1, ..., 6).$$

There is a standard method to determine the generality of such a coframing ω up to diffeomorphism (see [Bry14]). Our application of such a method involves three major steps.

Step 1. By applying $\mathrm{d}^2 = 0$ to (3.7), we find that P_{ij} are related among themselves and with the coefficients of their exterior derivatives. Repeating this, at a point, no new relations among the P_{ij} arise.

More explicitly, we can choose s expressions a^α ($\alpha = 1, ..., s$) from P_{ij}, find r expressions b^ρ ($\rho = 1, ..., r$), real analytic functions $F_i^\alpha : \mathbb{R}^{r+s} \to \mathbb{R}$ and $C^i_{jk} : \mathbb{R}^r \to \mathbb{R}$ satisfying $C^i_{jk} + C^i_{kj} = 0$, such that

(**A**) the equation (3.7), in general, takes the form

$$\mathrm{d}\omega^i = -\frac{1}{2} C^i_{jk}(a)\omega^j \wedge \omega^k; \tag{3.10}$$

(**B**) $\mathrm{d}a^\alpha$, in general, takes the form:

$$da^\alpha = F_i^\alpha(a, b)\omega^i; \tag{3.11}$$

moreover, applying $\mathrm{d}^2 = 0$ to (3.10) yields identities when we take into account both (3.10) and (3.11);

(**C**) there exist functions $G_j^\rho : \mathbb{R}^{r+s} \to \mathbb{R}$ such that applying $\mathrm{d}^2 = 0$ to (3.11) yields identities when we replace $\mathrm{d}b^\rho$ by $G_j^\rho \omega^j$ and take into account (3.10) and (3.11).

Step 2. For the *tableau of free derivatives* associated to (F_i^α), which is a subspace of $\mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^s)$ defined at each point of $\mathbb{R}^{r+s}$, compute its *Cartan characters* (an array of 6 integers $(s_1, s_2, ..., s_6)$) and the dimension δ of its *first prolongation*. For details, see [Bry14]. Moreover, in our case, we verify that $\sum_{i=1}^6 s_i = r$.

Step 3. Restricting to a domain $V \subset \mathbb{R}^{r+s}$ where the Cartan characters are constants, compare $s := \sum_{j=1}^6 js_j$ with δ. By *Cartan's inequality*, there are two possibilities: either $s = \delta$ (called the *involutive case*) or $s > \delta$.

In the involutive case, one can conclude that (see Theorem 3 of [Bry14]):

For any $(a_0, b_0) \in \mathbb{R}^{s+r}$ *there exists a coframing* ω *and functions* $a = (a^\alpha), b = (b^\rho)$ *defined on an open neighborhood of* $0 \in \mathbb{R}^6$ *that satisfy* (3.10)*,* (3.11) *and* $(a(0), b(0)) = (a_0, b_0)$*. Moreover, locally, such a coframing can be uniquely determined up to diffeomorphism by specifying* s_k *functions of* k *variables, where* s_k *is the last nonzero Cartan character.*

In the non-involutive case, which is the case we encounter, a natual step to take is to *prolong* (see [Bry14]) the system by introducing the derivatives of b^ρ, carry out similar steps as the above, and obtain new tableaux of free derivatives with Cartan characters $(\sigma_1, \sigma_2, ..., \sigma_6)$.

In practice, however, we do not actually prolong, for it is easy to show that, if s_k is the last nonzero character in $(s_1, ..., s_6)$, then $\sigma_j = 0$ $(j > k)$ and $\sigma_k \leqslant s_k$. Using this and the *Cartan-Kuranishi Theorem* ([BCG⁺13]), one can already conclude that the 'generality' of 1-adapted coframings is *bounded from above* by s_k functions of k variables. (In our case, $k = 3$ and $s_k = 6$.)

For the details of carrying out the steps above, see Appendix A. Most of our calculations are performed using Maple™. □

Remark 3. Clearly, two 1-adapted coframings that are equivalent under a diffeomorphism correspond to equivalent Bäcklund transformations. Because of this, the upper bound for the 'generality' of 1-adapted coframings in Theorem 3.3 applies to the 'generality' of generic rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Corollary 3.4. *There exist hyperbolic Monge-Ampère systems that are not related to any other hyperbolic Monge-Ampère system by a generic rank-*1 *Bäcklund transformation.*

Proof. One can apply the same method used the the proof of Theorem 3.3 to find the generality of *hyperbolic Euler-Lagrange systems* (see [BGG03]), which are hyperbolic Monge-Ampère systems. We can show that, locally, a hyperbolic Euler-Lagrange system can be determined uniquely by specifying 1 function of 5 variables.[3]

Note that a generic rank-1 Bäcklund transformation in consideration completely determines the two underlying hyperbolic Monge-Ampère systems (up to equivalence). The conclusion follows. □

3.3 Examples of Higher Cohomogeneity

Following the discussion in the previous section, let $U \subset N^6$ be a sufficiently small connected open subset. Let ω be a 1-adapted coframing defined on U in the sense of Definition 3.3. One can ask, *when we specify several structure invariants, can we classify the corresponding Bäcklund transformations, if any?*

In the rest of this section, we consider the case when $\epsilon_1 = \epsilon_2 = 1$ in (3.9), and A_1 and A_4 (or P_{81} and P_{84} in the new notation) in (3.7) are specified to be $A_1 = 1$ and $A_4 = -1$.

The following procedure is similar to that in Appendix A. All calculations below are performed with Maple™.

First, all coefficients in (3.7) are expressed in terms of the remaining 40 P_{ij}. Defining their derivatives P_{ijk} and applying the identity $\mathrm{d}^2 = 0$ to (3.7), we obtain a system of 106 polynomial equations for P_{ij} and P_{ijk}, which implies

$$\begin{aligned}
&P_{01} = P_{41}, \quad P_{02} = P_{42}, \quad P_{04} = P_{44}, \quad P_{06} = P_{46}, \quad P_{11} = 0, \quad P_{12} = 0,\\
&P_{16} = 2P_{46}, \quad P_{21} = -1, \quad P_{22} = -1, \quad P_{23} = 0, \quad P_{35} = 0, \quad P_{36} = -1,\\
&P_{51} = 0, \quad P_{52} = 0, \quad P_{54} = 2P_{44}, \quad P_{61} = 1, \quad P_{62} = -1, \quad P_{65} = 0,\\
&P_{73} = 0, \quad P_{74} = -1.
\end{aligned}$$

Using these relations and repeating the steps above, we obtain a system of 88

[3] This is not surprising, as, in our case, a Lagrangian is a function depending on 5 variables.

equations, which implies

$$P_{25} = -2(P_{41} + P_{44} - P_{42}), \quad P_{26} = P_{64}, \quad P_{63} = -2(P_{41} + P_{42} + P_{46}).$$

Using these and repeating, we obtain a system of 86 equations for the 17 P_{ij} remaining and 80 of their 102 derivatives. This system implies

$$P_{31} = P_{32}, \quad P_{41} = -P_{44} - P_{46}, \quad P_{42} = P_{44} - P_{46}, \quad P_{71} = -P_{72}.$$

Using these and repeating, we obtain a system of 85 equations for the 13 P_{ij} remaining and 64 of their derivatives. This system implies

$$P_{03} = -1, \quad P_{05} = 0, \quad P_{15} = 0, \quad P_{32} = P_{72}, \quad P_{34} = -1,$$
$$P_{43} = 0, \quad P_{45} = 1, \quad P_{53} = 0, \quad P_{76} = 1.$$

Using these and repeating, we obtain a system of 61 equations for $P_{72}, P_{44}, P_{46}, P_{64}$ and 22 of their derivatives. Solving this system leads to the two cases below.

Case 1: $P_{72} \neq 0$.

In this case, we have

$$P_{44} = 0, \quad P_{46} = 0.$$

Using these and applying $\mathrm{d}^2 = 0$ to the structure equations, we find

$$P_{64} = \frac{1}{P_{72}}.$$

Using this and applying $\mathrm{d}^2 = 0$ to the structure equations again, we find

$$\mathrm{d}(P_{72}) = 0.$$

It follows that the only primary invariant remaining, P_{72}, is a nonzero constant. The structure equations read

$$\begin{aligned}
\mathrm{d}\omega^1 &= \omega^1 \wedge (\omega^3 + \omega^5) + \omega^3 \wedge \omega^4 + \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^2 &= -\omega^2 \wedge (\omega^3 + \omega^5) + \omega^3 \wedge \omega^4 - \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^3 &= (\omega^1 + \omega^2 - P_{72}^{-1}\omega^6) \wedge \omega^4 + \omega^1 \wedge \omega^2,\\
\mathrm{d}\omega^4 &= -(P_{72}\omega^1 + P_{72}\omega^2 - \omega^6) \wedge \omega^3 + \omega^5 \wedge \omega^6,\\
\mathrm{d}\omega^5 &= -(\omega^1 - \omega^2 + P_{72}^{-1}\omega^4) \wedge \omega^6 + \omega^1 \wedge \omega^2,\\
\mathrm{d}\omega^6 &= (P_{72}\omega^1 - P_{72}\omega^2 + \omega^4) \wedge \omega^5 + \omega^3 \wedge \omega^4.
\end{aligned}$$

This, after the transformation

$$(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6) \mapsto (\sqrt{|P_{72}|}\omega^1, \sqrt{|P_{72}|}\omega^2, \omega^3, \sqrt{|P_{72}|}\omega^4, \omega^5, \sqrt{|P_{72}|}\omega^6),$$

can be readily seen to belong to the Case 3D in Clelland's classification ([Cle01]). According to [Cle01], if $P_{72} < 0$, then $(N, \mathcal{B})$ is a homogeneous Bäcklund transformation relating time-like surfaces of constant mean curvature $H = -P_{72}(P_{72}^2 + 1)^{-\frac{1}{2}}$ in $\mathbb{H}^{2,1}$; if $P_{72} > 0$, then $(N, \mathcal{B})$ is a homogeneous Bäcklund transformation relating certain surfaces in a 5-dimensional quotient space of $\mathrm{SO}^*(4)$.

Case 2: $P_{72} = 0$.

In this case, all coefficients in (3.7) are expressed in terms of P_{44}, P_{46}, and P_{64}. Applying $\mathrm{d}^2 = 0$ to the structure equations, no new relations between P_{44}, P_{46} and P_{64} arise. Furthermore, 16 of the 18 derivatives of P_{44}, P_{46} and P_{64} are expressed in terms of these three invariants; two derivatives, P_{644} and P_{646}, are free.

It is easy to check that Theorem 3 in [Bry14] applies to ω, the expressions $a = (P_{44}, P_{46}, P_{64})$, $b = (P_{644}, P_{646})$, and the functions C^i_{jk} and F^α_i determined during the calculation above. The corresponding *tableaux of free derivatives* is *involutive* with Cartan characters $(1, 1, 0, 0, 0, 0)$. We have thus proved the

Proposition 3.2. *Locally, a generic rank-*1 *Bäcklund transformation* $(N, \mathcal{B})$ *relating two hyperbolic Monge-Ampère systems with its* 1*-adapted coframing satisfying* $\epsilon_1 =$

$\epsilon_2 = 1$ *and* $A_1 = -A_4 = 1$ *can be uniquely determined by specifying* 1 *function of* 2 *variables.*

We can study Case 2 in greater detail. For convenience, we introduce the following new notation:

$$\begin{gathered} R := P_{44} + P_{46}, \quad S := P_{44} - P_{46}, \quad T := P_{64}; \\ T_4 := P_{644}, \quad T_6 := P_{646}. \end{gathered} \tag{3.12}$$

In the new notation, ω satisfies (3.7) and (3.9) where

$$\begin{aligned} &\alpha_0 = -R\omega^1 + S\omega^2 - \omega^3 + \tfrac{1}{2}(R+S)\omega^4 + \tfrac{1}{2}(R-S)\omega^6, \\ &\beta_0 = -R\omega^1 + S\omega^2 + \tfrac{1}{2}(R+S)\omega^4 + \omega^5 + \tfrac{1}{2}(R-S)\omega^6, \\ &\alpha_1 = (R-S)\omega^6, \qquad \alpha_2 = (R+S)\omega^5 + T\omega^6 - \omega^1 - \omega^2, \\ &\alpha_3 = -\omega^4 - \omega^6, \qquad \beta_1 = (R+S)\omega^4, \\ &\beta_2 = (R-S)\omega^3 + T\omega^4 + \omega^1 - \omega^2, \qquad \beta_3 = -\omega^4 + \omega^6, \end{aligned} \tag{3.13}$$

and $\epsilon_1 = \epsilon_2 = 1$, $A_1 = -A_4 = 1$.

Moreover, the exterior derivatives of R, S and T are

$$\begin{aligned} \mathrm{d}R &= -R^2\omega^1 + (RS-1)\omega^2 + R\omega^3 + \tfrac{1}{2}(R^2 + RS - 1)\omega^4 \\ &\quad + R\omega^5 + \tfrac{1}{2}(R^2 - RS + 1)\omega^6, \\ \mathrm{d}S &= -(RS-1)\omega^1 + S^2\omega^2 - S\omega^3 + \tfrac{1}{2}(S^2 + SR - 1)\omega^4 \\ &\quad - S\omega^5 + \tfrac{1}{2}(SR - S^2 - 1)\omega^6, \\ \mathrm{d}T &= 2(-RT + S)\omega^1 + 2(ST - R)\omega^2 + 2(R^2 - S^2)(\omega^3 + \omega^5) \\ &\quad + T_4\omega^4 + T_6\omega^6. \end{aligned} \tag{3.14}$$

Using these equations, we can study the symmetry of the corresponding Bäcklund transformation $(N, \mathcal{B})$. Let $U \subset N$ be the domain of a 1-adapted coframing ω. Let the map $\Phi : U \to \mathbb{R}^3$ be defined by

$$\Phi(p) = (R(p), S(p), T(p)).$$

Lemma 3.5. *The map Φ can never have rank 0. Moreover, it has*

- *rank 1 if and only if $2RS = 1$ and $T = R^2 + S^2$;*
- *rank 2 if and only if it does not have rank 1 and satisfies either (1) $2RS = 1$ or (2) $T_4 = (R+S)(T-1)$ and $T_6 = (R-S)(T+1)$;*
- *rank 3 if and only if it does not have rank 1 or 2.*

Proof. In $\mathrm{d}R \wedge \mathrm{d}S \wedge \mathrm{d}T$, the coefficients of $\omega^i \wedge \omega^j \wedge \omega^k$ are polynomials in R, S, T, T_4 and T_6. These coefficients have the common factor $2RS - 1$. Calculating with Maple™, we find that the coefficients of $\omega^i \wedge \omega^j \wedge \omega^k$ in $(2RS-1)^{-1}\mathrm{d}R \wedge \mathrm{d}S \wedge \mathrm{d}T$ all vanish if and only if $T_4 = (R+S)(T-1)$ and $T_6 = (R-S)(T+1)$. This justifies the conditions for having rank 2. The condition for rank 1 can be obtained by setting the coefficients of $\omega^i \wedge \omega^j$ in $\mathrm{d}R \wedge \mathrm{d}S$, $\mathrm{d}R \wedge \mathrm{d}T$ and $\mathrm{d}S \wedge \mathrm{d}T$ to be all zero. By (3.14), it is clear that $\mathrm{d}R \neq 0$ everywhere; hence, Φ cannot have rank 0. $\square$

Definition 3.4. In the current case, if the corresponding Bäcklund transformation $(U, \mathcal{B})$ has a symmetry whose orbits are of dimension $6 - k$, then it is said to have *cohomogeneity k*.

Case of cohomogeneity-1. This occurs precisely when $\mathrm{rank}(\Phi) = 1$. In this case, locally Φ is a submersion to either branch of the curve in $\mathbb{R}^3$ defined by $2RS = 1$ and $T = R^2 + S^2$. Expressing S and T in terms of R, we have, on $U \subset N$,

$$\mathrm{d}R = -R^2\omega^1 - \frac{1}{2}\omega^2 + R(\omega^3 + \omega^5) + \frac{1}{4}(2R^2 - 1)\omega^4 + \frac{1}{4}(2R^2 + 1)\omega^6. \tag{3.15}$$

It is clear that $\mathrm{d}R$ is nowhere vanishing. Since R is the only invariant, each constant value of R defines a 5-dimensional submanifold $N_R \subset N$, which has a Lie group structure. The Lie group structure can be determined by setting the right-hand-side of the equation (3.15) to be zero, obtaining, say,

$$\omega^1 = -\frac{1}{2R^2}\omega^2 + \frac{1}{R}\omega^3 + \frac{2R^2 - 1}{4R^2}\omega^4 + \frac{1}{R}\omega^5 + \frac{2R^2 + 1}{4R^2}\omega^6.$$

Substituting this into the structure equations, we obtain equations of $d\omega^i$ ($i = 2, ..., 6$), expressed in terms of $\omega^2, ..., \omega^6$ alone. These are the structure equations on each N_R. Let $X_1, X_2, ..., X_5$ be the vector fields tangent to N_R and dual to $\omega^2, ..., \omega^6$, such that $\omega^i(X_j) = \delta^i_{j+1}$ ($i - 1, j = 1, 2, ..., 5$). We obtain the Lie bracket relations:

$$[X_1, X_2] = 2X_1 + \frac{1}{R}(X_2 + X_4), \quad [X_1, X_3] = -\frac{1}{2R}X_1 - \frac{(2R^2 - 1)}{4R^2}(X_2 - X_4) + \frac{1}{R}X_3,$$

$$[X_1, X_4] = 2X_1 + \frac{1}{R}(X_2 + X_4), \quad [X_1, X_5] = \frac{1}{2R}X_1 + \frac{2R^2 + 1}{4R^2}(X_2 - X_4) + \frac{1}{R}X_5,$$

$$[X_2, X_3] = -X_1 - \frac{1}{R}X_2 - X_3 - X_5, \qquad [X_2, X_4] = 0,$$

$$[X_2, X_5] = -\frac{2R^2 - 1}{2R}X_2 + X_3 + \frac{2R^2 + 1}{2R}X_4 - X_5,$$

$$[X_3, X_4] = -\frac{2R^2 - 1}{2R}X_2 + X_3 + \frac{2R^2 + 1}{2R}X_4 - X_5,$$

$$[X_3, X_5] = \left(-R^2 + \frac{1}{2}\right) X_2 + RX_3 + \left(R^2 + \frac{1}{2}\right) X_4 - RX_5,$$

$$[X_4, X_5] = X_1 - X_3 + \frac{1}{R}X_4 - X_5.$$

Using these relations, it can be verified that X_i ($i = 1, ..., 5$) generate a 5-dimensional Lie algebra that is solvable but not nilpotent. The derived series has dimensions $(5, 3, 1, 0, ...)$. In fact, after introducing the following new basis

$$\mathbf{e}_1 = RX_1, \quad \mathbf{e}_2 = -\frac{1}{2}(X_2 - X_4), \quad \mathbf{e}_5 = \frac{1}{R}X_1 + \frac{1}{2R^2}(X_2 + X_4),$$

$$\mathbf{e}_3 = -\frac{1}{2R}X_1 - \frac{2R^2 - 1}{4R^2}(X_2 - X_4) + \frac{1}{R}X_3,$$

$$\mathbf{e}_4 = \frac{1}{2R}X_1 + \frac{2R^2 + 1}{R^2}(X_2 - X_4) + \frac{1}{R}X_5,$$

we obtain the Lie bracket relations:

$$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_3, \quad [\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_4, \qquad [\mathbf{e}_1, \mathbf{e}_5] = 2\mathbf{e}_5,$$

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_4, \quad [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_3, \quad [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_5,$$

where all $[\mathbf{e}_i, \mathbf{e}_j]$ not on this list are zero. An equivalent way of writing these relations is:

$$[\mathbf{e}_1 + i\mathbf{e}_2, \mathbf{e}_3 + i\mathbf{e}_4] = 2(\mathbf{e}_3 + i\mathbf{e}_4), \quad [\mathbf{e}_1 - i\mathbf{e}_2, \mathbf{e}_3 + i\mathbf{e}_4] = 0, \quad [\mathbf{e}_1 + i\mathbf{e}_2, \mathbf{e}_5] = 2\mathbf{e}_5,$$
$$[\mathbf{e}_3 + i\mathbf{e}_4, \mathbf{e}_3 - i\mathbf{e}_4] = -2i\mathbf{e}_5, \quad [\mathbf{e}_1 + i\mathbf{e}_2, \mathbf{e}_1 - i\mathbf{e}_2] = 0, \quad [\mathbf{e}_3 + i\mathbf{e}_4, \mathbf{e}_5] = 0.$$

It is then not hard to see that the Lie algebra $\bigoplus_{i=1}^5 \mathbb{R}\mathbf{e}_i$ is isomorphic to the Lie algebra generated by the real and imaginary parts of the vector fields

$$\partial_w, \quad e^{2w}(\partial_z + i\bar{z}\partial_\lambda), \quad e^{2(w+\bar{w})}\partial_\lambda$$

on $\mathbb{R} \times \mathbb{C}^2$ with coordinates $(\lambda; z, w)$. In fact, an isomorphism is induced by the correspondence

$$\mathbf{e}_1 + i\mathbf{e}_2 \mapsto \partial_w, \quad \mathbf{e}_3 + i\mathbf{e}_4 \mapsto e^{2w}(\partial_z + i\bar{z}\partial_\lambda), \quad \mathbf{e}_5 \mapsto e^{2(w+\bar{w})}\partial_\lambda.$$

Next, we describe the hyperbolic Monge-Ampère systems related by the Bäcklund transformation being considered. In particular, we prove the

Proposition 3.3. *A Bäcklund transformation in the current (cohomogeneity-*1*) case is an auto-Bäcklund transformation of a homogeneous Euler-Lagrange system.*

Proof. This proof is in two parts. First, we show that the underlying two hyperbolic Monge-Ampère systems are equivalent and are homogeneous. Second, we verify that they are hyperbolic *Euler-Lagrange systems* in the sense of [BGG03] by computing their local invariants.

Using the structure equations on $U \subset N$, if we let $(\theta^0, \theta^1, ..., \theta^4)$ be either

$$\left(S\omega^1, -R(\omega^1 - \omega^4) + \omega^3, S\omega^4, -R(\omega^1 - \omega^6) + \omega^5, S\omega^6\right) \tag{3.16}$$

or

$$\left(-R\omega^2, S(\omega^2 + \omega^4) - \omega^3, R\omega^4, S(\omega^2 - \omega^6) - \omega^5, -R\omega^6\right), \tag{3.17}$$

then one can verify (with Maple$^{\text{TM}}$) that θ^i $(i = 0, ..., 4)$, in both cases, satisfy the same structure equations

$$\begin{aligned}
\mathrm{d}\theta^0 &= -2(\theta^1 + \theta^3) \wedge \theta^0 + \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^1 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3, \\
\mathrm{d}\theta^2 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= \quad \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^3, \\
\mathrm{d}\theta^4 &= -\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4.
\end{aligned} \tag{3.18}$$

It is easy to verify that (3.18) are the structure equations on a 5-manifold M with a hyperbolic Monge-Ampère ideal $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$. It follows that the expressions (3.16) and (3.17) correspond to the pull-back of θ^i under two distinct submersions $\pi_1, \pi_2 : U \to M$. It is easy to see that $(U, \mathcal{B}; \pi_1, \pi_2)$ is an auto-Bäcklund transformation of the system $(M, \mathcal{I})$. Moreover, $(M, \mathcal{I})$ is *homogeneous*, since all coefficients in (3.18) are constants.

Next, we verify that $(M, \mathcal{I})$ is a *hyperbolic Euler-Lagrange system.* In [BGG03], a necessary and sufficient condition for a hyperbolic Monge-Ampère system to be Euler-Lagrange is proved, which is summarized in our Section 4.1 with a slight change of notation. In particular, $(M, \mathcal{I})$ is Euler-Lagrange if and only if an invariant tensor S_2 vanishes (see Proposition 4.2).

To compute S_2, we choose a new coframing (η^i) and 1-forms (ϕ_α) below

$$(\eta^0, \eta^1, \eta^2, \eta^3, \eta^4) = (\sqrt{2}\theta^0, \sqrt{2}\theta^1, \theta^1 + \theta^2 - \theta^0, \theta^3 + \theta^4 - \theta^0, \sqrt{2}(\theta^4 - \theta^0)),$$

$$\phi_0 = \frac{1}{\sqrt{2}}\eta^1 + \eta^3 - \frac{1}{\sqrt{2}}\eta^4, \quad \phi_1 = -\sqrt{2}\eta^0 - \eta^3 - \frac{1}{\sqrt{2}}\eta^4, \quad \phi_2 = \eta^0 + \sqrt{2}\eta^3, \tag{3.19}$$

$$\phi_3 = -2\eta^0 + \eta^1 - \frac{1}{\sqrt{2}}\eta^2 - \sqrt{2}\eta^3 - \eta^4, \quad \phi_4 = \phi_0 - \phi_1, \quad \phi_5 = -\eta^3 - \frac{1}{\sqrt{2}}\eta^4,$$

$$\phi_6 = \frac{1}{\sqrt{2}}\eta^3, \qquad \phi_7 = -\eta^0 + \eta^1 - \sqrt{2}\eta^2 - \sqrt{2}\eta^3, \qquad \phi_8 = \phi_0 - \phi_5.$$

These η^i $(i = 0, 1, ..., 4)$ and ϕ_α $(\alpha = 1, ..., 8)$ are chosen such that they satisfy the structure equations (4.1), and such that S_1 and S_2 are as simple as possible. One can verify that, under this choice,

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = 0.$$

This completes the proof. □

Now, one may wonder whether the homogeneous Monge-Ampère system $(M, \mathcal{I})$ considered in Proposition 3.3 has a symmetry of dimension greater than 5. Using the *method of equivalence*, we prove that it is not the case.

Proposition 3.4. *The hyperbolic Euler-Lagrange system in Proposition 3.3 has a symmetry of dimension* 5*. In addition, any such symmetry is induced from a symmetry of the Bäcklund transformation* $(N, \mathcal{B})$*.*

Proof. Let $(M, \mathcal{I})$ denote the Euler-Lagrange system being considered. To show that $(M, \mathcal{I})$ has a 5-dimensional symmetry, it suffices to show that there is a canonical way to determine a local coframing on M. This can be achieved by applying the *method of equivalence*. For details, see Appendix B.

By (3.16) and (3.17), it is easy to see that the fibers of $\pi_i : N \to M$ $(i = 1, 2)$ are everywhere transversal to the level sets of the functions R. The second half of the statement follows. □

To end the discussion of the cohomogeneity-1 case, we integrate the structure equations (3.18) to express the corresponding hyperbolic Euler-Lagrange system in local coordinates.

Proposition 3.5. *The hyperbolic Monge-Ampère system* $(M, \mathcal{I})$ *with the differential ideal* $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$*, where* θ^i *satisfy* (3.18)*, corresponds to the following hyperbolic Monge-Ampère PDE up to contact equivalence:*

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} = 0, \tag{3.20}$$

where $A = 2z_x + y$ *and* $B = 2z_y - x$.

Proof. Let $U \subset M$ be a domain on which θ^i are defined. One can verify, using the structure equations (3.18), that the 1-forms $\theta^1 - \theta^2 - \theta^3 - \theta^4$ and $\theta^1 + \theta^3$ are closed. Hence, by shrinking U if needed, there exist functions P, Q defined on U such that

$$\mathrm{d}P = -(\theta^1 + \theta^3), \quad \mathrm{d}Q = \theta^1 - \theta^2 - \theta^3 - \theta^4.$$

Moreover, if we let $\Theta = \theta^2 + i\theta^4$, then, by a straight-forward calculation, we obtain

$$\mathrm{d}\Theta = \mathrm{d}(P + iQ) \wedge \Theta.$$

It follows that there exist functions X, Y on U such that

$$\Theta = e^{P+iQ}\mathrm{d}(X + iY).$$

Equivalently, we have

$$\theta^2 = e^P(\cos Q\mathrm{d}X - \sin Q\mathrm{d}Y), \quad \theta^4 = e^P(\sin Q\mathrm{d}X + \cos Q\mathrm{d}Y).$$

Now we can express $\theta^1, ..., \theta^4$ completely in terms of the coordinates X, Y, P, Q. By computing the exterior derivative

$$\mathrm{d}(e^{-2P}\theta^0) = e^{-2P}(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4), \tag{3.21}$$

we notice that the right-hand-side of (3.21) is a symplectic form on a 4-manifold on which $\theta^1, ..., \theta^4$ are well-defined, by (3.18). Thus, by Darboux' Theorem, there locally exist functions x, y, p, q such that the right-hand-side of (3.21) is equal to $\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q$.

In fact, in $XYPQ$-coordinates, the right-hand-side of (3.21) is equal to

$$\mathrm{d}\left(\frac{e^{-P}}{2}(\cos Q + \sin Q) + \frac{Y}{2}\right) \wedge \mathrm{d}X + \mathrm{d}\left(\frac{e^{-P}}{2}(\cos Q - \sin Q) - \frac{X}{2}\right) \wedge \mathrm{d}Y.$$

As a result, we can set

$$x = X, \quad y = Y, \quad p = -\frac{e^{-P}}{2}(\cos Q + \sin Q) - \frac{Y}{2}, \quad q = -\frac{e^{-P}}{2}(\cos Q - \sin Q) + \frac{X}{2},$$

and write

$$e^{-2P}\theta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,$$

for some function z independent of x, y, p, q. From these expressions, it is clear that $2p + y$ and $2q - x$ cannot simultaneously vanish.

Now let $A = 2p+y$ and $B = 2q+x$. We can express $\theta^1 \wedge \theta^2$ in terms of x, y, z, p, q:

$$\frac{(A^2 - B^2)\mathrm{d}p \wedge \mathrm{d}y + (A+B)^2 \mathrm{d}x \wedge \mathrm{d}p - (A^2 - B^2)\mathrm{d}x \wedge \mathrm{d}q + (A-B)^2 \mathrm{d}y \wedge \mathrm{d}q}{(A^2 + B^2)^2}.$$

Multiplying this expression by $(A^2 + B^2)^2$ then subtracting $(A^2 + B^2)(\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q)$, we obtain

$$(A^2 - B^2)(\mathrm{d}p \wedge \mathrm{d}y - \mathrm{d}x \wedge \mathrm{d}q) + 2AB(\mathrm{d}x \wedge \mathrm{d}p - \mathrm{d}q \wedge \mathrm{d}y).$$

The vanishing of this 2-form on integral surfaces implies that z must satisfy the equation (3.20). □

Case of cohomogeneity-2. By Lemma 3.5, this case can only occur when $2RS = 1$ and $R^2 + S^2 = T$ do not both hold and either (1) $2RS = 1$ or (2) $T_4 = (R+S)(T-1)$, $T_6 = (R-S)(T+1)$ holds. We will focus on the latter case.

Proposition 3.6. *When Φ has rank 2, and when $T_4 = (R+S)(T-1)$ and $T_6 = (R-S)(T+1)$, the map $\Phi : N \to \mathbb{R}^3$ has its image contained in a surface that is defined by either $\dfrac{R^2 + S^2 - T}{2RS - 1}$ or its reciprocal being a constant.*

Proof. First note that $R^2 + S^2 - T$ and $2RS - 1$ cannot be both zero, for this would reduce to the cohomogeneity-1 case; hence the conclusion has meaning. To see that this statement is true, note that, in our current case, the pull-back of $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$ via Φ to N are linearly dependent. To be precise, the 1-form

$$\theta = -2(R^2S - S^3 + ST - R)\mathrm{d}R + 2(R^3 - RS^2 - RT + 2S)\mathrm{d}S + (2RS - 1)\mathrm{d}T$$

equals to zero when pulled back to N. Since the tangent map Φ_* has rank 2, this can only happen if $\theta \wedge \mathrm{d}\theta = 0$. It follows that θ is integrable. The primitives of $(2RS-1)^{-2}\theta$ and $(R^2+S^2-T)^{-2}\theta$ are, respectively, the function $\dfrac{R^2+S^2-T}{2RS-1}$ and its reciprocal, if $2RS-1$ and R^2+S^2-T are, respectively, nonzero. This completes the proof. □

We now study the symmetry of the Bäcklund transformation $(N, \mathcal{B})$ being considered. Let X_i $(i = 1, 2, ..., 6)$ be the dual vector fields of ω^i $(i = 1, 2, ..., 6)$, defined on $U \subset N$. Using the expressions of $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$, it is easy to see that the rank-4 distribution on U annihilated by $\mathrm{d}R, \mathrm{d}S$ and $\mathrm{d}T$ is spanned by the vector fields

$$Y_1 = RX_2 + SX_1 + X_5, \qquad Y_2 = \frac{1}{2}(X_1 - X_2) + X_4,$$

$$Y_3 = X_3 - X_5, \qquad Y_4 = \frac{1}{2}(X_1 + X_2) + X_6.$$

These vector fields generate a 4-dimensional Lie algebra $\mathfrak{l}$ with

$$[Y_1, Y_2] = \frac{R+S}{2}Y_3 + Y_4, \qquad [Y_1, Y_3] = 0, \qquad [Y_1, Y_4] = -Y_2 + \frac{R-S}{2}Y_3,$$

$$[Y_2, Y_3] = (R+S)Y_3 + 2Y_4, \quad [Y_2, Y_4] = Y_1 + (R-S)Y_2 + \left(\frac{1}{2} - T\right)Y_3 - (R+S)Y_4,$$

$$[Y_3, Y_4] = 2Y_2 - (R-S)Y_3.$$

It is easy to verify that $2Y_1 + Y_3$ belongs to the center of $\mathfrak{l}$. The quotient algebra $\mathfrak{q} = \mathfrak{l}/\mathbb{R}(2Y_1 + Y_3)$, with the basis $\mathbf{e}_1 = [Y_2]$, $\mathbf{e}_2 = [Y_3]$ and $\mathbf{e}_3 = [Y_4]$, satisfies

$$[\mathbf{e}_1, \mathbf{e}_2] = (R+S)\mathbf{e}_2 + 2\mathbf{e}_3,$$

$$[\mathbf{e}_1, \mathbf{e}_3] = -T\mathbf{e}_2 - (R+S)\mathbf{e}_3,$$

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1 + (S-R)\mathbf{e}_2.$$

According to the classification of 3-dimensional Lie algebras, see Lecture 2 in [Bry95] for example, to identify the Lie algebra $\mathfrak{q}$, it suffices to find a normal form of the matrix (note that it is symmetric)

$$C = \begin{pmatrix} 2 & S-R & 0 \\ S-R & T & R+S \\ 0 & R+S & 2 \end{pmatrix}$$

under the transformation $C \mapsto \det(A^{-1})ACA^T$, where $A \in \mathrm{GL}(3,\mathbb{R})$. Note that $\det(C) = -2(R^2+S^2-T)$. We have the following

Proposition 3.7. *If $R^2+S^2 < T$, then $\mathfrak{q}$ is isomorphic to $\mathfrak{so}(3,\mathbb{R})$. If $R^2+S^2 > T$, then $\mathfrak{q}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. If $R^2+S^2 = T$, then $\mathfrak{q}$ is isomorphic to the solvable Lie algebra with a basis x_1, x_2, x_3 satisfying $[x_2,x_3] = x_1$, $[x_3,x_1] = x_2$ and $[x_1,x_2] = 0$.*

Proof. After a transformation of the form above, C can be put in the form

$$C' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & T-R^2-S^2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

According to [Bry95], the conclusion follows immediately. □

Now consider the case when $\mathfrak{q}$ is solvable, i.e., when $R^2+S^2 = T$. By the cohomogeneity-2 assumption, we must have $2RS \neq 1$. We proceed to identify the Monge-Ampère systems related by such a Bäcklund transformation.

