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Lagrangian subvarieties of the moduli space of stable vector bundles on a regular algebraic surface with $p_g > 0$

Yun-Gang Ye[⋆]

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

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0 Introduction

Mukai [M1] showed that there is a nondegenerate symplectic structures on the moduli space of stable vector bundles on a K3 surface. Later Tyurin [T2] studied (generalized) symplectic structures on the moduli space of stable vector bundles on any smooth regular surface X with $p_g > 0$. In the work of Tyurin, a symplectic structure means a nonzero regular two form on the moduli spaces, in particular it may degenerate. In this paper, we define a Lagrangian subvariety of the moduli space to be a subvariety on the Zariski tangent space (at any point) of which the given symplectic two form is identically zero. Note that we do not impose any restriction on the dimension of a Lagrangian subvariety. This is because symplectic structures considered here may degenerate. The purpose of this paper is to use Brill-Noether theory for curves to construct explicitly a family of Lagrangian subvarieties of the moduli space of stable vector bundles on a regular surface with $p_g > 0$.

Let $\mathscr{M}$ be a generically smooth and irreducible component of the moduli space of Gieseker-stable bundles of rank $r + 1$ with respect to a fixed polarization D on a regular algebraic surface X with $p_g > 0$. By the boundedness of $\mathscr{M}$ (see [Ma]), after possibly twisted by the same negative line bundle $\mathscr{H}$ on X, we can assume that for any point $[E] \in \mathscr{M}$,

(i) E^* is generated by global sections.

(ii) $h^1(E^*) = h^2(E^*) = h^1(E) = 0$.

(iii) $h^1(\det E^*) = h^2(\det E^*) = 0$.

For any point $[E] \in \mathscr{M}$. Choose a $(r + 1)$ dimensional subspace $V \subset H^0(E^*)$, and consider an evaluation map $e_V : V \otimes \mathscr{O}_X \to E^*$. For a general V, we can make e_V degenerate exactly along a smooth curve $C \subset X$ and coker e_V is a line bundle

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⋆ Current address: Department of Mathematics, Purdue University, West Lafayette, IN 47907 – 1968, USA

A on C. Therefore E_V^* induces the following short exact sequences:

$$0 \longrightarrow E \longrightarrow V^* \otimes \mathcal{O}_V \xrightarrow{\lambda} L \longrightarrow 0$$

where $L = N_{C/X} \otimes A^*$. By condition (ii), we see that $V^* \cong H^0(L)$. Then it is clear that L is base-point-free complete g_d^r, i.e. $L \in V_d^r(C)$, where $d = c_2(E)$. By Porteous' formula, $C \in |\det(E^*)|$. Let $d = c_2(E)$. We denote by $V_d^r(C) \subset W_d^r(C)$ the space of all base-point-free complete linear systems of dimension r and degree d on C. Let $\varrho = g - (r+1)(g+r-d)$ be the Brill-Noether number. From now on, we fix a smooth curve C so constructed.

Let us also define:

$$V_d^r(X, C) = \{(C', L') \mid C' \in |C| \text{ is smooth and } L' \in V_d^r(C')\}.$$

Then there is a natural morphism $\pi : V_d^r(X, C) \to |C|_0 \subset |C|$, where $|C|_0$ consists of all smooth curve linearly equivalent to C.

Let $Z_d^r \subset V_d^r(X, C)$ be the set of all points $(C', L') \in V_d^r(X, C)$ such that the associated bundle $E' = E(C', L')$ is Gieseker-stable with respect to the fixed polarization D. Then Z_d^r is a non-empty Zariski open subet of $V_d^r(X, C)$ since the stability is an open condition and $(C, L) \in Z_d^r$. By associating (C', L') to $E' = E(C', L')$, we get a well-defined rational map $f : V_d^r(X, C) \to \mathcal{M}$ (see Lemma 2.1).

We will prove the following theorem in this paper.

Theorem 3.1. *Let $\mathcal{M}, X, C$ be as above. Let $\omega \in H^0(\Lambda^2 \Omega_M)$ be the symplectic structure obtained from an arbitrary two form $\omega_0 \in H^0(K_X)$ as described in [T2]. For a generic smooth curve $C_0 \in \pi(V_d^r(X, C))$, there is a smooth Zariski open subset $T_d^r(C_0) \subset W_d^r(C_0)$ such that $f(T_d^r(C_0) \subset \mathcal{M}$ is a smooth Lagrangian subvariety with respect to ω.*

Let us recall that according to Friedman [Fr], $\mathcal{M}$ is call *good* if there is an point $[E] \in \mathcal{M}$ such that $h^2(\text{ad } E) = 0$, where ad E is the traceless part of $E \otimes E^*$. If $\mathcal{M}$ is good, then it must be generically smooth due to Mukai-Artamkin Criterion (see Sect. 1).

It is necessary to know the dimension of the family of the Lagrangian subvarieties constructed above. The next theorem answers this question and it also tells us when $f|_{T_d^r(C_0)} : T_d^r(C_0) \to \mathcal{M}$ is etale.

Theorem 3.2. *Under the same assumptions as those in Theorem* 3.1.

(1) *Let $(C_0, L_0) \in V_d^r(X, C)$ be a generic point. The codimension of $\pi(V_d^r(X, C))$ in $|C|_0$ equals to* $\dim\ker(\mu_{L_0}) - h^2(\text{ad } E_0)$, *where $E_0 = f(C_0, L_0)$ and μ_{L_0} is the Petri's map for L_0. Therefore $\pi : V_d^r(X, C) \to |C|_0 \subset |C|$ is dominant if and only if there exists a point $(C_0, L_0) \in V_d^r(X, C)$,* $\dim \ker(\mu_{L_0}) = h^2(\text{ad } E_0)$.

(2) *If $\varrho \geqq 0$, then $\mathcal{M}$ is* good *if and only if there is a point $(C_0, L_0) \in V_d^r(X, C)$ such that $W_d^r(C_0)$ is smooth of dimension ϱ around L_0. If $\varrho \geqq 0$ and $\mathcal{M}$ is good, then $f|_{T_d^r(C_0)} : T_d^r(C_0) \to \mathcal{M}$ is etale, where $T_d^r(C_0)$ is the open subset of $W_d^r(C_0)$ in Theorem* 3.1.