If we let $(\theta^0, \theta^1, ..., \theta^4)$ be

$$(S\omega^1, -R(\omega^1-\omega^4)+\omega^3, S\omega^4, -R(\omega^1-\omega^6)+\omega^5), S\omega^6)$$

and let F be defined by

$$F = \frac{2RS-1}{S^2},$$

for which to have meaning we need to restrict to a domain on which $S \neq 0$, then the structure equations on N would imply

$$\begin{aligned}
\mathrm{d}\theta^0 &= \theta^0 \wedge (2\theta^1 - F\theta^2 + 2\theta^3 - F\theta^4) + \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^1 &= -F\theta^0 \wedge (\theta^2 + \theta^4) - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + (F+1)\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= -2F\theta^0 \wedge \theta^2 - \theta^1 \wedge \theta^2 - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + (1-F)\theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^4, \quad (3.22) \\
\mathrm{d}\theta^3 &= F\theta^0 \wedge (\theta^2 + \theta^4) + \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3 + (F-1)\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^4 &= -2F\theta^0 \wedge \theta^4 + \theta^1 \wedge \theta^2 - \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + (F+1)\theta^2 \wedge \theta^4 - \theta^3 \wedge \theta^4,
\end{aligned}$$

and

$$\mathrm{d}F = 2F^2(2\theta^0 - \theta^2 - \theta^4) + 2F(\theta^1 + \theta^3), \quad (F \neq 0). \tag{3.23}$$

It can be verified that, in the equations (3.22) and (3.23), the exterior derivative of the right-hand-sides are zero, by taking into account these equations themselves. By the construction of the θ^i $(i = 0, ..., 4)$, it follows that (3.22) and (3.23) are the structure equations of one of the Monge-Ampère systems being related by the Bäcklund transformation $(N, \mathcal{B})$.

Proposition 3.8. *The hyperbolic Monge-Ampère system* $(M, \mathcal{I})$ *with the differential ideal* $\mathcal{I} = \langle \theta^0, \theta^1 \wedge \theta^2, \theta^3 \wedge \theta^4 \rangle$*, where* θ^i *satisfy* (3.22) *and* (3.23)*, corresponds to the following hyperbolic Monge-Ampère PDE up to contact equivalence:*

$$(A^2 - B^2)(z_{xx} - z_{yy}) + 4ABz_{xy} + \epsilon(A^2 + B^2)^2 = 0. \tag{3.24}$$

where $A = z_x - y$*,* $B = z_y + x$*; and* $\epsilon = \pm 1$*.*

Proof. The proof is very similar to that of Proposition 3.5. First it is easy to verify that the 1-forms $F(-2\theta^0 + \theta^2 + \theta^4) - \theta^1 - \theta^3$ and $\theta^1 - \theta^2 - \theta^3 - \theta^4$ are closed. Consequently, there locally exist functions f, g such that

$$\mathrm{d}f = F(-2\theta^0 + \theta^2 + \theta^4) - \theta^1 - \theta^3, \quad \mathrm{d}g = \theta^1 - \theta^2 - \theta^3 - \theta^4.$$

Now the expression of $\mathrm{d}F$ can be written as $\mathrm{d}F = -2F\mathrm{d}f$. This implies that there exists a nonzero constant ϵ such that $F = \epsilon e^{-2f}$. Using the ambiguity in f, we can set $\epsilon = \pm 1$. In addition, if we let $\Theta = e^{-f}(\theta^2 + i\theta^4)$, it is easy to verify that Θ is integrable. To be explicit,

$$\mathrm{d}\Theta = i\ \mathrm{d}g \wedge \Theta.$$

Thus there exist functions X, Y such that $\Theta = e^{ig}(\mathrm{d}X + i\mathrm{d}Y)$. From this we have

$$\theta^2 = e^f(\cos g\ \mathrm{d}X - \sin g\ \mathrm{d}Y), \quad \theta^4 = e^f(\sin g\ \mathrm{d}X + \cos g\ \mathrm{d}Y).$$

Using these, differentiating $\theta^1 + \theta^3$ gives

$$\mathrm{d}(\theta^1 + \theta^3) = 2\epsilon\ \mathrm{d}X \wedge \mathrm{d}Y.$$

This implies that there exists a function Z, independent of X, Y, f, g, such that

$$\theta^1 + \theta^3 = \mathrm{d}Z - \epsilon(X\mathrm{d}Y - Y\mathrm{d}X).$$

Now, $\theta^0, \theta^1, ..., \theta^4$ can be completely expressed in terms of the functions X, Y, Z, f, g. In particular,

$$-2e^{-2f}\theta^0 = \epsilon(\mathrm{d}(Z+f)) + (-Y - e^{-f}(\sin g + \cos g))\mathrm{d}X + (X + e^{-f}(\sin g - \cos g))\mathrm{d}Y.$$

If we make the substitution

$$x = X, \quad y = Y, \quad z = \epsilon(Z+f), \quad p = e^{-f}(\cos g + \sin g) + Y, \quad q = e^{-f}(\cos g - \sin g) - X,$$

the contact form θ^0 is then a nonzero multiple of $\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$. The 2-form $\theta^3 \wedge \theta^4$, after replacing $\mathrm{d}z$ by $p\mathrm{d}x + q\mathrm{d}y$, can be expressed as

$$\theta^3 \wedge \theta^4 \equiv \frac{1}{8}e^{4f}\big[(A^2 - B^2)(\mathrm{d}p \wedge \mathrm{d}y - \mathrm{d}x \wedge \mathrm{d}q) + \\ (A+B)^2\mathrm{d}q \wedge \mathrm{d}y - (A-B)^2\mathrm{d}x \wedge \mathrm{d}p + \epsilon(A^2 + B^2)^2\mathrm{d}x \wedge \mathrm{d}y\big] \mod \theta^0,$$

where $A = p - y$, $B = q + x$. Note that, by construction, A, B cannot be simultaneously zero. The equation (3.24) follows. □

In the current case, there remain several obvious questions to investigate. *What is the Monge-Ampère system corresponding to $\langle\omega^2, \omega^3\wedge\omega^4, \omega^5\wedge\omega^6\rangle$? Are the Monge-Ampère systems being Bäcklund-related Euler-Lagrange? Is the Bäcklund transformation an auto-Bäcklund transformation?* Answers to these questions can be obtained in a very similar way as in the cohomogeneity-1 case. We thus have them summarized in the following remark, omitting the details of calculation.

Remark 4. **A.** Whenever $R \neq 0$,

$$(R\omega^2, S(\omega^2+\omega^4)-\omega^3, -R\omega^4, S(\omega^2-\omega^6)-\omega^5, R\omega^6)$$

form a well-defined coframing on a 5-manifold. The system $\langle\omega^2, \omega^3\wedge\omega^4, \omega^5\wedge\omega^6\rangle$ descends to correspond to the same equation (3.24) up to contact equivalence, where ϵ has the same sign as $2RS-1$. Since the function $2RS-1$ is defined on N, ϵ must be the same for the two underlying Monge-Ampère systems. The system $(N, \mathcal{B})$ is therefore an auto-Bäcklund transformation of the equation (3.24).

B. The hyperbolic Monge-Ampère system in Proposition 3.8 is Euler-Lagrange. In fact, by a transformation of the θ^i ($i = 0, ..., 4$), the structure equations (3.22) can be put in the form of (4.1) with

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = 0.$$

Applying the procedure described in Appendix B, we find the corresponding Monge-Ampère invariants

$$Q_{01}Q_{04} - Q_{02}Q_{03} = -\frac{1}{2}(1+F^2) \neq 0, \quad Q_{00} = 0.$$

Following from this and the expression of $\mathrm{d}F$, one can show (in a way that is similar to the proof of Proposition 3.4) that the underlying Euler-Lagrange system has a symmetry of dimension 4. Such a symmetry is induced from the symmetry of the Bäcklund transformation $(N, \mathcal{B})$.

4

Bäcklund Transformations and Monge-Ampère Invariants

Given two hyperbolic Monge-Ampère systems $(M_i, \mathcal{I}_i)$ $(i = 1, 2)$, suppose that we have chosen for each of them a 0-adapted coframing in the sense of Example 5 in Chapter 2. The problem of finding a nontrivial Bäcklund transformation relating these two Monge-Ampère systems is a problem of integration, that is, to find a 6-manifold $N \subset M_1 \times M_2$ with the obvious submersions $\pi_i : N \to M_i$ such that the differential ideal $\mathcal{B}$ generated by $\pi_1^*\mathcal{I}_1$ and $\pi_2^*\mathcal{I}_2$ makes $(N, \mathcal{B}; \pi_1, \pi_2)$ a Bäcklund transformation. From this point of view, it is desirable to express the obstructions to integrability in terms of the invariants of the two hyperbolic Monge-Ampère systems.

4.1 First Monge-Ampère Invariants

Let $(M, \mathcal{I})$ be a hyperbolic Monge-Ampère system. Let $\mathcal{G}_0$ be the G_0-structure on $(M, \mathcal{I})$ (Example 5, Chapter 2). In [BGG03], the reduction of $\mathcal{G}_0$ to a G_1-structure

$\mathcal{G}_1$ is performed such that the following structure equations hold on $\mathcal{G}_1$:

$$\mathrm{d}\begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} = -\begin{pmatrix}\phi_0 & 0 & 0 & 0 & 0\\ 0 & \phi_1 & \phi_2 & 0 & 0\\ 0 & \phi_3 & \phi_4 & 0 & 0\\ 0 & 0 & 0 & \phi_5 & \phi_6\\ 0 & 0 & 0 & \phi_7 & \phi_8\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} \tag{4.1}$$

$$+\begin{pmatrix}\omega^1\wedge\omega^2+\omega^3\wedge\omega^4\\ (V_1+V_5)\omega^0\wedge\omega^3+(V_2+V_6)\omega^0\wedge\omega^4\\ (V_3+V_7)\omega^0\wedge\omega^3+(V_4+V_8)\omega^0\wedge\omega^4\\ (V_8-V_4)\omega^0\wedge\omega^1+(V_2-V_6)\omega^0\wedge\omega^2\\ (V_3-V_7)\omega^0\wedge\omega^1+(V_5-V_1)\omega^0\wedge\omega^2\end{pmatrix},$$

where $\phi_0 = \phi_1 + \phi_4 = \phi_5 + \phi_8$, and $G_1 \subset G_0$ is the subgroup generated by

$$g = \begin{pmatrix} a & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & A & 0\\ \mathbf{0} & 0 & B\end{pmatrix},\ A, B \in \mathrm{GL}(2,\mathbb{R}),\ a = \det(A) = \det(B), \tag{4.2}$$

and

$$J = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 0 & I_2\\ \mathbf{0} & I_2 & 0\end{pmatrix} \in \mathrm{GL}(5,\mathbb{R}). \tag{4.3}$$

Definition 4.1. Let $(M, \mathcal{I})$ be a hyperbolic Monge-Ampère system. A 1-*adapted coframing*[1] of $(M, \mathcal{I})$ with domain $U \subset M$ is a section $\eta : U \to \mathcal{G}_1$.

Following [BGG03], we introduce the notation[2]

$$S_1 := \begin{pmatrix} V_1 & V_2\\ V_3 & V_4\end{pmatrix}, \quad S_2 := \begin{pmatrix} V_5 & V_6\\ V_7 & V_8\end{pmatrix}. \tag{4.4}$$

It is shown in [BGG03] that

Proposition 4.1. *Along each fiber of $\mathcal{G}_1$,*

$$S_i(u \cdot g) = aA^{-1}S_i(u)B, \quad (i = 1, 2) \tag{4.5}$$

[1] This is not to be confused with a 1-adapted coframing in the sense of Definition 3.3.

[2] Note that these S_i are half of those defined in [BGG03] with the same notation.

for any $g = \mathrm{diag}(a; A; B)$ *in the identity component of* G_1*. Moreover,*

$$S_1(u \cdot J) = \begin{pmatrix} -V_4 & V_2 \\ V_3 & -V_1 \end{pmatrix}, \quad S_2(u \cdot J) = \begin{pmatrix} V_8 & -V_6 \\ -V_7 & V_5 \end{pmatrix}. \tag{4.6}$$

Proposition 4.1 has a simple interpretation: the matrices S_1 and S_2 correspond to two invariant tensors under the G_1-action. More explicitly, one can verify that the quadratic form

$$\Sigma_1 := V_3\ \omega^1\omega^3 - V_1\ \omega^1\omega^4 + V_4\ \omega^2\omega^3 - V_2\ \omega^2\omega^4 \tag{4.7}$$

and the 2-form

$$\Sigma_2 := V_7\ \omega^1 \wedge \omega^3 - V_5\ \omega^2 \wedge \omega^3 + V_8\ \omega^1 \wedge \omega^4 - V_6\ \omega^2 \wedge \omega^4 \tag{4.8}$$

are G_1-invariant, which implies that Σ_1, Σ_2 are locally well-defined on $(M, \mathcal{I})$.

An infinitesimal version of Proposition 4.1 will be useful: for $i = 1, 2$,

$$\mathrm{d}S_i \equiv \begin{pmatrix} \phi_4 & -\phi_2 \\ -\phi_3 & \phi_1 \end{pmatrix} S_i + S_i \begin{pmatrix} \phi_5 & \phi_6 \\ \phi_7 & \phi_8 \end{pmatrix} \quad \mathrm{mod}\ \omega^0, \omega^1, ..., \omega^4. \tag{4.9}$$

An *Euler-Lagrange system*, in the classical calculus of variations, is the system of PDEs whose solutions correspond to the stationary points of a given first order functional. In [BGG03], an Euler-Lagrange system is formulated as a Monge-Ampère system[3]; moreover, it is shown:

Proposition 4.2. ([BGG03]) *A hyperbolic Monge-Ampère system is locally equivalent to an Euler-Lagrange system if and only if* S_2 *vanishes.*

Remark 5. Proposition 4.2 says that the property of being *Euler-Lagrange* is intrinsically defined.

Proposition 4.3. ([BGG03]) *A hyperbolic Monge-Ampère system corresponds to the wave equation* $z_{xy} = 0$ *(up to contact equivalence) if and only if* $S_1 = S_2 = 0$*.*

[3] See Definitions 1.3 and 1.4 of [BGG03]

Proposition 4.4. *A hyperbolic Monge-Ampère system* $(M, \mathcal{I})$*, where* $\mathcal{I}$ *is algebraically generated by* θ*,* $\omega^1 \wedge \omega^2$ *and* $\omega^3 \wedge \omega^4$*, locally corresponds to a PDE of the form* $z_{xy} = F(x, y, z, z_x, z_y)$ *(up to contact equivalence) if and only if each of the characteristic systems* $\mathcal{I}_{10} = \langle \theta, \omega^1, \omega^2 \rangle$ *and* $\mathcal{I}_{01} = \langle \theta, \omega^3, \omega^4 \rangle$ *admits a rank-*1 *integrable subsystem.*

Proof. One direction is obvious, since $\mathrm{d}x, \mathrm{d}y$ respectively belong to the two characteristic systems and are integrable. For the other direction, assume that $(M, \mathcal{I})$ has the property that each of $\mathcal{I}_{10}$ and $\mathcal{I}_{01}$ has a rank-1 integrable subsystem; and let $(\theta, \omega^1, ..., \omega^4)$ be a coframing satisfying $\mathrm{d}\theta \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \mod \theta$. For $\mathcal{I}_{10}$, this means that a certain linear combination $A\theta + B\omega^1 + C\omega^2$, where A, B, C (not all zero) are functions on M, is closed; hence, it locally equals to $\mathrm{d}x$ for some function x. Of course, B, C cannot both be zero, since θ is a contact form. Without loss of generality, assume that $B \neq 0$. Let $\hat{\omega}^1 = A\theta + B\omega^1 + C\omega^2 = \mathrm{d}x$, and $\hat{\omega}^2 = \frac{1}{B}\omega^2$. Similarly, there locally exist functions A', B', C' (assuming $B' \neq 0$) and y such that $\hat{\omega}^3 = A'\theta + B'\omega^3 + C'\omega^4 = \mathrm{d}y$; let $\hat{\omega}^4 = \frac{1}{B'}\omega^4$.

Now $\mathrm{d}\theta \equiv \mathrm{d}x \wedge \hat{\omega}^2 + \mathrm{d}y \wedge \hat{\omega}^4 \mod \theta$; hence the system $\langle \theta, \mathrm{d}x, \mathrm{d}y \rangle$ is Frobenius. As a result, there locally exists a function z such that $\langle \theta, \mathrm{d}x, \mathrm{d}y \rangle = \langle dz, \mathrm{d}x, \mathrm{d}y \rangle$. In other words, there exist functions g, p, q ($g \neq 0$) such that $\frac{1}{g}\theta = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$. This implies that $\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q \equiv \frac{1}{g}(\mathrm{d}x \wedge \hat{\omega}^2 + \mathrm{d}y \wedge \hat{\omega}^4) \mod \theta$. By Cartan's Lemma, there exists a function F such that

$$\hat{\omega}^2 \equiv g\mathrm{d}p - gF\mathrm{d}y \mod \mathrm{d}x, \theta; \qquad \hat{\omega}^4 \equiv g\mathrm{d}q - gF\mathrm{d}x \mod \mathrm{d}y, \theta.$$

The vanishing of θ and $\hat{\omega}^1 \wedge \hat{\omega}^2$ on integral surfaces then implies that locally the corresponding Monge-Ampère equation is equivalent to $z_{xy} = F(x, y, z, z_x, z_y)$. $\square$

Now we focus on hyperbolic Euler-Lagrange systems.

By Proposition 4.1, the sign of $\det(S_1)$ is independent of the choice of 1-adapted coframings. Hence, each hyperbolic Euler-Lagrange system belongs to one of the following three types.

Definition 4.2. Given a hyperbolic Euler-Lagrange system $(M, \mathcal{I})$, it is said to be

i. *positive* if $\det(S_1) > 0$;

ii. *negative* if $\det(S_1) < 0$;

iii. *degenerate* if $\det(S_1) = 0$.

Example 10. The hyperbolic Monge-Ampère systems in Proposition 3.5 and Proposition 3.8 in Chapter 3 are Euler-Lagrange of the positive type.

Example 11. In Example 4 of Chapter 2, let $K = -1$ for convenience. Recall that the differential ideal $\mathcal{I}$ on $\mathcal{O}$ is algebraically generated by

$$\theta = \omega^3, \quad \mathrm{d}\theta = -(\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2), \quad \Psi = \omega^1 \wedge \omega^2 + \omega_3^1 \wedge \omega_3^2.$$

Consider the change of basis

$$\omega^1 = -\eta^1 + \eta^3, \quad \omega^2 = \eta^2 + \eta^4, \quad \omega^3 = 2\eta^0, \quad \omega_3^1 = \eta^2 - \eta^4, \quad \omega_3^2 = \eta^1 + \eta^3.$$

In terms of η^i $(i = 0, ..., 4)$, we have $\mathcal{I} = \langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle$. It can be verified that $\eta = (\eta^0, \eta^1, ..., \eta^4)$, pulled back by a loal section $\sigma : U \to \mathcal{O}$ $(U \subset M$ open), is a local coframing 1-adapted to $(M, \mathcal{J})$ with $V_2 = V_3 = 1$ and all other V_i being zero. It follows that $(M, \mathcal{J})$ is a hyperbolic Euler-Lagrange system of the negative type.

Example 12. Consider an equation of the form $z_{xy} = f(z)$. (This is called an *f-Gordon equation.*) It corresponds to a hyperbolic Euler-Lagrange system of the degenerate type, for it is easy to verify that

$$\eta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \quad \eta^1 = \mathrm{d}x, \quad \eta^2 = \mathrm{d}p - f(z)\mathrm{d}y, \quad \eta^3 = \mathrm{d}y, \quad \eta^4 = \mathrm{d}q - f(z)\mathrm{d}x$$

is a 1-adapted coframing for the corresponding hyperbolic Monge-Ampère system. Under this coframing, $V_3 = -f'(z)$ and all other V_i are identically zero.

Remark 6. By Proposition 4.4, a hyperbolic Euler-Lagrange system of either the positive or the negative type is non-equivalent to any PDE of the form $z_{xy} = F(x, y, z, z_x, z_y)$. To see this, one can use (4.1) to show that, given a hyperbolic Monge-Ampère system $(M, \mathcal{I})$, if $S_2 = 0$ and S_1 nonsingular, then the *derived systems* (see [BCG$^+$13]) of each of the two characteristic systems of $(M, \mathcal{I})$ will terminate at zero.

4.2 The Bäcklund-Pfaff System

In this section, we show that given two hyperbolic Monge-Ampère systems, the existence of a *normal rank*-1 *Bäcklund transformation* (Definition 2.16) relating them can be formulated as the integrability of a rank-4 Pfaffian system.

To start with, let us fix some notation.

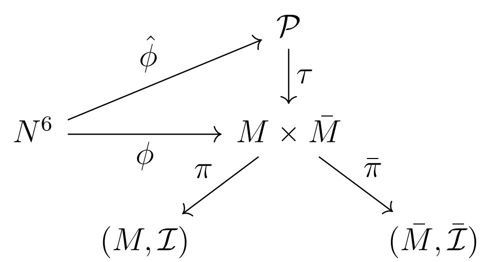
Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Monge-Ampère systems. Let $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$ be the respective G_1-structures (see Section 4.1). Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_4)$ and $\beta = (\beta_0, \beta_1, ..., \beta_4)$ be the tautological 1-forms on $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, respectively. And let

$$\begin{aligned}\mathcal{P} &:= \big\{(u, \bar{u}, s, t, \mu, \epsilon) \big| \ u \in \mathcal{G}_1, \bar{u} \in \bar{\mathcal{G}}_1; s, t \in \mathbb{R}^4; \mu \geqslant 1; \epsilon = \pm 1 \ (\epsilon\mu^2 \neq 1)\big\} \\ &\subset \mathcal{G}_1 \times \bar{\mathcal{G}}_1 \times \mathbb{R}^9 \times \{\pm 1\}.\end{aligned} \tag{4.10}$$

Proposition 4.5. *An immersed 6-manifold $\phi : N \to M \times \bar{M}$ is a normal rank*-1 *Bäcklund transformation if and only if N admits a lifting $\hat{\phi} : N \to \mathcal{P}$ that is an integral manifold of a rank-4 Pfaffian system $\mathcal{J}$ with the* independence condition $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_0 \wedge \beta_0 \neq 0$, *where $\mathcal{J}$ is generated by θ^i $(i = 1, ..., 4)$:*

$$\begin{aligned}\theta_1 &= \beta_1 - s_1\alpha_0 - t_1\beta_0 - \mu\alpha_1, \\ \theta_2 &= \beta_2 - s_2\alpha_0 - t_2\beta_0 - \mu\alpha_2, \\ \theta_3 &= \beta_3 - s_3\alpha_0 - t_3\beta_0 - \mu^{-1}\alpha_3, \\ \theta_4 &= \beta_4 - s_4\alpha_0 - t_4\beta_0 - \epsilon\mu^{-1}\alpha_4,\end{aligned} \tag{4.11}$$

at each $(u, \bar{u}, s_1, ..., s_4, t_1, ..., t_4, \mu, \epsilon) \in \mathcal{P}$.



Proof. By the construction of $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, on $\mathcal{P}$, we have

$$(\pi \circ \tau)^*(\mathcal{I}) = \langle \alpha_0, \alpha_1 \wedge \alpha_2, \alpha_3 \wedge \alpha_4 \rangle,$$

$$[\![(\pi \circ \tau)^*(\theta)]\!] \equiv [\![\alpha_0]\!],$$

and similarly for $\bar{\mathcal{I}}$ and the β_i. Now assume that ϕ admits a lifting $\hat{\phi}$ that integrates the Pfaffian system $\mathcal{J}$. It is easy to see that, on N,

$$\begin{aligned} \hat{\phi}^*([\![\alpha_1 \wedge \alpha_2, \alpha_3 \wedge \alpha_4]\!]) &\equiv \hat{\phi}^*([\![\beta_1 \wedge \beta_2, \beta_3 \wedge \beta_4]\!]) \\ &\equiv \hat{\phi}^*([\![\mathrm{d}\alpha_0, \mathrm{d}\beta_0]\!]) \mod \hat{\phi}^*\alpha_0, \hat{\phi}^*\beta_0. \end{aligned}$$

In the last equality, we have used the assumption that $\mu^2 \neq \epsilon$, which guarantees that the bundle $[\![\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4,\ \beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4]\!]$ has rank two modulo α_0, β_0 when pulled back to N. It follows that $\phi : N \to M \times \bar{M}$ defines a normal rank-1 Bäcklund transformation.

Conversely, suppose that $\phi : N \to M \times \bar{M}$ defines a normal rank-1 Bäcklund transformation. Let $\eta := (\eta_0, \eta_1, ..., \eta_4)$ (resp. $\xi := (\xi_0, \xi_1, ..., \xi_4)$) be a local 1-adapted coframing defined on a domain in M (resp. $\bar{M}$). We have, by definition,

$$\phi^*([\![\eta_1 \wedge \eta_2, \eta_3 \wedge \eta_4]\!]) \equiv \phi^*([\![\xi_1 \wedge \xi_2, \xi_3 \wedge \xi_4]\!]) \mod \phi^*\eta_0, \phi^*\xi_0.$$

By switching the pairs (η_1, η_2) and (η_3, η_4) if needed, we can assume that, on N,

$$\eta_1 \wedge \eta_2 \equiv \xi_1 \wedge \xi_2 \mod \eta_0, \xi_0,$$

$$\eta_3 \wedge \eta_4 \equiv \xi_3 \wedge \xi_4 \mod \eta_0, \xi_0,$$

where the pull-back symbol is dropped for convenience. Consequently, there exist 16 functions $s_1, ..., s_4, t_1, ..., t_4, u_1, ..., u_8$ defined on N such that, when restricted to N,

$$\begin{aligned} \xi_1 &= s_1\eta_0 + t_1\xi_0 + u_1\eta_1 + u_2\eta_2, \\ \xi_2 &= s_2\eta_0 + t_2\xi_0 + u_3\eta_1 + u_4\eta_2, \\ \xi_3 &= s_3\eta_0 + t_3\xi_0 + u_5\eta_3 + u_6\eta_4, \\ \xi_4 &= s_4\eta_0 + t_4\eta_0 + u_7\eta_3 + u_8\eta_4, \end{aligned} \tag{4.12}$$

where $u_1u_4 - u_2u_3$ and $u_5u_8 - u_7u_6$ are both nonvanishing. Moreover, since $[\![d\eta_0, d\xi_0]\!]$ has rank two modulo η_0, ξ_0, we have $u_1u_4 - u_2u_3 \neq u_5u_8 - u_6u_7$.

Using the flexibility of choosing the 1-adapted coframings η and ξ, we can normalize some of the u_i . To be specific, we can apply $\mathrm{SL}(2, \mathbb{R})$-actions to (η_1, η_2) and (η_3, η_4) to arrange u_2, u_3, u_6, u_7 to be zero, and to arrange that $u_4 = \pm u_1$, $u_8 = \pm u_5$, $(u_1, u_5 > 0)$. Then we transform $(\eta_0, \eta_1, ..., \eta_4)$ by a diagonal matrix in G_1 to arrange that $u_1 = u_4 > 0$ and $u_3 = u_1^{-1}$, $u_4 = \epsilon u_1^{-1}$, where $\epsilon = \pm 1$. Meanwhile, this action scales s_i $(i = 1, ..., 4)$. Finally, if $u_1 = u_4 < 1$, then we switch the pairs of indices $(1, 2)$ and $(3, 4)$ for ξ and η in (4.12), and multiply the new η_0, η_2, η_4 by ϵ.

It is easy to see that the resulting $(\eta, \xi, (s_i), (t_i), u_1, \epsilon)$ defines a map $\hat{\phi} : N \to \mathcal{P}$ that is a lifting of ϕ and integrates $\mathcal{J}$. □

In light of Proposition 4.5, we make the

Definition 4.3. The system $(\mathcal{P}, \mathcal{J})$ in Proposition 4.5 is called the 0-*refined Bäcklund-Pfaff system* for normal rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Definition 4.4. An integral manifold of the 0-refined Bäcklund-Pfaff system $(\mathcal{P}, \mathcal{J})$ is called a 0-*refined lifting* of the underlying Bäcklund transformation.

In these terms, Proposition 4.5 says that each normal Bäcklund transformation has a 0-refined lifting. Of course, given a normal Bäcklund transformation $\phi : N \to$

$M \times \bar{M}$, its 0-refined liftings are not unique.

Lemma 4.1. *Let $\phi : N \to M \times \bar{M}$ be a normal Bäcklund transformation. The functions $\mu \circ \hat{\phi}$ and $\epsilon \circ \hat{\phi}$ are independent of the choice of 0-refined liftings $\hat{\phi}$ of ϕ.*

Proof. Clearly, for different choices of $\hat{\phi}$, the 1-forms $\phi^*\alpha_0$ and $\phi^*\beta_0$ only change by scaling. On N, the quotient λ_1/λ_2 between the two solutions λ_1, λ_2 of the equation

$$(\mathrm{d}\beta_0 + \lambda \mathrm{d}\alpha_0)^2 \equiv 0 \mod \alpha_0, \beta_0$$

is independent of the scaling of α_0 and β_0 but may depend on the order of the pair (λ_1, λ_2). By (4.11), on N, either λ_1/λ_2 or λ_2/λ_1 must be equal to $\epsilon\mu^4$.

Since we have arranged $\mu \geqslant 1$ for 0-refined liftings, the ambiguity of ordering λ_1 and λ_2 can be removed by letting $\lambda_1/\lambda_2 = \epsilon\mu^4$. It follows that μ and ϵ are both independent of the lifting. □

Remark 7. (**A**) There is a simple geometric interpretation for the two possible values of ϵ. Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be as above. Suppose that $\phi : N \to M \times \bar{M}$ defines a normal rank-1 Bäcklund transformation. The 4-plane field $\mathcal{D} := [\![\alpha_0, \beta_0]\!]^\perp$ on N is independent of the choice of 0-refined liftings of ϕ. When restricted to $\mathcal{D}$, the 4-forms $(\mathrm{d}\alpha_0)^2$ and $(\mathrm{d}\beta_0)^2$ define two orientations. If $\epsilon = 1$, these two orientations are the same; if $\epsilon = -1$, they are distinct.

(**B**) If a Bäcklund transformation is normal, which we will always assume for the rest of this chapter, then $\epsilon = 1$ implies $\mu > 1$.

4.3 Obstructions to Integrability

In this section, we express the obstructions to integrability of $(\mathcal{P}, \mathcal{J})$ in terms of the invariants of the two hyperbolic Monge-Ampère systems.

For convenience, we introduce new notation below.

(**A**) Let $\eta_1 := \alpha_1,\ \eta_2 := \alpha_2,\ \eta_3 := \alpha_3,\ \eta_4 := \alpha_4,\ \eta_5 := \alpha_0,\ \eta_6 := \beta_0.$

(**B**) The components of the pseudo-connection 1-form on $\mathcal{G}_1$ (see (4.1)) are denoted by $\varphi_0, \varphi_1, ..., \varphi_8$; those on $\bar{\mathcal{G}}_1$ are denoted by $\psi_0, \psi_1, ..., \psi_8$.

On $\mathcal{P}$, differentiating the θ_i and reducing modulo $\theta_1, ..., \theta_4$ yields the following equations

$$\mathrm{d}\theta_i = -\pi_{i\alpha} \wedge \eta_\alpha + \tau_i \qquad \text{mod } \theta_1, ..., \theta_4, \tag{4.13}$$

where π_{ij} are linear combinations of the 1-forms in

$$\mathcal{S} := \{\mathrm{d}(s_1), ..., \mathrm{d}(s_4); \mathrm{d}(t_1), ..., \mathrm{d}(t_4); \mathrm{d}\mu; \varphi_0, ..., \varphi_3, \varphi_5, \varphi_6, \varphi_7; \psi_0, ..., \psi_3, \psi_5, \psi_6, \psi_7\},$$

and τ_i, called the *torsion*, are of the form

$$\tau_i = T_{ijk}\eta_j \wedge \eta_k, \quad T_{ijk} = -T_{ikj},$$

for some functions T_{ijk} defined on $\mathcal{P}$.

We can use a standard method (see [BCG$^+$13]) to obtain from (4.13) obstructions to integrability of $(\mathcal{P}, \mathcal{J})$. The key is to absorb the torsion terms (i.e., terms in τ_i) as much as possible by adding linear combinations of $\eta_1, ..., \eta_k$ to the 1-forms in $\mathcal{S}$.

In our case, the matrix $(\pi_{i\alpha})$ (called the *tableau*) takes the form

$$(\pi_{i\alpha}) = \begin{pmatrix} \pi_1 & \pi_2 & 0 & 0 & \pi_3 & \pi_4 \\ \pi_5 & \pi_6 & 0 & 0 & \pi_7 & \pi_8 \\ 0 & 0 & \pi_9 & \pi_{10} & \pi_{11} & \pi_{12} \\ 0 & 0 & \pi_{13} & \pi_{14} & \pi_{15} & \pi_{16} \end{pmatrix},$$

where $\pi_1, ..., \pi_{16}$ are 1-forms independent of the $\theta_1, ..., \theta_4, \eta_1, ..., \eta_6$ and independent among themselves. It is easy to see that all terms in τ_i can be absorbed except the $\eta_3 \wedge \eta_4$ terms in τ_1 and τ_2, and the $\eta_1 \wedge \eta_2$ terms in τ_3 and τ_4.

Calculation yields

$$\mathrm{d}\theta_1 \equiv -\frac{\mu^2 s_1 + \epsilon t_1}{\mu^2}\eta_3 \wedge \eta_4 \qquad \text{mod } \theta_1, ..., \theta_4, \eta_1, \eta_2, \eta_5, \eta_6,$$

$$\mathrm{d}\theta_2 \equiv -\frac{\mu^2 s_2 + \epsilon t_2}{\mu^2}\eta_3 \wedge \eta_4 \qquad \text{mod } \theta_1, ..., \theta_4, \eta_1, \eta_2, \eta_5, \eta_6,$$

$$\mathrm{d}\theta_3 \equiv -(\mu^2 t_3 - s_3)\eta_1 \wedge \eta_2 \qquad \mathrm{mod}\ \theta_1, ..., \theta_4, \eta_3, \eta_4, \eta_5, \eta_6,$$

$$\mathrm{d}\theta_4 \equiv -(\mu^2 t_4 - s_4)\eta_1 \wedge \eta_2 \qquad \mathrm{mod}\ \theta_1, ..., \theta_4, \eta_3, \eta_4, \eta_5, \eta_6.$$

As a result, we have proved

Lemma 4.2. *Integral manifolds of $(\mathcal{P}, \mathcal{J})$ only exist in the locus $\mathcal{L} \subset \mathcal{P}$ defined by the equations*

$$s_3 = -t_3\mu^2, \qquad s_4 = -t_4\mu^2,$$

$$t_1 = -\epsilon s_1\mu^2, \qquad t_2 = -\epsilon s_2\mu^2.$$

Definition 4.5. Let $\mathcal{P}_1 \subset \mathcal{P}$ be the locus defined by the equations

$$s_1 = t_1 = s_3 = t_3 = 0, \quad s_4 = -t_4\mu^2, \quad t_2 = -\epsilon s_2\mu^2.$$

Let $\mathcal{J}_1$ be the restriction of $\mathcal{J}$ to $\mathcal{P}_1$. The rank-4 Pfaffian system $(\mathcal{P}_1, \mathcal{J}_1)$ is called the 1-*refined Bäcklund-Pfaff system* for normal rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems.

Definition 4.6. An integral manifold of the 1-refined Bäcklund-Pfaff system $(\mathcal{P}_1, \mathcal{J}_1)$ is called a 1-*refined lifting* of the underlying Bäcklund transformation.

Proposition 4.6. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be as above. Any normal rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ admits a 1-refined lifting.*

Proof. By previous discussion, there exists a 0-refined lifting $\hat{\phi}$ of ϕ such that, when pulled-back via $\hat{\phi}$ to N,

$$\beta_1 = s_1\alpha_0 - \epsilon s_1\mu^2\beta_0 + \mu\alpha_1,$$

$$\beta_2 = s_2\alpha_0 - \epsilon s_2\mu^2\beta_0 + \mu\alpha_2,$$

$$\beta_3 = -t_3\mu^2\alpha_0 + t_3\beta_0 + \mu^{-1}\alpha_3,$$

$$\beta_4 = -t_4\mu^2\alpha_0 + t_4\beta_0 + \epsilon\mu^{-1}\alpha_4.$$

If $s_2 \neq 0$, transform the pairs (α_1, α_2) and (β_1, β_2) simultaneously using

$$h_1 = \begin{pmatrix} 1 & -s_1/s_2 \\ 0 & 1 \end{pmatrix}$$

If $s_2 = 0$ but $s_1 \neq 0$, transform the pairs (α_1, α_2) and (β_1, β_2) simultaneously using

$$h_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

such that the previou step applies. This will yield a 0-refined lifting with $s_1 = 0$. Further, we can apply simultaneous $\mathrm{SL}(2, \mathbb{R})$-actions on (a_3, a_4) and (b_3, b_4) to arrange $t_3 = 0$. $\square$

Now $\mathcal{J}_1$ is generated by

$$\begin{aligned} \theta_1 &= b_1 - \mu a_1, \\ \theta_2 &= b_2 - s_2 a_0 + \epsilon s_2 \mu^2 b_0 - \mu a_2, \\ \theta_3 &= b_3 - \mu^{-1} a_3, \\ \theta_4 &= b_4 + t_4 \mu^2 a_0 - t_4 b_0 - \epsilon \mu^{-1} a_4. \end{aligned}$$

Given a normal rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$, it is easy to see that whether the product $s_2 t_4$ locally vanishes is independent of the choice of 1-refined liftings of ϕ. The following proposition says that the case of $s_2 t_4 = 0$ is quite restricted when both $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ are Euler-Lagrange systems.

Proposition 4.7. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Euler-Lagrange systems. If there exists a normal rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ such that $s_2 t_4 = 0$ on a 1-refined lifting of ϕ, then both $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ must be equivalent to the system corresponding to the wave equation $z_{xy} = 0$.*

Proof. By the Euler-Lagrange assumption and Proposition 4.2, $S_2 = \bar{S}_2 = 0$. Let

$$S_1 = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}, \qquad \bar{S}_1 = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}.$$

By Proposition 4.3, it suffices to show that, on any integral manifold of the 1-refined Bäcklund-Pfaff system $(\mathcal{P}_1, \mathcal{J}_1)$, we have $S_1 = \bar{S}_1 = 0$.

First, we assume $s_2 = 0$. Restricted to the locus defined by $s_2 = 0$ in $\mathcal{P}_1$, the tableau $(\pi_{i\alpha})$ associated to $(\mathcal{P}_1, \mathcal{J}_1)$ satisfies

$$\pi_{i\alpha} = 0, \quad (i = 1, 2;\ \alpha = 3, ..., 6).$$

As a result, the $\eta^i \wedge \eta^j$ terms in $\mathrm{d}\theta_1$ and $\mathrm{d}\theta_2$, $i, j = 3, ..., 6$ and $i \neq j$, cannot be absorbed and the corresponding coefficients must vanish on integral manifolds of $(\mathcal{P}_1, \mathcal{J}_1)$. Calculating with Maple™, we find

$$\mathrm{d}\theta_1 \equiv \mu(V_1\eta_3 + V_2\eta_4) \wedge \eta_5 - \frac{1}{\mu}(W_1\eta_3 + \epsilon W_2\eta_4) \wedge \eta_6 + W_2 t_4 \mu^2 \eta_5 \wedge \eta_6,$$

$$\mathrm{d}\theta_2 \equiv \mu(V_3\eta_3 + V_4\eta_4) \wedge \eta_5 - \frac{1}{\mu}(W_3\eta_3 + \epsilon W_4\eta_4) \wedge \eta_6 + W_4 t_4 \mu^2 \eta_5 \wedge \eta_6,$$

both congruences being reduced modulo $\theta_1, ..., \theta_4, \eta_1, \eta_2$. It follows that $S_1 = \bar{S}_1 = 0$.

The case of $t_4 = 0$ is similar. □

Proposition 4.8. *On any* 1*-refined lifting of a normal rank-*1 *Bäcklund transformation relating two hyperbolic Euler-Lagrange systems, the following expressions must vanish*

$$\Phi_1 := -\mu^4 V_1 + \epsilon W_1, \qquad \Phi_2 := -\mu^4 V_2 + W_2,$$

$$\Phi_3 := \mu^4 W_4 - V_4, \qquad \Phi_4 := \mu^4 W_2 - V_2.$$

Proof. This is evident when such a Bäcklund transformations satisfies $s_2 t_4 = 0$. In fact, by Proposition 4.7, the functions V_i and W_i vanish identically on $\mathcal{P}_1$. Otherwise, restricting to the open subset of $\mathcal{P}_1$ defined by $s_2 t_4 \neq 0$, a calculation shows that the torsion of the Pfaffian system $\mathcal{J}_1$ can be absorbed only if Φ_i $(i = 1, ..., 4)$ are all zero. □

Corollary 4.3. *If two hyperbolic Euler-Lagrange systems are related by a normal rank-*1 *Bäcklund transformation with* $\epsilon = 1$*, then they are either both degenerate or both nondegenerate.*

Proof. We have noted above (see Remark 7) that, if a normal Bäcklund transformation satisfies $\epsilon = 1$, then $\mu \neq 1$. The vanishing of Φ_2 and Φ_4 on a 1-refined lifting of such a Bäcklund transformation then implies that, on such a lifting,

$$V_2 = W_2 = 0.$$

By the vanishing of Φ_1 and Φ_3, it is easy to see that the matrices S_1 and $\bar{S}_1$ are either both degenerate or both nondegenerate. □

Note that, in the proof of Corollary 4.3, the condition $V_2 = W_2 = 0$ is meaningful only if it is independent of the choice of 1-refined liftings of a Bäcklund transformation. To make this point explicit, we state the following proposition which shows how 1-refined liftings of a normal rank-1 Bäcklund transformation relate to each other.

Proposition 4.9. *Let* ϕ *be a normal rank-*1 *Bäcklund transformation relating two hyperbolic Monge-Ampère systems satisfying* $s_2 t_4 \neq 0$ *on its* 1*-refined liftings. There exists a subgroup* $H \subset G_1 \times G_1$ *such that any two* 1*-refined liftings of* ϕ *are related, in the* $\mathcal{G}_1 \times \bar{\mathcal{G}}_1$ *component, by an* H*-valued transformation. Moreover,*

i. *if* $\mu > 1$*, then* H *is the subgroup generated by elements of the form* $h = (g, g')$ *where*

$$g = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 & 0 \\ 0 & h_3 & h_0 h_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_4 & h_0 h_2^{-1} \end{pmatrix} \in G_1,$$

and g' *is the result of replacing* h_4 *in* g *by* ϵh_4*.*

ii. *if* $\mu = 1$*, then* H *is generated by the subgroup in case* **i** *and the element*

$$h_J = (J', J),$$

where J is as in (4.3) *and* $J' = \mathrm{diag}(-1, 1, -1, 1, -1)J$.

4.4 A Special Class of Bäcklund Transformations

In the previous section, we have seen that, to a normal rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$, we can associate a function $\mu : N \to [1, \infty)$ that is independent[4] of the 0-refined liftings of ϕ.

Definition 4.7. A normal rank-1 Bäcklund transformation relating two hyperbolic Monge-Ampère systems is said to be *special* if $\mu = 1$.

For the rest of this section, we will focus on special Bäcklund transformations relating two hyperbolic Euler-Lagrange systems. A motivation to this is that many classical Bäcklund transformations are of this type (cf. [Cle01], [CI13]).

By Proposition 4.8, given a special rank-1 Bäcklund transformation $\phi : N \to M \times \bar{M}$ relating two hyperbolic Euler-Lagrange systems, the following equalities must hold on any 1-refined lifting of ϕ:

$$\epsilon = -1, \quad W_1 = -V_1, \quad W_2 = V_2, \quad W_4 = V_4. \tag{4.14}$$

Now let $\mathcal{P}_s \subset \mathcal{P}_1$ be defined by the equations

$$\mu = 1, \qquad \epsilon = -1.$$

Proposition 4.10. *Any* 1*-refined lifting of a special rank-*1 *Bäcklund transformation relating two hyperbolic Euler-Lagrange systems is completely contained in* $\mathcal{P}_s$*. Moreover, on such a lifting, in addition to* (4.14)*, we have*

$$V_3 + W_3 + 2s_2t_4 = 0. \tag{4.15}$$

[4] In fact, it is easy to see that μ is an invariant of the corresponding Bäcklund transformation.

Proof. Restricting to $\mathcal{P}_s$, the generators θ_i of $\mathcal{J}_1$ satisfy equations of the form

$$\mathrm{d}\theta_i = -\pi_{i\alpha} \wedge \eta_\alpha + \tau_i.$$

The tableau $(\pi_{i\alpha})$ now takes the form

$$(\pi_{i\alpha}) = \begin{pmatrix} -\pi_1 & -\pi_2 & 0 & 0 & s_2\pi_8 & s_2\pi_8 \\ -\pi_3 & \pi_1 - \pi_7 & 0 & 0 & -s_2\pi_7 - \pi_9 & -\pi_9 \\ 0 & 0 & -\pi_4 & -\pi_5 & -t_4\pi_{10} & t_4\pi_{10} \\ 0 & 0 & -\pi_6 & \pi_7 - \pi_4 & t_4\pi_7 + \pi_{11} & -\pi_{11} \end{pmatrix},$$

where, reduced modulo $\eta_0, \eta_1, ..., \eta_4$,

$$\begin{aligned} &\pi_1 \equiv \varphi_1 - \psi_1, \quad \pi_4 \equiv \varphi_5 - \psi_5, \quad \pi_7 \equiv \varphi_0 - \psi_0, \quad && \pi_{10} \equiv \psi_6 \\ &\pi_2 \equiv \varphi_2 - \psi_2, \quad \pi_5 \equiv \varphi_6 + \psi_6, \quad \pi_8 \equiv \psi_2, && \pi_{11} \equiv -\mathrm{d}(t_4) + t_4\psi_5. \\ &\pi_3 \equiv \varphi_3 - \psi_3, \quad \pi_6 \equiv -\varphi_7 - \psi_7, \quad \pi_9 \equiv -\mathrm{d}(s_2) + s_2\psi_1, && \end{aligned}$$

By a calculation using Maple™, it is easy to see that the torsion τ_i can be absorbed only if the equations (4.14) and (4.15) hold. □

The equalities (4.14) and (4.15) tell us which Euler-Lagrange systems may be Bäcklund-related. In particular, by Propositions 4.9 and 4.1, it is easy to see that whether V_2 (hence W_2) vanishes is independent of the choice of 1-refined liftings. Thus, we may locally[5] classify special rank-1 Bäcklund transformations relating two hyperbolic Euler-Lagrange systems into the following three types:

Type I. $V_2 = 0$, $\det(S_1) = V_1V_4 \neq 0$;

Type II. $V_2 = 0$, $\det(S_1) = V_1V_4 = 0$;

Type III. $V_2 \neq 0$.

Lemma 4.4. (**A**) *A Type I special Bäcklund transformation cannot be an auto-Bäcklund transformation of a nondegenerate hyperbolic Euler-Lagrange system.*

[5] Namely, the conditions below hold on an entire open subset of N.

(**B**) *Two hyperbolic Euler-Lagrange systems related by a Type III special Bäcklund transformation cannot be both degenerate.*

Proof. Part (**A**) is immediate, because, in this case, on a 1-refined lifting, $\det(S_1) = -\det(\bar{S}_1) \neq 0$.

To prove Part (**B**), first apply Proposition 4.9 to show that, in this case, one can always find a 1-refined lifting on which $V_1 = V_4 = 0$. For such a 1-refined lifting, by (4.15), it is easy to see that the two Euler-Lagrange systems can be both degenerate only when $s_2 t_4 = 0$. By Proposition 4.7, both S_1 and $\bar{S}_1$ must vanish, which impossible since we have assumed $V_2 \neq 0$. □

Remark 8. Lemma 4.4 allows us identify the types of various known Bäcklund transformations. For example, any special Bäcklund transformation relating a pair of f-Gordon equations must appear as Type II (the classical auto-Bäcklund transformation of the sine-Gordon equation is of this type); the classical auto-Bäcklund transformation relating surfaces in $\mathbb{E}^3$ with a negative constant Guass curvature, which can be verified to be special, must be of Type III.

Now we focus on Type II. The analysis of Type I and III will be included in a future work.

Let $\phi : N \to M \times \bar{M}$ be a Type II special rank-1 Bäcklund transformation in the sense above. It is easy to see using Propositions 4.1 and 4.9 that whether the pair (V_1, V_4) vanishes is independent of the choice of 1-refined liftings of ϕ. It follows that ϕ must be one of the following two types.

Type IIa: on any 1-refined lifting of ϕ, $(V_1, V_4) = 0$;

Type IIb: on any 1-refined lifting of ϕ, $(V_1, V_4) \neq 0$.

4.4.1 Type IIa

In this case, (4.14) implies that

$$W_1 = W_2 = W_4 = 0.$$

If locally either V_3 or W_3 is zero, which is independent of the choice of 1-refined liftings, the underlying Bäcklund transformation must relate a hyperbolic Euler-Lagrange system with the system corresponding to $z_{xy} = 0$. See [CI$^+$09] for a classification of all hyperbolic Monge-Ampère systems that are rank-1 Bäcklund-related to the equation $z_{xy} = 0$.

Now suppose that V_3 and W_3 are both nonzero on any 1-refined lifting of ϕ.

Theorem 4.5. *If two hyperbolic Euler-Lagrange systems are related by a Type IIa special rank-*1 *Bäcklund transformation, then each of them corresponds (up to contact equivalence) to a second order PDE of the form*

$$z_{xy} = F(x, y, z, z_x, z_y).$$

Proof. By definition, any Type IIa special Bäcklund transformation admits a 1-refined lifting that is completely contained in the locus $\mathcal{P}_{\text{IIa}} \subset \mathcal{P}_1$ defined by the equations

$$\mu = 1, \quad \epsilon = -1, \quad V_1 = V_2 = V_4 = W_1 = W_2 = W_4 = 0.$$

Let $\mathcal{G}_2 \subset \mathcal{G}_1$ be the subbundle defined by $V_1 = V_2 = V_4 = 0$; similarly, let $\bar{\mathcal{G}}_2 \subset \bar{\mathcal{G}}_1$ be the subbundle defined by $W_1 = W_2 = W_4 = 0$. It is clear that $\mathcal{P}_{\text{IIa}}$ is the product of $\mathcal{G}_2$, $\bar{\mathcal{G}}_2$, and a space of parameters with coordinates (s_2, t_4).

By (4.9), on $\mathcal{G}_2$, there exist functions P_{ij} and V_{3j} such that

$$\varphi_2 = P_{20}\alpha_0 + P_{21}\alpha_1 + \cdots + P_{24}\alpha_4,$$

$$\varphi_6 = P_{60}\alpha_0 + P_{61}\alpha_1 + \cdots + P_{64}\alpha_4,$$

$$\mathrm{d}(V_3) = (\varphi_1 + \varphi_5)V_3 + V_{30}\alpha_0 + V_{31}\alpha_1 + \cdots + V_{34}\alpha_4.$$

Similarly, on $\bar{\mathcal{G}}_2$, there exist functions Q_{ij} and W_{3j} such that

$$\psi_2 = Q_{20}\beta_0 + W_{21}\beta_1 + \cdots + Q_{24}\beta_4,$$

$$\psi_6 = Q_{60}\beta_0 + W_{61}\beta_1 + \cdots + Q_{64}\beta_4,$$

$$\mathrm{d}(W_3) = (\psi_1 + \psi_5)W_3 + W_{30}\beta_0 + W_{31}\beta_1 + \cdots + W_{34}\beta_4.$$

There is freedom to add linear combinations of $\alpha_0, ..., \alpha_4$ (resp. $\beta_0, ..., \beta_4$) to φ_i (resp. ψ_i) without changing the form of the corresponding Monge-Ampère structure equations. Using this, we can arrange the following expressions to be zero:

$$P_{21},\ P_{22},\ P_{63},\ P_{64};\ Q_{21},\ Q_{22},\ Q_{63},\ Q_{64}.$$

Moreover, applying $\mathrm{d}^2 = 0$ to the Monge-Ampère structure equations yields

$$\mathrm{d}(\mathrm{d}\alpha_1) \equiv P_{24}V_3\alpha_0 \wedge \alpha_3 \wedge \alpha_4 \mod \alpha_1, \alpha_2,$$

$$\mathrm{d}(\mathrm{d}\alpha_2) \equiv V_{34}\alpha_0 \wedge \alpha_3 \wedge \alpha_4 \mod \alpha_1, \alpha_2,$$

$$\mathrm{d}(\mathrm{d}\alpha_3) \equiv P_{62}V_3\alpha_0 \wedge \alpha_1 \wedge \alpha_2 \mod \alpha_3, \alpha_4,$$

$$\mathrm{d}(\mathrm{d}\alpha_4) \equiv V_{32}\alpha_0 \wedge \alpha_1 \wedge \alpha_2 \mod \alpha_3, \alpha_4.$$

This implies that, on $\mathcal{G}_2$,

$$P_{24}V_3 = V_{34} = P_{62}V_3 = V_{32} = 0. \tag{4.16}$$

By a similar argument, one can show that, on $\bar{\mathcal{G}}_2$,

$$Q_{24}W_3 = W_{34} = Q_{62}W_3 = W_{32} = 0.$$

Restricted to $\mathcal{P}_{\mathrm{IIa}}$, the generators θ_i $(i = 1, ..., 4)$ of $\mathcal{J}_1$ satisfy equations of the form

$$\mathrm{d}\theta_i = -\pi_{i\alpha} \wedge \eta_\alpha + \tau_i.$$

The tableau takes the form

$$(\pi_{i\alpha}) = \begin{pmatrix} -\pi_1 & 0 & 0 & 0 & 0 & 0 \\ -\pi_2 & \pi_1 - \pi_5 & 0 & 0 & -s_2\pi_5 + \pi_6 & \pi_6 \\ 0 & 0 & -\pi_3 & 0 & 0 & 0 \\ 0 & 0 & \pi_4 & -\pi_3 + \pi_5 & t_4\pi_5 - \pi_7 & \pi_7 \end{pmatrix},$$

where, modulo η_i,

$$\pi_1 \equiv \varphi_1 - \psi_1, \quad \pi_3 \equiv \varphi_5 - \psi_5, \quad \pi_5 \equiv \varphi_0 - \psi_0, \qquad \pi_7 \equiv \mathrm{d}(t_4) - t_4\psi_5,$$
$$\pi_2 \equiv \varphi_3 - \psi_3, \quad \pi_4 \equiv \varphi_7 + \psi_7, \quad \pi_6 \equiv \mathrm{d}(s_2) - s_2\psi_1.$$

Assuming s_2, t_4 to be both nonzero and calculating with Maple$^{\mathrm{TM}}$, we find that the torsion can be absorbed only if the following expressions are zero

$$P_{20},\ P_{23},\ P_{24},\ P_{60},\ P_{61},\ P_{62};\ Q_{20},\ Q_{23},\ Q_{24},\ Q_{60},\ Q_{61},\ Q_{62}.$$

One can verify that, on the subbundle of $\mathcal{G}_2$ defined by the vanishing of $P_{20}, P_{23}, P_{24}, P_{60}, P_{61}$ and P_{62}, the following structure equations hold

$$\begin{aligned}
\mathrm{d}\alpha_0 &= -\varphi_0 \wedge \alpha_0 + \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4,\\
\mathrm{d}\alpha_1 &= -\varphi_1 \wedge \alpha_1,\\
\mathrm{d}\alpha_2 &= -\varphi_3 \wedge \alpha_1 + (\varphi_1 - \varphi_0) \wedge \alpha_2 + V_3\alpha_0 \wedge \alpha_3,\\
\mathrm{d}\alpha_3 &= -\varphi_5 \wedge \alpha_3,\\
\mathrm{d}\alpha_4 &= -\varphi_7 \wedge \alpha_3 + (\varphi_5 - \varphi_0) \wedge \alpha_4 + V_3\alpha_0 \wedge \alpha_1.
\end{aligned}$$

Clearly, the systems $\langle\alpha_1\rangle$ and $\langle\alpha_3\rangle$ are both integrable. It is a similar case for the structure equations on $\bar{\mathcal{G}}_2$. By Proposition 4.4, the proof is complete. □

4.4.2 Type IIb

In this case, on a 1-refined lifting of ϕ, either V_1 or V_4 vanishes. By Proposition 4.9 (in particular, using h_J), we can arrange $V_1 \neq 0$ and $V_4 = 0$ on a 1-refined lifting of ϕ. Such a 1-refined lifting can always be chosen to further satisfy $V_1 = 1$ and $V_3 = W_3 = 0$ or 1.

In the next proposition we show that the case of $V_1 = 1$ and $V_3 = W_3 = 0$ is impossible. Then we characterize the case when ϕ admits a 1-refined lifting for which $V_1 = 1, V_3 = W_3 = 1$.

Proposition 4.11. *Restricting to the locus in $\mathcal{P}_1$ defined by*

$$\mu = 1,\ \epsilon = -1,\ V_1 = -W_1 = 1,\ V_2 = V_3 = V_4 = W_2 = W_3 = W_4 = 0,$$

$\mathcal{J}_1$ is not integrable.

Proof. If there exists a 1-refined lifting of a special Bäcklund transformation such that $V_3 = W_3 = 0$, then the equality (4.15) enforces that $s_2 t_4 = 0$ on such a lifting. By Proposition 4.7, both Monge-Ampère systems must be contact equivalent to the wave equation $z_{xy} = 0$. In particular, V_i and W_i must all be zero on $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, respectively. This is a contradiction. □

Proposition 4.12. *Let $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ be two hyperbolic Euler-Lagrange systems. If $\phi : N \to M \times \bar{M}$ defines a Type IIb special rank-1 Bäcklund transformation relating $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$, then each of $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ must have a characteristic system that contains a rank-1 integrable subsystem.*

Proof. The idea is similar to that of Proposition 4.5. We restrict the differential ideal $\mathcal{J}_1$ to the locus $\mathcal{P}_{\text{IIb}} \subset \mathcal{P}_1$ defined by the equations

$$\mu = 1,\ \epsilon = -1,\ V_1 = -W_1 = V_3 = W_3 = 1,\ V_2 = V_4 = W_2 = W_4 = 0$$

and analyse the obstructions to integrability of the resulting rank-4 Pfaffian system.

By (4.9), on the subbundle of $\mathcal{G}_1$ defined by $V_1 = V_3 = 1$ and $V_2 = V_4 = 0$, there exist functions P_{ij} such that

$$\begin{aligned}
\varphi_2 &= P_{20}\alpha_0 + P_{21}\alpha_1 + \cdots + P_{24}\alpha_4 + \varphi_0 - \varphi_1 + \varphi_5,\\
\varphi_3 &= P_{30}\alpha_0 + P_{31}\alpha_1 + \cdots + P_{34}\alpha_4 + \varphi_1 + \varphi_5,\\
\varphi_6 &= P_{60}\alpha_0 + P_{61}\alpha_1 + \cdots + P_{64}\alpha_4.
\end{aligned}$$

Using the freedom in the choice of φ_i, we can arrange $P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{34}, P_{64}$ to be zero. By expanding $\mathrm{d}(\mathrm{d}\alpha_i) = 0$, we find

$$P_{24} = 0, \quad P_{61} = -P_{62}, \quad P_{63} = 0.$$

Similarly, on the subbundle of $\bar{\mathcal{G}}_1$ defined by $-W_1 = W_3 = 1$ and $W_2 = W_4 = 0$, there exist functions Q_{ij} such that

$$\begin{aligned}
\psi_2 &= Q_{20}\beta_0 + Q_{21}\beta_1 + \cdots + Q_{24}\beta_4 - \psi_0 + \psi_1 - \psi_5,\\
\psi_3 &= Q_{30}\beta_0 + Q_{31}\beta_1 + \cdots + Q_{34}\beta_4 - \psi_1 - \psi_5,\\
\psi_6 &= Q_{60}\beta_0 + Q_{61}\beta_1 + \cdots + Q_{64}\beta_4.
\end{aligned}$$

Using the freedom in the choice of ψ_i, we can arrange $Q_{21}, Q_{22}, Q_{23}, Q_{31}, Q_{32}, Q_{34}, Q_{64}$ to be zero. By expanding $\mathrm{d}(\mathrm{d}\beta_i) = 0$, we find

$$Q_{24} = 0, \quad Q_{61} = Q_{62}, \quad Q_{63} = 0.$$

Denote the restriction of $\mathcal{J}_1$ to $\mathcal{P}_{\mathrm{IIb}}$ as $\mathcal{J}_{\mathrm{IIb}}$. Calculating with Maple™, it is easy to see that the torsion of $(\mathcal{P}_{\mathrm{IIb}}, \mathcal{J}_{\mathrm{IIb}})$ can be absorbed only if the following expressions are zero:

$$s_2 t_4 + 1, \quad P_{20} - Q_{20}, \quad P_{60}, \quad P_{62}, \quad Q_{60}, \quad Q_{62}.$$

In particular, the vanishing of P_{60}, P_{62}, Q_{60} and Q_{62} implies that

$$\mathrm{d}\alpha_3 = -\varphi_5 \wedge \alpha_3, \qquad \mathrm{d}\beta_3 = -\psi_5 \wedge \beta_3.$$

The conclusion of the proposition follows. □

5

Homogeneous Rank-2 Bäcklund Transformations

Rank-2 Bäcklund transformations are interesting because they arise naturally in at least the following three ways[1]: one, constructed from a single or a 1-parameter family of rank-1 Bäcklund transformations; two, constructed as the composition (see Proposition 2.4) of two rank-1 Bäcklund transformations; three, arise as geometric examples such as the classical auto-Bäcklund transformation (1.5) of the hyperbolic Tzitzeica equation (1.4). An ultimate goal of studying Bäcklund transformations of higher ranks might be to answer the question: *How many more differential systems are Bäcklund-related when we allow Bäcklund tranformations of a higher rank?*

In this chapter, for rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems, we formulate the corresponding equivalence problem, and provide a partial classification in the homogeneous case.

5.1 Genericity Conditions and Structure Reduction

Let $(N, \mathcal{B}; \pi, \bar{\pi})$ be a rank-2 Bäcklund transformation relating two hyperbolic Monge-Ampère systems $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$. There exists a local coframing $(\theta, \eta^1, ..., \eta^4)$

[1] We are not claiming that these cases are disjoint.

defined on a domain $U_1 \subset M$ such that θ is a contact form and that the differential ideal $\mathcal{I}$ on U_1 can be written as $\mathcal{I} = \langle \theta, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\text{alg}}$. Similarly, we have $\bar{\mathcal{I}} = \langle \bar{\theta}, \bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4 \rangle_{\text{alg}}$ for a coframing $(\bar{\theta}, \bar{\eta}^1, ..., \bar{\eta}^4)$ defined on a domain $U_2 \subset \bar{M}$.

Definition 5.1. Let $\pi : N \to M$ be a submersion. A vector subbundle $J \subset T^*N$ is said to be *transversal* to π if, at each point $p \in N$, $(J_p)^\perp \cap \ker_p(\pi_*) = \mathbf{0} \subset T_pN$.

By Definitions 2.11 and 2.12 of Chapter 2, there exists a rank-two subbundle $J \subset B^1$ that is transversal to π and satisfy $B^1 = J \oplus \pi^*(I^1)$. Similarly, there exists a rank-two subbundle $\bar{J} \subset B^1$, transversal to $\bar{\pi}$, satisfying $B^1 = \bar{J} \oplus \bar{\pi}^*(\bar{I}^1)$. Clearly, $B^1 \subset T^*N$ has rank three.

Now we start to formulate the equivalence problem for rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. At several points of our analysis, we define *genericity conditions* to help us distinguish between cases.

To start with, we define the

> **First genericity condition**: $\pi^*\theta$ and $\bar{\pi}^*\bar{\theta}$, as sections of B^1, are linearly independent at each point of their common domain in N.

This first genericity condition will always be assumed in this chapter. As a result of this assumption, on an open subset $U \subset N$, there exists a 1-form $\gamma \in \Omega^1(U)$ such that $\pi^*\theta, \bar{\pi}^*\bar{\theta}, \gamma$ form local basis sections of the bundle $B^1 \to N$. Moreover, by the definition of an integrable extension, γ can be chosen in such a way that $\pi^*\bar{\theta}, \gamma$ (resp. $\pi^*\theta, \gamma$) restrict to each fiber of π (resp. $\bar{\pi}$) to be linearly independent.

We have thus obtained a coframing $(\pi^*\theta, \bar{\pi}^*\bar{\theta}, \gamma, \pi^*\eta^1, \pi^*\eta^2, \pi^*\eta^3, \pi^*\eta^4)$ on $U \subset N$. Dropping the pull-back symbol for convenience, we have, by Definition 2.12,

$$\mathcal{B} = \langle \theta, \bar{\theta}, \gamma, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle_{\text{alg}} = \langle \theta, \bar{\theta}, \gamma, \bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4 \rangle_{\text{alg}}.$$

Now we define the

Second genericity condition: $\mathrm{d}\theta$ and $\mathrm{d}\bar{\theta}$ are everywhere linearly independent modulo $\theta, \bar{\theta}, \gamma$.

We now assume the second genericity condition[2]. Consequently, as rank-2 vector bundles over U,

$$[\![\mathrm{d}\theta, \mathrm{d}\bar{\theta}]\!] \equiv [\![\eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4]\!] \equiv [\![\bar{\eta}^1 \wedge \bar{\eta}^2, \bar{\eta}^3 \wedge \bar{\eta}^4]\!] \qquad \mathrm{mod}\ \theta, \bar{\theta}, \gamma.$$

In addition, we have

$$\mathcal{B} = \langle \theta, \bar{\theta}, \gamma, \mathrm{d}\theta, \mathrm{d}\bar{\theta} \rangle_{\mathrm{alg}}.$$

Such a choice of coframing on $U \subset N$ can be normalized. In fact, one can start with a 0-adapted coframing $(\theta, \eta^1, ..., \eta^4)$ on U that satisfies the additional condition

$$\mathrm{d}\theta \equiv \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4 \qquad \mathrm{mod}\ \theta.$$

The same congruence then holds for the pull-backs of $\theta, \eta^1, ..., \eta^4$ under π. After the change of notations $\theta \mapsto \omega^0$, $\bar{\theta} \mapsto \bar{\omega}^0$, and $\eta^i \mapsto \omega^i$, we obtain, on U,

$$\mathrm{d}\omega^0 \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \qquad \mathrm{mod}\ \omega^0, \tag{5.1}$$

$$\mathrm{d}\bar{\omega}^0 \equiv A_1\omega^1 \wedge \omega^2 + A_2\omega^3 \wedge \omega^4 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \gamma, \tag{5.2}$$

$$\mathrm{d}\gamma \equiv A_3\omega^1 \wedge \omega^2 + A_4\omega^3 \wedge \omega^4 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \gamma, \tag{5.3}$$

where $A_1, ..., A_4$ are functions defined on U (at this point, A_i are essentially functions on $\pi(U)$). Here A_1, A_2 are non-vanishing because $\bar{\omega}^0 \wedge (\mathrm{d}\bar{\omega}^0)^2 \neq 0$. Moreover, $A_1 \neq A_2$ by the second genericity assumption.

As a result of the above, one can successively perform the following steps without violating the normalizations made in earlier steps:

1) add multiples of $\omega^0, \bar{\omega}^0$ to γ to arrange $A_3 = A_4 = 0$;

2) add multiples of ω^0 to $\omega^1, ..., \omega^4$ such that the congruence (5.2) holds modulo only $\bar{\omega}^0$ and γ;

[2] This assumption is to be removed in Section 5.3.

3) scale $\bar{\omega}^0$ to arrange $A_1 = 1$;

4) depending on the sign of A_2, scale ω^3 and ω^0 to put (5.1) and (5.2) in the form

$$\begin{aligned} \mathrm{d}\omega^0 &\equiv A\omega^1 \wedge \omega^2 + \ \omega^3 \wedge \omega^4 \qquad \mathrm{mod}\ \omega^0, \\ \mathrm{d}\bar{\omega}^0 &\equiv \omega^1 \wedge \omega^2 + \epsilon A\omega^3 \wedge \omega^4 \qquad \mathrm{mod}\ \bar{\omega}^0, \gamma, \end{aligned}$$

where $\epsilon = \pm 1$, $A > 0$, and $A^2 \neq \epsilon$.