If X is a K3 surface, then $\varrho \geqq 0$ holds automatically because in this case $\dim.\mathcal{M} = 2\varrho$, which is obtained by an easy computation using Riemann-Roch (see [L2]). And in this case $W_d^r(C_0)$ is a Lagrangian subvariety in the usual sense.

In general the condition that $\varrho \geqq 0$ is stronger then Bogomolov- Gieseker inequality for Chern classes of a stable bundle on an algebraic surface. Since E is D-stable of rank $r+1$, we have (see [L1]):

$$\frac{2r+2}{r}c_2(E) \geqq c_1^2(E).$$

The condition $\varrho \geqq 0$ is equivalent to a stronger inequality:

$$\frac{2r+2}{r}c_2(E) - c_1^2(E) \geqq 2r + 4 + K_X \cdot C.$$

The left side in the above inequality is stable under tensoring by a line bundle with E. Therefore, if the line bundle $\mathcal{H}$ (see the second paragraph above) we choose is too *negative*, then the above inequality does not hold, i.e., $\varrho < 0$. The real question is that if we fix c_1 and let c_2 be sufficiently large, can we find a negative line bundle $\mathcal{H}$ such that both the conditions (i)–(iii) and $\varrho \geqq 0$ hold?

The connections between linear systems on curves and vector bundles on algebraic surfaces have been well-known (see [L1] for an excellent survey). The author obtained his ideas for this paper when studying propagation problem on regular surfaces surface [Y]. Let C be a smooth curve on a K3 surface X and L be a line bundle on C of degree d and $h^0(L) = r+1$. Let $E = \ker(\lambda)$, where $\lambda : H^0(L) \otimes \mathcal{O}_X \to L$ is an evaluation map. Then E is vector bundle on X of rank $r+1$. Lazarsfeld [L1] proved that if C is a *general* number of a linear system $|C_0|$ such that C_0 generates Pic (X), then E is simple and $W_d^r(C)$ is smooth away from $W_d^{r+1}(C)$ of dimension ϱ, where $\varrho = g - (r+1)(g + r - d)$ is the Brill-Noether number of L. Let $\mathcal{M}_E$ is irreducible component of the moduli space of simple vector bundles that contains $[E]$. An result of Mukai [M1] says that $\mathcal{M}_E$ is smooth and symplectic of dimension 2ϱ, which is twice the dimension of $W_d^r(C)$. At least $W_d^r(C)$ has the right dimension to be a Lagrangian subvariety of $\mathcal{M}_E$. This is the starting point for this paper. After we finish dealing with K3 surfaces, we realized that the same method applies to any other regular surface with $p_g > 0$. However, in the case of an arbitrary regular surface, the symplectic structure on the moduli space may degenerate.

The paper is organized as follow. In the first section, we define symplectic structures on a variety, and Lagrangian subvarieties with respect to a symplectic structure on that variety. We also review results of Mukai and Tyurin. The second section is about Brill-Noether theory for a family of curves on a regular surface. Most of the materials in that section can be found in [Y]. The last section is devoted to the proofs of the main theorems of this paper, namely, Theorem 3.1 and Theorem 3.2. At the end of this paper, we discuss some problems arising naturally from this paper.

In this paper, all the schemes are defined over an algebraically closed field k with ch$(k) = 0$. Throughout this paper, X is always a smooth regular surface with $p_g > 0$, and $C \subset X$ is a smooth curve. Stability in this paper is always in the sense of Gieseker with respect to a fixed polarization D on X. Almost all of our notations are standard. Here are a few that we will use frequently. Other notations will be explained as they occur.

k: an algebraically closed field with $ch(k) = 0$.
Ω_M: the cotangent sheave of M.
T_M: the tangent sheave of M.
K_M or ω_M: the canonical bundle of M.
T_mM: the Zariski tangent space to M at a point $m \in M$.
ω: a symplectic two form on M, i.e., $\omega \in H^0(\Lambda^2\Omega_M)$.
p_g: the dimension of $H^0(K_X)$.
E^*: the dual of a vector bundle E.
$\det E$: the determinant line bundle of E.
$c_i(E)$: the i-th Chern class of E.
$N_{C/X}$: the normal bundle of C on X.
g^r_d: a linear system on a smooth curve C of degree d and dimension at least $r+1$.
ϱ: the Brill-Noether number of g^r_d, i.e., $\varrho = g-(r+1)(g+r-d)$, where g is the genus of C.
$V^r_d(C)$: the space of all base-point-free g^{rd}'s on C. Note that $V^r_d(C)$ is a Zariski open subset of $W^r_d(C)$.
D: a fixed polarization of X, i.e., an ample divisor on X.
$\mathcal{M}$: a generically smooth and irreducible component of the moduli space of Gieseker-stable bundles with respect to D.

1 Symplectic structures

In this section we will briefly review symplectic structures on moduli space of stable bundles. Detailed studies of these symplectic structures can be found in [M1] and [T2].

Let M be a smooth algebraic varieties over k. A (generalized) *symplectic structures* on M is a nonzero skew-symmetric two form $\omega \in H^0(\Lambda^2\Omega_M)$. A symplectic structures ω can also be thought of as a non-zero skew-symmetric homomorphism:

$$\omega : T_M \to \Omega_M$$

ω also induces a skew-symmetric pairing:

$$\omega : T_M \times T_M \to k\,.$$

We say ω is *nondegenerate* if it is *generically invertible* when viewed as a bundle homomorphism form T_M to Ω_M. We say ω is *everywhere nondegenerate* if it is invertible at *every point*. Everywhere nondegenerate symplectic structure in the generalized sense is the usual notion of symplectic structure. For example the unique everywhere nonzero two form on a K3 defines a everywhere nondegenerate symplectic structure on that K3 surface.

Now given a symplectic structure $\omega \in H^0(\Lambda^2\Omega_M)$ on M, a (not necessarily smooth or complete) subvariety $Y \subset M$ is called *Lagrangian* with respect to ω if at every point $y \in Y, \omega(X_1, X_2) = 0$ for any two vectors X_1 and X_2 in the Zariski tangent space of Y at y. When ω is everywhere nondegenerate and $Y = \dfrac{\dim(M)}{2}$, Y must be smooth and is a Lagrangian submanifold in the usual sense with respect to ω.