5) if needed, switch (ω^1, ω^2) with (ω^3, ω^4) and scale ω^0 and $\bar{\omega}^0$ to arrange $A \geqslant 1$;

6) add multiples of γ to ω^3, ω^4 such that

$$\mathrm{d}\omega^0 \equiv A\omega^1 \wedge \omega^2 + \ \omega^3 \wedge \omega^4 + (B_3\omega^3 + B_4\omega^4) \wedge \gamma \qquad \mathrm{mod}\ \omega^0, \tag{5.4}$$

$$\mathrm{d}\bar{\omega}^0 \equiv \omega^1 \wedge \omega^2 + \epsilon A\omega^3 \wedge \omega^4 + (B_1\omega^1 + B_2\omega^2) \wedge \gamma \qquad \mathrm{mod}\ \bar{\omega}^0, \tag{5.5}$$

where $B_1, ..., B_4$ are functions defined on U. Note, in particular, that there cannot be a $(\omega^0 \wedge \gamma)$-term in $\mathrm{d}\bar{\omega}^0$, because $\bar{\omega}^0 \wedge (\mathrm{d}\bar{\omega}^0)^3 = 0$.

Finally, there exist functions $C_0, C_1, ..., C_4$, $D_1, ..., D_4$ on U such that

$$\mathrm{d}\gamma \equiv C_0\omega^0 \wedge \bar{\omega}^0 + C_i\omega^i \wedge \omega^0 + D_i\omega^i \wedge \bar{\omega}^0 \qquad \mathrm{mod}\ \gamma. \tag{5.6}$$

Now we define the

> **Third genericity condition**: C_0 is locally non-vanishing; by multiplying γ by $\frac{1}{C_0}$, we arrange $C_0 = 1$.

Remark 9. The parameter ϵ is intrinsic to a rank-2 Bäcklund transformation. It determines whether $(\mathrm{d}\omega^0)^2$ and $(\mathrm{d}\bar{\omega}^0)^2$ define the same or the opposite orientations on the rank-4 distribution $(B^1)^\perp \subset TU$.

For the rest of this section, we will focus on the case when all three genericity conditions are satisfied and $\epsilon = 1$. In particular, this enforces $A > 1$ after the normalization above.

Definition 5.2. Let $(N, B; \pi, \bar{\pi})$ be a rank-2 Bäcklund transformation satisfying all three genericity conditions and $\epsilon = 1$. A coframing $(\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$ defined on a domain $U \subset N$ is said to be 0-*adapted* if it satisfies

$$[\![\omega^0]\!] = [\![\pi^*\theta]\!], \quad [\![\bar{\omega}^0]\!] = [\![\pi^*\bar{\theta}]\!], \quad [\![\omega^0, \bar{\omega}^0, \gamma]\!] = B^1 \tag{5.7}$$

and the equations

$$\mathrm{d}\omega^0 \equiv A\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + (B_3\omega^3 + B_4\omega^4) \wedge \gamma \qquad \mod \omega^0, \tag{5.8}$$

$$\mathrm{d}\bar{\omega}^0 \equiv \omega^1 \wedge \omega^2 + A\omega^3 \wedge \omega^4 + (B_1\omega^1 + B_2\omega^2) \wedge \gamma \qquad \mod \bar{\omega}^0, \tag{5.9}$$

$$\mathrm{d}\gamma \equiv \omega^0 \wedge \bar{\omega}^0 + C_i\omega^i \wedge \omega^0 + D_i\omega^i \wedge \bar{\omega}^0 \qquad \mod \gamma, \tag{5.10}$$

with $A > 1$.

Lemma 5.1. *For a rank-2 Bäcklund transformation $(N, \mathcal{B})$ satisfying all three genericity conditions and $\epsilon = 1$, the 0-adapted coframings are local sections of a G-structure $\mathcal{G}$ over N, where $G \subset \mathrm{GL}(7, \mathbb{R})$ is generated by*

$$g = \begin{pmatrix} \det(b) & 0 & 0 & 0 & 0 \\ 0 & \det(a) & 0 & 0 & 0 \\ 0 & 0 & \det(a)\det(b) & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \qquad \begin{matrix} \det(a) = \det(b) \neq 0, \\ a, b \in \mathrm{GL}(2, \mathbb{R}). \end{matrix} \tag{5.11}$$

Proof. Given a 0-adapted coframing ω define on $U \subset N$, it is easy to check that $\omega \cdot g = g^{-1}\omega$ remains a 0-adapted coframing for any $g : U \to G$.

Converserly, by (5.7) and $A > 1$, changing from one 0-adapted coframing to another does not change the order of the bundles $[\![\omega^0, \bar{\omega}^0, \gamma, \omega^1, \omega^2]\!]$ and $[\![\omega^0, \bar{\omega}^0, \gamma, \omega^3, \omega^4]\!]$. Consequently, if ω_1 and ω_2, both defined on $U \subset N$, are two 0-adapted coframings, then there exist a function $g : U \to \mathrm{GL}(7, \mathbb{R})$ of the form

$$g = \begin{pmatrix} \Psi & 0 & 0 \\ * & a & 0 \\ * & 0 & b \end{pmatrix},$$

where $\Psi \in \mathrm{GL}(3,\mathbb{R})$ is lower triangular, and $a, b \in \mathrm{GL}(2,\mathbb{R})$, such that $\omega_2 = \omega_1 \cdot g$. It is then not hard to see that, in order for ω_1, ω_2 to both satisfy (5.8), (5.9), and (5.10), g must be of the form (5.11). □

Concerning 0-adapted coframings, we prove the following

Lemma 5.2. (**i**) *Let* $\Xi_{10} = [\![\theta, \eta^1, \eta^2]\!]$ *and* $\Xi_{01} = [\![\theta, \eta^3, \eta^4]\!]$*, associated to* $(M, \mathcal{I})$*. As vector bundles over* $U_1 \subset M$*, their pull-backs via* π *are, up to ordering,* $[\![\omega^0, \omega^1, \omega^2]\!]$ *and* $[\![\omega^0, \omega^3 - B_4\gamma, \omega^4 + B_3\gamma]\!]$.

(**ii**) *Let* $\bar{\Xi}_{10} = [\![\bar{\theta}, \bar{\eta}^1, \bar{\eta}^2]\!]$ *and* $\bar{\Xi}_{01} = [\![\bar{\theta}, \bar{\eta}^3, \bar{\eta}^4]\!]$*, associated to* $(\bar{M}, \bar{\mathcal{I}})$*. As vector bundles over* $U_2 \subset \bar{M}$*, their pull-backs via* $\bar{\pi}$ *are, up to ordering,* $[\![\bar{\omega}^0, \omega^3, \omega^4]\!]$ *and* $[\![\bar{\omega}^0, \omega^1 - B_2\gamma, \omega^2 + B_1\gamma]\!]$.

Proof. We only prove (**i**); the proof of (**ii**) is identical.

By (5.8), the vector bundle associated to the Cartan system $C(\langle\omega^0\rangle)$ has a basis of sections ω^0, ω^1, ω^2, $\omega^3 - B_4\gamma$ and $\omega^4 + B_3\gamma$. Since $\pi^*(\mathcal{I}) \subset \mathcal{B}$, $\pi^*([\![\theta]\!]) = [\![\omega^0]\!]$ and $C(\mathcal{I}) = C(\langle\theta\rangle)$, the ideal generated by $\pi^*(\mathcal{I})$ must be contained in the intersection of $\mathcal{B}$ and $C(\langle\omega^0\rangle)$, which equals to

$$\langle \omega^0, \omega^1 \wedge \omega^2, (\omega^3 - B_4\gamma) \wedge (\omega^4 + B_3\gamma)\rangle_{\mathrm{alg}}. \tag{5.12}$$

It follows that the system generated by the sections of $\pi^*(\mathcal{I})$ is equal to $\mathcal{B} \cap C(\langle\omega^0\rangle)$. Comparing the characteristic systems lead to (**i**). □

By Lemma 5.2, the 1-forms $\omega^0, \omega^1, \omega^2, \omega^3 - B_4\gamma, \omega^4 + B_3\gamma$ are π-semi-basic. Moreover, the exterior differential system (5.12) is invariant in the fiber directions of π (see Definition 2.18).

To be explicit, let $X_0, \bar{X}_0, X_\gamma, X_1, ..., X_4$ be the dual vector fields of $\omega^0, \bar{\omega}^0, \gamma, \omega^1, \omega^2$, $\omega^3 - B_4\gamma, \omega^4 + B_3\gamma$; and let $Y_0, \bar{Y}_0, Y_\gamma, Y_1, ..., Y_4$ be the dual vector fields of $\omega^0, \bar{\omega}^0, \gamma, \omega^1 - B_2\gamma, \omega^2 + B_1\gamma, \omega^3, \omega^4$. One can show that any 0-adapted coframing defined on an open subset $U \subset N$ must satisfy the

Invariance Property:

(Let $\Gamma(K)$ denote the set of sections of a bundle $K \to U$; let $\sigma \in \Gamma(K)$ be an *arbitrary* element; let $\mathcal{L}_X$ denote the Lie derivative along a vector field X.)

When K is either $[\![\omega^0, \omega^1, \omega^2]\!]$ or $[\![\omega^0, \omega^3 - B_4\gamma, \omega^4 + B_3\gamma]\!]$, we have

$$\mathcal{L}_{\bar{X}_0}\sigma,\ \mathcal{L}_{X_\gamma}\sigma \in \Gamma(K), \quad \forall\sigma \in \Gamma(K); \tag{5.13}$$

When K is either $[\![\bar{\omega}^0, \omega^3, \omega^4]\!]$ or $[\![\bar{\omega}^0, \omega^1 - B_2\gamma, \omega^2 + B_1\gamma]\!]$, we have

$$\mathcal{L}_{Y_0}\sigma,\ \mathcal{L}_{Y_\gamma}\sigma \in \Gamma(K), \quad \forall\sigma \in \Gamma(K). \tag{5.14}$$

Proposition 5.1. *Let N be a 7-manifold. Suppose that there exists a coframing $\omega = (\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$ defined on a domain $U \subset N$ satisfying* (5.8)*,* (5.9)*, and* (5.10) *with $A > 1$. Suppose, in addition, that ω satisfies the* Invariance Property *described by* (5.13) *and* (5.14)*. Such a coframing ω is then* 0*-adapted to a rank-*2 *Bäcklund transformation satisfying all three genericity conditions with $\epsilon = 1$.*

Proof. The proof is in the same spirit as that of Proposition 3.1 of Chapter 3, so we omit the most of it. However, we need to show, for instance, that the system $\langle\omega^0, \omega^1, \omega^2, \omega^3 - B_4\gamma, \omega^4 + B_3\gamma\rangle$ is Frobenius, in order to justify the existence of rank-2 fibers. This amounts to showing that $[\bar{X}_0, X_\gamma]$ is annihilated by these 5 generating 1-forms. For example, we have

$$\omega^1([\bar{X}_0, X_\gamma]) = -\mathrm{d}\omega^1(\bar{X}_0, X_\gamma) = -(\bar{X}_0 \lrcorner \mathrm{d}\omega^1)(X_\gamma).$$

By (5.13), $\bar{X}_0 \lrcorner \mathrm{d}\omega^1$ is a linear combination of $\omega^0, \omega^1, \omega^2$, hence annihilates X_γ. Other cases are similar. □

A simple calculation shows that, under the transformation $u \mapsto u \cdot g = g^{-1}u \in \mathcal{G}$, where g is as in (5.11), the coefficients B_1, B_2, B_3, B_4 in (5.8) and (5.9) transform by

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u \cdot g) = \det(a)a^T \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u), \qquad \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u \cdot g) = \det(b)b^T \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u).$$

Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2\backslash\{0\}$, we can always, by choosing an appropriate 0-adapted coframing, reduce to one of the following 4 cases:

Case 1: $B_1 = B_3 = 1$, $B_2 = B_4 = 0$;

Case 2: $B_i = 0$ $(i = 1, ..., 4)$;

Case 3: $B_2, B_3, B_4 = 0$, $B_1 = 1$;

Case 3′: $B_1, B_2, B_4 = 0$, $B_3 = 1$.

Cases 3 and 3′ may be turned in to one another by switching the submersions π and $\bar{\pi}$. We thus consider them as essentially one case.

Geometrically, in Cases 1 and 3, the fibers of π and $\bar{\pi}$ are transversal; in Case 2, the fibers of π and $\bar{\pi}$ are non-transversal.

We will mainly focus on Case 1. In particular, we will prove that this seemingly most 'generic' case contains no homogeneous Bäcklund transformation. Case 2 turns out to be less interesting, in that locally N admits a 6-dimensional quotient which is a rank-1 Bäcklund transformation relating the same pair of hyperbolic Monge-Ampère systems. Case 3 is work in progress.

5.1.1 Case 1: $(B_1, B_2) = (B_3, B_4) = (1, 0)$

This reduces to a G_1-structure $\mathcal{G}_1 \subset \mathcal{G}$. Here, G_1 consists of elements $g \in G$ satisfying $a_{12} = b_{12} = 0$ and $a_{22}(a_{11})^2 = b_{22}(b_{11})^2 = 1$. In particular, G_1 is 3-dimensional.

We can apply the reproducing property (Proposition 2.2 of Chapter 2) to the equations (5.8)-(5.10) to obtain the structure equations on $\mathcal{G}_1$:

$$\mathrm{d}\begin{pmatrix}\omega^0\\ \bar{\omega}^0\\ \gamma\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 2\alpha & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\alpha & 0 & 0 & 0\\ 0 & 0 & 0 & \beta_1 & 2\alpha & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\alpha & 0\\ 0 & 0 & 0 & 0 & 0 & \beta_2 & 2\alpha\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \bar{\omega}^0\\ \gamma\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} + \begin{pmatrix}\Omega^0\\ \bar{\Omega}^0\\ \Gamma\\ \Omega^1\\ \Omega^2\\ \Omega^3\\ \Omega^4\end{pmatrix}, \quad (5.15)$$

where $[\![\alpha, \beta_1, \beta_2]\!] \subset T^*\mathcal{G}_1$ has rank 3 and is transversal to the fibers of $\mathcal{G}_1$, and

$$\begin{aligned}
\Omega^0 &= A\omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma) + (\bar{P}_0\bar{\omega}^0 + K\gamma + P_i\omega^i) \wedge \omega^0, \\
\bar{\Omega}^0 &= \omega^1 \wedge (\omega^2 + \gamma) + A\omega^3 \wedge \omega^4 + (Q_0\omega^0 + L\gamma + Q_i\omega^i) \wedge \bar{\omega}^0, \\
\Gamma &= \omega^0 \wedge \bar{\omega}^0 + C_i\omega^i \wedge \omega^0 + D_i\omega^i \wedge \bar{\omega}^0, \qquad (A > 1)
\end{aligned} \tag{5.16}$$

for functions $A, C_i, D_i, \bar{P}_0, K, P_i, Q_0, L, Q_i$ $(i = 1, ..., 4)$ defined on $\mathcal{G}_1$. In particular, Γ takes the form above because one can add any linear combination of $\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4$ to α without changing the form of (5.15) and (5.16).

Let $U \subset N$ be a sufficiently small domain. Let $\sigma : U \to \mathcal{G}_1$ be any section. The pull-back by σ of the tautological 1-form on $\mathcal{G}_1$ must then satisfy the *Invariance Property* (see (5.13) and (5.14)).

Lemma 5.3. (**A**) *If we express the 2-forms Ω^1, Ω^2 under the basis $\{\omega^0 \wedge \bar{\omega}^0, ..., \omega^3 \wedge \omega^4\}$, the coefficients of the following terms are zero:*

$$\gamma \wedge \omega^3, \quad \gamma \wedge \omega^4, \quad \gamma \wedge \bar{\omega}^0, \quad \bar{\omega}^0 \wedge \omega^3, \quad \bar{\omega}^0 \wedge \omega^4.$$

Proof. By construction, the vector fields $\bar{X}_0, X_\gamma$ are annihilated by the 1-forms $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4 + \gamma$ and are dual to $\bar{\omega}^0$ and γ. Using this and (5.15), we obtain

$$\begin{aligned}
&\mathcal{L}_{\bar{X}_0}\omega^0 \equiv 0 \mod \omega^0, \\
&\mathcal{L}_{\bar{X}_0}\omega^i \equiv \bar{X}_0 \lrcorner \Omega^i \mod \omega^0, \omega^1, \omega^2 \quad (i = 1, 2), \\
&\mathcal{L}_{X_\gamma}\omega^0 \equiv 0 \mod \omega^0, \\
&\mathcal{L}_{X_\gamma}\omega^i \equiv X_\gamma \lrcorner \Omega^i \mod \omega^0, \omega^1, \omega^2 \quad (i = 1, 2).
\end{aligned}$$

The conclusion then follows from the Invariance Property. □

By similar arguments, we can prove

Lemma 5.3. (**B**) *If we express the 2-forms Ω^1 and $\Omega^2 - \gamma \wedge C_2\omega^0 + (C_3\omega^3 + C_4\omega^4) \wedge \omega^0$ under the basis $\{\omega^0 \wedge \bar{\omega}^0, ..., \omega^3 \wedge \omega^4\}$, the coefficients of the terms in each pair below* *are the same:*

$$(\gamma \wedge \omega^0, \omega^2 \wedge \omega^0), \quad (\gamma \wedge \omega^3, \omega^2 \wedge \omega^3), \quad (\gamma \wedge \omega^4, \omega^2 \wedge \omega^4);$$

the coefficients of the following terms are zero:

$$\omega^0 \wedge \omega^3, \quad \omega^0 \wedge \omega^4.$$

(**C**) *In the expressions of* Ω^3 *and* Ω^4*, the coefficients of the following terms are zero:*

$$\gamma \wedge \omega^1, \quad \gamma \wedge \omega^2, \quad \gamma \wedge \omega^0, \quad \omega^0 \wedge \omega^1, \quad \omega^0 \wedge \omega^2.$$

(**D**) *In the expressions of* Ω^3 *and* $\Omega^4 - \gamma \wedge D_4\bar{\omega}^0 + (D_1\omega^1 + D_2\omega^2) \wedge \bar{\omega}^0$*, the coefficients of the terms in each pair below are the same:*

$$(\gamma \wedge \omega^1, \omega^4 \wedge \omega^1), \quad (\gamma \wedge \omega^2, \omega^4 \wedge \omega^2), \quad (\gamma \wedge \bar{\omega}^0, \omega^4 \wedge \bar{\omega}^0);$$

the coefficients of the following terms are zero:

$$\bar{\omega}^0 \wedge \omega^1, \quad \bar{\omega}^0 \wedge \omega^2.$$

By Lemma 5.3, $\Omega^1, ..., \Omega^4$ must take the form

$$\Omega^1 = T^1_0\omega^0 \wedge \bar{\omega}^0 + \omega^0 \wedge (T^1_1\omega^1 + T^1_2(\omega^2 + \gamma)) + \bar{\omega}^0 \wedge (\bar{T}^1_1\omega^1 + \bar{T}^1_2\omega^2)$$
$$+ \gamma \wedge (R^1_1\omega^1 + R^1_2\omega^2) + \frac{1}{2}T^1_{ij}\omega^i \wedge \omega^j, \tag{5.17}$$

$$\Omega^2 = T^2_0\omega^0 \wedge \bar{\omega}^0 + \omega^0 \wedge (C_3\omega^3 + C_4\omega^4 + T^2_1\omega^1 + T^2_2(\omega^2 + \gamma)) + \bar{\omega}^0 \wedge (\bar{T}^2_1\omega^1 + \bar{T}^2_2\omega^2)$$
$$+ \gamma \wedge (C_2\omega^0 + R^2_1\omega^1 + R^2_2\omega^2) + \frac{1}{2}T^2_{ij}\omega^i \wedge \omega^j, \tag{5.18}$$

$$\Omega^3 = T^3_0\bar{\omega}^0 \wedge \omega^0 + \bar{\omega}^0 \wedge (\bar{T}^3_3\omega^3 + \bar{T}^3_4(\omega^4 + \gamma)) + \omega^0 \wedge (T^3_3\omega^3 + T^3_4\omega^4)$$
$$+ \gamma \wedge (R^3_3\omega^3 + R^3_4\omega^4) + \frac{1}{2}T^3_{ij}\omega^i \wedge \omega^j, \tag{5.19}$$

$$\Omega^4 = T^4_0\bar{\omega}^0 \wedge \omega^0 + \bar{\omega}^0 \wedge (D_1\omega^1 + D_2\omega^2 + \bar{T}^4_3\omega^3 + \bar{T}^4_4(\omega^4 + \gamma)) + \omega^0 \wedge (T^4_3\omega^3 + T^4_4\omega^4)$$
$$+ \gamma \wedge (D_4\bar{\omega}^0 + R^4_3\omega^3 + R^4_4\omega^4) + \frac{1}{2}T^4_{ij}\omega^i \wedge \omega^j, \tag{5.20}$$

where $T^k_{ij} = -T^k_{ji}$, and $T^1_{23}, T^1_{24}, T^2_{23}, T^2_{24}, T^3_{14}, T^3_{24}, T^4_{14}, T^4_{24}$ are zero.

The coefficients in $\Omega^1, ..., \Omega^4$ are not all determined. In fact, by adding appropriate linear combinations of $\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4$ to β_1, we can arrange that

$$T^2_{12} = T^2_{13} = T^2_{14} = T^2_1 = \bar{T}^2_1 = R^2_1 = 0;$$

in a similar manner, by adjusting β_2, we can arrange that

$$\bar{T}^4_3 = T^4_3 = R^4_3 = T^4_{13} = T^4_{23} = T^4_{34} = 0.$$

As a standard step in the method of equivalence, we apply $\mathrm{d}^2 = 0$ to (5.15) and reduce modulo appropriate differential forms. From this, we obtain relations among the *torsion functions* (i.e., the coefficients of $\Omega^0, \bar{\Omega}^0, \Gamma, ..., \Omega^4$).

In particular, expanding the expressions

$$\mathrm{d}(\mathrm{d}\omega^0) \mod \omega^0, \omega^1, \qquad \mathrm{d}(\mathrm{d}\omega^0) \mod \omega^0, \omega^2,$$

$$\mathrm{d}(\mathrm{d}\bar{\omega}^0) \mod \bar{\omega}^0, \omega^3, \qquad \mathrm{d}(\mathrm{d}\bar{\omega}^0) \mod \bar{\omega}^0, \omega^4,$$

and

$$\mathrm{d}(\mathrm{d}\gamma) \mod \gamma, \omega^0, \bar{\omega}^0,$$

we find the following relations:

$$T^3_{23} = P_2, \quad T^1_{34} = 0, \quad T^3_{34} = -P_4 + K - R^3_3 - R^4_4, \quad D_4 = -\bar{P}_0 + \bar{T}^3_3 + \bar{T}^4_4,$$

$$T^3_{13} = P_1, \quad T^2_{34} = 0, \quad T^1_{14} = -Q_4, \quad T^3_{12} = 0,$$

$$T^1_{12} = -Q_2 + L - R^1_1 - R^2_2, \quad C_2 = -Q_0 + T^1_1 + T^2_2, \quad T^1_{13} = -Q_3, \quad T^4_{12} = 0,$$

$$D_2 = -\frac{1}{A}C_2, \quad C_4 = -\frac{1}{A}D_4, \quad D_1 = -\frac{1}{A}C_1, \quad C_3 = -\frac{1}{A}D_3,$$

In the equations above, we replace the functions on the left-hand-side by expressions on the right-hand-side. Then we compute

$$\mathrm{d}(\mathrm{d}\omega^0) \mod \omega^0, \qquad \mathrm{d}(\mathrm{d}\bar{\omega}^0) \mod \bar{\omega}^0.$$

From this, we find that the expression of $\mathrm{d}A$ is determined:

$$\frac{\mathrm{d}A}{A} = (Q_0 - T_3^3 - T_4^4)\omega^0 + (P_0 - \bar{T}_1^1 - \bar{T}_2^2)\bar{\omega}^0 - (P_1 - Q_1)\omega^1 - (P_2 - Q_2)\omega^2$$
$$+ (P_3 - Q_3)\omega^3 + (P_4 - Q_4)\omega^4 + (L - R_3^3 - R_4^4)\gamma,$$

with the extra condition: $K - R_1^1 - R_2^2 = L - R_3^3 - R_4^4$. This shows that the function A ($A > 1$) is an invariant of the G_1-structure.

Homogeneity Assumption: Now we assume that the underlying rank-2 Bäcklund transformation is *homogeneous*, that is, its symmetry group acts locally transitively. Making this assumption will imply that any local structure invariant is a constant.

Following from the homogeneity assumption, $\mathrm{d}A = 0$. This implies

$$P_3 = Q_3, \quad P_4 = Q_4, \quad Q_1 = P_1, \quad Q_2 = P_2,$$
$$K = R_1^1 + R_2^2, \quad \bar{P}_0 = \bar{T}_1^1 + \bar{T}_2^2, \quad L = R_3^3 + R_4^4, \quad Q_0 = T_3^3 + T_4^4.$$

Further differentiation of the structure equations yields

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv T_2^1 \omega^3 \wedge \omega^4 \wedge \gamma \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^1, \omega^2,$$
$$\mathrm{d}(\mathrm{d}\omega^1) \equiv (A\bar{T}_2^1 + T_2^1)\omega^3 \wedge \omega^4 \wedge \omega^2 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^1, \gamma$$
$$\mathrm{d}(\mathrm{d}\omega^2) \equiv \frac{1}{A}[A(-T_3^3 - T_4^4 + T_1^1)$$
$$- \bar{T}_3^3 - \bar{T}_4^4 + \bar{T}_1^1 + \bar{T}_2^2]\omega^3 \wedge \gamma \wedge \omega^4$$
$$+ T_0^2 \omega^3 \wedge (\omega^4 + \gamma) \wedge \bar{\omega}^0 \qquad \mathrm{mod}\ \omega^0, \omega^1, \omega^2,$$

and

$$\mathrm{d}(\mathrm{d}\omega^3) \equiv \bar{T}_4^3 \omega^1 \wedge \omega^2 \wedge \gamma \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^3, \omega^4,$$
$$\mathrm{d}(\mathrm{d}\omega^3) \equiv (AT_4^3 + \bar{T}_4^3)\omega^1 \wedge \omega^2 \wedge \omega^4 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^3, \gamma,$$
$$\mathrm{d}(\mathrm{d}\omega^4) \equiv \frac{1}{A}[A(-\bar{T}_1^1 - \bar{T}_2^2 + \bar{T}_3^3)$$
$$- T_1^1 - T_2^2 + T_3^3 + T_4^4]\omega^1 \wedge \gamma \wedge \omega^2$$
$$+ T_0^4 \omega^1 \wedge (\omega^2 + \gamma) \wedge \omega^0 \qquad \mathrm{mod}\ \bar{\omega}^0, \omega^3, \omega^4.$$

These congruences imply

$$T^1_2 = \bar{T}^1_2 = T^2_0 = \bar{T}^3_4 = T^3_4 = T^4_0 = 0,$$

$$T^1_1 = T^3_3 + T^4_4 + \frac{1}{A^2-1}(A\bar{T}^4_4 + T^2_2), \qquad \bar{T}^3_3 = \bar{T}^1_1 + \bar{T}^2_2 + \frac{1}{A^2-1}(AT^2_2 + \bar{T}^4_4).$$

Using these, we compute

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv T^1_0\omega^3 \wedge (\omega^4+\gamma) \wedge \bar{\omega}^0 \qquad \mathrm{mod}\ \omega^0, \omega^1, \omega^2,$$

$$\mathrm{d}(\mathrm{d}\omega^3) \equiv T^3_0\omega^1 \wedge (\omega^2+\gamma) \wedge \omega^0 \qquad \mathrm{mod}\ \bar{\omega}^0, \omega^3, \omega^4,$$

which implies

$$T^1_0 = T^3_0 = 0.$$

In addition, we have

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv (\mathrm{d}R^1_2 - 5R^1_2\alpha) \wedge \gamma \wedge \omega^2 \quad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^1, \omega^3, \omega^4, \tag{5.21}$$

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv \frac{R^1_2}{A^2-1}(A\bar{T}^4_4 + T^2_2)\gamma \wedge \omega^0 \wedge \omega^4 + \frac{R^1_2 D_3}{A}\gamma \wedge \omega^0 \wedge \omega^3 \quad \mathrm{mod}\ \omega^1, \omega^2, \tag{5.22}$$

$$\mathrm{d}(\mathrm{d}\omega^3) \equiv (\mathrm{d}R^3_4 - 5R^3_4\alpha) \wedge \gamma \wedge \omega^4 \quad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \omega^1, \omega^2, \omega^3, \tag{5.23}$$

$$\mathrm{d}(\mathrm{d}\omega^3) \equiv \frac{R^3_4}{A^2-1}(AT^2_2 + \bar{T}^4_4)\gamma \wedge \bar{\omega}^0 \wedge \omega^2 + \frac{R^3_4}{A}C_1\gamma \wedge \bar{\omega}^0 \wedge \omega^1 \quad \mathrm{mod}\ \omega^3, \omega^4. \tag{5.24}$$

Lemma 5.4. *R^1_2 and R^3_4 are both zero on an open subset of $\mathcal{G}_1$.*

Proof. Suppose that locally $R^1_2 \neq 0$. The equation (5.22) then implies

$$D_3 = 0, \qquad T^2_2 = -A\bar{T}^4_4.$$

Following from this, we have

$$\mathrm{d}(\mathrm{d}\gamma) \equiv -A\omega^3 \wedge \omega^4 \wedge \omega^0 \qquad \mathrm{mod}\ \gamma, \omega^1, \omega^2, \bar{\omega}^0,$$

which is impossible because $A > 1$. Therefore, $R^1_2 = 0$. An analogous argument leads to $R^3_4 = 0$. □

Note that there remains freedom to add a multiple of γ to α without changing the form of the structure equations. Using this, we can arrange

$$R^1_1 + R^2_2 = -(R^3_3 + R^4_4).$$

Recall that $K = R^1_1 + R^2_2$. We now remind the reader that all torsion coefficients are expressed in terms of the constant A and the functions

$$K, \quad P_1, \quad P_2, \quad Q_3, \quad Q_4, \quad \bar{T}^1_1, \quad \bar{T}^2_2, \quad T^3_3, \quad T^4_4,$$
$$C_1, \quad T^2_2, \quad \bar{T}^4_4, \quad D_3; \quad R^1_1, \quad R^3_3.$$

The torsion cannot be absorbed further.

By applying $d^2 = 0$ to the structure equations, we can find how[3] G_1 acts on these remaining torsion functions. For simplicity, we introduce the new notation:

$$F_1 := \frac{A}{A^2-1}(AT^2_2 + \bar{T}^4_4) \qquad F_3 := \frac{A}{A^2-1}(A\bar{T}^4_4 + T^2_2).$$

Infinitesimally, the G_1-action on the torsion functions can be expressed as:

$$\mathrm{d}D_3 \equiv -2\alpha D_3 + F_3\beta_2, \qquad \mathrm{d}F_3 \equiv \alpha F_3,$$
$$\mathrm{d}C_1 \equiv -2\alpha C_1 + F_1\beta_1, \qquad \mathrm{d}F_1 \equiv \alpha F_1,$$
$$\mathrm{d}P_1 \equiv \beta_1 P_2 - \alpha P_1, \qquad \mathrm{d}P_2 \equiv 2\alpha P_2,$$
$$\mathrm{d}Q_3 \equiv \beta_2 Q_4 - \alpha Q_3, \qquad \mathrm{d}Q_4 \equiv 2\alpha Q_4,$$
$$\mathrm{d}K \equiv 2\alpha K, \qquad \mathrm{d}R^1_1 \equiv 2\alpha R^1_1, \qquad \mathrm{d}R^3_3 \equiv 2\alpha R^3_3,$$
$$\mathrm{d}\bar{T}^1_1 \equiv \alpha\bar{T}^1_1, \quad \mathrm{d}\bar{T}^2_2 \equiv \alpha\bar{T}^2_2, \quad \mathrm{d}T^3_3 \equiv \alpha T^3_3, \quad \mathrm{d}T^4_4 \equiv \alpha T^4_4,$$

where all congruences are modulo the semi-basic 1-forms $\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4$.

[3] What we can obtain by applying $d^2 = 0$ to the structure equations is essentially an *infinitesimal* version of the G_1-action on the torsion components. Hence, it only tells us how the identity component of G_1 acts. Performing a structure reduction using the action by the identity component of G_1 thus may not distinguish two equivalent coframings that differ by a discrete transformation. However, this would not pose a problem, as we can examine whether there is such an equivalence at the end of our classification.

In the proof of Lemma 5.4, we have seen that, when (D_3, F_3) or (C_1, F_1) locally vanishes, we have a contradiction. In particular, D_3 cannot locally vanish, otherwise, by the equation (5.22), F_3 must vanish. For a similar reason, C_1 cannot locally vanish. Hence, for the pair of torsion functions (D_3, F_3) (and similarly for the pair (C_1, F_1)), there are two possibilities:

I. Locally on $\mathcal{G}_1$, $D_3 \neq 0$, $F_3 = 0$. This implies that G_1 scales D_3;

II. Locally on $\mathcal{G}_1$, $F_3 \neq 0$. By the G_1-action on F_3, it is easy to see that $F_3 \neq 0$ on the entire group fibers. In this case, it is possible to reduce to a G_2'-structure $\mathcal{G}_2' \subset \mathcal{G}_1$, defined by $D_3 = 0$. Restricting to $\mathcal{G}_2'$, the 1-form β_2 becomes semi-basic.

Lemma 5.5. *Case* **I** *is empty.*

Proof. If $F_3 = 0$, then D_3 scales under the G_1-action. We can then reduce to the subbundle defined by either $D_3 = 1$ or $D_3 = -1$. Now let $\mathcal{G}_2 \subset \mathcal{G}_1$ be defined by $D_3 = 1$.

On $\mathcal{G}_2$, α is semi-basic. In other words, there exist functions $H_0, \bar{H}_0, H_\gamma, H_1, ..., H_4$ on $\mathcal{G}_2$ such that

$$\alpha = H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + H_1\omega^1 + ... + H_4\omega^4.$$

In addition, the following functions are constant along each fiber of $\mathcal{G}_2$

$$K, \quad P_2, \quad Q_4, \quad R_1^1, \quad R_3^3, \quad \bar{T}_1^1, \quad \bar{T}_2^2, \quad T_3^3, \quad T_4^4, \quad F_1.$$

By the homogeneity assumption, they must be constants on $\mathcal{G}_2$.

We now compute

$$\begin{aligned} \mathrm{d}(\mathrm{d}\gamma) \equiv\ & \frac{1}{A}(2AH_2 + 2AP_2 - F_1H_3 + F_1Q_3)\omega^2 \wedge \omega^3 \wedge \bar{\omega}^0 \\ & + (1 - 2Q_4 + 2K - 2H_4)\omega^3 \wedge \omega^4 \wedge \bar{\omega}^0 \\ & + \frac{1}{A}F_1(H_4 - Q_4)\omega^4 \wedge \omega^2 \wedge \bar{\omega}^0 \qquad \mathrm{mod}\ \omega^0, \gamma, \omega^1. \end{aligned}$$

If $F_1 \neq 0$, we must have $Q_4 = H_4$ and $2K = 4H_4 - 1$. Using this, we compute

$$\mathrm{d}(\mathrm{d}\gamma) \equiv -\frac{1}{A}(A^2 - 1)\omega^0 \wedge \omega^3 \wedge \omega^4 \qquad \text{mod } \bar{\omega}^0, \gamma, \omega^1, \omega^2,$$

which is impossible since $A > 1$.

It follows that, $F_1 = 0$ on $\mathcal{G}_2$. With this, we compute

$$\mathrm{d}(\mathrm{d}\gamma) \equiv 2(P_2\omega^2 + Q_4\omega^4) \wedge \omega^0 \wedge \bar{\omega}^0 + \cdots \quad \text{mod } \gamma, \omega^1.$$

Therefore, $P_2 = Q_4 = 0$. Using this, we have

$$\mathrm{d}(\mathrm{d}\gamma) \equiv (1+2K-2H_4)\omega^3 \wedge \omega^4 \wedge \bar{\omega}^0 - \frac{1}{A}(A^2+2K-2H_4)\omega^0 \wedge \omega^3 \wedge \omega^4 \quad \text{mod } \gamma, \omega^1, \omega^0 \wedge \bar{\omega}^0.$$

This implies $A^2 = 1$, which is impossible, by assumption. □

By a similar argument, one can show that there is no homogeneous structure in the case when $C_1 \neq 0$ and $F_1 = 0$. This, together with Lemma 5.5, implies that the only case remaining is when both F_3 and F_1 are locally nonzero. In this case, one can reduce to the subbundle on which $D_3 = C_1 = 0$. There, β_1 and β_2 both become semi-basic. Further, we can reduce to an e-structure on which $F_3 = 1$. On this e-structure, F_1 is a nonzero constant, by homogeneity.

Lemma 5.6. *The case when both F_3 and F_1 are locally nonzero is empty.*

Proof. On the e-structure defined by $D_3 = C_1 = 0$ and $F_3 = 1$, the 1-forms α, β_1 and β_2 are semi-basic:

$$\alpha = H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + H_1\omega^1 + \ldots + H_4\omega^4,$$

$$\beta_1 = M_0\omega^0 + \bar{M}_0\bar{\omega}^0 + M_\gamma\gamma + M_1\omega^1 + \ldots + M_4\omega^4,$$

$$\beta_2 = N_0\omega^0 + \bar{N}_0\bar{\omega}^0 + N_\gamma\gamma + N_1\omega^1 + \ldots + N_4\omega^4.$$

Differentiation gives

$$\mathrm{d}(\mathrm{d}\gamma) \equiv -\frac{1}{A}(A^2 - H_3 + N_4 + Q_3)\omega^3 \wedge \omega^4 \wedge \omega^0 + (1 - H_3 + N_4 + Q_3)\omega^3 \wedge \omega^4 \wedge \bar{\omega}^0$$
$$\text{mod } \gamma, \omega^1, \omega^2, \omega^0 \wedge \bar{\omega}^0.$$

Clearly, $A^2 = 1$, which is a contradiction. $\square$

Combining Lemmas 5.5 and 5.6, the following theorem is immediate.

Theorem 5.7. *There exist no homogeneous rank-2 Bäcklund transformation satisfying all three genericity conditions, $\epsilon = 1$, $(B_1, B_2) \neq 0$, and $(B_3, B_4) \neq 0$.*

Remark 10. (**A**) In Theorem 5.7, the condition $\epsilon = 1$ can be removed. The case when $\epsilon = -1$ has a proof that is only a slight modification of the arguments above.

(**B**) It still remains to be answered whether there exists a (non-homogeneous) rank-2 Bäcklund transformation satisfying all three genericity conditions and $(B_1, B_2) \neq 0, (B_3, B_4) \neq 0$.

5.1.2 Case 2: $B_i = 0$ $(i = 1, ..., 4)$

In this case, we prove the

Proposition 5.2. *Suppose that $(N, \mathcal{B})$ is a rank-2 Bäcklund transformation (not necessarily homogeneous) satisfying all three genericity conditions, $\epsilon = 1$ and $B_i = 0$ $(i = 1, ..., 4)$, then $(N, \mathcal{B})$ admits a 6-dimensional quotient that is a rank-1 Bäcklund transformation relating the same pair of hyperbolic Monge-Ampère systems.*

Proof. Using previous notation, the vector fields X_γ, Y_γ coincide when $B_i = 0$ $(i = 1, ..., 4)$. Shrink N if needed, and let N' be the quotient of N by the flow of X_γ. The Invariance Property ((5.13) and (5.14)) then implies that the vector bundles $[\![\omega^0, \omega^1, \omega^2]\!]$, $[\![\omega^0, \omega^3, \omega^4]\!]$, $[\![\bar{\omega}^0, \omega^1, \omega^2]\!]$ and $[\![\bar{\omega}^0, \omega^3, \omega^4]\!]$ annihilate X_γ and are invariant under the flow of X_γ. Hence, their intersections are the pull-backs of vector subbundles of T^*N'. In particular, there locally exist 1-forms $\eta^0, \bar{\eta}^0, \eta^1, ..., \eta^4$ on N' such that the following relations hold (dropping the pull-back symbol): $[\![\omega^0]\!] = [\![\eta^0]\!]$, $[\![\bar{\omega}^0]\!] = [\![\bar{\eta}^0]\!]$, $[\![\omega^1, \omega^2]\!] = [\![\eta^1, \eta^2]\!]$, $[\![\omega^3, \omega^4]\!] = [\![\eta^3, \eta^4]\!]$. This implies that pull-back of the corresponding Monge-Ampère systems are respectively $\langle \eta^0, \eta^1 \wedge \eta^2, \eta^3 \wedge \eta^4 \rangle$

and $\langle\bar{\eta}^0, \eta^1\wedge\eta^2, \eta^3\wedge\eta^4\rangle$. It follows that $(N', \mathcal{B}')$, with $\mathcal{B}' = \langle\eta^0, \bar{\eta}^0, \eta^1\wedge\eta^2, \eta^3\wedge\eta^4\rangle$ and the obvious submersions to $M, \bar{M}$, defines a rank-1 Bäcklund transformation. $\square$

Remark 11. One can prove that the conclusion in Proposition 5.2 remains true when $\epsilon = -1$.

5.2 Assuming Genericity Conditions 1, 2

Without assuming the third genericity condition, we need a new definition of 0-adapted coframings.

Definition 5.3. Let $(N, B; \pi, \bar{\pi})$ be a rank-2 Bäcklund transformation (relating two hyperbolic Monge-Ampère systems) satisfying only the first two genericity conditions and $\epsilon = 1$. A coframing $(\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$ defined on an open subset $U \subset N$ is said to be 0-*adapted* if it satisfies

$$[\![\omega^0]\!] = [\![\pi^*\theta]\!], \quad [\![\bar{\omega}^0]\!] = [\![\pi^*\bar{\theta}]\!], \quad [\![\omega^0, \bar{\omega}^0, \gamma]\!] = B^1$$

and

$$d\omega^0 \equiv A\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + (B_3\omega^3 + B_4\omega^4) \wedge \gamma \qquad \text{mod } \omega^0, \tag{5.25}$$

$$d\bar{\omega}^0 \equiv \omega^1 \wedge \omega^2 + A\omega^3 \wedge \omega^4 + (B_1\omega^1 + B_2\omega^2) \wedge \gamma \qquad \text{mod } \bar{\omega}^0, \tag{5.26}$$

$$d\gamma \equiv C_i\omega^i \wedge \omega^0 + D_i\omega^i \wedge \bar{\omega}^0 \qquad \text{mod } \gamma, \tag{5.27}$$

with $A > 1$.

Lemma 5.8. *Given a rank-2 Bäcklund transformation $(N, \mathcal{B})$ (relating two hyperbolic Monge-Ampère systems) satisfying only the first two genericity conditions and $\epsilon = 1$, its 0-adapted coframings are local sections of a G-structure $\mathcal{G}$ over N, where $G \subset \mathrm{GL}(7, \mathbb{R})$ is generated by*

$$g = \begin{pmatrix} \det(b) & 0 & 0 & 0 & 0 \\ 0 & \det(a) & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \qquad \begin{array}{c} \det(a) = \det(b) \neq 0, c \neq 0 \\ a = (a_{ij}), b = (b_{ij}) \in \mathrm{GL}(2, \mathbb{R}). \end{array} \tag{5.28}$$

Proof. We omit this proof since the arguments are similar to Lemma 5.1. □

It is easy to show that, on $\mathcal{G}$,

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u \cdot g) = \frac{c}{\det(a)} a^T \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u), \qquad \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u \cdot g) = \frac{c}{\det(b)} b^T \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u).$$

It follows that one can normalize B_i $(i = 1, ..., 4)$ to be one of the following:

Case 1: $B_1 = B_3 = 1$, $B_2 = B_4 = 0$;

Case 2: $B_i = 0$ $(i = 1, ..., 4)$;

Case 3: $B_2, B_3, B_4 = 0$, $B_1 = 1$;

Case 3′: $B_1, B_2, B_4 = 0$, $B_3 = 1$.

Lemma 5.9. *A rank-2 Bäcklund transformation in the current case arises as a 1-parameter family of rank-1 Bäcklund transformations relating the same pair of hyperbolic Monge-Ampère systems if and only if γ is integrable.*

Proof. Suppose that γ is integrable. It is easy to see that each leaf of γ is a rank-1 Bäcklund transformation. Conversely, suppose that $(N, \mathcal{B})$ is constructed in the natural way from a 1-parameter family of rank-1 Bäcklund transformations. Let t be the parameter. Then we have that B^1 is generated by the pull-back of θ, $\bar{\theta}$ and $\mathrm{d}t$. It follows that, given a local 0-adapted coframing on N, there exists a linear combination $\psi = \gamma + \lambda\omega^0 + \mu\bar{\omega}^0$ that is integrable. Computing $\mathrm{d}\psi$ and reducing modulo $\omega^0, \bar{\omega}^0$ and γ, it is easy to see, by (5.25)-(5.27), that λ and μ are zero. It follows that γ must be integrable. □

We will only be interested in the case when γ is *not integrable*. In the current case, this is given by the condition: locally the functions C_i, D_i $(i = 1, ..., 4)$ are not all zero.

5.2.1 Case: $(B_1, B_2) = (B_3, B_4) = (1, 0)$

In this case, we reduce to a G_1-structure $\mathcal{G}_1 \subset \mathcal{G}$, where the subgroup $G_1 \subset G$ is defined by (maintaining the notation in (5.28))

$$a_{22} = b_{22} = c, \quad a_{12} = b_{12} = 0.$$

The structure equations on $\mathcal{G}_1$ can be written as

$$\mathrm{d}\omega = -\Phi \wedge \omega + \mathbf{T}, \tag{5.29}$$

where

$$\omega = (\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)^T, \quad \mathbf{T} = (\Omega^0, \bar{\Omega}^0, \Gamma, \Omega^1, ..., \Omega^4)^T,$$

$$\Phi = \begin{pmatrix} \alpha+\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha+\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & \phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & \phi \end{pmatrix}.$$

Here, by the reproducing property of the tautological 1-form ω and by adding an appropriate multiple of γ to ϕ, we can arrange

$$\begin{aligned} \Omega^0 &= A\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + \omega^3 \wedge \gamma + (\bar{P}_0\bar{\omega}^0 + K\gamma + P_i\omega^i) \wedge \omega^0, \\ \bar{\Omega}^0 &= \omega^1 \wedge \omega^2 + A\omega^3 \wedge \omega^4 + \omega^1 \wedge \gamma + (Q_0\omega^0 + L\gamma + Q_i\omega^i) \wedge \bar{\omega}^0, \\ \Gamma &= C_i\omega^i \wedge \omega^0 + D_i\omega^i \wedge \bar{\omega}^0. \end{aligned}$$

It is not hard to see that Lemma 5.2 and Lemma 5.3 apply to the current case. Following from this, $\Omega^1, ..., \Omega^4$ have expressions (5.17)-(5.20) where $T^k_{ij} = -T^k_{ji}$, and $T^1_{23}, T^1_{24}, T^2_{23}, T^2_{24}, T^3_{14}, T^3_{24}, T^4_{14}, T^4_{24}$ are all zero.

In (5.29), the torsion $\mathbf{T}$ can be further absorbed by adjusting Φ. In fact, by adding appropriate semi-basic 1-forms to α, we can arrange

$$T^1_1 = -T^3_3, \quad \bar{T}^1_1 = -\bar{T}^3_3, \quad T^1_{12} = T^3_{23}, \quad T^1_{14} = -T^3_{34}, \quad R^1_1 = -R^3_3, \quad T^1_{13} = T^3_{13} = 0.$$

There remains freedom of adding a multiple of γ to ϕ without changing the form of the structure equations. Using this, we can arrange

$$K = -L.$$

By adding appropriate semi-basic 1-forms to β_1 and β_2, we can arrange

$$T^2_1 = \bar{T}^2_1 = R^2_1 = T^2_{12} = T^2_{13} = T^2_{14} = \bar{T}^4_3 = T^4_3 = R^4_3 = T^4_{13} = T^4_{23} = T^4_{34} = 0.$$

By expanding the expressions

$$\mathrm{d}(\mathrm{d}\omega^0) \mod \omega^0, \omega^1, \gamma, \qquad \mathrm{d}(\mathrm{d}\omega^0) \mod \omega^0, \omega^2, \gamma,$$

$$\mathrm{d}(\mathrm{d}\bar{\omega}^0) \mod \bar{\omega}^0, \omega^3, \gamma, \qquad \mathrm{d}(\mathrm{d}\bar{\omega}^0) \mod \bar{\omega}^0, \omega^4, \gamma$$

and

$$\mathrm{d}(\mathrm{d}\gamma) \mod \gamma, \omega^0, \bar{\omega}^0,$$

we find the following relations

$$P_2 = AT^1_{34} + T^3_{23}, \quad P_0 = \bar{T}^3_3 + \bar{T}^4_4 - D_4, \quad P_1 = AT^2_{34},$$

$$Q_4 = AT^3_{12} + T^3_{34}, \quad Q_0 = T^2_2 - T^3_3 - C_2, \quad Q_3 = -AT^4_{12},$$

$$D_2 = -\frac{1}{A}C_2, \quad C_3 = -\frac{1}{A}D_3, \quad C_4 = -\frac{1}{A}D_4, \quad D_1 = -\frac{1}{A}C_1.$$

Further calculation yields

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\omega^0) \equiv{}& (L + T^3_{34} + R^3_3 + R^4_4 + P_4)\gamma \wedge \omega^3 \wedge \omega^4 + (AL + AR^2_2 - AR^3_3 + T^3_{12})\omega^1 \wedge \omega^2 \wedge \gamma \\
&+ (-AP_4 + AT^3_{34} + T^3_{12})\omega^1 \wedge \omega^2 \wedge \omega^4 - AT^2_{34}\omega^1 \wedge \omega^3 \wedge \gamma - 2AT^2_{34}\omega^1 \wedge \omega^3 \wedge \omega^4 \\
&- AT^1_{34}\omega^2 \wedge \omega^3 \wedge \gamma + (-AP_3 - T^4_{12})\omega^3 \wedge \omega^1 \wedge \omega^2 \\
&+ A(D_4 + \bar{T}^2_2 - 2\bar{T}^3_3 - \bar{T}^4_4)\bar{\omega}^0 \wedge \omega^1 \wedge \omega^2 + \mathrm{d}A \wedge \omega^1 \wedge \omega^2 \qquad \mod \omega^0,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\bar{\omega}^0) \equiv{}& (L - T^3_{23} + R^3_3 - R^2_2 - Q_2)\omega^2 \wedge \omega^1 \wedge \gamma + (-AL + AR^3_3 + AR^4_4 + T^1_{34})\omega^3 \wedge \omega^4 \wedge \gamma \\
&+ (-AQ_2 + AT^3_{23} + T^1_{34})\omega^2 \wedge \omega^3 \wedge \omega^4 + (-AQ_1 - T^2_{34})\omega^1 \wedge \omega^3 \wedge \omega^4 \\
&+ AT^4_{12}\omega^3 \wedge \omega^1 \wedge \gamma - AT^3_{12}\omega^4 \wedge \omega^1 \wedge \gamma \\
&+ A(C_2 - T^2_2 + 2T^3_3 + T^4_4)\omega^0 \wedge \omega^3 \wedge \omega^4 + \mathrm{d}A \wedge \omega^3 \wedge \omega^4 \qquad \mod \bar{\omega}^0.
\end{aligned}$$

This completely determines $\mathrm{d}A$ as a linear combination of the semi-basic 1-forms. Hence, A is a structure invariant.

Homogeneity Assumption: Now we assume that the underlying rank-2 Bäcklund transformation is *homogeneous*.

By this assumption, $\mathrm{d}A$ must be zero. The previous two congruences imply

$$P_3 = 0, \quad Q_1 = 0; \quad P_4 = -L, \quad Q_2 = L; \quad R^2_2 = -R^4_4, \quad R^3_3 = L - R^4_4;$$
$$T^2_2 = C_2 + 2T^3_3 + T^4_4, \quad \bar{T}^2_2 = -D_4 + 2\bar{T}^3_3 + \bar{T}^4_4;$$
$$T^1_{34} = 0, \quad T^2_{34} = 0, \quad T^3_{12} = 0, \quad T^3_{23} = L, \quad T^3_{34} = -L, \quad T^4_{12} = 0.$$

Furthermore, we compute

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv T^1_2\omega^3 \wedge \omega^4 \wedge \gamma + T^1_0\omega^3 \wedge (\omega^4 + \gamma) \wedge \bar{\omega}^0 \qquad \mod \omega^0, \omega^1, \omega^2,$$
$$\mathrm{d}(\mathrm{d}\omega^2) \equiv T^2_0\omega^3 \wedge (\omega^4 + \gamma) \wedge \bar{\omega}^0 + \frac{1}{A}(2AT^3_3 + AT^4_4 + D_4)\omega^3 \wedge \omega^4 \wedge \gamma \qquad \mod \omega^0, \omega^1, \omega^2,$$
$$\mathrm{d}(\mathrm{d}\omega^3) \equiv \bar{T}^3_4\omega^1 \wedge \omega^2 \wedge \gamma + T^3_0\omega^1 \wedge (\omega^2 + \gamma) \wedge \omega^0 \qquad \mod \bar{\omega}^0, \omega^3, \omega^4,$$
$$\mathrm{d}(\mathrm{d}\omega^4) \equiv T^4_0\omega^1 \wedge (\omega^2 + \gamma) \wedge \omega^0 + \frac{1}{A}(AD_4 - A\bar{T}^4_4 - C_2)\omega^1 \wedge \gamma \wedge \omega^2 \qquad \mod \bar{\omega}^0, \omega^3, \omega^4.$$

This implies that

$$T^1_0 = T^2_0 = T^3_0 = T^4_0 = T^1_2 = \bar{T}^3_4 = 0,$$
$$T^4_4 = -\frac{1}{A}(2AT^3_3 + D_4), \quad \bar{T}^4_4 = \frac{1}{A}(AD_4 - C_2).$$

Now, all torsion coefficients are expressed in terms of the constant A and the functions

$$L, \quad \bar{T}^3_3, \quad C_1, \quad C_2, \quad T^3_3, \quad D_3, \quad D_4, \quad \bar{T}^1_2, \quad T^3_4, \quad R^1_2, \quad R^3_4, \quad R^4_4.$$

Lemma 5.10. *Assume homogeneity. On $\mathcal{G}_1$, wherever $(D_3, D_4) \neq 0$ (resp., $(C_1, C_2) \neq 0$), one must have $R^1_2 = \bar{T}^1_2 = 0$ (resp., $T^3_4 = R^3_4 = 0$).*

Proof. By the structure equations,

$$A(\mathrm{d}(\mathrm{d}\omega^1)) \equiv R^1_2 \gamma \wedge \omega^0 \wedge (D_3\omega^3 + D_4\omega^4)$$
$$+ \bar{T}^1_2 \bar{\omega}^0 \wedge \omega^0 \wedge (D_3\omega^3 + D_4(\omega^4 + \gamma)) \mod \omega^1, \omega^2,$$
$$A(\mathrm{d}(\mathrm{d}\omega^3)) \equiv R^3_4 \gamma \wedge \bar{\omega}^0 \wedge (C_1\omega^1 + C_2\omega^2)$$
$$+ T^3_4 \omega^0 \wedge \bar{\omega}^0 \wedge (C_1\omega^1 + C_2(\omega^2 + \gamma)) \mod \omega^3, \omega^4.$$

The conclusion is immediate. □

Now we proceed to find how (infinitesimally) G_1 acts on the torsion functions.

First, expanding $\mathrm{d}(\mathrm{d}\gamma) = 0$, then reducing modulo $\{\bar{\omega}^0, \gamma, \omega^1, \omega^2, \omega^3\}$ and $\{\bar{\omega}^0, \gamma, \omega^1, \omega^2, \omega^4\}$, respectively, we obtain

$$dD_4 \equiv (\alpha + \phi)D_4 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4, \tag{5.30}$$

$$dD_3 \equiv 2\alpha D_3 + \beta_2 D_4 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.31}$$

Expanding $\mathrm{d}(\mathrm{d}\gamma) = 0$, then reducing modulo $\{\bar{\omega}^0, \gamma, \omega^1, \omega^3, \omega^4\}$ and $\{\bar{\omega}^0, \gamma, \omega^2, \omega^3, \omega^4\}$, respectively, we obtain

$$dC_2 \equiv (\alpha + \phi)C_2 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4, \tag{5.32}$$

$$dC_1 \equiv 2\alpha C_1 + \beta_1 C_2 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.33}$$

In particular, (5.30)-(5.33) imply that, if (D_3, D_4) (resp., (C_1, C_2)) is nonzero, then it is nonzero along the entire group fibers.

Expanding $\mathrm{d}(\mathrm{d}\omega^1) = 0$, then reducing modulo $\{\omega^0, \bar{\omega}^0, \omega^1, \omega^3, \omega^4\}$, we obtain

$$dR^1_2 \equiv (2\phi - \alpha)R^1_2 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4; \tag{5.34}$$

expanding $\mathrm{d}(\mathrm{d}\omega^1) = 0$, then reducing modulo $\{\omega^0, \omega^1, \omega^3, \omega^4, \gamma\}$, we obtain

$$d\bar{T}^1_2 \equiv 2\phi \bar{T}^1_2 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.35}$$

Expanding $\mathrm{d}(\mathrm{d}\omega^3) = 0$, then reducing modulo $\{\omega^0, \bar{\omega}^0, \omega^1, \omega^2, \omega^3\}$, we obtain

$$dR^3_4 \equiv (2\phi - \alpha)R^3_4 \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4; \tag{5.36}$$

expanding $d(d\omega^3) = 0$, then reducing modulo $\{\bar{\omega}^0, \omega^1, \omega^2, \omega^3, \gamma\}$, we obtain

$$dT^3_4 \equiv 2\phi T^3_4 \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.37}$$

We also have the following four congruences:

$$\begin{aligned}
d(d\omega^0) &\equiv -d(\alpha+\phi)\wedge\omega^0 + L\phi\wedge\gamma\wedge\omega^0 - dL\wedge\gamma\wedge\omega^0 \quad \text{mod } \bar{\omega}^0, \omega^1, ..., \omega^4,\\
d(d\bar{\omega}^0) &\equiv -d(\alpha+\phi)\wedge\bar{\omega}^0 - L\phi\wedge\gamma\wedge\bar{\omega}^0 + dL\wedge\gamma\wedge\bar{\omega}^0 \quad \text{mod } \omega^0, \omega^1, ..., \omega^4,\\
d(d\omega^1) &\equiv -dL\wedge\gamma\wedge\omega^1 + dR^4_4\wedge\gamma\wedge\omega^1 - d\alpha\wedge\omega^1\\
&\quad + (L - R^4_4)\phi\wedge\gamma\wedge\omega^1 + R^1_2\gamma\wedge\beta_1\wedge\omega^1 \qquad \text{mod } \omega^0, \bar{\omega}^0, \omega^2, \omega^3, \omega^4,\\
d(d\omega^3) &\equiv \quad dL\wedge\gamma\wedge\omega^3 - dR^4_4\wedge\gamma\wedge\omega^3 - d\alpha\wedge\omega^3\\
&\quad + (R^4_4 - L)\phi\wedge\gamma\wedge\omega^3 + R^3_4\gamma\wedge\beta_2\wedge\omega^3 \qquad \text{mod } \omega^0, \bar{\omega}^0, \omega^1, \omega^2, \omega^4.
\end{aligned}$$

As a consequence,

$$dL \equiv \phi L \quad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4, \tag{5.38}$$

and

$$dR^4_4 \equiv \phi R^4_4 + \frac{1}{2}(R^1_2\beta_1 - R^3_4\beta_2) \quad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.39}$$

Finally, by computing

$$\begin{aligned}
d(d\omega^1) &\equiv -d\alpha\wedge\omega^1 + (T^3_3(\alpha+\phi) - dT^3_3)\wedge\omega^0\wedge\omega^1 && \text{mod } \bar{\omega}^0, \gamma, \omega^2, \omega^3, \omega^4,\\
d(d\omega^3) &\equiv -d\alpha\wedge\omega^3 + (-T^3_3(\alpha+\phi) + dT^3_3 - T^3_4\beta_2)\wedge\omega^0\wedge\omega^3 && \text{mod } \bar{\omega}^0, \gamma, \omega^1, \omega^2, \omega^4,\\
d(d\omega^1) &\equiv -d\alpha\wedge\omega^1 + (\bar{T}^3_3(\phi+\alpha) - d\bar{T}^3_3 - \bar{T}^1_2\beta_1)\wedge\bar{\omega}^0\wedge\omega^1 && \text{mod } \omega^0, \gamma, \omega^2, \omega^3, \omega^4,\\
d(d\omega^3) &\equiv -d\alpha\wedge\omega^3 + (-\bar{T}^3_3(\alpha+\phi) + d\bar{T}^3_3)\wedge\bar{\omega}^0\wedge\omega^3 && \text{mod } \omega^0, \gamma, \omega^1, \omega^2, \omega^4,
\end{aligned}$$

we obtain

$$dT^3_3 \equiv (\alpha+\phi)T^3_3 + \frac{1}{2}\beta_2 T^3_4 \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4, \tag{5.40}$$

and

$$d\bar{T}^3_3 \equiv (\alpha+\phi)\bar{T}^3_3 - \frac{1}{2}\beta_1 \bar{T}^1_2 \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4. \tag{5.41}$$

Now we proceed to cases.

Case of either $D_4 \neq 0$ or $C_2 \neq 0$.

Lemma 5.11. *Assuming homogeneity, the case when $D_4 \neq 0$ is empty.*

Proof. If D_4 is nonzero, then, by Lemma 5.10, R^1_2 and $\bar{T}^1_2$ are zero. By (5.30) and (5.31), one can always reduce to the subbundle $\mathcal{G}_2 \subset \mathcal{G}_1$ defined by

$$D_4 = 1, \qquad D_3 = 0.$$

By (5.40), (5.41) and (5.32), the functions $T^3_3, \bar{T}^3_3$ and C_2 are constants along the fibers of $\mathcal{G}_2 \to N$; hence, they are constants, by the homogeneity assumption.

On $\mathcal{G}_2$, the 1-forms $\phi + \alpha$ and β_2 are semi-basic; thus, we can write

$$\alpha = -\phi + H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + \sum_{i=1}^{4} H_i\omega^i, \quad \beta_2 = N_0\omega^0 + \bar{N}_0\bar{\omega}^0 + N_\gamma\gamma + \sum_{i=1}^{4} N_i\omega^i,$$

where the new coefficient functions are defined on $\mathcal{G}_2$. By applying $\mathrm{d}^2 = 0$ to the structure equations, we find

$$\begin{aligned}
\mathrm{d}H_0 &\equiv 0, \\
\mathrm{d}\bar{H}_0 &\equiv 0, \\
\mathrm{d}H_\gamma &\equiv H_\gamma\phi, \\
\mathrm{d}H_1 &\equiv -H_1\phi + (H_2 - L)\beta_1, \\
\mathrm{d}H_2 &\equiv H_2\phi, \\
\mathrm{d}H_3 &\equiv -H_3\phi, \\
\mathrm{d}H_4 &\equiv H_4\phi;
\end{aligned}
\qquad
\begin{aligned}
\mathrm{d}N_0 &\equiv -2N_0\phi, \\
\mathrm{d}\bar{N}_0 &\equiv -2\bar{N}_0\phi, \\
\mathrm{d}N_\gamma &\equiv -N_\gamma\phi, \\
\mathrm{d}N_1 &\equiv -3N_1\phi + N_2\beta_1, \\
\mathrm{d}N_2 &\equiv -N_2\phi, \\
\mathrm{d}N_4 &\equiv -N_4\phi,
\end{aligned}$$

where all congruences are modulo the semi-basic 1-forms. (Note that N_3 never appears in the structure equations.) By homogeneity, $H_0, \bar{H}_0$ are constants.

Now, $H_\gamma, H_2, H_3, H_4, N_0, \bar{N}_0, N_\gamma, N_2, N_4, L, C_1, T^3_4, R^3_4, R^4_4$ are *relative invariants*[4] of $\mathcal{G}_2$ with nonzero weights in ϕ only. If they all vanish, then it can be verified that

[4] Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup, and let $\mathcal{G}$ be a G-structure over M. A G-equivariant function $f : \mathcal{G} \to \mathbb{R}$, where G acts on $\mathbb{R}$ linearly, is called a *relative invariant* of $\mathcal{G}$.

the structure equations are incompatible with the identity $\mathrm{d}^2 = 0$. On the other hand, instead of asking which of these relative invariants are nonzero and going into various cases, we can always choose a nonzero function U on $\mathcal{G}_2$, expressed in terms of these relative invariants, satisfying

$$\mathrm{d}U \equiv U\phi \qquad \bmod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4.$$

Then there exist constants $h_\gamma, h_2, h_3, h_4, n_0, \bar{n}_0, n_\gamma, n_2, n_4$ such that

$$\begin{aligned} H_\gamma &= h_\gamma U, & N_0 &= n_0 U^{-2},\\ H_2 &= h_2 U, & \bar{N}_0 &= \bar{n}_0 U^{-2},\\ H_3 &= h_3 U^{-1}, & N_\gamma &= n_\gamma U^{-1},\\ H_4 &= h_4 U, & N_2 &= n_2 U^{-1},\\ & & N_4 &= n_4 U^{-1}, \end{aligned}$$

and constants $\ell, c_1, t_4^3, r_4^3, r_4^4$ such that

$$L = \ell U, \quad C_1 = c_1 U^{-2}, \quad T_4^3 = t_4^3 U^2, \quad R_4^3 = r_4^3 U^3, \quad R_4^4 = r_4^4 U.$$

Since U is expressed in terms of the relative invariants and has weight 1 in ϕ, we can perform a structure reduction that leads to a subbundle $\mathcal{G}_3$ defined by $U = 1$.

On $\mathcal{G}_3$, ϕ is semi-basic, namely,

$$\phi = Z_0\omega^0 + \bar{Z}_0\bar{\omega}^0 + Z_\gamma\gamma + \sum_{i=1}^{4} Z_i\omega^i,$$

for $Z_0, \bar{Z}_0, ..., Z_4$ defined on $\mathcal{G}_3$. We find that

$$dZ_0 \equiv 0, \qquad d\bar{Z}_0 \equiv 0, \qquad dZ_\gamma \equiv 0,$$
$$dZ_1 \equiv Z_2\beta_1, \qquad dZ_2 \equiv 0, \qquad dZ_3 \equiv 0, \qquad dZ_4 \equiv 0,$$

all congruences being modulo the semi-basic 1-forms. By homogeneity, $Z_0, \bar{Z}_0, Z_\gamma, Z_2, Z_3, Z_4$ are constants. Taking this into account, it can be verified that the structure equations are incompatible with $\mathrm{d}^2 = 0$. □

Consequently, D_4 must be zero on $\mathcal{G}_1$. Moreover, note that the equations (5.25)-(5.27) allows us to switch $(\omega^0, \omega^1, \omega^2)$ with $(\bar{\omega}^0, \omega^3, \omega^4)$; applying this, (C_1, C_2, C_3, C_4) exchanges with (D_3, D_4, D_1, D_2); and (B_1, B_2) exchanges with (B_3, B_4). Since we are in a case when $(B_1, B_2) = (B_3, B_4)$, we can conclude from Lemma 5.11 that

Lemma 5.12. *Assuming homogeneity, C_2 must be zero on $\mathcal{G}_1$.*

Therefore, the functions D_3 and C_1 are relative invariants of $\mathcal{G}_1$, both having weight 2 in α.

Case of $D_3 \neq 0$.

Without loss of generality, we can assume that $D_3 \neq 0$ on $\mathcal{G}_1$. By Lemma 5.10, $R^1_2, \bar{T}^1_2$ must be zero.

Depending on the sign of D_3, we can reduce to the subbundle $\mathcal{G}_2$ defined by either $D_3 = 1$ or $D_3 = -1$. On $\mathcal{G}_2$, there exist functions $H_0, \bar{H}_0, H_\gamma, H_1, ..., H_4$ such that

$$\alpha = H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + \sum_{i=1}^{4} H_i\omega^i, \tag{5.42}$$

Moreover, the torsion functions on $\mathcal{G}_1$ restrict to $\mathcal{G}_2$ to satisfy

$$\begin{gathered}
\mathrm{d}L \equiv \phi L, \quad \mathrm{d}\bar{T}^3_3 \equiv \bar{T}^3_3\phi, \quad \mathrm{d}T^3_3 \equiv T^3_3\phi + \frac{1}{2}T^3_4\beta_2, \\
\mathrm{d}C_1 \equiv 0, \quad C_2 = 0, \quad D_3 = \pm 1, \quad D_4 = 0, \\
\bar{T}^1_2 = 0, \quad \mathrm{d}T^3_4 \equiv 2T^3_4\phi, \quad R^1_2 = 0, \quad \mathrm{d}R^3_4 \equiv 2R^3_4\phi, \quad \mathrm{d}R^4_4 \equiv R^4_4\phi - \frac{1}{2}R^3_4\beta_2,
\end{gathered} \tag{5.43}$$

where all congruences are modulo the semi-basic 1-forms. By homogeneity, C_1 must be a constant; $L, \bar{T}^3_3, T^3_4, R^3_4$ are now relative invariants. We have two cases:

I. $L, \bar{T}^3_3, T^3_4, R^3_4$ are all zero;

II. not all of $L, \bar{T}^3_3, T^3_4, R^3_4$ are zero.

Now consider the case when $\mathcal{G}_2$ is defined by $D_3 = 1$. A "+" sign will be used to indicate that we are in this case.