Let X be a smooth projective regular surface over k, i.e., $h^1(\mathcal{O}_X) = 0$. Let $\mathcal{M}$ be an irreducible component of the moduli space of stable bundles with respect to a fixed polarization D. Consider an arbitrary point $[E]$ in $\mathcal{M}$. Since E is stable, than it is, a priori, simple, i.e., $H^0(E \otimes E^*) \cong k$. Let ad$E$ be the traceless part of the $E \otimes E^*$. Then $E \otimes E^* = \mathrm{ad}\, E \oplus \mathcal{O}_X$. Let us recall the following special case of Mukai-Artamkin [M1, Ar] criterion about the smoothness of $\mathcal{M}$.

Mukai-Artamkin Criterion. *If $H^2(\mathrm{ad}E) = 0$, then $\mathcal{M}$ is smooth at $[E]$ of dimension $h^1(E \otimes E^*)$.*

In the case of rank two vector bundle, Donaldson [D] showed that $\mathcal{M}$ is *good* if $c_1 = 0$ and c_2 is sufficiently large. Recently Zuo [Z] generalized Donaldson's result to the case of rank two vector bundles with an arbitrary c_1. The higher rank case is still open as far as the author's knowledge goes.

As for symplectic structures on $\mathcal{M}$ we have the following theorem of Mukai [M1] and Tyurin [T2].

Theorem 1.1 (Mukai-Tyurin). *There is an homomorphism τ:*

$$H^0(\Lambda^2 \Omega_X) \xrightarrow{\tau} H^0(\Lambda^2 \Omega_{\mathcal{M}}).$$

That is, given a nonozero holomorphic two form on X, there is a symplectic structure on $\mathcal{M}$. Moreover, if X is a K3 surface, the unique symplectic structure obtained in this way is everywhere nondegenerate.

The above theorem of Mukai-Tyurin can be roughly shown in the following way. Consider the product pairing for $E \otimes E^*$,

$$H^1(E \otimes E^*) \times H^1(E \otimes E^*) \longrightarrow H^2(E \otimes E^*) \xrightarrow{\mathrm{tr}^2} H^2(\mathcal{O}_X) \qquad (1.1)$$

where tr^2 is induced by the trace homomorphism $\mathrm{tr} : E \otimes E^* \to \mathcal{O}_X$. Any two form $\omega_0 \in H^0(K_X)$ induces a homomorphism $H^2(\mathcal{O}_X) \to k$. Therefore for any two form ω_0 on X, we get a pairing:

$$\begin{array}{ccccc} H^1(E \otimes E^*) & \times & H^1(E \otimes E^*) & \longrightarrow & k \\ \| & & \| & & \\ T_{[E]}\mathcal{M} & & T_{[E]}\mathcal{M} & \longrightarrow & k\,. \end{array} \qquad (1.2)$$

This pairing gives a simplectic structure ω on $T_{[E]}\mathcal{M}$. These symplectic structures at various points in $\mathcal{M}$ can be glued together to give a global symplectic structure on $\mathcal{M}$ (see [M1, T2] for details).

2 Linear systems on smooth curves on X

The purpose of this section is to review constructions of the moduli space of linear systems on smooth curves within a given linear system on X. We will also study its relationships with the moduli space of stable vector bundles over X.

Let $\mathcal{M}$ be a generically smooth and irreducible component of the moduli space of stable bundles of rank $r+1$ with respect to D. By the boundedness of $\mathcal{M}$ (see [Ma]), after possibly twisted by the same very negative line bundle on X, we can assume that for any point $[E] \in \mathcal{M}$,

(i) E^* is generated by global sections.

(ii) $h^1(E^*) = h^2(E^*) = h^1(E) = 0$.

(iii) $h^1(\det E^*) = h^2(\det E^*) = 0$.

Since $p_g > 0$, K_X is effective. By Serre duality, (ii) implies that $h^0(E) = 0$. If X is a K3 surface, then the fact that $h^1(E^*) = 0$ implies that $h^1(E) = 0$.

Now fix a point $[E] \in \mathcal{M}$. Choose a $(r+1)$ dimensional subspace $V \subset H^0(E^*)$, and consider an evaluation map e_V:

$$V \otimes \mathcal{O}_X \xrightarrow{e_V} E^* \tag{2.1}$$

For a general V, we can make e_V degenerate exactly along a smooth curve $C \subset X$ and coker e_V is a line bundle A on C. The dual of (2.1) implies that:

$$0 \longrightarrow E \longrightarrow V^* \otimes \mathcal{O}_X \xrightarrow{\lambda} L \longrightarrow 0$$

where $L = N_{C/X} \otimes A^*$. By condition (ii), we see that $V^* \cong H^0(L)$. Then it is clear that L is base-point-free complete g^r_d, i.e., $L \in V^r_d(C)$, where $d = c_2(E)$. By Porteous' formula, $C \in |\det(E^*)|$. Therefore we have the following exact sequence:

$$0 \longrightarrow E \longrightarrow H^0(L) \otimes \mathcal{O}_X \xrightarrow{\lambda} L \longrightarrow 0. \tag{2.2}$$

Note that if we don't assume that $H^1(E) = 0$, then we only have an injective homomorphism $V^* \hookrightarrow H^0(L)$ rather than an isomorphism. Because of this L may be an incomplete g^r_d.

From now on we will fix a smooth curve $C \in |\det(E^*)|$ and a base-point-free g^r_d denoted by L on C such that (2.2) is exact. Let g be the genus of C. Since E is constructed directly from (C, L), we sometimes denote E by $E(C, L)$.

Let us define:

$$V^r_d(X, C) = \{(C', L') \mid C' \in |C| \text{ is smooth and } L' \in V^r_d(C')\}.$$

Fix a $(r+1)$-dimensional vector space W. Sometimes it is more convenient to work with $P^r_d(X, C)$, which parametrizes all the triples (C', L', λ'), where:

(1) $(C', L') \in V^r_d(X, C)$.