$\boxed{+\mathbf{I}.}$ If $L, \bar{T}_3^3, T_4^3, R_4^3$ are identically zero on $\mathcal{G}_2$, then, by (5.43), T_3^3, R_4^4 are relative invariants. It is easy to verify that

$$\begin{aligned} dH_0 &\equiv H_0\phi, \\ d\bar{H}_0 &\equiv \bar{H}_0\phi, \\ dH_\gamma &\equiv H_\gamma\phi, \end{aligned} \qquad \begin{aligned} dH_1 &\equiv H_2\beta_1, \\ dH_2 &\equiv H_2\phi, \\ dH_3 &\equiv H_4\beta_2, \\ dH_4 &\equiv H_4\phi, \end{aligned} \tag{5.44}$$

modulo the semi-basic 1-forms. We can always choose U to be a function defined on $\mathcal{G}_2$, satisfying

$$dU \equiv U\phi, \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4,$$

in the following manner: noting that $T_3^3, R_4^4, H_0, H_\gamma, \bar{H}_0, H_2, H_4$ are relative invariants with nonzero weights in ϕ only, if they are all zero, we simply choose $U = 0$; otherwise, we choose U be an appropriate combination of these relative invariants satisfying the equation above and the condition $U \neq 0$. There are thus two subcases to consider:

$+\mathbf{I1}$. $T_3^3, R_4^4, H_0, H_\gamma, \bar{H}_0, H_2, H_4$ are all zero;

$+\mathbf{I2}$. not all of $T_3^3, R_4^4, H_0, H_\gamma, \bar{H}_0, H_2, H_4$ are zero.

$\boxed{+\mathbf{I1}.}$ In this case, by (5.44) and the homogeneity assumption, H_1 and H_3 are constants on $\mathcal{G}_2$. We compute

$$\begin{aligned} \mathrm{d}(\mathrm{d}\gamma) &= -d\phi \wedge \gamma + (1 - C_1)\omega^1 \wedge \omega^3 \wedge \gamma \\ &\quad - \frac{2}{A}(AC_1H_3 + H_1)\omega^3 \wedge \omega^1 \wedge \omega^0 + \frac{2}{A}(AH_1 + C_1H_3)\omega^3 \wedge \omega^1 \wedge \bar{\omega}^0. \end{aligned}$$

Since $A > 1$, we must have

$$H_1 = C_1H_3 = 0; \tag{5.45}$$

in particular, either C_1 or H_3 is zero.

Lemma 5.13. *C_1 is nonzero.*

Proof. If $C_1 = 0$ on $\mathcal{G}_2$, then

$$\mathrm{d}(\mathrm{d}\omega^0) = -\mathrm{d}\phi \wedge \omega^0 + \omega^3 \wedge \omega^1 \wedge \omega^0, \tag{5.46}$$

$$\mathrm{d}(\mathrm{d}\bar{\omega}^0) = -\mathrm{d}\phi \wedge \bar{\omega}^0 + \omega^3 \wedge \omega^1 \wedge \bar{\omega}^0, \tag{5.47}$$

$$\mathrm{d}(\mathrm{d}\gamma) = -\mathrm{d}\phi \wedge \gamma + \omega^1 \wedge \omega^3 \wedge \gamma. \tag{5.48}$$

The equation (5.46) and (5.47) imply that, on $\mathcal{G}_2$, there exists a function K such that

$$d\phi = \omega^3 \wedge \omega^1 + K\omega^0 \wedge \bar{\omega}^0,$$

which, however, is incompatible with (5.48). The conclusion follows. □

By Lemma 5.13 and (5.45), we must have $H_3 = 0$ on $\mathcal{G}_2$. Applying $\mathrm{d}^2 = 0$ to the structure equations yields

$$\mathrm{d}(\mathrm{d}\omega^0) = -\mathrm{d}\phi \wedge \omega^0 + (1 - C_1)\omega^3 \wedge \omega^1 \wedge \omega^0, \tag{5.49}$$

$$\mathrm{d}(\mathrm{d}\bar{\omega}^0) = -\mathrm{d}\phi \wedge \bar{\omega}^0 + (1 - C_1)\omega^3 \wedge \omega^1 \wedge \bar{\omega}^0, \tag{5.50}$$

$$\mathrm{d}(\mathrm{d}\gamma) = -\mathrm{d}\phi \wedge \gamma + (1 - C_1)\omega^1 \wedge \omega^3 \wedge \gamma. \tag{5.51}$$

Among these three equations, (5.49) and (5.50) imply that there exists a function K on $\mathcal{G}_2$ satisfying

$$\mathrm{d}\phi = (1 - C_1)\omega^3 \wedge \omega^1 + K\omega^0 \wedge \bar{\omega}^0;$$

in order for (5.51) to hold, we must have

$$C_1 = 1, \qquad K = 0.$$

As a result, ϕ is (determined and) integrable. Restricting to a leaf of $\langle\phi\rangle$, we have $\phi = 0$. Such a leaf then can be regarded as a bundle (over an open subset $U \subset N$)

on which the following structure equations are satisfied ($A > 1$ being a constant)

$$\begin{aligned}
\mathrm{d}\omega^0 &= A\omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma),\\
\mathrm{d}\bar{\omega}^0 &= \omega^1 \wedge (\omega^2 + \gamma) + A\omega^3 \wedge \omega^4,\\
\mathrm{d}\gamma &= \omega^1 \wedge \left(\omega^0 - \frac{1}{A}\bar{\omega}^0\right) + \omega^3 \wedge \left(\bar{\omega}^0 - \frac{1}{A}\omega^0\right),\\
\mathrm{d}\omega^1 &= 0,\\
\mathrm{d}\omega^2 &= -\beta_1 \wedge \omega^1 - \frac{1}{A}\omega^0 \wedge \omega^3,\\
\mathrm{d}\omega^3 &= 0,\\
\mathrm{d}\omega^4 &= -\beta_2 \wedge \omega^3 - \frac{1}{A}\bar{\omega}^0 \wedge \omega^1.
\end{aligned} \tag{5.52}$$

Remark 12. In terms of the method of equivalence, one can check that the structure equation (5.52) has constant torsion and involutive tableau, the Cartan characters of the tableau being $(2, 0, ..., 0)$. By a theorem of Cartan, any two Bäcklund transformations in this case are equivalent; the symmetry of such a Bäcklund transformation depends on 2 functions of 1 variable.

We have thus proven:

Proposition 5.3. *Up to equivalence, there is a unique local model for a rank-*2 *homogeneous Bäcklund transformation in Case* (+**I1**).

Remark 13. We will see later that the Bäcklund transformation corresponding to (5.52) is an auto-Bäcklund transformation of the linear equation $z_{xy} = z$.

+I2. In this case, by the construction of U, there exist constants $t^3_3, r^4_4, h_0, \bar{h}_0, h_\gamma, h_2, h_4$, such that

$$T^3_3 = t^3_3 U, \quad R^4_4 = r^4_4 U,$$
$$H_0 = h_0 U, \quad \bar{H}_0 = \bar{h}_0 U, \quad H_\gamma = h_\gamma U, \quad H_2 = h_2 U, \quad H_4 = h_4 U.$$

Moreover, it is easy to see that one can reduce to the subbundle $\mathcal{G}_3$ defined by $U = 1$.

On $\mathcal{G}_3$, there exist functions $Z_0, \bar{Z}_0, Z_\gamma, Z_1, ..., Z_4$ such that

$$\phi = Z_0\omega^0 + \bar{Z}_0\bar{\omega}^0 + Z_\gamma\gamma + \sum_{i=1}^{4} Z_i\omega^i,$$

By applying $d^2 = 0$ to the structure equations, one can verify that

$$\begin{aligned} &\mathrm{d}Z_0 \equiv 0, \qquad &&\mathrm{d}Z_1 \equiv Z_2\beta_1,\\ &\mathrm{d}\bar{Z}_0 \equiv 0, &&\mathrm{d}Z_2 \equiv 0,\\ &\mathrm{d}Z_\gamma \equiv 0, &&\mathrm{d}Z_3 \equiv Z_4\beta_2,\\ & &&\mathrm{d}Z_4 \equiv 0, \end{aligned} \tag{5.53}$$

modulo semi-basic 1-forms. Therefore, by homogeneity, $Z_0, \bar{Z}_0, Z_\gamma, Z_2, Z_4$ are constants. Using the structure equations, we find

$$\mathrm{d}(\mathrm{d}\gamma) \equiv -2h_2\omega^2 \wedge \omega^3 \wedge \bar{\omega}^0 - 2h_4\omega^4 \wedge \omega^3 \wedge \bar{\omega}^0 \qquad \bmod \omega^0, \gamma, \omega^1.$$

Evidently, h_2 and h_4 must be zero. By (5.44) and the homogeneity assumption, H_1 and H_3 are constants.

By expanding

$$\mathrm{d}(\mathrm{d}\omega^0),\ \mathrm{d}(\mathrm{d}\bar{\omega}^0),\ \mathrm{d}(\mathrm{d}\gamma),\ \mathrm{d}(\mathrm{d}\omega^2),\ \mathrm{d}(\mathrm{d}\omega^4) \quad \text{all mod } \omega^1, \omega^3,$$

$$\mathrm{d}(\mathrm{d}\omega^1) \quad \bmod \omega^3, \qquad \mathrm{d}(\mathrm{d}\omega^3) \quad \bmod \omega^1$$

we find that either Z_2, Z_4 are both zero, or $h_0, \bar{h}_0, h_\gamma, r^4_4, t^3_3$ are all zero.

However, noting that h_2, h_4 are already zero, we cannot have $h_0, \bar{h}_0, h_\gamma, r^4_4, t^3_3$ to be all zero in the current case (by the assumption for Case (+**I2**)).

Now assume that Z_2, Z_4 are both zero. In this case, Z_1, Z_3 become invariants, by (5.53), and are thus constants by the homogeneity assumption. Expanding $\mathrm{d}(\mathrm{d}\omega^0), \mathrm{d}(\mathrm{d}\bar{\omega}^0), \mathrm{d}(\mathrm{d}\gamma), \mathrm{d}(\mathrm{d}\omega^1), \mathrm{d}(\mathrm{d}\omega^3)$ and

$$\mathrm{d}(\mathrm{d}\omega^2) \quad \bmod \omega^1, \quad \mathrm{d}(\mathrm{d}\omega^4) \quad \bmod \omega^3$$

we find

$$h_0 = \bar{h}_0 = h_\gamma = Z_0 = \bar{Z}_0 = Z_\gamma = H_1 = H_3 = r^4_4 = t^3_3 = 0, \quad C_1 = 1.$$

Together with the condition $h_2 = h_4 = 0$, this brings us back to Case (+**I1**).

$\boxed{+\mathbf{II}.}$ In this case, one can always choose a function U, defined on $\mathcal{G}_2$ and expressed in terms of $L, \bar{T}^3_3, T^3_4$ and R^3_4, that satisfies

$$\mathrm{d}U \equiv U\phi \qquad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4.$$

Following from this and (5.43), there exist constants $\ell, \bar{t}^3_3, t^3_4, r^3_4$, not all zero, such that

$$L = \ell U, \quad \bar{T}^3_3 = \bar{t}^3_3 U, \quad T^3_4 = t^3_4 U^2, \quad R^3_4 = r^3_4 U^2.$$

One can reduce to the subbundle $\mathcal{G}_3$ defined by $U = 1$. On $\mathcal{G}_3$, there exist functions $Z_0, \bar{Z}_0, Z_\gamma, Z_1, ..., Z_4$ such that

$$\phi = Z_0\omega^0 + \bar{Z}_0\bar{\omega}^0 + Z_\gamma\gamma + \sum_{i=1}^{4} Z_i\omega^i.$$

How the torsion functions vary on the fibers of $\mathcal{G}_3 \to N$ can be found, as usual, by applying $\mathrm{d}^2 = 0$ to the structure equations. Moreover, by taking into account the homogeneity assumption, we find

(i) $\bar{H}_0, H_2, H_4, Z_0, \bar{Z}_0, Z_2, Z_4$ are constants;

(ii) there exist constants h_0, z_γ, h_γ such that

$$H_0 = h_0 - T^3_3, \qquad H_\gamma = h_\gamma + R^4_4, \qquad Z_\gamma = z_\gamma - R^4_4; \tag{5.54}$$

(iii) H_1, H_3, Z_1, Z_3 satisfy, modulo the semi-basic 1-forms,

$$\begin{aligned} \mathrm{d}H_1 &\equiv (H_2 - \ell)\beta_1, & \mathrm{d}Z_1 &\equiv \beta_1 Z_2, \\ \mathrm{d}H_3 &\equiv (H_4 + \ell)\beta_2, & \mathrm{d}Z_3 &\equiv \beta_2 Z_4. \end{aligned} \tag{5.55}$$

Calculation yields

$$\mathrm{d}(\mathrm{d}\gamma) \equiv t^3_4 \omega^0 \wedge \omega^4 \wedge \bar{\omega}^0 \qquad \mod \omega^1, \omega^2, \omega^3, \gamma,$$

$$\mathrm{d}(\mathrm{d}\gamma) \equiv \frac{r^3_4}{A} \omega^0 \wedge \gamma \wedge \omega^4 \qquad \mod \omega^1, \omega^2, \omega^3, \bar{\omega}^0.$$

Hence, t^3_4 and r^3_4 must be zero. By (5.43) and homogeneity, it follows that T^3_3 and R^4_4 are constants on $\mathcal{G}_3$. Using these and expanding the following expressions

$$\mathrm{d}(\mathrm{d}\omega^0),\ \mathrm{d}(\mathrm{d}\bar{\omega}^0),\ \mathrm{d}(\mathrm{d}\gamma),\ \mathrm{d}(\mathrm{d}\omega^2),\ \mathrm{d}(\mathrm{d}\omega^4) \quad \text{all} \mod \omega^1, \omega^3,$$

$$\mathrm{d}(\mathrm{d}\gamma) \mod \gamma, \quad \mathrm{d}(\mathrm{d}\omega^2) \mod \omega^3, \quad \mathrm{d}(\mathrm{d}\omega^3) \mod \omega^1,$$

and

$$\mathrm{d}(\mathrm{d}\omega^2) \mod \omega^1, \omega^2, \qquad \mathrm{d}(\mathrm{d}\omega^2) \mod \omega^3, \omega^4,$$

we find (with Maple™):

$$Z_2 = Z_4 = H_1 = 0.$$

This, with (5.55), implies that Z_1, Z_3 are constants and that $H_2 = \ell$. Taking these into account, one can show, by applying the identity $\mathrm{d}^2 = 0$ to the structure equations, that ℓ and $\bar{t}^3_3$ must be zero. This violates the assumption of the current case. This completes analysis for the case of $D_3 > 0$ on $\mathcal{G}_1$.

The case when $D_3 < 0$ is similar to the case when $D_3 > 0$. The only nonempty case is ($-$**I1**), which leads to $C_1 = -1$ and the structure equations

$$\begin{aligned}
d\omega^0 &= -\phi \wedge \omega^0 + A\omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma),\\
d\bar{\omega}^0 &= -\phi \wedge \bar{\omega}^0 + \omega^1 \wedge (\omega^2 + \gamma) + A\omega^3 \wedge \omega^4,\\
d\gamma &= -\phi \wedge \gamma - \omega^1 \wedge \left(\omega^0 - \frac{1}{A}\bar{\omega}^0\right) - \omega^3 \wedge \left(\bar{\omega}^0 - \frac{1}{A}\omega^0\right),\\
d\omega^1 &= 0,\\
d\omega^2 &= -\beta_1 \wedge \omega^1 - \phi \wedge \omega^2 + \frac{1}{A}\omega^0 \wedge \omega^3,\\
d\omega^3 &= 0,\\
d\omega^4 &= -\beta_2 \wedge \omega^3 - \phi \wedge \omega^4 + \frac{1}{A}\bar{\omega}^0 \wedge \omega^1,
\end{aligned} \tag{5.56}$$

where ϕ is integrable.

Remark 14. One can show that the structure equations (5.56) corresponds to an auto-Bäcklund transformation of the partial differential equation $z_{xy} = z$.

5.2.2 Integration of the structure equations

Equation (5.52). Replace β_1 by ω^4 and β_2 by ω^2 in (5.52). It is easy to check that $\mathrm{d}^2 = 0$ automatically holds. We may thus regard the result as the structure equations on some open $U \subset N$.

Since ω^1, ω^3 are closed, there locally exist functions x, y such that

$$\omega^1 = \mathrm{d}x, \qquad \omega^3 = \mathrm{d}y.$$

Since $\mathrm{d}\omega^0$ and $\mathrm{d}\bar{\omega}^0$ are both closed and both congruent to zero modulo $\mathrm{d}x, \mathrm{d}y$, there exist functions $z_1, p_1, q_1,\ z_2, p_2, q_2$ such that

$$\omega^0 = \mathrm{d}z_1 - p_1\mathrm{d}x - q_1\mathrm{d}y,$$

$$\bar{\omega}^0 = \mathrm{d}z_2 - p_2\mathrm{d}x - q_2\mathrm{d}y.$$

Then the equation of $\mathrm{d}\omega^0$ implies

$$\big(A\omega^2 - \mathrm{d}p_1\big) \wedge \mathrm{d}x + \big(\omega^4 + \gamma - \mathrm{d}q_1\big) \wedge \mathrm{d}y = 0. \tag{5.57}$$

Similarly, the equation of $\mathrm{d}\bar{\omega}^0$ implies

$$\big(\omega^2 + \gamma - \mathrm{d}p_2\big) \wedge \mathrm{d}x + \big(A\omega^4 - \mathrm{d}q_2\big) \wedge \mathrm{d}y = 0. \tag{5.58}$$

Following from (5.57) and (5.58), there exist functions $s_1, s_2, s_3,\ t_1, t_2, t_3$ satisfying

$$\begin{cases} \omega^2 = \dfrac{1}{A}(\mathrm{d}p_1 + s_1\mathrm{d}x + s_2\mathrm{d}y), \\ \omega^4 + \gamma = \mathrm{d}q_1 + s_2\mathrm{d}x + s_3\mathrm{d}y; \end{cases} \tag{5.59}$$

and

$$\begin{cases} \omega^2 + \gamma = \mathrm{d}p_2 + t_1\mathrm{d}x + t_2\mathrm{d}y, \\ \omega^4 = \dfrac{1}{A}(\mathrm{d}q_2 + t_2\mathrm{d}x + t_3\mathrm{d}y). \end{cases} \tag{5.60}$$

Writing γ as $(\omega^2+\gamma)-\omega^2$ and $(\omega^4+\gamma)-\omega^4$, then using (5.59) and (5.60), we obtain

$$\left(\mathrm{d}p_2-\frac{1}{A}\mathrm{d}p_1\right)+\left(t_1-\frac{s_1}{A}\right)\mathrm{d}x+\left(t_2-\frac{s_2}{A}\right)\mathrm{d}y= \qquad (5.61)$$
$$\left(\mathrm{d}q_1-\frac{\mathrm{d}q_2}{A}\right)+\left(s_2-\frac{t_2}{A}\right)\mathrm{d}x+\left(s_3-\frac{t_3}{A}\right)\mathrm{d}y.$$

Furthermore, the equation of $\mathrm{d}\omega^2$ leads to

$$(\mathrm{d}s_1+\mathrm{d}q_2+t_3\mathrm{d}y)\wedge\mathrm{d}x+(\mathrm{d}s_2+\mathrm{d}z_1-p_1\mathrm{d}x)\wedge\mathrm{d}y=0; \qquad (5.62)$$

the equation of $\mathrm{d}\omega^4$ leads to

$$(\mathrm{d}t_2+\mathrm{d}z_2-q_2\mathrm{d}y)\wedge\mathrm{d}x+(\mathrm{d}t_3+\mathrm{d}p_1+s_1\mathrm{d}x)\wedge\mathrm{d}y=0. \qquad (5.63)$$

Applying $\mathrm{d}^2=0$ to (5.62) and (5.63) implies

$$\mathrm{d}(p_1+t_3)\wedge\mathrm{d}x\wedge\mathrm{d}y=\mathrm{d}(q_2+s_1)\wedge\mathrm{d}x\wedge\mathrm{d}y=0.$$

As as result, there exist functions $f(x,y)$, $g(x,y)$ defined on U such that

$$\begin{cases} p_1+t_3=f(x,y),\\ q_2+s_1=g(x,y).\end{cases}$$

Now (5.62) can be written as

$$\left(\mathrm{d}(s_2+z_1)-\left(\frac{\partial g}{\partial y}+f\right)\mathrm{d}x\right)\wedge\mathrm{d}y=0,$$

which implies that, by fixing an x_0,

$$s_2=-z_1+\int_{x_0}^{x}\left(\frac{\partial g}{\partial y}(\sigma,y)+f(\sigma,y)\right)\mathrm{d}\sigma+h(y),$$

for a function $h(y)$. With a similar argument, one can obtain, by fixing a y_0,

$$t_2=-z_2+\int_{y_0}^{y}\left(\frac{\partial f}{\partial x}(x,\tau)+g(x,\tau)\right)\mathrm{d}\tau+\tilde{h}(x),$$

for a function $\tilde{h}(x)$. Since we have

$$\omega^0 = \mathrm{d}z_1 - p_1\mathrm{d}x - q_1\mathrm{d}y,$$

$$\omega^1 \wedge \omega^2 = \mathrm{d}x \wedge \frac{1}{A}(\mathrm{d}p_1 + s_2\mathrm{d}y),$$

$$\omega^3 \wedge (\omega^4 + \gamma) = \mathrm{d}y \wedge (\mathrm{d}q_1 + s_2\mathrm{d}x),$$

it is clear that x, y, p_1, q_1, z_1 are local coordinates on M and that the PDE (up to contact equivalence) corresponding to $(M, \mathcal{I})$ must be of the form

$$\frac{\partial^2 u}{\partial x \partial y} = u - \int_{x_0}^{x} \left(\frac{\partial g}{\partial y}(\sigma, y) + f(\sigma, y) \right) \mathrm{d}\sigma + h(y),$$

for some functions $f(x, y), g(x, y)$ and $h(y)$. Similarly, the PDE (up to contact equivalence) corresponding to $(\bar{M}, \bar{\mathcal{I}})$ must be of the form

$$\frac{\partial^2 v}{\partial x \partial y} = v - \int_{y_0}^{y} \left(\frac{\partial f}{\partial x}(x, \tau) + g(x, \tau) \right) \mathrm{d}\tau + \tilde{h}(x),$$

for the $f(x, y), g(x, y)$ as above, and some $\tilde{h}(y)$.

We claim that one can make an appropriate choice of coordinates such that $f, g, h, \tilde{h}$ are all zero. As a result, each of $(M, \mathcal{I})$ and $(\bar{M}, \bar{\mathcal{I}})$ corresponds (up to equivalence) to the following linear PDE:

$$z_{xy} = z.$$

To see why such a choice is possible, consider the following change of variables

$$z_1 \mapsto z_1 + \alpha(x, y), \qquad z_2 \mapsto z_2 + \beta(x, y), \tag{5.64}$$

where $\alpha(x, y), \beta(x, y)$ are analytic functions to be determined. This implies the corresponding change of the variables (in order to preserve ω^0, $\bar{\omega}^0$,...,ω^4)

$$\begin{aligned}
p_1 &\mapsto p_1 + \alpha_x, & p_2 &\mapsto p_2 + \beta_x, \\
q_1 &\mapsto q_1 + \alpha_y, & q_2 &\mapsto q_2 + \beta_y, \\
s_1 &\mapsto s_1 - \alpha_{xx}, & t_1 &\mapsto t_1 - \beta_{xx}, \\
s_2 &\mapsto s_2 - \alpha_{xy}, & t_2 &\mapsto t_2 - \beta_{xy}, \\
s_3 &\mapsto s_3 - \alpha_{yy}, & t_3 &\mapsto t_3 - \beta_{yy}.
\end{aligned}$$

In particular,

$$p_1 + t_3 \mapsto (\alpha_x - \beta_{yy}) + p_1 + t_3, \qquad q_2 + s_1 \mapsto (\beta_y - \alpha_{xx}) + q_2 + s_1.$$

Thus, by solving for α, β in the following system

$$\begin{cases} \beta_{yy} - \alpha_x = f(x, y), \\ \alpha_{xx} - \beta_y = g(x, y), \end{cases} \tag{5.65}$$

one can find a set of coordinates under which f, g become zero. The system (5.65) can be turned into the form

$$\begin{cases} \beta_{yy} - \alpha_x = 0, \\ \alpha_{xx} - \beta_y = \tilde{g}(x, y), \end{cases} \tag{5.66}$$

by making the change of variable $\alpha \mapsto \alpha - \int_{x_0}^{x} f(\tau, y)\mathrm{d}\tau$. The system (5.66) is easily solved once we have solved the equation

$$\beta_{xyy} - \beta_y = \tilde{g}(x, y),$$

which is essentially

$$\sigma_{xy} - \sigma = \tilde{g}(x, y), \tag{5.67}$$

after the substitution $\sigma = \beta_y$. By Cartan-Kähler theory, a regular solution of (5.67) depends on 2 functions of 1 variable. This justifies the normalization: $f = g = 0$.

In addition, under the transformation (5.64), we have

$$s_2 + z_1 \mapsto s_1 + z_1 + (\alpha - \alpha_{xy}), \qquad t_2 + z_2 \mapsto t_2 + z_2 + (\beta - \beta_{xy}).$$

It is clear that, since f, g are now zero, in order for $h, \tilde{h}$ to become zero after (5.64) without affecting f and g, we need α, β to satisfy the PDE system

$$\begin{cases} \beta_{yy} - \alpha_x = 0, \\ \alpha_{xx} - \beta_y = 0, \\ \alpha - \alpha_{xy} = h(y), \\ \beta - \beta_{xy} = \tilde{h}(x). \end{cases}$$

An obvious solution is $\alpha = h(y)$ and $\beta = \tilde{h}(x)$. This justifies normalizing $f, g, h, \tilde{h}$ to be zero.

Now, since $t_3 = -p_1,\ s_1 = -q_2,\ s_2 = -z_1,\ t_2 = -z_2$, the left-hand-side of (5.61) becomes

$$\gamma = \mathrm{d}p_2 - \frac{1}{A}\mathrm{d}p_1 + \left(t_1 + \frac{1}{A}q_2\right)\mathrm{d}x - \left(z_2 - \frac{1}{A}z_1\right)\mathrm{d}y. \tag{5.68}$$

On the other hand, the equation of $\mathrm{d}\gamma$ implies

$$\begin{aligned}\mathrm{d}\gamma = &\left(\frac{q_2}{A} - q_1 + p_2 - \frac{p_1}{A}\right)\mathrm{d}x \wedge \mathrm{d}y \\ &+ \mathrm{d}x \wedge \left(\mathrm{d}z_1 - \frac{1}{A}\mathrm{d}z_2\right) + \mathrm{d}y \wedge \left(\mathrm{d}z_2 - \frac{1}{A}\mathrm{d}z_1\right).\end{aligned} \tag{5.69}$$

Since $\mathrm{d}\gamma$ is closed, it is evident (noting that A is a constant) that locally there exists a function $k(x, y)$ such that

$$\frac{q_2}{A} - q_1 + p_2 - \frac{p_1}{A} = k(x, y), \tag{5.70}$$

which is precisely the extra equation (the other two being $x_1 = x_2 = x,\ y_1 = y_2 = y$) needed to define a 7-submanifold $N \subset M \times \bar{M}$.

Using (5.69), (5.70) and the exterior derivative of (5.68), we obtain

$$\mathrm{d}\left(t_1 + \frac{1}{A}q_2 + z_1 - \frac{1}{A}z_2\right) \wedge \mathrm{d}x + k(x, y)\mathrm{d}y \wedge \mathrm{d}x = 0.$$

Following from this,

$$t_1 = -\frac{1}{A}q_2 - z_1 + \frac{1}{A}z_2 - K(x, y),$$

where $K(x, y)$ is a function satisfying $\partial_y K(x, y) = k(x, y)$. The function $k(x, y)$ is not arbitrary. In fact, expanding (5.61), using the expressions for t_1, s_1, t_2, s_2 and t_3, and using (5.70), one can obtain

$$\mathrm{d}(k(x, y)) - K(x, y)\mathrm{d}x + \left(-z_2 + \frac{1}{A}z_1 - s_3 - \frac{1}{A}p_1\right)\mathrm{d}y = 0.$$

Consequently, $k_x = K$ holds; in particular, we have $k_{xy} = k$ and

$$s_3 = k_y(x, y) + z_2 - \frac{1}{A} z_1 + \frac{1}{A} p_1.$$

Again, we ask whether the function $k(x, y)$ can be normalized by a change of local coordinates. It amounts to solving the following linear system of PDEs

$$\begin{cases} \beta_{yy} - \alpha_x = 0, \\ \alpha_{xx} - \beta_y = 0, \\ \alpha - \alpha_{xy} = 0, \\ \beta - \beta_{xy} = 0, \\ A^{-1}(\beta_y - A\alpha_y + A\beta_x - \alpha_x) = k(x, y). \end{cases} \tag{5.71}$$

It is clear that $k_{xy} = k$ is an integrability condition for (5.71).

Using the Frobenius Theorem, one can show that solutions of (5.71) exist and depend on 5 constants. Consequently, a Bäcklund transformation corresponding to the structure equations (5.52) is contact equivalent to one that relates solutions of the equation $z_{xy} = z$. Such a Bäcklund transformation is locally determined by the equation

$$\frac{q_2}{A} - q_1 + p_2 - \frac{p_1}{A} = 0, \quad (\text{const. } A > 1); \tag{5.72}$$

and the 1-form

$$\gamma = \mathrm{d}p_2 - \frac{1}{A}\mathrm{d}p_1 + \left(-z_1 + \frac{1}{A} z_2\right) \mathrm{d}x - \left(z_2 - \frac{1}{A} z_1\right) \mathrm{d}y. \tag{5.73}$$

Equation (5.56). The integration of (5.56) is similar to that of (5.52). Replace β_1 by ω^4 and β_2 by $-\omega^2$. Applying $\mathrm{d}^2 = 0$ to (5.56) yields only identities. We can show that there exist functions $x, y; z_1, p_1, q_1; z_2, p_2, q_2; s_1, s_2, s_3; t_1, t_2, t_3$ on U satisfying (5.59), (5.60) and (5.61). After a contact transformation of the coordinates, without changing x, y, we obtain

$$t_3 = p_1, \qquad s_1 = -q_2, \qquad s_2 = z_1, \qquad t_2 = z_2, \qquad t_1 = -\frac{q_2}{A} + z_1 - \frac{1}{A} z_2$$

and the equation

$$\frac{q_2}{A} - q_1 + p_2 - \frac{p_1}{A} = 0. \tag{5.74}$$

Now γ can be written as

$$\gamma = \mathrm{d}p_2 - \frac{1}{A}\mathrm{d}p_1 + \left(z_1 - \frac{1}{A}z_2\right)\mathrm{d}x + \left(z_2 - \frac{1}{A}z_1\right)\mathrm{d}y. \tag{5.75}$$

The corresponding Bäcklund transformation, up to contact equivalence, relates solutions of the equation $z_{xy} = -z$. We can perform an additional contact transformation, sending x to $-x$, so that p_1 is replaced by $-p_1$, p_2 by $-p_2$, and so on. Under the new coordinates, (5.74) becomes

$$\frac{q_2}{A} - q_1 - p_2 + \frac{p_1}{A} = 0, \tag{5.76}$$

and γ becomes

$$\gamma = -dp_2 + \frac{1}{A}dp_1 - \left(z_1 - \frac{1}{A}z_2\right)dx + \left(z_2 - \frac{1}{A}z_1\right)dy. \tag{5.77}$$

It is easy to see that the corresponding Bäcklund transformation is an auto-Bäcklund transformation of the equation $z_{xy} = z$ (up to equivalence).

5.3 Assuming Genericity Condition 1

If a rank-2 Bäcklund transformation satisfies only the first genericity condition, it then admits a coframing $(\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$ defined on an open subset $U \subset N$ and satisfying the following congruences

$$d\omega^0 \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \qquad \text{mod } \omega^0, \tag{5.78}$$

$$d\bar{\omega}^0 \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma, \tag{5.79}$$

$$d\gamma \equiv A_1\omega^1 \wedge \omega^2 + A_2\omega^3 \wedge \omega^4 \qquad \text{mod } \omega^0, \bar{\omega}^0, \gamma. \tag{5.80}$$

We can further refine such a coframing by performing the following steps successively:

1) add a suitable multiple of ω^0 to γ to arrange $A_2 = -A_1$;

2) add appropriate multiples of ω^0 to $\omega^1, .., \omega^4$ such that the congruence (5.79) still holds when modulo only $\bar{\omega}^0$ and γ;

3) add appropriate multiples of γ to ω^3, ω^4 such that the following congruences hold (note that $\mathrm{d}\bar{\omega}^0$ cannot have a $\omega^0 \wedge \gamma$ term since $\bar{\omega}^0 \wedge (\mathrm{d}\bar{\omega}^0)^3 = 0$.):

$$\mathrm{d}\omega^0 \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + (B_3\omega^3 + B_4\omega^4) \wedge \gamma \qquad \mod \omega^0, \tag{5.81}$$

$$\mathrm{d}\bar{\omega}^0 \equiv \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + (B_1\omega^1 + B_2\omega^2) \wedge \gamma \qquad \mod \bar{\omega}^0, \tag{5.82}$$

where $B_1, ..., B_4$ are functions defined on U.

Whether the functions $B_1, ..., B_4$ are all zero is an intrinsic property. To be specific, it is easy to see that $B_i = 0$ $(i = 1, ..., 4)$ precisely when the rank-2 distributions $(C(\langle\omega^0\rangle))^\perp$ and $(C(\langle\bar{\omega}^0\rangle))^\perp$ are everywhere non-transversal. In this case, an analogous result as Proposition 5.2 holds.

From now on, we will assume that B_i $(i = 1, ..., 4)$ are *not* all zero.

For (5.80), locally there are two possibilities:

i. A_1 vanishes on an open subset $U \subset N$;

ii. A_1 is nonvanishing on an open subset $U \subset N$,

These two cases are intrinsically distinguished by whether the first derived system of $B^1 \subset T^*U$ has rank 1 or 2.

From now on, we will assume Case **ii**. In this case, one can scale γ to arrange $A_1 = 1$, so that the following congruence holds:

$$\mathrm{d}\gamma \equiv \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 \qquad \mod \omega^0, \bar{\omega}^0, \gamma. \tag{5.83}$$

Definition 5.4. Let $(N, \mathcal{B}; \pi, \bar{\pi})$ be a rank-2 Bäcklund transformation satisfying only the first genericity condition. Moreover, suppose that the first derived system of B^1

has rank 1. A coframing $(\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$ defined on an open subset $U \subset N$ satisfying

$$[\![\omega^0]\!] = [\![\pi^*\theta]\!], \quad [\![\bar{\omega}^0]\!] = [\![\pi^*\bar{\theta}]\!], \quad [\![\omega^0, \bar{\omega}^0, \gamma]\!] = B^1$$

and the equations (5.81), (5.82) and (5.83) will be called 0-*adapted*.

In terms of a 0-adapted coframing $(\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4)$, locally the systems

$$\Xi_{10} = [\![\omega^0, \bar{\omega}^0, \gamma, \omega^1, \omega^2]\!] \quad \text{and} \quad \Xi_{01} = [\![\omega^0, \bar{\omega}^0, \gamma, \omega^3, \omega^4]\!]$$

are well-defined up to ordering. From now on, we only consider those local 0-adapted coframings that respect a fixed order of these two systems. Next, we show that such coframings are the local sections of a G-structure with $G \subset \mathrm{GL}(7, \mathbb{R})$ being the subgroup consisting of elements of the form

$$g = \begin{pmatrix} rI_3 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad r \in \mathbb{R};\ a, b \in \mathrm{GL}(2, \mathbb{R});\ \det(a) = \det(b) = r \neq 0, \tag{5.84}$$

where I_3 is the 3×3 identity matrix.

Lemma 5.14. *Let $(N, \mathcal{B})$ be a rank-2 Bäcklund transformation satisfying only the first genericity condition. Let $G \subset \mathrm{GL}(7, \mathbb{R})$ be as above. Any 0-adapted coframing defined on an open subset $U \subset N$ and respecting a fixed order of the systems Ξ_{10}, Ξ_{01} is a section $U \to \mathcal{G}$, where $\mathcal{G}$ is a G-structure on N.*

Proof. It is easy to see that, if ω is a 0-adapted coframing in the sense above, then $\omega \cdot g$ is also a 0-adapted coframing for any function $g : U \to G$.

Now we prove the converse. Suppose that ω is a 0-adapted coframing defined on U. We determine the most general form of a function $g : U \to \mathrm{GL}(7, \mathbb{R})$ so that $\omega \cdot g$ remains 0-adapted.

First note that ω^0 and $\bar{\omega}^0$ are determined up to scaling. Moreover, since a 0-adapted coframing respects a fixed order of the systems Ξ_{10} and Ξ_{01}, $\omega^1 \wedge \omega^2$ and

$\omega^3 \wedge \omega^4$ (both modulo $\omega^0, \bar{\omega}^0$ and γ) are determined up to scaling. As a result, a transformation $g \in \mathrm{GL}(7, \mathbb{R})$ that makes $\omega \cdot g$ a 0-adapted coframing must take the form

$$g = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & r & 0 & 0 & 0 & 0 \\ f_1 & f_5 & f_9 & a_{11} & a_{12} & 0 & 0 \\ f_2 & f_6 & f_{10} & a_{21} & a_{22} & 0 & 0 \\ f_3 & f_7 & f_{11} & 0 & 0 & b_{11} & b_{12} \\ f_4 & f_8 & f_{12} & 0 & 0 & b_{21} & b_{22} \end{pmatrix}, \qquad \begin{aligned} r &= \det(a_{ij}) \\ &= \det(b_{ij}) \neq 0. \end{aligned}$$

Any such g can be written as the product of of two matrices of the same form, one, say g_1, with all the f_i being zero, the other, say g_2, with $r = 1$ and $(a_i), (b_i)$ both being the 2×2 identity matrix.

It is clear that $\omega \cdot g_1$ is 0-adapted.

In order for $\omega \cdot g_2$ to be 0-adapted, by (5.81) and (5.82), $f_9, f_{10}, f_{11}, f_{12}$ must be zero; by (5.83), c_1 and c_2 must be the negative of each other, say,

$$c_1 = -c_2 = c.$$

Now (5.81) implies that

$$\begin{aligned}(B_3\omega^3 + B_4\omega^4) \wedge \gamma = \bar{\omega}^0 \wedge (f_5\omega^2 - f_6\omega^1 + f_7\omega^4 - f_8\omega^3) \\ + (\tilde{B}_3\omega^3 + \tilde{B}_4\omega^4) \wedge (\gamma - c\bar{\omega}^0) + (\tilde{B}_3 f_7 + \tilde{B}_4 f_8)\bar{\omega}^0 \wedge \gamma.\end{aligned}$$

for some functions $\tilde{B}_3, \tilde{B}_4$. Reducing modulo γ, it is clear that f_5, f_6, f_7, f_8 are zero. Immediately, either $c = 0$ or $B_3 = B_4 = \tilde{B}_3 = \tilde{B}_4 = 0$. Similarly, (5.82) implies that f_1, f_2, f_3, f_4 are all zero. In addition, either $c = 0$ or $B_1 = B_2 = 0$.

To conclude, we have either $B_i = 0$ $(i = 1, ..., 4)$ or $c = 0$. By assumption, B_i $(i = 1, ..., 4)$ are not all zero. It follows that $c = 0$ and g is of the form (5.3). □

Furthermore, one can show that, on $\mathcal{G}$,

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u \cdot g) = a^T \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(u), \qquad \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u \cdot g) = b^T \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}(u).$$

Using this transformation, one can reduce to one of the following two cases[5]:

Case 1: $B_1 = B_3 = 1$, $B_2 = B_4 = 0$;

Case 2: $B_2, B_3, B_4 = 0$, $B_1 = 1$.

We now focus on Case 1; the analysis for Case 2 is work in progress.

5.3.1 Case: $(B_1, B_2) = (B_3, B_4) = (1, 0)$

Let G and $\mathcal{G}$ be as above. It is easy to see that the subbundle $\mathcal{G}_1 \subset \mathcal{G}$ defined by $B_1 = B_3 = 1$, $B_2 = B_4 = 0$ is a G_1-structure on N, where $G_1 \subset G$ is 3-dimensional and consists of those elements of G that satisfy the extra conditions:

$$a_{22} = b_{22} = r, \quad a_{12} = b_{12} = 0, \quad a_{11} = b_{11} = 1.$$

The structure equations on $\mathcal{G}_1$ can be written as

$$\mathrm{d}\begin{pmatrix} \omega^0 \\ \bar{\omega}^0 \\ \gamma \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = -\begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & \phi \end{pmatrix} \wedge \begin{pmatrix} \omega^0 \\ \bar{\omega}^0 \\ \gamma \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \Omega^0 \\ \bar{\Omega}^0 \\ \Gamma \\ \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix},$$

where, after adding a linear combination of the semi-basic 1-forms to ϕ, one can arrange

$$\Omega^0 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + \omega^3 \wedge \gamma + (\bar{P}_0 \bar{\omega}^0 + K\gamma + P_i \omega^i) \wedge \omega^0,$$

$$\bar{\Omega}^0 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + \omega^1 \wedge \gamma + (Q_0 \omega^0 + L\gamma + Q_i \omega^i) \wedge \bar{\omega}^0,$$

$$\Gamma = \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4 + C_i \omega^i \wedge \omega^0 + D_i \omega^i \wedge \bar{\omega}^0.$$

[5] One can verify that the classical Bäcklund transformation (1.5) relating solutions of the hyperbolic Tzitzeica equation (1.4) belongs Case 1. By computing the corresponding structure invariants, one can show that it is non-homogeneous.

Lemmas 5.2 and 5.3 apply to the current case without change. Following from this, $\Omega^1, ..., \Omega^4$ have expressions (5.17)-(5.20) where $T^k_{ij} = -T^k_{ji}$ for all $k, i, j = 1, ..., 4$, and $T^1_{23}, T^1_{24}, T^2_{23}, T^2_{24}, T^3_{14}, T^3_{24}, T^4_{14}, T^4_{24}$ are zero.

Furthermore, one can add a linear combination of the semi-basic 1-forms to α to arrange

$$T^2_1 = \bar{T}^2_1 = R^2_1 = T^2_{12} = T^2_{13} = T^2_{14} = 0.$$

By adjusting β, we can arrange

$$\bar{T}^4_3 = T^4_3 = R^4_3 = T^4_{13} = T^4_{23} = T^4_{34} = 0$$

Finally, by adding a suitable multiple of γ to ϕ, we can arrange

$$K = -L.$$

The torsion cannot be absorbed further.

Applying $\mathrm{d}^2 = 0$ to the structure equations, we obtain the congruences

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\omega^0) \equiv\ & (-P_3 - 1 - T^4_{12} - T^1_{13})\omega^3 \wedge \omega^1 \wedge \omega^2 + (-P_4 - T^1_{14} + T^3_{12})\omega^4 \wedge \omega^1 \wedge \omega^2 \\
& + (-P_4 - L - T^3_{34} - R^3_3 - R^4_4)\omega^4 \wedge \omega^3 \wedge \gamma + (-P_1 + T^3_{13} - T^2_{34})\omega^1 \wedge \omega^3 \wedge \omega^4 \\
& + (-P_2 + T^3_{23} + T^1_{34})\omega^2 \wedge \omega^3 \wedge \omega^4 + (L + T^3_{12} + R^1_1 + R^2_2)\gamma \wedge \omega^1 \wedge \omega^2 \\
& + (-\bar{P}_0 + \bar{T}^1_1 + \bar{T}^2_2)\bar{\omega}^0 \wedge \omega^1 \wedge \omega^2 + (-\bar{P}_0 - D_4 + \bar{T}^3_3 + \bar{T}^4_4)\bar{\omega}^0 \wedge \omega^3 \wedge (\omega^4 + \gamma) \\
& + (-P_1 + T^3_{13})\omega^1 \wedge \omega^3 \wedge \gamma + (-P_2 + T^3_{23})\omega^2 \wedge \omega^3 \wedge \gamma \qquad \mathrm{mod}\ \omega^0,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\bar{\omega}^0) \equiv\ & (-Q_1 + 1 + T^3_{13} - T^2_{34})\omega^1 \wedge \omega^3 \wedge \omega^4 + (-Q_4 - T^1_{14} + T^3_{12})\omega^4 \wedge \omega^1 \wedge \omega^2 \\
& + (-Q_2 + L - T^1_{12} - R^1_1 - R^2_2)\omega^2 \wedge \omega^1 \wedge \gamma + (-Q_3 - T^4_{12} - T^1_{13})\omega^3 \wedge \omega^1 \wedge \omega^2 \\
& + (-Q_2 + T^3_{23} + T^1_{34})\omega^2 \wedge \omega^3 \wedge \omega^4 + (-L + T^1_{34} + R^3_3 + R^4_4)\gamma \wedge \omega^3 \wedge \omega^4 \\
& + (-Q_0 + T^3_3 + T^4_4)\omega^0 \wedge \omega^3 \wedge \omega^4 + (-Q_0 + T^1_1 + T^2_2 - C_2)\omega^0 \wedge \omega^1 \wedge (\omega^2 + \gamma) \\
& + (-Q_3 - T^1_{13})\omega^3 \wedge \omega^1 \wedge \gamma + (-Q_4 - T^1_{14})\omega^4 \wedge \omega^1 \wedge \gamma \qquad \mathrm{mod}\ \bar{\omega}^0.
\end{aligned}$$

This implies the following relations

$$T^1_{12} = -Q_2 + 2L, \quad T^1_{13} = -P_3 - 1, \quad T^1_{14} = -P_4, \quad T^1_{34} = 0; \quad T^2_{34} = 0,$$

$$T^3_{34} = -P_4 - 2L, \quad T^3_{13} = Q_1 - 1, \quad T^3_{23} = Q_2, \quad T^3_{12} = 0, \quad T^4_{12} = 0,$$

$$T^2_2 = -T^1_1 + Q_0 + C_2, \qquad T^4_4 = Q_0 - T^3_3, \qquad \bar{T}^2_2 = \bar{P}_0 - \bar{T}^1_1, \qquad \bar{T}^4_4 = -\bar{T}^3_3 + \bar{P}_0 + D_4,$$

$$Q_4 = P_4, \qquad P_1 = Q_1 - 1, \qquad P_2 = Q_2, \qquad Q_3 = P_3 + 1,$$

$$R^2_2 = -L - R^1_1, \qquad R^4_4 = L - R^3_3.$$

Taking this into account, we compute

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\omega^1) &\equiv T^1_0 \bar{\omega}^0 \wedge \omega^3 \wedge (\omega^4 + \gamma) + T^1_2 \gamma \wedge \omega^3 \wedge \omega^4 && \mod \omega^0, \omega^1, \omega^2, \\
\mathrm{d}(\mathrm{d}\omega^2) &\equiv T^2_0 \bar{\omega}^0 \wedge \omega^3 \wedge (\omega^4 + \gamma) + (C_4 - T^1_1 + Q_0)\gamma \wedge \omega^3 \wedge \omega^4 && \mod \omega^0, \omega^1, \omega^2, \\
\mathrm{d}(\mathrm{d}\omega^3) &\equiv T^3_0 \omega^0 \wedge \omega^1 \wedge (\omega^2 + \gamma) + \bar{T}^3_4 \gamma \wedge \omega^1 \wedge \omega^2 && \mod \bar{\omega}^0, \omega^3, \omega^4, \\
\mathrm{d}(\mathrm{d}\omega^4) &\equiv T^4_0 \omega^0 \wedge \omega^1 \wedge (\omega^2 + \gamma) + (-D_2 + P_0 - \bar{T}^3_3)\gamma \wedge \omega^1 \wedge \omega^2 && \mod \bar{\omega}^0, \omega^3, \omega^4
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\gamma) \equiv\ & (-D_2 - C_2 - Q_2)\omega^2 \wedge \omega^3 \wedge \omega^4 + (-C_3 - D_3 + P_3 + 1)\omega^3 \wedge \omega^1 \wedge \omega^2 \\
& + (-D_4 - C_4 + P_4)\omega^4 \wedge \omega^1 \wedge \omega^2 + (-D_1 - C_1 - Q_1 + 1)\omega^1 \wedge \omega^3 \wedge \omega^4 \\
& \mod \omega^0, \bar{\omega}^0, \gamma.
\end{aligned}$$

As a result,

$$T^1_0 = T^2_0 = T^1_2 = T^3_0 = T^4_0 = \bar{T}^3_4 = 0,$$

$$T^1_1 = Q_0 - C_4, \qquad \bar{T}^3_3 = \bar{P}_0 - D_2,$$

$$Q_2 = -D_2 - C_2, \qquad P_4 = D_4 + C_4, \qquad Q_1 = 1 - C_1 - D_1, \qquad P_3 = D_3 + C_3 - 1.$$

Now the torsion coefficients are expressed in terms of the 19 functions : C_i $(i = 1, ..., 4)$, D_i $(i = 1, ..., 4)$, $\bar{T}^1_1, \bar{T}^1_2$, R^1_1, R^1_2, T^3_3, T^3_4, R^3_3, R^3_4, P_0, Q_0 and L. Applying $\mathrm{d}^2 = 0$ to the structure equations, we find the G_1-action on these torsion functions to be, infinitesimally,

$$\mathrm{d}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 0 & \alpha \\ 0 & \phi \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad \mathrm{d}\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} \equiv \begin{pmatrix} 0 & \beta \\ 0 & \phi \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \end{pmatrix},$$

$$\mathrm{d}\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \equiv \begin{pmatrix} 0 & \alpha \\ 0 & \phi \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \qquad \mathrm{d}\begin{pmatrix} D_3 \\ D_4 \end{pmatrix} \equiv \begin{pmatrix} 0 & \beta \\ 0 & \phi \end{pmatrix} \begin{pmatrix} D_3 \\ D_4 \end{pmatrix},$$

$$\mathrm{d}\begin{pmatrix} \bar{T}_1^1 \\ \bar{T}_2^1 \end{pmatrix} \equiv \begin{pmatrix} \phi & \alpha \\ 0 & 2\phi \end{pmatrix} \begin{pmatrix} \bar{T}_1^1 \\ \bar{T}_2^1 \end{pmatrix}, \qquad \mathrm{d}\begin{pmatrix} T_3^3 \\ T_4^3 \end{pmatrix} \equiv \begin{pmatrix} \phi & \beta \\ 0 & 2\phi \end{pmatrix} \begin{pmatrix} T_3^3 \\ T_4^3 \end{pmatrix},$$

$$\mathrm{d}\begin{pmatrix} R_1^1 \\ R_2^1 \end{pmatrix} \equiv \begin{pmatrix} \phi & \alpha \\ 0 & 2\phi \end{pmatrix} \begin{pmatrix} R_1^1 \\ R_2^1 \end{pmatrix}, \qquad \mathrm{d}\begin{pmatrix} R_3^3 \\ R_4^3 \end{pmatrix} \equiv \begin{pmatrix} \phi & \beta \\ 0 & 2\phi \end{pmatrix} \begin{pmatrix} R_3^3 \\ R_4^3 \end{pmatrix},$$

and

$$\mathrm{d}L \equiv L\phi, \qquad \mathrm{d}\bar{P}_0 \equiv \bar{P}_0\phi, \qquad \mathrm{d}Q_0 \equiv Q_0\phi,$$

where all congruences are modulo the semi-basic 1-forms $\omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4$.

Clearly, each element in $\mathcal{R} := \{C_2, C_4, D_2, D_4, \bar{T}_2^1, T_4^3, R_2^1, R_4^3, L, \bar{P}_0, Q_0\}$ is a relative invariant with nonzero weights only in ϕ. There are two cases:

I. All functions in $\mathcal{R}$ are all zero;

II. Not all functions in $\mathcal{R}$ are zero.

Homogeneity Assumption: Now we assume that the underlying rank-2 Bäcklund transformation is *homogeneous*.

I. In this case, by the homogeneity assumption, C_1, C_3, D_1, D_3 must be constants; then $\bar{T}_1^1, R_1^1, T_3^3, R_3^3$ become relative invariants, with nonzero weights only in ϕ.

We apply $\mathrm{d}^2 = 0$ to the equations of $\mathrm{d}\omega^0$ and $\mathrm{d}\bar{\omega}^0$, obtaining

$$\begin{aligned}\mathrm{d}(\mathrm{d}\omega^0) = & -\mathrm{d}\phi \wedge \omega^0 - R_1^1(C_1 + D_1)\gamma \wedge \omega^1 \wedge \omega^0 + ((C_3 + D_3 - 1)R_3^3 - T_3^3)\gamma \wedge \omega^3 \wedge \omega^0 \\ & + (2C_1 + D_1 + C_3)\omega^0 \wedge \omega^1 \wedge \omega^3 - (C_1 + D_1)\bar{T}_1^1\omega^0 \wedge \bar{\omega}^1 \wedge \omega^1, \\ \mathrm{d}(\mathrm{d}\bar{\omega}^0) = & -\mathrm{d}\phi \wedge \bar{\omega}^0 + R_3^3(D_3 + C_3)\gamma \wedge \omega^3 \wedge \bar{\omega}^0 - ((D_1 + C_1 - 1)R_1^1 + \bar{T}_1^1)\gamma \wedge \omega^1 \wedge \bar{\omega}^0 \\ & + (2D_3 + C_3 + D_1)\omega^3 \wedge \omega^1 \wedge \bar{\omega}^0 + (D_3 + C_3)T_3^3\omega^0 \wedge \omega^3 \wedge \bar{\omega}^0.\end{aligned}$$

From this we deduce

$$\bar{T}_1^1 = R_1^1, \qquad T_3^3 = -R_3^3, \qquad C_1 + C_3 = -(D_1 + D_3).$$

By writing D_3 as $-(C_1+C_3+D_1)$, we have

$$\begin{aligned} \mathrm{d}\phi &= -W\omega^0\wedge\bar{\omega}^0 - (C_1+D_1)\gamma\wedge(R^1_1\omega^1+R^3_3\omega^3) \\ &\quad - (C_1+D_1)(R^1_1\bar{\omega}^0\wedge\omega^1 - R^3_3\omega^0\wedge\omega^3) + (2C_1+D_1+C_3)\omega^1\wedge\omega^3, \end{aligned} \tag{5.85}$$

for a function W.

Taking this into account, we compute

$$\begin{aligned} \mathrm{d}(\mathrm{d}\gamma) &= W\omega^0\wedge\bar{\omega}^0\wedge\gamma + C_1R^1_1\gamma\wedge\omega^1\wedge(\omega^0-\bar{\omega}^0) + C_1R^1_1\omega^0\wedge\bar{\omega}^0\wedge\omega^1 \\ &\quad + (C_1+C_3+D_1)R^3_3\gamma\wedge\omega^3\wedge(\omega^0-\bar{\omega}^0) - (C_1+C_3+D_1)R^3_3\omega^0\wedge\bar{\omega}^0\wedge\omega^3 \\ &\quad + (2(C_1)^2 - 2C_1C_3 + 2C_1D_1 - 2C_3D_1 + C_1 + C_3)\omega^1\wedge\omega^3\wedge\omega^0 \\ &\quad + (2(C_1)^2 + 2C_1C_3 + 6C_1D_1 + 2C_3D_1 + 4(D_1)^2 - C_1 - C_3)\omega^1\wedge\omega^3\wedge\bar{\omega}^0 \\ &\quad - 2(2C_1+D_1+C_3)\omega^1\wedge\omega^3\wedge\gamma. \end{aligned} \tag{5.86}$$

Therefore, $W=0$, and, by the vanishing of the coefficients of $\omega^1\wedge\omega^3\wedge\gamma$, $\omega^1\wedge\omega^3\wedge\omega^0$ and $\omega^1\wedge\omega^3\wedge\bar{\omega}^0$, we must have

$$-C_1 = D_1 = C_3.$$

As a result, ϕ is closed, by (5.85).

To continue, we need to ask whether the functions R^1_1 and R^3_3 are both zero. This leads to two subcases:

I1. If R^1_1, R^3_3 are zero, then, after replacing the constant C_3 by λ and restricting to a leaf of ϕ, the structure equations become

$$\begin{aligned} \mathrm{d}\omega^0 &= \omega^1\wedge\omega^2 + \omega^3\wedge(\omega^4+\gamma-\omega^0), \\ \mathrm{d}\bar{\omega}^0 &= \omega^1\wedge(\omega^2+\gamma+\bar{\omega}^0) + \omega^3\wedge\omega^4, \\ \mathrm{d}\gamma &= \omega^1\wedge\omega^2 - \omega^3\wedge\omega^4 - \lambda(\omega^1-\omega^3)\wedge(\omega^0-\bar{\omega}^0), \\ \mathrm{d}\omega^1 &= 0, \\ \mathrm{d}\omega^2 &= -\alpha\wedge\omega^1 + \lambda\omega^0\wedge\omega^3, \\ \mathrm{d}\omega^3 &= 0, \\ \mathrm{d}\omega^4 &= -\beta\wedge\omega^3 + \lambda\bar{\omega}^0\wedge\omega^1. \end{aligned} \tag{5.87}$$

Applying $\mathrm{d}^2 = 0$ to $\mathrm{d}\omega^0, \mathrm{d}\bar{\omega}^0, \mathrm{d}\gamma, \mathrm{d}\omega^1$ and $\mathrm{d}\omega^3$ leads to identities. Applying $\mathrm{d}^2 = 0$ to $\mathrm{d}\omega^2$ and $\mathrm{d}\omega^4$ yields

$$(\mathrm{d}\alpha - \lambda\omega^2 \wedge \omega^3) \wedge \omega^1 = 0,$$

$$(\mathrm{d}\beta - \lambda\omega^4 \wedge \omega^1) \wedge \omega^3 = 0.$$

I2. Suppose that R^1_1, R^3_3 are not both zero. If R^1_1 is nonzero, then let U be R^1_1, otherwise, let U be R^3_3. Such a function U, being nonzero, satisfies

$$dU \equiv U\phi, \quad \mathrm{mod}\ \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4.$$

By homogeneity, there exist constants r^1_1, r^3_3 (one of which equals to 1), such that

$$R^1_1 = r^1_1 U, \qquad R^3_3 = r^3_3 U.$$

Let $\mathcal{G}_2 \subset \mathcal{G}_1$ be defined by $U = 1$. On $\mathcal{G}_2$, there exist functions $H_0, \bar{H}_0, H_\gamma, H_1, ..., H_4$ such that

$$\phi = H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + \sum_{i=1}^{4} H_i\omega^i.$$

Applying $\mathrm{d}^2 = 0$ to the structure equations, it is easy to find

$$\begin{array}{lll} \mathrm{d}H_0 \equiv 0, & & \\ \mathrm{d}\bar{H}_0 \equiv 0, & \mathrm{d}H_1 \equiv \alpha H_2, & \mathrm{d}H_3 \equiv \beta H_4, \\ & \mathrm{d}H_2 \equiv 0, & \mathrm{d}H_4 \equiv 0, \\ \mathrm{d}H_\gamma \equiv 0, & & \end{array}$$

where all congruences are modulo the semi-basic 1-forms. As a result of this and the homogeneity assumption, $H_0, \bar{H}_0, H_\gamma, H_2, H_4$ are constants. Moreover, we have

$$\mathrm{d}(\mathrm{d}\omega^1) \equiv -r^1_1(H_2\omega^2 + H_3\omega^3 + H_4\omega^4) \wedge \gamma \wedge \omega^1 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0,$$

$$\mathrm{d}(\mathrm{d}\omega^3) \equiv -r^3_3(H_1\omega^1 + H_2\omega^2 + H_4\omega^4) \wedge \gamma \wedge \omega^3 \qquad \mathrm{mod}\ \omega^0, \bar{\omega}^0.$$

Since we have assumed that r^1_1 and r^3_3 are not both zero, it follows that

$$H_2 = H_4 = 0.$$

As a result, H_1 and H_3 must be constants. Using this and expanding $\mathrm{d}(\mathrm{d}\omega^0), \mathrm{d}(\mathrm{d}\bar{\omega}^0)$, $\mathrm{d}(\mathrm{d}\gamma), \mathrm{d}(\mathrm{d}\omega^1)$, $\mathrm{d}(\mathrm{d}\omega^3)$, and

$$\mathrm{d}(\mathrm{d}\omega^2) \mod \omega^1, \qquad \mathrm{d}(\mathrm{d}\omega^4) \mod \omega^3,$$

we find that $H_0, \bar{H}_0, H_\gamma, H_1$ and H_3 must all be zero.

Taking these into account, replacing r_1^1 by λ and r_3^3 by μ; the structure equations now take the form

$$\begin{aligned}
\mathrm{d}\omega^0 &= \omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma - \omega^0), \\
\mathrm{d}\bar{\omega}^0 &= \omega^1 \wedge (\omega^2 + \gamma + \bar{\omega}^0) + \omega^3 \wedge \omega^4, \\
\mathrm{d}\gamma &= \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4, \\
\mathrm{d}\omega^1 &= \lambda(\gamma + \bar{\omega}^0) \wedge \omega^1, \\
\mathrm{d}\omega^2 &= -\alpha \wedge \omega^1 - \lambda(\gamma + \bar{\omega}^0) \wedge \omega^2, \\
\mathrm{d}\omega^3 &= \mu(\gamma - \omega^0) \wedge \omega^3, \\
\mathrm{d}\omega^4 &= -\beta \wedge \omega^3 - \mu(\gamma - \omega^0) \wedge \omega^4,
\end{aligned} \tag{5.88}$$

where either λ or μ equals to 1. Applying $\mathrm{d}^2 = 0$ to the equations of $\mathrm{d}\omega^0, \mathrm{d}\bar{\omega}^0, \mathrm{d}\gamma, \mathrm{d}\omega^1$ and $\mathrm{d}\omega^3$ yields identities; applying it to the equations of $\mathrm{d}\omega^2$ and $\mathrm{d}\omega^4$ yields

$$\begin{aligned}
(\mathrm{d}\alpha + \lambda(\gamma + \bar{\omega}^0) \wedge (2\alpha + \omega^2)) \wedge \omega^1 = 0, \\
(\mathrm{d}\beta + \mu(\gamma - \omega^0) \wedge (2\beta - \omega^4)) \wedge \omega^3 = 0.
\end{aligned}$$

II. In this case, we can find a nonzero function U, expressed in terms of the functions in $\mathcal{R}$, satisfying

$$\mathrm{d}U \equiv U\phi, \quad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4.$$

By the homogeneity assumption, there exist constants $c_2, c_4, d_2, d_4, \ell, \bar{p}_0, q_0, \bar{t}_2^1, r_2^1, t_4^3, r_4^3$ such that

$$\begin{array}{lll}
L = \ell U, & C_2 = c_2 U, & \bar{T}_2^1 = \bar{t}_2^1 U^2, \\
\bar{P}_0 = \bar{p}_0 U, & C_4 = c_4 U, & R_2^1 = r_2^1 U^2, \\
Q_0 = q_0 U, & D_2 = d_2 U, & T_4^3 = t_4^3 U^2, \\
 & D_4 = d_4 U, & R_4^3 = r_4^3 U^2.
\end{array}$$

Since U is a relative invariant with weight 1 in ϕ, we can perform a structure reduction to $\mathcal{G}_2 \subset \mathcal{G}_1$ defined by

$$U = 1.$$

On $\mathcal{G}_2$, there exist functions $H_0, \bar{H}_0, H_\gamma, H_1, ..., H_4$ such that

$$\phi = H_0\omega^0 + \bar{H}_0\bar{\omega}^0 + H_\gamma\gamma + \sum_{i=1}^{4} H_i\omega^i.$$

By applying $\mathrm{d}^2 = 0$ to the structure equations, we find

$$\begin{aligned} &\mathrm{d}H_0 \equiv 0, \\ &\mathrm{d}\bar{H}_0 \equiv 0, \\ &\mathrm{d}H_\gamma \equiv 0, \end{aligned} \qquad \begin{aligned} &\mathrm{d}H_1 \equiv \alpha H_2, \quad \mathrm{d}H_3 \equiv \beta H_4, \\ &\mathrm{d}H_2 \equiv 0, \qquad\ \ \mathrm{d}H_4 \equiv 0, \end{aligned}$$

where all congruences are modulo the semi-basic 1-forms. By the homogeneity assumption, $H_0, \bar{H}_0, H_\gamma,\ H_2, H_4$ are constants.

Now, depending on whether the structure can be further reduced, we need to consider three subcases:

II1. $c_2, d_2, \bar{t}^1_2, r^1_2, H_2$ are all zero;

II2. $c_4, d_4, t^3_4, r^3_4, H_4$ are all zero;

II3. Not all of $c_2, d_2, \bar{t}^1_2, r^1_2, H_2$, and not all of $c_4, d_4, t^3_4, r^3_4, H_4$ are zero.

II1. In this case, by homogeneity, $C_1, D_1, \bar{T}^1_1, R^1_1, H_1$ must be constants. Taking this into account and using the structure equations, we expand $\mathrm{d}(\mathrm{d}\omega^0),\ \mathrm{d}(\mathrm{d}\bar{\omega}^0),\ \mathrm{d}(\mathrm{d}\gamma),\ \mathrm{d}(\mathrm{d}\omega^1),\ \mathrm{d}(\mathrm{d}\omega^3)$ and reduce modulo ω^3; expand $\mathrm{d}(\mathrm{d}\omega^2)$, modulo ω^1, ω^3; and expand $\mathrm{d}(\mathrm{d}\omega^4)$, modulo ω^3, ω^4. Using Maple™, we verify that these calculations imply

$$\ell = \bar{p}_0 = q_0 = c_4 = d_4 = 0, \qquad r^3_4 = -t^3_4.$$

By the assumption of the current case, the only possibility is when t^3_4 is nonzero. As a result, one can reduce to the subbundle $\mathcal{G}_3$ defined by

$$T^3_3 = 0.$$

On $\mathcal{G}_3$, the functions C_3, D_3, R^3_3, H_3 reduce to constants. Moreover, there exist functions $N_0, \bar{N}_0, N_\gamma,\ N_1, ..., N_4$ on $\mathcal{G}_3$ such that

$$\beta = N_0\omega^0 + \bar{N}_0\bar{\omega}^0 + N_\gamma\gamma + \sum_{i=1}^{4} N_i\omega^i.$$

By differentiation and using the homogeneity assumption, we find that among these new functions, $N_0, \bar{N}_0, N_\gamma, N_2, N_4$ are constants, and that

$$\mathrm{d}N_1 \equiv \alpha N_2, \quad \mod \omega^0, \bar{\omega}^0, \gamma, \omega^1, ..., \omega^4.$$

Moreover, we have

$$\mathrm{d}(\mathrm{d}\omega^4) \equiv -N_2 t^3_4\ \omega^0 \wedge \omega^2 \wedge \omega^4 \qquad \mod \bar{\omega}^0, \gamma, \omega^1, \omega^3.$$

Since t^3_4 is nonzero (by assumption), N_2 must be zero, and N_1 is therefore a constant. No further structure reduction is possible.

Now, expanding using Maple™ $\mathrm{d}(\mathrm{d}\omega^0),\ \mathrm{d}(\mathrm{d}\bar{\omega}^0),\ \mathrm{d}(\mathrm{d}\gamma),\ \mathrm{d}(\mathrm{d}\omega^1),\ \mathrm{d}(\mathrm{d}\omega^3)$ and

$$\mathrm{d}(\mathrm{d}\omega^2) \quad \mod \omega^1, \qquad \mathrm{d}(\mathrm{d}\omega^4) \quad \mod \omega^1.$$

we find that either

$$\{H_\gamma, \ell, C_1, C_3, D_1, D_3, H_0, \bar{H}_0, H_1, H_3, H_4, \bar{N}_0, N_1, N_2, R^3_3, c_4, d_4, \bar{p}_0, q_0\} = \{0\},$$
$$N_0 = -N_\gamma, \qquad N_4 = 1, \qquad R^1_1 = \bar{T}^1_1, \qquad r^3_4 = -t^3_4;$$

or

$$\{\ell, C_1, C_3, D_1, D_3, \bar{H}_0, H_1, H_3, \bar{N}_0, N_1, N_2, R^1_1, R^3_3, \bar{T}^1_1, c_4, d_4, \bar{p}_0, q_0\} = \{0\},$$
$$H_\gamma = -H_0 = \frac{1}{2}H_4, \qquad N_\gamma = -N_0 = \frac{1}{2}, \qquad N_4 = 1, \qquad r^3_4 = -t^3_4.$$

By a change of notation, the former case provides the structure equations

$$\begin{aligned}
\mathrm{d}\omega^0 &= \omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma - \omega^0),\\
\mathrm{d}\bar{\omega}^0 &= \omega^1 \wedge (\omega^2 + \gamma + \bar{\omega}^0) + \omega^3 \wedge \omega^4,\\
\mathrm{d}\gamma &= \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4,\\
\mathrm{d}\omega^1 &= \lambda(\bar{\omega}^0 + \gamma) \wedge \omega^1,\\
\mathrm{d}\omega^2 &= -\alpha \wedge \omega^1 - \lambda(\bar{\omega}^0 + \gamma) \wedge \omega^2,\\
\mathrm{d}\omega^3 &= \mu(\omega^0 - \gamma) \wedge \omega^4,\\
\mathrm{d}\omega^4 &= \tau(\omega^0 - \gamma) \wedge \omega^3 + \omega^3 \wedge \omega^4,
\end{aligned} \tag{5.89}$$

where λ, μ, τ are constants with μ being nonzero. These structure equations verify the identity $\mathrm{d}^2 = 0$ except that

$$\mathrm{d}(\mathrm{d}\omega^2) = -(\mathrm{d}\alpha + \lambda(\bar{\omega}^0 + \gamma) \wedge (2\alpha + \omega^2)) \wedge \omega^1.$$

The latter case provides the structure equations

$$\begin{aligned}
\mathrm{d}\omega^0 &= -\lambda(\gamma + 2\omega^4) \wedge \omega^0 + \omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma - \omega^0),\\
\mathrm{d}\bar{\omega}^0 &= -\lambda(\gamma - \omega^0 + 2\omega^4) \wedge \bar{\omega}^0 + \omega^1 \wedge (\omega^2 + \gamma + \bar{\omega}^0) + \omega^3 \wedge \omega^4,\\
\mathrm{d}\gamma &= \lambda(\omega^0 - 2\omega^4) \wedge \gamma + \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4,\\
\mathrm{d}\omega^1 &= 0,\\
\mathrm{d}\omega^2 &= -\alpha \wedge \omega^1 - \lambda(\gamma - \omega^0 + 2\omega^4) \wedge \omega^2,\\
\mathrm{d}\omega^3 &= -\mu(\gamma - \omega^0) \wedge \omega^4,\\
\mathrm{d}\omega^4 &= -\frac{1}{2}(\gamma - \omega^0 + 2\omega^4) \wedge \omega^3 - \lambda(\gamma - \omega^0) \wedge \omega^4,
\end{aligned} \tag{5.90}$$

where λ, μ are constants with μ being nonzero. These structure equations verify the identity $\mathrm{d}^2 = 0$ except that

$$\mathrm{d}(\mathrm{d}\omega^2) = (-\mathrm{d}\alpha + \lambda\alpha \wedge (\gamma - \omega^0 + 2\omega^4)) \wedge \omega^1.$$

II2. The analysis of this case is similar to that of Case (**II1**), so we omit the details. In fact, the only two possible solutions in this case can be turned into (5.89) and (5.90), respectively, by switching $(\omega^0, \omega^1, \omega^2)$ and $(\bar{\omega}^0, \omega^3, \omega^4)$ then changing the signs of ω^3 and ω^4. Therefore, we will not regard these as new solutions.