(2) λ' is a surjective homomorphism of $\mathcal{O}_X$-sheaves:

$$W \otimes \mathcal{O}_X \xrightarrow{\lambda'} L' \to 0$$

inducing an isomorphism $W \cong H^0(L')$, two such homomorphisms identified if they differ only by multiplication by a non-zero scalar.

$P^r_d(X, C)$ was first introduced by Lazarsfeld in [L2]. It is clear that $P^r_d(X, C)$ is a natural principal GL $(r+1)$-bundle over $V^r_d(X, C)$. Note that $P^r_d(X, C)$, therefore $V^r_d(X, C)$, is not empty since $(C, L, \lambda) \in P^r_d(X, C)$ for any evaluation map λ :

$H^0(L)\otimes\mathcal{O}_X \to L$. Note that it was proved in [L1] that locally $V_d^r(X,C)$ is birational to the Grassmannian bundle Gr $(r+1, p_*\mathbb{E})$ over the moduli space $\mathcal{M}$, where $\mathbb{E}$ is a local universal bundle over $\mathcal{M}$, and $p : \mathcal{M}\times X \to \mathcal{M}$ is the projection.

Lemma 2.1. *There is a surjective rational map f from $V_d^r(X,C)$ to $\mathcal{M}$.*

Proof. Let (C', L', λ') be an arbitrary point in $P_d^r(X,C)$. We define $E(C', L', \lambda') =$ ker (λ'). Note that two different λ''s give the same vector bundle. For this reason, we sometimes simply denote $E(C', L', \lambda')$ by $E(C', L')$ forgetting the last component.

Claim. There is a nonempty Zariski open set U_d^r of $P_d^r(X,C)$ such that for every $(C', L', \lambda') \in U_d^r$ the vector bundle $E(C', L', \lambda')$ is stable and it satisfied conditions (i)–(iii) *above.*

Proof of the claim. For simplicity we let $E' = E(C', L', \lambda')$. Let Q be the Quot scheme of $W\otimes\mathcal{O}_X$ in the sense of Grothendieck [Gr]. Q parametrizes all quotients of $W\otimes\mathcal{O}_X$. It is clear that $P_d^r(X,C)$ is a Zariski open subset of Q. Since Q is a fine moduli space, there is an universal quotient sheaf $\mathcal{G}$ on $P_d^r(X,C)$. Therefore we have a natural homomorphism of $\mathcal{O}_{X\times P_d^r(X,C)}$-sheaves:

$$0 \longrightarrow \mathcal{E} \longrightarrow W\otimes\mathcal{O}_{X\times P_d^r(X,C)} \xrightarrow{\Lambda} \mathcal{G} \longrightarrow 0 \tag{2.3}$$

where $\mathcal{E}$ is the kernel of Λ. $\mathcal{E}$ is the universal kernel bundle of Q. Let p_2 be the projection of $X\times P_d^r(X,C)$ to $P_d^r(X,C)$. Then for any point $(C', L', \lambda') \in P_d^r(X,C)$, the restriction of $\mathcal{E}$ to $p_2^{-1}(C', L', \lambda')$ is $E(C', L', \lambda')$. Therefore, by the openness of stability and the semicontinuity theorem, there is a Zariski open set U_d^r of $P_d^r(X,C)$ such that for every $(C', L', \lambda') \in U_d^r$ the vector bundle $E(C', L', \lambda')$ is stable and it satisfied conditions (i)–(iii) above. Since $[E] \in U_d^r$ is nonempty, U_d^r is nonempty. This proves the claim.

For any $(C', L', \lambda') \in U_d^r$, let us define

$$\tilde{f}(C', L', \lambda') = E(C', L', \lambda').$$

This gives a morphism from U_d^r to $\mathcal{M}$, hence a rational map from $P_d^r(X,C)$ to $\mathcal{M}$. Since different λ''s give the same bundle E', the rational map $\tilde{f}$ actually factors through $V_d^r(X,C)$, hence it induces a rational map denoted by f from $V_d^r(X,C)$ to $\mathcal{M}$. Since every bundle in $\mathcal{M}$ satisfies (i)–(iii) above, every bundle in $\mathcal{M}$ can be constructed from some $(C', L') \in V_d^r(X,C)$ via sequence (2.2). Hence the morphism f defined above is actually surjective. □

By the way it is constructed, U_d^r consists of all triples $(C', L', \lambda') \in P_d^r(X,C)$ such that the associated bundle $E(C', L', \lambda')$ is stable and (i) and (ii) are satisfied.

By abuse of notation, we simply assume $U_d^r = P_d^r(X,C)$ in the rest of this paper.

We can define a space similar to $P_d^r(X,C)$ in the following way. For a smooth curve C, we define $P_d^r(C)$ to be the space that parametrizes the set of all pairs (L,λ), where

(1) L is a base-point-free g_d^r on C.

(2) λ is a surjective homomorphism of $\mathcal{O}_C$-sheaves:

$$W\otimes\mathcal{O}_C \xrightarrow{\lambda} L \longrightarrow 0$$

inducing an isomorphism $W \cong H^0(L)$, two such homomorphisms identified if they differ only by multiplication by a non-zero constant.

It is clear that $P^r_d(C)$ is a principal GL $(r+1)$-bundle over $V^r_d(C)$. To each $(L, \lambda) \in P^r_d(C)$, we can associate a vector bundle $F(L)$ on C, which is defined by $F(L) =$ ker λ. For simplicity, we sometimes write $F(L)$ simply as F. Just as $P^r_d(X, C)$, we can realize $P^r_d(C)$ as a Zariski open subset of the Quot scheme for $W \otimes \mathcal{O}_C$. Theorem 5.3 in [Gr] easily implies that following lemma.

Lemma 2.2 *The Zariski tangent space to $P^r_d(X, C)$ at a point (C, L, λ) (respectively, $P^r_d(C)$ at (L, λ)) can be canonically identified with $H^0(L \otimes E^*)$ (respectively, $H^0(L \otimes F^*)$) and the space of obstructions can be identified with $H^1(L \otimes E^*)$ (respectively, $H^1(L \otimes F^*)$).*

The bundles $E(C, L)$ and $F(L)$ will play very important roles in what follows. It is worthwhile to see how they are related (see [Y] for details).