II3. In this case, not all of $c_2, d_2, \bar{t}_2^1, r_2^1, H_2$ are zero, and not all of $c_4, d_4, t_4^3, r_4^3, H_4$ are zero. Thus, it is always possible to reduce to an e-structure. Now let

$$\alpha = M_0\omega^0 + \bar{M}_0\bar{\omega}^0 + M_\gamma\gamma + \sum_{i=1}^{4} M_i\omega^i,$$

$$\beta = N_0\omega^0 + \bar{N}_0\bar{\omega}^0 + N_\gamma\gamma + \sum_{i=1}^{4} N_i\omega^i,$$

where all the new coefficients are constants, by the homogeneity assumption. Applying $\mathrm{d}^2 = 0$ to the structure equations and computing with Maple™, we find that the structure equations must take the form

$$\begin{aligned}
\mathrm{d}\omega^0 &= \omega^1 \wedge \omega^2 + \omega^3 \wedge (\omega^4 + \gamma - \omega^0),\\
\mathrm{d}\bar{\omega}^0 &= \omega^1 \wedge (\omega^2 + \gamma + \bar{\omega}^0) + \omega^3 \wedge \omega^4,\\
\mathrm{d}\gamma &= \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4,\\
\mathrm{d}\omega^1 &= (\bar{\omega}^0 + \gamma) \wedge (V\omega^1 + \lambda\omega^2),\\
\mathrm{d}\omega^2 &= -(\bar{\omega}^0 + \gamma) \wedge (M\omega^1 + V\omega^2) - \omega^1 \wedge \omega^2,\\
\mathrm{d}\omega^3 &= (\omega^0 - \gamma) \wedge (W\omega^3 + \mu\omega^4),\\
\mathrm{d}\omega^4 &= (\omega^0 - \gamma) \wedge (N\omega^3 - W\omega^4) + \omega^3 \wedge \omega^4,
\end{aligned} \tag{5.91}$$

where M, N, V, W, λ, μ are constants with λ, μ being nonzero. Applying $\mathrm{d}^2 = 0$ to (5.91) yields identities.

5.3.2 Integration of the Structure Equations

Equations (5.87). Suppose that there exists a coframing defined on a small enough domain $U \subset N$ satisfying the structure equations (5.87). Since ω^1 and ω^3 are closed, there exist functions x, y on U such that

$$\omega^1 = \mathrm{d}x, \qquad \omega^3 = \mathrm{d}y.$$

The equations of $\mathrm{d}\omega^0$ and $\mathrm{d}\bar{\omega}^0$ then imply the existence of new functions z, p, q, w, s, t that satisfy

$$\omega^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \qquad \bar{\omega}^0 = \mathrm{d}w - s\mathrm{d}x - t\mathrm{d}y.$$

By the equation of $\mathrm{d}\omega^0$, we have

$$-\mathrm{d}p \wedge \mathrm{d}x - \mathrm{d}q \wedge \mathrm{d}y = \mathrm{d}\omega^0 = -\omega^2 \wedge \mathrm{d}x - (\omega^4 + \gamma - \mathrm{d}z + p\mathrm{d}x + q\mathrm{d}y) \wedge \mathrm{d}y,$$

which implies that

$$\omega^2 \equiv \mathrm{d}p + h\mathrm{d}y \mod \mathrm{d}x, \tag{5.92}$$

$$\omega^4 + \gamma \equiv \mathrm{d}z + \mathrm{d}q + (-p + h)\mathrm{d}x \mod \mathrm{d}y, \tag{5.93}$$

for a new function h. Similarly, by the equation of $\mathrm{d}\bar{\omega}^0$, we obtain

$$-\mathrm{d}s \wedge \mathrm{d}x - \mathrm{d}t \wedge \mathrm{d}y = \mathrm{d}\bar{\omega}^0 = -(\omega^2 + \gamma + \mathrm{d}w - s\mathrm{d}x - t\mathrm{d}y) \wedge \mathrm{d}x - \omega^4 \wedge \mathrm{d}y,$$

which implies that

$$\omega^2 + \gamma \equiv -\mathrm{d}w + \mathrm{d}s + (t + g)\mathrm{d}y \mod \mathrm{d}x,$$

$$\omega^4 \equiv \mathrm{d}t + g\mathrm{d}x \mod \mathrm{d}y,$$

for a new function g.

Now, using the expressions of ω^2, ω^4 and their exterior derivatives, we have

$$\mathrm{d}h \wedge \mathrm{d}y \equiv \mathrm{d}\omega^2 \equiv \lambda \mathrm{d}z \wedge \mathrm{d}y \mod \mathrm{d}x,$$

$$\mathrm{d}g \wedge \mathrm{d}x \equiv \mathrm{d}\omega^4 \equiv \lambda \mathrm{d}w \wedge \mathrm{d}x \mod \mathrm{d}y.$$

As a result, there exist functions $H(x,y)$ and $G(x,y)$ such that

$$h = \lambda z + H(x,y), \qquad g = \lambda w + G(x,y). \tag{5.94}$$

We claim that one can always arrange $H = G = 0$ by a change of coordinates that preserves the expressions of $\omega^0, \bar{\omega}^0$. In fact, consider the change of variable

$$z \mapsto z + \zeta(x,y).$$

By the expression of ω^0, we have, correspondingly,

$$p \mapsto p + \zeta_x, \qquad q \mapsto q + \zeta_y;$$

and, by the expression of ω^2,

$$h \mapsto h - \zeta_{xy};$$

finally, by the expression of h,

$$H \mapsto H - \lambda\zeta - \zeta_{xy}.$$

Locally, given an analytic function $H(x,y)$, it is always possible to solve the equation $H = \zeta_{xy} + \lambda\zeta$. This shows that we can always put H to be equal to zero by an appropriate choice of the coordinates. By a similar argument, it is easy to see that a transformation of the form

$$w \mapsto w + \xi(x,y)$$

leads to

$$G \mapsto G - \lambda\xi - \xi_{xy}.$$

Hence, we can put $G = 0$ by an appropriate choice of the functions w, s and t.

With the simplified expressions of h and g, we have

$$\begin{aligned} \mathrm{d}\gamma = {} & \mathrm{d}x \wedge (\mathrm{d}p - \lambda \mathrm{d}z + \lambda \mathrm{d}w) - \mathrm{d}y \wedge (\mathrm{d}t - \lambda \mathrm{d}z + \lambda \mathrm{d}w) \\ & + \lambda(z + w + q - t + p - s)\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Moreover, comparing the expressions of $\omega^4, \omega^4 + \gamma$ and $\omega^2, \omega^2 + \gamma$ yields:

$$\begin{aligned} \gamma &= \mathrm{d}z + \mathrm{d}q - \mathrm{d}t - (p + \lambda w - \lambda z)\mathrm{d}x + M\mathrm{d}y \\ &= -\mathrm{d}w + \mathrm{d}s - \mathrm{d}p + (t + \lambda w - \lambda z)\mathrm{d}y + N\mathrm{d}x, \end{aligned} \tag{5.95}$$

for two new functions M and N. By (5.95), we have

$$\mathrm{d}(z + w + q - s + p - t) - (p + \lambda w - \lambda z)\mathrm{d}x - (t + \lambda w - \lambda z)\mathrm{d}y = N\mathrm{d}x - M\mathrm{d}y.$$

This shows that $z + w + q - t + p - s$ is a function of x, y only, say, $\eta(x, y)$. The functions M and N are then:

$$M = -\eta_y + t + \lambda w - \lambda z, \qquad N = \eta_x - p - \lambda w + \lambda z.$$

Now, we can compute the exterior derivative of $\gamma = \mathrm{d}z + \mathrm{d}q - (p + \lambda w - \lambda z)\mathrm{d}x + (-\eta_y + t + \lambda w - \lambda z)\mathrm{d}y$. This is compatible with the equation for $\mathrm{d}\gamma$ if and only if $\eta(x, y)$ satisfies

$$\eta_{xy} + \lambda\eta = 0.$$

If we make the following change of variables

$$z \mapsto z + \zeta(x, y), \qquad w \mapsto w + \xi(x, y)$$

such that the functions H and G both remain zero, i.e., such that ζ, ξ satisfy the equations $\zeta_{xy} + \lambda\zeta = 0$ and $\xi_{xy} + \lambda\xi = 0$, then η would change by

$$\eta \mapsto \eta + (\zeta + \zeta_x + \zeta_y) + (\xi - \xi_x - \xi_y).$$

Now let $\tilde{\eta} = \eta + (\xi - \xi_x - \xi_y)$. It suffices to ask whether the following system of partial differential equations has solutions, given that $\tilde{\eta}$ satisfies $\tilde{\eta}_{xy} + \lambda\tilde{\eta} = 0$:

$$\begin{cases} \zeta_{xy} + \lambda\zeta = 0, \\ \zeta + \zeta_x + \zeta_y = -\tilde{\eta}. \end{cases} \tag{5.96}$$

One can apply the Frobenius Theorem to (5.96), and conclude that its solutions depend on 2 constants. Consequently, one can put $\eta = 0$ by an appropriate choice of coordinates.

We have shown that the Bäcklund transformation corresponding to the structure equations (5.87) is an auto-Bäcklund transformation relating solutions of the equation $z_{xy} + \lambda z = 0$ (see (5.92) and (5.94)). It is not hard to see that such a Bäcklund transformation locally correspond to the system (up to contact equivalence):

$$\begin{cases} z + z_x + z_y + w - w_x - w_y = 0, \\ z_{xy} + \lambda z = w_{xy} + \lambda w, \end{cases}$$

where the second equation is deduced from the first and the vanishing of γ.

Equations (5.88)**,** (5.89)**,** (5.90) **and** (5.91). All these structure equations except (5.90) satisfy the property

$$\mathrm{d}(\omega^1 \wedge \omega^2) = \mathrm{d}(\omega^3 \wedge \omega^4) = 0.$$

However, for (5.90), note that $\omega^0 - \gamma - 2\omega^4$ is closed. Thus, there exists a function h such that $\omega^0 - \gamma - 2\omega^4 = \mathrm{d}h$. After the change of variables

$$\eta_\gamma = e^{-\lambda h}\gamma, \qquad \eta^0 = e^{-\lambda h}\omega^0, \qquad \bar{\eta}^0 = e^{-\lambda h}\bar{\omega}^0,$$

$$\eta^1 = \omega^1, \qquad \eta^3 = \omega^3, \qquad \eta^2 = e^{-\lambda h}\omega^2, \qquad \eta^4 = e^{-\lambda h}\omega^4.$$

the equations (5.90) become

$$\begin{aligned} \mathrm{d}\eta^0 &= \eta^1 \wedge \eta^2 - \eta^3 \wedge (\eta^4 + e^{-\lambda h}\mathrm{d}h), \\ \mathrm{d}\bar{\eta}^0 &= \eta^1 \wedge (\eta^2 + \eta_\gamma) + \eta^1 \wedge \bar{\eta}^0 + \eta^3 \wedge \eta^4, \\ \mathrm{d}\eta_\gamma &= \eta^1 \wedge \eta^2 - \eta^3 \wedge \eta^4, \\ \mathrm{d}\eta^1 &= 0, \qquad \mathrm{d}\eta^3 = \mu e^{\lambda h}\mathrm{d}h \wedge \eta^4, \\ \mathrm{d}\eta^2 &= -\hat{\alpha} \wedge \eta^1, \quad \mathrm{d}\eta^4 = \frac{1}{2}e^{-\lambda h}\mathrm{d}h \wedge \eta^3. \end{aligned} \tag{5.97}$$

Clearly, in (5.97), $\eta^1 \wedge \eta^2$ and $\eta^3 \wedge \eta^4$ are closed.

Proposition 5.4. *The structure equations* (5.88), (5.89), (5.97) *and* (5.91) *all represent rank-*2 *auto-Bäcklund transformations of the wave equation* $z_{xy} = 0$.

Proof. We only provide details for (5.88). The argument works identically for the other structure equations.

In (5.88), the pull-back of $\mathcal{I}$ corresponds to the system $\langle \omega^0, \Phi, \Psi \rangle$ where $\Phi = \omega^1 \wedge \omega^2$, $\Psi = \omega^3 \wedge (\omega^4 + \gamma - \omega^0)$ are decomposable 2-forms. These forms satisfy $\mathrm{d}\Phi = 0$ and $\mathrm{d}\omega^0 = \Phi + \Psi$. It follows that $\mathrm{d}\Psi = 0$ as well. Hence, there locally exist functions x, y, p, q such that $\Phi = \mathrm{d}x \wedge \mathrm{d}p$, $\Psi = \mathrm{d}y \wedge \mathrm{d}q$. It follows that $\mathrm{d}(\omega^0 + p\mathrm{d}x + q\mathrm{d}y) = 0$. Thus, locally, there exists a function z such that $\omega^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$. Clearly, $\mathcal{I}$ corresponds to the wave equation $z_{xy} = 0$. The argument for $\bar{\mathcal{I}}$ is similar. □

5.4 Summary

We summarize our main results in this chapter in the following

Theorem 5.15. *Let* $(N, \mathcal{B})$ *be a homogeneous rank-*2 *Bäcklund transformation relating two hyperbolic Monge-Ampère systems.*

(i) *It is impossible for* $(N, \mathcal{B})$ *to satisfy all three genericity conditions with* (B_1, B_2) *and* (B_3, B_4) *being both nonzero (Theorem 5.7).*

(ii) *Suppose that* $(N, \mathcal{B})$ *satisfies only the first two genericity conditions with* $\epsilon = 1$ *and with the relative invariants* (B_1, B_2) *and* (B_3, B_4) *being both nonzero. If* $(N, \mathcal{B})$ *does not arise as a* 1*-parameter family of rank-*1 *Bäcklund transformations in the sense of Lemma 5.9, then* $(N, \mathcal{B})$ *is an auto-Bäcklund transformation of the linear equation* $z_{xy} = z$ *(see* (5.72), (5.73), (5.76) *and* (5.77)*).*

(iii) *If* $(N, \mathcal{B})$ *satisfies only the first genericity condition, with the first derived system of* B^1 *having rank* 2 *and* (B_1, B_2), (B_3, B_4) *being both nonzero, then it is an auto-Bäcklund transformation of a linear equation of the form* $z_{xy} = \lambda z$ *(see Section 5.3.2).*

Of course, Theorem 5.15 does not exhaust all cases of homogeneous rank-2 Bäcklund transformations relating two hyperbolic Monge-Ampère systems. In particular, we still need to investigate the following cases:

1. when all three genericity conditions are satisfied, and exactly one of the relative invariants (B_1, B_2) and (B_3, B_4) is zero;

2. when only the first two genericity conditions are satisfied, $\epsilon = -1$ and both (B_1, B_2) and (B_3, B_4) are zero;

3. when only the first two genericity conditions are satisfied, and exactly one of the relative invariants (B_1, B_2) and (B_3, B_4) is zero;

4. when only the first genericity condition is satisfied, with the first derived system of B^1 having rank 1, and exactly one of (B_1, B_2), (B_3, B_4) is zero;

5. when only the first genericity condition is satisfied, with the first derived system of B^1 having rank 2.

However, based on our classification so far, one may conjecture that those homogeneous Bäcklund transformations (relating two hyperbolic Monge-Ampère systems) that are 'genuinely' rank-2 are quite few.

6

Conclusion

The purpose of this short chapter is to briefly summarize the main results in this thesis and to give several directions for future research.

1. In this thesis, we have proved an upper bound for the generality of *generic* rank-1 Bäcklund transformations relating two hyperbolic Monge-Ampère systems (Theorem 3.3). *What is a corresponding result in the non-generic case? What is the generality of Bäcklund transformations of higher ranks? Given a hyperbolic Monge-Ampère system, is it related to a system of the same type by a Bäcklund transformation, regardless of the rank?*

2. We have proved several results that tell us which hyperbolic Monge-Ampère systems may be related by a rank-1 Bäcklund transformation of a particular type (Theorem 4.5 and Proposition 4.12). *Is there an efficient way to tell whether a given hyperbolic Monge-Ampère system is not related to any other hyperbolic Monge-Ampère system by a rank-*1 *Bäcklund transformation?*

3. Our classification of homogeneous rank-2 Bäckund transformations relating two hyperbolic Monge-Ampère systems so far is summarized in Theorem 5.15. *Is there any non-homogeneous rank-*2 *Bäcklund transformation arising in a similar way as* *the classical one relating solutions of the hyperbolic Tzitzeica equation?*

4. In this thesis, our study of Bäcklund transformations is local. *What can be said about the global structure of Bäcklund transformations?*

5. *How to tell whether a Bäcklund transformation relating two hyperbolic Monge-Ampère systems can arise as the composition of two Bäcklund transformations of lower ranks?*

6. *Apply the methods used in this thesis to study Bäcklund transformations relating equations of broader classes (elliptic, parabolic, etc.).*

Appendix A

Calculations for Theorem 3.3

This Appendix supplements the proof of Theorem 3.3 by providing more calculation details. Most calculations below are computed with Maple™.

First consider the case when, on U, $\epsilon_1 = \epsilon_2 = 1$. Since P_{24}, P_{33}, P_{66}, P_{75} never appear in the equation (3.7), we can set them all to zero. Since P_{14} and P_{23} only appear in the term $(P_{14} - P_{23})\omega^3 \wedge \omega^4$, we can set $P_{14} = 0$. For similar reasons, we can set $P_{13}, P_{55}, P_{56} = 0$. For convenience, we rename A_1 as P_{81} and A_4 as P_{84}. Now there are 42 functions P_{ij} remaining, and they are determined.

For each P_{ij}, there exist functions P_{ijk} defined on U satisfying

$$\mathrm{d}(P_{ij}) = P_{ijk}\omega^k.$$

We call these P_{ijk} the *derivatives* of P_{ij}.

Now, applying $\mathrm{d}^2 = 0$ to the equation (3.7), we obtain 106 polynomial equations expressed in terms of all 42 P_{ij} and 186 of all 252 P_{ijk}. These equations imply:

$$P_{01} = P_{41} - P_{51}, \quad P_{02} = P_{42} - P_{52}, \quad P_{03} = -P_{43} - P_{81} + P_{53}, \quad P_{04} = -P_{44} + P_{54},$$

$$P_{05} = -P_{45} - P_{84} + P_{15}, \quad P_{11} = -P_{51}, \quad P_{12} = -P_{52}, \quad P_{21} = P_{84}, \quad P_{22} = -1,$$

$$P_{35} = 0, \quad P_{36} = -1, \quad P_{61} = 1, \quad P_{62} = -P_{81}, \quad P_{73} = 0, \quad P_{74} = -1.$$

With these relations, all coefficients in (3.7) can be expressed in terms of 27 P_{ij}. Repeating the steps above by defining the derivatives P_{ijk} (now 162 in all) and applying $\mathrm{d}^2 = 0$ to (3.7), we obtain a system of 91 polynomial equations, which imply

$$P_{31} = -P_{32}P_{84} - P_{15} - P_{34} - 2P_{43} - 2P_{45} + P_{53} + P_{76} - P_{81} - P_{84},$$

$$P_{72} = -P_{71}P_{81} - P_{15} + P_{34} + 2P_{43} + 2P_{45} - 3P_{53} - P_{76} + P_{81} + P_{84}.$$

Using these relations and repeating the steps above, we obtain

$$P_{06} = P_{16} - P_{46}.$$

Now all coefficients in (3.7) are expressed in terms of 24 P_{ij}.

Now, corresponding to the remaining 24 P_{ij} are 144 derivatives P_{ijk}. Applying $\mathrm{d}^2 = 0$ to (3.7) yields a system of 88 equations, expressed in terms of the 24 P_{ij} and 122 of the 144 derivatives P_{ijk}. This system can be solved for P_{ijk}; in the solution, all P_{ijk} are expressed explicitly in terms of 24 P_{ij} and 64 P_{ijk} that are 'free'.

Let $a = (a^\alpha)$ $(\alpha = 1, ..., 24)$ stand for the 24 remaining P_{ij}; let $b = (b^\rho)$ $(\rho = 1, ..., 64)$ stand for the 64 'free' P_{ijk}. We already have

$$\mathrm{d}\omega^i = -\frac{1}{2}C^i_{jk}(a)\omega^j \wedge \omega^k, \tag{A.1}$$

$$\mathrm{d}a^\alpha = F^\alpha_i(a, b)\omega^i, \tag{A.2}$$

for some real analytic functions F^α_i and C^i_{jk} satisfying $C^i_{jk} + C^i_{kj} = 0$.

Now compute the exterior derivatives

$$\mathrm{d}(F^\alpha_i(a, b)\omega^i), \quad \alpha = 1, ..., 24,$$

and take into account (A.1) and (A.2). From this we obtain 2-forms Ω^α that are linear combinations of $\mathrm{d}b^\rho \wedge \omega^i$ and $\omega^i \wedge \omega^j$. Let $\hat{\Omega}^\alpha$ denote the part of Ω^α consisting of linear combinations of $\mathrm{d}b^\rho \wedge \omega^i$ only. Replacing $\mathrm{d}b^\rho$ in $\hat{\Omega}^\alpha$ by $G^\rho_i\omega^i$ defines a linear map

$$\phi : \mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^{64}) \to \Lambda^2(\mathbb{R}^6)^* \otimes \mathbb{R}^{24}$$

at each point of U.

Let $[\Omega]$ denote the equivalence class of (Ω^α) in the cokernel of ϕ. One can show that $[\Omega]$ must vanish and that its vanishing leads to a system of 35 equations for a and b. This system can be solved for 12 of the 64 components of b. Apply such a solution and update a^α, b^ρ and the functions F_i^α accordingly.

It is not hard to verify, using Maple™, that the updated a^α $(\alpha = 1, ..., 24)$, b^ρ $(\rho = 1, ..., 52)$, C^i_{jk} and F_i^α satisfy the conditions (**A**)-(**C**) in **Step 1**.

For **Steps 2 and 3**, calculation shows that the *tableaux of free derivatives* has Cartan characters

$$(s_1, s_2, s_3, s_4, s_5, s_6) = (24, 22, 6, 0, 0, 0)$$

and the dimension of its first prolongation

$$\delta = 64 < s_1 + 2s_2 + 3s_3 + 4s_4 + 5s_5 + 6s_6 = 86.$$

The cases when, on U, ϵ_1 and ϵ_2 take other values follow similar steps. In each of these cases, the last nonzero Cartan character, computed at a correspond stage, is $s_3 = 6$.

Appendix B

Invariants of an Euler-Lagrange System

This Appendix supplements the proof of Proposition 3.4.

We start with the G_1-structure $\pi : \mathcal{G}_1 \to M$ of a hyperbolic Monge-Ampère system $(M, \mathcal{I})$ (see [BGG03] or Section 4.1). Assume that $S_2 = 0$ (i.e., the Euler-Lagrange case).

Recall that the 2×2-matrix $S_1 : \mathcal{G}_1 \to \mathfrak{gl}(2, \mathbb{R})$ is equivariant under the G_1-action. By (4.5) and (4.6), it is easy to see that, when $\det(S_1(u)) > 0$ (resp., $\det(S_1(u)) < 0$) at $u \in \mathcal{G}_1$, the same is true for $\det(S_1(u \cdot g))$ for all $g \in G_1$, and the matrix $S_1(u)$ lies in the same G_1-orbit as $\mathrm{diag}(1, 1)$ (resp., $\mathrm{diag}(1, -1)$).

Now assume that $\det(S_1) > 0$ holds on $\pi^{-1}U \subset \mathcal{G}$ for some domain $U \subset M$. By the discussion above, we can reduce to a subbundle $\mathcal{H} \subset \mathcal{G}_1$ defined by $S_1 = \mathrm{diag}(1, 1)$.

It is easy to see that $\mathcal{H}$ is an H-structure on U where

$$H = \left\{ \left(\begin{array}{ccc} \epsilon & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \epsilon A \end{array} \right) \middle| \, \epsilon = \pm 1, \; A \in \mathrm{GL}(2, \mathbb{R}), \; \det(A) = \epsilon \right\} \subset G_1$$

is a (disconnected) 3-dimensional Lie subgroup. Let the restriction of $\pi : \mathcal{G}_1 \to M$

to $\mathcal{H}$ be denoted by the same symbol π.

One can verify that, restricted to $\mathcal{H}$, the 1-forms $\phi_7 - \phi_3$, $\phi_6 - \phi_2$, $\phi_5 - \phi_1$ and ϕ_0 in equation (4.1) become semi-basic relative to $\pi : \mathcal{H} \to U$. Hence, there exist functions Q_{ij} defined on $\mathcal{H}$ such that

$$\begin{aligned} \phi_7 &= \phi_3 + Q_{7i}\omega^i, \qquad & \phi_6 &= \phi_2 + Q_{6i}\omega^i, \\ \phi_5 &= \phi_1 + Q_{5i}\omega^i, \qquad & \phi_0 &= Q_{0i}\omega^i, \end{aligned} \tag{B.1}$$

where the summations are over $i = 0, 1, ..., 4$. There are ambiguities in these Q_{ij} as we can modify them without changing the form of the structure equation (4.1). Using such ambiguities, we can arrange that

$$Q_{71} = Q_{73} = Q_{62} = Q_{64} = Q_{51} = Q_{52} = Q_{53} = Q_{54} = 0; \tag{B.2}$$

the remaining Q_{ij} are then determined.

Applying $d^2 = 0$ to (4.1) and reducing appropriately, we obtain

$$\begin{aligned} d^2\omega^1 &\equiv \quad (Q_{63} - Q_{04})\omega^0 \wedge \omega^3 \wedge \omega^4 \quad \mathrm{mod}\ \omega^1, \omega^2, \\ d^2\omega^2 &\equiv \quad (Q_{03} - Q_{74})\omega^0 \wedge \omega^3 \wedge \omega^4 \quad \mathrm{mod}\ \omega^1, \omega^2, \\ d^2\omega^3 &\equiv \quad (Q_{02} + Q_{61})\omega^0 \wedge \omega^1 \wedge \omega^2 \quad \mathrm{mod}\ \omega^3, \omega^4, \\ d^2\omega^4 &\equiv (-Q_{01} - Q_{72})\omega^0 \wedge \omega^1 \wedge \omega^2 \quad \mathrm{mod}\ \omega^3, \omega^4. \end{aligned}$$

This implies that

$$Q_{61} = -Q_{02}, \quad Q_{63} = Q_{04}, \quad Q_{72} = -Q_{01}, \quad Q_{74} = Q_{03}.$$

Now all coefficients in the structure equation (4.1) are expressed in terms of Q_{0i} $(i = 0, 1, ..., 4)$ and Q_{j0} $(j = 5, 6, 7)$. By applying $d^2 = 0$ to (4.1), it is not hard to verify that, reduced modulo $\omega^0, \omega^1, ..., \omega^4$, the following congruences hold:

$$d\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix} \equiv \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & -\phi_1 \end{pmatrix}\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}, \quad d(Q_{00}) \equiv 0,$$

$$d\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix} \equiv \begin{pmatrix} 0 & \phi_3 & -\phi_2 \\ 2\phi_2 & -2\phi_1 & 0 \\ -2\phi_3 & 0 & 2\phi_1 \end{pmatrix}\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}. \tag{B.3}$$

The equations (B.3) tell us how the remaining Q_{ij} transform under the action by the identity component of H. Moreover, it is easy to compute directly from (4.1) to verify that

$$\begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}(u \cdot h_0) = \begin{pmatrix} -Q_{01} & Q_{03} \\ Q_{02} & -Q_{04} \end{pmatrix}(u), \quad Q_{00}(u \cdot h_0) = -Q_{00}(u), \quad \text{(B.4)}$$

$$\begin{pmatrix} Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}(u \cdot h_0) = \begin{pmatrix} -Q_{50} \\ Q_{60} \\ Q_{70} \end{pmatrix}(u), \quad h_0 = \mathrm{diag}(-1,-1,1,1,-1) \in H$$

hold for any $u \in \mathcal{H}$.

Note that H is generated by its identity component and h_0. Combining (B.3) and (B.4), it is easy to see that $Q_{01}Q_{04} - Q_{02}Q_{03}$ and $|Q_{00}|$ are local invariants of the underlying Euler-Lagrange system.

Moreover, using (B.3) and (B.4), it is easy to see that the H-orbit of

$$q(u) := \begin{pmatrix} Q_{01} & Q_{03} \\ Q_{02} & Q_{04} \end{pmatrix}(u), \quad u \in \mathcal{H}$$

consists of all 2-by-2 matrices with the same determinant as $q(u)$. Now we are ready to prove the

Lemma B.1. *If* $\det(q) \neq 0$ *on* $\mathcal{H}$*, then there is a canonical way to define a coframing on* U*.*

Proof. If the function $L := \det(q)$ is nonvanishing on U, one can reduce to the subbundle $\mathcal{H}_1$ of $\mathcal{H}$ defined by $q = \mathrm{diag}(L, 1)$. It is easy to see that each fiber of $\mathcal{H}_1$ over U contains a single element. □

Remark 15. As a result of Lemma B.1, if $\det(q) \neq 0$ on U, then the corresponding hyperbolic Euler-Lagrange system has a symmetry of dimension at most 5. This is a consequence of applying the *Frobenius Theorem*.

Now we proceed to complete proving Proposition 3.4. Recall that the coframing $(\eta^0, \eta^1, ..., \eta^4)$ and the ϕ_α in (3.19) verify the equation (4.1), $S_1 = \mathrm{diag}(1,1)$, and $S_2 = 0$. Moreover, we have chosen the ϕ_α to satisfy (B.2), where Q_{ij} are computed using (B.1). By (3.19), it is immediate that

$$Q_{00} = Q_{02} = 0, \quad Q_{01} = -Q_{04} = \frac{1}{\sqrt{2}}, \quad Q_{03} = 1,$$

$$Q_{70} = 1, \quad Q_{60} = -1, \quad Q_{50} = \sqrt{2}.$$

Clearly, $\det(q) = Q_{01}Q_{04} - Q_{02}Q_{03} = -\frac{1}{2} \neq 0$. By Lemma B.1 and Remark 15, the hyperbolic Euler-Lagrange system considered in Proposition 3.4 has a symmetry of dimension at most 5. Since such an Euler-Lagrange system is homogeneous, it follows that its symmetry has dimension 5.

Bibliography

[AF15] Ian M Anderson and Mark E Fels. Bäcklund transformations for Darboux integrable differential systems. *Selecta Mathematica*, 21(2):379–448, 2015.

[Bac83] AV Backlund. Om ytor med konstant negativ krokning. *Lunds Universitets Arsskrift*, 19, 1883.

[BCG+13] Robert Bryant, Shiing-Shen Chern, Robert Gardner, Hubert Goldschmidt, and Phillip Griffiths. *Exterior differential systems*, volume 18. Springer Science & Business Media, 2013.

[BGG03] Robert Bryant, Phillip Griffiths, and Daniel Grossman. *Exterior Differential Systems and Euler-Lagrange Partial Differential Equations*. University of Chicago Press, 2003.

[BGH95] Robert Bryant, Phillip Griffiths, and Lucas Hsu. Hyperbolic exterior differential systems and their conservation laws, part I. *Selecta Mathematica*, 1(1):21–112, 1995.

[Bry95] Robert Bryant. An introduction to Lie groups and symplectic geometry. *IAS/Park City Mathematics Series, Vol.1*, 1995.

[Bry02] Robert Bryant. Lectures on exterior differential systems. 2002.

[Bry14] Robert Bryant. Notes on exterior differential systems. *arXiv preprint arXiv:1405.3116*, 2014.

[CI+09] Jeanne N Clelland, Thomas A Ivey, et al. Bäcklund transformations and Darboux integrability for nonlinear wave equations. *Asian Journal of Mathematics*, 13(1):15–64, 2009.

[CI13] Jeanne N Clelland and Thomas A Ivey. Geometric characterizations of Bäcklund transformations of sine-Gordon type. *arXiv preprint arXiv:1311.4016*, 2013.

[Cle01] Jeanne N Clelland. Homogeneous Bäcklund transformations of hyperbolic Monge-Ampère systems. *arXiv preprint math/0111208*, 2001.

[CT80] Shiing-Shen Chern and Chuu-Lian Terng. An analogue of Bäcklund's theorem in affine geometry. *Rocky Mountain J. Math.*, 10(1), 1980.

[Gar89] Robert B Gardner. *The method of equivalence and its applications*. SIAM, 1989.

[Gou25] E Goursat. Le probleme de Bäcklund (memorial des sciences mathematiques. fasc. vi). *Paris: Gauthier-Villars*, 1925.

[NC82] JJC Nimmo and DG Crighton. Bäcklund transformations for nonlinear parabolic equations: the general results. *Proc. R. Soc. Lond. A*, 384(1787):381–401, 1982.

[RS02] Colin Rogers and Wolfgang Karl Schief. *Bäcklund and Darboux transformations: geometry and modern applications in soliton theory*, volume 30. Cambridge University Press, 2002.

[Tzi08] M Georges Tzitzéica. Sur une nouvelle classe de surfaces. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 25(1):180–187, 1908.

[Tzi09] M Georges Tzitzéica. Sur une nouvelle classe de surfaces (2ème partie). *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 28(1):210–216, 1909.

Biography

Yuhao Hu was born on Jan 8, 1991 in Jingdezhen, China. During 2008-2012, he was a student at the Shing-Tung Yau class of mathematics at Zhejiang University, where he obtained a Bachelor of Science degree in June, 2012. After graduation, he went to Duke University for PhD studies in mathematics. He conducted research in differential geometry and the geometry of differential equations under the supervision of Professor Robert Bryant. After graduating from Duke, he will work as a postdoc with Professor Jeanne Clelland at the University of Colorado at Boulder. As a hobby, he plays the piano.