First of all, we have the following exact sequence:

$$0 \longrightarrow F \longrightarrow H^0(L) \otimes \mathcal{O}_C \stackrel{\lambda}{\longrightarrow} L \longrightarrow 0. \tag{2.4}$$

It is easy to derive from (2.2) and (2.4) that we have the following exact sequence:

$$0 \longrightarrow L \otimes N^*_{C/X} \longrightarrow E \otimes \mathcal{O}_C \stackrel{\alpha}{\longrightarrow} F \longrightarrow 0. \tag{2.5}$$

Taking duals of the above sequence, we get:

$$0 \longrightarrow F^* \stackrel{\alpha^*}{\longrightarrow} E^* \otimes \mathcal{O}_C \stackrel{\beta}{\longrightarrow} L^* \otimes N_{C/X} \longrightarrow 0 \tag{2.6}$$

where α^* is the dual of α in (2.5). There is a natural commutative diagram:

$$\begin{array}{ccc} P^r_d(X, C) & \stackrel{\pi}{\longrightarrow} & |C| \\ {\scriptstyle p}\downarrow & & \| \\ V^r_d(X, C) & \stackrel{\pi}{\longrightarrow} & |C| \end{array} \tag{2.7}$$

where π is defined by forget the last two components of a given triple $(C', L', \lambda') \in P^r_d(X, C)$. We denote the induced map from $V^r_d(X, C)$ to $|C|$ also by π when there is no danger of confusion. We let p be the projection from $P^r_d(X, C)$ to $V^r_d(X, C)$. In these notations, $\pi(C, L) = C$. The fiber of π in $P^r_d(X, C)$ over a smooth member $C' \in |C|$ is $P^r_d(C')$, and the corresponding fiber in $V^r_d(X, C)$ is $V^r_d(C')$.

The following diagram will be referred to frequently later in this paper.

$$\begin{array}{ccc} P^r_d(X, C) & \stackrel{\tilde{f}}{\longrightarrow} & \mathcal{M} \\ {\scriptstyle p}\downarrow & & \| \\ V^r_d(X, C) & \stackrel{f}{\longrightarrow} & \mathcal{M} \\ {\scriptstyle \pi}\downarrow & & \\ |C|. & & \end{array} \tag{2.8}$$

Recall from [Y] that we say a linear system L on C *propagates* if π is dominant around the point (C, L), and it propagates *infinitesimally* if the differential of π at (C, L) is surjective. The problem that when a given linear system propagates is called the propagation problem. This problem has been studied by many mathematicians. We refer readers to [Y] for a detailed description of the history of the problem. Vector bundles E and F constructed above have been proved to be very useful to studies the propagation problem and some other problems in algebraic geometry. [L1] is an excellent survey article on this vector bundle techniques. However we show in this paper that the other way is also fruitful, namely, we can use Brill-Noether theory to study vector bundles on surfaces.

3 Proofs of the theorems

Now fix a smooth point $[E] \in \mathscr{M}$. As we saw in the previous section, we can assume that E satisfies conditions (i)–(iii). Then there is a smooth curve $C \in |\det(E^*)|$ and a base-point free g^r_d (denoted by L) on C such that E and (C, L) are related via exact sequence (2.2). From now on, let us fix a pair (C, L) which gives rise to $E = E(C, L)$.

We will adopt the same notations as those in the previous sections. By Lemma 2.1, there is a natural rational map f from $V^r_d(X, C)$ to $\mathscr{M}$ such that $f(C, L) = E(C, L)$.

The purpose of this section is to prove the following two theorems.

Theorem 3.1. *Let $\mathscr{M}, X, C$ be as above. Let $\omega \in H^0$ be the symplectic structure obtained from an arbitrary two form $\omega_0 \in H^0(K_X)$ as described in [T2]. For a generic smooth curve $C_0 \in \pi(V^r_d(X, C))$, there is a smooth Zariski open subset $T^r_d(C_0) \subset W^r_d(C_0)$ such that $f(T^r_d(C_0)) \subset \mathscr{M}$ is a smooth Lagrangian subvariety with respect to ω.*

Theorem 3.2. *Under the same assumptions of Theorem 3.1.*

(1) *Let $(C_0, L_0) \in V^r_d(X, C)$ be a generic point. The codimension of $\pi(V^r_d(X, C))$ in $|C|_0$ equals to* dim ker $(\mu_{L_0}) - h^2(\text{ad } E_0)$, *where $E_0 = f(C_0, L_0)$ and μ_{L_0} is the Petri's map for L_0. Therefore $\pi : V^r_d(X, C) \longrightarrow |C_0| \subset |C|$ is dominant if and only if there exists a point $(C_0, L_0) \in V^r_d(X, C)$,* dim ker $(\mu_{L_0}) = h^2(\text{ad } E_0)$.

(2) *If $\varrho \geq 0$, then $\mathscr{M}$ is* good *if and only if there is a point $(C_0, L_0) \in V^r_d(X, C)$ such that $W^r_d(C_0)$ is smooth of dimension ϱ around L_0. If $\varrho \geq 0$ and $\mathscr{M}$ is good, then $f|_{T^r_d(C_0)} : T^r_d(C_0) \longrightarrow \mathscr{M}$ is etale, where $T^r_d(C_0)$ is the open subset of $W^r_d(C_0)$ in Theorem* 3.1.

To prove these two theorems, we need several lemmas and propositions.

For the proof of the following proposition, we do not need to assume those cohomology to be vanishing as we did in the introduction.

Proposition 3.3. *Let $E = f(C, L)$, i.e., E is obtained via sequence* (2.2). *Let σ be the following pairing:*

$$H^1(E \otimes E^*) \times H^1(E \otimes E^*) \to H^2(E \otimes E^*) \xrightarrow{\text{tr}^2} H^2(\mathcal{O}_X).$$

Then for any two Zariski tangent vectors X_1 and X_2 in $df_*(T_L V^r_d(C))$, we have:

$$\sigma(X_1, X_2) = 0$$

where $df_* : T_L V^r_d(C) \to T_{[E]}\mathscr{M} \cong H^1(E \otimes E^*)$ *is the differential of the map defined in Lemmas 2.1.*

Proof. Consider the following exact sequence:

$$0 \longrightarrow E \otimes E^* \longrightarrow H^0(L) \otimes E^* \longrightarrow L \otimes E^* \longrightarrow 0. \tag{3.1}$$

This above exact sequence is obtained from (2.2) by tensoring E^*. Take cohomology on (3.1), we get a coboundary map ∂_0:

$$H^0(L \otimes E^*) \xrightarrow{\partial_0} H^1(E \otimes E^*). \tag{3.2}$$

The coboundary map ∂_0 in (3.2) is nothing but the differential of $\tilde{f}$, i.e., $d\tilde{f}_*$. Consider the following rational maps:

$$P^r_d(C) \hookrightarrow P^r_d(X, C) \xrightarrow{\tilde{f}} \mathscr{M}.$$

The composed map is denoted by $f : P^r_d(C) \to \mathscr{M}$. Let $df_* : TV^r_d(C) \to T\mathscr{M}$ be the differential of f. The differential of these maps at the point (C, L, λ) are:

$$H^0(L \otimes F^*) \xrightarrow{\alpha^*_L} H^0(L \otimes E^*) \xrightarrow{d\tilde{f}_*} H^1(E \otimes E^*) \tag{3.3}$$

where α^*_L is induced by the homomorphism α^* in (2.6). It is clear that $df_* = d\tilde{f}_* \circ \alpha^*_L$. Therefore for any $e \in df_*(T_L V^r_d(C)) \subset H^1(E \otimes E^*)$, we can associate a map $f_e : E \to L$, and f_e factors through F.

Now fix a $e_0 \in df_*(T_L V^r_d(C))$. Since e_0 comes from $df_*(T_L V^r_d(C))$, f_{e_0} factors through F. Therefore there is map $g_{e_0} : F \to L$ such that $f_{e_0} = \alpha \circ g_{e_0}$, where $\alpha : E \to F$ is the canonical homomorphism.

By abuse of notations, for a homomorphism $h : \mathscr{F}_1 \to \mathscr{F}_2$ between two $\mathscr{O}_X$-modules, we denote by h_* or h^* (depending on being covariant or contravariant) any other homomorphism induced by h through any algebraic operations, e.g., tensor product, taking cohomology, etc.

From f_{e_0} we can construct a $\mathscr{O}_X$-module $\mathscr{N}^{\cdot}$ that fits into the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E \otimes E^* & \longrightarrow & H^0(L) \otimes E^* & \longrightarrow & E^* \otimes L & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow f_{e_0} & & \\
0 & \longrightarrow & E \otimes E^* & \longrightarrow & \mathscr{N}^{\cdot} & \longrightarrow & E \otimes E^* & \longrightarrow & 0.
\end{array} \tag{3.4}$$

Taking cohomology of (3.4), we get:

$$\begin{array}{ccc}
H^1(L \otimes E^*) & \xrightarrow{\partial_0} & H^2(E \otimes E^*) \\
\uparrow f_{e_0 *} & & \| \\
H^1(E \otimes E^*) & \xrightarrow{\partial_{e_0}} & H^2(E \otimes E^*).
\end{array} \tag{3.5}$$

By Theorem 9.1 in [Mc], $\partial_{e_0}(e) = \sigma_0(e_0, e)$, for any $e \in H^1(e \otimes E^*)$, where σ_0 is the following product pairing:

$$H^1(E \otimes E^*) \times H^1(E \otimes E^*) \longrightarrow H^2(E \otimes E^*).$$

Therefore by the commutativity of (3.5), we have for any $e \in H^1(E \otimes E^*)$:

$$\sigma(e_0, e) = \mathrm{tr}^2 \circ \partial_0 \circ f_{e_0 *}(e). \tag{3.6}$$

Let $\alpha_* : E \otimes E^* \to F \otimes E^*$ be the map induced by α in (2.5). Also let $\beta_* : F \otimes E^* \to F \otimes L^* \otimes N_{C/X}$ be the map induced by β in (2.6). Denote $\beta_* \circ \alpha_*$ by γ. Now g_{e_0} induces the following commutative diagram:

$$\begin{array}{ccc}
H^1(E \otimes E^*) & = & H^1(E \otimes E^*) \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \gamma} \\
H^1(F \otimes E^*) & \xrightarrow{\beta_*} & H^1(F \otimes L^* \otimes N_{C/X}) \\
\downarrow{\scriptstyle g_{e_0^*}} & & \downarrow{\scriptstyle g_{e_0^*}} \\
H^1(L \otimes E^*) & \xrightarrow{\beta_*} & H^1(N_{C/X}).
\end{array} \tag{3.7}$$

Claim 1. For any $e \in H^1(E \otimes E^*)$, the following is true:

$$\sigma(e_0, e) = \delta_0 g_{e_0^*} \circ \gamma(e) \tag{3.8}$$

where $\delta : H^1(N_{C/X}) \to H^2(\mathcal{O}_X)$ is a coboundary map associated with the following exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to N_{C/X} \to 0. \tag{3.9}$$

Proof of Claim 1: Consider the following diagram:

$$\begin{array}{ccc}
 & & \mathrm{Ext}^1_X(L \otimes E^*, E \otimes E^*) \\
 & & \downarrow{\scriptstyle \mathrm{tr}_*} \\
\mathrm{Ext}^1_X(N_{C/X}, \mathcal{O}_X) & \xrightarrow{\beta^*} & \mathrm{Ext}^1_X(L \otimes E^*, \mathcal{O}_X)
\end{array} \tag{3.10}$$

where tr_* is induced by the trace map $\mathrm{tr} : E \otimes E^* \to \mathcal{O}_X$, and β^* is induced by β. Since the trace map admits a splitting, tr_* is *surjective*. Let e_2 be the extension class of (3.9). Therefore there is an extension class $e_1 \in \mathrm{Ext}^1_X(K \otimes E^*, E \otimes E^*)$ such that $\mathrm{tr}_*(e_1) = \beta^*(e_2)$. Therefore there is an $\mathcal{O}_X$-module $\mathcal{F}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E\otimes E^* & \longrightarrow & \mathcal{F} & \longrightarrow & E^*\otimes L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{tr.}} & & \downarrow & & \downarrow{\scriptstyle \beta_*} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(C) & \longrightarrow & N_{C/X} & \longrightarrow & 0 \end{array} \tag{3.11}$$

where the first row of (3.11) gives the extension class e_1.

Now diagram (3.11) implies that:

$$\begin{array}{ccc} H^1(L\otimes E^*) & \xrightarrow{\beta_*} & H^1(N_{C/X}) \\ \downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \delta} \\ H^2(E\otimes E^*) & \xrightarrow{\mathrm{tr}^2} & H^2(\mathcal{O}_X) \end{array} \tag{3.12}$$

We see easily now that (3.6), (3.7) and (3.12) imply that for any $e\in H^1(E\otimes E^*)$:

$$\sigma(e_0, e) = \delta\circ g_{e_{0*}}\circ\gamma(e)\,. \tag{3.13}$$

Hence Claim 1 is proved.

We need one more claim to finish the proof.

Claim 2. If e belongs to $df_*(T_L V_d^r(C))\subset H^1(E\otimes E^*)$, *then* $\gamma(e)=0$.

Note that the above claim implies the proposition immediately thanks to Claim 1.

Proof of Claim 2. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F\otimes F^* & \to & H^0(L)\otimes F^* & \to & L\otimes F^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F\otimes E^* & \to & H^0(L)\otimes\mathcal{O}_C\otimes E^* & \to & L\otimes E^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F\otimes L^*\otimes N_{C/X} & \to & H^0(L)\otimes L^*\otimes N_{C/X} & \to & N_{C/X} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0\,. & & \end{array} \tag{3.14}$$

The above commutative diagram is deduced from sequences (2.4), (2.5) and (2.6).

Now the above commutative diagram implies that:

$$\begin{array}{ccc} H^0(L\otimes F^*) & \longrightarrow & H^1(F\otimes F^*) \\ \downarrow{\scriptstyle \alpha_L^*} & & \downarrow \\ H^0(L\otimes E^*) & \xrightarrow{\partial_1} & H^1(F\otimes E^*) \\ \downarrow & & \downarrow{\scriptstyle \beta_*} \\ H^0(N_{C/X}) & \longrightarrow & H^1(F\otimes L^*\otimes N_{C/X}) \end{array} \tag{3.15}$$

where α_L^* is induced by $\alpha : E \to F$. As we have seen in (3.3) that α_L^* can be identified as the differential of the embedding map $P_d^r(C) \hookrightarrow P_d^r(X, C)$ at $(L, \lambda) \in P_d^r(C)$. Let ∂_1 is the coboundary map $H^0(L \otimes E^*) \to H^1(F \otimes E^*)$. It is clear that ∂_1 factors through $H^1(E \otimes E^*)$, i.e., $\partial_1 = \alpha_* \circ d\widetilde{f}_*$. Hence, after composing with β_* and α_L^*, we get:

$$\beta_* \circ \partial_1 \circ \alpha_L^* = \beta_* \circ \alpha_* \circ d\widetilde{f}_* \circ \alpha_L^* .$$

Note that $\beta_* \circ \partial_1 \circ \alpha_L^* = 0$, and $d\widetilde{f}_* \circ \alpha_L^*$ is simply the differential of the map $\widetilde{f} : P_d^r(C) \to \mathcal{M}$ at L. Since $\widetilde{f}$ factors through $V_d^r(C)$, an element $e \in H^1(E \otimes E^*)$ belongs to $df_*(T_L V_d^r(C)) \subset H^1(E \otimes E^*)$ if and only if $\beta_* \circ \alpha_*(e) = 0$, i.e., $\gamma(e) = 0$. Hence Claim 2 is proved, so is the proposition. □

To prove Theorem 3.1 we need two additional lemmas.

Lemma 3.4. *There is a smooth curve $C_0 \in |C|$ and a smooth nonempty Zariski open subset $T_d^r(C_0) \subset V_d^r(C_0)$ such that for every $L_0 \in T_d^r(C_0)$, $df_* : T_{L_0} T_d^r(C_0) \to T_{E_0} f(T_d^r(C_0))$ is surjective, where $E_0 = f(C_0, L_0)$.*

Proof. By our assumption (ii) at the beginning of Sect. 2, $h^1(E^*) = h^2(E^*) = 0$. Therefore sequence (3.1) implies that the map $\partial_0 : H^1(L \otimes E^*) \to H^2(E \otimes E^*)$ is isomorphic. Since $\mathcal{M}$ is smooth at $[E]$, $P_d^r(X, C)$ is smooth at (C, L, λ) for any evaluation map λ by Lemma 2.2. The generic smoothness theorem [Ha, Corollary 10.7, Chap. III] for $\pi : P_d^r(X, C) \to \pi(P_d^r(X, C)) \subset |C|$ implies that there is a smooth curve $C_0 \in |C|$ such that $V_d^r(C_0)$ is generically smooth. Then the generic smoothness theorem for $f : V_d^r(C_0) \to f(V_d^r(C_0)) \subset \mathcal{M}$ implies the lemma immediately. □

Proof of Theorem 3.1. Theorem 3.1 is an immediate consequence of Proposition 3.3 and Lemma 3.4. □

Proof of Theorem 3.2. Sequence (2.6) for any pair (C, L) induces the following short exact sequence:

$$0 \longrightarrow L \otimes F^* \longrightarrow L \otimes E^* \longrightarrow N_{C/X} \longrightarrow 0 . \tag{3.16}$$

The associated homomorphism $H^0(L \otimes E^*) \to H^0(N_{C/X})$ is nothing but the differential of π (see (2.7)) at the point (C, L, λ), where $\lambda : H^0(L) \otimes \mathcal{O}_X \to L$ is an evaluation map.

Since $\mathcal{M}$ is generically smooth, so does $V_d^r(X, C)$ as we proved in Lemma 3.4. By Generic Smoothness Theorem, co-dimension of $\pi(V_d^r(X, C))$ is the same as the co-dimension of the image of $d\pi_*|_{(C_0, L_0)}$ for a generic point $(C_0, L_0) \in V_d^r(X, C)$. Let $F_0 = F(L_0)$, and $E_0 = E(C_0, L_0)$. However the codimension of $d\pi_*|_{(C_0, L_0)}$ is:

$$\text{codimension of } \operatorname{im}(d\pi_*|_{(C_0, L_0)}) = h^1(L_0 \otimes F_0^*) + h^1(N_{C/X}) - h^1(L \otimes E_0^*)$$

by the exactness of the corresponding sequence (3.16) for (C_0, L_0).

By condition (ii) at the beginning of Sect. 2, $h^1(E_0^*) = h^2(E_0^*) = 0$. Now sequence (3.1) for (C_0, L_0) implies that $h^1(L_0 \otimes E_0^*) = h^2(E_0 \otimes E_0^*) = h^2(\text{ad } E_0) + h^2(\mathcal{O}_X)$. By the condition (iii) in Sect. 2, $h^1(\mathcal{O}_X(C_0)) = 0$. Now sequence (3.9) for the curve C_0 gives $h^2(\mathcal{O}_X) = h^1(N_{C_0/X})$. By the way F_0 is defined (see the corresponding exact sequence (2.4) for L_0), it is clear that

$$\ker \mu_{L_0} \cong H^0(F_0 \otimes L_0^* \otimes \omega_{C_0}) \cong H^1(L_0 \otimes F_0^*)^*$$

where μ_{L_0} is the Petri's map for L_0. Therefore the codimension $\pi(V_d^r(X,C))$ is $\dim\ker(\mu_{L_0}) - h^2(\text{ad } E_0)$ for a generic point (C_0, L_0). This implies the first part of Theorem 3.2.

Let us now prove the second part of Theorem 3.2. When $\varrho \geqq 0$, the morphism $\pi : V_d^r(X,C) \to |C|$ is dominant by the existence theorem of Kleinman and Laksov [KL]. By what we just prove, for a generic $(C_0, L_0) \in V_d^r(X,C), \dim\ker(\mu_{L_0}) = h^2(\text{ad } E_0)$. Note that $W_d^r(C_0)$ is smooth of dimension ϱ around L_0 if and only if $\ker(\mu_{L_0}) = 0$ (see [ACGH p. 198]). Therefore if $\varrho \geqq 0$, then $\mathscr{M}$ is *good* if and only if there is a point $(C_0, L_0) \in V_d^r(X,C)$ such that $W_d^r(C_0)$ is smooth of dimension ϱ around L_0.

Finally, we have to show that when $\varrho \geqq 0$, $f|_{T_d^r(C_0)} \to \mathscr{M}$ is etale. By shrinking $T_d^r(C_0)$ if necessary, this is equivalent to showing that the differential $df_*|_{T_d^r(C_0)}$ is an isomorphic. By Lemma 3.4, we can make $df_*|_{T_d^r(C_0)}$ surjective. Let $L_0 \in T_d^r(C_0)$ be an arbitrary closed point. Denote by $df_*|_{L_0}$ the restriction of $df_*|_{T_d^r(C_0)}$ to the point $L_0 \in T_d^r(C_0)$. Note that $df_*|_{L_0}$ is nothing but the Kodaira-Spencer map $\kappa_{L_0} : T_{L_0}W_d^r(C_0) \to H^1(E_0 \otimes E_0^*)$. This map was originally introduced by Tyurin [T1] when he studied propagation problem on a K3 surface. As we saw in formula (5.34) of [Y] that

$$\dim\ker(df_*|_{L_0}) = h^0(F_0 \otimes L_0^* \otimes N_{C_0/X}) + 1 - h^0(E_0 \otimes E_0^*)\,. \tag{3.17}$$

We should point out that (5.34) of [Y] was proved for a K3 surface. But the same is true for any regular surface X if we replace ω_C by $N_{C/X}$.

Since E is stable, hence it is simple, i.e., $h^0(E_0 \otimes E_0^*) = 1$. Therefore (3.17) implies that $\dim\ker(df_*|_{L_0}) = h^0(F_0 \otimes L_0^* \otimes N_{C_0/X})$.

Since $p_g > 0$, $h^0(F_0 \otimes L_0^* \otimes N_{C_0/X}) \leqq h^0(F_0 \otimes L_0^* \otimes \omega_{C_0}) = \dim\ker(\mu_{L_0})$. Therefore

$$\dim\ker(df_*|_{L_0}) \leqq \dim\ker(\mu_{L_0}) = 0\,. \tag{3.18}$$

Hence $df_*|_{L_0}$ is also injective. Therefore $df_*|_{L_0}$ is in fact isomorphic. Since L_0 is an arbitrary point in $T_d^r(C_0)$, $df_*|_{T_d^r(C_0)}$ is isomorphic. Hence $f|_{T_d^r(C_0)}$ is etale. This finishes the proof of the second of Theorem 3.2. Hence we are done. □

Remarks. (1) We have shown above that if $\varrho \geqq 0$, then f is etále on $T_d^r(C_0)$. We would hope that it is actually *isomorphic*. We have no idea how to prove or disprove it.

(2) In fact, what we have done above can be generalized to moduli space of *simple* vector bundles. Even though for simple vector bundles there is no boundness result, Theorem 3.1 and Theorem 3.2 still hold locally around a point in the moduli space.

(3) Tyurin [T2] also studied *Poisson structures* on the moduli space of simple vector bundles on a regular surface X with $h^0(-K_X) > 0$. We believe that we can get a result similar to Theorem 3.1 about the Poisson structures using the same techniques in this paper.

(4) It is known [M2] that if gcd $(r+1, C\cdot D, g+r-d) = 1$, the moduli space $\mathscr{M}$ is actually compact. However, the Lagrangian subvarieties constructed in this paper are not compact due to the facts that not every (C', L') gives a stable bundle and not every g_d^r is base-point-free. It would be very interesting to know how to extend the map f to those points (C', L') where either L' is not free, or $E(C', L')$ is not stable with respect to the fixed polarization D.

(5) A question related to (4) is under what conditions, a given pair (C, L), i.e., a smooth curve on X together with a base-point free g^r_d, gives a stable bundle $E(C, L)$ on X with respect to a fixed polarization D. Lazarsfeld [L2] gives a sufficient conditions for $E(C, L)$ to be simple, namely, $|C|$ contains no reducible member. Obvious, there are some inequalities involving Chern numbers of $E(C, L)$ have to be satisfied.

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